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# The almost closed range condition

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ABSTRACT. The almost closed range condition is presented and we explain how this notion can be used to give a topological interpretation of the space of $L^2$ harmonic forms on the Hilbert schemes of 2 and 3 points on $\mathbb{C}^2$.

*À Jacques*

## 1. Introduction

When $(M, g)$ is a compact manifold the celebrated theorem of Hodge and de Rham says that the spaces of $L^2$ harmonic forms on $M$ are isomorphic to the cohomology spaces of $M$; that is, if we denote by

$$\mathcal{H}^k(M, g) = \{\alpha \in L^2_g(\Lambda^k M),\ d\alpha = d^*\alpha = 0\}$$

the space of $L^2$ harmonic $k$-forms,[1] then we have a natural isomorphism

$$\mathcal{H}^k(M, g) \simeq H^k(M, \mathbb{R}).$$

When $(M, g)$ is noncompact but complete, the spaces of $L^2$ harmonic forms have an interpretation in terms of reduced $L^2$ cohomology. A general and naive question is to understand how we can give some topological interpretation for these spaces of $L^2$ harmonic forms. There are many results, as well as predictions and conjectures, in this direction. For instance, Zucker's conjecture [32] about locally symmetric Hermitian spaces, eventually solved by E. Looijenga, L. Saper and M. Stern [18; 27] and extended by A. Nair [22], and the recent result of L. Saper [25; 26], as well as results for manifolds with flat ends [6], manifolds with cylindrical end [2], and negatively curved manifolds with finite volume [17; 30; 31]. Also, $L^2$ harmonic forms have some significance

[1] Here $d^*$ is the formal adjoint of the exterior differentiation operator $d$ for the $L^2$ structure induced by the metric $g$.

in modern physics and there are several predictions based on a duality arising in string theory: for instance there is Sen's conjecture about the moduli space of magnetic monopoles [28] and the Vafa–Witten conjecture about Nakajima's quiver manifolds [29; 13].

When $M$ has a locally finite open covering $M = \bigcup_\alpha U_\alpha$ admitting a partition of unity with bounded gradient such that on any of $M$, $U_\alpha$, $U_\alpha \cap U_\beta$, $U_\alpha \cap U_\beta \cap U_\gamma, \dots$ the $L^2$-range of $d$ is closed, then we can sometimes use sheaf cohomology to obtain a topological interpretation of the space of $L^2$ harmonic forms. However, this is not always possible. Several tools have been developed in order to circumvent this difficulty. For instance, pseudodifferential calculi have been used successfully in several situations [19; 20; 21; 14]. Here we present the notion of almost closed range for $d$ which was introduced in [7]. We give some general results (including a Mayer–Vietoris sequence) that are true if this almost closed range condition is satisfied. We also explain that this condition has to be used with some care. In order to illustrate how this notion is used in [7], we explain the arguments (and amongst them the almost closed range condition) leading to the topological interpretation of the space of $L^2$ harmonic forms on the Hilbert schemes of 2 and 3 points on $\mathbb{C}^2$.

## 2. $L^2$ cohomology

We start with basic definitions, to present the setting and fix notation.

**2.1. Definitions.** Let $(M^n, g)$ be an oriented Riemannian manifold. We endow it with a smooth positive measure $\mu\, d\mathrm{vol}_g$ (where $\mu$ is a positive smooth function), so that we can define the space $L^2_\mu(\Lambda^k M)$ of differential $k$-forms which are in $L^2_\mu(\Lambda^k M)$. This is a Hilbert space when endowed with the norm

$$\|\alpha\|^2_\mu := \int_M |\alpha(x)|^2_g \; \mu\, d\mathrm{vol}_g(x).$$

The associated Hermitian scalar product will be denoted by $\langle\, .\, ,\, .\, \rangle_\mu$.

We introduce the space $Z^k_\mu(M)$ of $L^2_\mu$ $k$-forms that are weakly closed:

$$Z^k_\mu(M) := \left\{\alpha \in L^2_\mu(\Lambda^k M) : \textstyle\int_M \alpha \wedge d\varphi = 0 \text{ for all } \varphi \in C^\infty_0(\Lambda^{n-1-k} M)\right\}$$

The space $Z^k_\mu(M)$ is in fact a subspace of the (maximal) domain of $d$,

$$Z^k_\mu(M) \subset \mathcal{D}^k_\mu(d),$$

where $\mathcal{D}^k_\mu(d)$ is the maximal domain of $d$ on $L^2_\mu(\Lambda^k M)$. This is the space of $\alpha \in L^2_\mu(\Lambda^k M)$ such that there is a constant $C$ with

$$\left| \int_M \alpha \wedge d\varphi \right| \le C \|\varphi\|_\mu \quad \text{for all } \varphi \in C^\infty_0(\Lambda^{n-1-k} M).$$

When $\alpha \in \mathcal{D}^k_\mu(d)$ we can define $d\alpha \in L^2_\mu(\Lambda^{k+1}M)$ by duality:

$$\int_M d\alpha \wedge \varphi = (-1)^{k+1} \int_M \alpha \wedge d\varphi \quad \text{for all } \varphi \in C_0^\infty(\Lambda^{n-1-k}M).$$

By definition we have $d\mathcal{D}^{k-1}_\mu(d) \subset Z^k_\mu(M)$, but $d\mathcal{D}^{k-1}_\mu(d)$ is not necessarily a closed subspace of $L^2_\mu(\Lambda^k M)$. If we introduce the space $B^k_\mu(M) = \overline{d\mathcal{D}^{k-1}_\mu(d)}$, then $B^k_\mu(M) \subset Z^k_\mu(M)$ since $Z^k_\mu(M)$ is a closed subspace of $L^2_\mu(\Lambda^k M)$.

DEFINITION 2.1. The $k$-th reduced $L^2_\mu$ cohomology space is the quotient

$$\mathbb{H}^k_\mu(M) = \frac{Z^k_\mu(M)}{B^k_\mu(M)}.$$

The $k$-th $L^2_\mu$ cohomology space is the quotient

$$\frac{Z^k_\mu(M)}{d\mathcal{D}^{k-1}_\mu(d)},$$

which is not a Hilbert space but satisfies other good properties of a cohomology theory; for instance, a Mayer–Vietoris sequence holds for $L^2_\mu$ cohomology.

