The Yamabe problem on stratified spaces

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Abstract

We introduce new invariants of a Riemannian singular space, the local Yamabe and Sobolev constants, and then go on to prove a general version of the Yamabe theorem under that the global Yamabe invariant of the space is strictly less than one or the other of these local invariants. This rests on a small number of structural assumptions about the space and of the behavior of the scalar curvature function on its smooth locus. The second half of this paper shows how this result applies in the category of smoothly stratified pseudomanifolds, and we also prove sharp regularity for the solutions on these spaces. This sharpens and generalizes the results of Akutagawa and Botvinnik [3] on the Yamabe problem on spaces with isolated conic singularities.

Introduction

Our aim in this paper is to study a version of the Yamabe problem on a class of compact Riemannian singular spaces satisfying a small list of general structural axioms which we call 'almost smooth metric-measure spaces'. This approach emphasizes the centrality of Sobolev inequality, and indeed relies on little else. Our main existence result is the analogue of that part of the resolution of this problem on compact smooth manifolds (M, g) obtained through the work of Yamabe, Trudinger and Aubin, [7], [29], [33]. In that original setting, the work of these authors established the existence of a smooth positive function minimizing the Yamabe functional

$$Q_g(u) = \frac{\int_M \left(|\nabla u|^2 + \frac{n-2}{4(n-1)} \operatorname{Scal}_g u^2 \right) dV_g}{\left(\int_M u^{\frac{2n}{n-2}} dV_g \right)^{\frac{n-2}{n}}}, \tag{0.1}$$

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where Scal_g is the scalar curvature of the metric g, provided the infimum of this functional, the so-called Yamabe invariant (sometimes also called the Yamabe constant or conformal Yamabe invariant) of that conformal class $Y(M, [g])$, is strictly less than the corresponding invariant of the round sphere. In some papers on this subject, the energy Q_g is replaced by $((4(n-1)/(n-2))Q_g$. The geometric meaning of this functional is that if u is a minimizer, or indeed any critical point, then the conformally related metric $\widetilde{g} = u^{\frac{4}{n-2}}g$ has constant scalar curvature on any open set where $u > 0$. We refer to the well-known survey paper by Lee and Parker [16], as well as [27], [28], for all details on the complete existence theory in the setting of smooth compact manifolds.

The singular spaces (M, g, μ) we are interested in here are typically Riemannian pseudomanifolds, and in particular Riemannian smoothly stratified spaces with iterated edge metrics, endowed with a measure which is a smooth positive multiple of the Riemannian volume form dV_g. However, as indicated already, we require only a few structural assumptions and so our main existence theorem holds in much more general settings. In a companion to this paper we explore this direction further, extending this method to handle rather general semi-Riemannian spaces, for example. The ability to allow for a more general measure μ is perhaps useful, but plays essentially no role in any of the arguments below, and for reasons of notational simplicity, we often omit μ altogether from the discussion. The spaces here are 'mostly smooth' in that they possess an open dense set Ω which is a smooth n-dimensional manifold carrying a Riemannian metric. Infinitesimally, every point in Ω looks the same as every other. However, that is not true if one includes the singular points. To accomodate this, we replace the global Yamabe invariant by a new invariant which we call the *local* Yamabe invariant $Y_\ell(M, [g])$. Briefly, this is just the infimum over all points $p \in M$ of the Yamabe invariants of arbitrarily small balls around p, where we minimize the standard energy functional amongst functions on these balls which vanish on the outer boundaries, but not necessarily near the singular set of M. We also introduce the corresponding local Sobolev invariant $S_\ell(M, g)$. Our main existence theorem states that under various sets of conditions on the scalar curvature Scal_g (which we regard as a function computed in the usual way on the smooth domain Ω), if the *global* Yamabe invariant $Y(M, [g])$ is *strictly* less than $Y_\ell(M, [g])$ (or, in some versions of the result, than $S_\ell(M, g)$), then Q_g admits a strictly positive minimizer u. In certain cases we prove that this minimizer u is strictly positive, but show by example that this need not be the case if the hypotheses are relaxed.

Lest this criterion seem too abstract, observe that by conformal invariance, the local Yamabe invariant at a smooth point is equal to the Yamabe invariant of the round sphere; this is essentially what is known as Aubin's inequality. It is important

that Y_ℓ involves the limits as $r \to 0$ of the Yamabe invariants $Y(B_r(p), g)$, rather than their values at any fixed $r > 0$; this means that local curvature invariants play a smaller role in Y_ℓ. An invariant of this nature has been used previously for spaces (M, g) with isolated conic singularities. In that setting, if p is a conic point, so that some neighbourhood $\mathcal{U}$ of p in M is modelled by a cone over a compact smooth Riemannian manifold (Z, h), then the local Yamabe invariant at p is the same as the so-called cylindrical Yamabe invariant $Y(\mathbb{R} \times Z, [dt^2 + h])$ which plays an important role in the work of the first author and Botvinnik [3], see also [1] for a discussion of this problem on orbifolds. It is proved there that Q_g has a minimizer provided

$$-\infty < Y(M, [g]) < \min_j \{Y(S^n, [g_0]), Y(\mathbb{R} \times Z_j, [dt^2 + h_j])\}, \qquad (0.2)$$

where $(C(Z_j), dx^2 + x^2 h_j)$, $j = 1, \ldots, N$, are the local models for the conic points of (M, g). We note that $Y(\mathbb{R} \times Z_j, [dt^2 + h_j]) \le Y(S^n, [g_0])$ is always true. Also, implicit here is the fact that the cylinder $(\mathbb{R} \times Z, dt^2 + h)$ and the cone $(C(Z), dx^2 + x^2 h)$ are mutually conformal. We note that there are many examples of conic spaces where one does know that (0.2) holds, see also [15] for some results of the case where $R_g < 0$ (rather than $Y(M, [g]) < 0$), when an existence result is obtained in some cases using barriers.

Our main existence theorem states that if (M, g, μ) is an almost smooth metric-measure space which satisfies the first three properties listed in the definition at the beginning of §1.1 below as well as one of the three possible hypotheses on Scal_g, then $Q_{g,\mu}$ attains its minimum. In certain of these cases, we also prove that the minimizing function u is strictly positive on M. The proof is divided into two parts: the proof of existence is obtained through a variant of the original method appearing in the work of Trudinger and Aubin, and the surprising fact is that this original proof may be adapted quite simply to this general setting. However, in order to accomodate some of the natural geometric applications later, we present an alternate proof of the step which uses Moser iteration to give a uniform upper bound for the minimizing sequence, by another argument related to some old ideas of Varopoulos. The proof that, in certain cases, the minimizer is strictly positive uses some ideas developed by Gursky.

In the second part of this paper we expand on the theme and setting of [3] by considering in more detail the case where (M, g) is a smoothly stratified Riemannian pseudomanifold, also known as an iterated edge space. We identify the local Yamabe invariants at all point $p \in M$ as higher versions of the cylindrical/conic Yamabe invariants discussed above; these are simply the global Yamabe invariants for the model spaces $\mathbb{R}^k \times C(Z)$, or (conformally) equivalently, $\mathbb{H}^{k+1} \times Z$, where Z is a compact iterated edge space with lower singular 'depth' than the original

space M. The special cases of these invariants when $Z = S^{n-k-1}$ play an interesting role in the work of Ammann, Dahl and Humbert [5], [6], where quantitative estimates of the change of the σ-Yamabe invariant (which is the supremum of the Yamabe constants over all conformal classes) under surgeries are obtained. Finally, using the more specialized analytic tools available for the study of PDE on smoothly stratified spaces, we prove sharp regularity results about the behaviour of the minimizer u (or indeed any solution of the Yamabe equation) at the singular strata of M.

The final resolution of the Yamabe problem on smooth manifolds by Schoen, described in [16], devolves to showing that $Y(M, [g]) < Y(S^n, [g_0])$ except when (M, g) is conformal to (S^n, g_0). One might hope for some analogue of this result here. For example, a natural conjecture is that if (M, g) has only isolated conic singularities, then $Y(M, [g]) < Y_\ell(M, [g])$ unless (M, g) is conformal to the cylinder $(\mathbb{R} \times Z, dt^2 + h)$. Unfortunately, this is now known to be false! Indeed, a recent paper by Viaclovsky [32] exhibits a manifold with orbifold singularity which does not admit any (incomplete) orbifold metric of constant scalar curvature, or equivalently, any finite energy critical point of Q_g. Because the existence theory in [3] and [1] would guarantee a minimizer unless the local and global Yamabe invariants are the same, we conclude that these invariants must be equal for the spaces Viaclovsky considers. There are no known examples beyond the cylinder where $Y_\ell(M, [g]) = Y(M, [g])$ holds and Q_g also has a critical point. It remains a tantalizing mystery to determine whether there is some rigidity phenomenon here.

1   The general existence theorem

As discussed above, our main existence theorem yields a minimizer of the functional Q_g on a rather broad class of Riemannian singular spaces. We state and prove this result in this section. We first explain the precise geometric and analytic hypotheses, then define the local Yamabe and Sobolev invariants and describe their relationship to the (global) Sobolev constant of the space in question. We also give a number of auxiliary technical facts, including the compactness of the embedding $W^{1,2} \hookrightarrow L^{2p/(p-2)}$ for $p > n$, and that the finiteness of the Sobolev constant implies discreteness of the spectrum of the (Friedrichs extension of the) Laplacian. We also review the standard Moser iteration argument to obtain a uniform upper bound for the subcritical solutions and give a different proof based on a different

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(Morrey-type) assumption on the scalar curvature. Existence of the minimizer then follows the lines of the original Trudinger/Aubin argument. The positive lower bound for the minimizer uses an argument due to Gursky [14].

1.1 Almost smooth metric-measure spaces

Suppose that (M, d, μ) is a compact metric-measure (MM) space which is 'almost smooth' in the sense that there is an open dense subset $\Omega \subset M$ which is a smooth n-dimensional manifold, and a smooth Riemannian metric g on Ω which induces the same metric space structure as d on Ω, and hence by density on all of M. (It is not hard to check that the arguments in this section only require that g be $W^{2,q}$ for some $q > n/2$, but for simplicity we do not work in this generality.) We also assume that the measure $d\mu$ is a smooth positive multiple of the volume form, i.e. $d\mu = h^2 dV_g$ for some $h \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}^0(M)$ which is strictly positive. Since g and d induce the same distance, we refer to the triple (M, g, μ) and omit mention of d. Note that the metric balls $B(p, r)$ coincide with geodesic balls provided $B(p, r) \subset \Omega$.

We shall assume the following properties of the space (M, g, μ).

$\boxed{Hypotheses}$:

i) Let $W^{1,2}(M; d\mu)$ denote the Sobolev space which is the completion of the space of Lipschitz function $\mathrm{Lip}(M)$ with respect to the usual norm; then we assume that $\mathcal{C}_0^1(\Omega)$ is dense in $W^{1,2}(M; d\mu)$. Notice that this precludes the existence of codimension one boundaries.

ii) Hausdorff n-dimensional measure is absolutely continuous with respect to $d\mu$, and both of these measures are Ahlfors n-regular, i.e.

$$C^{-1} r^n \leq \mu(B(p, r)) \leq C r^n$$

for some $C > 0$, and for every $p \in M$ and $r \leq \mathrm{diam}(M)$. Later, for simplicity, we often write $\mu(B(p, r)) = V(p, r)$.

iii) The Sobolev inequality holds: there exist $A, B > 0$ such that

$$\|f\|_{\frac{2n}{n-2}}^2 \leq A \int_M |df|^2 \, d\mu + B \int_M |f|^2 \, d\mu \tag{1.1}$$

for all $f \in W^{1,2}(M; d\mu)$.

iv) Finally, the scalar curvature function satisfies at least one of the following properties:

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a) $\mathrm{Scal}_g \in L^q(M, d\mu)$ for some $q > n/2$;

b) For some $q > 1$, there exist $\alpha \in [0, 2)$ and $C > 0$ such that for every point $p \in M$,

$$\sup_{r>0} \frac{r^{\alpha q}}{V(p, r)} \int_{B(p,r)} |\mathrm{Scal}_g|^q d\mu \leq C. \qquad (1.2)$$

c) $\mathrm{Scal}_g^- := \min\{\mathrm{Scal}_g, 0\} \in L^q(M, d\mu)$ for some $q > n/2$;

We henceforth assume that conditions i) - iii) and at least one of iv) a)-c), are satisfied, unless explicitly stated otherwise. We call a triplet (M, g, μ) which satisfies these properties an almost smooth metric measure space.

The condition iv) b) (combined with the condition ii)) states that Scal_g lies in the Morrey regularity class $\mathcal{M}_\lambda^q$ where $\lambda = n - \alpha q \in (0, n]$. We state now an important fact about functions which lie in this class which will be used in several places below. The proof is deferred to the end of §1.3 simply because it involves techniques which are discussed there for other reasons, but does not rely on any of the intervening results.

Lemma 1.1. *Suppose that the function V satisfies (1.2). Then for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for all $\phi \in W^{1,2}(M)$,*

$$\int |V||\phi|^2 \, d\mu \leq \epsilon \int |d\phi|^2 \, d\mu + C_\epsilon \int |\phi|^2 \, d\mu.$$

Remark 1.2. *The extra generality of allowing the measure $d\mu$ to be a smooth multiple of dV_g rather than just the volume form itself, plays very little role here. For simplicity in this paper, we usually assume that $d\mu = dV_g$. Although the analysis in this paper goes through for more general measures, the conclusions then are no longer strictly within the realm of conformal geometry.*

1.2 Yamabe and Sobolev constants

For any open set $\mathcal{U} \subset M$, we define its Sobolev and Yamabe constants,

$$
\begin{aligned}
S(\mathcal{U}) &= \inf\{\textstyle\int |d\varphi|^2 \, d\mu : \varphi \in W_0^{1,2}(\mathcal{U}), \|\varphi\|_{\frac{2n}{n-2}} = 1\}, \text{ and} \\
Y(\mathcal{U}) &= \inf\{\textstyle\int (|d\varphi|^2 + \frac{n-2}{4(n-1)} \mathrm{Scal}_g \, \varphi^2) \, d\mu : \\
&\qquad \varphi \in W_0^{1,2}(\mathcal{U} \cap \Omega), \|\varphi\|_{\frac{2n}{n-2}} = 1\},
\end{aligned}
$$

respectively. We also define the *local* Sobolev constant and *local* Yamabe invariant of (M, g, μ) by

$$S_\ell(M, g) = \inf_{p \in M} \lim_{r \to 0} S(B(p, r)), \quad Y_\ell(M, [g]) = \inf_{p \in M} \lim_{r \to 0} Y(B(p, r)).$$

It should be pointed out that the *Aubin-type inequality* for the Yamabe invariant $Y(M, [g])$ still holds:

$$Y(M, [g]) \leq Y_\ell(M, [g]).$$

All these quantities depend on g and μ, but we often suppress this, and even explicit mention of M, in the notation.

