Seminario Matematico e Fisico di Milano :

The constant scalar curvature equation in some singular spaces.

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Plane of the talk

- The scalar curvature

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- Uniformisation of the scalar curvature : the Yamabe problem

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- Perspectives

The scalar curvature : geometric point of view

If (M^n, g) is a Riemannian manifold, then the scalar curvature

$$\mathrm{Scal}_g \colon M \to \mathbb{R}$$

measures a deviation of the metric to be Euclidean :

$$\mathrm{vol}_g B(x,r) = \mathrm{vol}\,\mathbb{B}^n(r)\,\left(1 - \frac{1}{6(n+2)}\mathrm{Scal}_g(x) r^2 + \mathcal{O}\left(r^4\right)\right)$$

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where

$$\mathrm{vol}\,\mathbb{B}^n(r) = \omega_n r^n$$

is the volume of an Euclidean ball of radius r.

The scalar curvature : 2D

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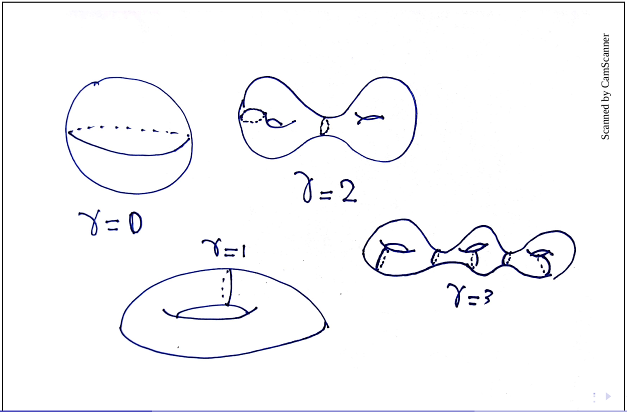
We have the Gauss-Bonnet theorem :

$$\int_M \mathrm{Scal}_g = 4\pi\chi(M) = 8\pi(1-\gamma)$$



The scalar curvature : 2D-Uniformisation

And by the uniformisation theorem of Koebe and Poincaré, there is always a conformal factor $f \in \mathcal{C}^\infty(M)$ such that for $\tilde{\mathbf{g}} = e^{2f} g$:

$$\mathrm{Scal}_{\tilde{\mathbf{g}}} = \mathrm{constant}.$$

The scalar curvature : PDE point of view

In higher dimension, the scalar curvature gives only few information on the topology.

The scalar curvature : PDE point of view

But we have a nice transformation rule under conformal change of the scalar curvature. If $\dim M \geq 3$ and $\tilde{\mathbf{g}} = u^{\frac{4}{n-2}} g$ then

$$\frac{4(n-1)}{n-2}\Delta_g u + \mathrm{Scal}_g u = \mathrm{Scal}_{\tilde{\mathbf{g}}}\, u^{\frac{n+2}{n-2}}.$$

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Where in coordinates $g = \sum g_{i,j} dx_i dx_j$ and $\left(g^{i,j}\right) = \left(g_{i,j}\right)^{-1}$ and $\Theta = \det\left(g_{i,j}\right)$:

$$\Delta_g u = -\frac{1}{\sqrt{\Theta}} \sum_{i,j} \frac{\partial}{\partial x_j}\left(\sqrt{\Theta}\, g^{i,j} \frac{\partial u}{\partial x_i}\right)$$

$$\Delta_g u = -\sum_{i,j} g^{i,j} \frac{\partial^2 u}{\partial x_j \partial x_i} + l.o.t.$$

The scalar curvature : PDE point of view

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In particular on $(\mathbb{R}^n, \mathrm{eucl})$:

$$\Delta_{\mathrm{eucl}} u = -\sum_i \frac{\partial^2 u}{\partial x_i^2}$$

The scalar curvature : PDE point of view

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Another interpretation of the Laplacian : $\forall \varphi \in \mathcal{C}^\infty(M)$:

$$\int_M \varphi\, \Delta_g\, \varphi \mathrm{dvol}_g = \int_M |d\varphi|_g^2\, \mathrm{dvol}_g$$

The scalar curvature : PDE point of view

Hence finding a conformal metric $\tilde{\mathbf{g}} = u^{\frac{4}{n-2}} g$ with constant scalar curvature amounts to find $u \in \mathcal{C}^\infty(M)$, $u > 0$ solution of the non linear PDE :

$$\boxed{\frac{4(n-1)}{n-2}\Delta_g u + \mathrm{Scal}_g u = \mathrm{const}\, u^{\frac{n+2}{n-2}}.}$$

The scalar curvature : Variational point of view

It turns out that this equation has a variational formulation :
u solves the equation $\frac{4(n-1)}{n-2}\Delta_g u + \mathrm{Scal}_g u = \mathrm{const}\, u^{\frac{n+2}{n-2}}$.

if and only if u is a critical point of the functional :

$$u \mapsto \mathcal{Q}_g(u) := \frac{\int_M \frac{4(n-1)}{n-2}|du|_g^2 + \mathrm{Scal}_g u^2}{\left(\int_M u^{\frac{2n}{n-2}}\right)^{1-\frac{2}{n}}}$$

We also have : if $\tilde{\mathbf{g}} = u^{\frac{4}{n-2}} g$

$$\mathcal{Q}_g(u) = \frac{1}{(\mathrm{vol}(M, \tilde{\mathbf{g}}))^{1-\frac{2}{n}}} \int_M \mathrm{Scal}_{\tilde{\mathbf{g}}}\, \mathrm{dvol}_{\tilde{\mathbf{g}}}$$

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This is the Einstein-Hilbert functional.

The scalar curvature : Variational point of view

As a particular critical point, one can look for a minimum of this functional $\mathcal{Q}_g$ this is the Yamabe problem.

The Yamabe problem

We introduce :

Yamabe invariant

$$Y(M,[g]) = \inf_{u\neq 0} \mathcal{Q}_g(u)$$

It is conformal invariant, it is associated to the conformal class

$$[g] = \left\{ \mathbf{\tilde{g}} = u^{\frac{4}{n-2}} g;\ \ u \in \mathcal{C}^\infty(M), u > 0 \right\}$$

We also have

$$Y(M,[g]) = \inf_{\mathbf{\tilde{g}}\in[g],\mathrm{vol}(M,\mathbf{\tilde{g}})=1} \int_M \mathrm{Scal}_{\mathbf{\tilde{g}}}\, \mathrm{dvol}_{\mathbf{\tilde{g}}}$$

The Yamabe problem

The Yamabe problem is to find a $u \in \mathcal{C}^\infty(M), u > 0$ that realizes $Y(M, [g])$:

$$Y(M, [g]) = \mathcal{Q}_g(u)$$

then necessary the metric $\tilde{\mathbf{g}} = u^{\frac{4}{n-2}} g$ has constant scalar curvature.

The Yamabe problem

This problem has been solved by Yamabe, Trudinger, Aubin and Schoen. As an abstract of the story, we have

- (Aubin, 1974) : we always have

$$Y(M, [g]) \leq Y(\mathbb{S}^n, [\mathrm{rounded}]).$$

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Hence there is some $u \in \mathcal{C}^\infty(M), u > 0$ such that $\tilde{\mathbf{g}} = u^{\frac{4}{n-2}} g = \mathrm{rounded}$ hence hjas constant scalar curvature.

Some stratified space : the surface of a cube

We are looking for the geometry of the surface of a cube :

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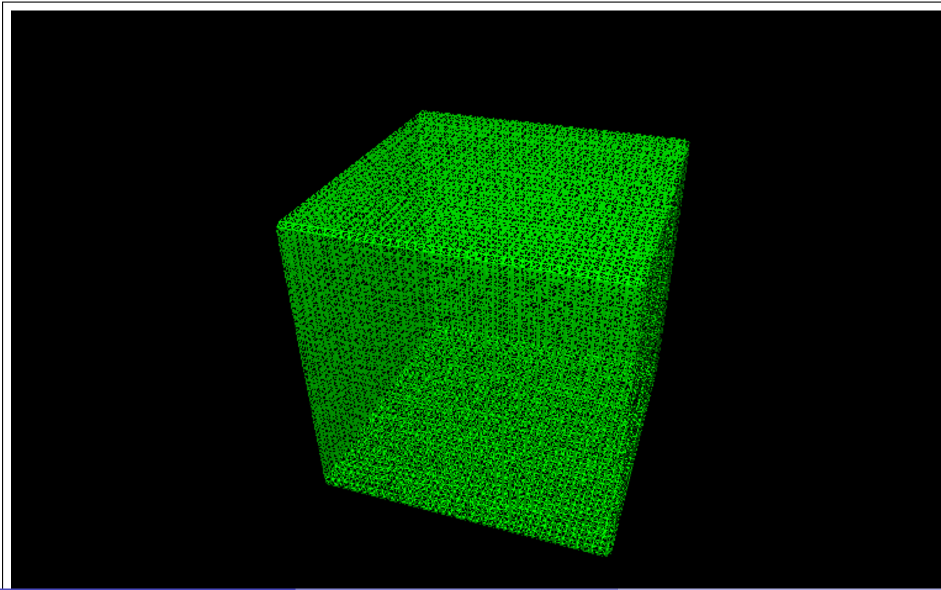

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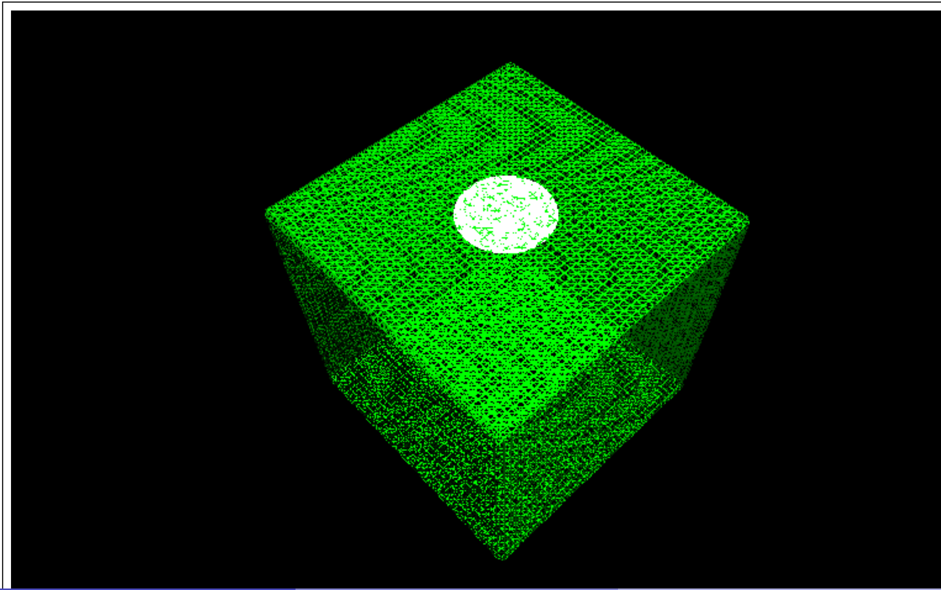

