Ends and L^2 harmonic 1-forms

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- IV. Minimal hypersurfaces.

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- III. Topology and harmonics forms.
- IV. Minimal hypersurfaces.
- V. Hyperbolic manifolds.

I. Ends and topology.

Definition :

A manifold M is said to have only one end, if for any compact subset $K \subset M$, its exterior $M \setminus K$ has only one unbounded connected component.

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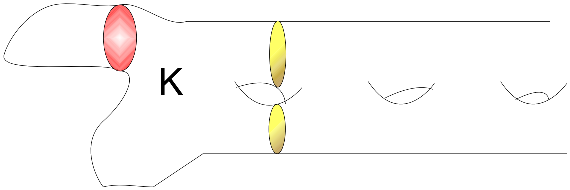
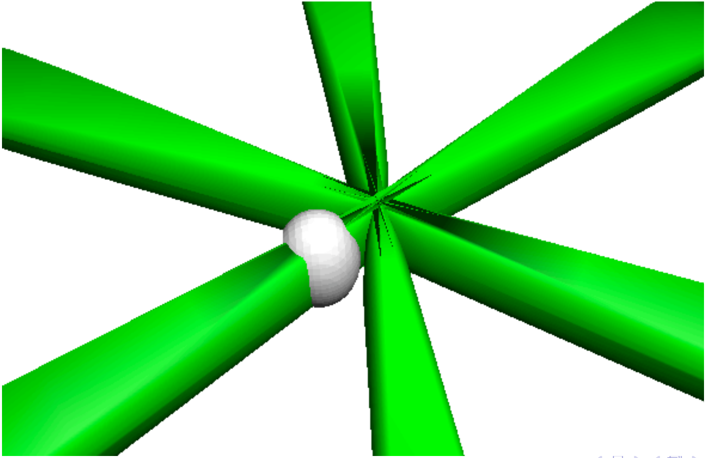
A manifold M is said to have only one end, if for any compact subset $K \subset M$, its exterior $M \setminus K$ has only one unbounded connected component. That is to say only one not relatively compact connected component.

For instance $\mathbb{R}^{n\geq 2}$ has only one end (i.e. only one way to go to infinity).



Definition :

A manifold M is said to have at least **N** ends, if there is a compact subset $K \subset M$, such that its exterior $M \setminus K$ has at least **N** unbounded connected components.



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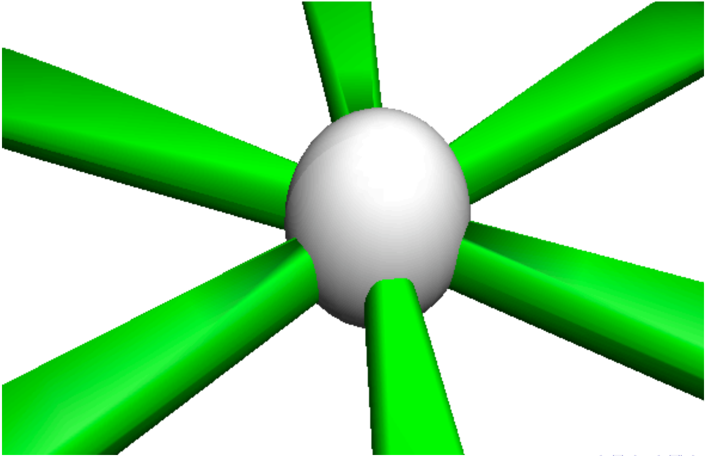


I. Ends and topology/cohomology

Proposition :

If a (connected) manifold M has at least 2 ends then

$$H_c^1(M) \neq \{0\}.$$

Recall that

$$H_c^1(M) = \frac{\{\alpha \in C_0^\infty(T^*M), d\alpha = 0\}}{dC_0^\infty(M)}.$$