Lemma 1.3. *If* Scal_g *satisfies either iv) a) or iv) b), then* $Y_\ell(M, [g]) = S_\ell(M, g)$.

Proof. Assume first that iv) a) holds, i.e. that $\mathrm{Scal}_g \in L^q$ for some $q > n/2$. By the Hölder inequality,

$$|Y(\mathcal{U}) - S(\mathcal{U})| \leq \frac{n-2}{4(n-1)} \| \mathrm{Scal}_g \|_q \ \mathrm{vol}(\mathcal{U})^{\frac{2q-n}{nq}},$$

and thus, since $q > n/2$,

$$\inf_{p \in M} \lim_{r \to 0} S(B(p, r)) = \inf_{p \in M} \lim_{r \to 0} Y(B(p, r))$$

for any p.

For the other case, we invoke Lemma 1.1 as follows. Fix any $\epsilon > 0$ and choose C_ϵ accordingly. Then

$$|Y(B(p, r) - S(B(p, r))| \leq \inf_\phi \int_{B(p,r)} \frac{n-2}{4(n-1)} | \mathrm{Scal}_g | |\phi|^2$$
$$\leq \epsilon S(B(p, r)) + C_\epsilon \inf_\phi \int_{B(p,r)} |\phi|^2 \leq \epsilon S(B(p, r)) + C_\epsilon r^2,$$

where the Hölder inequality and the normalization $\|\phi\|_{2n/(n-2)} = 1$ are used to get the last term. Letting $r \searrow 0$ and taking the infimum over all $p \in M$ shows that $|Y_\ell - S_\ell| \leq \epsilon S_\ell$, and since this is true for all $\epsilon > 0$, we see that $Y_\ell(M) = S_\ell(M)$, as claimed. $\square$

The fact that, under these hypotheses, $Y_\ell(M)$ is the same as the local Sobolev constant $S_\ell(M)$, leads to an important criterion for the positivity of the local Yamabe invariant.

Proposition 1.4. *Let* (M, g) *satisfy hypotheses i) - iii).*

a) *For any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that*

$$(S_\ell - \epsilon)\|f\|_{\frac{2n}{n-2}}^2 \le \int_\Omega |df|^2 \, d\mu + C_\epsilon \int_\Omega |f|^2 \, d\mu \qquad (1.3)$$

for all $f \in W^{1,2}(M)$.

b) *If Scal_g satisfies either iv) a) or iv) b), then $Y_\ell(M) > 0$ if and only if the Sobolev inequality (1.1) holds on (M, g, μ). If these conditions are true, then for any $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that*

$$(Y_\ell - \epsilon)\|f\|_{\frac{2n}{n-2}}^2 \le \int_\Omega |df|^2 \, d\mu + C_\epsilon \int_\Omega |f|^2 \, d\mu \qquad (1.4)$$

for all $f \in W^{1,2}(M)$.

Proof. Let us first address b). If (1.1) holds, then rearranging and using the Hölder inequality, we see that

$$S(\mathcal{U}) \ge \frac{1}{A}\left(1 - B(\mathrm{vol}\,\mathcal{U})^{2/n}\right),$$

which implies a positive lower bound for $S_\ell(M) = Y_\ell(M)$.

On the other hand, suppose $Y_\ell(M) > 0$. Fixing any $\delta \in (0, 1)$, for each $p \in M$ there is a radius $r_p > 0$ such that

$$\min\{Y(B(p, r_p)), S(B(p, r_p))\} \ge (1 - \delta)Y_\ell(M). \qquad (1.5)$$

Since M is compact, there is a finite covering $M = \bigcup_i B(p_i, r_i)$ with $r_i = \frac{1}{2}r_{p_i}$. Hence if $s = \min r_i$, then (1.5) is true for every $p_i \in M$ with r_p replaced by s.

Now choose a partition of unity $\{\rho_i\}$ subordinate to this covering such that each $\sqrt{\rho_i} \in \mathrm{Lip}(M)$. If $f \in W^{1,2}(M)$ and $\chi \in \mathrm{Lip}(M)$, then $\chi f \in W^{1,2}(M)$. Note that this follows from the density assumption i) and the fact that it holds on Ω. Thus for $f \in W^{1,2}(M)$,

$$(1-\delta)Y_\ell(M)\|f\|_{\frac{2n}{n-2}}^2 = (1-\delta)Y_\ell(M)\|f^2\|_{\frac{n}{n-2}}$$

$$\le (1-\delta)Y_\ell(M)\sum_i \|\rho_i f^2\|_{\frac{n}{n-2}} \le \sum_i \|d(\sqrt{\rho_i}f)\|_2^2$$

$$\le \sum\left((1+\epsilon')\|\sqrt{\rho_i}df\|_2^2 + C_{\epsilon'}\|fd\sqrt{\rho_i}\|_2^2\right) = (1+\epsilon')\|df\|_2^2 + C\|f\|_2^2$$

where C depends on the ρ_i and on $C_{\epsilon'}$. This gives (1.4) with $\epsilon = 2\delta Y_\ell(M)$ if we choose ϵ' appropriately.

The proof of a) is the same. $\qquad\square$

Remark 1.5. *Using the Hölder inequality and Lemma 1.1, we also deduce from any one of the hypotheses iv) a), b) or c) that*

$$Y(M, [g]) > -\infty.$$

The Sobolev inequality has another important consequence.

Proposition 1.6. *If the Sobolev inequality holds on (M, g), then the inclusion*

$$W^{1,2}(M) \longrightarrow L^{\frac{2p}{p-2}}(M)$$

is compact for any $p \in (n, \infty)$.

Proof. We first write

$$u - e^{-t(-\Delta+1)}u = -\int_0^t \frac{d}{ds} e^{-s(-\Delta+1)}u \, ds = \int_0^t e^{-s(-\Delta+1)}(-\Delta+1)u \, ds$$

$$= \int_0^t e^{-\frac{1}{2}s(-\Delta+1)}(-\Delta+1)^{\frac{1}{2}} e^{-\frac{1}{2}s(-\Delta+1)}(-\Delta+1)^{\frac{1}{2}}u \, ds.$$

We shall show that the $L^{2p/(p-2)}$ norm of this difference is bounded by $t^\beta \|u\|_{1,2}$ for some $\beta > 0$. This proves that inclusion mapping is approximated in the operator norm topology by a sequence of compact mappings, which implies that it must be compact.

To do this, we need three facts. First, if $u \in W^{1,2}(M)$, then

$$\int (|du|^2 + |u|^2) = \langle(-\Delta+1)u, u\rangle = \|(-\Delta+1)^{\frac{1}{2}}u\|_2^2.$$

Next, it is an easy consequence of the spectral theorem that

$$\left\|\sqrt{-\Delta+1}\, e^{-s(-\Delta+1)}\right\|_{L^2 \to L^2} \leq Cs^{-\frac{1}{2}}.$$

Finally, we claim that, if $q = 2p/(p-2) < 2n/(n-2)$, then

$$\left\|e^{-s(-\Delta+1)}\right\|_{L^2 \to L^q} \leq \frac{C}{s^{\frac{n}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}} = \frac{C}{s^{n/2p}}, \quad s > 0.$$

This is proved by interpolation as follows. There is a standard estimate that

$$\left\|e^{-s(-\Delta+1)}\right\|_{L^r \to L^r} \leq 1,$$

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uniformly in s for any $r \geq 2$. This follows by a simpler interpolation from the case $r = 2$ (spectral theorem) and $r = \infty$ (easy direct argument). We can obtain the same estimate for $1 < r < 2$ either by duality or noting that this also holds for $r = 1$ and interpolating again. On the other hand, it is known [26] that the Sobolev inequality (1.4) implies that

$$\left\| e^{-s(-\Delta + 1)} \right\|_{L^1 \to L^\infty} \leq \frac{C}{s^{n/2}}, \quad s > 0.$$

Thus if we interpolate between this $L^1 \to L^\infty$ estimate and the $L^r \to L^r$ estimate with $r = 1 - 1/p$, then a bit of arithmetic proves the claim.

Putting these three estimates together, we conclude that

$$\left\| u - e^{-t(-\Delta + 1)} u \right\|_q \leq C \int_0^t s^{-\frac{1}{2} - \frac{n}{2p}} \, ds \, \|u\|_{1,2} = C t^{\frac{1}{2} - \frac{n}{2p}} \, \|u\|_{1,2},$$

which decays as required. $\qquad\square$

Corollary 1.7. *Let $-\Delta$ be the self-adjoint operator obtained as the Friedrichs extension from the semi-bounded quadratic form*

$$\langle \nabla f, \nabla f \rangle = \int_M |\nabla f|^2$$

over the core domain $\mathcal{C}_0^\infty(\Omega)$. Then $-\Delta$ has discrete spectrum.

Proof. It suffices to show that the Friedrichs domain of $-\Delta$ is compactly contained in $L^2(M; d\mu)$. However, this domain is simply $W^{2,2}(M; d\mu) \cap W_0^{1,2}(M; d\mu) \subset W^{1,2}(M; d\mu)$, which by the previous result is compactly contained in $L^{\frac{2p}{p-2}}$ for any $p > n$, which in turn continuously includes in L^2. $\qquad\square$

1.3 Uniform boundedness of subsolutions

Let $(M, g, d\mu)$ be an almost smooth metric measure space, as considered above. We now present two different methods which lead to the uniform boundedness of nonnegative functions u which satisfy $\Delta u \geq V u$. The first is simply the adaptation of the Moser iteration method to this setting, and assumes that V satisfies either hypothesis iv) a), or c). This does not cover all the geometric cases we wish to consider, so we then prove a stronger result assuming that V satisfies iv) b). This second result subsumes the first one, but we describe both proofs since the former is the more traditional method and certain constructions in its proof will be used later.

1.3.1 Moser iteration

We now review the classical Moser iteration method with enough detail to make clear that all steps work on almost smooth MM spaces. (In fact, Moser iteration works in greater generality still, see [8].)

Proposition 1.8. *Let* $u \in W^{1,2}(M)$ *be nonnegative and satisfy* $\Delta u - Vu \geq 0$, *where* $V \in L^q$ *for some* $q > n/2$. *Then* $u \in L^\infty$ *and*

$$\|u\|_\infty \leq C(\|V\|_q)\|u\|_2,$$

where the constant C *depends only on* $n, q, \|V\|_q$ *and the constants* A, B *from the Sobolev inequality.*

Remark 1.9. *As usual, the differential inequality is to be interpreted weakly, i.e.*

$$\int (du, d\varphi) \leq - \int Vu\varphi. \tag{1.6}$$

for any $\varphi \in W^{1,2}(M)$ *with* $\varphi \geq 0$. *Notice that the right hand side of* (1.6) *is well defined because* $V \in L^{n/2}$ *and, from the Sobolev inequality,* $u\varphi \in L^{n/(n-2)}$.

Proof. We follow the standard proof [13, Theorem 8.15] as soon as we verify the chain rule:

Claim: If $u \in W^{1,2}(M)$ and $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfies $f' \in L^\infty$ then $f \circ u \in W^{1,2}(M)$ and
$$d(f \circ u) = f' \circ u \, . \, du.$$

To prove this claim, note that by [13, Theorem 7.5], we have

$$\int (d(f \circ u), d\varphi) = \int (du, d\varphi) \, f' \circ u \quad \text{for all } \varphi \in \mathcal{C}^1_0(\Omega),$$

and the result follow from the density of $\mathcal{C}^1_0(\Omega)$ in $W^{1,2}$.

For $\alpha > 1$, define

$$f_\alpha(x) = \begin{cases} x^\alpha & \text{if } 0 \leq x \leq \alpha^{-\frac{1}{\alpha-1}} \\ x + (\alpha^{-\frac{\alpha}{\alpha-1}} - \alpha^{-\frac{1}{\alpha-1}}) & \text{if } \alpha^{-\frac{1}{\alpha-1}} \leq x; \end{cases} \tag{1.7}$$

the cutoff and additive constant are chosen so that $f_\alpha(x)$ is $\mathcal{C}^1$ and convex. Next, for any $L \geq 1$, let
$$\phi_{\alpha,L}(x) = L^\alpha f_\alpha\left(\frac{x}{L}\right).$$

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Note that $\phi_{\alpha,L}(x) = x^\alpha$ on larger and larger intervals as $L \to \infty$. Furthermore, if we define $G_{\alpha,L}(x) = \int_0^x \phi_{\alpha,L}'(t)^2 \, dt$, then a laborious computation gives

$$\phi_{\alpha,L}(x) \le x^\alpha \quad \text{and} \quad x G_{\alpha,L}(x) \le \frac{\alpha^2}{2\alpha - 1} \left(\phi_{\alpha,L}(x) \right)^2, \tag{1.8}$$

for all $x \ge 0$.

Inserting $\varphi = G_{\alpha,L}(u)$ into (1.6) and using (1.8), we obtain

$$\int |d\phi_{\alpha,L}(u)|^2 = \int G_{\alpha,L}'(u) |du|^2 = \int (du, d\varphi)$$

$$\le \int V u \, G_{\alpha,L}(u) \le \frac{\alpha^2}{2\alpha - 1} \int V \phi_{\alpha,L}(u)^2.$$

Use both the Sobolev and Hölder inequalities to get

$$\|\phi_{\alpha,L}(u)\|^2_{L^{\frac{2n}{n-2}}} \le \int \left(\frac{\alpha^2}{2\alpha - 1} A V + B \right) \left(\phi_{\alpha,L}(u) \right)^2$$

$$\le \frac{\alpha^2}{2\alpha - 1} C \left(\int \left(\phi_{\alpha,L}(u) \right)^{\frac{2q}{q-1}} \right)^{\frac{q-1}{q}},$$

with $C = A\|V\|_q + B(\operatorname{vol} M)^{1/q}$. Letting $L \to \infty$ yields

$$\left(\int u^{\frac{2\alpha n}{n-2}} \right)^{\frac{n-2}{n}} \le \frac{C\alpha^2}{2\alpha - 1} \left(\int u^{\frac{2\alpha q}{q-1}} \right)^{\frac{q-1}{q}}. \tag{1.9}$$

This is of course only interesting if the right side is finite.

