J. London Math. Soc. (2) 65 (2002) 757–768
DOI: 10.1112/S0024610702003125

A TOPOLOGICAL CRITERION FOR THE EXISTENCE OF HALF-BOUND STATES

GILLES CARRON

Abstract

The following theorem is proved: if (M^{4n+1}, g) is a complete Riemannian manifold and $\Sigma \subset M$ is an oriented hypersurface partitioning M and with non-zero signature, then the spectrum of the Hodge–deRham Laplacian is $[0, \infty[$. This result is obtained by a new Callias-type index. This new formula links half-bound harmonic forms (that is, nearly L^2 but not in L^2) with the signature of Σ.

0. *Introduction*

In this paper, we obtain the following result.

Theorem 0.1. *If (M^{4n+1}, g) is a complete Riemannian manifold and $\Sigma \subset M$ is an oriented hypersurface partitioning M and with non-zero signature, then the spectrum of the Hodge–deRham Laplacian is $[0, \infty[$.*

Our proof also gives a similar result for the Dirac operator of a spin Riemannian manifold, and we recover the following result of J. Roe [**18**].

Theorem 0.2. *If (M^{2n+1}, g) is a complete spin Riemannian manifold, such that there is a compact oriented hypersurface Σ in M with non-zero $\hat{A}$ genus, then the spectrum of the Dirac operator is $\mathbf{R}$.*

In [**12**], N. Higson gives a Callias-type index formula which implies J. Roe's result. Another important corollary of Higson's paper is a beautiful and short proof of the cobordism invariance of the index.

A Callias-type index formula is a formula for the L^2 index of an operator of the type $D + ia$, where $D : C^\infty(E) \longrightarrow C^\infty(E)$ is a Dirac-type operator on a non-compact manifold and a is a symmetric endomorphism of E. This formula says that this index is an index of a Dirac operator on a hypersurface and on a bundle built with the eigenspaces of a. We refer to the papers of Callias [**6**] and of Bott and Seeley [**4**] on Euclidean space and of Anghel [**2**], Bunke [**5**] and Rade [**17**] on Riemannian manifolds. Our proof of Theorems 0.1 and 0.2 also relies upon a new Callias-type index formula. Under the assumptions of Theorem 0.1, we give a formula which links the signature of Σ with the dimension of the space of harmonic forms which are not square integrable but which have slow decay at infinity. In many examples, these almost (or extended) L^2 harmonic forms appear as zero energy resonances or equivalently in the singular part of the resolvent of the Hodge–deRham Laplacian near 0.

Received 1 October 1999; final revision 10 July 2001.

2000 *Mathematics Subject Classification* 58J10, 58J20, 58J32.

Our index theorem also provides a bound on the Novikov–Shubin type invariant.

THEOREM 0.3. *Let (M^{4n+1}, g) be a complete Riemannian manifold and $\Sigma \subset M$ be an oriented hypersurface partitioning M, with non-zero signature. If we denote by $e^{-t\Delta}$ the heat operator associated with the Hodge–deRham Laplacian then*

$$\limsup_{t\to\infty} \frac{\log \|e^{-t\Delta}(x,x)\|_{L^\infty(M)}}{\log(t)} \leqslant 1.$$

1. *Preliminaries*

In this section, we describe the results of [**10**] and [**7**], which will be needed to prove Theorems 0.1 and 0.3.

1.1. *Non-parabolicity at infinity*

In [**10**], we introduced the following definition.

DEFINITION 1.1. A Dirac-type operator $D : C^\infty(E) \longrightarrow C^\infty(E)$ on a complete Riemannian manifold (M, g) is called *non-parabolic at infinity* if there is a compact set K of M such that for any bounded open subset U of $M - K$ there is a constant $C(U) > 0$ with the inequality

$$\text{for all } \sigma \in C_0^\infty(M-K, E), \quad C(U)\int_U |\sigma|^2 \leqslant \int_{M-K} |D\sigma|^2. \tag{1.1}$$

This definition came from potential theory and N. Anghel's characterization of Dirac-type operators not having zero in their spectrum. The main property of these operators is the following [**10**].

PROPOSITION 1.2. *If $D : C^\infty(E) \longrightarrow C^\infty(E)$ is non-parabolic at infinity then*

$$\dim\{\sigma \in L^2(E),\ D\sigma = 0\} < \infty.$$

Let $W(E)$ be the Sobolev space obtained by completion of $C_0^\infty(E)$ with the norm

$$\sigma \mapsto \int_K |\sigma|^2 + \int_M |D\sigma|^2.$$

Then this space is continuously embedded into H^1_{loc} and

$$D : W(E) \longrightarrow L^2(E)$$

is a Fredholm operator.