I. Ends and topology/cohomology

$$H^1_c(M) = \frac{\{\alpha \in C_0^\infty(T^*M), d\alpha = 0\}}{dC_0^\infty(M)}.$$

Indead, in that case, there is a compact set $K \subset M$ such that

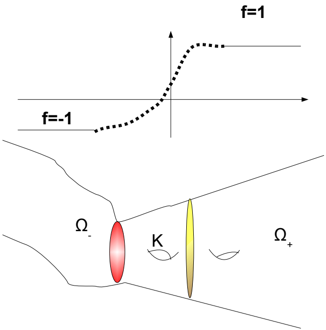
$$M \setminus K = \Omega_- \cup \Omega_+,$$

($\Omega_\pm$ unbounded). Choose $f \in C^\infty(M)$ such that

$$f = \pm 1, \text{ on } \Omega_\pm$$

Then $\alpha = df$ is closed with compact support in K and

$$\alpha \notin dC_0^\infty(M)$$



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In particular, we have the implication

$$H^1_c(M) = \{0\} \Rightarrow M \text{ has only one end.}$$

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Moreover, the kernel of the natural map from cohomology with compact support to absolute cohomology is in degree 1 isomorphic to :

$$\ker\left(H^1_c(M) \to H^1(M)\right) \simeq \mathbb{R}^{\text{Number of ends}}/\mathbb{R}.$$

I. Ends and topology/cohomology

In fact, there is a kind of reciproque of this fact :

Proposition: (C-Pedon)

If M is a connected manifold such that M and all its two fold cover have only one end then

$$H_c^1(M, \mathbb{Z}) = \{0\}.$$

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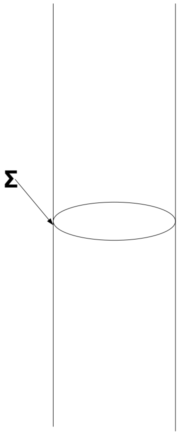
It is relatively easy to explain the homological argument : Assume that $\Sigma \subset M$ is a compact hypersurface. We want to show that Σ bounded a compact domain in M.

I. Ends and topology/cohomology

We have two cases : **In the first one :**

$M \setminus \Sigma$ has two connected components



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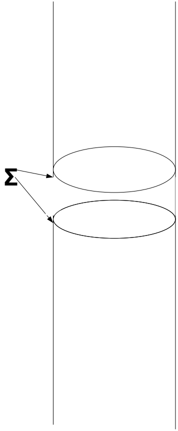
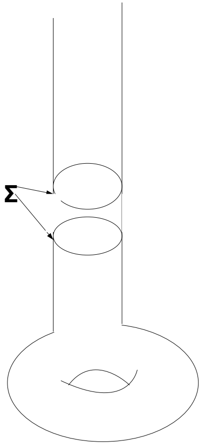


I. Ends and topology/cohomology

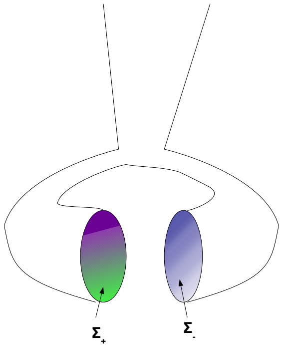
We have two cases : **In the first one :**

$M \setminus \Sigma$ has two connected components, but M is assumed to have only one end one of this connected component is relatively compact : so Σ bounded a compact domain.



I. Ends and topology/cohomology

In the second case :
$M \setminus \Sigma$ is connected :



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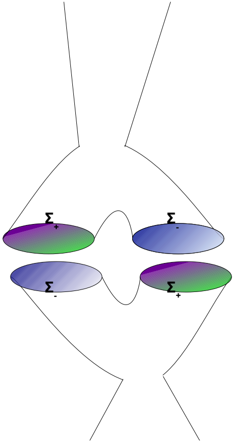
In the second case :
$M \setminus \Sigma$ is connected :then we can cut M along Σ and build a two fold cover of M with two ends:



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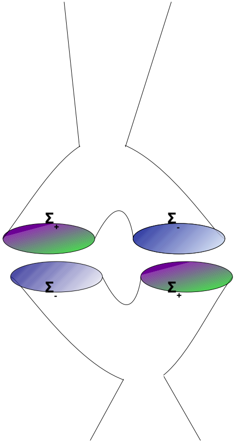
In the second case :
$M \setminus \Sigma$ is connected :then we can cut M along Σ and build a two fold cover of M with two ends: By hypothesis, this can not happen.