We are given the initial choice of q through the potential V, with $q > n/2$. Thus $r := 2q/(q-1) < 2n/(n-2)$ and we can choose α sufficiently close to 1 so that $\alpha r < 2n/(n-2)$ as well. Since $W^{1,2} \hookrightarrow L^{\alpha r}$, the right hand side of (1.9) is finite, and hence so is the left, i.e. $u \in L^{\kappa \alpha r}$ where

$$\kappa := \frac{n}{n-2} \frac{q-1}{q} > 1.$$

Furthermore, we can rewrite (1.9) as

$$\|u\|_{\kappa \alpha r} \le (C_1 \alpha)^{\frac{1}{2\alpha}} \|u\|_{\alpha r}, \tag{1.10}$$

where $C_1 = (C\alpha/(2\alpha - 1))$. Note that (1.10) is valid for any $\alpha_j \ge \alpha$ so long as $u \in L^{\alpha_j r}$, since $\alpha_j/(2\alpha_j - 1)$ is uniformly bounded. Now set $\alpha_j = \kappa^j \alpha$, and apply (1.10) inductively to obtain that

$$\|u\|_{\kappa^N \alpha r} \le \left(\prod_{j=0}^{N-1} \left(C_1 \kappa^j \alpha \right)^{\frac{1}{2\kappa^j}} \right) \|u\|_{\alpha r}$$

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for any $N \geq 1$. Finally, note that the constant here is bounded independently of N; indeed

$$\log \prod_{j=0}^{N-1} \left(C_1 \kappa^j \alpha \right)^{\frac{1}{2\kappa j}} = \sum_{j=0}^{N-1} \left(\frac{\log C_1 \alpha}{2\kappa^j} + \frac{\log \kappa^j}{2\kappa^j} \right) \leq C_2 \frac{\kappa}{\kappa - 1}.$$

Thus, taking the limit as $N \to \infty$, we obtain finally that

$$\|u\|_\infty \leq C(q, n, A, B, \|V\|_q) \|u\|_{1,2}.$$

$\square$

1.3.2 Varopoulos' method

For the second method, we now suppose that V satisfies the Morrey condition iv) b) for some $q > 1$ and $0 \leq \alpha < 2$. To compare this hypothesis with the hypothesis in Proposition 1.8, observe simply that by the Hölder inequality, if $V \in L^{n/\alpha}$, then (1.2) holds (with Scal_g replaced by V) provided $q < n/\alpha$.

Next, the existence of the Sobolev inequality (1.4) implies the Gaussian upper bound

$$e^{t\Delta}(x, y) \leq C \frac{1}{t^{n/2}} e^{-\frac{d(x,y)^2}{5t}} \tag{1.11}$$

for the Schwartz kernel of the heat operator $e^{t\Delta}$, see [12]. Thus if $G(x, y)$ denotes the Green kernel associated to $-\Delta + 1$, then

$$0 < G(x, y) \leq \frac{C}{d^{n-2}(x, y)}$$

provided $d(x, y) \leq C$.

Our goal, as before, is to prove the following.

Theorem 1.10. *Assume that u is a nonnegative function in $W^{1,2} \cap L^p$ for some $p > q^*$ such that*

$$\Delta u \geq V u$$

where V satisfies (1.2). *Then*

$$\|u\|_\infty \leq C \left(\sup_x \sup_{r>0} \frac{r^{q\alpha}}{\mathrm{Vol}(x, r)} \int_{B(x,r)} |V|^q \, d\mu \right) \|u\|_p.$$

(In other words, the constant C depends in some possibly nonlinear way on the quantity in parentheses.)

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The key point is to rewrite the differential inequality as $(-\Delta + 1)u \leq (-V + 1)u$ and then, using that G is positivity preserving, $u \leq G \circ (-V + 1)u$. Clearly $-V + 1$ satisfies (1.2) if V does, so for simplicity we replace $-V + 1$ by V. We then establish the following mapping properties.

Theorem 1.11. *Let L, G and V be as above. Then $G \circ V$ is bounded as a mapping*

$$L^p(M, d\mu) \to L^\infty(M, d\mu) \quad \text{when } p > \frac{n}{2 - \alpha}$$

$$L^p(M, d\mu) \to L^{\frac{pn}{n - (2 - \alpha)p}}(M, d\mu), \quad \text{when } p \in \left(q^*, \frac{n}{2 - \alpha} \right), \quad (q^* = q/(q - 1))$$

Proof. Define the Stieljes measure $d\nu_x(t)$ associated to the nondecreasing function

$$t \mapsto \nu_x(t) = \int_{B(x,t)} |V(y)| |f(y)| d\mu(y),$$

and then write

$$
\begin{aligned}
|(G \circ V)f(x)| &\leq \int_M \frac{C}{d^{n-2}(x, y)} |V(y)| |f(y)| d\mu(y) \\
&\leq \int_0^D \frac{C}{r^{n-2}} d\nu_x(r) \\
&= \frac{C}{D^{n-2}} \nu_x(D) + (n - 2) \int_0^D \frac{C}{r^{n-1}} \nu_x(r) dr,
\end{aligned}
\tag{1.12}
$$

where $D = \operatorname{diam} M$. We write the right side of this chain of inequalities as $T_0(f) + T_\infty(f)$, and prove the boundedness properties for these operators separately.

The estimate of the first of these is trivial. Indeed, for any $p > q^*$,

$$\nu_x(D) \leq C \|V\|_q \|f\|_p \implies \|T_0(f)\|_\infty \leq C \|f\|_p.$$

We can thus concentrate on T_∞.

By the Hölder inequality,

$$\nu_x(r) \leq C r^{-\alpha} \|f\|_p V(x, r)^{1 - \frac{1}{p}} \leq C \|f\|_p r^{n - \alpha - \frac{n}{p}}.$$

Thus if $p > \frac{n}{2 - \alpha}$, then

$$\|T_\infty(f)\|_\infty = \sup_x \int_0^D \frac{C}{r^{n-1}} \nu_x(r) dr \leq C' \|f\|_p \int_0^D r^{1 - \alpha - \frac{n}{p}} dr \leq C'' \|f\|_p.$$

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In the second case, when $q^* < p < \frac{n}{2-\alpha}$, this integral no longer converges near 0, so we use instead a classical cutting argument from harmonic analysis, replacing the estimate for $\nu_x(r)$ for r small by a different one.

Define

$$v(x) = \mathcal{M}\left(|f|^{q^*}\right)^{\frac{1}{q^*}}(x),$$

where $\mathcal{M}(f)$ is the maximal function, defined for any L^1_{loc} function f by

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{V(x,r)} \int_{B(x,r)} |f|\, d\mu.$$

Hypothesis ii), Ahlfors n-regularity, implies volume doubling, i.e. $V(x,2r) \leq CV(x,r)$ for all $x \in M$ and $r > 0$, and from this it is easy to deduce that the maximal function, is bounded $L^1 \to L^1_{\mathrm{weak}}$,

$$\mu\left\{\mathcal{M}(f) > \lambda\right\} \leq \frac{C}{\lambda}\|f\|_1.$$

Since $\mathcal{M}$ is also (trivially) bounded $L^\infty \to L^\infty$, by interpolation we see that it is also bounded $L^s \to L^s$ for any $1 < s \leq \infty$. This will be invoked below.

From the definition of v,

$$\nu_x(r) \leq Cr^{\frac{n}{q}-\alpha}V(x,r)^{\frac{1}{q^*}}v(x) \leq Cr^{n-\alpha}v(x),$$

and hence

$$
\begin{aligned}
T_\infty(f)(x) &\leq \int_0^{\lambda(x)} \frac{C}{r^{n-1}}\nu_x(r)dr + \int_{\lambda(x)}^D \frac{C}{r^{n-1}}\nu_x(r)dr \\
&\leq \int_0^{\lambda(x)} \frac{C}{r^{n-1}}r^{n-\alpha}v(x)dr + \int_{\lambda(x)}^D \frac{C}{r^{n-1}}r^{n-\alpha-\frac{n}{p}}\|f\|_p dr \\
&\leq C\lambda(x)^{2-\alpha}v(x) + C\lambda(x)^{2-\alpha-\frac{n}{p}}\|f\|_p
\end{aligned}
$$

for any $0 < \lambda(x) < D$. The optimal choice of $\lambda(x)$ satisfies

$$v(x) = \lambda(x)^{-\frac{n}{p}}\|f\|_p,$$

so inserting this yields

$$
\begin{aligned}
T_\infty(f)(x) &\leq C\|f\|_p^{(2-\alpha)\frac{p}{n}} v(x)^{1-(2-\alpha)\frac{p}{n}} \implies \\
|T_\infty(f)(x)|^{\frac{pn}{n-(2-\alpha)p}} &\leq C\|f\|_p^{\frac{(2-\alpha)p^2}{n-(2-\alpha)p}} v(x)^p.
\end{aligned}
\tag{1.13}
$$

Finally, using the boundedness of the maximal function on L^s, $s = p/q^*$, we obtain

$$\|v\|_p \leq \|f\|_p,$$

whence, after some arithmetic,

$$\|T_\infty(f)\|_{\frac{pn}{n-(2-\alpha)p}} \leq C\|f\|_p.$$

This is the desired estimate. $\qquad\square$

This result implies Theorem 1.10 quite directly. Indeed, reverting back to the original potential, we have already noted that $0 < u \leq G \circ (-V+1)u$ and $u \in L^p$. If $p > n/(2-\alpha)$, then the first part of Theorem 1.11 bounds $\|u\|_\infty$ immediately. On the other hand, if we only know that $p > q^*$, then the second part of this Theorem shows that $G \circ (-V+1)u$ lies in L^{p_1} where $p_1 = p(n/(n-(2-\alpha)p))$. It is easy to check that there exists $\epsilon > 0$ such that $p_1/p \geq 1 + \epsilon$ for any $p > q^*$, which means that we can iterate this procedure, obtaining successively that $u \in L^{p_j}$ for an increasing sequence p_j with $p_j \geq p(1+\epsilon)^j$. Hence $p_N > n/(2-\alpha)$ for some N, so that at the next step $u \in L^\infty$.

We conclude this section with the

Proof of Lemma 1.1 We begin by noting that under the assumptions of this Lemma, the heat kernel bound (1.11) holds, hence the Schwartz kernel $K_\mu(x,y)$ corresponding to

$$\left(-\Delta + 1 + \mu^2\right)^{-1/2} = C \int_0^\infty t^{-1/2} e^{-t(-\Delta+1+\mu^2)} \, dt$$

satisfies

$$0 < K_\mu(x,y) \leq C \frac{e^{-\delta\mu d(x,y)}}{d^{n-1}(x,y)} \ . \tag{1.14}$$

for some $\delta > 0$. Moreover, defining $\varphi_\mu(r) := C e^{-\delta\mu r}/r^{n-1}$, then

$$-\varphi'_\mu(r) \leq C' \frac{e^{-\frac{\delta}{2}\mu r}}{r^n}.$$

In order to prove this Lemma, it is known (see [25, Theorem X.18]), that it suffices to show that the operator

$$A_\mu := \left(-\Delta + 1 + \mu^2\right)^{-1/2} |V|^{\frac{1}{2}} : L^2(X,\mu) \to L^2(X,\mu),$$

satisfies

$$\lim_{\mu \to +\infty} \|A_\mu\|_{L^2 \to L^2} = 0.$$

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Define $q^* \in (1,2)$ by $\frac{1}{2q} + \frac{1}{q^*} = 1$. Then for $u \in L^2(X,\mu)$, we set

$$v(x) := \left(\sup_{r>0} \frac{1}{V(x,r)} \int_{B(x,r)} |u|^{q^*} d\mu \right)^{\frac{1}{q^*}}.$$

Because $q^* \in (1,2)$, we have that $\|v\|_2 \leq C\|u\|_2$.

Now introduce the Stieljes measure $\widetilde{\nu}_x$ associated to the nondecreasing function

$$r \mapsto \int_{B(x,r)} |V|^{\frac{1}{2}} |u| \, d\mu$$

so that

$$|(A_\mu u)(x)| \leq C \int_0^D \varphi_\mu(r) \, d\widetilde{\nu}_x(r) = C\varphi_\mu(D)\widetilde{\nu}_x(D) - C \int_0^D \varphi_\mu'(r)\widetilde{\nu}_x(r) \, dr,$$

$D = \operatorname{diam} M$. Using the Hölder inequality and the fact that $|V|^{\frac{1}{2}} \in L^2$ (since $V \in L^q$ for some $q > 1$), we get

$$\widetilde{\nu}_x(D) \leq C\|u\|_2, \qquad \text{and} \qquad \widetilde{\nu}_x(r) \leq Cr^{n-\alpha/2}v(x).$$

This gives

$$|(A_\mu u)(x)| \leq C\varphi_\mu(D)\|u\|_2 + C \int_0^D e^{-\frac{\delta}{2}\mu r} r^{-\alpha/2} dr \, v(x)$$
$$\leq C\varphi_\mu(D)\|u\|_2 + C\mu^{\frac{\alpha}{2}-1}v(x),$$

which proves finally that

$$\|A_\mu\|_{L^2 \to L^2} \leq C\varphi_\mu(D) + C\mu^{\frac{\alpha}{2}-1}$$

Since $\alpha < 2$, we deduce the result. $\qquad\square$

1.4 Existence of the minimizer

We are now in a position to prove the basic existence result.

Theorem 1.12. *Let (M,g,μ) be a compact almost smooth metric measure space, in particular satisfying hypotheses i) - iii) in §1.1, and such that Scal_g satisfies either iv) a), b) or c). Supposing that Scal_g satisfies iv) a) or b), we then assume*

$$Y(M,[g]) < Y_\ell(M,[g]),$$

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while if Scal_g *satisfies iv) c), then our assumption becomes*

$$Y(M, [g]) < S_\ell(M, g).$$

Then there exists a function $u \in W^{1,2}(M) \cap L^\infty(M)$ *such that* $\|u\|_{\frac{2n}{n-2}} = 1$ *and*

$$Y(M, [g]) = \int \left(|du|^2 + \frac{n-2}{4(n-1)} \mathrm{Scal}_g \, u^2 \right) d\mu.$$

Hence on the smooth locus $\Omega \subset M$,

$$\Delta u - \frac{n-2}{4(n-1)} \mathrm{Scal}_g \, u + \frac{n-2}{4(n-1)} Y(M, [g]) u^{\frac{n+2}{n-2}} = 0.$$

Proof. We follow the lines of the classical proof of Trudinger and Aubin. Since $W^{1,2}(M) \hookrightarrow L^{2p/(p-2)}$ is compact when $p > n$, the minimum value

$$Y_p = \inf \left\{ \int \left(|d\varphi|^2 + \frac{n-2}{4(n-1)} \mathrm{Scal}_g \, \varphi^2 \right) d\mu \right\}$$

over all $\varphi \in W^{1,2}(M; d\mu)$ with $\|\varphi\|_{\frac{2p}{p-2}}^2 = 1$ is attained by some function u_p. The usual arguments from the calculus of variations show that $u_p \geq 0$ and

$$\Delta u_p - \frac{n-2}{4(n-1)} \mathrm{Scal}_g \, u_p + Y_p \, u_p^{\frac{p+2}{p-2}} = 0. \tag{1.15}$$

It follows from Theorem 1.10 that $u_p \in L^\infty$. Indeed, (1.15) implies that $\Delta u_p \geq V u_p$, where $V = c_n \mathrm{Scal}_g^- - Y_p u^{4/(p-2)}$. Under any of the hypotheses a)-c), Scal_g^- satisfies (1.2); on the other hand, setting $s = p/2q$, we have

$$r^{-n} \int_{B(x,r)} |u_p^{\frac{4}{p-2}}|^q \, d\mu \leq \|u_p\|_{\frac{2p}{p-2}}^{1/s} \, r^{n/s^*-n} = r^{-\alpha q}$$

where $\alpha = 2n/p < 2$, so the second summand in V satisfies (1.2) as well. Hence we may apply this theorem as claimed.

