

L^2 HARMONIC FORMS ON NON-COMPACT RIEMANNIAN MANIFOLDS

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First, I want to present some questions on L^2 -harmonic forms on non-compact Riemannian manifolds. Second, I will present an answer to an old question of J. Dodziuk on L^2 -harmonic forms on manifolds with flat ends. In fact some of the analytical tools presented here apply in other situations (see [C4]).

1. THE SPACE OF HARMONIC FORMS

Let (M^n, g) be a complete Riemannian manifold. We denote by $\mathcal{H}^k(M, g)$ or $\mathcal{H}^k(M)$ its space of L^2 -harmonic k -forms, that is to say the space of L^2 k -forms which are closed and coclosed:

$$\mathcal{H}^k(M) = \{\alpha \in L^2(\Lambda^k T^*M), d\alpha = \delta\alpha = 0\},$$

where

$$d : C_0^\infty(\Lambda^k T^*M) \longrightarrow C_0^\infty(\Lambda^{k+1} T^*M)$$

is the exterior differentiation operator and

$$\delta : C_0^\infty(\Lambda^{k+1} T^*M) \longrightarrow C_0^\infty(\Lambda^k T^*M)$$

its formal adjoint. The operator d does not depend on g but δ does; δ is defined with the formula:

$$\forall \alpha \in C_0^\infty(\Lambda^k T^*M), \forall \beta \in C_0^\infty(\Lambda^{k+1} T^*M), \int_M \langle d\alpha, \beta \rangle = \int_M \langle \alpha, \delta\beta \rangle.$$

The operator $(d + \delta)$ is elliptic hence the elements of $\mathcal{H}^k(M)$ are smooth and the L^2 condition is only a decay condition at infinity.

2. IF THE MANIFOLD M IS COMPACT WITHOUT BOUNDARY

If M is compact without boundary, then these spaces have finite dimension, and we have the theorem of Hodge-DeRham : the spaces $\mathcal{H}^k(M)$ are isomorphic to the real cohomology groups of M :

$$\mathcal{H}^k(M) \simeq H^k(M, \mathbb{R}).$$

Hence the dimension of $\mathcal{H}^k(M)$ is a homotopy invariant of M , i.e. it does not depend on g . A corollary of this and of the Chern-Gauss-Bonnet formula is :

$$\chi(M) = \sum_{k=0}^n (-1)^k \dim \mathcal{H}^k(M) = \int_M \Omega^g,$$

where Ω^g is the Euler form of (M^n, g) ; for instance if $\dim M = 2$ then $\Omega^g = \frac{KdA}{2\pi}$, where K is the Gaussian curvature and dA the area form.

3. WHAT IS TRUE ON A NON-COMPACT MANIFOLD

Almost nothing is true in general:

The space $\mathcal{H}^k(M, g)$ can have infinite dimension and the dimension, if finite, can depend on g . For instance, if M is connected we have

$$\mathcal{H}^0(M) = \{f \in L^2(M, d\text{vol}_g), f = \text{constant}\}.$$

$$\begin{aligned} \text{Hence } \mathcal{H}^0(M) &= \mathbb{R} & \text{if } \text{vol } M < \infty, \\ \text{and } \mathcal{H}^0(M) &= \{0\} & \text{if } \text{vol } M = \infty. \end{aligned}$$

For instance if \mathbb{R}^2 is equipped with the euclidean metric, we have $\mathcal{H}^0(\mathbb{R}^2, \text{eucl}) = \{0\}$, and if \mathbb{R}^2 is equipped with the metric $g = dr^2 + r^2 e^{-2r} d\theta^2$ in polar coordinates, then $\mathcal{H}^0(\mathbb{R}^2, g) = \mathbb{R}$. We have also that $\mathcal{H}^k(\mathbb{R}^n, \text{eucl}) = \{0\}$, for any $k \leq n$. But if we consider the unit disk in \mathbb{R}^2 equipped with the hyperbolic metric $4|dz|^2/(1-|z|^2)^2$ then it is isometric to the metric $g_1 = dr^2 + \sinh^2 r d\theta^2$ on \mathbb{R}^2 . And then we have $\dim \mathcal{H}^1(\mathbb{R}^2, g_1) = \infty$. As a matter of fact if $P(z) \in \mathbb{C}[z]$ is a polynomial, then $\alpha = P'(z)dz$ is a L^2 harmonic form on the unit disk for the hyperbolic metric (this comes from the conformal invariance, see 5.2). So we get an injection $\mathbb{C}[z]/\mathbb{C} \rightarrow \mathcal{H}^1(\mathbb{R}^2, g_1)$. However, the spaces $\mathcal{H}^k(M, g)$ satisfy the following two properties:

- These spaces have a reduced L^2 cohomology interpretation:
Let $Z_2^k(M)$ be the kernel of unbounded operator d acting on $L^2(\Lambda^k T^*M)$, or equivalently

$$Z_2^k(M) = \{\alpha \in L^2(\Lambda^k T^*M), d\alpha = 0\},$$

where the equation $d\alpha = 0$ has to be understood in the distribution sense i.e. $\alpha \in Z_2^k(M)$ if and only if

$$\forall \beta \in C_0^\infty(\Lambda^{k+1} T^*M), \int_M \langle \alpha, \delta\beta \rangle = 0.$$

That is to say $Z_2^k(M) = (\delta C_0^\infty(\Lambda^{k+1} T^*M))^\perp$. The space $L^2(\Lambda^k T^*M)$ has the following Hodge-DeRham-Kodaira orthogonal decomposition

$$L^2(\Lambda^k T^*M) = \mathcal{H}^k(M) \oplus \overline{dC_0^\infty(\Lambda^{k-1} T^*M)} \oplus \overline{\delta C_0^\infty(\Lambda^{k+1} T^*M)},$$

where the closure is taken with respect to the L^2 topology.; We also have

$$Z_2^k(M) = \mathcal{H}^k(M) \oplus \overline{dC_0^\infty(\Lambda^{k-1} T^*M)},$$

hence we have

$$\mathcal{H}^k(M) \simeq Z_2^k(M) / \overline{dC_0^\infty(\Lambda^{k-1} T^*M)}.$$

A corollary of this identification is the following:

Proposition 3.1. *The space $\mathcal{H}^k(M, g)$ are quasi-isometric invariant of (M, g) . That is to say if g_1 and g_2 are two complete Riemannian metrics such that for a $C > 1$ we have*

$$C^{-1}g_1 \leq g_2 \leq Cg_1,$$

then $\mathcal{H}^k(M, g_1) \simeq \mathcal{H}^k(M, g_2)$.

In fact, the spaces $\mathcal{H}^k(M, g)$ are biLipschitz-homotopy invariants of (M, g) .

- The finiteness of $\dim \mathcal{H}^k(M, g)$ depends only of the geometry of ends :

Theorem 3.2. (*J. Lott, [L]*) *The spaces of L^2 -harmonic forms of two complete Riemannian manifolds, which are isometric outside some compact set, have simultaneously finite or infinite dimension.*

4. A GENERAL PROBLEM

In view of the Hodge-DeRham theorem and of J. Lott's result, we can ask the following questions :

- (1) What geometrical condition on the ends of M insure the finiteness of the dimension of the spaces $\mathcal{H}^k(M)$?
Within a class of Riemannian manifold having the same geometry at infinity :
- (2) What are the links of the spaces $\mathcal{H}^k(M)$ with the topology of M and with the geometry "at infinity" of (M, g) ?
- (3) And what kind of Chern-Gauss-Bonnet formula could we hope for the L^2 -Euler characteristic

$$\chi_{L^2}(M) = \sum_{k=0}^n (-1)^k \dim \mathcal{H}^k(M) \quad ?$$

There are many articles dealing with these questions. I mention only three of them :

- (1) In the pioneering article of Atiyah-Patodi-Singer ([A-P-S]), the authors considered manifolds with cylindrical ends : that is to say there is a compact subset K of M such that $M \setminus K$ is isometric to the Riemannian product $\partial K \times]0, \infty[$. Then they show that the dimension of the space of L^2 -harmonic forms is finite ; and that these spaces are isomorphic to the image of the relative cohomology in the absolute cohomology. These results were used by Atiyah-Patodi-Singer in order to obtain a formula for the signature of compact manifolds with boundary.
- (2) In [M, M-P], R. Mazzeo and R. Phillips give a cohomological interpretation of the space $\mathcal{H}^k(M)$ for geometrically finite real hyperbolic manifolds .
- (3) The solution of the Zucker's conjecture by Saper and Stern ([S-S]) shows that the spaces of L^2 harmonic forms on hermitian locally symmetric space with finite volume are isomorphic to the middle intersection cohomology of the Borel-Serre compactification of the manifold.