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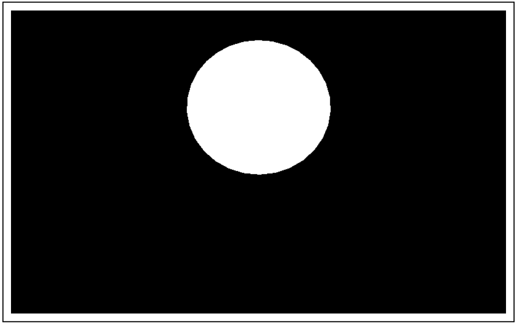

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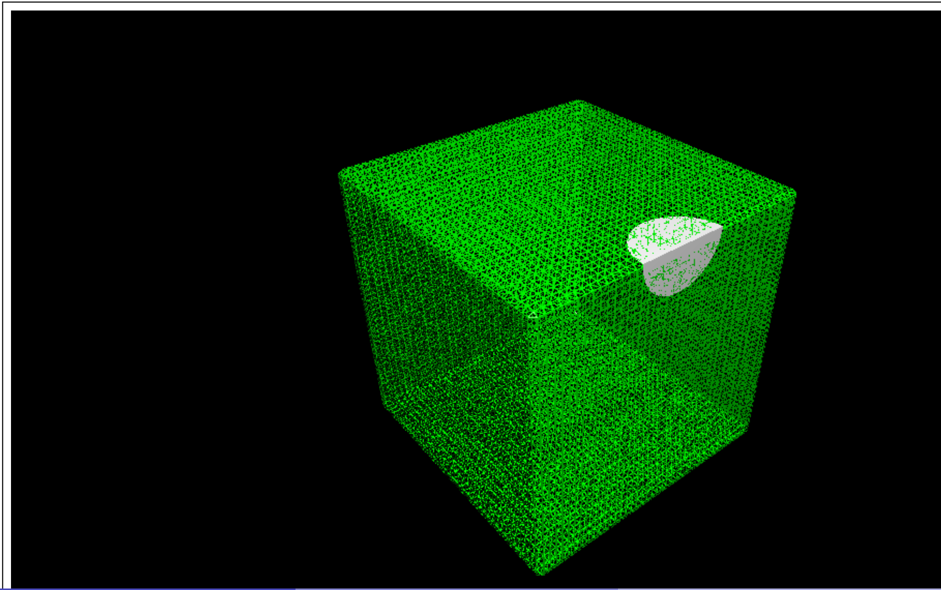

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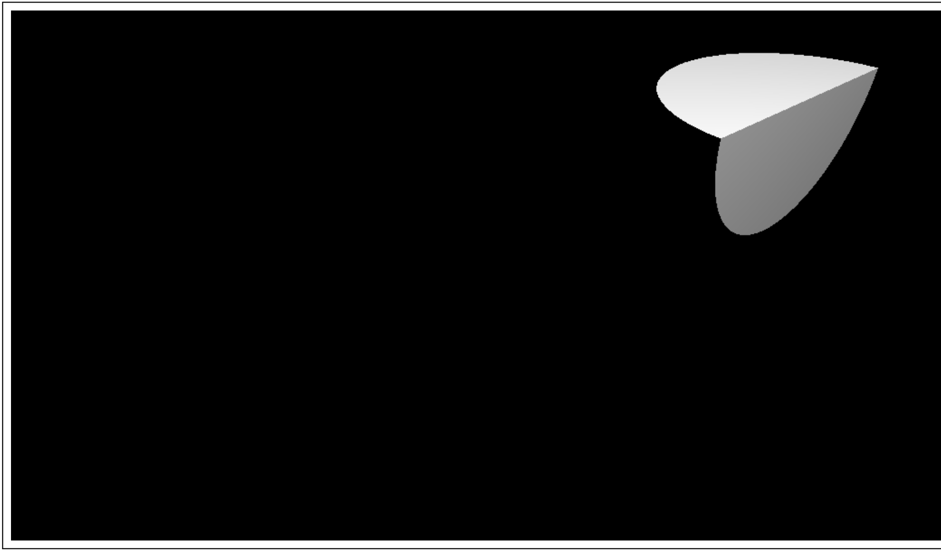

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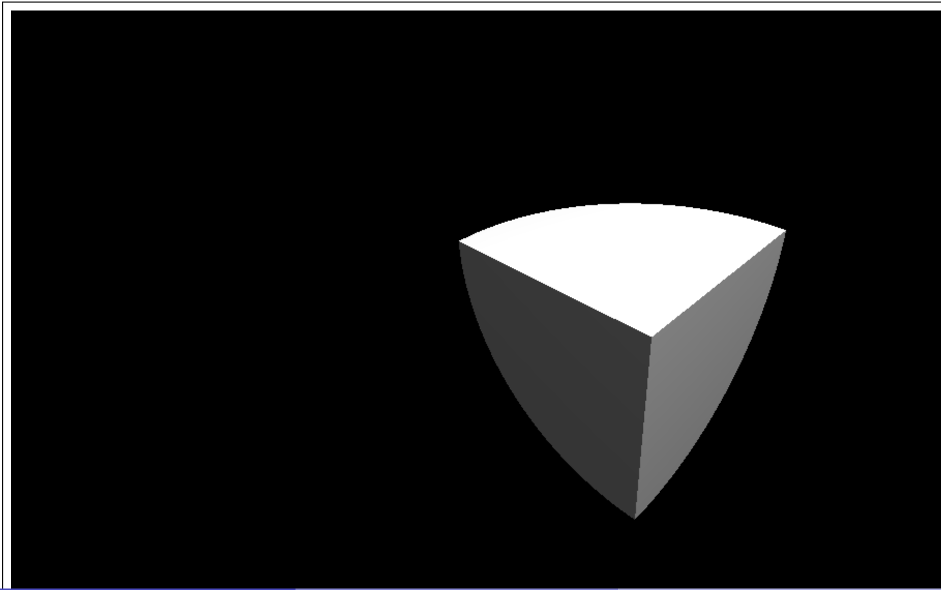
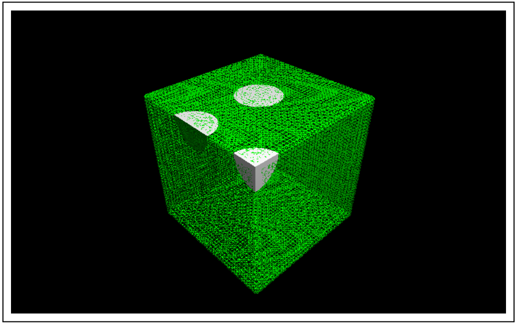
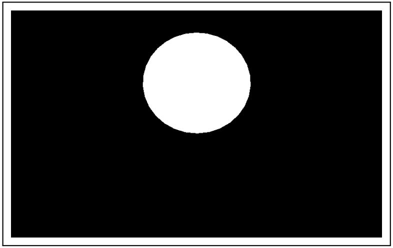




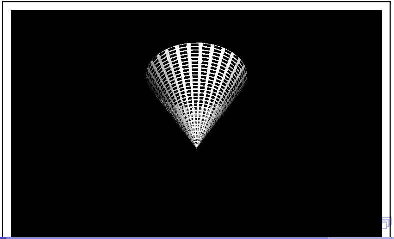
Some stratified space : the surface of a cube

Summary : the surface of a cube has a decomposition $X \supset X_0$, where

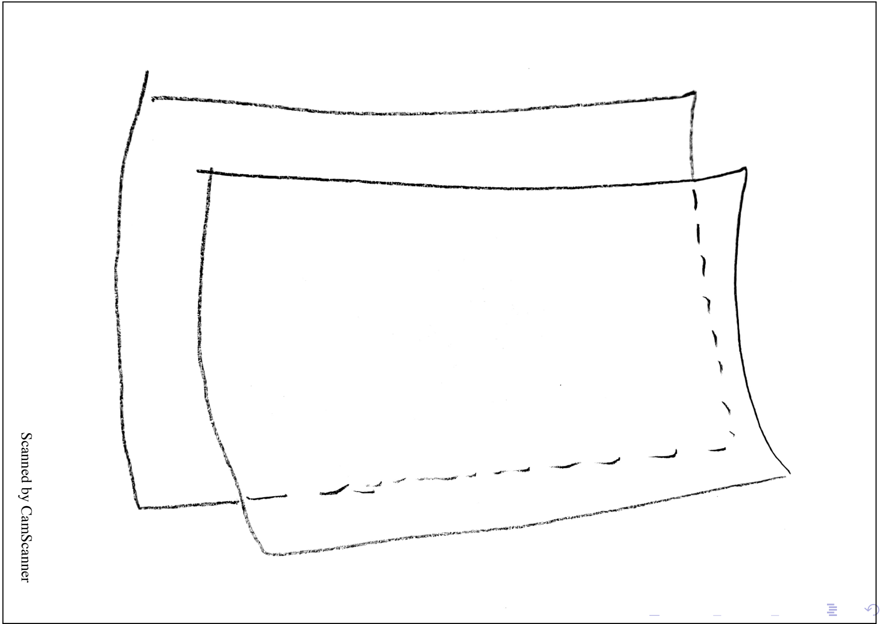
- near each point of $X \setminus X_0$, the geometry is Euclidean



- X_0 is the collection of 8 vertex and near each of these point the geometry is a cone over a circle of length $3\pi/2$



Some stratified space : the surface of a pillow



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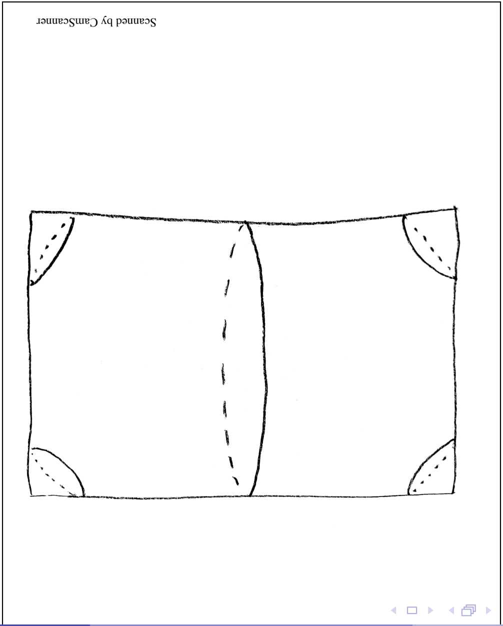

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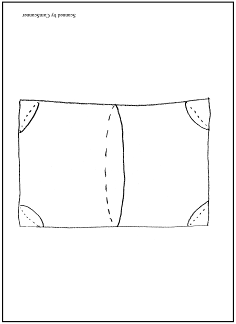

Summary : the surface of a pillow has a decomposition $X \supset X_0$, where

- near each point of $X \setminus X_0$, the geometry is Euclidean
- X_0 is the collection of 4 vertex and near each of these point the geometry is a cone over a circle of length π