Now, $\lim_{p \to n} Y_p = Y(M, [g]) < Y_\ell$, so for any sufficiently small $\epsilon > 0$, $Y_p \leq Y_\ell - \epsilon$ provided p is sufficiently close to n. We now may as well replace $Y_\ell(M)$ by $S_\ell(M)$, and argue assuming that $\mathrm{Scal}_g^- \in L^q$ for some $q > n/2$.

Since $\|u_p\|_{1,2}$ is uniformly bounded, we can choose a subsequence $p_j \to n$ such that u_{p_j} converges to some function u, weakly in $W^{1,2}$ and strongly in L^q

for all $q \in [1, 2n/(n-2))$. Our goal is to show that some further subsequence converges strongly in $L^{\frac{2n}{n-2}}$. For if this is the case, then we can pass to the limit in

$$\int \langle du_p, d\varphi \rangle + \frac{n-2}{4(n-1)} \operatorname{Scal}_g u_p \, \varphi = Y_p \int \varphi \, u_p^{\frac{p+2}{p-2}} \qquad (1.16)$$

to conclude that $u \in W^{1,2} \cap L^\infty$ is a weak, and hence strong, solution of the equation with $\|u\|_{\frac{2n}{n-2}} = 1$; setting $\varphi = u$ into (1.16) with $p = n$ then gives that $Q_g(u) = Y(M, [g])$, as desired.

To accomplish this, recall the function $G_{\alpha,L}$ introduced in §1.3.1; inserting $\varphi = G_{\alpha,L}(u_p)$ with $\alpha \simeq 1$ into (1.16), and using the properties of these functions, gives

$$\int |d\phi_{\alpha,L}(u_p)|^2 \leq \frac{\alpha^2}{2\alpha - 1}\Big(c(n) \int |\operatorname{Scal}_g^-| |\phi_{\alpha,L}(u_p)|^2 + Y_p^+ \int u_p^{\frac{4}{p-2}} |\phi_{\alpha,L}(u_p)|^2\Big),$$
$$(1.17)$$

where $Y_p^+ = \max\{Y_p, 0\}$ and $c(n) = (n-2)/4(n-1)$. Next, using this in the Sobolev inequality (1.4), with ϵ replaced by $\epsilon/2$, yields that

$$(S_\ell - \epsilon/2)\|\phi_{\alpha,L}(u_p)\|_{\frac{2n}{n-2}}^2 \leq \int (C_{\epsilon/2} + c(n)| \operatorname{Scal}_g^-|) \, |\phi_{\alpha,L}(u_p)|^2$$
$$+ \frac{\alpha^2}{2\alpha - 1} Y_p^+ \int u_p^{\frac{4}{p-2}} |\phi_{\alpha,L}(u_p)|^2.$$

By the Hölder inequality and the normalization of u_p,

$$\int \phi_{\alpha,L}(u_p)^2 u_p^{\frac{4}{p-2}} \leq \|\phi_{\alpha,L}(u_p)\|_{\frac{2n}{n-2}}^2 \|u_p\|_{\frac{2p}{p-2}}^{2/p} (\operatorname{Vol} M)^{2(p-n)/pn}$$
$$= \|\phi_{\alpha,L}(u_p)\|_{\frac{2n}{n-2}}^2 (\operatorname{vol} M)^{2(p-n)/pn}.$$

Furthermore, if $|\alpha - 1|$ and $|p - n|$ are both sufficiently small, then

$$\frac{\alpha^2}{2\alpha - 1}(\operatorname{Vol} M)^{2(p-n)/pn} \leq 1 + \epsilon'.$$

Now choose ϵ' so that $(1 + \epsilon')Y_p^+ \leq S_\ell - 3\epsilon/4$. Rearranging the inequality above, we obtain

$$\frac{\epsilon}{4}\|\phi_{\alpha,L}(u_p)\|_{\frac{2n}{n-2}}^2 \leq C\|\phi_{\alpha,L}(u_p)\|_2^2 + c(n) \int |\operatorname{Scal}_g^-| |\phi_{\alpha,L}(u_p)|^2.$$

We handle this last term in two different ways, depending on whether Scal_g satisfies iv)a) or c), or else iv) b). In the former cases, the Hölder inequality estimates this term by $\|\operatorname{Scal}_g^-\|_q \|\phi_{\alpha,L}(u_p)\|_{2q/(q-1)}^2$. Since $2\alpha < 2p/(p-2)$ and

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$\alpha q/(q-1) < 2p/(p-2)$ uniformly as $p \searrow n$, we can then pass to a limit as $L \to \infty$ to conclude that

$$\|u_p\|_{\frac{2n\alpha}{n-2}} \leq C$$

for some C which is independent of p. If Scal_g satisfies iv) b), then we use Lemma 1.1 already in (1.17) to absorb this term at the expense of an extra factor of $(1 + \epsilon')$ in front of the Y_p^+, but we can then proceed exactly as before to reach the same conclusion.

To conclude, observe that by Proposition 1.8, Theorem 1.10, and using (1.16), we obtain that $\|u_p\|_\infty \leq C$ with C independent of p. This leads immediately to the strong convergence of u_p to u in $L^{2n/(n-2)}$. $\qquad\square$

Remark 1.13. *The existence of a minimizer $u \in W^{1,2}$ can also be proved using the almost optimal Sobolev inequality and a useful trick of Brezis and Lieb [9]. However, it still requires the same amount of work as above to prove that this minimizer is bounded.*

1.5   The lower bound

We now show that when Scal_g satisfies either iv) a) or b), then the minimizer obtained in the last subsection is strictly positive. As we show by explicit example Remark 2.6 later, this fails when Scal_g only satisfies iv) c). This lower bound is attained by adapting an argument of Gursky [14, Lemma 4.1].

Lemma 1.14. *For any ball $B(p,r)$ properly contained in M, there is a constant $C > 0$ such that*

$$\|\varphi\|_{L^2(M)} \leq C \left(\|d\varphi\|_{L^2(M)} + \|\varphi\|_{L^2(B)} \right) \quad \forall \, \varphi \in W^{1,2}(M).$$

Proof. Let $B' = \frac{1}{2}B$ be the ball with the same center and half the radius. Since $\mathrm{vol}\, B' > 0$, there is a Poincaré inequality on the complement of B',

$$\|\varphi\|_2 \leq C\|d\varphi\|_{L^2}. \quad \forall \, \varphi \in W_0^{1,2}(M \setminus B').$$

Hence if $\rho(x)$ is a Lipschitz cutoff function which equals 1 in B' and vanishes outside B, then

$$\begin{aligned}
\|\varphi\|_{L^2(M)} &\leq \|\varphi\|_{L^2(B)} + \|(1-\rho)\varphi\|_{L^2(M)} \\
&\leq \|\varphi\|_{L^2(B)} + C\|d[(1-\rho)\varphi]\|_{L^2(M \setminus B')} \\
&\leq C' \left(\|d\varphi\|_{L^2(M)} + \|\varphi\|_{L^2(B)} \right)
\end{aligned}$$

as claimed. $\qquad\square$

Proposition 1.15. *Assume that* Scal_g *satisfies either iv) a) or b). Let u be the minimizing solution obtained in the last subsection. Then* $\inf_M u > 0$.

Proof. We know that for every $\varphi \in W^{1,2}(M)$,

$$\int_M \langle du, d\varphi \rangle + \int_M \frac{n-2}{4(n-1)} \mathrm{Scal}_g\, u\varphi = Y \int_M u^{\frac{n+2}{n-2}} \varphi.$$

It is easy to check that for $\epsilon, \delta > 0$ the function

$$\varphi = (\epsilon + u)^{-1-2\delta} \in W^{1,2}.$$

Inserting this into the identity above gives

$$-\frac{1+2\delta}{\delta^2} \int_M \left| d(\epsilon + u)^{-\delta} \right|^2 = -\int_M \frac{n-2}{4(n-1)} \mathrm{Scal}_g\, u\varphi + Y \int_M u^{\frac{n+2}{n-2}} \varphi,$$

and hence

$$\int_M \left| d(\epsilon + u)^{-\delta} \right|^2 \le \frac{\delta^2}{1+2\delta} C \left[\int_M (|\,\mathrm{Scal}_g\,| + u^{\frac{4}{n-2}})(\epsilon + u)^{-2\delta} \right] \qquad (1.18)$$

with $C = \max\left\{ \frac{n-2}{4(n-1)}, |Y| \right\}$. By the Hölder inequality, and assuming that Scal_g satisfies iv) a),

$$\int_M \left| d(\epsilon + u)^{-\delta} \right|^2 \le \frac{\delta^2}{1+2\delta} C' \left[\int_M (\epsilon + u)^{-\frac{2\delta n}{n-2}} \right]^{1-\frac{2}{n}},$$

where $C' = C \left[\|\,\mathrm{Scal}_g\,\|_{L^{n/2}} + \|u\|_{L^{2n/(n-2)}} \right]$. If Scal_g satisfies iv) b) instead, then we handle the first term on the right using Lemma 1.1 in an obvious way, and end up with the same inequality.

Applying the Sobolev inequality to the function $(u+\epsilon)^{-\delta}$ and using the Lemma above and this inequality, we conclude that for δ small enough,

$$\left(\int_M (\epsilon + u)^{-\frac{2\delta n}{n-2}} \right)^{1-\frac{2}{n}} \le C \int_B (\epsilon + u)^{-2\delta}.$$

Assuming that $\overline{B} \subset \Omega$, then by the known upper bound on u and the Harnack inequality, there is a $c > 0$ such that $u \ge c > 0$ on B. Letting $\epsilon \to 0$ in the estimate above, we see that

$$\int_M u^{-\frac{2\delta n}{n-2}} \le C.$$

To conclude the proof, note that the convexity of $x \mapsto x^\delta$ implies that

$$\Delta(u^{-\delta}) \ge \left(-\delta \frac{n-2}{4(n-1)} \mathrm{Scal}_g + \delta Y u^{\frac{4}{n-2}} \right) u^{-\delta} := V u^{-\delta}$$

This function V satisfies (1.2) (resp. $V \in L^q$), so Theorem 1.10 (resp. Proposition 1.8) gives that $\|u^{-\delta}\|_\infty < \infty$, i.e. $\inf u > 0$. $\qquad\square$

2 The Yamabe problem on stratified spaces

We now specialize the results of the last section to the setting of spaces with smooth stratifications, also called iterated edge spaces, with corresponding adapted iterated edge metrics. We begin by reviewing some aspects of the differential topology and metric structure of these spaces, then prove that all such spaces satisfy a Sobolev inequality and the other hypotheses from §1.1, and then formulate the precise existence theorem in this setting.

2.1 Smoothly stratified spaces

We now briefly review the definition of smoothly stratified pseudomanifolds. Further details can be found in in the foundational monograph of Verona [31] and the exposition by Pflaum [24]. Basic definitions vary between sources, and the recent paper [4] provides a clarification and unified presentation of some of this material; we follow the notation and development of [4, §2] and refer to it for all further details, in particular, for a proof that this class of spaces coincides with the class of iterated edge spaces considered by Cheeger [11], cf. also [19].

Let X be a compact stratified space. By definition, X admits a disjoint decomposition into strata, $X = \sqcup \Sigma_j$, where each Σ_j is a (possibly disconnected, possibly open) manifold of dimension j. There are a set of axioms describing how the strata fit together, key amongst which is that each connected component of Σ_j has a tubular neighbourhood $\mathcal{U}$ which is the total space of a smooth bundle over that component with fibre a truncated cone $C(Z_j)$. Here Z_j is itself a compact stratified space and is called the link of that cone bundle. There is a natural filtration of X in terms of 'depth' of singularities. Thus compact smooth manifolds are said to have depth 0, and if Z is a compact space of depth k, then a space which has a neighbourhood which is a truncated cone or a bundle of truncated cones with link Z has depth $k + 1$. This depth filtration is different than the filtration of X determined by the closures $\overline{\Sigma_j}$ since a stratum of high codimension can have low depth (for example, an isolated conic singularity only has depth 1). An interesting subtlety is that the fibration of each tubular neighbourhood is required to have a smooth trivialization, but it is not a priori obvious what the proper class of smooth maps and diffeomorphisms between stratified spaces should be. This is precisely the point where the various treatments cited above differ. The definition we give is inductive: once a suitable definition of a stratified diffeomorphism between spaces of depth j has been given, one declares the suspension of such a diffeomorphism, i.e. the radial extension of that diffeomorphism to the cone over that space, to be smooth. This extends the definition to spaces of depth $j + 1$. (This smoothness hypothesis excludes many spaces that constitute the standard broader

class of stratified spaces, where these local trivializations are only required to be continuous; see [4] for more on this.) A smoothly stratified space X is a pseudomanifold if the stratum of maximal dimension is dense in X. In distinction to [4], we allow X to have strata of codimension one, but since hypersurface boundaries play a somewhat different role in our main theorem, we say that X is an iterated edge space (or smoothly stratified pseudomanifold) with boundary in this case.