Our aim is to circumvent the fact that in general we have problems computing reduced $L^2_\mu$ from local calculations because the Mayer–Vietoris exact sequence does not hold in the reduced setting.

### 2.2. Some general properties of reduced $L^2_\mu$ cohomology

***Quasi-isometry invariance.*** The first general fact is a consequence of the definition: $L^2_\mu$ (reduced or not) cohomology spaces depend only on the $L^2_\mu$ topology; hence if $g_0$ and $g_1$ are two Riemannian metrics such that $\varepsilon g_0 \le g_1 \le g_0/\varepsilon$ for a certain $\varepsilon > 0$, and if $\mu_0, \mu_1$ are positive smooth functions such that $\mu_0/\mu_1$ and $\mu_1/\mu_0$ are bounded, then

$$\mathbb{H}^k_{\mu_0}(M, g_0) = \mathbb{H}^k_{\mu_1}(M, g_1).$$

***Smooth forms in $L^2$ cohomology.*** Using de Rham's smoothing operator of (see [10] and also [9]), we can show that reduced and nonreduced $L^2_\mu$ cohomology can be computed using only smooth forms; that is,

$$\mathbb{H}^k_\mu(M) \simeq \frac{Z^k_\mu(M) \cap C^\infty(\Lambda^k M)}{\overline{d\mathcal{D}^{k-1}_\mu(d) \cap C^\infty(\Lambda^k M)} \cap C^\infty(\Lambda^k M)}.$$

This smoothing argument also shows that if $M$ is a closed manifold, reduced and nonreduced $L^2_\mu$ cohomology are both isomorphic to de Rham cohomology.

The smoothing operator gives additional results in the following setting: Assume that $M$ is an open subset in a manifold $N$ such that near every point of the boundary $p \in \partial M = \overline{M} \setminus M$ there is a submersion $x = (x_1, \dots, x_k) : U \to \mathbb{R}^k$ on a neighborhood of $p$ such that $x(p) = 0$ and $U \cap M = \{x_1 > 0, \dots, x_k > 0\}$. Such a manifold will be called a manifold with corners. Consider a Riemannian metric on $M$ which extends smoothly to $N$. Moreover, assume that $(\overline{M}, g)$ is *metrically complete*, that is, for any $o \in M$ and $r > 0$ then the closure $\overline{B(o,r) \cap M}$ is compact in $\overline{M}$. This is automatically the case when $g$ extends to a smooth geodesically complete metric on $N$.

Then we can define two spaces of smooth forms: $C_0^\infty(\Lambda^k M)$ is the set of smooth forms with compact support in $M$ and $C_0^\infty(\Lambda^k \overline{M})$ is the set of smooth forms with compact support in $\overline{M}$. This is illustrated in Figure 1. Then a smoothing argument shows:

PROPOSITION 2.2. *If $(M, g)$ is a Riemannian manifold with corner whose closure is metrically complete, then $C_0^\infty(\Lambda^{k-1} \overline{M})$ is dense in $\mathcal{D}_\mu^{k-1}(d)$ when the domain of $d$ is endowed with the graph norm*:

$$\alpha \mapsto \sqrt{\|\alpha\|_\mu^2 + \|d\alpha\|_\mu^2}.$$

**2.3. Harmonic forms and $L_\mu^2$ cohomology.** When the Riemannian manifold $(M, g)$ is *geodesically complete* (hence boundaryless), reduced $L_\mu^2$ cohomology has an interpretation in terms of appropriate harmonic $L_\mu^2$ forms. We introduce $d_\mu^*$, the formal $L_\mu^2$ adjoint of the operator $d$; it is defined through the integration by parts formula

$$\langle d\alpha, \beta \rangle_\mu = \langle \alpha, d_\mu^* \beta \rangle_\mu \quad \text{for all } \alpha \in C_0^\infty(\Lambda^k M) \text{ and } \beta \in C_0^\infty(\Lambda^{k+1} M),$$

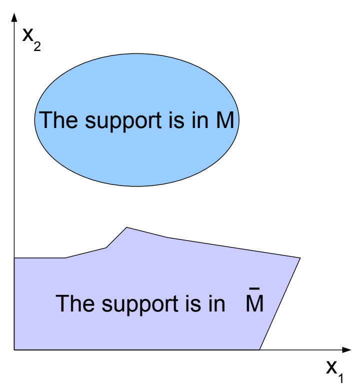


**Figure 1.** The support of an element of $C_0^\infty(\Lambda^k M)$ and the support of an element of $C_0^\infty(\Lambda^k \overline{M})$. Here $\overline{M}$ is the positive quadrant $\{x_1 \geq 0, x_2 \geq 0\}$ in the plane.

Then we have

$$\begin{aligned}\mathbb{H}^k_\mu(M) &\simeq \left\{\alpha \in L^2_\mu(\Lambda^k M) : d\alpha = d^*_\mu\alpha = 0\right\}\\ &= \left\{\alpha \in L^2_\mu(\Lambda^k M) : (d^*_\mu d + d d^*_\mu)\alpha = 0\right\}.\end{aligned}$$

## 3. The almost closed range condition

*From now we assume that $(M, g)$ is a manifold with corner whose closure is metrically complete.*

**3.1. Good primitives.** A natural question, which leads to a better understanding of reduced $L^2_\mu$ cohomology, is how an $L^2_\mu$ smooth closed form

$$\alpha \in L^2(\Lambda^k M) \cap C^\infty(\Lambda^k \overline{M})$$

can be zero in reduced $L^2_\mu$ cohomology.

A result of de Rham [10, Theorem 24] implies that for such an $\alpha$, there is always a $\beta \in C^\infty(\Lambda^k \overline{M})$ such that

$$\alpha = d\beta,$$

but this $\beta$ will not generally be in $L^2$. By Proposition 2.2, the vanishing of the reduced $L^2_\mu$ cohomology class of $\alpha$ is equivalent to the existence of a sequence of smooth forms $\beta_j \in C^\infty_0(\Lambda^{k-1} \overline{M})$ such that

$$d\beta = L^2_\mu\text{-}\lim_{j\to\infty} d\beta_j.$$

Hence the problem is to understand what growth conditions on the primitive $\beta$ imply the existence of such a sequence of smooth compactly supported forms, $(\beta_j)$.