Some stratified space : the surface of a *berlingot*

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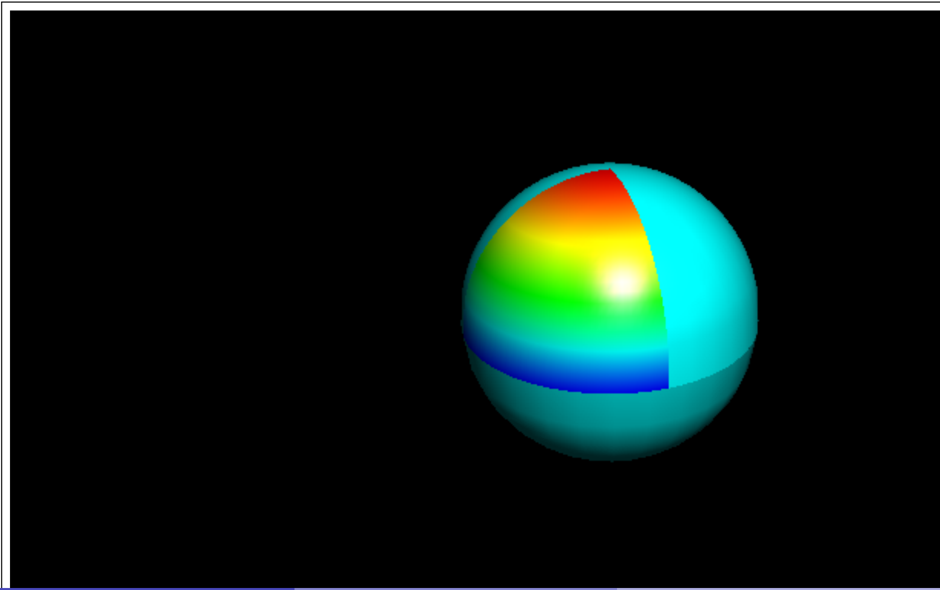

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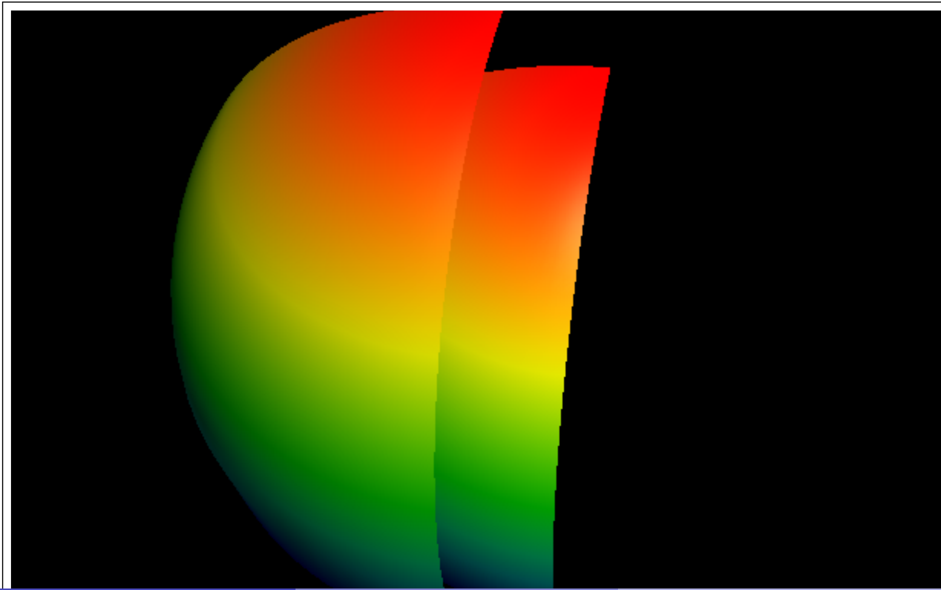

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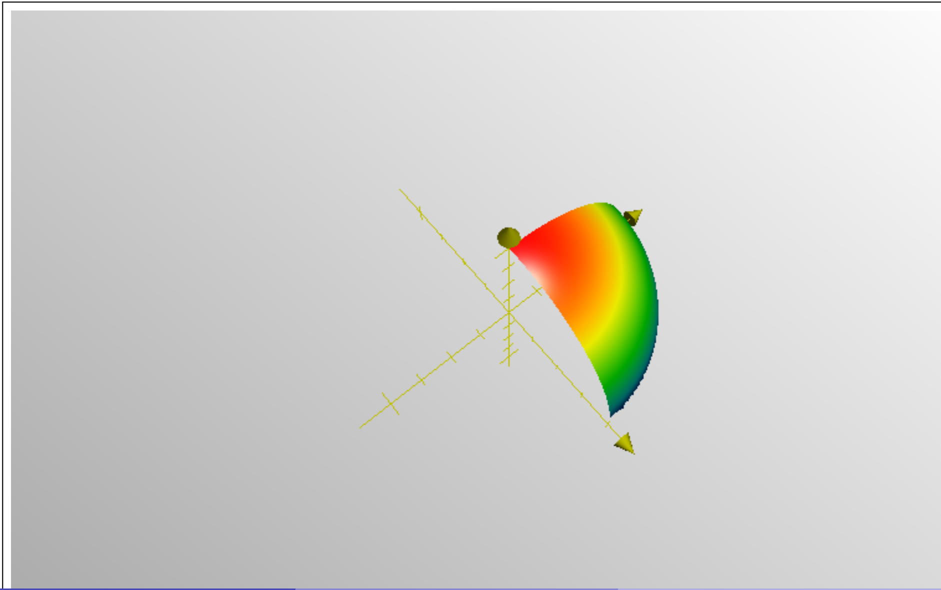

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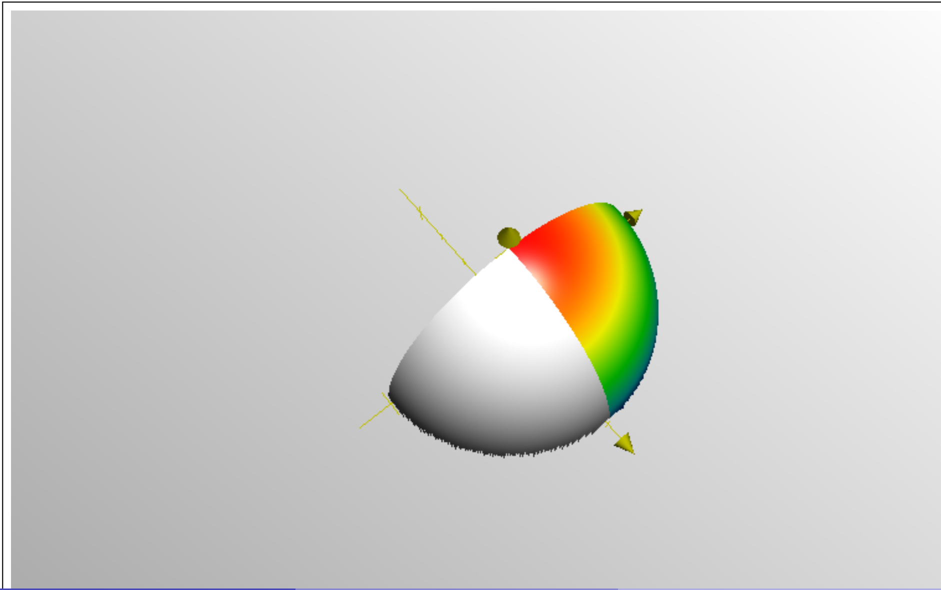

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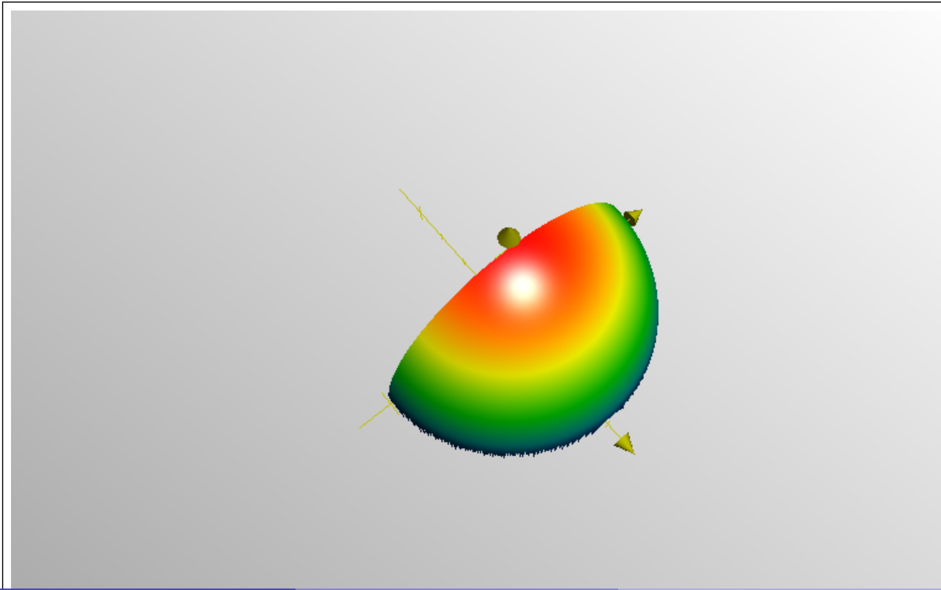

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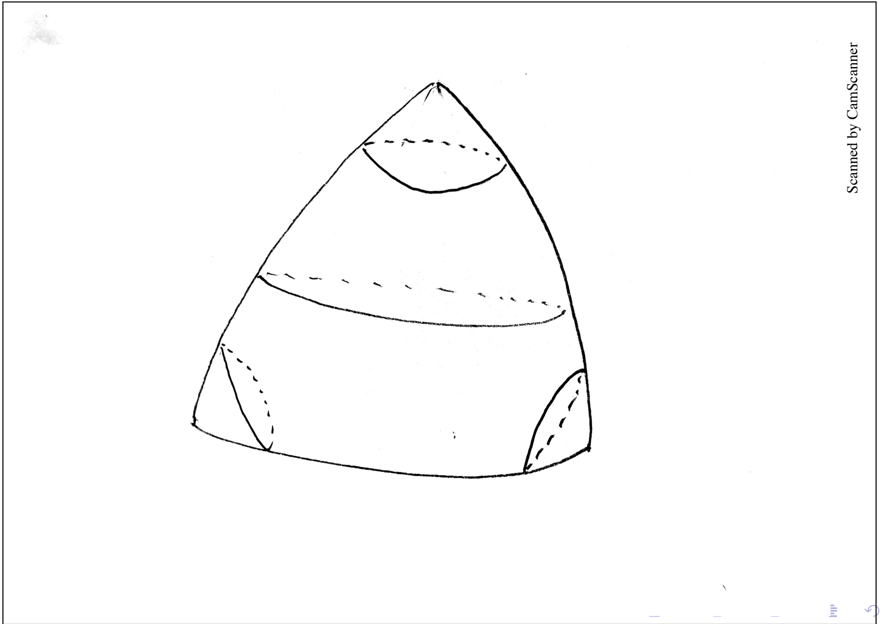

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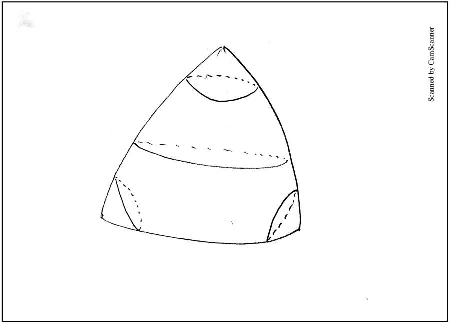