5. AN EXAMPLE

I want now to discuss the L^2 Gauss-Bonnet formula through one example. The sort of L^2 Gauss-Bonnet formula one might expect is a formula of the type

$$\chi_{L^2}(M) = \int_K \Omega^g + \text{terms depending only on } (M - K, g),$$

where $K \subset M$ is a compact subset of M ; i.e. $\chi_{L^2}(M)$ is the sum of a local term $\int_K \Omega^g$ and of a boundary (at infinity) term. Such a result will imply a relative index formula :

If (M_1, g_1) and (M_2, g_2) are isometric outside compact set $K_i \subset M_i$, $i = 1, 2$, then

$$\chi_{L^2}(M_1) - \chi_{L^2}(M_2) = \int_{K_1} \Omega^{g_1} - \int_{K_2} \Omega^{g_2}.$$

It had been shown by Gromov-Lawson and Donnelly that when zero is not in the essential spectrum of the Gauss-Bonnet operator $d + \delta$ then this relative index formula is true ([G-L, Do]). For instance, by the work of Borel and Casselman [BC], the Gauss-Bonnet operator is a Fredholm operator if M is an even dimensional locally symmetric space of finite volume and negative curvature.

In fact such a relative formula is not true in general. The following counterexample is given in [C2]:

(M_1, g_1) is the disjoint union of two copies of the euclidean plane and (M_2, g_2) is two copies of the euclidean plane glued along a disk. As these surface are oriented with infinite volume, we have $i = 1, 2$:

$$\mathcal{H}^0(M_i, g_i) = \mathcal{H}^2(M_i, g_i) = \{0\}.$$

And we also have $\mathcal{H}^1(M_1, g_1) = \{0\}$. Moreover

Lemma 5.1. $\mathcal{H}^1(M_2, g_2) = \{0\}$.

This comes from the conformal invariance of this space. Indeed, it is a general fact:

Proposition 5.2. *If (M^n, g) is a Riemannian manifold of dimension $n = 2k$, and if $f \in C^\infty(M)$ then*

$$\mathcal{H}^k(M, g) = \mathcal{H}^k(M, e^{2f}g).$$

Proof.— As a matter of fact the two Hilbert spaces $L^2(\Lambda^k T^*M, g)$ and $L^2(\Lambda^k T^*M, e^{2f}g)$ are the same: if $\alpha \in \Lambda^k T^*M$, then

$$\|\alpha\|_{e^{2f}g}^2(x) = e^{-2kf(x)} \|\alpha\|_g^2(x)$$

and $d\text{vol}_{e^{2f}g} = e^{-2kf} d\text{vol}_g$.

We have

$$\begin{aligned} \mathcal{H}^k(M, e^{2f}g) &= Z_2^k(M, e^{2f}g) \cap \overline{dC_0^\infty(\Lambda^{k-1}T^*M)} \\ \text{and } \mathcal{H}^k(M, g) &= Z_2^k(M, g) \cap \overline{dC_0^\infty(\Lambda^{k-1}T^*M)}. \end{aligned}$$

As the two Hilbert space $L^2(\Lambda^k T^*M, g)$ and $L^2(\Lambda^k T^*M, e^{2f}g)$ are the same, these two spaces are the same. **Q.E.D**

But (M_2, g_2) is conformally equivalent to the 2-sphere with two points removed. A L^2 harmonic form on the 2-sphere with two points removed extends smoothly on the sphere. The sphere has no non trivial L^2 harmonic 1-form, hence Lemma 5.1 follows.

