BIRKHOFF NORMAL FORM FOR PDEs WITH TAME MODULUS

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Abstract

We prove an abstract Birkhoff normal form theorem for Hamiltonian Partial Differential Equations. The theorem applies to semilinear equations with nonlinearity satisfying a property that we call of Tame Modulus. Such a property is related to the classical tame inequality by Moser. In the nonresonant case we deduce that any small amplitude solution remains very close to a torus for very long times. We also develop a general scheme to apply the abstract theory to PDEs in one space dimensions and we use it to study some concrete equations (NLW,NLS) with different boundary conditions. An application to a nonlinear Schrödinger equation on the d-dimensional torus is also given. In all cases we deduce bounds on the growth of high Sobolev norms. In particular we get lower bounds on the existence time of solutions.

Contents

1 Introduction

During the last fifteen years remarkable results have been obtained in perturbation theory of integrable partial differential equation. In particular the existence of quasiperiodic solutions has been proved through suitable extensions of KAM theory (see [Kuk87, Kuk93, Kuk00, Way90, Cra00, KP03, CW93, Bou03]). However very little is known on the behavior of the solutions lying outside KAM tori. In the finite dimensional case a description of such solutions is provided by Nekhoroshev's theorem whose extension to PDEs is at present a completely open problem. In the particular case of a neighbourhood of an elliptic equilibrium point, the Birkhoff normal form theorem provides a quite precise description of the dynamics. In the present paper we give an extension of the Birkhoff normal form theorem to the infinite dimensional case and we apply it to some semilinear PDEs in one or more space dimensions.

To start with consider a finite dimensional Hamiltonian system $H = H_0 + P$ with quadratic part

$$H_0 = \sum_{j=1}^{n} \omega_j \frac{p_j^2 + q_j^2}{2} \ , \quad \omega_j \in \mathbb{R}$$

and P a smooth function having a zero of order at least three at the origin. The classical Birkhoff normal form theorem states that, for each $r \geq 1$, there exits a real analytic symplectic transformation $\mathcal{T}_r$ such that

$$H \circ \mathcal{T}_r = H_0 + \mathcal{Z} + \mathcal{R}_r, \tag{1.1}$$

where the remainder $\mathcal{R}_r$ has a zero of order $r+3$ and $\mathcal{Z}$ is a polynomial of degree $r+2$. Provided the frequencies are nonresonant, $\mathcal{Z}$ depends on the actions $I_j = (p_j^2 + q_j^2)/2$ only, and one says that the Hamiltonian (1.1) is in integrable Birkhoff normal form up to order $r+3$. As a dynamical consequence, solutions with initial data of size $\varepsilon \ll 1$ remain at distance 2ε from the origin for times of order $\varepsilon^{-(r+1)}$, and the actions remain almost constant during the same lapse of time. Moreover any solution remains ε^{r_1} close to a torus of maximal dimension up to times of order ε^{-r_2}, with $r_1 + r_2 = r + 2$.

The proof of such a standard theorem is obtained by applying a sequence of canonical transformations that eliminate order by order the non normalized part of the nonlinearity. Remark that, since the space of homogeneous polynomials of any finite order is finite dimensional, only a finite number of monomials have to be removed. Of course this is no more true in the infinite dimensional case.

To generalize Birkhoff normal form theory to infinite dimensional systems, the main difficulty consists in finding a nonresonance property that is satisfied in quite general situations and that allows to remove from the nonlinearity all the relevant non-normalized monomials. This problem has been solved by the first author in a particular case, namely the nonlinear wave equation (see [Bam03b]), by remarking that most of the monomials are "not relevant", since their vector field is already small. Moreover all the remaining monomials can be eliminated using a suitable nonresonance condition.

In the present paper we generalize the procedure of [Bam03b] to obtain an abstract result that can be applied to a wide class of PDEs. The idea is again that a large part of the nonlinearity is 'not relevant', but we show here that this is due to the so called tame inequality, namely

$$\|uv\|_{H^s} \le C_s \left(\|u\|_{H^s} \|v\|_{H^1} + \|v\|_{H^s} \|u\|_{H^1} \right) \ , \tag{1.2}$$

on which we base our construction. The point is that, if $u = \sum_{|j|\ge N} u_j e^{\mathrm{i}jx}$, then, by (1.2) one has

$$\left\|u^2\right\|_{H^s} \le \frac{\|u\|_{H^s}^2}{N^{s-1}}$$

which is small provided N and/or s are large. This remark allows to forget a large part of the nonlinearity. However in order to exploit (1.2) one has to show that the tame property is not lost along the iterative normalization procedure. We prove that this is the case for a class of nonlinearities that we call of *tame modulus* (see definition 2.6).

In our abstract theorem we show that, if the nonlinearity has tame modulus, then it is always possible to put H in the form (1.1), where the remainder term R_r is of "order" $r + 5/2$ and $\mathcal{Z}$ is a polynomial of degree $r + 2$ containing only monomials which are "almost resonant" (see definition 2.12). If the frequencies fulfill a nonresonance condition similar to the second Melnikov condition, then H can be put in integrable Birkhoff normal form, i.e. $\mathcal{Z}$ depends on the actions only. We will also show how to deduce some informations on the dynamics when some resonances or almost resonances are present (see sections 3.2.2, 3.5 below).

When applying this abstract Birkhoff normal form result to PDEs, one meets two difficulties: (1) to verify the tame modulus condition; (2) to study the structure of $\mathcal{Z}$ in order to extract dynamical information (e.g. by showing that it depends on the actions only).

Concerning (1), in the context of PDEs with Dirichlet or periodic boundary conditions, we obtain a simple condition ensuring tame modulus. Thus we obtain that typically all local functions and also convolution type nonlinearities have the tame modulus property.

The structure of $\mathcal{Z}$ depends on the resonance relations among the frequencies. Their study is quite difficult since, at variance with respect to the case of KAM theory for PDEs, infinitely many frequencies are actually involved. Here we prove that in the case of NLW and NLS in one space dimensions with Dirichlet boundary conditions $\mathcal{Z}$ is typically integrable. Thus all the small initial data give rise to solutions that remain small and close to a torus for times of order ε^{-r}, where ε measures the size of the initial datum and r is an arbitrary integer (see corollary 2.16 for precise statements).

In the case of periodic boundary conditions the nonresonance condition is always violated. This is due to the fact that the periodic eigenvalues of a Sturm Liouville operator are asymptotically double. Nevertheless we prove that, roughly speaking, the energy transfers are allowed only between pairs of modes corresponding to almost double eigenvalues (see theorem 3.16 for a precise statement in the case of periodic NLW equation). In particular we deduce an estimate of higher Sobolev norms as in the Dirichlet case.

We also study a system of coupled NLS's with indefinite energy, getting the same result as in the case of the wave equation with periodic boundary conditions. In particular we thus obtain long time existence of solutions somehow in the spirit of the almost global existence of [Kla83].

Finally we apply our result to NLS in higher dimension. However, in order to be able to control the frequencies, we have to assume that the linear part is very simple from a spectral point of view. Namely the linear operator we consider reads $-\Delta u + V * u$, involving a convolution operator instead of the more natural product operator. In this context we again obtain long time existence and bounds on the energy exchanges among the modes (see theorem 3.26).

The first normal form result for Hamiltonian PDEs has been obtained by Kuksin and Pöschel in [KP96]. The normal form of [KP96] (which holds for nonresonant systems) is obtained by fixing an integer n and then transforming H to

$$H_0 + \mathcal{Z}_{KP} + \mathcal{R}$$

where $\mathcal{R}$ is a remainder, and $\mathcal{Z}_{KP}$ depends on the action $I_1, ..., I_n$, but, concerning the variables with index larger than n it is only known that it has a zero of second order. Thus such a normal form only shows that the approximate system in which the remainder is neglected admits finite dimensional invariant tori (of dimension n) which are linearly stable. On the contrary, in the nonresonant case, our normal form depends only on the actions *in all the variables* and therefore it gives informations on the dynamics corresponding to all initial data.

In [KP96] Bourgain added the idea that any initial datum which has a small H^s norm is close, in $H^{s/2}$ norm, to a linearly stable finite dimensional torus. Thus, in [Bou96] the author got a long time existence result and a qualitative description of the solution (see also [Bam99b]). Along the same line of research one finds the paper [Bou04] where the author present a result of almost global existence for the nonlinear wave and Schrödinger equations. We remark that, due to the fact that our normal form has the same properties of the standard finite dimensional Birkhoff normal form we get here a more precise result. Moreover, we recall that the results of [KP96, Bou96] hold for the case of Dirichlet boundary conditions in one space dimensions, while our techinique also allows to deal with the case of periodic boundary conditions and with some higher dimensional cases. This seems difficult with the techniques of [KP96, Bou96].

Other related results were obtained by Bourgain [Bou00] who showed that, in perturbations of the integrable NLS, one has that most small amplitude solutions remain close to tori for long times. Such a result is proved by exploiting the idea that most monomials in the nonlinearity are already small (as in the present paper and in [Bam03b]). However the technique of [Bou00] seems to be quite strongly related to the particular problem dealt with in that paper. We also mention the papers [DS04b, DS04a] by Delort and Szeftel where one step of normal form is performed for the Klein Gordon equation on $\mathbb{T}^d$ and on Zoll

Manifolds, models to which our general theorem does not apply.

Finally, other Birkhoff normal form results were presented in [Bam03a, Bam03c]. Those results apply to much more general equations (quasilinear equations in higher space dimensions) but the kind of normal form obtained in those papers only allows to describe the dynamics up to times of order ϵ^{-1}. This is due to the fact that in the normal form of [Bam03a, Bam03c] the remainder $\mathcal{R}_r$ has a vector field which is unbounded and greatly reduces regularity. On the contrary in the present paper the result obtained allows to control the dynamics for much longer times.

We point out that the study of the structure of the resonances that we do in the paper is based on an accurate study of the eigenvalues of Sturm–Liouville operators. A typical difficulty met here is related to the fact that the eigenvalues of Sturm–Liouville operators have very rigid asymptotic properties. In order to prove our nonresonance type conditions we use ideas from degenerate KAM theory (as in [Bam03a, Bam03b]) and ideas from the paper [Bou96].