It is clear that if $\beta \in L^2_\mu$ then the class of $\alpha = d\beta$ is zero in $\mathbb{H}^k_\mu(M)$. A natural way to obtain a more general condition is to find a sequence of cut-off[2] Lipschitz functions $\chi_j$ satisfying the following conditions:

- $\chi_j$ tends to 1 uniformly on the compact sets of $\overline{M}$.
- $L^2_\mu - \lim_{j\to\infty} d\chi_j \wedge \beta = 0$.

We always have $\chi_j\beta \in \mathcal{D}^{k-1}_\mu(d)$ and $d(\chi_j \wedge \beta) = d\chi_j \wedge \beta + \chi_j d\beta$; hence these conditions would imply $\alpha = d\beta = L^2_\mu - \lim_{j\to\infty} d\beta_j$. The notion of parabolic weights, which we're about to introduce is used to describe the regulation required on the growth of a primitive $\beta$ at infinity needed for this idea to work.

[2]i.e., with compact support on $\overline{M}$.

### 3.2. Parabolic weights

DEFINITION 3.1. A positive function $w : M \to (0, +\infty)$ is called a *parabolic weight* when there is a function $\psi : (0, +\infty) \to (0, +\infty)$ such that

- for a fixed point $o \in M$ and $r(x) = d(o, x)$ we have $w \geq \dfrac{1}{\psi^2(r)}$, and
- $\displaystyle\int_1^\infty \frac{dr}{\psi(r)} = +\infty$.

LEMMA 3.2. *Assume that $\alpha \in Z^k_\mu(M)$ satisfies $\alpha = d\beta$, with $\beta \in L^2_{w\mu}$ for some parabolic weight $w$. Then the reduced $L^2_\mu$ class of $\alpha$ is zero.*

We will not give a proof of this result, but we note that the parabolic condition makes possible the choice of a good sequence of cutoff functions as described in the last subsection. We will instead explain where this definition comes from. Parabolicity for a weighted Riemannian manifold $(M, g, \nu)$ has several equivalent definitions in terms of Brownian motion, capacity, and existence of positive Green functions [1; 12]. We will only give the following definition:

DEFINITION 3.3. The weighted Riemannian manifold $(M, g, \nu)$ is called parabolic if there is a sequence of cut-off Lipschitz functions $\chi_j$ such that

- $\chi_j$ tends to 1 uniformly on the compact set of $\overline{M}$, and
- $\lim_{j\to\infty} \int_M |d\chi_j|^2 \nu d \operatorname{vol}_g = 0$.

Here is a well known criterion that implies parabolicity.

PROPOSITION 3.4. *Let $o$ be a fixed point in the weighted Riemannian manifold $(M, g, \nu)$ and let*

$$L(r) = \int_{\partial B(o,r)} \nu \, d\sigma_g \quad \text{and} \quad V(r) = \int_{B(o,r)} \nu \, d\operatorname{vol}_g .$$

*If*

$$\int_1^\infty \frac{dr}{L(r)} = +\infty \quad \text{or} \quad \int_1^\infty \frac{r dr}{V(r)} = +\infty,$$

*then $(M, g, \nu)$ is parabolic.*

By definition, the parabolicity of the weighted manifold $(M, g, |\beta|^2\mu)$ implies that the reduced $L^2_\mu$ class of $d\beta = \alpha$ is zero. When $w$ is a parabolic weight, we define the space $\mathcal{C}^{k-1}_{w,\mu}(M)$ to be the set of $\beta \in L^2_{w\mu}(\Lambda^{k-1}M)$ such that the weak differential of $\beta$ is in $L^2_\mu$:

$$\mathcal{C}^{k-1}_{w,\mu}(M) := \{\beta \in L^2_{w\mu}(\Lambda^{k-1}M) : \ d\beta \in L^2_\mu\}.$$

Then the parabolicity of $w$ implies that $d : \mathcal{C}^{k-1}_{w,\mu}(M) \to B^k_\mu(M)$ is a bounded operator.

### 3.3. The almost closed range properties

DEFINITION 3.5. We say that the $L^2_\mu$ range of $d$ is *almost closed in degree $k$ with respect to $w$* when $w$ is a parabolic weight and

$$d\mathcal{C}^{k-1}_{w,\mu}(M) = B^k_\mu(M).$$

In other words, the $L^2_\mu$ range of $d$ is almost closed in degree $k$ with respect to $w$ if and only if every $L^2_\mu$ closed forms $\alpha$ which is zero in reduced $L^2_\mu$ cohomology has a $L^2_{w\mu}$ primitive.

***An example.*** Let $\Sigma$ be an $(n-1)$-dimensional compact manifold with boundary, endowed with a smooth Riemannian metric $h$ that extends smoothly to $\partial\Sigma$. The truncated cone $C_1(\Sigma) = (1, +\infty) \times \Sigma$ endowed with the conical metric

$$(dr)^2 + r^2 h$$

is a manifold with corner whose closure is metrically complete. Then we have:

PROPOSITION 3.6. *Consider the weight $\mu_a(r,\theta) = r^{2a}$. The $L^2_{\mu_a}$ cohomology of $C_1(\Sigma)$ is given by*

$$\mathbb{H}^k_{\mu_a}(C_1(\Sigma)) = \begin{cases} \{0\} & \text{if } k \le \frac{n}{2} + a, \\ H^k(\Sigma) & \text{if } k > \frac{n}{2} + a. \end{cases}$$

*Introduce the two (parabolic) weights $\overline{w} = 1/r^2$ and $w = 1/(r^2 \log^2(r+1))$.*

- *If $k \neq n/2 + a$ then the $L^2_{\mu_a}$ range of $d$ is almost closed in degree $k$ with respect to $\overline{w}$.*
- *If $k = n/2 + a$ then in general the $L^2_{\mu_a}$ range of $d$ is almost closed in degree $k$ with respect to $w$.*
- *If $k = n/2 + a$ and $H^{\frac{n}{2}+a-1}(\Sigma) = \{0\}$, the $L^2_{\mu_a}$ range of $d$ is almost closed in degree $k$ with respect to $\overline{w}$.*

***The good news: a Mayer–Vietoris exact sequence.*** The almost closed range is convenient because it implies a short Mayer–Vietoris exact sequence for reduced $L^2$ cohomology:

PROPOSITION 3.7. *Assume that $M = U \cup V$ and that $U$, $V$ and $U \cap V$ are manifolds with corners whose closures are metrically complete. Assume that for a parabolic weight $w : M \to (0, +\infty)$, the $L^2_\mu$ range of $d$ is almost closed in degree $k$ with respect to $w$ on $M$, $U$, $V$ and $U \cap V$, and that the sequence*

$$\{0\} \to \mathcal{C}^{k-1}_{w,\mu}(M) \xrightarrow{r^*} \mathcal{C}^{k-1}_{w,\mu}(U) \oplus \mathcal{C}^{k-1}_{w,\mu}(V) \xrightarrow{\delta} \mathcal{C}^{k-1}_{w,\mu}(U \cap V) \to \{0\}$$

*is exact. Then we have the short Mayer–Vietoris exact sequence*

$$\mathbb{H}^{k-1}_{w\mu}(U) \oplus \mathbb{H}^{k-1}_{w\mu}(V) \xrightarrow{\delta} \mathbb{H}^{k-1}_{w\mu}(U \cap V) \xrightarrow{b} \mathbb{H}^k_\mu(M) \xrightarrow{r^*} \mathbb{H}^k_\mu(U) \oplus \mathbb{H}^k_\mu(V) \xrightarrow{\delta} \mathbb{H}^k_\mu(U \cap V).$$

We will not give the proof of this result. In fact, the argument is relatively straightforward. We have only to follow the proof of the exactness of the Mayer–Vietoris sequence in de Rham cohomology in the compact case; the hypotheses made here are the ones that are necessary to adapted these classical arguments.

***The bad news.*** There is a difficulty with the assumption

$$\mathcal{C}^{k-1}_{w,\mu}(U) \oplus \mathcal{C}^{k-1}_{w,\mu}(V) \xrightarrow{\delta} \mathcal{C}^{k-1}_{w,\mu}(U \cap V) \to \{0\}.$$

For instance, let $C_1(\Sigma)$ be a truncated cone over a compact manifold with boundary $(\Sigma, h)$. Now when $\Sigma = \widetilde{U} \cup \widetilde{V}$, where $\widetilde{U}, \widetilde{V}, \widetilde{U} \cap \widetilde{V}$ are open with smooth boundaries, then if we let $U = C_1(\widetilde{U})$ and $V = C_1(\widetilde{V})$, we have that for the parabolic weight $\overline{w} = 1/r^2$, the sequence

$$\mathcal{C}^{k-1}_{\overline{w},1}(U) \oplus \mathcal{C}^{k-1}_{\overline{w},1}(V) \xrightarrow{\delta} \mathcal{C}^{k-1}_{\overline{w},1}(U \cap V) \to \{0\}$$

is exact. However, for the parabolic weight $w = 1/(r^2 \log^2(r+1))$, the sequence

$$\mathcal{C}^{k-1}_{w,1}(U) \oplus \mathcal{C}^{k-1}_{w,1}(V) \xrightarrow{\delta} \mathcal{C}^{k-1}_{w,1}(U \cap V) \to \{0\}$$

is not (necessarily) exact.

From Proposition 3.6, we see that on a truncated cone we cannot always use only the weight $\overline{w}$. Thus we'll have some difficulties using this exact sequence.

### 3.4. Comparison with other notions

***With the nonparabolicity condition.*** In [4] and [5], we introduced the notion of nonparabolicity at infinity for the Dirac operator on a complete Riemannian manifold and used it in [6] to compute the $L^2$ cohomology of manifolds with flat ends. This condition is an extended Fredholmness condition. Specialized to the case of the Gauss–Bonnet operator, $d + d^*_\mu$, this condition is satisfied in the following case:

PROPOSITION 3.8. *Assume $(M, g)$ is a complete Riemannian manifold and there is a weight $w : M \to (0, +\infty)$ and a compact set $K \subset M$ such that*

$$\|\alpha\|_{w\mu} \leq \|(d + d^*_\mu)\alpha\|_\mu \quad \textit{for all } \alpha \in C^\infty_0(\Lambda^k(M \setminus K)),$$

*Then the reduced $L^2_\mu$ cohomology of $M$ is finite-dimensional. Moreover, for any $\alpha \in L^2_\mu(\Lambda^k M)$, there exist $h \in L^2(\Lambda^k M)$ such that $dh$ and $d^*_\mu h$ vanish, $\beta \in L^2_{w\mu}(\Lambda^{k-1} M)$, and $\gamma \in L^2_{w\mu}(\Lambda^{k+1} M)$, such that*

$$\alpha = h + d\beta + d^*_\mu \gamma.$$

*Finally, for any $\alpha \in B^k_\mu(M)$, there is a $\beta \in L^2_{w\mu}(\Lambda^{k-1} M)$ such that $\alpha = d\beta$.*

Hence, under the assumptions of the Proposition 3.8, if $w$ is a parabolic weight the $L^2_\mu$ range of $d$ is almost closed in degree $k$ with respect to $w$. There is a closely related result about the almost closed range condition [7]:

PROPOSITION 3.9. *Assume that $(M, g)$ is a complete Riemannian manifold and that $w : M \to (0, +\infty)$ is a parabolic weight. Suppose there are a positive constant $C$ and a compact set $K \subset M$ such that*

$$C\|\alpha\|^2_{w\mu} \le \|d\alpha\|^2_\mu + \|d^*_{w\mu}\alpha\|^2_{w\mu} \quad \textit{for all } \alpha \in C^\infty_0(\Lambda^k(M \setminus K)). \tag{3-1}$$

*Then*

- *the $L^2_\mu$ range of $d$ is almost closed in degree $k$ with respect to $w$;*
- *the $L^2_{w\mu}$ range of $d$ is almost closed in degree $k-1$ with respect to $w$;*
- *the space $H^{k-1}_{w\mu}(M)$ is finite-dimensional.*

*Moreover, these three properties imply the existence of a positive constant $C$ and a compact set $K \subset M$ such that the inequality* (3-1) *holds.*

The first proposition is in fact a statement about the operator $d + d^*_\mu$, whereas the second is a statement about $d$.