The surfaces (M_1, g_1) and (M_2, g_2) are isometric outside some compact set but

$$\chi_{L^2}(M_1) - \int_{M_1} \frac{K_{g_1} dA_{g_1}}{2\pi} = 0 - 0 = 0$$

whereas

$$\chi_{L^2}(M_2) - \int_{M_2} \frac{K_{g_2} dA_{g_2}}{2\pi} = - \int_{M_2} \frac{K_{g_2} dA_{g_2}}{2\pi} = -2.$$

Hence the relative index formula is not true in general. A corollary of this argument is the following

Corollary 5.3. *If (S, g) is a complete surface with integrable Gaussian curvature, according to a theorem of A. Huber [H], we know that such a surface is conformally equivalent to a compact surface \bar{S} with a finite number of points removed, then*

$$\dim \mathcal{H}^1(S, g) = b_1(\bar{S}).$$

6. MANIFOLDS WITH FLAT ENDS

In (1982, [D]), J. Dodziuk asked the following question: according to Vesentini ([V]) if M is flat outside a compact set, the spaces $\mathcal{H}^k(M)$ are finite dimensional. Do they admit a topological interpretation ?

My aim is to present an answer to this question. For the detail, the reader may look at [C4]:

6.1. Visentini's finiteness result.

Theorem 6.1. *Let (M, g) be a complete Riemannian manifold such that for a compact set $K_0 \subset M$, the curvature of (M, g) vanishes on $M - K_0$. Then for every p*

$$\dim \mathcal{H}^p(M, g) < \infty.$$

We give here a proof of this result; this proof will furnish some analytical tools to answer J. Dodziuk's question.

We begin to define a Sobolev space adapted to our situation:

Definition 6.2. *Let D be a bounded open set containing K_0 , let $W_D(\Lambda T^*M)$ be the completion of $C_0^\infty(\Lambda T^*M)$ for the quadratic form*

$$\alpha \mapsto \int_D |\alpha|^2 + \int_M |(d + \delta)\alpha|^2 = N_D^2(\alpha).$$

Proposition 6.3. *The space W_D doesn't depend on D , that is to say if D and D' are two bounded open sets containing K_0 , then the two norms N_D and $N_{D'}$ are equivalent. We write W for W_D .*

Proof.– The proof goes by contradiction. We notice that with the Bochner-Weitzenböck formula:

$$\forall \alpha \in C_0^\infty(\Lambda T^*M), \int_M |(d + \delta)\alpha|^2 = \int_M |\nabla \alpha|^2 + \int_{K_0} |\alpha|^2.$$

Hence, by standard elliptic estimates, the norm N_D is equivalent to the norm

$$Q_D(\alpha) = \sqrt{\int_M |\nabla \alpha|^2 + \int_D |\alpha|^2}.$$

If D and D' are two connected bounded open set containing K_0 , such that $D \subset D'$ then $Q_D \leq Q_{D'}$. Hence if Q_D and $Q_{D'}$ are not equivalent there is a sequence $(\alpha_l)_{l \in \mathbb{N}} \in C_0^\infty(\Lambda T^*M)$, such that $Q_{D'}(\alpha_l) = 1$ whereas $\lim_{l \rightarrow \infty} Q_D(\alpha_l) = 0$. This implies that the sequence $(\alpha_l)_{l \in \mathbb{N}}$ is bounded in $W^{1,2}(D')$ and $\lim_{l \rightarrow \infty} \|\nabla \alpha_l\|_{L^2(M)} = 0$. Hence we can extract a subsequence converging weakly in $W^{1,2}(D')$ and strongly in $L^2(\Lambda T^*D')$ to a $\alpha_\infty \in W^{1,2}(D')$. We can suppose this subsequence is $(\alpha_l)_l$.

We must have $\nabla \alpha_\infty = 0$ and $\alpha_\infty = 0$ on D and $\|\alpha_\infty\|_{L^2(D')} = 1$. This is impossible. Hence the two norms Q_D and $Q_{D'}$ are equivalent. **Q.E.D**

We have the corollary

Corollary 6.4. *The inclusion $C_0^\infty \longrightarrow W_{loc}^{1,2}$ extends by continuity to a injection $W \longrightarrow W_{loc}^{1,2}$.*