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2 Statement of the abstract result

We will study a Hamiltonian system of the form

$$H(p,q) := H_0(p,q) + P(p,q) \ , \tag{2.1}$$

$$H_0 := \sum_{l\geq 1} \omega_l \frac{(p_l^2+q_l^2)}{2}, \tag{2.2}$$

where the real numbers ω_l play the role of frequencies and P has a zero of order at least three at the origin. The formal Hamiltonian vector field of the system is $X_H := (-\frac{\partial H}{\partial q_k}, \frac{\partial H}{\partial p_k})$. Define the Hilbert space $\ell^2_s(\mathbb{R})$ of the sequences $x \equiv \{x_l\}_{l\geq 1}$ with $x_l \in \mathbb{R}$ such that

$$\|x\|_s^2 := \sum_{l\geq 1} l^{2s}|x_l|^2 < \infty \tag{2.3}$$

and the scale of phase spaces $\mathcal{P}_s(\mathbb{R}) := \ell^2_s(\mathbb{R}) \oplus \ell^2_s(\mathbb{R}) \ni (p,q)$. We endow $\mathcal{P}_s(\mathbb{R})$ with the standart symplectic structure $\sum_{j\geq 1} dq_j \wedge dp_j$ and we assume that P is of class C^∞ from $\mathcal{P}_s(\mathbb{R})$ into $\mathbb{R}$ for any s large enough. In this context, the Hamiltonian equations associated to H read

$$\begin{cases} \dot{q}_j = \omega_j p_j + \frac{\partial P}{\partial p_j}, & j \geq 1 \\ \dot{p}_j = -\omega_j q_j - \frac{\partial P}{\partial q_j}, & j \geq 1 \end{cases}$$

We will denote by $z \equiv (z_l)_{l\in\bar{\mathbb{Z}}}$, $\bar{\mathbb{Z}} := \mathbb{Z} - \{0\}$ the set of all the variables, where

$$z_{-l} := p_l \ , \quad z_l := q_l \quad l \geq 1 \ .$$

and by $B_{\mathbb{R},s}(R)$ the open ball centered at the origin and of radius R in $\mathcal{P}_s(\mathbb{R})$. Often we will simply write

$$\mathcal{P}_s \equiv \mathcal{P}_s(\mathbb{R}) \ , \quad B_s(R) \equiv B_{\mathbb{R},s}(R) \ .$$

Remark 2.1. In section 3, depending on the concrete example we will consider, the index l will run in $\mathbb{N}$, $\mathbb{Z}$ or even $\mathbb{Z}^d$. In this abstract section we will consider indexes in $\mathbb{N}$. Clearly one can always reduce to this case relabelling the indexes.

2.1 Tame maps

Let $f : \mathcal{P}_s \to \mathbb{R}$ be a homogeneous polynomial of degree r; we recall that f is continuous and also analytic if and only if it is bounded, namely if there exists C such that

$$|f(z)| \leq C \|z\|_s^r \ , \quad \forall z \in \mathcal{P}_s \ .$$

To the polynomial f it is naturally associated a symmetric r-linear form $\tilde{f}$ such that

$$\tilde{f}(z, ..., z) = f(z) \ . \tag{2.4}$$

More explicitly, write

$$f(z) = \sum_{|j|=r} f_j z^j \ , \quad j = (..., j_{-l}, ..., j_{-1}, j_1, ..., j_l, ...) \ , \tag{2.5}$$

$$z^j := ...z_{-l}^{j_{-l}} ... z_{-1}^{j_{-1}} z_1^{j_1} ... z_l^{j_l} ... \ , \quad |j| := \sum_l |j_l| \ , \tag{2.6}$$

then

$$\tilde{f}(z^{(1)}, ..., z^{(r)}) = \sum_{|j|=r} f_{j_1,...,j_r} z_{j_1}^{(1)} ... z_{j_r}^{(r)} \tag{2.7}$$

The multilinear form $\tilde{f}$ is bounded i.e.

$$|\tilde{f}(z^{(1)}, ..., z^{(r)})| \leq C \left\| z^{(1)} \right\|_s ... \left\| z^{(r)} \right\|_s$$

(and analytic) if and only if f is bounded.

Given a polynomial vector field $X : \mathcal{P}_s \to \mathcal{P}_s$ homogeneous of degree r we write it as

$$X(z) = \sum_{l \in \bar{\mathbb{Z}}} X_l(z) \mathbf{e}_l$$

where $\mathbf{e}_l \in \mathcal{P}_s$ is the vector with all components equal to zero but the $l-$th one which is equal to 1. Thus $X_l(z)$ is a real valued homogeneous polynomial of degree r. Consider the r-linear symmetric form $\tilde{X}_l$ and define $\tilde{X} := \sum_l \tilde{X}_l \mathbf{e}_l$, so that

$$\tilde{X}(z, ..., z) = X(z) \ . \tag{2.8}$$

Again $X : \mathcal{P}_s \to \mathcal{P}_s$ is analytic if and only if it is bounded. Then the same is true for $\tilde{X}$.

Definition 2.2. Let $X : \mathcal{P}_s \to \mathcal{P}_s$ be a homogeneous polynomial of degree r; let $s \geq 1$, then we say that X is *an s–tame map* if there exists a constant C_s such that

$$\left\|\tilde{X}(z^{(1)},...,z^{(r)})\right\|_s \leq C_s \sum_{l=1}^{r} \left\|z^{(1)}\right\|_1 .... \left\|z^{(l-1)}\right\|_1 \left\|z^{(l)}\right\|_s \left\|z^{(l+1)}\right\|_1 ... \left\|z^{(r)}\right\|_1$$
$$\forall z^{(1)},...,z^{(r)} \in \mathcal{P}_s \tag{2.9}$$

If a map is s–tame for any $s \geq 1$ then it will be said to be tame. □

Example 2.3. Given a sequence $z \equiv (z_l)_{l\in\mathbb{Z}}$ consider the map

$$(z_l) \mapsto (w_k) \ , \quad w_k := \sum_l z_{k-l} z_l \tag{2.10}$$

this is a tame s map for any $s \geq 1$. To see this define the map[1]

$$\mathcal{P}_s \ni \{z_l\} \mapsto u(x) := \sum_l z_l e^{\mathrm{i}lx} \in H^s(\mathbb{T}) \tag{2.11}$$

where the Sobolev space $H^s(\mathbb{T})$ is the space of periodic functions of period 2π having s weak derivatives of class L^2. Then the map (2.10) is transformed into the map

$$u \mapsto u^2$$

The corresponding bilinear form is $(u, v) \mapsto uv$ and, by Moser inequality, one has

$$\|uv\|_{H^s} \leq C_s \left(\|u\|_{H^s} \|v\|_{H^1} + \|v\|_{H^s} \|u\|_{H^1}\right) \tag{2.12}$$

□

Example 2.4. According to definition 2.2, any bounded linear map $A : \mathcal{P}_s \to \mathcal{P}_s$ is s-tame. □

2.2 The modulus of a map

Definition 2.5. Let f be a homogeneous polynomial of degree r. Following Nikolenko [Nik86] we define its *modulus* $\lfloor f \rceil$ by

$$\lfloor f \rceil (z) := \sum_{|j|=r} |f_j| z^j \tag{2.13}$$

where f_j is defined by (2.5). We remark that in general the modulus of a bounded analytic polynomial can be an unbounded densely defined polynomial.

Analogously the modulus of a vector field X is defined by

$$\lfloor X \rceil (z) := \sum_{l\in\bar{\mathbb{Z}}} \lfloor X_l \rceil (z) \mathbf{e}_l \ . \tag{2.14}$$

with X_l the l-th component of X. □

[1] In fact, for $z \in \mathcal{P}_s$ the index l runs in $\mathbb{Z} - \{0\}$, but one can reorder the indexes so that they run in $\mathbb{Z}$.

Definition 2.6. A homogeneous polynomial vector field X is said to have s–*tame modulus* if its modulus $\lfloor X \rceil$ is an s–tame map. The set of polynomial functions f whose Hamiltonian vector field has s–tame modulus will be denoted by T^s_M, and we will write $f \in T^s_M$. If $f \in T^s_M$ for any $s > 1$ we will write $f \in T_M$ and say that it has tame modulus. □

Remark 2.7. The property of having tame modulus depends on the coordinate system. □

Remark 2.8. Consider a Hamiltonian function f then it is easy to see that its vector field X_f has tame modulus if and only if the Hamiltonian vector field of $\lfloor f \rceil$ is tame. □

Example 2.9. Consider again the map of example 2.3: In this case the map $X_k(z)$ is simply given by (2.10) and can be written in the form

$$X_k(z) = \sum_{j_1,j_2} \delta_{j_2,k-j_1} z_{j_1} z_{j_2}$$

so that in this case $\lfloor X \rceil = X$, and therefore this map has tame modulus. □

Example 2.10. Let $(W_k)_{k\in\bar{\mathbb{Z}}}$ be a given bounded sequence, and consider the map $z \mapsto X(z) := (W_k z_k)_{k\in\bar{\mathbb{Z}}}$. Its modulus is $\lfloor X \rceil (z) := \{|W_k| z_k\}_{k\in\bar{\mathbb{Z}}}$, which is linear and bounded as a map from $\mathcal{P}_s$ to $\mathcal{P}_s$. Therefore, according to example 2.4 it has tame modulus. □

2.3 The theorem

To define what we mean by normal form we introduce the complex variables

$$\xi_l := \frac{1}{\sqrt{2}}(p_l + \mathrm{i}q_l) \ ; \ \eta_l := \frac{1}{\sqrt{2}}(p_l - \mathrm{i}q_l) \quad l \geq 1 \ , \tag{2.15}$$

in which the symplectic form takes the form $\sum_l \mathrm{i}\ d\xi_l \wedge d\eta_l$.