***With more classical cohomology theory.*** Let $(X, g)$ be a complete Riemannian manifold, fix a degree $k$ and assume that we have a sequence of weights $w_l$ (which will depend on $k$ in general) such that $w_k = 1$ and, for all degrees $l$, the $L^2_{w_l\mu}$ range of $d$ is almost closed in degree $l$ with respect to $w_{l-1}/w_l$. Then we consider the complex

$$\cdots \to \mathcal{C}^{l-1}_{w_{l-1}/w_l, w_l\mu}(X) \xrightarrow{d} \mathcal{C}^{l}_{w_l/w_{l+1}, w_{l+1}\mu}(X) \xrightarrow{d} \cdots$$

When the cohomology of this complex can be computed from a local computation (that is, when there is a Poincaré lemma characterizing the cohomology of this complex), then the $L^2_\mu$ cohomology of $X$ can be obtained from the degree $k$ cohomology space of this complex. This method has been used successfully by T. Hausel, E. Hunsicker and R. Mazzeo in [14] to obtain a topological interpretation of the $L^2_{\mu=1}$ cohomology of manifolds with fibered cusp ends or with fibered boundary ends. However, in this case the proof is not simple and the authors have to face the same kind of difficulty as the one we encountered on page 24, essentially because the choice of primitive sometimes doesn't lead to a complex whose cohomology follows from local computations. In this paper, the authors had to compare the $L^2_{\mu=1}$ cohomology with two other weighted cohomologies, $L^2_\mu$, $L^2_{1/\mu}$, with $\mu = r^\varepsilon$ or $\mu = e^{\varepsilon r}$ where $r$ is the function given by distance to a fixed point; this comparison is made with an adapted pseudodifferential calculus.

## 4. The QALE geometry of the Hilbert scheme of 2 or 3 points

We will now describe the QALE geometry of the Hilbert scheme of 2 and 3 points on $\mathbb{C}^2$. The Hilbert scheme of $n$ points on $\mathbb{C}^2$, denoted by $\mathrm{Hilb}_0^n(\mathbb{C}^2)$, is a crepant resolution of the quotient of $(\mathbb{C}^2)_0^n = \{q \in (\mathbb{C}^2)^n, \sum_j q_j = 0\}$ by the action of the symmetric group $S_n$, which acts by permutation of the indices:

$$\sigma \in S_n, q \in (\mathbb{C}^2)_0^n,\ \sigma.q = (q_{\sigma^{-1}(1)}, q_{\sigma^{-1}(2)}, \dots, q_{\sigma^{-1}(n)}).$$

Hence we have a resolution of singularities map

$$\pi : \mathrm{Hilb}_0^n(\mathbb{C}^2) \to (\mathbb{C}^2)_0^n / S_n.$$

**4.1. The case of 2 points.** For $n = 2$, we have

$$(\mathbb{C}^2)_0^n = \{(x, -x) : x \in \mathbb{C}^2\};$$

hence

$$(\mathbb{C}^2)_0^2 / S_2 \simeq \mathbb{C}^2 / \{\pm \mathrm{Id}\}.$$

Now the crepant resolution of $\mathbb{C}^2/\{\pm \mathrm{Id}\}$ is $T^*\mathbb{P}^1(\mathbb{C})$; indeed, we have that the cotangent bundle of $\mathbb{P}^1(\mathbb{C})$ is the set of pairs $(L, \xi)$ where $L$ is a line in $\mathbb{C}^2$ and $\xi : \mathbb{C}^2 \to \mathbb{C}^2$ a linear map such that the range of $\xi$ is contained in $L$ and such that the kernel of $\xi$ contains $L$. That is, $\xi$ induces a linear map $\overline{\xi} : \mathbb{C}^2/L \to L$. In particular, $T^*\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{C})$ is identified with the set

$$\{\xi \in \mathcal{M}_2(\mathbb{C}^2) : \xi \neq 0, \xi \circ \xi = 0\} = \left\{\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) : a^2 = bc, (a, b, c) \neq (0, 0, 0)\right\},$$

through the identification $\xi \mapsto (\mathrm{Im}\, \xi, \xi)$. This space is diffeomorphic to the quotient $(\mathbb{C}^2 \setminus \{0\})/\{\pm \mathrm{Id}\}$ through the map

$$\pm(x, y) \mapsto \begin{pmatrix} xy & y^2 \\ x^2 & -xy \end{pmatrix}.$$

$T^*\mathbb{P}^1(\mathbb{C})$ carries a remarkable metric, the Eguchi–Hanson metric, which is Kähler and Ricci flat [11; 3]. Moreover, this metric on $T^*\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{C}) \simeq \mathbb{C}^2/\{\pm \mathrm{Id}\}$ is asymptotic to the Euclidean metric. Such a metric is called asymptotically locally euclidean (ALE in short).

**4.2. The case of 3 points.** We can also understand the geometry of $\mathrm{Hilb}_0^3(\mathbb{C}^2)$. Outside a compact set, $(\mathbb{C}^2)_0^3/S_3$ is a truncated cone over $\mathbb{S}^7/S_3$ and the singular set of the quotient $\mathbb{S}^7/S_3$ pulls back to $\mathbb{S}^7$ as a disjoint union of 3 sub-spheres $\mathbb{S}^3$ given by the intersection of $\mathbb{S}^7$ with the collision planes given by

$$P_{i,j} = \{(q_1, q_2, q_3) \in (\mathbb{C}^2)_0^3 : q_i = q_j\}, \quad \text{where } i < j.$$

This is illustrated in Figure 2, left. These three spheres are interchanged by the action of $S_3$; hence the singular set of $\mathbb{S}^7/S_3$ is a sphere $\mathbb{S}^3$ and the geometry of

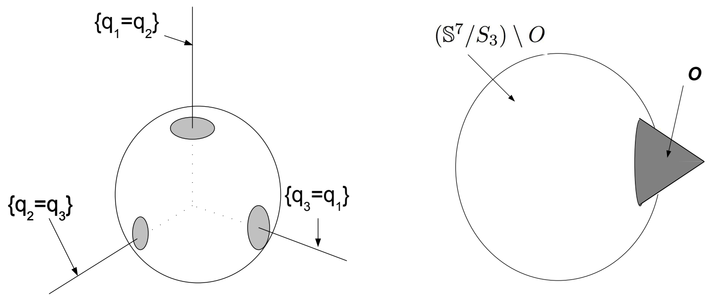


**Figure 2.** Left: The three collision planes in $(\mathbb{C}^2)^3_0$. Right: $\mathbb{S}^7/S_3$.

$\mathbb{S}^7/S_3$ near the singular set is the one of $\mathbb{B}^2/\{\pm\,\mathrm{Id}\}\times\mathbb{S}^3$, where $\mathbb{B}^2$ is the unit ball in $\mathbb{C}^2$.