We remark that the domain of the Gauss-Bonnet operator $\mathcal{D}(d + \delta) = \{\alpha \in L^2, (d + \delta)\alpha \in L^2\}$ is in W . As a matter of fact, because (M, g) is complete $\mathcal{D}(d + \delta)$ is the completion of $C_0^\infty(\Lambda T^*M)$ equipped with the quadratic form

$$\alpha \mapsto \int_M |\alpha|^2 + \int_M |(d + \delta)\alpha|^2.$$

This norm is larger than the one used for defining W . Hence $\mathcal{D}(d + \delta) \subset W$. As a corollary we get that a L^2 harmonic form is in W . The Visentini's finiteness result will follow from:

Proposition 6.5. *The operator $(d + \delta) : W \longrightarrow L^2$ is Fredholm. That is to say its kernel and its cokernel have finite dimension and its image is closed.*

Proof.— Let A be the operator $(d + \delta)^2 + 1_D$, where

$$1_D(\alpha)(x) = \begin{cases} \alpha(x) & \text{if } x \in D \\ 0 & \text{if } x \notin D \end{cases}$$

We have

$$N_D(\alpha)^2 = \langle A\alpha, \alpha \rangle.$$

So the operator $A^{-1/2} = \int_0^\infty e^{-tA} \frac{dt}{\sqrt{\pi t}}$ realizes an isometry between L^2 and W . It is enough to show that the operator $(d + \delta)A^{-1/2} = B$ is Fredholm on L^2 . But

$$B^*B = A^{-1/2}(d + \delta)^2 A^{-1/2} = \text{Id} - A^{-1/2}1_D1_D A^{-1/2}.$$

The operator $1_D A^{-1/2}$ is the composition of the operator $A^{-1/2} : L^2 \longrightarrow W$ then of the natural injection from W to $W_{loc}^{1,2}$ and finally of the map 1_D from $W_{loc}^{1,2}$ to L^2 . D being a bounded set, this operator is a compact one by the Rellich compactness theorem. Hence $1_D A^{-1/2}$ is a compact operator. Hence, B has a closed range and a finite dimensional kernel. So the operator $(d + \delta) : W \longrightarrow L^2$ has a closed range and a finite kernel. But the cokernel of this operator is the orthogonal space to $(d + \delta)C_0^\infty(\Lambda T^*M)$ in L^2 . Hence the cokernel of this operator is the L^2 kernel of the Gauss-Bonnet operator. We notice that this space is included in the space of the W kernel of $(d + \delta)$. Hence it has finite dimension. **Q.E.D**

We also get the following corollary :

Corollary 6.6. *There is a Green operator $G : W \longrightarrow L^2$, such that*

$$\text{on } L^2, (d + \delta)G = \text{Id} - P^{L^2}$$

where P^{L^2} is the orthogonal projection on $\oplus \mathcal{H}^k(M)$.

$$\text{On } W, G(d + \delta) = \text{Id} - P^W$$

where P^W is the W orthogonal projection on $\ker_W(d + \delta)$.

Moreover, $\alpha \in Z_2^k(M)$ is L^2 cohomologous to zero if and only if there is a $\beta \in W(\Lambda^{k-1}T^*M)$ such that $\alpha = d\beta$.

6.2. A long exact sequence. In the DeRham cohomology, we have a long exact sequence linking the cohomology with compact support and the absolute cohomology. And this exact sequence is very useful to compute the DeRham cohomology groups. In L^2 cohomology, we can also define this sequence but generally it is not an exact sequence.

Let $\mathcal{O} \subset M$ be a bounded open subset, we can define the sequence:

$$(6.1) \quad \dots \longrightarrow \mathcal{H}^k(M \setminus \mathcal{O}, \partial\mathcal{O}) \xrightarrow{e} \mathcal{H}^k(M) \xrightarrow{j^*} H^k(\mathcal{O}) \xrightarrow{b} \mathcal{H}^{k+1}(M \setminus \mathcal{O}, \partial\mathcal{O}) \longrightarrow \dots$$

Here

$$\mathcal{H}^k(M \setminus \mathcal{O}, \partial\mathcal{O}) = \{h \in L^2(\Lambda^k T^*(M \setminus \mathcal{O})), dh = \delta h = 0 \text{ and } i^* h = 0\},$$

where $i : \partial\mathcal{O} \longrightarrow M \setminus \mathcal{O}$ is the inclusion map, and