Summary : the surface of a berlingot has a decomposition $X \supset X_0$, where

- near each point of $X \setminus X_0$, the geometry is Euclidean
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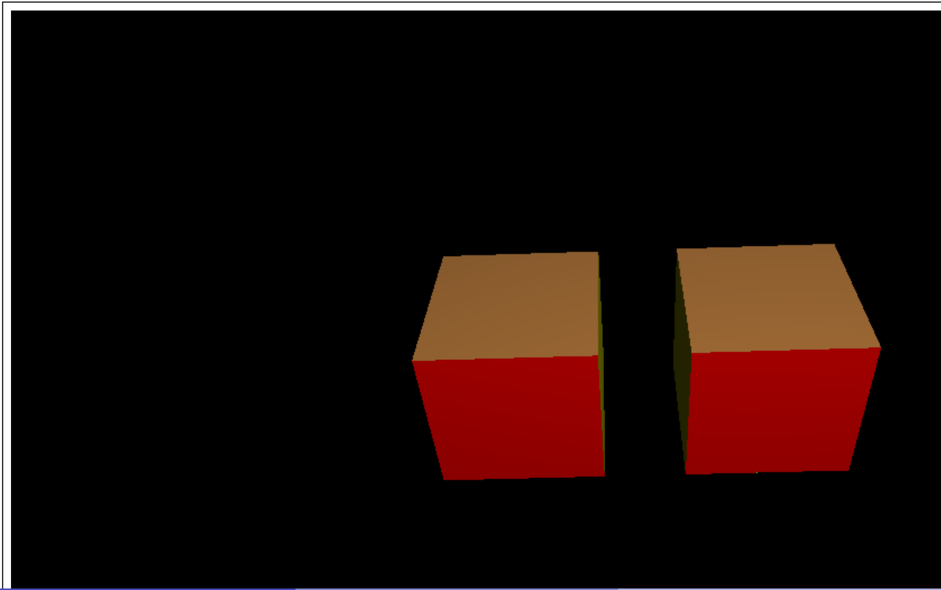
Some stratified space : the double solid cube



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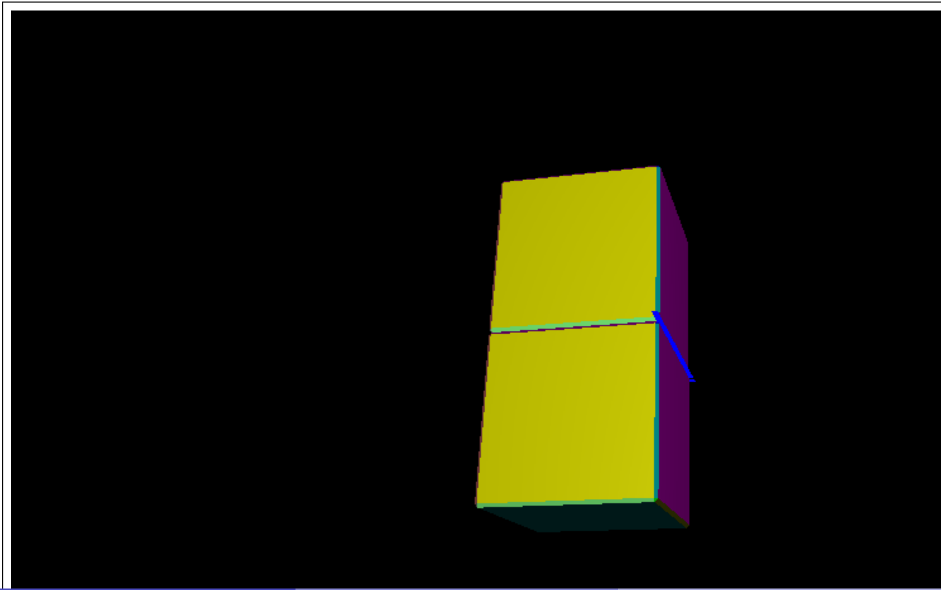

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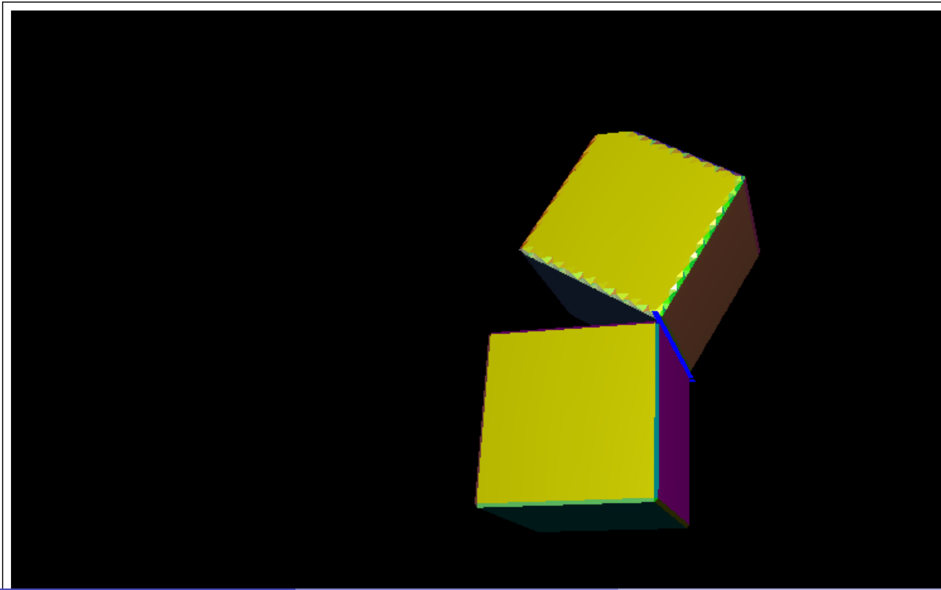

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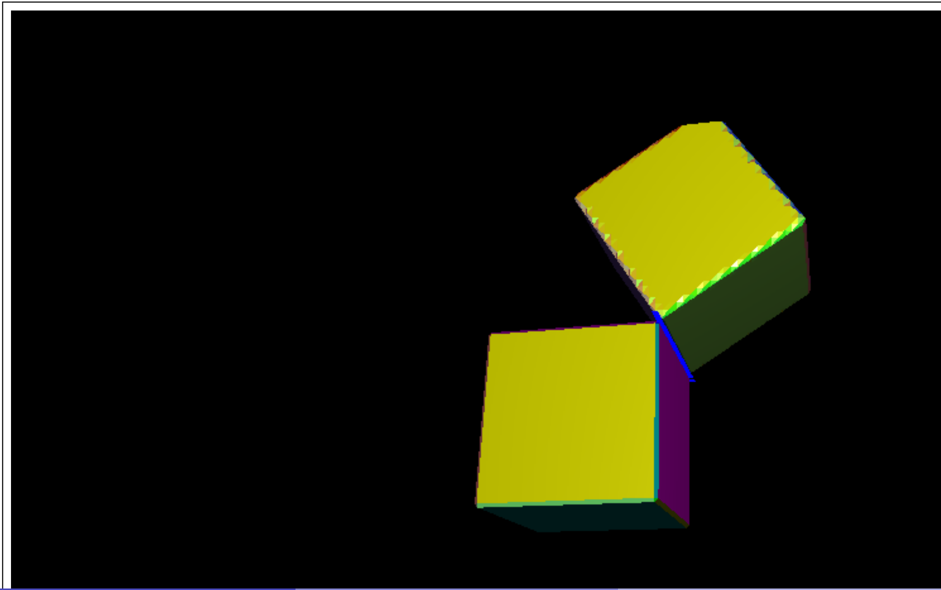

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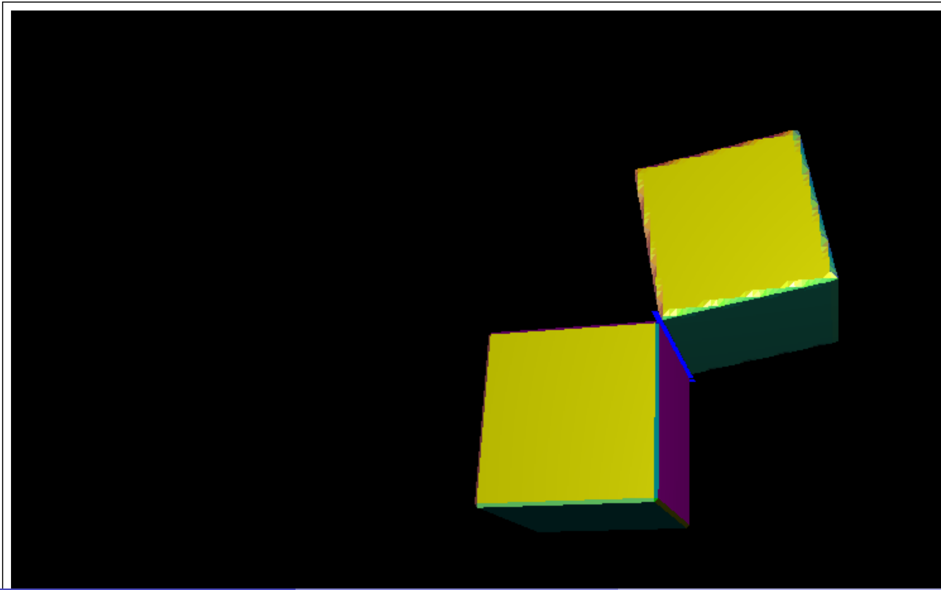

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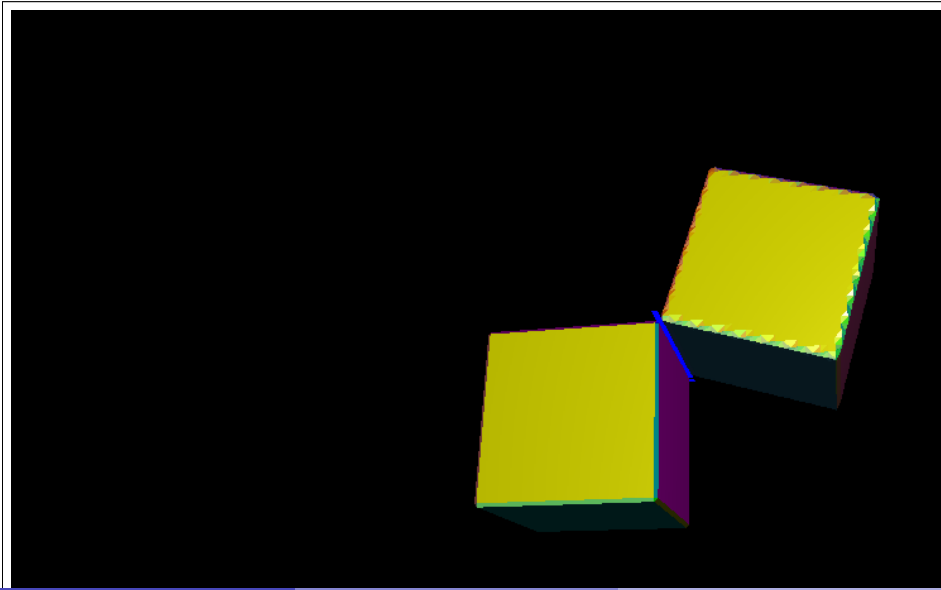