Hence, as illustrated in Figure 2, right, outside a compact set, $(\mathbb{C}^2)^3_0/S_3$ is the union of

- $U$, a truncated cone over $(\mathbb{S}^7/S_3)\setminus O$, where $O$ is an $\varepsilon$-neighborhood of the singular set (hence homeomorphic to $\mathbb{B}^2/\{\pm\,\mathrm{Id}\}\times\mathbb{S}^3$),
- $\big\{(x,v)\in\mathbb{C}^2/\{\pm\,\mathrm{Id}\}\times\mathbb{C}^2 : |x|^2+|v|^2>1,\ |x|<\varepsilon|v|\big\}$.

The geometry at infinity of $\mathrm{Hilb}^3_0(\mathbb{C}^2)$ is the union of two open sets:

- $U$: a truncated cone over $(\mathbb{S}^7/S_3)\setminus O$, and
- $V=\big\{(y,v)\in T^*\mathbb{P}^1(\mathbb{C})\times(\mathbb{C}^2\setminus\mathbb{B}^2) : |y|<\varepsilon|v|\big\}$, where $|y|$ is the pullback to $T^*\mathbb{P}^1(\mathbb{C})$ of the Euclidean distance;

that is, there is a compact set $K\subset\mathrm{Hilb}^3_0(\mathbb{C}^2)$ such that $\mathrm{Hilb}^3_0(\mathbb{C}^2)\setminus K=U\cup V$.

In [16], D. Joyce constructed a hyperkähler metric $g$ on $\mathrm{Hilb}^3_0(\mathbb{C}^2)$ (in particular this is a Kähler and Ricci flat metric) which is quasi-asymptotically locally euclidean (QALE) asymptotic to $(\mathbb{C}^2)^3_0/S_3$, which means that

- on $U$, the truncated cone over $(\mathbb{S}^7/S_3)\setminus O$, we have for all $l\in\mathbb{N}$
$$\nabla^l(g-\mathrm{eucl})=O\Big(\frac{1}{r^{4+l}}\Big),$$
where $r$ is the radial function on this cone.
- On $V$ we have, for all $l\in\mathbb{N}$,
$$\nabla^l\big(g-(g_{\mathrm{Hilb}^2_0(\mathbb{C}^2)}+\mathrm{eucl})\big)=O\left(\frac{1}{|y|^{2+l}|v|^2}\right).$$

## 5. $L^2$ cohomology of the Hilbert scheme of 2 or 3 points

We now explain how we can compute the $L^2$ cohomology[3] of the Hilbert Scheme of 2 or 3 points with the almost closed range condition.

**5.1. The case of 2 points.** We use Proposition 3.6, which computed the $L^2$ cohomology of truncated cones. Since $\mathrm{Hilb}_0^2(\mathbb{C}^2) = T^*\mathbb{P}^1(\mathbb{C})$ has real dimension 4, is oriented and has infinite volume, we easily get that the $L^2$ cohomology of $\mathrm{Hilb}_0^2(\mathbb{C}^2)$ is zero in degrees 0 and 4. Moreover, the Ricci curvature of the Eguchi–Hansen metric is zero, so the Bochner formula implies that for any $L^2$ harmonic 1-form $\alpha$, we get

$$0 = \int_{\mathrm{Hilb}_0^2(\mathbb{C}^2)} |d\alpha|^2 + |d^*\alpha|^2 = \int_{\mathrm{Hilb}_0^2(\mathbb{C}^2)} |\nabla\alpha|^2.$$

Hence the space of $L^2$ harmonic 1-forms is trivial and it remains only to compute the $L^2$ cohomology of $\mathrm{Hilb}_0^2(\mathbb{C}^2)$ in degree 2. The main point is this:

LEMMA 5.1. *The natural map from cohomology with compact support to $L^2$ cohomology is surjective in degree* 2:

$$H_c^2(\mathrm{Hilb}_0^2(\mathbb{C}^2)) \to \mathbb{H}^2(\mathrm{Hilb}_0^2(\mathbb{C}^2)) \to \{0\}.$$

PROOF. As a matter of fact, if $\alpha \in Z^2_{\mu=1}(\mathrm{Hilb}_0^2(\mathbb{C}^2))$ is a $L^2$ closed 2-form, its restriction to the neighborhood $U$ of infinity[4] is exact, according to Proposition 3.6. Moreover, for $w$ as in that proposition, we can find $\beta \in \mathcal{C}^1_{w,1}(U)$ a primitive of $\alpha|_U$:

$$\text{on } U,\ \alpha = d\beta.$$

If $\bar\beta \in \mathcal{C}^1_{w,1}(\mathrm{Hilb}_0^2(\mathbb{C}^2))$ is an extension of $\beta$ then because $w$ is a parabolic weight, $\alpha - d\overline{\beta}$ and $\alpha$ have the same $L^2$ cohomology class and $\alpha - d\overline{\beta}$ has compact support. □

The Hodge and Poincaré dualities imply that we also have an injective map from $L^2$ cohomology to absolute cohomology:

$$\{0\} \to \mathbb{H}^2(\mathrm{Hilb}_0^2(\mathbb{C}^2)) \to H^2(\mathrm{Hilb}_0^2(\mathbb{C}^2)).$$

But the natural map from cohomology with compact support to absolute cohomology is an isomorphism in degree 2; hence we have the following isomorphism:

[3] $L^2$ cohomology refers to reduced $L^2_{\mu=1}$ cohomology. From now on we will avoid the subscript 1 when dealing with spaces related to $L^2_{\mu=1}$ cohomology.

[4] The ALE condition says that on $U$, the Eguchi–Hanson metric and the Euclidean metric (for which $U$ will be a truncated cone over $\mathbb{S}^3/\{\pm \mathrm{Id}\}$) are quasi-isometric. Hence by 2.2, the $L^2$ cohomology of the two metrics are the same.