- e is the extension by zero map: to $h \in \mathcal{H}^k(M \setminus \mathcal{O}, \partial\mathcal{O})$ it associates the L^2 cohomology class of \bar{h} , where $\bar{h} = 0$ on \mathcal{O} and $\bar{h} = h$ on $M \setminus \mathcal{O}$. It is well defined because of the Stokes formula: if $\beta \in C_0^\infty(\Lambda^{k+1} T^* M)$, then

$$\langle \bar{h}, \delta\beta \rangle = \langle dh, \beta \rangle_{L^2(\mathcal{O})} - \int_{\partial\mathcal{O}} i^* h \wedge i^* * \beta = 0$$

- j^* is associated to the inclusion map $j : \mathcal{O} \longrightarrow M$; to $h \in \mathcal{H}^k(M)$ it associates $[j^* h]$, the cohomology class of $h|_{\mathcal{O}}$ in $H^k(\mathcal{O})$.
- b is the coboundary operator: if $[\alpha] \in H^k(\mathcal{O})$, and if $\bar{\alpha}$ is a smooth extension of α , with compact support, then $b[\alpha]$ is the orthogonal projection of $d\bar{\alpha}$ on $\mathcal{H}^{k+1}(M \setminus \mathcal{O}, \partial\mathcal{O})$. The map b is well defined, that is to say, it does not depend on the choice of α nor of its extension.

It is relatively easy to check that

$$j^* \circ e = 0, \quad b \circ j^* = 0 \text{ and } e \circ b = 0;$$

Hence we have the inclusion:

$$\text{Im } e \subset \text{Ker } j^*, \quad \text{Im } j^* \subset \text{Ker } b \text{ and } \text{Im } b \subset \text{Ker } e.$$

In [C1], we observed that

Proposition 6.7. *The equality $\text{ker } b = \text{Im } j^*$ always holds.*

This comes from the the long exact sequence in DeRham cohomology. Moreover, we have the following:

Proposition 6.8. *On a manifold with flat ends, the equality $\text{Im } b = \text{Ker } e$ always holds.*

Proof.– As a matter of fact, if $h \in \text{Ker } e$ then by (6.6) we get a $\beta \in W$, such that $h = d\beta$ on M . Hence $h = b[\beta|_{\mathcal{O}}]$. **Q.E.D**

The last fact requires more analysis:

Theorem 6.9. *If (M, g) is a complete manifold with flat ends and if for every end E of M we have*

$$\lim_{r \rightarrow \infty} \frac{\text{vol } E \cap B_x(r)}{r^2} = \infty,$$

then the long sequence (6.1) is exact.

6.3. Hodge theorem for manifolds with flat ends. With the help of the geometric description of flat ends due to Eschenburg and Schroeder ([E-S], see also [G-P-Z]), we can compute the L^2 -cohomology on flat ends. Then with the long sequence (6.1), we can give an answer to J. Dodziuk's question; for sake of simplicity, we give here only the result for manifolds with one flat end.

Theorem 6.10. *Let (M^n, g) be a complete Riemannian manifold with one flat end E . Then*

- (1) *If (M^n, g) is parabolic, that is to say if the volume growth of geodesic ball is at most quadratic*

$$\lim_{r \rightarrow \infty} \frac{\text{vol } B_x(r)}{r^2} < \infty,$$

then we have

$$\mathcal{H}^k(M, g) \simeq \text{Im} (H_c^k(M) \longrightarrow H^k(M)).$$

- (2) *If (M^n, g) is non-parabolic (i.e. if $\lim_{r \rightarrow \infty} \frac{\text{vol } B_x(r)}{r^2} = \infty$), then the boundary of E has a finite covering diffeomorphic to the product $S^{\nu-1} \times T$ where T is a flat $(n - \nu)$ -torus. Let $\pi : T \longrightarrow \partial E$ the induced immersion, then*

$$\mathcal{H}^k(M, g) \simeq H^k(M \setminus E, \ker \pi^*),$$

where $H^k(M \setminus E, \ker \pi^)$ is the cohomology associated to the subcomplex of differential forms on $M \setminus E : \ker \pi^* = \{\alpha \in C^\infty(\Lambda^\bullet T^*(M \setminus E)), \pi^* \alpha = 0\}$.*

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