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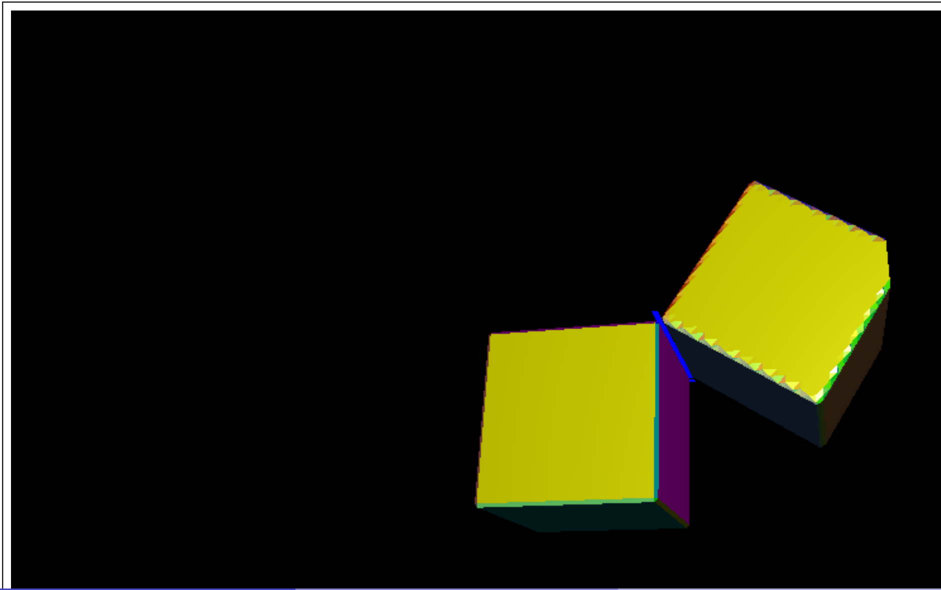

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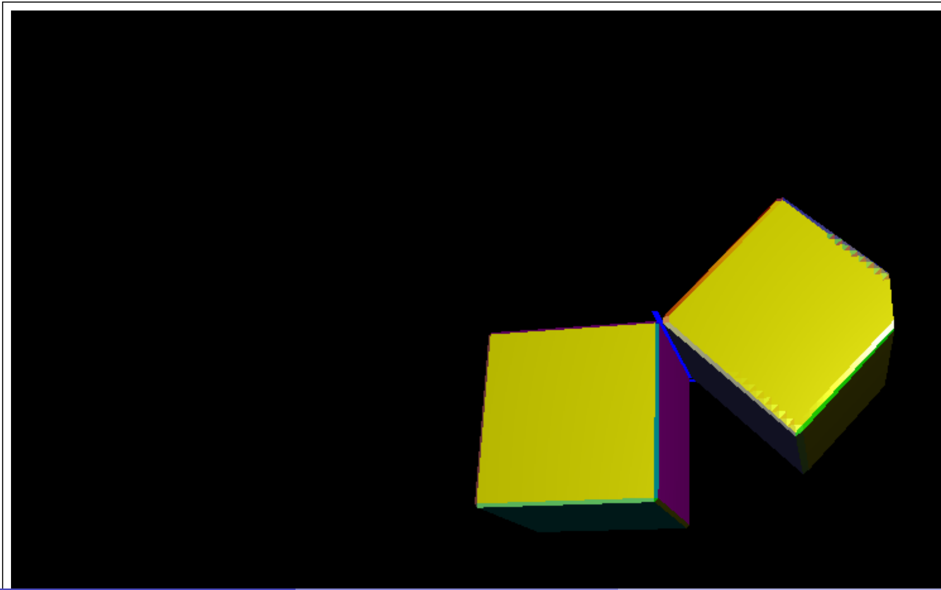

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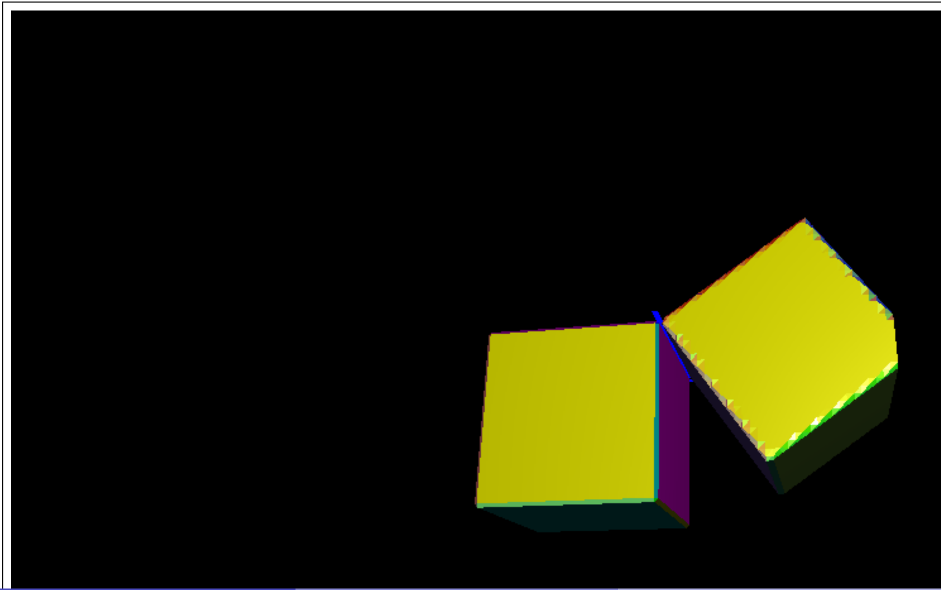

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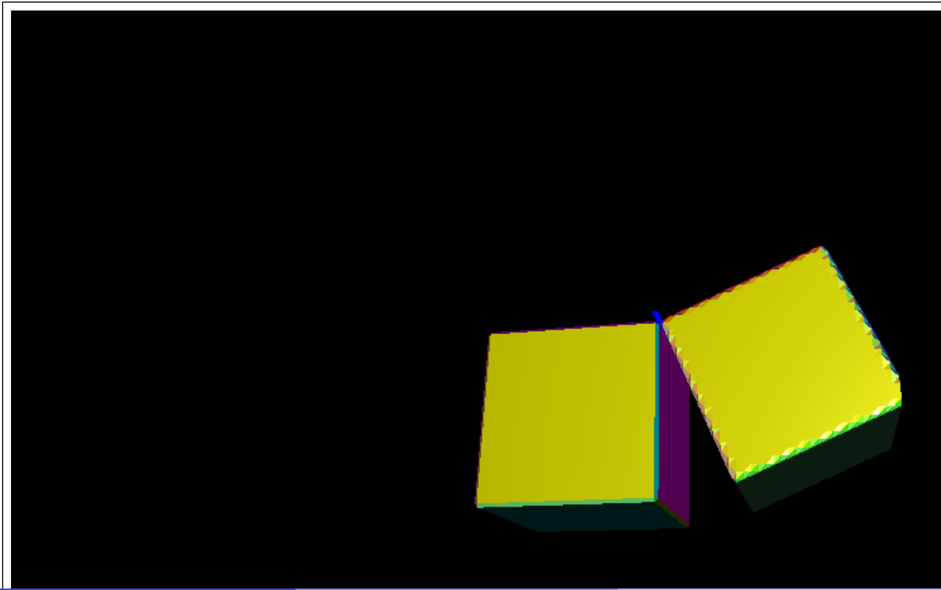

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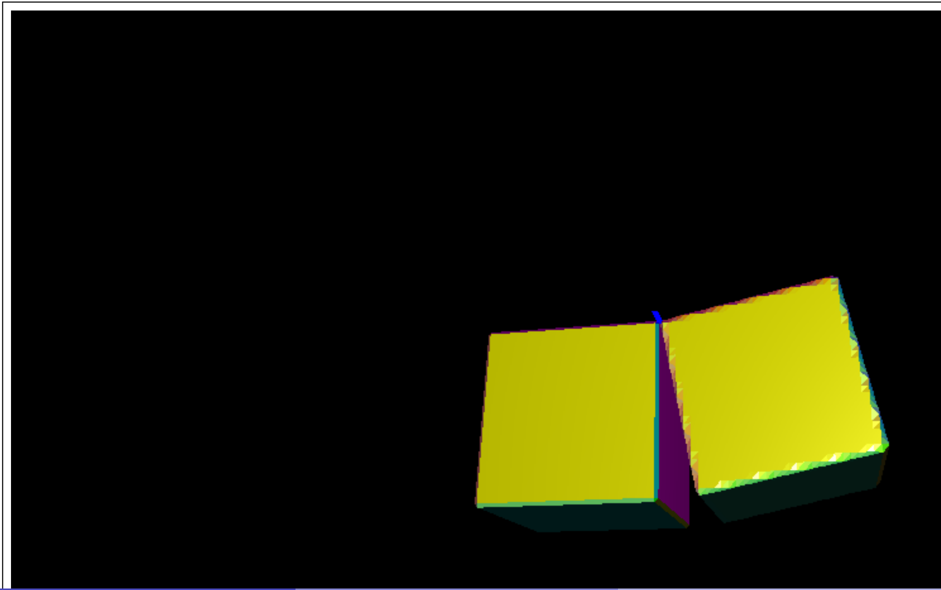

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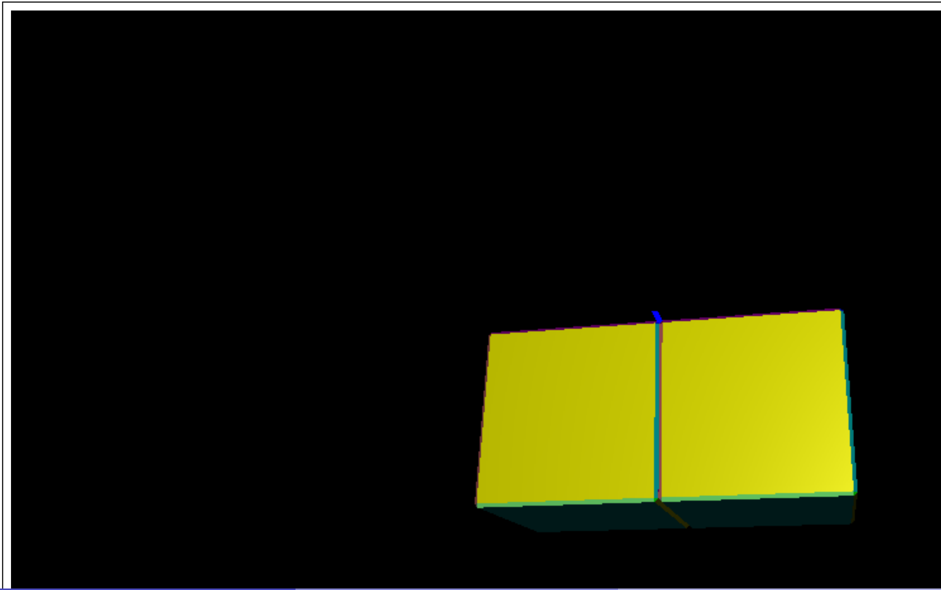