II. Ends and harmonic function

We assume now that (M, g) is a complete connected Riemannian manifold with at least two ends, hence there is a compact set set $K \subset M$ such that

$$M \setminus K = \Omega_- \cup \Omega_+,$$

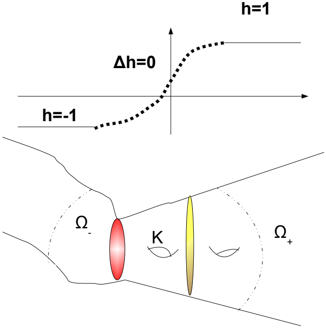
with $\Omega_\pm$ unbounded. We want to build a **harmonic** function h on (M, g) such that

$$\lim_{x \to \infty_{\Omega_\pm}} h(x) = \pm 1$$

II. Ends and harmonic function :Construction

We could try to build this function by exhaustion :
when $o \in M$ is a fixed point and $R > 0$ large enough (so that $K \subset B(o,R)$) then we can find a unique solution h_R to the Dirichlet problem

$$\begin{cases} \Delta h_R = 0 & \text{on } B(o,R) \\ h_R = \pm 1 & \text{on } \partial B(o,R) \cap \Omega_{\pm} \end{cases}$$



We extend h_R with

$$h_R = \pm 1 \text{ on } \Omega_{\pm} \setminus B(o,R)$$

II. Ends and harmonic function :Construction

Properties of h_R

By the maximum principle, we have

$$-1 \leq h_R \leq 1.$$

Moreover h_R realizes

$$\inf \left\{ \int_M |df|_g^2 d\mathrm{vol}_g , f \in W^{1,2}_{loc}, f = \pm 1 \text{ on } \Omega_\pm \setminus B(o,R) \right\}.$$

Hence $R \mapsto \int_{B(o,R)} |dh_R|_g^2 d\mathrm{vol}_g = \|dh_R\|_{L^2}^2$ is non increasing.

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Hence $R \mapsto \int_{B(o,R)} |dh_R|_g^2 d\mathrm{vol}_g = \|dh_R\|_{L^2}^2$ is non increasing. Hence we can find a sequence (h_{R_k}) which converge uniformly on compact set to a harmonic function h_∞ such that

$$-1 \leq h_\infty \leq 1.$$

$$\int_M |dh_\infty|_g^2 d\mathrm{vol}_g < \infty$$

II. Ends and harmonic function : First example : $\mathbb{R}$

In that cases , on $[-R, R]$

$$h_R(t) = \frac{t}{R}$$

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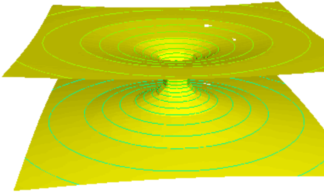
and $h_\infty = 0$

II. Ends and harmonic function : Second example : $\mathbb{R}^3 \# \mathbb{R}^3$

We consider two copies of the Euclidean space $\mathbb{R}^3$ glued together. This manifold is diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$ and is endowed with the metric

$$(dt)^2 + (1 + t^2)(d\sigma)^2$$



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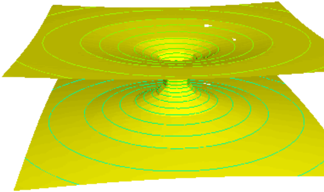