Remark 2.11. In these complex variables the actions are given by

$$I_j = \xi_j \eta_j \ .$$

□

Consider a polynomial function $\mathcal{Z}$ and write it in the form

$$\mathcal{Z}(\xi,\eta) = \sum_{k,l\in\mathbb{N}^{\mathbb{N}}} \mathcal{Z}_{kl} \eta^k \xi^l \ . \tag{2.16}$$

Definition 2.12. Fix two positive parameters γ and α, and a positive integer N. A function $\mathcal{Z}$ of the form (2.16) will be said to be in (γ, α, N)–normal form with respect to ω if $\mathcal{Z}_{kl} \neq 0$ implies

$$|\omega \cdot (k-l)| < \frac{\gamma}{N^\alpha} \qquad \text{and} \qquad \sum_{j\geq N+1} k_j + l_j \leq 2 \ . \tag{2.17}$$

□

In other words, a monomial $\mathcal{Z}_{kl}\eta^k\xi^l$ is in normal form if, it contains at most three modes of large indexes (i.e. larger than $N+1$), and it is almost resonant, i.e. its associated small divisor are very small or equivalently its Poisson bracket with H_0 is very small.

Consider the formal Taylor expansion of $P = P(p,q)$, namely

$$P = P_3 + P_4 + ...$$

with P_j homogeneous of degree j. We assume

(H) For any $s \geq 1$ the vector field X_P is of class C^∞ from a neighbourhood of the origin in $\mathcal{P}_s(\mathbb{R})$ to $\mathcal{P}_s(\mathbb{R})$. Moreover, for any $j \geq 3$ one has $P_j \in T_M$, namely its Hamiltonian vector field has tame modulus. □

Given three numbers $R > 0$, $r \geq 1$ and $\alpha \in \mathbb{R}$ define the functions

$$N_*(r,\alpha,R) := R^{-\frac{1}{2(r+2)\alpha}} \ , \quad s_*(r,\alpha) := 2\alpha r(r+2) + 2 \ . \tag{2.18}$$

Theorem 2.13. *Let H be the Hamiltonian given by* (2.1)*,* (2.2)*, with P satisfying* (H)*. Fix positive γ, and α. Then for any $r \geq 1$, and any $s \geq s_* = s_*(r,\alpha)$ there exists a positive number R_s with the following properties: for any $R < R_s$ there exists an analytic canonical transformation $\mathcal{T}_R : B_s(R/3) \to B_s(R)$ which puts the Hamiltonian in the form*

$$(H_0 + P) \circ \mathcal{T}_R = H_0 + \mathcal{Z} + \mathcal{R} \tag{2.19}$$

where $\mathcal{Z} \in T_M$ is a polynomial of degree at most $r+2$ which is in (γ,α,N_*)–normal form with respect to ω, and $\mathcal{R} \in C^\infty(B_{\mathbb{R},s}(R/3))$ is small, precisely it fulfills the estimate

$$\sup_{\|(p,q)\|_s \leq R/3} \|X_{\mathcal{R}}(p,q)\|_s \leq C_s R^{r+\frac{3}{2}} \ . \tag{2.20}$$

Finally the canonical transformation fulfills the estimate

$$\sup_{\|z\|_s \leq R/3} \|z - \mathcal{T}_R(z)\|_s \leq C_s R^2 \ . \tag{2.21}$$

Exactly the same estimate is fulfilled by the inverse canonical transformation. The constant C_s does not depend on R.

The proof of this abstract theorem is postponed to section 4.

Assuming that the frequencies are nonresonant one can easily get dynamical informations. Precisely, let r be a positive integer, assume

(r-NR) There exist $\gamma > 0$, and $\alpha \in \mathbb{R}$ such that for any N large enough one has

$$\left| \sum_{j\geq 1} \omega_j k_j \right| \geq \frac{\gamma}{N^\alpha} \ , \tag{2.22}$$

for any $k \in \mathbb{Z}^\infty$, fulfilling $0 \neq |k| := \sum_j |k_j| \leq r+2$, $\sum_{j>N} |k_j| \leq 2$. □

Remark 2.14. If the frequency vector satisfies assumption (r-NR) then any polynomial of degree $r+2$ which is in (γ, N, α)-normal form with respect to ω depends only on the actions $\xi_j\eta_j = \frac{1}{2}(p_j^2 + q_j^2)$, $j \geq 1$. $\square$

We thus have the following

Corollary 2.15. *Fix r, assume (H,r-NR), and consider the system $H_0 + P$, then the normal form $\mathcal{Z}$ depends on the actions only.*

In this case one has

Corollary 2.16. *Fix r, assume (H,r-NR), and consider the system $H_0 + P$. For any $s \geq s_* = s_*(r, \alpha)$ (where α is defined in (2.22)) there exist ϵ_s and c such that if the initial datum belongs to $\mathcal{P}_s$ and fulfills*

$$\epsilon := \|z(0)\|_s < \epsilon_s \tag{2.23}$$

one has

(i) $\|z(t)\|_s \leq 2\epsilon$, *for* $|t| \leq \dfrac{c}{\epsilon^{r+1/2}}$

(ii) $|I_j(t) - I_j(0)| \leq \dfrac{\epsilon^3}{j^{2s}}$, *for* $|t| \leq \dfrac{c}{\epsilon^{r+1/2}}$

iii) There exists a smooth torus $\mathbb{T}_0$ with the following properties: For any $s_1 < s - 1/2$ there exists C_{s_1} such that

$$d_{s_1}(z(t), \mathbb{T}_0) \leq C_{s_1} \epsilon^{\frac{r_1}{2}+1} \ , \quad \text{for} \quad |t| \leq \frac{1}{\epsilon^{r - r_1 + \frac{1}{2}}} \tag{2.24}$$

where $r_1 \leq r$ and $d_{s_1}(.,.)$ is the distance in $\mathcal{P}_{s_1}$.

Proof. Define $R := 8\epsilon$ and use theorem 2.13 to construct the normalizing canonical transformation $z = \mathcal{T}_R(z')$. Denote by I'_j the actions expressed in the variables z'. Define the function $\mathcal{N}(z') := \|z'\|_s^2 \equiv \sum_j j^{2s} I'_j$. By (2.21) one has $\mathcal{N}(z'(0)) \leq R^2/62$ (provided R, i.e. ϵ is small enough). One has

$$\frac{d\mathcal{N}}{dt}(z') = \{\mathcal{R}; \mathcal{N}\}(z')$$

and therefore, as far as $\mathcal{N}(z'(t)) < R^2/9$, one has

$$\left|\frac{d\mathcal{N}}{dt}(z')\right| \leq C R^{r+5/2} = C' \epsilon^{r+5/2} \ . \tag{2.25}$$

Denote by T_f the escape time of z' from $B_s(R/3)$. Remark that for all times smaller than T_f, (2.25) holds. So one has

$$\frac{R^2}{9} = \mathcal{N}(z'(T_f)) \leq \mathcal{N}(z'(0)) + C' \epsilon^{r+5/2} T_f$$

which (provided the constants are chosen suitably) shows that $T_f > C\epsilon^{-(r+1/2)}$. Going back to the original variables one gets the estimate (i). To come to the estimate (ii) just remark that

$$|I_j(t) - I_j(0)| \leq |I_j(t) - I'_j(t)| + |I'_j(t) - I'_j(0)| + |I'_j(0) - I_j(0)|$$

and that $j^{2s} I_j$ is a smooth function on $\mathcal{P}_s$ and therefore, using (2.21) together with (i), it is easy to estimate the first and the last terms at r.h.s. The middle term is estimated by computing the time derivative of $j^{2s} I'_j$ with the Hamiltonian and remarking that it is of order $\epsilon^{r+5/2}$.

Denote by $\bar{I}_j := I_j(0)$ the initial actions, in the normalized coordinates. Up to the considered times

$$\left|I_j(t) - \bar{I}_j\right| \leq \frac{C\epsilon^{2r_1}}{j^{2s}} \ . \tag{2.26}$$

Define the torus

$$\mathbb{T}_0 := \left\{z \in \mathcal{P}_s \ : \ I_j(z) = \bar{I}_j \ , j \geq 1\right\}$$

One has

$$d_{s_1}(z(t), \mathbb{T}_0) \leq \left[\sum_j j^{2s_1} \left|\sqrt{I_j(t)} - \sqrt{\bar{I}_j}\right|^2\right]^{1/2} \tag{2.27}$$

Notice that for $a, b \geq 0$ one has,

$$\left|\sqrt{a} - \sqrt{b}\right| \leq \sqrt{|a-b|} \ .$$

Thus, using (2.26), one has that

$$[d_{s_1}(z(t), \mathbb{T}_0)]^2 \leq \sum_j \frac{j^{2s}|I_j(t) - \bar{I}_j|}{j^{2(s-s_1)}} \leq \sup j^{2s}|I_j(t) - \bar{I}_j| \sum \frac{1}{j^{2(s-s_1)}}$$

which is convergent provided $s_1 < s - 1/2$ and gives iii). □

Remark 2.17. It can be shown that, if the vector field of the nonlinearity has tame modulus when considered as a map from $\mathcal{P}_s$ to $\mathcal{P}_{s+\tau}$ with some positive τ (as in the case of the nonlinear wave equation), then both the vector field of $\mathcal{Z}$ and of $\mathcal{R}$ are regularizing, in the sense that they map $\mathcal{P}_s$ into $\mathcal{P}_{s+\tau}$. Moreover the estimates (2.20), (2.21), are substituted by

$$\sup_{\|(p,q)\|_s \leq R/3} \|X_{\mathcal{R}}(p,q)\|_{s+\tau} \leq C_s R^{r+\frac{3}{2}} \ . \tag{2.28}$$

$$\sup_{\|z\|_s \leq R/3} \|z - \mathcal{T}_R(z)\|_{s+\tau} \leq C_s R^2 \ , \tag{2.29}$$

and the estimate (2.24) holds with $s_1 < s + (\tau - 1)/2$. □

3 Applications

3.1 An abstract model of Hamiltonian PDE

In this section we present a general class of Hamiltonian PDEs to which theorem 2.13 applies. We focus only on the one dimensional case but the discussion of the tame modulus property can be easily generalized to higher dimension.

Denote by $\mathbb{T}$ the torus, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and consider the space $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ endowed by the symplectic form

$$\Omega\left((p_1, q_1), (p_2, q_2)\right) := \langle J(p_1, q_1); (p_2, q_2)\rangle_{L^2\times L^2}$$

where

$$J\,(p, q) = (-q, p)$$

With this symplectic structure the Hamilton equations associated to a Hamiltonian function, $H : L^2 \times L^2 \supset D(H) \to \mathbb{R}$, read

$$\begin{cases} \dot{p} = -\nabla_q H \\ \dot{q} = \nabla_p H \end{cases}$$

with ∇_q and ∇_p denoting the L^2 gradient with respect to the q and the p variables respectively.