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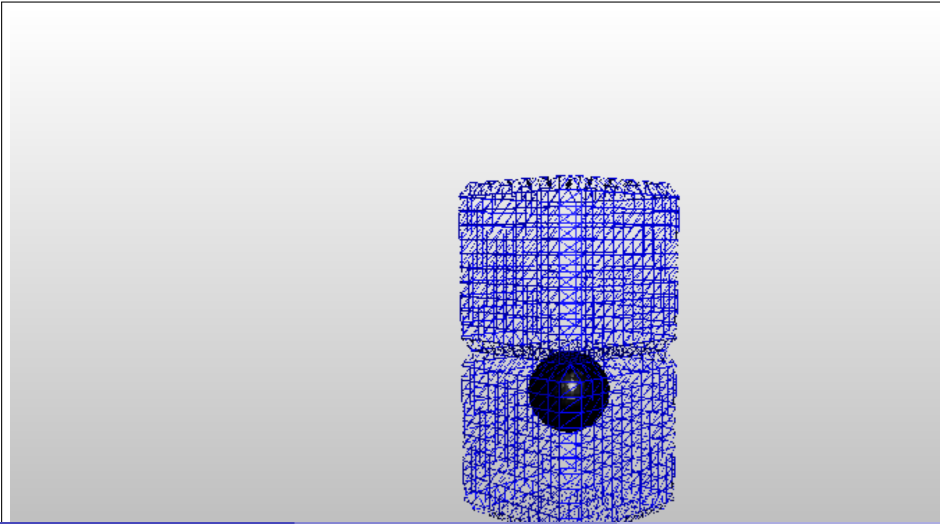
The geometry of the double solid cube is the following : at a point interior or on a face of the cube, the geometry is Euclidean.



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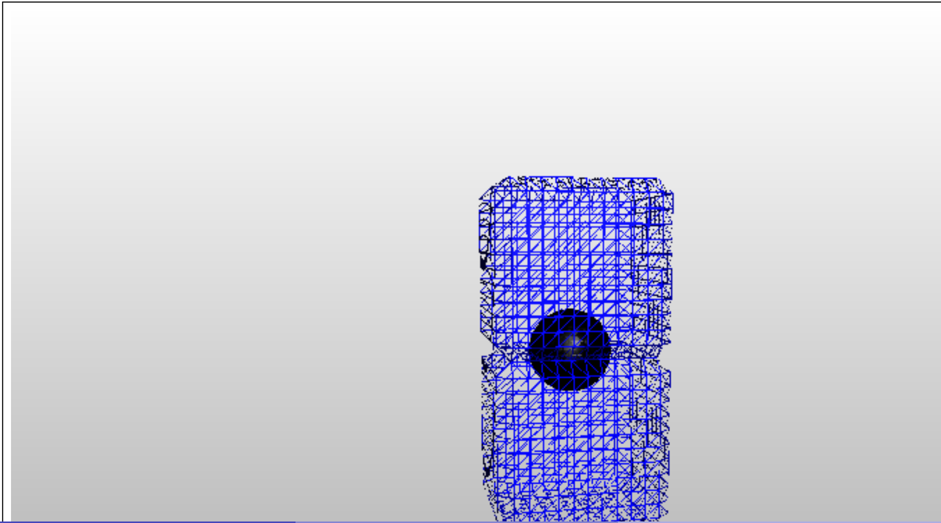

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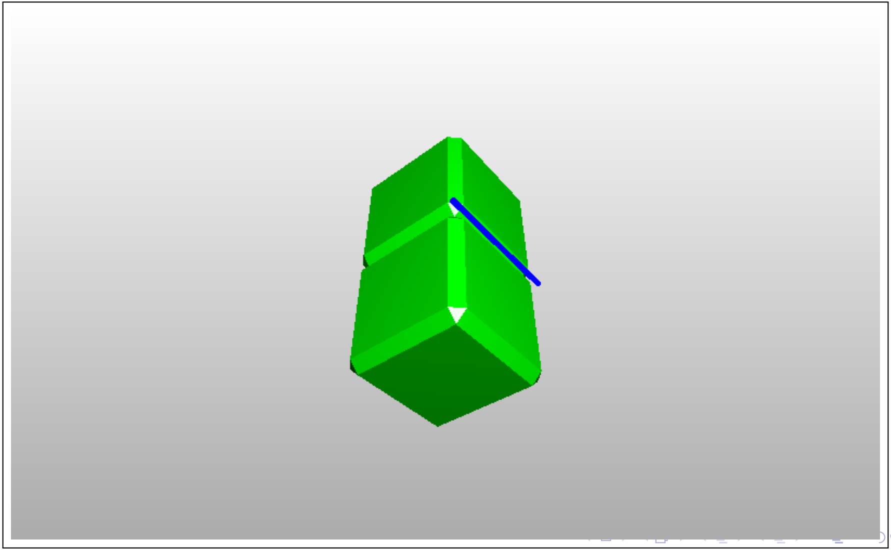

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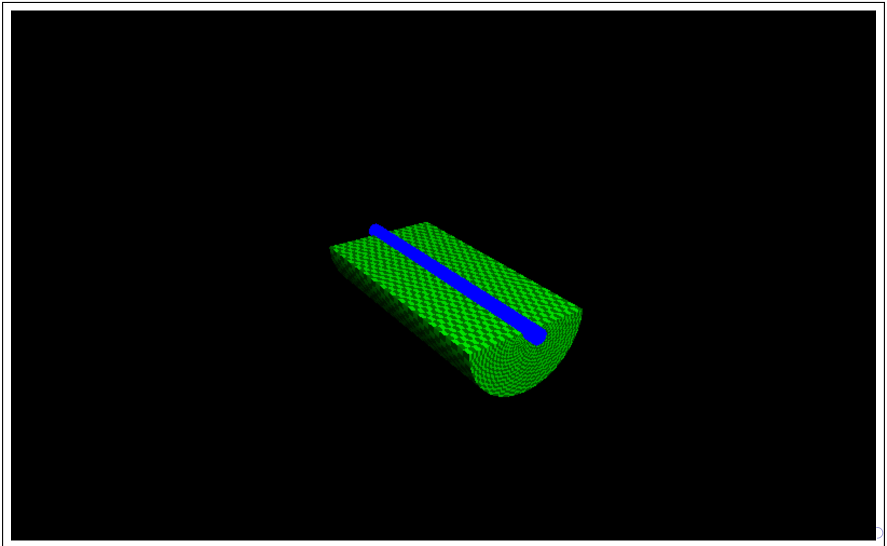

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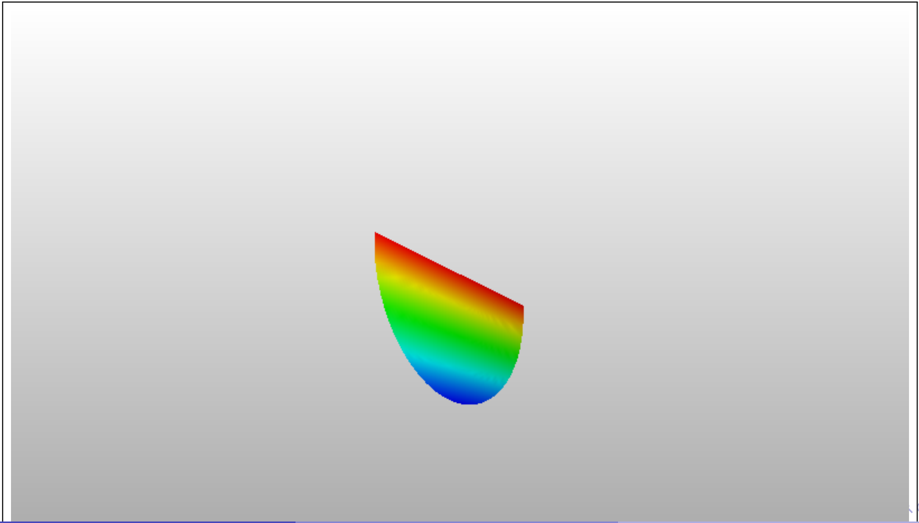

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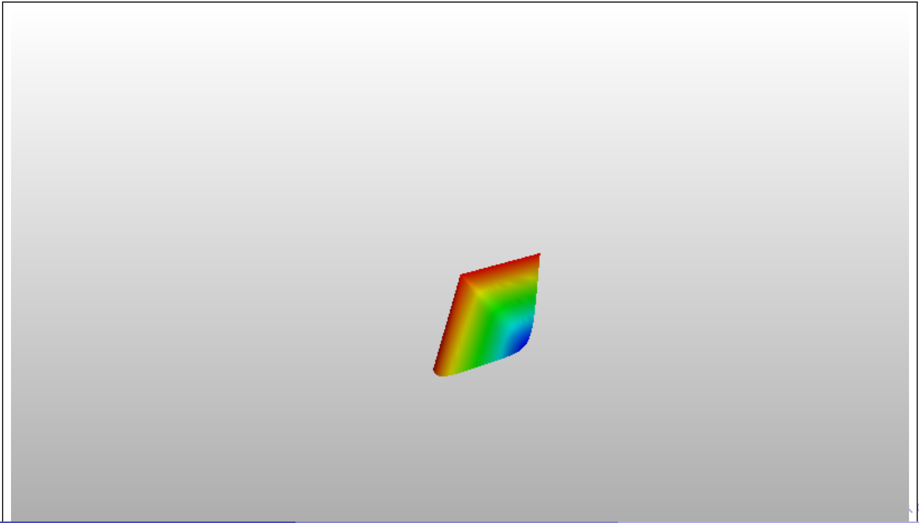

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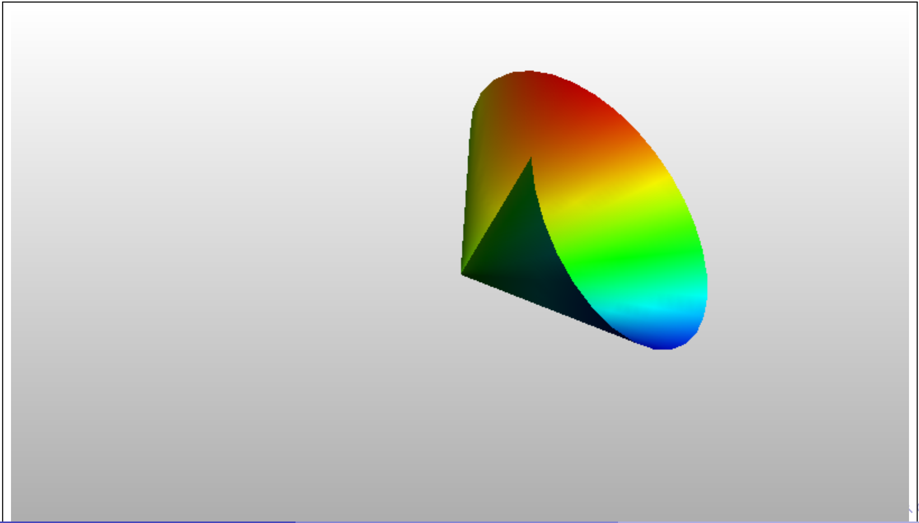