THEOREM 5.2. *For* $\mathrm{Hilb}_0^2(\mathbb{C}^2) = T^*\mathbb{P}^1(\mathbb{C})$ *endowed with the Eguchi–Hansen metric, we have*

$$\mathbb{H}^k(\mathrm{Hilb}_0^2(\mathbb{C}^2)) = \begin{cases} \{0\} & \text{if } k \neq 2, \\ \mathbb{R} \simeq H_c^2(\mathrm{Hilb}_0^2(\mathbb{C}^2)) \simeq H^2(\mathrm{Hilb}_0^2(\mathbb{C}^2)) & \text{if } k = 2. \end{cases}$$

REMARK 5.3. In fact, there is a general result about the $L^2$ cohomology of manifolds with conical ends: *Suppose* $(X^n, g)$ *is a Riemannian manifold with conical ends* (*meaning that there is a compact set with smooth boundary* $K \subset X$ *such that* $(X \setminus K, g)$ *is isometric to the truncated cone* $C_1(\partial K)$). *Let* $w$ *be a smooth function on* $X$ *such that*

$$w(r, \theta) = r^{-2}(1 + \log r)^{-2}. \quad \text{on } X \setminus K \simeq C_1(\partial K).$$

*Then, on* $(X, g)$, *the* $L^2$ *range of* $d$ *is almost closed in any degree with respect to* $w$ *and the* $L^2$ *cohomology of* $(X, g)$ *is given by*

$$\mathbb{H}_\mu^k(X) = \begin{cases} H_c^k(X) & \text{if } k < n/2, \\ \mathrm{Im}\big(H_c^k(X) \to H^k(X)\big) & \text{if } k = n/2, \\ H^k(X) & \text{if } k > n/2. \end{cases}$$

There are different proofs of this result. The first one uses the scattering calculus developed by Melrose; see [21, Theorem 4] and [14, Theorem 1A]. The second uses the almost closed range condition, the computation of the $L^2$ and weighted $L^2_w$ cohomologies of a truncated cone, and the Mayer–Vietoris sequence 3.7 [7, Theorem 4.11]. For the case of the Eguchi–Hanson metric, this topological interpretation of the space of $L^2$ cohomology can also be obtained using explicit computation of harmonic forms because this metric has an SU(2) invariance; hence the harmonic equation reduces to an ODE [15, section 5.5].

### 5.2. The case of 3 points

***A vanishing result outside degree 4.*** According to [8], the QALE metric on $\mathrm{Hilb}_0^3(\mathbb{C}^2)$ constructed by D. Joyce coincides with the one of H. Nakajima, who showed in [23] that $\mathrm{Hilb}_0^3(\mathbb{C}^2)$ can be endowed with a hyperkähler metric using the hyperkähler reduction of a Euclidean quaternionic space.[5] According to N. Hitchin, the $L^2$ cohomology of a hyperkähler reduction of a Euclidean quaternionic space is trivial except perhaps for the degree equal to the middle (real) dimension [15]. Hence in our case, we only need to compute the $L^2$ cohomology of $\mathrm{Hilb}_0^3(\mathbb{C}^2)$ in degree 4.

[5] This is a general fact for all the Hilbert schemes of points in $\mathbb{C}^2$, $\mathrm{Hilb}_0^n(\mathbb{C}^2)$, $n \geq 2$.

***The result in degree 4.*** It is again true that for $\mathrm{Hilb}_0^3(\mathbb{C}^2)$, the natural map from cohomology with compact support to absolute cohomology is an isomorphism in degree 4 and moreover these spaces have dimension 1. We have:

LEMMA 5.4. *There exists a compact set $K \subset \mathrm{Hilb}_0^3(\mathbb{C}^2)$ such that $\mathrm{Hilb}_0^3(\mathbb{C}^2)$ retracts on $K$ and such that any $L^2$ closed 4-form $\alpha$ on $\mathrm{Hilb}_0^3(\mathbb{C}^2) \setminus K$ has a primitive $\beta \in L^2_w$ with $w = 1/(r\log(r+1))^2$. In particular, on $\mathrm{Hilb}_0^3(\mathbb{C}^2) \setminus K$, the $L^2$ range of $d$ is almost closed in degree 4 with respect to $w$.*

Using this and the same arguments as in the case of 2 points, we obtain:

THEOREM 5.5. *For $\mathrm{Hilb}_0^3(\mathbb{C}^2)$ endowed with the QALE metric described in (4.2) we have*:

$$\mathbb{H}^k(\mathrm{Hilb}_0^3(\mathbb{C}^2)) = \begin{cases} \{0\} & \text{if } k \neq 4, \\ \mathbb{R} \simeq H^4_c(\mathrm{Hilb}_0^3(\mathbb{C}^2)) \simeq H^4(\mathrm{Hilb}_0^3(\mathbb{C}^2)) & \text{if } k = 4. \end{cases}$$

THE PROOF OF LEMMA 5.4. Let $K \subset \mathrm{Hilb}_0^3(\mathbb{C}^2)$ be a compact set such that $\mathrm{Hilb}_0^3(\mathbb{C}^2) \setminus K = U \cup V$, where

- $U$ is a truncated cone over $(\mathbb{S}^7/S_3) \setminus O$, and
- $V = \{(y,v) \in T^*\mathbb{P}^1(\mathbb{C}) \times (\mathbb{C}^2 \setminus \mathbb{B}^2) : |y| < \varepsilon|v|\}$, where $|y|$ is the pullback to $T^*\mathbb{P}^1(\mathbb{C})$ of the Euclidean distance.

The main point in the proof of Lemma 5.4 is the following result concerning the $L^2$ cohomology of $V$:

LEMMA 5.6. *$\mathbb{H}^4(V) = \{0\}$ and $Z^4(V) = d\mathcal{C}^3_{w,1}(V)$. That is, any $L^2$ closed 4-form $\alpha$ on $V$ has a primitive $\varphi \in L^2_w(\Lambda^3 V)$, that is, $\alpha = d\varphi$.*

We will only sketch the proof of this lemma.