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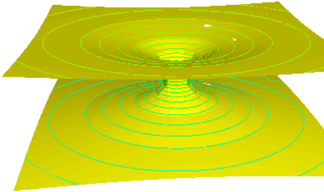


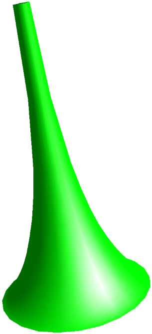
Then on $[-R, R] \times \mathbb{S}^2$, we have $h_R(t, \sigma) = \frac{\arctan(t)}{\arctan(R)}$. Hence

$$h_\infty(t, \sigma) = \frac{2 \arctan(t)}{\pi}.$$

II. Ends and harmonic function :Third example : $\mathbb{R} \times \mathbb{S}^1$

We consider $\mathbb{R} \times \mathbb{S}^1$ and with the metric

$$(dt)^2 + e^{2t}(d\sigma)^2$$



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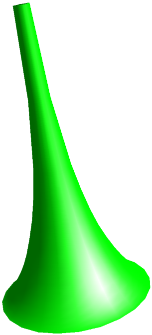

Then on $[-R, R] \times \mathbb{S}^1$, we have $h_R(t,\sigma) = \frac{\cosh(R) - e^{-t}}{\sinh(R)}$. Hence

$$h_\infty(t,\sigma) = 1.$$

II. Ends and harmonic function : the spectral gap condition

If we assume that the Laplace operator on (M,g) has a spectral gap : the best constant λ_0 in the inequality

$$\lambda_0 \int_M f^2 \leq \int_M |df|^2, \ \forall f \in C_0^\infty(M)$$

is positive : $\lambda_0 > 0$.

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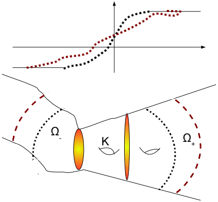
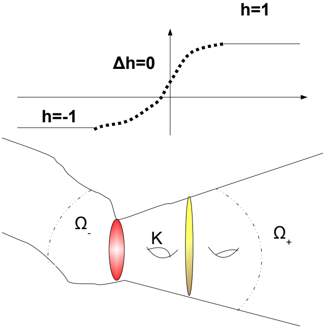
is positive : $\lambda_0 > 0$. Then we get : for $R > R_0$

$$\begin{aligned} \lambda_0 \int_M |h_R - h_{R_0}|^2 &\leq \int_M |d(h_R - h_{R_0})|^2 \\ &\leq 2 \int_M |dh_R|^2 + 2 \int_M |dh_{R_0}|^2 \\ &\leq 4 \int_M |dh_{R_0}|^2 \end{aligned}$$



II. Ends and harmonic function : the spectral gap condition

Letting R tends to ∞, we get that $h_\infty - h_{R_0} \in L^2$. Now because $h_{R_0} = \pm 1$ on $\Omega_\pm \setminus B(o, R_0)$. We can say that if both Ω_+ and Ω_- have infinite volume then h_∞ is non trivial.



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- In the third one :

$$\left(\mathbb{R} \times \mathbb{S}^1, (dt)^2 + e^{2t}(d\sigma)^2)\right)$$

we have a spectral gap but the end $(-\infty, 0) \times \mathbb{S}^1$ has finite volume

II. Ends and harmonic function : Sobolev inequality

If we replace the spectral gap inequality :

$$\lambda_0 \int_M f^2 \leq \int_M |df|^2, \ \forall f \in C_0^\infty(M)$$

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by a Sobolev inequality

$$\mu \left(\int_M f^{2\tau} \right)^{\frac{1}{\tau}} \leq \int_M |df|^2, \ \forall f \in C_0^\infty(M).$$

where $\tau \geq 1$. Then the same argument leads to the existence of a harmonic function associated to a pair of ends with infinite volume.