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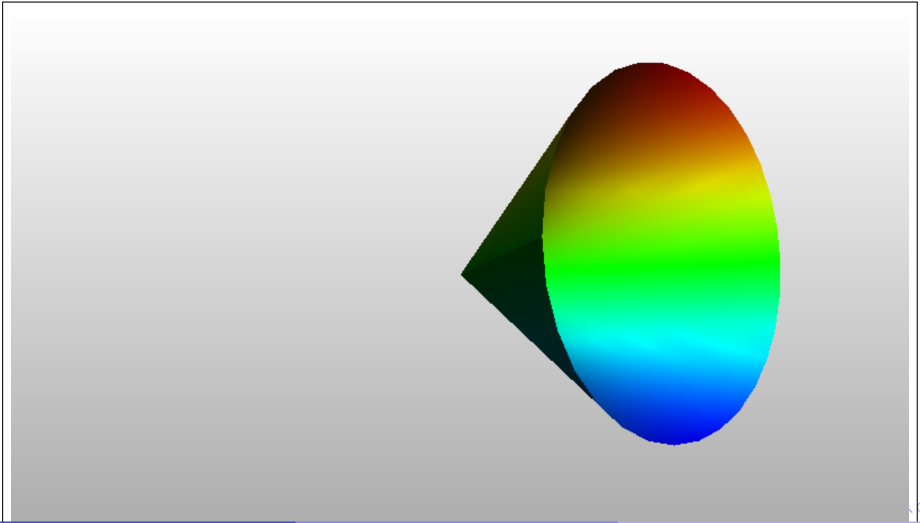

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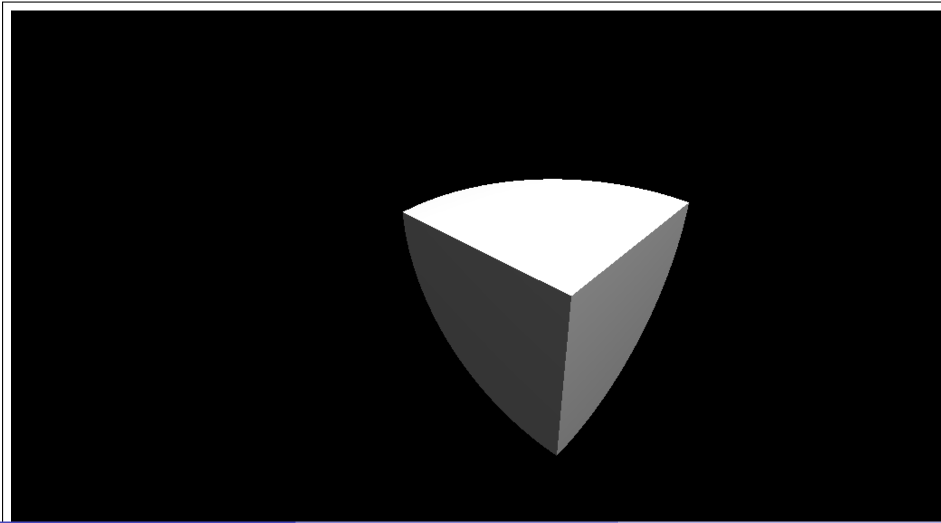

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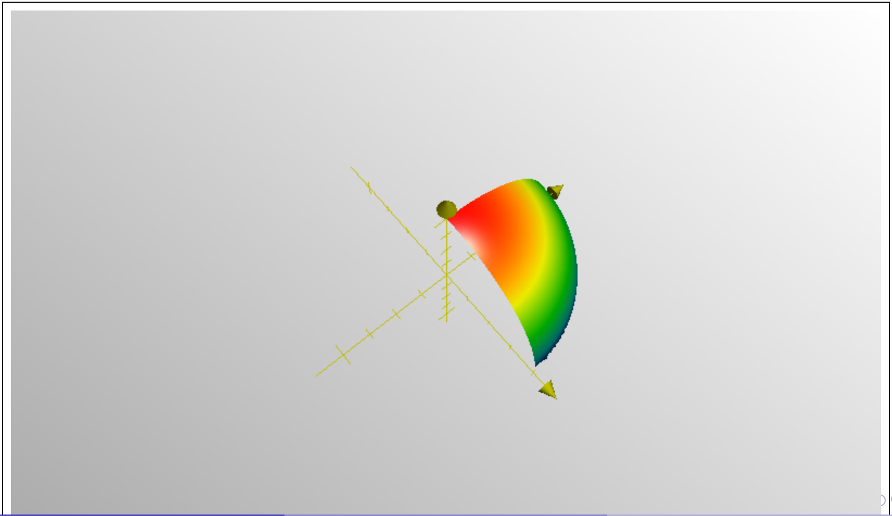

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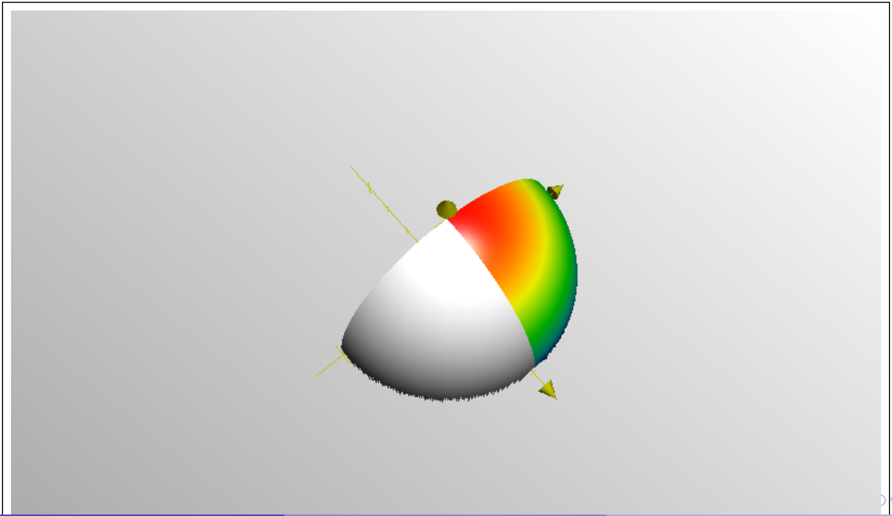

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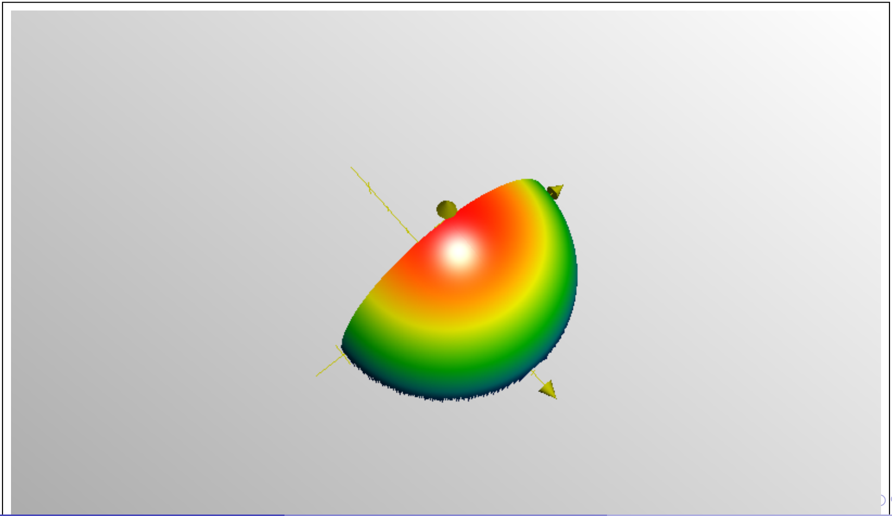

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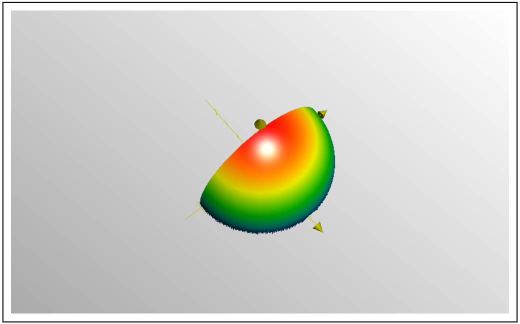
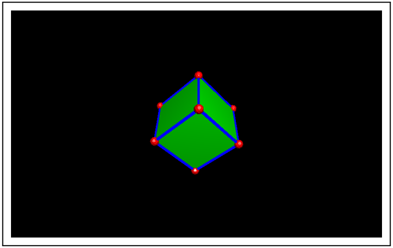


This is a cone over the berlingot !

Some stratified space : the double solid cube

Summary : The double solid cube has a decomposition $X \supset X_1 \subset X_0$:



- On $X \setminus X_1$ the geometry is Euclidean
- $X_1 \setminus X_0$ is the union of 12 unit segments and at a point on $X_1 \setminus X_0$, the geometry is the product of an interval with a cone whose link has length π.
- X_0 consists of 8 points and the geometry near these points looks like a cone over a berlingot.

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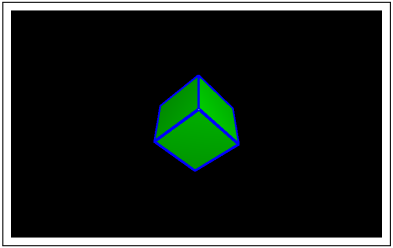

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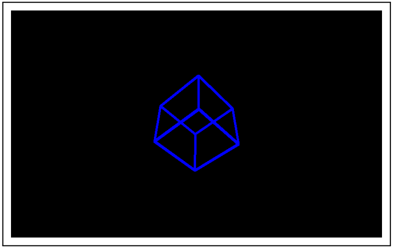

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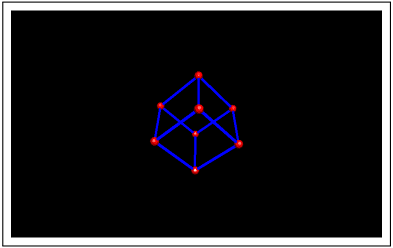

What are stratified spaces ?

The basics object are cone over metric space : if Σ is a complete metric space with distance d_Σ, the cone $C(\Sigma)$ over Σ is the completion of the product $(0,\infty)\times\Sigma$ with the distance for
$p=(t,x),\;q=(s,y)\in(0,\infty)\times\Sigma$

$$d(p,q)=\begin{cases} t+s & \text{if} \quad d_Y(x,y)\geq\pi \\ \sqrt{t^2+s^2-2ts\cos d_Y(x,y)} & \text{if} \quad d_Y(x,y)\leq\pi \end{cases}$$

We have only to blown down $\{0\}\times\Sigma$ to a point (the vertex of the cone) from $[0+\infty)\times\Sigma$.

What are stratified spaces ?