Let A be a self-adjoint operator on $L^2(\mathbb{T})$ with pure point spectrum $(\omega_j)_{j\in\mathbb{Z}}$. Denote by φ_j, $j \in \mathbb{Z}$, the associated eigenfunctions, i.e.

$$A\varphi_j = \omega_j \varphi_j \ .$$

The sequence $(\varphi_j)_{j\in\mathbb{Z}}$ defines a Hilbert basis of $L^2(\mathbb{T})$.

We use this operator to define the quadratic part of the Hamiltonian. Precisely we put

$$H_0 := \frac{1}{2}\left(\langle Ap, p\rangle_{L^2} + \langle Aq, q\rangle_{L^2}\right) \tag{3.1}$$

$$= \sum_{j\in\mathbb{Z}} \omega_j \frac{p_j^2 + q_j^2}{2} \tag{3.2}$$

where q_j is the component of q on φ_j and similarly for p_j. Remark that here the indexes run in $\mathbb{Z}$, so that the set $\bar{\mathbb{Z}}$ has to be substituted by the disjoint union of two copies of $\mathbb{Z}$.

Concerning the normal modes φ_j of the quadratic part, we assume they are well localized with respect to the exponentials: Consider the Fourier expansion of φ_j,

$$\varphi_j(x) = \sum_{k\in\mathbb{Z}} \varphi_j^k e^{\mathrm{i}kx} \ ,$$

we assume

(S1) For any $n > 0$ there exists a constant C_n such that for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$

$$\left|\varphi_j^k\right| \le \max_{\pm} \frac{C_n}{(1+|k\pm j|)^n} \ . \tag{3.3}$$

□

Example 3.1. If $A = -\partial_{xx}$, then $\varphi_j(x) = \sin jx$ for $j > 0$ and $\varphi_j(x) = \cos(-jx)$ for $j \le 0$, are well localized with respect to the exponentials. □

Example 3.2. Let $A = -\partial_{xx} + V$, where V is a C^∞, 2π periodic potential. Then $\varphi_j(x)$ are the eigenfunctions of a Sturm Liouville operator. By the theory of Sturm Liouville operators (cf [Mar86, PT87]) it is well localized with respect to the exponentials (cf [CW93, KM01]). Moreover, if V is even then one can order the eigenfunctions in such a way that φ_j is odd for strictly positive j and even for negative j. □

Remark 3.3. Under assumption (S1) the space $H^s(\mathbb{T})$ coincides with the space of the functions $q = \sum_j q_j \varphi_j$ with $(q_j) \in \ell^2_s$. □

On the symplectic space $\mathcal{B}_s := H^s(\mathbb{T}) \times H^s(\mathbb{T})$ consider the Hamiltonian system, $H = H_0 + P$ with H_0 defined by (3.1) and P a C^∞ function which has a zero of order at least three at the origin.

While it is quite hard to verify the tame modulus property when the basis φ_j is general, it turns out that it is quite easy to verify it using the basis of the complex exponentials. So it is useful to reduce the general case to that of the Fourier basis. Let Φ be the isomorphism between $\mathcal{P}_s$ and $\mathcal{B}_s$ given by

$$\mathcal{P}_s \ni (p_k, q_k) \mapsto \Phi(p_k, q_k) := (\sum_k p_k e^{\mathrm{i}kx}, \sum_k q_k e^{\mathrm{i}kx}) \tag{3.4}$$

Then any polynomial vector field on $\mathcal{B}_s$ induces a polynomial vector field on $\mathcal{P}_s$.

Definition 3.4. A polynomial vector field $X : \mathcal{B}_s \to \mathcal{B}_s$ will be said to have *tame modulus with respect to the exponentials* if the polynomial vector field $\Phi^{-1} X \Phi$ has tame modulus. □

Example 3.5. By examples 2.9, 2.10 and the result of lemma B.1 ensuring that the composition of maps having tame modulus has tame modulus, one has that given a C^∞ periodic function g, the Hamiltonian functions

$$H_1(p,q) = \int_{\mathbb{T}} g(x) p(x)^{n_1} q(x)^{n_2} dx \ , \tag{3.5}$$

$$H_2(p,q) = \int_{\mathbb{T}\times\mathbb{T}} p(x)^{n_1} q(x)^{n_2} g(x-y) p(y)^{n_3} q(y)^{n_4} dxdy \tag{3.6}$$

have a vector field with tame modulus with respect to the exponentials. □

The main result we will use to verify the tame modulus property is

Theorem 3.6. *Let $X : \mathcal{B}_s \to \mathcal{B}_s$ be a polynomial vector field having tame modulus with respect to the exponentials; assume (S1), then X has tame modulus.*

The proof is detailed in appendix B.

Example 3.7. By example 3.5 given a C^∞ periodic function g and a C^∞ function f from $\mathbb{R}^2$ or $\mathbb{R}^4$ into $\mathbb{R}$, the Hamiltonian functions

$$P_1(p,q) = \int_{\mathbb{T}} g(x) f(p(x), q(x)) dx \ , \tag{3.7}$$

$$P_2(p,q) = \int_{\mathbb{T}\times\mathbb{T}} g(x-y) f(p(x), p(y), q(x), q(y)) dx dy \tag{3.8}$$

satisfy hypothesis (H). □

Remark 3.8. All the above theory extends in a simple way to the case where the space $H^s(\mathbb{T})$ is substituted by $H^s(\mathbb{T}) \times ... \times H^s(\mathbb{T})$, or by $H^s(\mathbb{T}^d)$, cases needed to deal with equations in higher space dimensions and systems of coupled partial differential equations.

To deal with Dirichlet boundary conditions, we will consider the space

$$\mathcal{H}_s := \text{Span}\left((\varphi_j)_{j\geq 1}\right) \ . \tag{3.9}$$

Example 3.9. Consider again examples 3.1, and 3.2 with V an even function. In these cases the function space $\mathcal{H}_s$ is the space of the functions that extend to H^s skew–symmetric periodic functions of period 2π. Equivalently $\mathcal{H}_s$ is the space of the functions $q \in H^s([0,\pi])$ fulfilling the compatibility conditions

$$q^{(2j)}(0) = q^{(2j)}(\pi) = 0 \ , \quad 0 \leq j \leq \frac{s-1}{2} \ , \tag{3.10}$$

i.e. a generalized Dirichlet condition. Actually, recall that for V even, the Dirichlet eigenvalues of $A = -\partial_{xx} + V$ are periodic eigenvalues. Thus, due to our choice of the labeling, the sequence of eigenvalues $(\omega_j)_{j\geq 1}$ corresponds to the Dirichlet spectrum of A and $(\varphi_j)_{j\geq 1}$ are the corresponding Dirichlet eigenfunctions. □

Concerning the tame modulus property, one has in this case

Corollary 3.10. *Assume that the subspace $\mathcal{H}_s \times \mathcal{H}_s \subset \mathcal{B}_s$ is mapped into itself by the nonlinearity X_P and that $H_0 + P$ fulfills assumption (H) in $\mathcal{B}_s$. Then the restriction of $H_0 + P$ to $\mathcal{H}_s \times \mathcal{H}_s$ fulfills assumption (H).*

3.2 Nonlinear wave equation

As a first concrete application we consider a nonlinear wave equation

$$u_{tt} - u_{xx} + V(x)u = g(x,u) \ , \quad x \in \mathbb{T} \ , \ t \in \mathbb{R} \ , \tag{3.11}$$

where V is a C^∞, 2π periodic potential and $g \in C^\infty(\mathbb{T} \times \mathcal{U})$, $\mathcal{U}$ being a neighbourhood of the origin in $\mathbb{R}$.

Define the operator $A := (-\partial_{xx} + V)^{1/2}$, and introduce the variables (p,q) by

$$q := A^{1/2} u \ , \quad p := A^{-1/2} u_t \ ,$$

then the Hamiltonian takes the form $H_0 + P$, with H_0 of the form (3.1) and P given by

$$P(q) = \int_{\mathbb{T}} G(x, A^{-1/2}q)dx \sim \sum_{j\geq 3} \int_{\mathbb{T}} G_j(x)(A^{-1/2}q)^j dx \tag{3.12}$$

where $G(x,u) \sim \sum_{j\geq 3} G_j(x)u^j$ is such that $\partial_u G = -g$ and $\sim$ denotes the fact that the r.h.s. is the asymptotic expansion of the l.h.s.

The frequencies are the square roots of the eigenvalues of the Sturm–Liouville operator

$$-\partial_{xx} + V \tag{3.13}$$

and the normal modes φ_j are again the eigenfunctions of (3.13). In particular, due to example 3.2 they fulfill (S1).

Proposition 3.11. *The nonlinearity P has tame modulus.*

Proof. Denote by P_j the j-th Taylor polynomial of P, then

$$X_{P_j} = \left(-A^{-1/2}G_j(x)(A^{-1/2}q)^j, 0\right)$$

which is the composition of three maps, namely

$$q \overset{1}{\mapsto} A^{-1/2}q \overset{2}{\mapsto} G_j(x)(A^{-1/2}q)^j \overset{3}{\mapsto} -A^{-1/2}G_j(x)(A^{-1/2}q)^j$$

The first and the third ones are smoothing linear maps, which therefore have tame modulus, and the second one has tame modulus with respect to the exponentials. By lemma B.1 the thesis follows. □

Thus the system can be put in (γ, α, N_*)–normal form. To deduce dynamical informations we need to know something on the frequencies.