Furthermore, the inclusions $H^1(E) \subset W(E) \subset H^1_{\text{loc}}(E)$ are true; so that any L^2 harmonic spinor is a $W(E)$-harmonic spinor, that is, we have

$$\ker_{L^2} D \subset \ker_W D.$$

This concept provides a unified framework for index theorems on non-compact manifolds. Let us give some examples (see [**7**, **10**]).

(i) If $D : C^\infty(E) \longrightarrow C^\infty(E)$ is Fredholm on its L^2 domain, or according to

N. Anghel [1], there exists a compact set K of M and a constant $\Lambda > 0$ such that

$$\text{for all } \sigma \in C_0^\infty(M-K,E), \quad \Lambda \int_{M-K} |\sigma|^2 \leqslant \int_{M-K} |D\sigma|^2,$$

then obviously D is non-parabolic at infinity and $W(E) = H^1(E) = \{\sigma \in L^2,\ D\sigma \in L^2\}$. For instance, the Dirac operator of a Riemannian spin manifold with uniformly positive scalar curvature at infinity is non-parabolic at infinity.

(ii) If (M,g) is a complete Riemannian manifold whose curvature vanishes at infinity then its Gauss–Bonnet operator is non-parabolic at infinity.

(iii) The Dirac operator of a complete Riemannian spin manifold with non-negative scalar curvature at infinity is non-parabolic at infinity.

In these two last cases, the Sobolev space W is the completion of $C_0^\infty(E)$ with the norm $\sigma \mapsto \|\nabla\sigma\|_{L^2(M)} + \|\sigma\|_{L^2(K)}$, where ∇ is the Levi–Civita connexion and where K is a compact outside which the curvature is zero.

(iv) If (M,g) is a manifold with one cylindrical end, that is, there is a compact subset K of M such that $M-K$ is isometric to the Riemannian product $\partial K \times]0,\infty[$, then all geometric Dirac-type operators on M are non-parabolic at infinity, and the Sobolev space W is

$$\left\{\sigma \in H^1_{\text{loc}}(E), D\sigma \in L^2,\ \frac{\sigma}{1+\text{dist}(x,K)} \in L^2\right\}.$$

These operators have been studied by Atiyah, Patodi and Singer in order to give a formula for the signature of compact manifold with boundary [3]. Harmonic spinors in W are there called extended L^2 harmonic spinors. This is why we call the index of the operator $D : W(E) \longrightarrow L^2(E)$ the extended index. We note:

$$\text{ind}_e\, D = \dim \ker_W D - \dim \ker_{L^2} D.$$

1.2. *The Dirac–Neumann operator*

In [7], we developed a theory in order to give a formula for the index of $D : W(E) \longrightarrow L^2(E)$. Our analysis relies upon the resolution of the Dirichlet problem: assume that $D : C^\infty(E) \longrightarrow C^\infty(E)$ is a Dirac-type operator on a complete Riemannian manifold (Ω, g) with compact boundary Σ, and that $D : C^\infty(E) \longrightarrow C^\infty(E)$ is *non-parabolic*, that is to say for any bounded open subset U of Ω there is a constant $C(U) > 0$ with the inequality

$$C(U)\int_U |\sigma|^2 \leqslant \int_\Omega |D\sigma|^2, \quad \forall\, \sigma \in C_0^\infty(\Omega, E). \tag{1.2}$$

THEOREM 1.3. *For each $\sigma \in C^\infty(\Sigma, E)$, there is a unique $\mathscr{E}(\sigma) \in C^\infty(\overline{\Omega}, E) \cap W$ such that*

$$\begin{cases} D^2\mathscr{E}(\sigma) = 0 & \text{on } \Omega \\ \mathscr{E}(\sigma) = \sigma & \text{on } \Sigma. \end{cases}$$

This allowed us to define the Dirac–Neumann operator

$$T : C^\infty(\Sigma, E) \longrightarrow C^\infty(\sigma, E)$$

$$\sigma \mapsto D(\mathscr{E}\sigma)|_\Sigma.$$

As the harmonic extension of $D(\mathcal{E}\sigma)|_\Sigma$ is $D(\mathcal{E}\sigma)$, $T \circ T = 0$, and moreover, we have the following theorem.