Definition 2.1. *For each $k \geq 0$, define the class $\mathcal{I}_k$ of compact iterated edge spaces of depth k as follows:*

- *An element of $\mathcal{I}_0$ is a compact smooth manifold;*

- *A space X lies in $\mathcal{I}_k$ if it has a decomposition $X = X' \cup X''$, where X'' is an element of $\mathcal{I}_{k-1}$ with a codimension one boundary along the intersection $X' \cap X''$ and each component of X' is the total space of a cone bundle over a compact base space B with fibre a truncated cone $C(Z)$ for some $Z \in \mathcal{I}_{k-1}$. (The common boundary $\partial X' \cap \partial X''$ is the total space of a bundle over the same base B with fibre Z.)*

- *Any element $X \in \mathcal{I}_k$ has a well-defined dimension, where in the decomposition above, $\dim X'' = \dim X' = \dim B + \dim Z + 1$.*

Note that if $X \in \mathcal{I}_k$, then its stratum Y of maximal depth k is necessarily a compact smooth manifold.

Every iterated edge space X carries a class of adapted iterated edge metrics, which are also defined inductively. Thus assuming that we have described the class of admissible iterated edge metrics on all iterated edge spaces of depth $k-1$, let X be an iterated edge space of depth k. If Y is the stratum of depth k and $\mathcal{U}$ the tubular neighbourhood around Y, then we can assume that the structure of the metric g on X has been described on $X \setminus \mathcal{U}$. In particular, if x is the radial coordinate on the conic fibres of $\mathcal{U}$, then $\partial \mathcal{U} = \{x = 1\}$ is an iterated edge space of depth $k-1$, which is the total space of a fibration over Y with fibre Z. Let G be an admissible metric on $\partial \mathcal{U}$, which we assume has been defined by the inductive hypothesis. Then it is of the form $\pi^* h + k$, where $\pi : \partial \mathcal{U} \to Y$ is the fibration, h is an ordinary Riemannian metric on Y and k is a symmetric 2-tensor on $\partial \mathcal{U}$ which restricts to an admissible metric on each fibre Z. We also assume that k is totally degenerate on a subspace of dimension ℓ at each point, $\ell = \dim Y$. Now define the metric g on $\mathcal{U}$ by coning off each fibre. In other words, we set

$$g_0 = dx^2 + \pi^* h + x^2 k,$$

where x is the radial function on each conical fibre. An admissible metric on $\mathcal{U}$ is any metric which has the form $g_0 + \kappa$ where κ is polyhomogeneous on the

resolution of this neighbourhood and such that $|\kappa|_{g_0}$ decays at some rate x^γ. The notion of polyhomogeneous regularity will be defined in the next section, so for the moment consider this only as an appropriate smoothness condition. It is possible to consider finite regularity metrics too, but for simplicity we shall not do so. Finally, a metric g on all of X is admissible if it is admissible in a sense defined via the inductive hypothesis away from $\mathcal{U}$, and which takes this form in $\mathcal{U}$.

To make this more explicit, let $\mathcal{V} \times Z$ be a local trivialization of $\partial \mathcal{U}$, where $\mathcal{V} \subset \mathbb{R}^\ell$ is an open ball, and Z is the depth $k-1$ link, and introduce a coordinate system $y \in \mathcal{V}$ as well as local coordinates z on the smooth stratum of Z. Then we can write

$$G = \sum_{i,j=1}^{\ell} h_{ij}(y)dy^i dy^j + \sum_{i=1}^{\ell} \sum_{p=1}^{n-\ell-1} b_{ip}(y,z)dy^i dz^p + \sum_{p,q=1}^{n-\ell-1} k_{pq}(y,z)dz^p dz^q,$$

where the second and third sums here constitute the tensor k. Thus

$$g_1 = dx^2 + \sum_{i,j=1}^{\ell} h_{ij}(x,y)dy^i dy^j +$$

$$x^2 \sum_{i=1}^{\ell} \sum_{p=1}^{n-\ell-1} b_{ip}(x,y,z)dy^i dz^p + x^2 \sum_{p,q=1}^{n-\ell-1} k_{pq}(x,y,z)dz^p dz^q,$$

where h_{ij}, b_{ip} and k_{pq} are smooth for $0 \le x \le 1$.

Pick any point $p \in Y$, the depth k stratum, and use coordinates so that $x = 0$, $y = 0$ and $z = z_0$ at p. Define the dilations $D_\lambda : (x,y,z) \to (\lambda x, \lambda y, z)$, and consider the family of metrics $g_\lambda := D_{1/\lambda}^* g_1$. Then, as $\lambda \to \infty$,

$$\lambda^2 g_\lambda \longrightarrow dx^2 + \sum_{i,j=1}^{\ell} h_{ij}(0)dy^i dy^j + x^2 \sum_{p,q=1}^{n-\ell-1} k_{pq}(0,z)dz^p dz^q$$

which is a product metric on the space $\mathbb{R}^\ell \times C(Z)$ (where $\mathbb{R}^\ell$ is identified with $T_p Y$). Note that the metric k on Z in this product decomposition depends on the basepoint $p \in Y$, whereas $h(p)$ is simply the Euclidean metric in some linear change of coordinates. We summarize this by saying that $|dy|^2 + k(y)$ is the model iterated edge metric for g at $y \in Y$. Observe that the perturbation κ disappears in this same rescaling limit. An important consequence of this metric structure is that we can choose local coordinates (x,y,z) near any fibre Z_y corresponding to $y \in Y$ such that the scalar Laplacian takes the form

$$\Delta_g = \partial_x^2 + x^{-1}A(x,y,z)\partial_x + x^{-2}\Delta_{k(x,y,z)} + \Delta_{h(x,y,z)} + E,$$

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where all coefficients are smooth, or at least bounded and polyhomogeneous, with $A(0, y, z) \equiv n - \ell - 1$, and where E is a higher order error term of first order in the sense that it is a sum of smooth multiples of the vector fields $x\partial_x$, $x\partial_y$ and ∂_z.

There are slightly less restrictive types of metrics which one can handle without too much more difficulty; for example, one could allow terms like $xdy^i dz^p$, or (for the final metric, after the perturbation κ is added), terms like $dxdy^i$ or $xdxdz^p$, but again for simplicity we do not do so here.

2.2 Sobolev inequalities

We next show that the Sobolev inequality (1.1) holds on any iterated edge space with adapted metric.

Proposition 2.2. *Let (M, g) be an iterated edge space, possibly with boundary, with admissible metric g as defined in the last subsection. Denote by Ω its principal open dense stratum. Then the Sobolev inequality (1.1) is valid for all $u \in \mathcal{C}_0^\infty(\Omega)$, and hence for all $u \in W_0^{1,2}(M)$.*

Proof. We reduce the problem of verifying (1.1) on an iterated edge space (M, g) of depth k using the following observations. First, the Sobolev inequality is localizable; in other words, if (1.1) holds on every set in a finite open cover $\{\mathcal{U}_\alpha\}$ of M, then using a partition of unity we can show that it holds on all of M. Now decompose $M = M' \cup \mathcal{U}$ where $\mathcal{U}$ is the tubular neighbourhood around the maximal depth stratum in M and M' is an iterated edge space of depth $k - 1$ with boundary. We may assume by induction that (1.1) holds for all functions with support in M', so it suffices to verify this inequality for functions with support in $\mathcal{U}$. Localizing further, we can restrict attention to functions supported in a local trivialization $\mathcal{V} \times C_1(Z)$ of $\mathcal{U}$, where $\mathcal{V} \subset \mathbb{R}^\ell$ is an open ball and Z is a compact space of depth strictly less than k. Finally, noting that (1.1) is stable under quasi-isometric changes of metric, we may assume that g is the product metric $|dy|^2 + dx^2 + x^2 h_Z$ on $\mathbb{R}^\ell \times C(Z)$.

We now recall the fact that (1.1) holds on a space (W, g_W) if and only if the heat kernel $H^W(t, w, w')$ for the scalar Laplacian satisfies

$$H^W(t, w, w') \leq C' t^{-n/2} \tag{2.1}$$

for all $w, w' \in W$ and $0 < t < 1$, where $n = \dim W$. (Indeed, [26, Theorem 4.1.3] states that the Nash inequality is equivalent to this heat kernel estimate; the equivalence of the Nash inequality with (1.1) is treated in [26, Ch. 3]; alternately, [21] shows that the Sobolev inequality implies the heat kernel bound, while by [30], the heat kernel bound implies the Sobolev inequality.)

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We apply this in two separate ways. First, since Z is a compact iterated edge space of depth less than k, (1.1) holds on Z; hence $H^Z(t, z, z') \leq Ct^{-m/2}$ where $m = \dim Z = n - \ell - 1$. Using this, we shall show that

$$H^{C(Z)}(t, x, z, x', z') \leq Ct^{-(m+1)/2}. \tag{2.2}$$

Since the corresponding heat kernel bound on $\mathbb{R}^\ell$ is standard, and since heat kernels multiply for Riemannian products, we see that

$$H^{\mathbb{R}^\ell \times C_1(Z)}(t, y, x, z, y', x', z') =$$
$$H^{\mathbb{R}^\ell}(t, y, y') H^{C(Z)}(t, x, z, x', z') \leq Ct^{-\ell/2 - (n-\ell)/2} = Ct^{-n/2}.$$

Hence (1.1) holds on $\mathcal{V} \times C_1(Z)$.

It remains to verify (2.2). Denote by $H^{a,b}$ the heat kernel on the conic nappe $C_{a,b}(Z) = \{(x, z) : a \leq x \leq b\}$, with Dirichlet conditions at the boundaries. Note that (1.1) holds on $C_{1,2}(Z)$ with respect to the product metric, hence by quasi-isometry invariance, it also holds with respect to the conic metric. Therefore,

$$H^{1,2}(t, x, z, x', z') \leq Ct^{-(m+1)/2}. \tag{2.3}$$

Now recall the basic scaling property of the heat kernel. For any $\lambda > 0$, the heat kernels $H^{\lambda,2\lambda}$ and $H^{1,2}$ are related to one another by

$$H^{\lambda,2\lambda}(\lambda^2 t, \lambda x, y, \lambda x', y') \lambda^{m+1} = H^{1,2}(t, x, y, x', y').$$

Using (2.3) and changing variables, we obtain

$$H^{\lambda,2\lambda}(t, x, y, x', y') \leq Ct^{-(m+1)/2}.$$

The squared $L^{2n/(n-2)}$ norm on the left in (1.1) and the squared L^2 norm of ∇u on the right both scale the same way, but the squared L^2 norm of u scales differently. Thus when we apply this for the sequence $\lambda = 2^{-j}$, and assemble the pieces using a dyadic partition of unity $\{\chi(2^j x)\}$, where χ is supported on $1/2 \leq x \leq 4$, then we conclude that

$$\left(\int_{C_{0,1}(Z)} u^{\frac{2n}{n-2}} \, dV_g \right)^{\frac{n-2}{n}} \leq C \left(\int_{C_{0,1}(Z)} |\nabla u|^2 \, dV_g + \int_{C_{0,1}(Z)} x^{-2} u^2 \, dV_g \right),$$

which is valid for all $u \in \mathcal{C}_0^\infty(C_{0,1}(Z) \setminus \{0\})$.

Moreover, we claim that there is a Poincaré-Hardy inequality in this setting, i.e.

$$\frac{(m-1)^2}{4} \int_{C(Z)} x^{-2} u^2 \, dV_g \leq \int_{C(Z)} |\nabla u|^2 \, dV_g.$$

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This is standard when Z is a compact smooth manifold, but since Δ_Z has discrete spectrum by virtue of Corollary 1.7 and the inductive hypothesis, we can reduce to the individual eigenspaces, where it becomes the usual Hardy inequality on $\mathbb{R}^+$. As an alternate path to proving this we could use the argument in [10], which uses integration by parts and hence requires only the density of functions with compact support in the smooth locus.

We have now verified (2.2), and hence have proved that (1.1) holds for all iterated edge spaces of depth k. □

Appealing to Lemma 2.4 below and combining the result above with Proposition 1.4, we obtain the

Corollary 2.3. *Let (M, g) be a compact iterated edge space with adapted metric which satisfies one of the conditions in Lemma 2.4 so that at least one of the hypotheses iv) a) or iv) b) hold. Then the local Yamabe constant $Y_\ell(M, [g])$ is strictly positive.*

In the next subsection we identify this local Yamabe constant somewhat more explicitly.

2.3 Existence of Yamabe metrics

We now turn to the problem of finding minimizers for the functional Q_g in this setting of iterated edge spaces. The main issue now is to understand when the hypotheses iv) a), b) or c) hold so that we can apply Theorem 1.12.

We first describe the local Yamabe invariant of an iterated edge space (M, g). Let $p \in M$. If p lies in the depth 0 stratum, i.e. is a smooth point, then the local Yamabe invariant at p is just $Y(S^n)$. If p lies on a depth k stratum Σ, then as described at the end of §2.1, the rescaled limit of the metric g equals $dx^2 + dy^2 + x^2 k_p$, where k_p is the metric on the link Z at p and dy^2 is the Euclidean metric on $\mathbb{R}^\ell$, $\ell = \dim \Sigma$. Note that this is conformally equivalent to the product metric $g_{\mathbb{H}^{\ell+1}} + k_p$ on $\mathbb{H}^{\ell+1} \times Z$, and hence

$$Y(\mathbb{R}^\ell \times C(Z), [dx^2 + dy^2 + x^2 k_p]) = Y(\mathbb{H}^{\ell+1} \times Z, [g_{\mathbb{H}^{\ell+1}} + k_p]).$$

This generalizes the fact that the Yamabe invariant of the cone $C(Z)$ and the cylinder $\mathbb{R} \times Z$ are the same. In any case, enumerating the depth j strata as $\{\Sigma_j\}$, and denoting the link around Σ_j by Z_j, then we have proved that

$$Y_\ell(M, [g]) = \min_j \inf_{p \in Y_j} \{Y(\mathbb{R}^\ell \times C(Z_j), [dy^2 + dx^2 + x^2 (k_{Z_j})_p])\}. \qquad (2.4)$$

Now consider the hypotheses in §1.1. The verification of i) is a straightforward exercise using cutoff functions and mollifications, which we leave to the reader.

The Ahlfors n-regularity is even easier. We have verified in §2.2 that the Sobolev inequality (1.1) holds; this is condition iii). On the other hand, the hypotheses iv) a)-c) require more careful attention. Indeed, as we now show, these hypotheses are valid for a rather limited set of iterated edge metrics.