II. Ends and harmonic function : Sobolev inequality

The Sobolev inequality

$$\mu\left(\int_M f^{2\tau}\right)^{\frac{1}{\tau}} \leq \int_M |df|^2,\ \forall f \in C_0^\infty(M)$$

is true on

- $\mathbb{R}^{n>2}$ with $\tau = \frac{n}{n-2}$.
- our second example $\mathbb{R}^3 \# \mathbb{R}^3$ satisfies the Sobolev inequality with $\tau = 3$.

II. Ends and harmonic function : Sobolev inequality

But with the Sobolev inequality : $\mu\left(\int_M f^{2\tau}\right)^{\frac{1}{\tau}} \leq \int_M |df|^2, \ \forall f \in C_0^\infty(M)$, ends have infinite volume:

De Giogi-Nash-Moser iteration scheme:

If u is an harmonic function on a ball $B(x,r)$, $p \geq 2$ and $m := \frac{2\tau}{(\tau-1)}$ then

$$|u(x)| \leq \frac{C(\tau,p,\mu)}{r^{\frac{m}{p}}} \|u\|_{L^p(B(x,r))},$$

In particular with the function $u=1$, we get

$$\mathrm{vol} B(x,r) \geq C(\tau,\mu) r^m.$$

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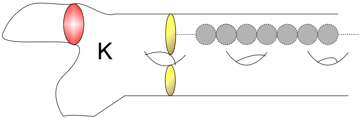
This implies that ends have infinite volume :



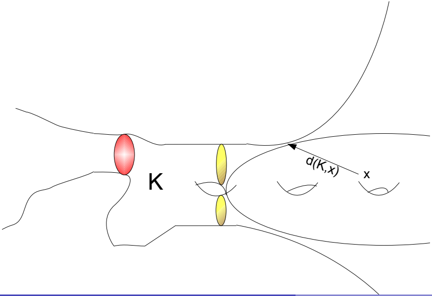
II. Ends and harmonic function : Sobolev inequality

Recall that when u is an harmonic function on a ball $B(x,r)$ and $p \geq 2$ then

$$|u(x)| \leq \frac{C(\tau, p\mu)}{r^{\frac{m}{p}}} \|u\|_{L^p(B(x,r))}, m := \frac{2\tau}{(\tau - 1)}$$

Then because the function $h_\infty - (\pm 1)$ is harmonic and $h_\infty - (\pm 1) \in L^{2\tau}(\Omega_\pm)$ we get

$$1 - (\pm h_\infty(x)) \leq \frac{C}{d(o,x)^{\frac{m-2}{2}}} \text{ on } \Omega_\pm$$



II. Ends and harmonic function : Conclusion

Theorem

Assume that (M, g) is a complete Riemannian manifold which satisfies

- for a $\mu > 0$ and $\tau \geq 1$,

$$\mu \left(\int_M f^{2\tau} \right)^{\frac{1}{\tau}} \leq \int_M |df|^2, \ \forall f \in C_0^\infty(M).$$

- There is a compact set $K \subset M$ such $M \setminus K = \Omega_- \cup \Omega_+$ with both $\Omega_\pm$ unbounded ; when $\tau = 1$ assume moreover that $\mathrm{vol}\Omega_\pm = \infty$.

Then there is a non trivial harmonic function h_∞ such that $dh_\infty \in L^2$ and when $\tau > 1$

$$\lim_{x \to \infty_{\Omega_\pm}} h_\infty(x) = \pm 1$$

III. Topology and harmonics forms

Definition

When (M, g) is a complete Riemannian manifold, a metric analogue of $H^1_c(M)$ is the degree 1 space of (reduced) L^2-cohomology :

$$\mathbb{H}^1(M) = \frac{\{\alpha \in L^2(T^*M), d\alpha = 0\}}{\overline{dC_0^\infty(M)}},$$

where the closure is for the L^2 topology.

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where the closure is for the L^2 topology.

