Hecke operators for even unimodular lattices

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(joint work with Gaëtan Chenevier)

An even unimodular lattice of dimension n is a free $\mathbb{Z}$-module L of dimension n, equipped with a quadratic form $\mathrm{q} : L \to \mathbb{Z}$, non-degenerate over $\mathbb{Z}$ (*i.e.* such that the associated bilinear form induces an isomorphism $L \cong \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$) and positive definite. Such an L can be thought of as a lattice in the euclidean vector space $V := \mathbb{R} \otimes_{\mathbb{Z}} L$, satisfying the two following properties:

– one has $x.x \in 2\mathbb{Z}$ for all x in L (hence $x.y \in \mathbb{Z}$ for all x and y in L);

– the lattice L has covolume 1 in V.

This observation explains the terminology.

One denotes by X_n the set of isomorphism classes of n-dimensional even unimodular lattices. It is well-known that the set X_n is finite and that it is non-empty if and only if n is divisible by 8. The set X_n has been determined for $n \leq 24$:

– X_8 has only one element E_8 (related to the root system $\mathbf{E}_8$);

– X_{16} has two elements E_{16} (related to the root system $\mathbf{D}_{16}$) and $\mathrm{E}_8 \oplus \mathrm{E}_8$;

– X_{24} was determined by Niemeier in 1968, it has 24 element (one of them is the famous Leech lattice, the 23 other ones are again related to root systems).

One shows that X_{32} has more than 8×10^7 elements (Serre says playfully in his *cours d'arithmétique* that they have not been listed!).

Let p be a prime. Let V be an euclidean vector space of dimension n. Two even unimodular lattices L and L' in V are said to be p-neighbours (in the sense of M. Kneser) if $L \cap L'$ is of index p in L (and L').

In this context, the Hecke operator T_p is the endomorphism of the free $\mathbb{Z}$-module $\mathbb{Z}[\mathrm{X}_n]$, generated by the set X_n, defined by the formula

$$\mathrm{T}_p[L] = \sum_{L' \ p\text{-neighbour of } L} [L'] \qquad .$$

Theorem. *The matrix of the endomorphism* T_p *in the basis* $(\mathrm{E}_{16}, \mathrm{E}_8 \oplus \mathrm{E}_8)$ *is*

$$\frac{p^4-1}{p-1} \; (\, (p^{11}+p^7+p^4+1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{p^{11}-\tau(p)+1}{691} \begin{bmatrix} -286 & 405 \\ 286 & -405 \end{bmatrix}) \qquad ,$$

τ *denoting the Ramanujan function.*

One will explain how to prove this theorem using the theory of Hecke operators for Siegel modular forms and one will describe the ingredients involved in the analog of the above formula in dimension 24.