THEOREM 1.4. *T is a pseudo-differential operator of order 1, and the adjoint of T is $T^* = nTn$, where n is the Clifford multiplication by the unit inward normal vector field of Σ in Ω. The operator $\mathcal{D} = T + T^*$ is elliptic and its principal symbol is*

$$\sigma(\mathcal{D})(x,\xi) = 2i\xi, \quad \xi \in T_x^*(\Sigma).$$

Hence, T defines an elliptic complex on $E \to \partial M$. The cohomology of T was interpreted in [7].

PROPOSITION 1.5. *Via restriction along the boundary we have the identification*

$$\{\sigma \in W \cap C^\infty(\overline{M},E),\ D\sigma = 0\} \simeq \ker T;$$

$$\{\sigma \in L^2 \cap C^\infty(\overline{M},E),\ D\sigma = 0\} \simeq \operatorname{Im} T.$$

The quotient $\ker T/\operatorname{Im} T$ is then isomorphic to the space of harmonic spinors in W modulo those in L^2. In fact a harmonic spinor σ is in W if and only if there is a sequence $(\sigma_k) \in C_0^\infty(\overline{M},E)$ such that

$$\begin{cases} \lim_k \sigma_k = \sigma & \text{in } H^1_{\mathrm{loc}} \\ \lim_k D\sigma_k = 0 & \text{in } L^2. \end{cases}$$

The complex defined by T is elliptic, therefore we have a Hodge-type decomposition: if $\mathcal{H}_\infty(M) = \ker \mathcal{D} = \ker T \cap \ker T^*$ then

$$\ker T = \mathcal{H}_\infty(M) \oplus \operatorname{Im} T.$$

This finally yields the identification $\mathcal{H}_\infty(M) \simeq \ker T/\operatorname{Im} T$.

As an example, we describe the case of a geometric operator on a Riemannian manifold with one cylindrical end. Assume that outside some compact set K, $(M - K, g)$ is isometric to the Riemannian product $]0,+\infty[\times\partial K$, and that the Hermitian bundle $E|_{M-K}$ is the pull-back of some Hermitian bundle E on ∂K. Assume moreover that the Dirac operator D takes the following form on $]0,+\infty[\times\partial K$:

$$D = \gamma\left(\frac{\partial}{\partial r} + A\right),$$

where $A : C^\infty(\partial K, E) \longrightarrow C^\infty(\partial K, E)$ is a Dirac-type operator on ∂K and γ is the Clifford multiplication by the unit outward normal vector to $\{t\} \times \partial K \subset]0,t] \times \partial K$. A is an elliptic self-adjoint operator on ∂K, so we have the following spectral decomposition:

$$L^2(\partial K, E) = \bigoplus_{\lambda \in \mathrm{Sp} A} \mathbf{C}\varphi_\lambda,$$

where

$$A\varphi_\lambda = \lambda\varphi_\lambda,$$

$$\int_{\partial K} |\varphi_\lambda|^2 = 1.$$

Now every $\sigma \in C^\infty(\partial K, E)$ can be expanded in Fourier series with respect to this decomposition

$$\sigma(\theta) = \sum_{\lambda \in \mathrm{Sp}A} \sigma_\lambda \varphi_\lambda(\theta),$$

and the harmonic extension of such a σ is given by

$$\mathscr{E}(\sigma)(r,\theta) = \sum_{\lambda \in \mathrm{Sp}A} \sigma_\lambda e^{-|\lambda| r} \varphi_\lambda(\theta).$$

Therefore the operator T is given by

$$(T\sigma)(\theta) = \sum_{\lambda \in \mathrm{Sp}A} (\lambda - |\lambda|)\, \sigma_\lambda\, \gamma.\varphi_\lambda(\theta),$$

and we compute:

$$\ker T = \bigoplus_{\lambda \in \mathrm{Sp}A,\ \lambda \geqslant 0} \mathbf{C}\varphi_\lambda$$

$$\operatorname{Im} T = \bigoplus_{\lambda \in \mathrm{Sp}A,\ \lambda > 0} \mathbf{C}\varphi_\lambda.$$

Thus in this example, we have

$$\ker A = \mathscr{H}_\infty(M) \simeq \ker T / \operatorname{Im} T.$$

In [**7**], we proved the following theorem.