This space has a interpretation in term of harmonic (i.e. closed and coclosed) forms

$$\mathbb{H}^1(M) \simeq \mathcal{H}^1(M) = \{\alpha \in L^2(T^*M), d\alpha = 0, d^*\alpha = 0\}.$$

Locally a harmonic 1−form is the differential of a harmonic function.

$$\Delta h = 0 \text{ and } dh \in L^2 \Rightarrow dh \in \mathcal{H}^1(M).$$

III. Topology and harmonics forms

There is a natural map :

$$H^1_c(M) \to \mathbb{H}^1(M)$$

Sending a closed smooth form with compact support to its orthogonal projection onto the space of harmonic forms. The injectivity of this map will imply the existence of harmonic function with L^2 gradient associated to pair of ends.

III. Topology and harmonics forms :$H^1_c(M)$ versus L^2 harmonic 1-forms

Proposition, C. and C.-Pedon, 2003

Assume that (M,g) is a complete Riemannian manifold which satisfies

- for a $\mu > 0$ and $\tau \geq 1$,

$$\mu \left(\int_M f^{2\tau} \right)^{\frac{1}{\tau}} \leq \int_M |df|^2, \ \forall f \in C_0^\infty(M).$$

- when $\tau = 1$ assume moreover that all ends of (M,g) has infinite volume.

Then the natural map :

$$H^1_c(M) \to \mathbb{H}^1(M)$$

is injective.

III. Topology and harmonics forms : proof of the proposition

If $\alpha \in C_0^\infty(T^*M)$ satisfies $\mathrm{d}\alpha = 0$ is mapped to zero in $\mathbb{H}^1(M) = \frac{\{\alpha \in L^2(T^*M), d\alpha = 0\}}{\overline{dC_0^\infty(M)}}$. Then by definition $\alpha \in \overline{dC_0^\infty(M)}$:

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$$\alpha = L^2 - \lim_{k\to\infty} df_k.$$

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Our hypothesis is the Sobolev 's inequality :

$$\mu \left(\int_M f^{2\tau} \right)^{\frac{1}{\tau}} \leq \int_M |df|^2, \ \forall f \in C_0^\infty(M).$$

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Hence, there is $f_\infty \in L^{2\tau}$ such that

$$L^{2\tau} - \lim_{k\to\infty} f_k = f_\infty \text{ so } \alpha = df_\infty.$$

Moreover, f_∞ is locally constant outside the support of α, our hypothesis implies that f_∞ has compact support.

The Bochner formula

Hence a vanishing result for the space of these harmonic functions or for the space of L^2 harmonic 1-forms implies that such a manifold has only one end.

The Bochner formula

Hence a vanishing result for the space of these harmonic functions or for the space of L^2 harmonic 1-forms implies that such a manifold has only one end.

A classical way to obtain such a vanishing result is from the Bochner formula : If h satisfies

$$\Delta h = 0 \text{ and } dh \in L^2$$

then $\alpha = dh$ satisfies

$$\int_M \langle \nabla^* \nabla \alpha + \mathrm{Ricci}\, \alpha, \alpha \rangle d\mathrm{vol}_g = \int_M |\nabla \alpha|^2 + \mathrm{Ricci}\,(\alpha, \alpha) d\mathrm{vol}_g = 0$$

The Bochner formula

Hence a vanishing result for the space of these harmonic functions or for the space of L^2 harmonic 1-forms implies that such a manifold has only one end.

A classical way to obtain such a vanishing result is from the Bochner formula : If h satisfies

$$\Delta h = 0 \text{ and } dh \in L^2$$

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Moreover, the Kato inequality implies that for $f := |\alpha|$ and with $\mathrm{Ricci}_-(x) :=$ the lowest eigenvalue of the Ricci tensor at x :

$$\int_M \left[f \Delta f + \mathrm{Ricci}_- f^2 \right] d\mathrm{vol}_g \leq \int_M |\nabla \alpha|^2 + \mathrm{Ricci}\,(\alpha, \alpha) d\mathrm{vol}_g = 0.$$

Hence if the operator $\Delta + \mathrm{Ricci}_-$ is non negative with no L^2-kernel, we have such a vanishing result.