Consider $\alpha \in Z^4_{\mu=1}(V)$. The set $U \cap V$ is a truncated cone over the product $\mathbb{S}^3/\{\pm \mathrm{Id}\} \times \mathbb{S}^3 \times (\varepsilon, 2\varepsilon)$. The third Betti number of this product is not zero; hence by Proposition 3.6, there is a $\psi \in L^2_w(\Lambda^3(U \cap V))$ such that

$$\alpha = d\psi \quad \text{on } U \cap V.$$

We cannot extend $\psi$ to $V$ as an element of $\mathcal{C}^3_{w,1}(V)$ but only as an element $\overline{\psi} \in \mathcal{C}^3_{w,\rho}(V)$ where $\rho(y,v) = 1/\log^2(|v|+1)$. Then

$$\alpha - d\overline{\psi} \in Z^4_\rho(V)$$

and, because this form is zero on $V \cap U$, it can be extended to the whole $T^*\mathbb{P}^1(\mathbb{C}) \times (\mathbb{C}^2 \setminus \mathbb{B}^2)$. This extension will be also denoted by $\alpha - d\overline{\psi}$. It is still a closed form; that is,

$$\alpha - d\overline{\psi} \in Z^4_\rho\big(T^*\mathbb{P}^1(\mathbb{C}) \times (\mathbb{C}^2 \setminus \mathbb{B}^2)\big).$$

Now using a Künneth-type argument and the computation of the $L^2_\rho$ cohomology of $\mathbb{C}^2 \setminus \mathbb{B}^2$, it can be shown that the $L^2_\rho$ cohomology of $T^*\mathbb{P}^1(\mathbb{C}) \times (\mathbb{C}^2 \setminus \mathbb{B}^2)$ vanishes in degree 4 and that if we introduce the weight $\overline{w}_1(y, v) = 1/(1+|y|)^2$, there are

$$u \in \mathcal{C}^3_{\overline{w},\rho}(T^*\mathbb{P}^1(\mathbb{C}) \times (\mathbb{C}^2 \setminus \mathbb{B}^2)) \quad \text{and} \quad v \in \mathcal{C}^3_{\overline{w}_1,\rho}(T^*\mathbb{P}^1(\mathbb{C}) \times (\mathbb{C}^2 \setminus \mathbb{B}^2))$$

such that

$$\alpha - d\overline{\psi} = du + dv.$$

If we let $\varphi = \overline{\psi} + u + v$, then because on $V$ we have $\varepsilon\overline{w} \leq \overline{w}_1$, we conclude that $\varphi \in L^2_w(\Lambda^3 V)$ and

$$\alpha = d\varphi.$$

With this result, we can finish the proof of Lemma 5.4: we cannot use the Mayer–Vietoris exact sequence because of the log factor in the weight $w$ (see middle of page 24). However, we will use some of the arguments leading to the proof of the exactness of the Mayer–Vietoris sequence.

Let $\alpha$ be a closed $L^2$ 4-form outside $K$. Because $U$ is a truncated cone, we know that there is $\varphi_U \in \mathcal{C}^3_{w,1}(U)$ such that $\alpha = d\varphi_U$ on $U$, and by Lemma 5.6, there is a $\varphi_V \in \mathcal{C}^3_{w,1}(V)$ such that on $V$

$$\alpha = d\varphi_V.$$

Now on the intersection $U \cap V$ the difference $\varphi_U - \varphi_V$ is a closed $L^2_w$ 3-form. But $U \cap V$ is a truncated cone over $\mathbb{S}^3/\{\pm \mathrm{Id}\} \times \mathbb{S}^3 \times (\varepsilon, 2\varepsilon)$ and there is an analogue of Proposition 3.6 for the $L^2_w$ cohomology, the threshold now being $n/2-1 = 3$ in our case. But the second Betti number of $\mathbb{S}^3/\{\pm \mathrm{Id}\} \times \mathbb{S}^3 \times (\varepsilon, 2\varepsilon)$ is zero; hence on $U \cap V$, $\varphi_U - \varphi_V$ has a primitive $\eta \in \mathcal{C}^2_{\overline{w},w}(U \cap V)$ which can be extend to a $\overline{\eta} \in \mathcal{C}^2_{\overline{w},w}(U)$. Now we can define $\psi$

$$\psi = \begin{cases} \varphi_U + d\overline{\eta} & \text{on } U, \\ \varphi_V & \text{on } V. \end{cases}$$

By construction, we have $\alpha = d\psi$ and $\psi \in L^2_w\big(\Lambda^3(\mathrm{Hilb}^3_0(\mathbb{C}^2) \setminus K)\big)$. □

**5.3. Conclusion.** In the physics literature, $\mathrm{Hilb}^n_0(\mathbb{C}^2)$ is associated to the moduli space of instantons on noncommutative $\mathbb{R}^4$ [24]. One motivation for the study of $L^2$ cohomology of $\mathrm{Hilb}^n_0(\mathbb{C}^2)$ comes from a question of C. Vafa and E. Witten: in [29], see also the nice survey of T. Hausel [13], the following conjecture is formulated (note that $2(n-1) = \frac{1}{2}\dim_{\mathbb{R}} \mathrm{Hilb}^n_0(\mathbb{C}^2)$):

$$\mathbb{H}^k = \begin{cases} \{0\} & \text{if } k \neq 2(n-1), \\ \mathrm{Im}\big(H^k_c(\mathrm{Hilb}^n_0(\mathbb{C}^2)) \to H^k(\mathrm{Hilb}^n_0(\mathbb{C}^2))\big) & \text{if } k = 2(n-1). \end{cases} \tag{5-1}$$

However, Vafa and Witten have said that "unfortunately, we do not understand the prediction of $S$-duality on noncompact manifolds precisely enough to fully exploit them." According to N. Hitchin's vanishing result [15], the first part of this conjecture is true. The result above says that this is true for $n = 2, 3$.

## 6. Other results and perspectives

In [7], the $L^2$ cohomology of certain QALE spaces is computed. The proof uses the same general idea given in 5.2, but the argumentation is considerably longer and we cannot in general use the same vanishing result. Quasi Asymptotically Locally Euclidean (QALE) geometry is defined by induction. A QALE manifold asymptotic to $\mathbb{C}^n/\Gamma$ (where $\Gamma$ is a finite subgroup of $\mathrm{SU}(n)$) is a manifold whose geometry at infinity is the union of a piece that looks (up to a finite cover) like a subset of the product $Y \times \mathbb{C}^p$, where $Y$ is a QALE manifold asymptotic to $\mathbb{C}^{n-p}/A$ for some $A$ a finite subgroup of $\mathrm{SU}(n-p)$. In [7], we computed the $L^2$ cohomology of QALE spaces where the singular space $\mathbb{C}^n/\Gamma$ has only two singular strata ($\{0\}$ and a finite union of linear subspaces). In order to prove the Vafa–Witten conjecture (5-1), it is important to be able to understand the $L^2$ cohomology of more general QALE spaces. The almost closed range condition has been a interesting tool for doing this for the case of $\mathrm{Hilb}_0^3(\mathbb{C}^2)$. We hope that it will also be useful in other situations and for the other Hilbert schemes of points.

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