THEOREM 1.6. *If the Dirac-type operator $D : C^\infty(E) \longrightarrow C^\infty(E)$ is non-parabolic at infinity, and if K is a compact subset of M with smooth boundary Σ, outside of which estimates* (1.2) *hold, then*

$$\operatorname{ind}_{\mathrm{e}} D = \dim \frac{\{\sigma \in W(M,E),\ D\sigma = 0\}}{\{\sigma \in L^2(M,E),\ D\sigma = 0\}} = \frac{\dim \mathscr{H}_\infty(M-K)}{2}.$$

This theorem is the generalisation of a formula in a paper by Atiyah, Patodi and Singer [**3**, (3.25)]. Here the quotient space in the left-hand side is the space of harmonic spinors in W modulo those in L^2. In the case of a manifold with one cylindrical end, W. Müller showed that extended L^2 harmonic spinors appear as zero energy resonances, that is to say they appear in the singular part of the resolvant $\zeta \mapsto (D^2 - \zeta^2)^{-1}$, at $\zeta = 0$ [**15**]. Such zero energy resonances are called half-bound states in quantum mechanics; and in the general case of a non-parabolic at infinity Dirac-type operator, we will also call a harmonic spinor in W but not in L^2 a half-bound state. The absence of such states for D implies that the cohomology defined with the Dirac–Neumann operator T is trivial. According to our first example, we can assert the following.

PROPOSITION 1.7. *If $D : C^\infty(E) \longrightarrow C^\infty(E)$ is a Dirac-type operator which is Fredholm on its L^2-domain, then D is non-parabolic at infinity and it has no half-bound states:* $\operatorname{ind}_{\mathrm{e}} D = 0$.

1.3. *Novikov–Shubin invariants and non-parabolicity at infinity*

We give a new spectral condition which implies that a Dirac-type operator which is non-parabolic at infinity has no half-bound states.

PROPOSITION 1.8. *Assume that $D : C^\infty(E) \longrightarrow C^\infty(E)$ is a Dirac-type operator such that there is an $\alpha > 2$ and a locally bounded function $C(x)$, $x \in M$ with*

$$\|e^{-tD^2}(x,x)\| \leqslant C(x)t^{-\alpha/2}, \quad \forall\, t \geqslant 1, x \in M,$$

then D is non-parabolic at infinity and it has no bound or half-bound states, so that $\operatorname{ind}_e D = 0$.

Proof. We claim that under these assumptions the integral

$$G = \int_0^\infty De^{-tD^2}\,dt$$

defines a bounded operator from L^2 to H^1_{loc} and moreover

$$DG = \mathrm{Id}_{L^2}, \quad GD = \mathrm{Id}_{C_0^\infty}.$$

The second equality implies that D is non-parabolic at infinity and also that D has no L^2-kernel. Moreover if we define $W(E)$ as the completion of the space $C_0^\infty(E)$ with respect to the norm

$$\sigma \mapsto \|D\sigma\|_{L^2},$$

then the injection of $C_0^\infty(E)$ into H^1_{loc} extends by continuity to an injection of $W(E)$ into H^1_{loc}; since $D : W \longrightarrow L^2$ is an isometry, D has no W-kernel.

We have to prove the convergence; for this we cut the integral at $t = 1$

$$G = \int_0^1 De^{-tD^2}\,dt + \int_1^\infty De^{-tD^2}\,dt.$$

The first integral converges because of standard parabolic estimates, the second is convergent in the norm topology in the space of bounded operators from L^2 to L^∞_{loc}. As a matter of fact, the spectral theorem shows that

$$\|De^{-t/2D^2}\|_{L^2\to L^2} \leqslant \sqrt{\frac{2}{t}};$$

moreover if $f \in L^2(E)$, we have

$$\begin{aligned}\|De^{-tD^2}f(x)\| &= \|e^{-t/2D^2}De^{-t/2D^2}f(x)\|\\ &\leqslant \sqrt{\|e^{-tD^2}(x,x)\|}\sqrt{\frac{2}{t}}\|f\|\\ &\leqslant \sqrt{2C(x)}t^{-1/2-\alpha/4}\|f\|.\end{aligned}$$

This proves the convergence of the second integral. The two equalities are consequences of our hypothesis and of the formula

$$D\int_0^T De^{-tD^2}\,dt = \mathrm{Id} - e^{-TD^2}.$$

□

REMARK 1.9. The non-parabolicity at infinity property depends only on the geometry at infinity, so $D : C^\infty(E) \longrightarrow C^\infty(E)$ is non-parabolic at infinity when there is a positive function with compact support χ so that there is an $\alpha > 2$ and a locally bounded function $C(x)$, $x \in M$ with

$$\|e^{-t(D^2+\chi)}(x,x)\| \leqslant C(x)t^{-\alpha/2}, \quad \forall\, t \geqslant 1, x \in M.$$

Remark 1.10. By the Karamata theorem, our assumption is equivalent to the following on E the spectral resolution of D^2

$$\|E([0,\lambda],x,x)\| \leqslant \tilde{C}(x)\lambda^{\alpha/2}, \quad \forall \lambda \in [0,1],\ \forall x \in M.$$