IV. Minimal hypersurfaces

When $M^{n>2} \subset \mathbb{R}^{n+1}$ is a minimal hypersurface, then according to J.Michael & L.Simon the Sobolev inequality holds on M (with the induced metric).

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Recall that the Bochner formula implies that any L^2 harmonic 1-form α satisfies :

$$\int_M (\Delta|\alpha| - |A|^2|\alpha|)|\alpha| d\mathrm{vol}_g \leq \int_M |d|\alpha||^2 - |A|^2|\alpha|^2 + \frac{1}{n}|A|^2|\alpha|^2 d\mathrm{vol}_g \leq 0$$

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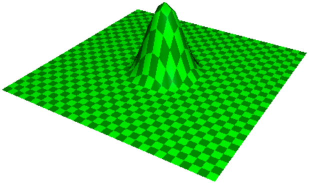
Recall that a minimal hypersurface is a critical point for the volume functional

$$\varphi \in C_0^\infty(M) \mapsto V(\varphi),$$

where $V(\varphi)$ is the volume of the normal graph of φ over M.



We have

$$V''(0)(\varphi,\varphi) = \int_M \left[|d\varphi|^2 - |A|^2\varphi^2\right] d\mathrm{vol}_g.$$

IV. Minimal hypersurfaces

From these ideas, we can obtain the following result

Theorem

- If the minimal hypersurface $M^n \subset \mathbb{R}^{n+1}$ is stable ($\Delta - |A|^2 \geq 0$), then M^n has only one ends (H.Cao &Y.Shen & S.Zhu, 1997) and $H^1_c(M) = \{0\}$.

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- If the minimal hypersurface $M^n \subset \mathbb{R}^{n+1}$ has finite index (the operator $\Delta - |A|^2$ has a finite number of negative eigenvalues), then M^n has a finite number of ends (P. Li & J. Wang, 2002) and $\dim H^1_c(M) < \infty$ (with the above argument and a result of S.Pigola & M.Rigoli & A.Setti, 2008).

IV. Minimal hypersurfaces

- On a minimal hypersurface with finite total scalar curvature :

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- There are many of such minimal hypersurfaces (S. Fakhi & F. Pacard, 2000)
- But stable minimal hypersurface with finite total scalar curvature are hyperplane (M. do Carmo & C-K. Peng (1980), P. Bérard ($n \leq 5$, 1991) and Y-B. Shen & X-H. Zhu (1997))

V. Hyperbolic manifolds

When $\Gamma\backslash\mathbb{H}^n$ is a hyperbolic manifold (with $\Gamma \subset SO(n,1)$ discrete, torsion free subgroup), the bottom of the spectrum of $\Gamma\backslash\mathbb{H}^n$ is related to a dynamical invariant of Γ : the critical exponent :

$$\delta(\Gamma) = \limsup_{R\to\infty} \frac{\mathrm{Card}\ (\Gamma.o \cap B(o,R))}{R}$$

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If we introduce the limit set of Γ to be the set

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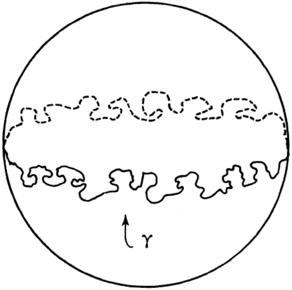
then the quotient

$$\Gamma \backslash \left(\mathbb{H}^n \cup (\mathbb{S}^\infty \setminus \Lambda(\Gamma))\right)$$

is a manifold with boundary.