For $p = (t, x),\ q = (s, y) \in (0, \infty) \times \Sigma$, the distance has to be interpreted as follow :

- If $d_Y(x, y) \geq \pi$ then the shortest geodesic between p and q is the union of the to ray

$$[0, t] \times \{x\} \cup [0, s] \times \{y\}.$$

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$$(u, v) = (r\cos(\theta), r\sin(\theta)) \mapsto (r, \gamma(\theta))$$

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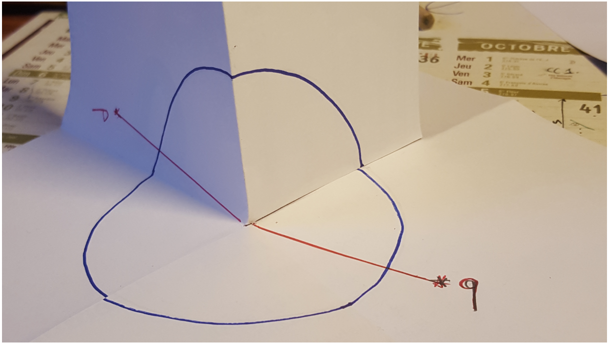

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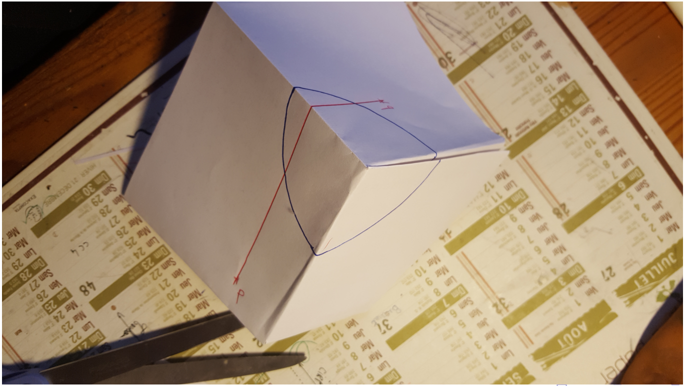

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A stratified space is a compact metric space (X, d) with a stratification

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such that

- near each point $x \in X \setminus X_{n-2} = X_{\mathrm{reg}}$, the geometry is Riemannian.

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Some remarks :

- It is an inductive definition by induction.

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- It is an inductive definition by induction.
- Near $x \in X_k \setminus X_{k-1}$, the $\mathbb{R}^k$-*directions* are tangent to the stratum X_k.

What are stratified spaces ?

The local/infinitesimal geometry can be described by the tangent cone :

Tangent cone

If $x \in X$, one defines

$$T_x X = \lim_{\varepsilon \to 0} \left(X, \frac{d}{\varepsilon}, x \right).$$

It means that the geometry of $T_x X$ is obtained after zooming around $x \in X$. In our case, when $X \supset X_{n-2} \supset \cdots \supset X_1 \supset X_0$

- If $x \in X \setminus X_{n-2} = X_{\mathrm{reg}}$, then $T_x X = \mathbb{R}^n$.

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Tangent cone are stratified spaces.

Yamabe problem on stratified space

If X is a stratified space, its regular part is a Riemannian manifold (X_{reg}, g) , it has a well defined scalar curvature :

$$\mathrm{Scal}\colon X_{\mathrm{reg}} \longrightarrow \mathbb{R}.$$

We can still defined the Yamabe invariant :

$$Y(X) = \inf_{u \in \mathcal{C}_0^\infty(X_{\mathrm{reg}})} \mathcal{Q}_g(u) = \inf_{u \in \mathcal{C}_0^\infty(X_{\mathrm{reg}})} \frac{\int_{X_{\mathrm{reg}}} \frac{4(n-1)}{n-2} |du|_g^2 + \mathrm{Scal}_g u^2}{\left(\int_{X_{\mathrm{reg}}} u^{\frac{2n}{n-2}}\right)^{1-\frac{2}{n}}}$$

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- With this definition $Y(\mathbb{R}^n) = Y(\mathbb{S}^n, [\mathrm{rounded}])$.

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- Does such $u \in L^\infty$ and $\frac{1}{u} \in L^\infty$, such that the geometry given by g and $u^{\frac{2n}{n-2}} g$ are Lipschitz equivalent.
- Does such u extends continuously to X.

Yamabe problem on stratified space : results I

local Yamabe invariant

If X is a stratified space, one defines $Y_{\mathrm{loc}}(X) = \inf_x Y(T_x X)$.

Hence for a smooth Riemannian manifold (M, g) :

$$Y_{\mathrm{loc}}(M) = Y(\mathbb{S}^n, [\mathrm{rounded}]).$$

Yamabe problem on stratified space : results I

Theorem (Akutagawa-C-Mazzeo : 2014)

For a stratified space X of dimension n with Riemannian metric g on the regular part X_{reg}

- $Y(X) > -\infty \iff Y_{\mathrm{loc}}(X) > -\infty$
- $Y(X) \leq Y_{\mathrm{loc}}(X)$.
- If $\mathrm{Scal} \in L^{p>\frac{n}{2}}$ then $Y(X) > -\infty$.
- If $Y(X) < Y_{\mathrm{loc}}(X)$ and $\mathrm{Scal} \in L^{p>\frac{n}{2}}$, then there is some $u \in \mathcal{C}^\infty(X_{\mathrm{reg}})$ with $du \in L^2, u \in L^{\frac{2n}{n-2}}$ such that $\mathcal{Q}_g(u) = Y(X)$ hence the metric $u^{\frac{2n}{n-2}} g$ has constant scalar curvature. Moreover u has a positive $\mathcal{C}^\alpha$ extension to X.

Yamabe problem on stratified space : about regularity of solutions

The regularity issue is similar to the one on domains with corner. If $\Omega \subset \mathbb{R}^2$, one look for regularity of solution of the equation

$$\Delta h = Vh \text{ and } h = 0 \text{ on } \partial\Omega.$$

If $\partial\Omega$ has a corner at $p \in \partial\Omega$ with angle π/k, then

$$h\left(p + re^{i\theta}\right) = b\, r^k \sin(k\theta) + l.o.t$$

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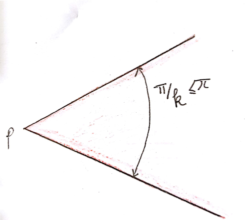
Hence if the angle is smaller than π then h has a Lipschitz-extension to $\bar{\Omega}$.



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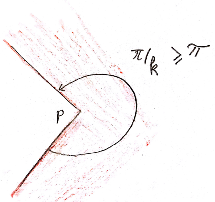
Where as if the angle is larger than π ($\pi/k \geq \pi$) then h has a k-Hölder extension to $\bar{\Omega}$.



Yamabe problem on stratified space : about regularity of solutions

Theorem (Mondello : 2016)

If the Ricci curvature of the stratified space is bounded, then the solution of the Yamabe problem is necessary Lipschitz.

Yamabe problem on stratified space : about regularity of solutions

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If the Ricci curvature of the stratified space is bounded, then the solution of the Yamabe problem is necessary Lipschitz.

This condition implies that all tangent cone are in fact Ricci flat (on their regular part).

Yamabe problem on stratified space : computation of local Yamabe invariant

For any $x \in X$, one defined its volume density by

$$\theta(x) = \lim_{r \to 0+} \frac{\mathrm{vol} B(x,r)}{r^n}.$$

Theorem (Mondello : 2016)

If the Ricci curvature of the stratified space is bounded, then

$$Y_{\mathrm{loc}}(X) = \inf_{x \in X} n(n-1)\gamma(n)\ \theta(x)^{2/n}$$

Where $\gamma(n)$ is a explicit computable constant (given in term of the Gamma function) with

$$\gamma(n)\,(\mathrm{vol}\ \mathbb{B}^n)^{2/n} = (\mathrm{vol}\ \mathbb{S}^n)^{2/n}\,.$$

Yamabe problem on stratified space : rigidity

A natural question is about the equality case $Y(X) = Y_{\text{loc}}(X)$. We know that for smooth space this implies that the manifold is conformal to the rounded sphere. However the picture here is more complicated and perhaps not totally solvable.

Theorem (Viaclovsky : 2012)

There is a stratified spaces X^4 with metric g only one singular point x where the geometry looks like $\mathbb{R}^4/\{\pm \mathrm{Id}\}$ such that

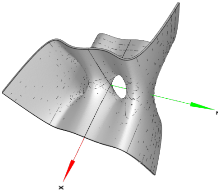
$$Y(X) = Y_{\text{loc}}(X) = \frac{1}{\sqrt{2}} Y(\mathbb{S}^4, [\text{rounded}])$$

and there is **no** metric $\tilde{g} = u^2 g$ with u Lipschitz with constant scalar curvature.

Yamabe problem on stratified space : rigidity

These examples are build from non compact Ricci flat space called (Asymptotically Locally Euclidean or ALE space). The most famous has been discovered by Eguchi et Hanson. It is a complete Ricci flat metric on the 4-manifold

$$Y = \{(x, y, z) \in \mathbb{C}^3, x^2 + y^2 + z^2 = 1\}$$



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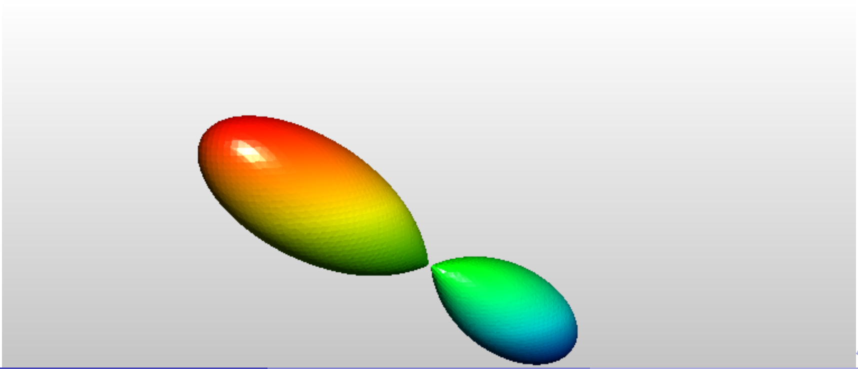
On can make a stereographic compactification of Y to obtain a stratified space (in fact an orbifold) with one singular point where the geometry looks like $\mathbb{R}^4/\{\pm \mathrm{Id}\}$



Other examples can be build from the other Ricci flat ALE space.

A4 ALE gravitational instanton
€63.38

A1 ALE gravitation instanton
€34.32

Others perpectives

Since J.Lott & C. Villani, K-T. Sturm, one has a good definition of what can be a dimension-Ricci curvature lower bound on mesured metric space. This curvature dimension inequalitiesare defined from a convexity of an entropy functional of the space of probability measure. This condition is called the $RCD^\star(K,n)$ condition, for Riemannian manifold (M^d,g) this condition is equivalent to $d\le n$ and $\mathrm{Ricci}\ge Kg$. It turns out that stratified spaces furnished a large class on new examples where such conditions is valid :

Theorem (J. Bertrand, C. Ketterer, I. Mondello, T. Richard : 2018)

Assume that X is a stratified space with the following condition :

- $\mathrm{Ricci}\ge Kg$ on X_{reg}
- For all $x\in X$, $\theta(x)\le \mathrm{vol}\,\mathbb{B}^n$

Then (X,d,dvol_g) satisfies the $RCD^\star(K,n)$ conditions