In fact the best possible exponent α is linked with the Novikov–Shubin invariants: if (M,g) is the universal covering of a compact manifold $\overline{M}$ and if $D = d + \delta$ is the Gauss–Bonnet operator acting on differential forms on M, and if $F \subset M$ is a fundamental domain of the covering $M \to \overline{M}$, then

$$\beta = \inf\left\{\alpha \mid \int_F \operatorname{trace}_{\Lambda T_x^* M} e^{-tD^2}(x,x)\,dx = O(t^{-\alpha/2})\right\}$$

is a Novikov–Shubin invariant of $\overline{M}$, it does not depend on the metric [**16**], nor on the differential structure [**14**] and it is a homotopy invariant of $\overline{M}$ [**11**].

According to the calculus done by M. Rumin [**19**] and by L. Schubert [**20**] we have the following corollary.

Corollary 1.11. *If $n > 1$ then the Gauss–Bonnet operator of the Heisenberg group*

$$H_{2n+1} = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & \dots & x_n & z \\ 0 & 1 & 0 & \dots & 0 & y_1 \\ . & .. & . & \dots & . & . \\ . & .. & \dots & . & . & . \\ . & .. & \dots & 0 & 1 & y_n \\ 0 & .. & . & \dots & . & 1 \end{pmatrix} ;\ x_1,\dots,x_n,y_1,\dots,y_n,z \in \mathbf{R} \right\}$$

with a left-invariant metric is non-parabolic at infinity and has no half-bound states.

Proof. It is shown in [**19**] and [**20**] that the hypothesis of our Proposition 1.8 holds with $\alpha = n+1$ for the Gauss–Bonnet operator of the Heisenberg group H_{2n+1} with a left-invariant metric. □

2. *Application to the Dirac operator*

In this part, we will prove a new Callias index type formula. Such a formula usually deals with non-self-adjoint Dirac-type operators and relates their indices to an index on a compact hypersurface. Here we relate half-bound states to an index on a compact hypersurface.

2.1. *Callias-type theorem*

Theorem 2.1. *Let (M^{2n+1}, g) be a complete Riemannian spin manifold with compact boundary. If the Dirac operator is non-parabolic (that is, satisfies the estimates (1.2)), then we have*

$$\hat{\mathrm{A}}(\partial M) = \frac{1}{i}\operatorname{trace}_{\mathscr{H}_\infty} n$$

where n is Clifford multiplication by the unit inward normal vector field of ∂M in M.

Proof. This follows directly from the analysis described in the first section: as $T^* = nTn$, the operator $\mathscr{D} = T + T^* = T + nTn$ anticommutes with Clifford multiplication by n. However the splitting of the spinor bundle of M along ∂M with respect to the eigenspaces of n is precisely the splitting with respect to the chirality

$$S|_{\partial M} = S^+ \oplus S^- = \ker\{n - i\mathrm{Id}\} \oplus \ker\{n + i\mathrm{Id}\}.$$

Hence $\mathscr{D}$ splits as

$$\begin{pmatrix} 0 & \mathscr{D}^- \\ \mathscr{D}^+ & 0 \end{pmatrix}.$$

Now the operator

$$\mathscr{D}^+ : C^\infty(\partial M, S^+) \longrightarrow C^\infty(\partial M, S^-)$$

is an elliptic pseudo-differential operator of order 1 and up to a constant it has the same principal symbol as the Dirac operator of ∂M:

$$\not{D} : C^\infty(\partial M, S^+) \longrightarrow C^\infty(\partial M, S^-);$$

hence they have the same index, that is, the Â-genus $\hat{\mathrm{A}}(\partial M)$, but

$$\mathrm{ind}\,\mathscr{D}^+ = \dim\{\sigma \in \mathscr{H}_\infty \mid n\sigma = i\sigma\} - \dim\{\sigma \in \mathscr{H}_\infty \mid n\sigma = -i\sigma\} = \frac{1}{i}\mathrm{trace}_{\mathscr{H}_\infty} n.$$

□

Remark 2.2. If $a \in C^\infty(M, \mathrm{Sym}(S))$ and $D = \not{D} + a$ satisfies the estimates (1.2), then the same formula links the half-bound states of $\not{D} + a$ and $\hat{\mathrm{A}}(\partial M)$.

2.2. *Topological consequences*

We can now give some corollaries of this theorem.