3.2.1 Dirichlet boundary conditions

First remark that if both V and $G(x,u)$ are even, then the eigenfunctions and the eigenvalues can be ordered according to example 3.9, and moreover the space $\mathcal{H}_s \times \mathcal{H}_s$ is invariant under X_P, so that assumption (H) holds also for the system with Dirichlet boundary conditions. The nonresonance condition (r-NR) is satisfied for almost all the potentials in the following sense: Write $V = V_0 + m$, with V_0 having zero average. Let λ_j be the sequence of the eigenvalues of $-\partial_{xx} + V_0$, then the frequencies are

$$\omega_j := \sqrt{\lambda_j + m} \tag{3.14}$$

Let $m_0 := \min_j \lambda_j$, then the following theorem holds

Theorem 3.12. *Consider the sequence $\{\omega_j\}_{j>0}$ given by* (3.14)*, for any $\Delta > m_0$ there exists a set $\mathcal{I} \subset (m_0, \Delta)$ of full measure such that, if $m \in \mathcal{I}$ then for any $r \geq 1$ assumption (r-NR) holds.*

Theorem 3.12 was proved in ref. [Bam03b]; in section 5 we will reproduce the main steps of the proof.

So, in the case of Dirichlet boundary, conditions it is immediate to conclude that for m in the set $\mathcal{I}$ corollary 2.16 applies to the equation (3.11). Moreover it is easy to verify that we are in the situation of remark 2.17 with $\tau = 1$ and therefore (2.24) holds for $s_1 < s$.

Remark 3.13. Such a result was already obtained in [Bam03b].

3.2.2 Periodic boundary conditions

In the case of periodic boundary conditions the frequencies are again of the form (3.14) with λ_j being the periodic eigenvalues of the operator $-\partial_{xx} + V_0$. We label them in such a way that, for $j > 0$, λ_j are the Dirichlet eigenvalues and, for $j \leq 0$, λ_j are the Neumann eigenvalues.

The situation is more complicated than in the Dirichlet case since asymptotically $\omega_j \sim \omega_{-j}$ and we cannot hope condition (r-NR) to be satisfied. Actually, for analytical V one has

$$|\omega_j - \omega_{-j}| \leq Ce^{-\sigma|j|}$$

and thus $|\omega_j - \omega_{-j}|$ cannot be bounded from below by $1/N^\alpha$ as soon as $|j| \geq C \ln N$. The forthcoming theorem essentially states that for typical small V this is the only case where condition (r-NR) is not satisfied.

Consider a potential V of the form

$$V(x) = m + \sum_{k\geq 1} v_k \cos kx \tag{3.15}$$

we will use the values $(v_k)_{k\geq 1}$ and the value of the mass m as random variables. More precisely, having fixed a positive Δ and a positive σ, for any $R > 0$ we consider the probability space

$$\mathcal{V}_R := \left\{ (m, (v_k)_{k\geq 1}) \ : \ m' := \Delta^{-1} m \in [0,1] \ , \ v'_k := R^{-1} e^{\sigma k} v_k \in [-\frac{1}{2}, \frac{1}{2}] \right\} \tag{3.16}$$

endowed by the product probability measure on (m', v'_k). We will identify V with the coefficients (m, v_k).

Theorem 3.14. *For any positive r there exist $\alpha > 0$, $R > 0$ and a set $\mathcal{S}_r \subset \mathcal{V}_R$ of full measure such that for any $V \in \mathcal{S}_r$ there exists a positive γ, and a positive b such that for any $N \geq 1$*

$$|\sum_{j\in\mathbb{Z}} \omega_j k_j| \geq \frac{\gamma}{N^\alpha} \ , \tag{3.17}$$

for any $k \in \mathbb{Z}^{\mathbb{Z}}$, fulfilling $0 \neq |k| \leq r + 2$, $\sum_{|j|>N} |k_j| \leq 2$ except if

$$(k_j = 0 \ \textit{for} \ |j| \leq b \ln N) \ \textit{and} \ \ (k_j + k_{-j} = 0 \ \ \textit{for} \ |j| > b \ln N) \ .$$

The proof is postponed to section 5.2.

As the assumption (r-NR) is no more satisfied, corollary 2.15 does not apply. However one has

Lemma 3.15. *Assume $V \in \mathcal{S}_r$. Let $k,\ j \in \mathbb{N}^{\mathbb{Z}}$ with $|k-l| \le r+2$, let $\xi^k\eta^l$ be a monomial in (γ,α,N)–normal form for the system* (3.11) *with periodic boundary conditions, then one has*

$$\left\{\xi_j\eta_j;\xi^k\eta^l\right\} = 0\ , \quad \textit{for } |j| \le b\ln N \tag{3.18}$$

$$\left\{\xi_j\eta_j+\xi_{-j}\eta_{-j};\xi^k\eta^l\right\} = 0\ , \quad \textit{for } |j| > b\ln N \tag{3.19}$$

Proof. Denote $J := b\ln N$ and $\gamma_j := \omega_j - \omega_{-j}$. Assume that b is so large that $|\gamma_j| < \gamma/2N^\alpha\ \forall j > J$. Let $\xi^k\eta^l$ be in (γ,α,N)–normal form; denote $K := (k-l) \in \mathbb{Z}^{\mathbb{Z}}$. By definition of normal form one has

$$\frac{\gamma}{N^\alpha} > \left|\sum_{j\in\mathbb{Z}}\omega_jK_j\right| = \left|\sum_{|j|\le J}\omega_jK_j + \sum_{j>J}\omega_j(K_j+K_{-j}) - \sum_{j>J}\gamma_jK_{-j}\right|$$

from which

$$\left|\sum_{|j|\le J}\omega_jK_j + \sum_{j>J}\omega_j(K_j+K_{-j})\right| < \frac{\gamma}{2N^\alpha}$$

then, by theorem 3.14 one has $K_j = 0$ for $|j| \le J$ and $K_j + K_{-j} = 0$ for $j > J$. As a consequence the normal form commutes with I_j for all $|j| \le J$. Write

$$\xi^k\eta^l = \prod_{j\in\mathbb{Z}}\xi_j^{k_j}\eta_j^{l_j} = \left(\prod_{|j|\le J}\xi_j^{k_j}\eta_j^{l_j}\right)\prod_{j>J}\xi_j^{k_j}\eta_j^{l_j}\xi_{-j}^{k_{-j}}\eta_{-j}^{l_{-j}}$$

For $J > J$ compute now

$$\begin{aligned}
&\left\{\xi_j\eta_j+\xi_{-j}\eta_{-j};\xi^k\eta^l\right\}\\
=\ &\left(\prod_{|i|\le J}\xi_i^{k_i}\eta_i^{l_i}\right)\left(\prod_{n>J,n\ne j}\xi_n^{k_n}\eta_n^{l_n}\xi_{-n}^{k_{-n}}\eta_{-n}^{l_{-n}}\right)\\
&\left\{\xi_j\eta_j+\xi_{-j}\eta_{-j};\xi_j^{k_j}\eta_j^{l_j}\xi_{-j}^{k_{-j}}\eta_{-j}^{l_{-j}}\right\}\\
=\ &\left(\prod_{|i|\le J}\xi_i^{k_i}\eta_i^{l_i}\right)\left(\prod_{n>J,n\ne j}\xi_n^{k_n}\eta_n^{l_n}\xi_{-n}^{k_{-n}}\eta_{-n}^{l_{-n}}\right)\\
&(-\mathrm{i})\left[(l_j-k_j)+(l_{-j}-k_{-j})\right]\xi_j^{k_j}\eta_j^{l_j}\xi_{-j}^{k_{-j}}\eta_{-j}^{l_{-j}}\\
=\ &-\mathrm{i}(K_j+K_{-j})\xi^k\eta^l = 0\ .
\end{aligned}$$

□

This allows to get dynamical consequences. Define $J_j := I_j + I_{-j}$ then one has

Theorem 3.16. *Consider the wave equation* (3.11) *with periodic boundary conditions, fix r, assume $V \in \mathcal{S}_r$. For any s large enough, there exists $\varepsilon_s > 0$ and $C_s > 0$ such that if the initial datum $(u_0, \dot{u}_0)$ belongs to $H^s(\mathbb{T}) \times H^{s-1}(\mathbb{T})$ and fulfills $\varepsilon := \|u_0\|_s + \|\dot{u}_0\|_{s-1} < \varepsilon_s$ then*

$$\|u(t)\|_s + \|\dot{u}(t)\|_{s-1} \leq 2\varepsilon \text{ for all } |t| \leq C_s \varepsilon^{-r} .$$

Further there exists C'_s such that for all $|t| \leq C_s \varepsilon^{-r}$ one has

$$|I_j(t) - I_j(0)| \leq \frac{1}{|j|^{2s}} \varepsilon^3 \quad \text{for } |j| \leq -C'_s \ln \varepsilon$$
$$|J_j(t) - J_j(0)| \leq \frac{1}{|j|^{2s}} \varepsilon^3 \quad \text{for } j > -C'_s \ln \varepsilon .$$

Roughly speaking, the last property means that energy transfers are allowed only between modes of index j and $-j$ with j large.

Remark 3.17. In theorem 3.16, the sobolev index s has to be greater than some $s_0 := 2\alpha r(r+2) + 2$ with α defined in (3.17) and hence independent of $V \in \mathcal{S}_r$. Then the bound on the initial data, ε_s, depends also on r and V (or more precisely on γ). The explicit dependence of ε_s with respect to these parameters is dificult to determine. Actually, $\varepsilon_s = \frac{1}{8} R_s$ where R_s is defined in theorem 2.13 and depends on all the non explicit constant intervening in section 4. Nevertheless one easily verifies than, as we could expect, ε_s is a decreasing function of s, r and γ.

3.3 NLS in 1 dimension

We will consider here only the case of Dirichlet boundary conditions. However, to fit the scheme of section 3.1 we start with the periodic system.

Consider the nonlinear Schrödinger equation

$$-\mathrm{i}\dot{\psi} = -\psi_{xx} + V\psi + \frac{\partial g(x, \psi, \psi^*)}{\partial \psi^*} , \quad x \in \mathbb{T}, \quad t \in \mathbb{R} \tag{3.20}$$

where V is a C^∞, 2π periodic potential. We assume that $g(x, z_1, z_2)$ is $C^\infty(\mathbb{T} \times \mathcal{U})$, $\mathcal{U}$ being a neighbourhood of the origin in $\mathbb{C} \times \mathbb{C}$. We also assume that g has a zero of order three at $(z_1, z_2) = (0, 0)$ and that $g(x, z, z^*) \in \mathbb{R}$.