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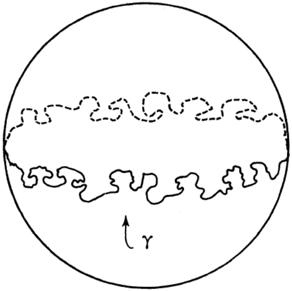

Theorem, (Sullivan)

When Γ is convex-cocompact, then

$$\delta(\Gamma) = \dim_{\mathrm{Hausdorff}} \Lambda(\Gamma).$$

V. Hyperbolic manifolds

Theorem (Sullivan)

If $\delta(\Gamma) < n-1$ then on $\Gamma\backslash\mathbb{H}^n$, we have the spectral gap estimate

$$\lambda_0 \int f^2 \leq \int |df|^2, \ \forall f \in C_0^\infty(\Gamma\backslash\mathbb{H}^n)$$

where

$$\lambda_0 = \begin{cases} \left(\frac{n-1}{2}\right)^2 & \text{if } \delta(\Gamma) \leq \frac{n-1}{2} \\ (n-1-\delta(\Gamma))\,\delta(\Gamma) & \text{if } \delta(\Gamma) \geq \frac{n-1}{2} \end{cases}$$

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According to our discussion, when $\delta(\Gamma) < n - 1$ and $\Gamma\backslash\mathbb{H}^n$ has two ends of infinite volume then we can find a non trivial harmonic function with L^2 gradient associate to this pair of ends.

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then all ends have infinite volume.

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Theorem (P. Li & J. Wang, 2001)

If $(M^{n>2}, g)$ is a complete Riemannian manifold with $\mathrm{Ricci} \geq -(n-1)g$ and $\lambda_0 \geq (n-2)$, then either M has only one end with infinite volume or (M^n, g) is isometric to the warped product

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Ideas of the proof when $n \geq 4$:

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- Then $g \in L^2$, hence $\Delta g = (n-2)g$.
- Analyse the equality case in $\Delta g \leq (n-2)g$ and deduce the geometry of M (in fact $g = \frac{C}{\cosh^{n-2}(t)}$).

V. Hyperbolic manifolds

A corollary for hyperbolic manifold :

Corollary

If $\Gamma \subset SO(n,1)$ discrete, torsion free subgroup, then

- If $\delta(\Gamma) < n-2$ then $\Gamma\backslash\mathbb{H}^n$ has only one end with infinite volume.
- If $\delta(\Gamma) = n-2$ and $\Gamma\backslash\mathbb{H}^n$ has at least two ends with infinite volume, then there is a totally geodesic hypersurface in $\mathbb{H}^n$, that is a copy of $\mathbb{H}^{n-1} \subset \mathbb{H}^n$ stable by Γ such that $\Gamma\backslash\mathbb{H}^{n-1}$ is compact.

This nice result has a certain familiarity with other results.

V. Hyperbolic manifolds

- (R. Bowen,1979) Assume that $\Gamma_0 \subset SO(2,1)$ is a cocompact group and that that $\Gamma \subset SO(3,1)$ is convex cocompact and isomorphic to Γ_0, then

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- (P.Pansu, M.Bourdon, C-B.Yue, H.Izeki, G.Besson & G.Courtois & S.Gallot [1996-1998]) Assume that $\Gamma_0 \subset SO(n-1,1)$ is a cocompact group and that that $\Gamma \subset SO(n,1)$ is convex cocompact and isomorphic to Γ_0, then
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The above rigidity result is more general :

- If $\Gamma_0 \subset SO(n-k,1)$ is a cocompact group isomorphic to a convex cocompact group $\Gamma \subset SO(n,1)$, then
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- (H. Izeki 1995, C. & E.Pedon, 2003) If $\Gamma \subset SO(n,1)$ satisfies $\delta(\Gamma) < n-1-k$, then

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- (X. Wang, 2002) If $\Gamma \subset SO(n,1)$ is convex-cocompact and satisfies $\frac{n-1}{2} < \delta(\Gamma) \leq n-1-k$ and
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