Corollary 2.3. *Under the assumption of Theorem* 2.1, *if there are no half-bound states then*

$$\hat{\mathrm{A}}(\partial M) = 0.$$

For instance, if $\not{D}$ is invertible, for example, if M has uniformly positive scalar curvature at infinity, we recover part of a result of M. Lesch: the $\hat{A}$-genus of the boundary is vanishing [**13**, Theorem 4.3.6].

Corollary 2.4. *If (M^{2n+1}, g) is a complete Riemannian spin manifold with compact boundary, if for a $\nu > 4$ we have the Sobolev inequality*

$$\mu_\nu(M)\left(\int_M |u|^{2\nu/(\nu-2)}(x)\,dx\right)^{1-2/\nu} \leqslant \int_M |du|^2(x)\,dx, \quad \forall\, u \in C_0^\infty(M),$$

and if the negative part of the scalar curvature is in $L^{\nu/2}$, then

$$\hat{\mathrm{A}}(\partial M) = 0.$$

As a matter of fact, it is shown in this case in [**8**] that every harmonic spinor in $W = H_0^1(S)$ is in fact in L^2. Similar results can be given by using the heat kernel on functions rather than a Sobolev inequality, see [**9**, Theorem 3.7].

2.3. *Spectral consequences*

We also recover the following result of J. Roe.

COROLLARY 2.5. *If (M^{2n+1}, g) is a complete Riemannian spin manifold without boundary, such that there is a compact hypersurface Σ in M with non-zero $\hat{A}$ genus, then the spectrum of the Dirac operator is* $\mathbf{R}$.

Proof. If a real number λ does not belong to the spectrum of the Dirac operator $\not{D}$, then the operator $\not{D} - \lambda$ is an invertible operator, so that the complex on $C^\infty(\Sigma, S)$ defined by the Dirac–Neumann operator associated to $\not{D} - \lambda$ has no cohomology. By Remark 2.2, $\hat{\mathrm{A}}(\partial M) = 0$. □

With Remark 2.2, we can assert the following.

COROLLARY 2.6. *If M^{2n+1} is a non-compact spin manifold without boundary, and Σ is a compact hypersurface in M with non-zero $\hat{A}$ genus, then for every complete metric on M and every $a \in C^\infty(M, \mathrm{Sym}(S))$, we have*

$$\mathrm{Spectrum}(\not{D} + a) = \mathbf{R}.$$

In fact, we have more than Roe's result.

COROLLARY 2.7. *Let (M^{2n+1}, g) be a complete Riemannian spin manifold without boundary, and Σ be a compact hypersurface in M with non-zero $\hat{A}$ genus. If E is the spectral resolution of the Dirac operator then for every real λ,*

$$\limsup_{\mu \to 0} \frac{\log \|E([\lambda - \mu, \lambda + \mu], x, x)\|_{L^\infty}}{\log(1/\mu)} \leqslant 1.$$

In other words, if there are no bound states of energy λ^2 then there is enough spectrum near λ.

3. *Application to the signature operator*

We are going to show analogous results for the signature. Whereas most of the results of the previous section were well known, the results here are new.

3.1. *Callias-type theorem*

THEOREM 3.1. *Let (M^{4n+1}, g) be a complete oriented Riemannian manifold with compact boundary whose signature operator is non-parabolic (that is, satisfies the estimates (1.2)). Then*

$$2\,\mathrm{Sign}(\partial M) = \mathrm{trace}_{\mathscr{H}_\infty} \omega_0,$$

where ω_0 is Clifford multiplication by the unit oriented volume form of ∂M, that is, if $(e_1, \ldots, e_{4n})$ is an oriented orthonormal basis of ∂M, then ω_0 is the Clifford multiplication by $e_1 \cdot e_2 \ldots e_{4n}$.

Proof. We start by recalling that $\mathscr{D} = T^* + T$ is an elliptic pseudo-differential operator of order 1 on $\mathrm{Cl}(M)|_{\partial M}$ and that $\mathscr{H}_\infty = \ker T \cap \ker T^* = \ker \mathscr{D}$ is the space of boundary values of the half-bound states. First we claim that $\mathscr{D}$ is anticommuting with ω_0: as a matter of fact $\mathscr{D}$ is anticommuting with n and moreover it is commuting with the Clifford multiplication by $\omega_0 n$, because this last operator is globally defined on M, is parallel and is commuting with the signature operator. Second, $\omega_0 \cdot \omega_0 = 1$, so that with respect to the splitting

$$\mathrm{Cl}(M)|_{\partial M} = \mathrm{Cl}(M)^+|_{\partial M} \oplus \mathrm{Cl}(M)^-|_{\partial M} = \ker\{\omega_0 - \mathrm{Id}\} \oplus \ker\{\omega_0 + \mathrm{Id}\},$$