The Hamiltonian function of the system is

$$H = \int_{-\pi}^{\pi} \frac{1}{2} \left(|\psi_x|^2 + V|\psi|^2 \right) + g(x, \psi(x), \psi^*(x)) dx \tag{3.21}$$

Define p and q as the real and imaginary parts of ψ, namely write $\psi = p + \mathrm{i}q$. Then the operator A is the Sturm–Liouville operator $-\partial_{xx} + V$ with periodic boundary conditions, the frequencies ω_j are the corresponding eigenvalues and the normal modes φ_j are the corresponding eigenfunctions.

Then property (S1) is a consequence of Sturm Liouville theory (see example 3.2) and it is easy to verify that property (H) holds for the periodic system. To deal with Dirichlet boundary conditions we have to ensure the invariance of the space $\mathcal{H}_s$ under the vector field of the equation (cf. corollary 3.10). To this end we assume

$$V(x) = V(-x) \ , \quad g(-x,-z,-z^*) = g(x,z,z^*) \ . \tag{3.22}$$

Then (H) holds true in the Dirichlet context and thus theorem 2.13 applies and the Hamiltonian 3.21 can be put in (γ, α, N)- normal form in $\mathcal{H}_s \times \mathcal{H}_s$. We are going to prove that for typical small V such a normal form is integrable.

Fix $\sigma > 0$ and, for any positive R define the space of the potentials, by

$$\mathcal{V}_R := \left\{ V(x) = \sum_{k\geq 1} v_k \cos kx \mid v_k' := v_k R^{-1} e^{\sigma k}, \ v_k \in \left[-\frac{1}{2}, \frac{1}{2}\right] \text{ for } k \geq 1 \right\} \tag{3.23}$$

that we endow with the product normalized probability measure. We remark that any potential in $\mathcal{V}_R$ has size of order R, is analytic and has zero average. We also point out that the choice of zero average was done for simplicity since the average does not affect the resonance relations among the frequencies.

Theorem 3.18. *For any r there exists a positive R and a set $\mathcal{S} \subset \mathcal{V}_R$ such that property (r-NR) holds for any potential $V \in \mathcal{S}$ and $|\mathcal{V}_R - \mathcal{S}| = 0$.*

The proof of this theorem is postponed to section 5.

Thus corollary 2.16 holds and every small amplitude solution remains small for long times and approximatively lies on a finite dimensional torus.

Remark 3.19. We remark that the theory applies also to Hartree type equations of the form

$$-\mathrm{i}\psi_t = -\psi_{xx} + V\psi + (W * |\psi|^2)\psi \ . \tag{3.24}$$

3.4 Coupled NLS in 1 dimension

As an example of a system of coupled partial differential equations we consider a pair of NLS equations. From the mathematical point of view the interest of this example is that, since it does not have a positive definite energy, nothing is a priori known about global existence of its solutions in any phase space. So, consider the Hamiltonian

$$H = \int_{\mathbb{T}} \left[|\psi_x|^2 - |\phi_x|^2 + V_1 |\psi|^2 - V_2 |\phi|^2 - g(x,\psi,\psi^*,\phi,\phi^*) \right] dx \tag{3.25}$$

The corresponding equations of motion have the form

$$-\mathrm{i}\dot{\psi} = -\psi_{xx} + V_1\psi - \partial_{\psi^*} g \tag{3.26}$$

$$\mathrm{i}\dot{\phi} = -\phi_{xx} + V_2\phi + \partial_{\phi^*} g \tag{3.27}$$

Assume as in the previous sections that the potentials and the function g are of class C^∞, then condition (H) holds for the system with periodic boundary conditions. Assuming also that the potentials V_1, V_2 and g are even in each of the variables, one has that the space of odd functions is invariant and therefore the system with Dirichlet boundary conditions fulfills also condition (H).

We concentrate now on the case of Dirichlet boundary conditions.

The frequencies are given by

$$\omega_j := \lambda_j^1 \ , \quad \omega_{-j} := -\lambda_j^2 \ , \quad j \geq 1$$

where λ_j^1, and λ_j^2, are the Dirichlet eigenvalues of $-\partial_{xx} + V_1$ and $-\partial_{xx} + V_2$ respectively.

Assume that the potentials vary in the probability space obtained by taking two copies of the space (3.23). As in the case of periodic NLW (cf. subsection 3.2.2), $\omega_j \sim \omega_{-j}$ and we cannot hope condition (r-NR) to be satisfied. Actually one has $|\omega_j - \omega_{-j}| \leq C/|j|^2$ and, by adapting the proof of theorem 3.18 in the spirit of theorem 3.14, one gets

Theorem 3.20. *For any positive r there exist $\alpha > 0$, $R > 0$ and a set $\mathcal{S}_{\psi,\phi} \subset \mathcal{V}_R \times \mathcal{V}_R$ such that*

i) for any $(V_1, V_2) \in \mathcal{S}_{\psi,\phi}$ there exist a positive γ and a positive C such that for any $N \geq 1$

$$|\sum_{j \in \bar{\mathbb{Z}}} \omega_j k_j| \geq \frac{\gamma}{N^\alpha} \ , \tag{3.28}$$

for any $k \in \mathbb{Z}^{\bar{\mathbb{Z}}}$, fulfilling $0 \neq |k| \leq r+2$, $\sum_{|j|>N} |k_j| \leq 2$ except if

$$(k_j = 0 \ \text{for} \ |j| \leq CN^{\sqrt{2\alpha}}) \ \text{and} \ \ (k_j + k_{-j} = 0 \ \ \text{for} \ |j| > CN^{\sqrt{2\alpha}}) \ .$$

ii) $|\mathcal{V}_R \times \mathcal{V}_R - \mathcal{S}_{\psi,\phi}| = 0$.

The proof which is a simple variant of the proof theorem 3.18 using also some of the techniques for the proof of theorem 3.14 is omitted.

In particular one deduces long time existence of solutions and long time stability of the zero equilibrium point:

Theorem 3.21. *Consider the system (3.26,3.27) and fix $r \geq 1$. Provided $V_1 \times V_2 \in \mathcal{S}_{\psi,\phi}$, for any s large enough there exists a positive ϵ_s such that if the initial datum $(\psi_0, \phi_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ fulfills*

$$\epsilon := \|\psi_0\|_{H^s} + \|\phi_0\|_{H^s} \leq \epsilon_s$$

then the corresponding solution exists until the time ϵ^{-r} and fulfills

$$\|\psi(t)\|_{H^s} + \|\phi(t)\|_{H^s} \leq 2\epsilon$$

Notice that remark 3.17 also applies to theorem 3.21.

3.5 NLS in arbitrary dimension

Consider the following non linear Schrödinger equation in dimension $d \geq 1$

$$-\mathrm{i}\psi_t = -\Delta\psi + V * \psi + \frac{\partial g(x, \psi, \psi^*)}{\partial \psi^*}\ , \quad x \in [-\pi, \pi]^d, t \in \mathbb{R} \tag{3.29}$$

with periodic boundary conditions.

As in the previous sections we assume that $g(x, z_1, z_2)$ is $C^\infty(\mathbb{T}^d \times \mathcal{U})$, $\mathcal{U}$ being a neighbourhood of the origin in $\mathbb{C} \times \mathbb{C}$. We also assume that g has a zero of order three at $(z_1, z_2) = (0, 0)$ and that $g(x, z, z^*) \in \mathbb{R}$.

Fix $m > d/2$ and $R > 0$, then the potential V is chosen in the space $\mathcal{V}$ given by

$$\mathcal{V} = \{V(x) = \sum_{k \in \mathbb{Z}^d} v_k e^{\mathrm{i}k\cdot x} \mid v'_k := v_k(1 + |k|)^m/R \in [-1/2, 1/2] \text{ for any } k \in \mathbb{Z}^d\} \tag{3.30}$$

that we endow with the product probability measure. In contrast with the previous cases, here R is arbitrary (it does not need to be small).

In this section we denote by $H^s \equiv H^s(\mathbb{T}^d; \mathbb{C})$ the Sobolev space of order s on the d-dimensional torus $\mathbb{T}^d$, and by $\|\cdot\|_s$ the usual norm on H^s. Notice also that in this section all the indexes run in $\mathbb{Z}^d$.

The NLS equation (3.29) is Hamiltonian with Hamiltonian function given by

$$H(\psi, \psi^*) = \int_{\mathbb{T}^d} \left(|\nabla\psi|^2 + (V * \psi)\psi^* + g(x, \psi, \psi^*)\right) dx.$$

It is convenient to introduce directly the variables ξ, η by

$$\psi(x) = \left(\frac{1}{2\pi}\right)^{d/2} \sum_{k \in \mathbb{Z}^d} \xi_k e^{\mathrm{i}k\cdot x}\ , \quad \psi^*(x) = \left(\frac{1}{2\pi}\right)^{d/2} \sum_{k \in \mathbb{Z}^d} \eta_k e^{-\mathrm{i}k\cdot x},$$

so the Hamiltonian reads

$$H(\xi, \eta) = \sum_{k \in \mathbb{Z}^d} \omega_k \xi_k \eta_k \ + \int_{\mathbb{T}^d} g(x, \psi, \psi^*) d^d x.$$

where the linear frequencies are given by $\omega_k = |k|^2 + v_k$.

It is immediate to realize that the nonlinearity has tame modulus so that theorem 2.13 applies with an adapted definition of (γ, α, N)–normal form.

Remark that, if $|l| = |j| \to \infty$ then $\omega_j - \omega_l \to 0$ as $|l| \to \infty$. Thus property (r-NR) is violated. The following theorem ensures this is the only case where it happens.

Theorem 3.22. *There exists a set $\mathcal{S} \subset \mathcal{V}$ of measure 1 such that, for any $V \in \mathcal{S}$ the following property holds. For any positive r there exist positive constants $\gamma,\ \alpha$, such that for any $N \geq 1$*

$$|\sum_{j \in \mathbb{Z}^d} \omega_j k_j| \geq \frac{\gamma}{N^\alpha}\ , \tag{3.31}$$

for any $k \in \mathbb{Z}^{\mathbb{Z}^d}$, fulfilling $0 \neq |k| \leq r+2$, $\sum_{|j|>N} |k_j| \leq 2$ except if

$$(k_j = 0 \text{ for } |j| \leq N^{\sqrt{\alpha/m}}) \text{ and } (\sum_{|j|=K} k_j = 0 \text{ for all } K > N^{\sqrt{\alpha/m}}) \ .$$

The proof, which exploits the fact that $\omega_k = |k|^2 + v_k$, is a simple variant of the techniques introduced in sect. 5 and is omitted.