We begin with some general remarks. It is clear from the structure of adapted iterated edge metrics that if Σ_j is any stratum of depth j and x_j is the radial distance function in the tubular neighbourhood of Σ_j, then the scalar curvature Scal_g can blow up no faster than x_j^{-2}. If $\dim Z_j = f_j$, so $\dim \Sigma_j = \ell_j = n - f_j - 1$, then $dV_g \approx x_j^{f_j} dx_j dV_{h_j} dV_{k_j}$ near Σ_j, where h_j is a smooth metric on Σ_j pulled back to the tubular neighbourhood and k_j restricts to a metric on the (depth $j - 1$) link Z_j. Assuming that g is smooth in the variable x_j, then

$$\mathrm{Scal}_g = \frac{A_0^{(j)}}{x_j^2} + \frac{A_1^{(j)}}{x_j} + \mathcal{O}(1).$$

To correlate this with the hypotheses iv) a) - c), note that $1/x_j^2 \in L^q$ implies $q < (f_j + 1)/2$, and hence we can never take $q > n/2$ as in iv) a). Similarly, the Morrey condition requires that for some $q > 1$,

$$r^{-n} \int_{B_r} x_j^{-2q+f_j} \, dx_j dy dz = \mathcal{O}(r^{-n+f_j-2q+1+\ell_j}) = \mathcal{O}(r^{-2q}),$$

which is (1.2) with $\alpha = 2$ and hence does not fit into our hypotheses. Suppose, however, that the coefficient of x_j^{-2} in this expansion vanishes. Then $x_j^{-1} \in L^q$ provided $q < f_j + 1$, and hence we can take $q > n/2$ and see that iv) a) is satisfied provided $f_j + 1 > n/2$. Similarly, the Morrey condition holds because we only need choose $q > 1$, which is always possible since $f_j + 1 > 1$, and for such a q we then have

$$r^{-n} \int_{B_r} x_j^{-q+f_j} \, dx_j dy dz = \mathcal{O}(r^{-n+f_j-q+1+\ell_j}) = \mathcal{O}(r^{-q}),$$

which is (1.2) with $\alpha = 1$. This proves the

Lemma 2.4. *The scalar curvature* Scal_g *satisfies iv) a) if and only if then* $A_0^{(j)} = 0$ *for all j and in addition* $A_j^{(1)} = 0$ *whenever* $f_j \leq (n-2)/2$. *The scalar curvature* Scal_g *satisfies iv) b) (for some $q > 1$ and $0 \leq \alpha < 2$) if and only if* $A_0^{(j)} = 0$ *for all j. Finally,* Scal_g *satisfies iv) c) if and only if* $A_0^{(j)} \geq 0$ *for all j and* $A_1^{(j)} \geq 0$ *when* $f_j \leq (n-2)/2$.

It is clear from this that the terms in Scal_g which blow up like $1/x_j^2$ are the most problematic. The $1/x_j$ terms always fit within hypothesis iv) b).

We will also need to consider metrics with a polyhomogeneous expansion, which include noninteger powers of x or terms like $x^\gamma (\log x)^\ell$, $\ell \in \mathbb{N}_0$. For any such metric, the scalar curvature function also has an expansion and there is an obvious extension of this lemma which requires the vanishing or nonnegativity of the coefficient of any term $x^\gamma (\log x)^\ell$ where $n\gamma/2 + f_j \leq 1$.

Isolated conic points

We first examine the simplest case: an isolated conic singularity, where $\dim \Sigma_j = 0$ and $f_j = n - 1$. For simplicity drop the index j, but to be consistent with later notation, we still use $f = n - 1$. A well-known formula [23, p.69] shows that an exact warped product conic metric $g = dx^2 + x^2 k$ has $\mathrm{Scal}_g = x^{-2}(\mathrm{Scal}_k - f(f-1))$. More generally, if k depends smoothly on x, then

$$\mathrm{Scal}_g = \frac{\mathrm{Scal}_{k(0)} - f(f-1)}{x^2} + \mathcal{O}(x^{-1}) \tag{2.5}$$

This leading coefficient vanishes if and only if $\mathrm{Scal}_{k(0)} = f(f-1)$, which indicates a very strong geometric and topological obstruction: if the scalar curvature of (M, g) is bounded, then in particular the link $(Z, k(0))$ must have positive Yamabe invariant.

We now study whether it is possible to remove the singular terms in the expansion of Scal_g using a conformal change. If $\widehat{g} = w^{\frac{4}{n-2}} g$, then

$$\mathrm{Scal}_{\widehat{g}} = -c(n)^{-1} w^{-\frac{n+2}{n-2}}(\Delta_g w - c(n)\,\mathrm{Scal}_g\, w), \quad c(n) = \frac{n-2}{4(n-1)}. \tag{2.6}$$

Thus if we introduce the expansion in x of Δ_g and Scal_g, we obtain that

$$\mathrm{Scal}_{\widehat{g}} \sim -c(n)^{-1} w^{-\frac{n+2}{n-2}} \times$$
$$\left(\partial_x^2 + \frac{n-1}{x}\partial_x + \frac{1}{x^2}(\Delta_{k_0} - c(n)(\mathrm{Scal}_{k(0)} - f(f-1))) + \frac{1}{x}E \right) w.$$

The error term $E \sim E_0 + xE_1 + \ldots$ is a second order differential operator composed of a sum of smooth multiples of products of the vector fields $x\partial_x$ and ∂_z, and also includes the terms beyond the leading one in the expansion for Scal_g.

From this we see that a necessary and sufficient condition for the coefficient of x^{-2} to vanish is that

$$(\Delta_{k(0)} - c(n)\,\mathrm{Scal}_{k(0)})w_0 = -c(n)f(f-1)w_0,$$

where w_0 is the restriction of w to $x = 0$, i.e. the leading term in the expansion of w. We denote the operator which appears on the left here by $\mathcal{L}^n_{k(0)}$; it is a special element of the family of operators

$$\mathcal{L}^m_{k_0} = \Delta_{k_0} - c(m) \operatorname{Scal}_{k(0)}, \quad c(m) = \frac{m-2}{4(m-1)}, \tag{2.7}$$

for any value of m. Note that $\mathcal{L}^f_{k(0)}$ is simply the conformal Laplacian of $(Z, k(0))$. The positivity of the operator $-\mathcal{L}^n_{k(0)}$ plays an important role in the main existence theorem of [3], as we now recall.

Theorem 2.5 ([3]). *Suppose that (M^n, g) is a space with isolated conic singularities, and that at each conic point p, the operator $-\mathcal{L}^n_{k(0)}$ on the link $(Z, k(0))$ has all eigenvalues strictly positive. Suppose too that $Y(M, [g]) < Y_\ell(M, [g])$. Then there exists a function u on M which minimizes Q_g and is such that $\widehat{g} = u^{\frac{4}{n-2}} g$ remains incomplete. Conversely, there exists a minimizer u such that $\widehat{g}$ is incomplete only if $-\mathcal{L}_{k(0)} > 0$.*

We do not assert that u is bounded, nor that the new constant scalar curvature metric is conic. We shall explain shortly why u may fail to be bounded; in the next section we describe the polyhomogeneous regularity of u which makes clear that $\widehat{g}$ is in fact still conic.

We can recover part of this theorem immediately from Theorem 1.12. Indeed, if the lowest eigenvalue of $-\mathcal{L}^n_{k(0)}$ is *exactly* $c(n)f(f-1)$, then we can choose w so that w_0 is the eigenfunction corresponding to this lowest eigenvalue, so that w_0 is strictly positive, and then $\operatorname{Scal}_{\widehat{g}}$ blows up only like x^{-1}, hence lies in L^q for $q \in (n/2, n)$. We thus obtain the existence of a bounded, strictly positive function u which minimizes $Q_{\widehat{g}}$. Clearly $u^{4/(n-2)} \widehat{g}$ is quasi-isometric to $\widehat{g}$ and thence to g.

In order to prove existence whenever $-\mathcal{L}^n_{k(0)}$ is positive, fix $\delta > 0$, to be specified below, and define $g_\delta = x^{2\delta-2} g$. The change of variables $\xi = x^\delta / \delta$ gives the transform

$$g_\delta = x^{2\delta-2}(dx^2 + x^2 k) = d\xi^2 + \xi^2 \delta^2 k,$$

so g_δ is still conic, but its link metric has been scaled by δ^2. Oserve also that $-\mathcal{L}^n_{\delta^2 k(0)} = -\delta^{-2} \mathcal{L}^n_{k(0)}$. This means that if we first replace g by g_δ and then set $\widehat{g}_\delta = w^{4/(n-2)} g_\delta$, then we can make the coefficient of ξ^{-2} vanish provided that $\delta^{-2} \lambda_0(-\mathcal{L}^n_{k(0)}) = c(n)f(f-1)$, which determines the value of δ, and w_0 is the corresponding eigenfunction. We are then in a position to apply Theorem 1.12 again, this time with $\widehat{g}_\delta$ as the background metric. The solution metric is quasi-isometric to g_δ, and hence conic.

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The converse statement is an easy consequence of these same calculations, at least once we show that a minimizer u (or indeed any positive solution of the corresponding Euler-Lagrange equation) has a polyhomogeneous expansion as $x \to 0$, which we do in the next section. Thus we have now given an independent proof of Theorem 2.5.

The condition that $\lambda_0(-\mathcal{L}^n_{k(0)}) > 0$ is actually stronger than the condition that $(Z, k(0))$ is Yamabe positive. Indeed, referring back to the family of operators (2.7), an easy calculation shows that if $p < q$, then there are positive constants $A = A(p, q)$ and $B = B(p, q)$ such that

$$- \mathcal{L}^p_{k(0)} = A(-\mathcal{L}^q_{k(0)}) + B(-\Delta_{k(0)}) \implies -\mathcal{L}^p_{k(0)} \geq A(-\mathcal{L}^q_{k(0)}). \qquad (2.8)$$

In particular, taking $p = f$ and $q = n$, then the positivity of $-\mathcal{L}^n_{k(0)}$ implies that the conformal Laplacian of $(Z, k(0))$ is positive, which is well-known to imply the existence of a conformally equivalent (constant) positive scalar curvature metric.

Remark 2.6. *When the scalar curvature of $k(0)$ is constant and greater than* $(n-1)(n-2)$

$$\mathrm{Scal}_{k(0)} = (n-1)(n-2)(1+\delta^2),$$

then the scalar curvature of the metric g is positive near the conical points. Hence the scalar curvature satisfies the condition iv)-c), but the above argument shows that the minimizer of the Yamabe problem is not uniformly positive. It will decay like $x^{\frac{n-2}{2}\delta}$. Therefore the hypotheses of the Proposition 1.15 can not be relaxed.

Simple edges

We next suppose that M has only simple edges, i.e. that each singular stratum Σ_j is a compact smooth manifold of dimension $n - r_j$. For simplicity we assume that there is only one such stratum and drop the index j. A tubular neighbourhood of Σ is a cone bundle with compact smooth link Z^f, and in this neighbourhood, $g \sim dx^2 + x^2 k + \pi^* h$, where h is a metric on Σ, π the projection from this neighbourhood onto Σ and k a symmetric 2-tensor so that $dx^2 + x^2 k$ pulls back to an asymptotically conic metric on each conical fibre of the tubular neighbourhood.

Lemma 2.7. *If g has a smooth expansion as $x \to 0$, then*

$$\mathrm{Scal}_g = \frac{\mathrm{Scal}_{k(0,y)} - f(f-1)}{x^2} + \frac{A_1(y,z)}{x} + \mathcal{O}(1). \qquad (2.9)$$

This is slightly less obvious than in the isolated conic case and can be verified by direct calculation. It can also be proved by observing that since Scal_g has an expansion with initial term x^{-2}, if we dilate the coordinates via $(x, y, z) \rightarrow (\lambda x, \lambda y, z)$ (around some fixed basepoint $y_0 \in \Sigma$) and let g_λ be the corresponding pulled back metric, then the coefficient of the leading term of homogeneity -2 in λ must be the limit of $\lambda^2 \text{Scal}_{g_\lambda}$ as $\lambda \rightarrow \infty$. However, it is evident that $\lambda^{-2} g_\lambda$ converges to the product metric $dx^2 + x^2 k(y_0, z) + dy^2$ on $C(Z) \times \mathbb{R}^{n-r}$, which has scalar curvature exactly equal to $x^{-2}(\text{Scal}_{k(0)} - f(f - 1))$.

Let us now investigate whether it is possible to conformally transform away the singular term of order $1/x^2$ in the expansion for Scal_g at Σ. As we have already shown, the existence of the singular term A_1/x can be handled using hypothesis iv) b).

Replace g by $\widehat{g} = w^{\frac{4}{n-2}} g$ and proceed with exactly the same formal calculation as in the isolated conic case. There are several important differences in this setting. First, it is still clearly necessary that $\lambda_0\big(-\mathcal{L}^n_{k(0)(y)}\big) \equiv c(n)f(f-1)$ and that $w_0(y, z)$ must lie in this eigenspace for every y. In particular, this eigenvalue must be independent of $y \in \Sigma$, which is a strong rigidity statement. Assuming this, we can thus eliminate the x^{-2} term. Note that we may – and indeed we shall later need to – let w_0 depend nontrivially but smoothly on y. This does not interfere with this calculation since although E now contains y derivatives, these are accompanied by a nonnegative power of x, hence the derivatives of w_0 can be regarded as junk terms in the expansion and can be solved away.

Applying Theorem 1.12 and the fact that we have arranged that Scal_g satisfies iv) b), we have now proved the

Theorem 2.8. *Let (M, g) have at most simple edge singularities. Assume that $c(n)f(f-1) = \lambda_0\big(-\mathcal{L}^n_{k_j(0)}\big)$ along each singular stratum Σ_j. Suppose in addition that $Y(M, [g]) < Y_\ell(M, [g])$. Then there exists a bounded and strictly positive function u which minimizes $Q_{\widehat{g}}$. The metric $u^{4/(n-2)}\widehat{g}$ is quasi-isometric to the initial metric g.*

Unlike the conic case, we cannot go further and still remain within the class of iterated edge metrics. Indeed, if we were to multiply g by the conformal factor $x^{2\delta-2}$, then this factor would also multiply π^*h; if $\delta < 1$, the corresponding metric would have infinite diameter, while if $\delta > 1$ then the entire edge Σ would be collapsed to a point. In either case, we would leave the category of smoothly stratified spaces and iterated edge metrics.

We shall not carry out the detailed study of when we can modify g conformally to ensure the weaker condition iv) c), that $(\text{Scal}_{\widehat{g}})_- \in L^q$ for some $q > n/2$. The conditions are not particularly explicit, and the solution u is not bounded away from 0 so that the solution metric is again not of iterated edge type.

The general case

We now come to the general case where (M, g) is a smoothly stratified space with iterated edge metric. As we shall explain, the conditions needed to obtain a solution of the Yamabe problem in this category are even more restrictive than in the simple edge case.

We begin with a statement of the simplest case, which follows immediately from Theorem 1.12.