$\mathscr{D}$ splits as

$$\begin{pmatrix} 0 & \mathscr{D}^- \\ \mathscr{D}^+ & 0 \end{pmatrix}.$$

Since $\mathrm{Cl}(M)|_{\partial M} \simeq \mathrm{Cl}(\partial M)\hat{\otimes}\mathrm{Cl}_1$, where Cl_1 is the sub-algebra of $\mathrm{Cl}(M)$ generated by n, $\mathrm{Cl}(M)^+|_{\partial M} = \mathrm{Cl}(\partial M)^+\hat{\otimes}\mathrm{Cl}_1$ and the same for $\mathrm{Cl}(M)^-|_{\partial M}$, because ω_0 and n are commuting. If D is the Cl_1-ification of the signature operator of ∂M to $C^\infty(\partial M, \mathrm{Cl}(M))$ then $D : C^\infty(\partial M, \mathrm{Cl}^+(M)) \longrightarrow C^\infty(\partial M, \mathrm{Cl}^-(M))$ and $\mathscr{D}^+ : C^\infty(\partial M, \mathrm{Cl}^+(M)) \longrightarrow C^\infty(\partial M, \mathrm{Cl}^-(M))$ have up to a constant factor the same principal symbol, so they have the same index: $\mathrm{ind}\,\mathscr{D}^+ = 2\,\mathrm{ind}_{\mathrm{Cl}_1} D = 2\mathrm{Sign}(\partial M)$. One concludes with:

$$\mathrm{ind}\,\mathscr{D}^+ = \dim\{\sigma \in \mathscr{H}_\infty |\, \omega_0\sigma = \sigma\} - \dim\{\sigma \in \mathscr{H}_\infty |\, \omega_0\sigma = -\sigma\} = \mathrm{trace}_{\mathscr{H}_\infty}\omega_0.$$

□

REMARK 3.2. If $a \in C^\infty(M, \mathrm{Sym}(\Lambda T^*M))$ and $D = d+\delta+a$ satisfies the estimates (1.2), then the same formula links the half-bound states of $d+\delta+a$ and the signature of ∂M.

3.2. *Topological consequences*

COROLLARY 3.3. *Under the hypothesis of Theorem* 3.1*, if there are no half-bound states then* $\mathrm{Sign}(\partial M) = 0$.

REMARK 3.4. This result can be shown in an alternative and more classical way: with the assumptions of Corollary 3.3, our result in **[9]** is that an exact sequence links the cohomology of ∂M and the (reduced) L^2-cohomology of M:

$$\ldots \to H_2^{2n}(M) \to H^{2n}(\partial M) \to H_2^{2n+1}(M, \partial M) \to \ldots$$

where $H_2^{2n}(M)$ is the absolute reduced L^2-cohomology space of M or alternatively the space of harmonic L^2 form whose normal components vanish along ∂M, and where $H_2^{2n+1}(M, \partial M)$ is the relative reduced L^2-cohomology space of M or alternatively the space of harmonic L^2 form whose tangential components vanish along ∂M. These two spaces are isomorphic via the star Hodge operator. We can show that the image of $H^{2n}(M)$ into $H_2^{2n}(\partial M)$ is a Lagrangian for the intersection form $(\alpha, \beta) \mapsto \int_{\partial M} \alpha \wedge \beta$, for this we have to use the Stokes formula

$$\int_{\partial M} \alpha \wedge \beta = \int_M d(\alpha \wedge \beta)$$

which is valid for every $\alpha, \beta \in L^2$ with $d\alpha, d\beta \in L^2$.

As in the previous section (Corollary 2.4), we can assert the following.

COROLLARY 3.5. *If (M^{4n+1}, g) is a complete oriented Riemannian manifold with compact boundary, if for a $\nu > 4$ we have the Sobolev inequality*

$$\mu_\nu(M)\left(\int_M |u|^{2\nu/(\nu-2)}(x)\,dx\right)^{1-2/\nu} \leqslant \int_M |du|^2(x)\,dx, \quad \forall\, u \in C_0^\infty(M),$$

and if the Riemannian curvature of (M, g) is in $L^{\nu/2}$, then

$$\mathrm{Sign}(\partial M) = 0.$$

For instance there is no such metric on $P^2(\mathbf{C}) \times [0, \infty[$, and in dimension 5, this implies that ∂M bounds a compact manifold.

3.3. *Spectral consequences*

We give here results similar to Corollary 2.7.