As a consequence theorem 2.13 does not allow to eliminate monomial of the form $I_{k_1} \dots I_{k_r} \xi_j \eta_l$ with large $|j|$ and $|l|$. Nevertheless, defining for $M > N^{\sqrt{\alpha/m}}$ $J_M = \sum_{|k|^2=M} I_k$, we have

Theorem 3.23. *Consider the equation (3.29), and assume $V \in \mathcal{S}$. Fix $r \geq 1$, then for any s large enough, there exist $\varepsilon_s > 0$ and $C_s > 0$ such that if the initial datum $\psi(\cdot, 0)$ belongs to $H^s(\mathbb{T}^d)$ and fulfills $\varepsilon := \|\psi(\cdot,0)\|_s < \varepsilon_s$ then*

$$\|\psi(\cdot,t)\|_s \leq 2\varepsilon \text{ for all } |t| \leq C_s \varepsilon^{-r} \ .$$

Furthermore there exists an integer $N \sim \epsilon^{-\frac{1}{2r\alpha}}$ (where α is defined in theorem 3.22) such that

$$|I_j(t) - I_j(0)| \leq \frac{C_s}{|j|^{2s}} \varepsilon^3 \quad \text{ for } |j| \leq N^{\sqrt{\alpha/m}}$$
$$|J_M(t) - J_M(0)| \leq \frac{C_s}{M^{2s}} \varepsilon^3 \quad \text{ for } M > N^{\sqrt{\alpha/m}} \ .$$

Roughly speaking, the last property means that energy transfers are allowed only among modes having indexes with equal large modulus. Notice that remark 3.17 also applies to theorem 3.23.

If the nonlinearity does not depend on x something more can be concluded. To come to this point consider the following

Definition 3.24. Given a monomial $\xi_{j_1} \dots \xi_{j_{r_1}} \eta_{l_1} \dots \eta_{l_{r_2}}$, its momentum is defined by $j_1 + \dots + j_{r_1} - (l_1 + \dots + l_{r_2})$. □

It is easy to see that if the function g does not depend on x then the nonlinearity contains only monomials with zero momentum. Moreover this property is conserved by our iteration procedure defined in section 4.2. Therefore the normal form has zero momentum and the following corollary holds:

Corollary 3.25. *If the function g does not depend on x then the normal form $\mathcal{Z}$ of the system depends on the actions only.*

Proof. By theorem 3.22 the only non integrable term that the normal form may contain are of the form

$$I_{k_1} \dots I_{k_l} \xi_j \eta_i$$

with $|i| = |j|$, but the momentum $j - i$ of such a term must vanish and therefore one must have $i = j$. □

Thus the following theorem holds

Theorem 3.26. *Consider the d-dimensional NLS equation* (3.29) *with periodic boundary conditions and g independent of x. Assume $V \in \mathcal{S}$. Fix r, then, for s large enough, there exist $\varepsilon_s > 0$ and $c_s > 0$ such that the following properties hold :*

If $\psi(t)$ is the solution of the Cauchy problem (3.29) *with initial datum $\psi_0 \in H^s$ satisfying $\varepsilon := \|\psi_0\|_s \leq \varepsilon_s$ then for all*

$$|t| \leq \frac{c_s}{\varepsilon^{r+1/2}}$$

the solution ψ satisfies

$$\|\psi(t)\|_s \leq 2\varepsilon \ , \quad |I_k(t) - I_k(0)| \leq \frac{1}{|k|^{2s}}\varepsilon^3 \ .$$

One also has that corollary 2.16 applies and therefore any initial datum which is smooth and small enough give rise to a solution which is ϵ^{r_1} close to a torus up to times ϵ^{-r_2}.

The results of this section were announced in [BG03].

4 Proof of the Normal Form Theorem

First of all we fix a number r_* (corresponding to the one denoted by r in the statement of theorem 2.13) determining the order of normalization we want to reach, precisely it represents the number of steps of normalization we will perform. In the following we will use the notation

$$a \preceq b$$

to mean: There exists a positive constant C independent of R and of N (but dependent on r_*, s, γ and α), such that

$$a \leq Cb \ .$$

The proof is based on the iterative elimination of nonresonant monomials. In order to improve by one the order of the normalized part of the Hamiltonian we will use a canonical transformation generated by Lie transform, namely the time 1 flow of a suitable auxiliary Hamiltonian function. So, first of all we recall some facts about Lie transform, and we introduce some related tools.

Consider an auxiliary Hamiltonian function χ and the corresponding Hamilton equations $\dot{z} = X_\chi(z)$. Denote by X_χ^t the corresponding flow and by $\mathcal{T} := X_\chi^1 \equiv X_\chi^t\big|_{t=1}$ the time 1 flow.

Definition 4.1. The canonical transformation $\mathcal{T}$ will be called the *Lie transform* generated by χ. □

Given an analytic function g, consider the transformed function $g \circ \mathcal{T}$. Using the relation

$$\frac{d}{dt}(g \circ X_\chi^t) = \{\chi; g\} \circ X_\chi^t \ ,$$

it is easy to see that, at least formally, one has

$$g \circ \mathcal{T} = \sum_{l=0}^{\infty} g_l \ , \tag{4.1}$$

with g_l defined by

$$g_0 = g \ , g_l = \frac{1}{l} \{\chi; g_{l-1}\} \ , \quad l \geq 1 \ . \tag{4.2}$$

Then, provided one is able to show the convergence of the series, (4.1) gets a rigorous meaning.

We will use (4.1) to show that, if g and χ have s–tame modulus then the same is true also for $g \circ \mathcal{T}$. To this end, since infinite sums are involved, we have to introduce a suitable notion of convergence.

4.1 Properties of the functions with tame modulus

In this section we will use only the complex variables ξ, η defined by (2.15). When dealing with such variables, we will continue to denote by z a phase point (i.e. $z = (\ldots, \xi_l, \ldots, \xi_1, \eta_1, \ldots, \eta_l, \ldots)$) but we will use complex spaces, so in this context $\mathcal{P}_s$ will denote the complexification of the phase space and $B_s(R)$ the complex ball of radius R centered at the origin.

Definition 4.2. Let X be an s–tame vector field homogeneous of degree r as a polynomial in z; the infimum of the constants C_s such that the inequality

$$\begin{aligned}\left\| \tilde{X}(z^{(1)}, ..., z^{(r)}) \right\|_s \quad &\leq \quad C_s \frac{1}{r} \sum_{l=1}^{r} \left\| z^{(1)} \right\|_1 .... \left\| z^{(l-1)} \right\|_1 \left\| z^{(l)} \right\|_s \left\| z^{(l+1)} \right\|_1 ... \left\| z^{(r)} \right\|_1 \\ &\qquad \forall z^{(1)}, ..., z^{(r)} \in \mathcal{P}_s \end{aligned} \tag{4.3}$$

holds will be called *tame norm s of X* (or tame s norm). Such a norm will be denoted by $|X|_s^T$. □

Definition 4.3. Let $f \in T_M^s$ be a homogeneous polynomial. The tame norm s of $X_{\lfloor f \rceil}$ will be denoted by $|f|_s$. □

It is useful to introduce a simple notation for the r.h.s. of (4.3), so we will write

$$\left\| (z^{(1)}, ..., z^{(r)}) \right\|_{s,1} := \frac{1}{r} \sum_{l=1}^{r} \left\| z^{(1)} \right\|_1 .... \left\| z^{(l-1)} \right\|_1 \left\| z^{(l)} \right\|_s \left\| z^{(l+1)} \right\|_1 ... \left\| z^{(r)} \right\|_1 \tag{4.4}$$

Moreover, we will often denote by $w \equiv (z^{(1)}, ..., z^{(r)})$ a multivector. Thus the quantity (4.4) will be simply denoted by $\|w\|_{s,1}$.

Remark 4.4. The tame s norm of a homogeneous polynomial f of degree $r+1$ is given by

$$|f|_s := \sup \frac{\left\| \tilde{X}_{\lfloor f \rceil}(w) \right\|_s}{\|w\|_{s,1}} \tag{4.5}$$

where the sup is taken over all the multivectors

$$w = (z^{(1)}, ..., z^{(r)})$$

such that $z^{(l)} \neq 0$ for any l, and $\|w\|_{s,1}$ is defined by (4.4). □

Remark 4.5. Since all the components of the multilinear form $\widetilde{\lfloor X_f \rceil}$ are positive, the above supremum need to be taken only on the positive 'octant' on which all the components of each of the vectors $z^{(l)}$ are positive. □

Remark 4.6. If $f \in T^s_M$ is a homogeneous polynomial of degree $r+1$ then one has

$$\|X_f(z)\|_s \leq \left\| X_{\lfloor f \rceil}(z) \right\|_s \leq |f|_s \, \|z\|_1^{r-1} \, \|z\|_s \tag{4.6}$$

□

Definition 4.7. Let $f \in T^s_M$ be a non homogeneous polynomial. Consider its Taylor expansion

$$f = \sum f_r$$

where f_r is homogeneous of degree r. For $R > 0$ we will denote

$$\langle |f| \rangle_{s,R} := \sum_{r \geq 2} |f_r|_s R^{r-1} \ . \tag{4.7}$$

Such a definition extends naturally to the set of analytic functions such that (4.7) is finite. The set of the functions of class T^s_M with a finite $\langle |f| \rangle_{s,R}$ norm will be denoted by $T_{s,R}$. □

Definition 4.8. Let f be an analytic function whose vector field is analytic as a map from $B_s(R)$ to $\mathcal{P}_s$. We denote

$$\|X_f\|_{s,R} := \sup_{\|z\|_s \leq R} \|X_f(z)\|_s$$

□

Remark 4.9. With the above definitions, for any $f \in T_{s,R}$, one has

$$\|X_f\|_{s,R} \leq \left\| X_{\lfloor f \rceil} \right\|_{s,R} \leq \langle |f| \rangle_{s,R} \tag{4.8}$$

□

Remark 4.10. The norm $\langle |f| \rangle_{s,R}$ makes the space $T_{s,R}$ a Banach space. □

A key property for the proof of theorem 2.13 is related to the behaviour of functions of class T^s_M with respect to the decomposition of the phase variables into "variables with small index" and "variables with large index". To be precise we fix some notations. Corresponding to a given N we will denote by $\bar{z} \equiv (\bar{\xi}_j, \bar{\eta}_j)_{j\leq N} = (\xi_j, \eta_j)_{j\leq N}$ the first N canonical variables and by $\hat{z} \equiv (\hat{\xi}_j, \hat{\eta}_j)_{j>N} = (\xi_j, \eta_j)_{j>N}$ the remaining ones.