Theorem 2.9. *Let (M, g) be a compact smoothly stratified space with iterated edge metric g. Suppose that along each stratum Σ_j, the link metric (Z_j, k_j) has $\mathrm{Scal}_{k_j} \equiv f_j(f_j - 1)$, and in addition, that $Y(M, [g]) < Y_\ell(M, [g])$. Then there exists a bounded, strictly positive function u which minimizes Q_g, and hence $u^{\frac{4}{n-2}} g$ is an iterated edge metric with constant scalar curvature.*

The regularity theorem in the next section will show that u is polyhomogeneous, so that this solution metric is indeed an iterated edge metric in the strict sense of the word.

As in the conic and simple edge cases, we might also seek conditions on the initial metric g so that there is *some* conformally related metric $\widehat{g}$ which satisfies the hypotheses of Theorem 1.12. As in those cases, the idea is to choose the conformal factor w to kill the appropriate singular terms at each stratum.

The calculations we have done above may be carried out almost exactly as before, and lead to the following necessary conditions: for any stratum Σ with link (Z^f, k), we assume that

1) The operator $-\mathcal{L}_k^n$ on Z has discrete spectrum;

2) The operator $-\mathcal{L}_k^n$ has lowest eigenvalue $c(n)f(f-1)$ at every point of Σ;

The first hypothesis, on the discreteness of the spectrum, may be surprising. The fact that the scalar Laplacian Δ_k itself is essentially self-adjoint and has discrete spectrum is a consequence of Corollary 1.7 and Proposition 2.2. However, the extra term $c(n) \mathrm{Scal}_k$ may blow up like $1/r^2$ on approach to any of the singular strata of Z itself, which changes the indicial roots. It is not hard to find examples of spaces (Z, k), even with just isolated conic singularities, where $-\mathcal{L}_k^n$ is not even semi-bounded, which simply amounts to the fact that $c(n) \mathrm{Scal}_k$ diverges to $-\infty$ like $-c/r^2$ with leading coefficient larger c than the permissible Hardy estimate bound $(f-1)^2/4$. This question is closely related to the problems studied in [20], see also [10] and [2]

One further point which requires explanation is that in using condition 2), we use a conformal factor w which has leading coefficient along Σ equal to the eigenfunction w_0 for $-\mathcal{L}_k^n$ corresponding to the eigenvalue $c(n)f(f-1)$. In order to stay

with the class of iterated edge metrics, it is necessary that w_0 be bounded above and strictly positive, and this may fail. Indeed, it is easy to construct examples of operators $-\Delta_k + V$ on Z with V blowing up like $1/r^2$, where the ground state eigenfunction either vanishes at the singular set of Z or else blows up at some rate. Fortunately, the fact that this does not occur follows from the hypotheses we have already made.

Proposition 2.10. *If (M, g) satisfies conditions 1) - 2) along each singular stratum Σ, and if (Z, k) is any link, then the eigenfunction w_0 for the ground state eigenvalue of $-\mathcal{L}_k^n$ is bounded and strictly positive.*

Proof. The assertion follows from the regularity theory for eigenfunctions, reviewed in the next section, and an indicial root computation. Let Σ' be any singular stratum of Z with corresponding link (Z', k'), $\dim Z' = f'$. If r is the radial variable to this stratum, then near Σ',

$$\mathcal{L}_k^n = \partial_r^2 + \frac{f'}{r}\partial_r + \frac{1}{r^2}(\Delta_{k'} - c(n)(\mathrm{Scal}_{k'} - f'(f'-1))) + \Delta_{\Sigma'} + E',$$

where E' contains all higher order terms (including higher order terms in the expansion of Scal_k). The indicial roots of this operator are then equal to

$$\nu_j^{\pm} = -\frac{f'-1}{2} \pm \sqrt{\frac{(f'-1)^2}{4} + \mu_j},$$

where the μ_j are the eigenvalues of $-(\Delta_{k'} - c(n)(\mathrm{Scal}_{k'} - f'(f'-1))) = -\mathcal{L}_{k'}^n - c(n)f'(f'-1)$. By assumption, $\mu_0 = 0 < \mu_1 \leq \cdots$, hence

$$\nu_0^+ = 0 < \nu_1^+ < \ldots, \text{ and } \nu_0^- = 1 - f' > \nu_1^- > \ldots.$$

By the aforementioned regularity theory, $w_0 \sim cr^{\nu_j+}\phi_j + \ldots$ near Σ', where ϕ_j is the eigenfunction corresponding to ν_j^+. However, w_0 must remain strictly positive in the interior of Z by the standard maximum principle arguments, hence $j = 0$ and $w_0 \sim c\phi_0$ which shows that it remains bounded and strictly positive near this stratum. $\qquad\square$

To conclude this section we observe finally that assuming the conditions 1) and 2) on (M, g), if (Z, k) is any link, then by (2.8), the conformal Laplacian $-\mathcal{L}_k^f$ is strictly positive.

3   Regularity

The final goal of this paper is to study the regularity of the minimizers of the functional Q_g obtained in the last section when (M, g) is an iterated edge space.

The techniques here are nonvariational, so the results below apply to any positive solution of

$$\Delta_g u - c(n)\operatorname{Scal}_g u + c(n)\Lambda u^{\frac{n+2}{n-2}} = 0, \qquad (3.1)$$

assuming that u satisfies a natural growth condition so that $u^{\frac{4}{n-2}}g$ remains quasi-isometric to g, and which is satisfied for the solutions constructed in §2.3. Note that if $\|u\|_{\frac{2n}{n-2}} = 1$, then $\Lambda = \operatorname{Scal}_g \geq Y(M,[g])$. We shall prove that u is conormal along each of the singular strata, and has (at least) a partially polyhomogeneous expansion. We explain this below.

As in §2.3, we first prove regularity when M has only isolated conic singularities. The steps in this case are quite elementary, but rely on a certain number of definitions concerning the function spaces and the b-calculus of pseudodifferential operators. With these preliminaries, the proof of regularity in this case is only a few lines. We then prove regularity for spaces with simple edges, and here we can quote known results about the pseudodifferential edge calculus from [17]. For the general case we need only mimic one small part of this edge calculus to be able to deduce what we need.

3.1 Conic singularities

Our first goal is to prove the

Proposition 3.1. *Let (M, g) be a compact space with only isolated conic singularities. Assume that g is a polyhomogeneous conic metric. Suppose that u is a solution of (3.1) which is positive on the regular part of M and which satisfies $u < Cx^{-(n-2)/2+\epsilon}$ for some $\epsilon > 0$ near each conic point, where x is the radial distance to the conic tip. Then u is polyhomogeneous as $x \to 0$. If the link (Z, k) satisfies the simplest condition, that $R_k \equiv (n-1)(n-2)$, then the expansion of u takes the form $u \sim c_0 + c_1(z)x^{\nu_1} + \ldots$, where c_0 is a positive constant. If we only have that the lowest eigenvalue of $-\mathcal{L}_k^n$ is positive, then $u \sim x^{(\delta-1)(n-2)/2}(c_0(z) + c_1(z)x^{\nu_1'} + \ldots)$, where $c_0(z)$ is strictly positive and is the ground state eigenfunction of $-\mathcal{L}_k^n$ and δ is the constant described in §2.3. The exponents ν_1, ν_1', etc., which appear in this expansion are determined by the higher eigenvalues of $-\mathcal{L}_k^n$.*

This regularity is local near each conic tip, but we emphasize that it is global with respect to the links (Z, k). The main issue is to prove that the solution is conormal (see below); its polyhomogeneity and the precise form of its expansion are then formal consequences.

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Rather than analyzing (3.1) directly, we rewrite it relative to the background metric $\widetilde{g} = x^{-2}g = (x^{(2-n)/2})^{4/(n-2)}$, yielding

$$(\Delta_{\widetilde{g}} - c(n)R_{\widetilde{g}})v + c(n)\Lambda v^{\frac{n+2}{n-2}} = 0, \tag{3.2}$$

where the original solution $u = x^{(2-n)/2}v$. Note that by the transformation properties of the conformal Laplacian, $R_{\widetilde{g}} = R_k + \cdots$, hence

$$\left((x\partial_x)^2 + \Delta_k - c(n)R_k + E\right)v + c(n)\Lambda v^{\frac{n+2}{n-2}} = 0; \tag{3.3}$$

here E contains all higher order error terms from both $\Delta_{\widetilde{g}}$ and $\mathrm{Scal}_{\widetilde{g}}$. Note that our assumption that $|u| \leq Cx^{(2-n)/2+\epsilon}$ becomes $|v| \leq Cx^\epsilon$, which is much easier to work with. (In fact, it is precisely because we are able to work with solutions which decay that this argument is easier than the corresponding regularity theorem in [18].)

Before embarking on the proof, we recall several facts, first about the function spaces which will be used and then about parametrices in the b-calculus. For simplicity, assume that M has only one conic point and that the radial function x is extended globally and is strictly positive elsewhere on M.

Definition 3.2. *Decompose M as $M' \sqcup C_1(Z)$, where the second factor is the truncated cone over Z with coordinates $z \in Z$ and $x \in (0, 1]$. It is most natural to work relative to the complete metric $\widetilde{g}$, and in this geometry, $-\log x$ is the distance function on the asymptotically cylindrical end.*

i) *The space $\mathcal{C}_b^{k,\gamma}(M)$ consists of all functions v which lie in the ordinary Hölder space $\mathcal{C}^{k,\gamma}$ on M', and in addition satisfy $(x\partial_x)^j \partial_z^\alpha v \in \mathcal{C}_b^{0,\gamma}$, where the latter space is defined using the seminorm*

$$[v]_{b;0,\gamma} := \sup_{\substack{(x,z)\neq(x',z') \\ 1/2 \leq x/x' \leq 2}} \frac{|v(x,z) - v(x',z')|}{dist_{\widetilde{g}}((x,z),(x',z'))^\gamma}.$$

We also define

$$x^\mu \mathcal{C}^{k,\gamma}(M) = \{v = x^\mu \widetilde{v} : \widetilde{v} \in \mathcal{C}_b^{k,\gamma}(M)\}.$$

ii) *For any $\nu \in \mathbb{R}$, let $\mathcal{A}^\nu(M) = \bigcap_{k\geq 0} x^\nu \mathcal{C}^{k,\gamma}(M)$. This is the space of conormal functions. Next, define the space of polyhomogeneous functions $\mathcal{A}_{\mathrm{phg}}(M)$ to consist of all conormal functions v which admit complete asymptotic expansions with smooth coefficients, and write $\mathcal{A}_{\mathrm{phg}}^\nu$ for all polyhomogeneous functions with leading term $x^{\nu_0}\phi(z)$ for some ν_0 with (real part)*

greater than or equal to ν. Note that $x^\nu \log x \in \mathcal{A}_{\mathrm{phg}}^{\nu-\epsilon}$ for any $\epsilon > 0$. Finally, let $\nu < \nu'$ be any pair of real numbers, and define $\mathcal{A}_{\mathrm{phg}}^{\nu,\nu'}(M) = \mathcal{A}_{\mathrm{phg}}^\nu(M) + \mathcal{A}^{\nu'}(M)$; thus v is in this space if it has a partial polyhomogeneous expansion with initial term bounded by x^ν and with conormal 'remainder' vanishing like $x^{\nu'}$.

As a first step in the proof of Proposition 3.1, note that since $\widetilde{g}$ has locally uniformly controlled geometry and since v is uniformly bounded, we obtain directly from classical Hölder estimates that $v \in \mathcal{A}^\epsilon(M)$.

Using this in (3.3), we have

$$((x\partial_x)^2 + \Delta_k - c(n)R_k + E)v = c(n)\Lambda v^{\frac{n+2}{n-2}} \in \mathcal{A}^{\tau\epsilon}, \qquad (3.4)$$

where $\tau = \frac{n+2}{n-2}$ (and we assume that $\tau\epsilon \leq 1$ for simplicity).

The conformal Laplacian $L_{\widetilde{g}}$ is an example of an elliptic b-operator, and the operator on the left in this last equation is its asymptotic model at $x = 0$ and called its indicial operator, $I(L_{\widetilde{g}})$. This indicial operator can be analyzed quite directly using the Mellin transform in the x variable. To this end, introduce the indicial roots of $L_{\widetilde{g}}$; these are the values ν for which there exists a function $\phi \in \mathcal{C}^\infty(Z)$ such that

$$I(L_{\widetilde{g}})x^\nu\phi = 0 \Leftrightarrow L_{\widetilde{g}}x^\nu\phi = \mathcal{O}(x^{\nu+\epsilon'})$$

for some $\epsilon' > 0$. It is easy to see in this case, by separation of variables, that the indicial roots are given by

$$\nu_j^\pm := \pm\sqrt{\lambda_j}, \quad \text{where } \mathrm{spec}(-\mathcal{L}_k) = \{\lambda_j\}. \qquad (3.5)$$

The coefficient function ϕ for any such indicial root equals the corresponding eigenfunction ϕ_j. Since the lowest eigenvalue of $-\mathcal{L}_k^n = -\Delta_k + c(n)R_k$ is strictly positive, we have that $\ldots \leq \nu_1^- < \nu_0^- < 0 < \nu_0^+ < \nu_1^+ \leq \ldots$. The indicial roots are the precise rates of growth or decay of approximate solutions of $L_{\widetilde{g}}w = 0$.

The b-calculus is merely a systematized method for passing from information about the indicial operator to the corresponding information about $L_{\widetilde{g}}$ itself. We quote some results from this theory, referring to [17] for a careful development of this b-calculus as well as the more general edge calculus which will be invoked below.

Proposition 3.3. ([17, Theorem 4.4]) *For $k \in \mathbb{N}$ and $0 < \gamma < 1$, the mapping*

$$L_{\widetilde{g}} : x^\nu \mathcal{C}_b^{k+2,\gamma}(M) \longrightarrow x^\nu \mathcal{C}_b^{k,\gamma}(M)$$

is Fredholm if and only if $\nu \neq \nu_j^\pm$ for any j.

Proposition 3.4. ([17, Proposition 3.28]) *Let $f \in \mathcal{A}^{\nu'}(M)$ and suppose that $L_{\widetilde{g}} v = f$, where $v \in \mathcal{A}^{\nu}(M)$ for some $\nu < \nu'$. Then $v \in \mathcal{A}^{\nu,\nu'}_{\mathrm{phg}}(M)$, or in other words, v has a partial expansion*

$$
v = \sum_{j=0}^{N} \sum_{p=0}^{N'_j} x^{\mu_j} (\log x)^p v_{jp}(z) + \widetilde{v},
$$

where $\widetilde{v} \in \mathcal{A}^{\nu'}$ and the μ_j lie in the interval (ν, ν'). Moreover, if $f \in \mathcal{A}^{\nu,\nu'}_{\mathrm{phg}}(M)$, then $v \in \mathcal{A}^{\nu,\nu'}_{\mathrm{phg}}(M)$ and if $f \in \mathcal{A}^{\nu}_{\mathrm{phg}}$, then $v \in \mathcal{A}^{\nu}_{\mathrm{phg}}$.