COROLLARY 3.6. *If (M^{4n+1}, g) is a complete Riemannian manifold (without boundary) and $\Sigma \subset M$ is an oriented hypersurface with non-zero signature, then the spectrum of the Hodge–deRahm Laplacian is $[0, \infty[$.*

Moreover, if E is the spectral resolution of the Hodge–deRahm Laplacian then

$$\limsup_{\mu \to 0} \frac{\log \|E([0, \mu], x, x)\|_{L^\infty}}{\log(1/\mu)} \leqslant 1.$$

REMARK 3.7. If $a \in C^\infty(M, \mathrm{Sym}(\Lambda T^*M))$, the same theorem holds with the operator $(d + \delta + a)^2$ instead of the Hodge–deRham Laplacian $(d + \delta)^2$. Moreover, the first part of this result can be recovered in the same way as Theorem 0.3, by a Callias-type index formula: we apply the Callias index formula of [**2**, **5**, **17**] for the operator $D + t\omega$, where t is a real number not in the spectrum of the signature operator D and ω is the skew-adjoint operator which is the Clifford multiplication by $e_1 \ldots e_{4n+1}$. Unfortunately the other part of the corollary cannot be recovered by this method.

Acknowledgements. I want to thank Marc Herzlich for his help, and in particular for his careful revision of my English text.

References

1. N. Anghel, 'An abstract index theorem on non-compact Riemannian manifolds', *Houston J. Math.* 19 (1993) 223–237.
2. N. Anghel, 'On the index of Callias type operators', *Geom. Funct. Anal.* 3 (1995) 432–438.
3. M. F. Atiyah, V. K. Patodi and I. M. Singer, 'Spectral asymmetry and Riemannian geometry I', *Math. Proc. Cambridge Philos. Soc.* 77 (1975) 43–69.
4. R. Bott and R. Seeley, 'Some remarks on the paper of Callias', *Comm. Math. Phys.* 62 (1978) 236–245.
5. U. Bunke, 'A K-theoretic relative index theorem and Callias-type Dirac operator', *Math. Ann.* 303 (1995) 241–279.
6. C. Callias, 'Axial anomalies and index theorems on open spaces', *Comm. Math. Phys.* 62 (1978) 213–234.
7. G. Carron, 'Théorèmes de l'indice sur les variétés non-compactes' *J. Reine Angew. Math.* 541 (2001) 81–115.
8. G. Carron, 'Une suite exacte en L^2-cohomologie', *Duke Math. J.* 95 (1998) 343–372.
9. G. Carron, 'L^2-cohomology and Sobolev's inequalities', Prépublication 227, l'ENS Lyon, 1998.

10. G. Carron, 'Un théorème de l'indice relatif', *Pacific J. Math.* 198 (2001) 81–107.
11. M. Gromov and M. A. Shubin, 'Von Neumann spectra near zero', *Geom. Funct. Anal.* 1 (1991) 375–404.
12. N. Higson, 'A note on the cobordism invariance of the index', *Topology* 30 (1991) 439–443.
13. M. Lesch, 'Operators of Fuchs type, conical singularities, and asymptotic methods', *Teubner-Texte Math.* 136 (1997).
14. J. Lott, 'Heat kernels on covering spaces and topological invariants', *J. Differential Geom.* 35 (1992) 471–510.
15. W. Müller 'L^2-index and resonances', *Geometry and analysis on manifolds* (*Katata/Kyoto*, 1987), Lecture Notes in Mathematics 1339 (Springer, Berlin, 1988) 203–211.
16. S. P. Novikov and M. A. Shubin, 'Morse theory and Von Neumann invariant of non-simply-connected manifold', *Uspekhi Mat. Nauk* 41 (1986) 222–223 (Russian).
17. J. Rade, 'Callias' index theorem, elliptic boundary conditions and cutting and gluing', *Comm. Math. Phys.* 161 (1994) 51–61.
18. J. Roe, 'Partitioning non-compact manifolds and the dual Toeplitz problem', *Operator algebras and applications* (Cambridge University Press, 1989) 187–228.
19. M. Rumin, 'Sub-Riemannian limit of the differential form spectrum of contact manifolds', *Geom. Funct. Anal.* 10 (2000) 407–452.
20. L. Schubert, 'The Laplacian on p-forms on the Heisenberg group', preprint, dg-ga/9807148, 1998.

Département de Mathématiques
UMR 6629
Université de Nantes
2 rue de la Houssinière
F-44322 Nantes Cedex 03
France

gilles.carron@math.univ-nantes.fr