Remark 3.5. *Once we know that $v \in \mathcal{A}^{\nu,\nu'}_{\mathrm{phg}}$ for some ν, ν', we can determine the exponents μ_j which appear in its expansion by a formal computation with the equation $L_{\widetilde{g}} v = f$. In particular, if the link metric $k(x)$ depends smoothly on x, then all μ_j are of the form $\nu_j^{\pm} + \ell$, where $\nu_j^{\pm}$ is an indicial root and $\ell \in \mathbb{N}_0$.*

These are proved by constructing a parametrix G for $L_{\widetilde{g}}$, which is a pseudodifferential operator, depending on the choice of (nonindicial!) weight ν. The fundamental mapping results for this class of operators, proved in [17, §3] give that

$$
G : x^{\nu} \mathcal{C}_b^{k-2,\gamma}(M) \longrightarrow x^{\nu} \mathcal{C}_b^{k,\gamma}(M), \tag{3.6}
$$

$$
G : \mathcal{A}^{\nu'}(M) \longrightarrow \mathcal{A}^{\nu,\nu'}_{\mathrm{phg}}(M), \tag{3.7}
$$

$$
G : \mathcal{A}^{\nu}_{\mathrm{phg}}(M) \longrightarrow \mathcal{A}^{\nu}_{\mathrm{phg}}(M) \tag{3.8}
$$

are all bounded mappings, for any $k \in \mathbb{N}_0$, and $\nu < \nu'$ with $\nu \notin \{\nu_j^{\pm}\}$.

Proof of Proposition 3.1: Appealing directly to (3.4) and applying Proposition 3.4, we deduce that $v \in \mathcal{A}^{\epsilon,\tau\epsilon}_{\mathrm{phg}}$. Using this on the right side of this equation gives $v \in \mathcal{A}^{\epsilon,\tau^2\epsilon}_{\mathrm{phg}}$, and bootstrapping further, we obtain that $v \in \mathcal{A}_{\mathrm{phg}}$. The precise form of its expansion can then be determined by substituting this expansion into the equation. In particular, the leading exponent in the expansion is equal to one of the positive indicial roots ν_j^+, i.e.

$$
v \sim \phi_j(z) x^{\nu_j^+} + \mathcal{O}(x^{\nu_j^+ + \epsilon'}).
$$

Since $v > 0$ when $x > 0$ and the eigenfunction ϕ_j changes sign unless $j = 0$, this expansion must start with $\phi_0(z) x^{\nu_0^+}$.

In the simplest case, where $R_k \equiv (n-1)(n-2)$, we can easily see that $\nu_0^+ = (n-2)/2$ and $\phi_0(z)$ is a positive constant. In particular, $u = x^{(2-n)/2} v$ is bounded and strictly positive. In the more general case where we only assume that

$-\mathcal{L}_k^n > 0$, we conclude that $u \sim \phi_0(z) x^{(2-n)/2 + \nu_0^+}$, where now ϕ_0 is variable but still strictly positive. Recalling the conformal change $g \mapsto g_\delta = x^{2\delta - 2} g$ introduced in §2.3 which allows one to reduce to the case that $\nu_0^+ = (n-2)/2$ with respect to the new radial coordinate $\xi = x^\delta / \delta$ provided one chooses δ correctly, we have that the solution is bounded and strictly positive relative to the background metric g_δ. Note that in the first case, the solution metric is still exact conic, while in the second case, it is conformally exact conic. $\qquad \square$

3.2 Simple edges

We next present the corresponding proof of regularity when (M, g) has simple edges. In this case we can take advantage of the construction of parametrices and their mapping properties in the edge calculus. There is one important new step in the proof, beyond what was needed in the conic case, but then it proceeds exactly as before.

We begin with exactly the same initial conformal change, replacing g by $\widetilde{g} = x^{-2} g$, with the corresponding changes of conformal Laplacian and the Yamabe equation. We write the solution metric as $v^{4/(n-2)} \widetilde{g}$, so that the initial hypothesis is that $0 \leq v \leq C x^\epsilon$. Using the uniform geometry of $\widetilde{g}$ we deduce immediately that v is infinitely differentiable with respect to the geometry of this metric, or equivalently, in local coordinates,

$$|(x\partial_x)^j (x\partial_y)^\alpha \partial_z^\beta v| \leq C_{j,\alpha,\beta} x^\epsilon$$

for all j, α, β. Note however that this is not sufficient to assert conormality yet because these estimates do not control the y derivatives, i.e. the derivatives tangent to the singular stratum, as $x \to 0$. Obtaining this control requires the parametrix for the conformal Laplacian $L_{\widetilde{g}}$ as constructed in [17]. The difference between this operator and the one for the conic problem is the inclusion of the tangential part of the Laplacian $x^2 \Delta_y$; thus when rewriting the Yamabe equation in the form (3.4), the error term E on the left includes $x^2 \Delta_y v$, which at this stage we cannot guarantee vanishes any faster than x^ϵ.

The indicial roots of $L_{\widetilde{g}}$ are defined exactly as in the conic case above, and in particular are given by exactly the same formulæ, with the important difference that the eigenvalues λ_j of $-\mathcal{L}_k^n$ can vary with y. Because of this, we do not expect solutions to have discrete asymptotics, i.e. polyhomogeneity, in the sense above, and we shall be satisfied with a more limited partial polyhomogeneity result. Recall, however, our key hypothesis that the lowest eigenvalue of $-\mathcal{L}_k^n$ is equal to $c(n) f(f-1)$, and in particular is independent of y. Let $\nu_0^\pm = \pm \sqrt{c(n) f(f-1)}$ be the corresponding indicial roots. Fix $0 < \nu < \nu_0^+$. By [17, Theorem 6.1], there

exists a parametrix G for $L_{\tilde{g}}$ with the properties: $G \circ L_{\tilde{g}} = \mathrm{Id} - Q$, where Q is a finite rank operator which maps into $\mathcal{A}^{\nu_0^+ + \epsilon'}$, and the analogues of (3.6), (3.7) and (3.8) are all valid provided $\nu' \leq \nu_1^+$ and furthermore, that we replace $\mathcal{C}_b^{k,\gamma}$ by the Hölder spaces $\mathcal{C}_e^{k,\gamma}$ based on derivatives with respect to $x\partial_x$, $x\partial_y$ and ∂_z. The final extra property we need is that the commutator $[\partial_y, G] = G_1$ is an operator in the edge calculus with exactly the same mapping properties as G itself; this is [17, Theorem 3.30].

We now use all of this information as follows. First let us apply the parametrix G to $L_{\tilde{g}}v + c(n)\Lambda v^{(n+2)/(n-2)} = 0$, to obtain that

$$v = G(c(n)\Lambda v^{\frac{n+2}{n-2}}) + Qv.$$

We wish to use this equation to prove that v is conormal. Once this is done, the existence of the partial expansion follows directly from (3.7) and bootstrapping. Applying ∂_y to each side gives

$$\partial_y v = G(c(n)\Lambda \partial_y v^{\frac{n+2}{n-2}}) + G_1(c(n)\Lambda v^{\frac{n+2}{n-2}}) + \partial_y Qv.$$

The final term Qv is already conormal so presents no difficulties here. Applying (3.6) to G_1, we see that this term lies in $x^\epsilon \mathcal{C}_e^{k,\gamma}$ for all k. Finally, write $\partial_y v^{(n+2)/(n-2)} = cv^{4/(n-2)}\partial_y v$, and recall that $v^{4/(n-2)} \leq Cx^{4\epsilon/(n-2)}$. In the simplest version of this argument, $\epsilon \geq (n-2)/4$ so that $v \leq Cx$. Note that this is satisfied if we assume that our original solution u is bounded, which is a more natural assumption once we leave the setting of isolated singularities; for simplicity we assume that this is the case. Then $|v^{4/(n-2)}\partial_y v| \leq |x\partial_y v|$, hence this term lies in $\cap_k x^\epsilon \mathcal{C}_e^{k,\gamma}$, and finally, $\partial_y v \in \cap_k x^\epsilon \mathcal{C}_e^{k,\gamma}$. Iterating this argument gives eventually that $\partial_y^\alpha \in \cap_k x^\epsilon \mathcal{C}_e^{k,\gamma}$ for every multi-index α, so that $v \in \mathcal{A}^\epsilon$. Proceeding as explained above shows that v, and hence u, has a partial expansion.

3.3 The general case

The final step in our proof of regularity is to extend these arguments to handle the case when M is a general smoothly stratified space and g is an iterated edge metric satisfying the hypotheses i) - iii) of §2.3. We also assume that the solution u is bounded. We shall be rather brief here since we have covered almost all of the main points of the argument already.

Suppose that M is a space of depth k. By induction and the fact that this regularity theorem is localizable, we may assume that u has the appropriate regularity, i.e. partial polyhomogeneity, everywhere except possibly along the singular strata of highest depth, and so we can focus on these. Indeed, we can focus on the

equation (3.1) in a neighbourhood of the form $\mathcal{U} \times C_1(Z)$ where $\mathcal{U} \subset \mathbb{R}^\ell_y$, Z is a compact smoothly stratified space of depth $k - 1$, where again the regularity result is known by induction, and $C_1(Z)$ is the truncated cone over Z. Let x be the radial variable on this cone. As before, we conformally transform this problem, writing the equation in terms of the new partially completed background metric $\widetilde{g} = x^{-2}g$, so $u = x^{(2-n)/2}v$. As an important part of our inductive hypothesis, we assume the operator $-\mathcal{L}_k^n$ on Z has discrete spectrum. This allows us to define indicial roots and analyze the indicial operator exactly as before.

We use function spaces $\mathcal{C}_{\text{ice}}^{k,\gamma}$ based on derivatives with respect to $x\partial_x$, $x\partial_y$ and vector fields V which are tangent to the fibre Z and all of its singular strata. We have, by local elliptic regularity and induction, that $v \in x^{(n-2)/2}\mathcal{C}_{\text{ice}}^{k,\gamma}$ for all k, so the remaining job is to prove that v is conormal, that every tangential derivative ∂_y^α decays at this same rate. We do this by applying exactly the same commutator argument, but unfortunately there is no ready-made iterated edge calculus to which we can appeal. Fortunately we need very few consequences of such a calculus and can deduce these from a somewhat primitive parametrix construction. This is carried out in more detail in [4]. Recall that we wish to construct an operator G such that, with $L_{\widetilde{g}}$ equal to the conformal Laplacian for $\widetilde{g}$, $G \circ L_{\widetilde{g}} = I - Q$ where Q maps into $\mathcal{A}^\nu(M)$, which has mapping properties analogous to (3.6), (3.7), (3.8), and finally, so that the commutator $[\partial_y, G]$ enjoys the same mapping properties.

For simplicity let N denote the localized space $\mathcal{U} \times C_1(Z)$ blown up fibrewise at the vertex of each cone. Thus N is the product $\mathcal{U} \times [0, 1) \times Z$. We construct G just as in [17] by regarding its Schwartz kernel $G(x, y, z, \widetilde{x}, \widetilde{y}, \widetilde{z})$ as a distribution on the space N_{ice}^2 obtained from $N \times N$ by blowing up the fibre diagonal at the corner $\{x = \widetilde{x} = 0\}$. In fact, this space is identified with the space $(\mathcal{U} \times [0, 1))_0^2 \times Z \times Z$, where the first factor is the 0-double space of $\mathcal{U} \times [0, 1)$, as constructed and used in [17]. This has three boundary components, the left and right faces, corresponding to $\widetilde{x} \to 0$ and $\widetilde{x} \to 0$, and the front face ff, which covers $\{x = \widetilde{x} = 0, y = \widetilde{y}\}$ and is the face created in the blowup. The key point is that the lift of G to this space is conormal and partially polyhomogeneous at all faces. Its leading coefficient at ff is precisely the inverse for the so-called normal operator

$$N(L_{\widetilde{g}}) = (x\partial_x)^2 + x^2\Delta_y + \mathcal{L}_k^n$$

which is globally defined on $\mathbb{R}^+ \times \mathbb{R}^\ell \times Z$. The invertibility of this normal operator on $x^\nu \mathcal{C}_{\text{ice}}^{k,\gamma}$ is the main ellipticity hypothesis, and is proved exactly as in the simple edge case, relying on the fact that $-\mathcal{L}_k^n$ is strictly positive. We analyze this normal operator by taking the Fourier transform in y, thus reducing it to

$$(x\partial_x)^2 + \mathcal{L}_k^n - x^2|\eta|^2,$$

and then rescaling, setting $t = x|\eta|$, to arrive at

$$(t\partial_t)^2 + \mathcal{L}_k^n - t^2,$$

which is an operator on $\mathbb{R}^+ \times Z$. The inverse for this can be analyzed as in [17], using mainly that $\mathcal{L}_k^n$ has discrete spectrum. We denote the Schwartz kernel of this inverse by $\widehat{G}(t, z, \widetilde{t}, \widetilde{z})$. Rescaling and taking the inverse Fourier transform, we see that the Schwartz kernel of the inverse of the normal operator equals

$$\int_{\mathbb{R}^\ell} e^{i(y-\widetilde{y})\eta} G_0(x|\eta|, \widetilde{x}|\eta|, z, \widetilde{z})|\eta|^q \, d\eta$$

for an appropriately chosen q.

The mapping properties for this inverse are deduced exactly as in the simple edge case. The final fact concerning the commutator $[\partial_y, G]$ can now be proved just as in the edge calculus, where this ultimately reduces to the fact that the commutator of the globally defined translation-invariant vector field ∂_y on $\mathbb{R}^\ell$ commutes exactly with the explicit inverse for the normal operator written above.

We have been (extremely) sketchy in the development in this last section. There are several reasons for this. The first is simply that while the idea is very close to that used in the simple edge case, it would still take considerable space to write out these details fully, and given the relatively minor importance of this final result, we have chosen not to do so. The sketch above is intended to provide a guide for anyone with a reasonable familiarity with the edge calculus. Finally, we point out that there are certainly other proofs that one might carry out to prove this regularity which would be more elementary in the sense that they do not explicitly use blowups and pseudodifferential operators, but which would require a substantial amount of verification of elementary details nonetheless.

As described in the introduction, in a companion piece to this paper we give the full details of a proof of rather different sort of regularity statement which requires very little regularity of the background iterated edge metric, and which shows that the solution u to the Yamabe equation enjoys some Hölder continuity properties.

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