We have the the following important

Lemma 4.11. *Fix N and consider the decomposition $z = \bar{z} + \hat{z}$, as above. Let $f \in T^s_M$ be a polynomial of degree less or equal than $r_* + 2$. Assume that f has a zero of order three in the variables $\hat{z}$, then one has*

$$\left\|X_f\right\|_{s,R} \preceq \frac{\langle|f|\rangle_{s,R}}{N^{s-1}} \ . \tag{4.9}$$

The proof of this lemma is based on two facts: (i) if a sequence $z \in \mathcal{P}_s$ has only components with large index (i.e. larger than N), then its $\mathcal{P}_1$ norm is bounded by its $\mathcal{P}_s$ norm divided by N^{s-1}; (ii) according to the tame property, in the estimate of $\left\|X_f\right\|_{s,R}$ the quantity $\left\|\hat{z}\right\|_1$ appears at least once. The actual proof is slightly complicated due to the different behaviour of the different components of the Hamiltonian vector field with respect to the variables with small and large index. For this reason it is deferred to the appendix A.

Lemma 4.12. *Let $f, g \in T^s_M$ be homogeneous polynomials of degrees $n+1$ and $m+1$ respectively, then one has $\{f; g\} \in T^s_M$ with*

$$\left|\{f; g\}\right|_s \leq (n+m)|f|_s|g|_s \tag{4.10}$$

The proof is based on the following three facts: (i) the vector field of the Poisson brackets of two functions is the commutator of the vector fields of the original functions; (ii) in case of polynomials the commutator can be computed in terms of the composition of the multilinear functions associated to the original polynomials; (iii) the composition of two tame multilinear functions is still a tame multilinear function. However the proof requires some attention due to the moduli and the symmetrization required in the definition of the class T^s_M. For this reason it is deferred to the appendix A.

Lemma 4.13. *Let $h, g \in T_{s,R}$, then for any positive $d < R$ one has $\{f; g\} \in T_{s,R-d}$ and*

$$\langle|\{h; g\}|\rangle_{s,R-d} \leq \frac{1}{d} \langle|h|\rangle_{s,R} \langle|g|\rangle_{s,R} \ . \tag{4.11}$$

Proof. Write $h = \sum_j h_j$ and $g = \sum_k g_k$ with h_j homogeneous of degree j and similarly for g, we have

$$\{h; g\} = \sum_{j,k} \{h_j; g_k\} \ .$$

Now each term of the series is estimated by

$$\begin{aligned}\langle|\{h_j;g_k\}|\rangle_{s,R-d} &= |\{h_j;g_k\}|_s\,(R-d)^{j+k-3}\\ &\le |h_j|_s\,|g_k|_s(j+k-2)(R-d)^{j+k-3}\\ &\le |h_j|_s\,|g_k|_s\frac{1}{d}R^{j+k-2}=\frac{1}{d}\langle|h_j|\rangle_{s,R}\langle|g_k|\rangle_{s,R}\ ,\end{aligned}$$

where we used the inequality

$$k(R-d)^{k-1}<\frac{R^k}{d}\ , \tag{4.12}$$

which holds for any positive R and $0<d<R$. Then the thesis follows. □

We estimate now the terms of the series (4.1,4.2) defining the Lie transform.

Lemma 4.14. *Let $g\in T_{s,R}$ and $\chi\in T_{s,R}$ be two analytic functions; denote by g_n the functions defined recursively by (4.2); then, for any positive $d<R$, one has $g_n\in T_{s,R-d}$, and the following estimate holds*

$$\langle|g_n|\rangle_{s,R-d}\le\langle|g|\rangle_{s,R}\left(\frac{e}{d}\langle|\chi|\rangle_{s,R}\right)^n\ . \tag{4.13}$$

Proof. Fix n, and denote $\delta:=d/n$, we look for a sequence $C_l^{(n)}$ such that

$$\langle|g_l|\rangle_{s,R-\delta l}\le C_l^{(n)}\ ,\quad\forall l\le n\ .$$

By (4.11) this sequence can be defined by

$$C_0^{(n)}=\langle|g|\rangle_{s,R}\ ,\quad C_l^{(n)}=\frac{1}{l\delta}C_{l-1}^{(n)}\langle|\chi|\rangle_{s,R}=\frac{n}{ld}C_{l-1}^{(n)}\langle|\chi|\rangle_{s,R}\ .$$

So one has

$$C_n^{(n)}=\frac{1}{n!}\left(\frac{n\langle|\chi|\rangle_{s,R}}{d}\right)^n\langle|g|\rangle_{s,R}\ .$$

Using the inequality $n^n<n!e^n$, which is easily verified by writing the iterative definition of $n^n/n!$, one has the thesis. □

Remark 4.15. Let χ be an analytic function with Hamiltonian vector field which is analytic as a map from $B_s(R)$ to $\mathcal{P}_s$, fix $d<R$. Assume $\|X_\chi\|_{s,R}<d$ and consider the time t flow T^t of X_χ. Then, for $|t|\le1$, one has

$$\sup_{\|z\|_s\le R-d}\left\|T^t(z)-z\right\|_s\le\|X_\chi\|_{s,R}\ . \tag{4.14}$$

□

Now using remark 4.15 and the formula $X_{g\circ T^t}(z)=dT^{-t}(T^t(z))X_g(T^t(z))$, we obtain as in [Bam99a] lemma 8.2

Lemma 4.16. *Consider* χ *as above and let* $g : B_s(R) \to \mathbb{C}$ *be an analytic function with vector field analytic in* $B_s(R)$*, fix* $0 < d < R$ *assume* $\|X_\chi\|_{s,R} \leq d/3$*, then, for* $|t| \leq 1$*, one has*

$$\|X_{g\circ T^t}\|_{s,R-d} \leq 2\,\|X_g\|_{s,R}$$

The following lemma solves the so-called homological equation (4.15) in T^s_M. We recall that $H_0 = \sum_{j\geq 1}\omega_j\xi_j\eta_j$.

Lemma 4.17. *Let* f *be a polynomial in* T^s_M *which is at most quadratic in the variables* $\hat{z}$*. There exists* $\chi, \mathcal{Z} \in T_{s,R}$ *with* $\mathcal{Z}$ *in* (γ,α,N)*–normal form such that*

$$\{H_0;\chi\} + \mathcal{Z} = f\ . \tag{4.15}$$

Moreover $\mathcal{Z}$ *and* χ *fulfill the estimates*

$$\langle|\chi|\rangle_{s,R} \leq \frac{N^\alpha}{\gamma}\langle|f|\rangle_{s,R}\ ,\quad \langle|\mathcal{Z}|\rangle_{s,R} \leq \langle|f|\rangle_{s,R} \tag{4.16}$$

Proof. Expanding f, in Taylor series, namely

$$f(\xi,\eta) = \sum_{j,l} f_{jl}\xi^j\eta^l$$

and similarly for χ and $\mathcal{Z}$, the equation (4.15) becomes an equation for the coefficients of f, χ and $\mathcal{Z}$, namely

$$\mathrm{i}\omega\cdot(j-l)\chi_{jl} + \mathcal{Z}_{jl} = f_{jl}$$

We define

$$\mathcal{Z}_{jl} := f_{jl}\ ,\quad j,l \quad \text{such that} \quad |\omega\cdot(j-l)| < \frac{\gamma}{N^\alpha} \tag{4.17}$$

$$\chi_{jl} := \frac{f_{jl}}{\mathrm{i}\omega\cdot(j-l)}\ ,\quad j,l \quad \text{such that} \quad |\omega\cdot(j-l)| \geq \frac{\gamma}{N^\alpha}\ . \tag{4.18}$$

By construction, $\mathcal{Z}$ and χ are in T^s_M. Further, since f is at most quadratic in the variables $\hat{z}$ one has $\sum_{k>N}(j_k+l_k) \leq 2$ and thus $\mathcal{Z}$ is in (γ,α,N)–normal form. The estimates (4.16) immediately follows from the definition of the norm. □

Lemma 4.18. *Let* $\chi \in T_{s,R}$ *be the solution of the homological equation (4.15) with* $f \in T^s_M$*. Denote by* $H_{0,n}$ *the functions defined recursively as in (4.2); for any positive* $d < R$*, one has* $H_{0,n} \in T_{s,R-d}$*, and the following estimate holds*

$$\langle|H_{0,n}|\rangle_{s,R-d} \leq 2\,\langle|f|\rangle_{s,R}\left(\frac{e}{d}\langle|\chi|\rangle_{s,R}\right)^n\ . \tag{4.19}$$

Proof. The idea of the proof is that, using the homological equation one gets $H_{0,1} = \mathcal{Z} - f \in T^s_M$. Then proceeding as in the proof of lemma 4.14 one gets the result. □

4.2 The main lemma and conclusion of the proof

The main step of the proof is a proposition allowing to increase by one the order of the perturbation. As a preliminary step expand P in Taylor series up to order $r_* + 2$:

$$P = P^{(1)} + \mathcal{R}_* \ , \quad P^{(1)} := \sum_{l=1}^{r_*} P_l \tag{4.20}$$

where P_l is homogeneous of degree $l + 2$ and $\mathcal{R}_*$ is the remainder of the Taylor expansion.

Remark 4.19. From assumption (H) it follows that, for R small enough, one has

$$\left\langle \left| P^{(1)} \right| \right\rangle_{s,R} \preceq R^2 \ , \tag{4.21}$$

$$\left\| X_{\mathcal{R}_*} \right\|_{s,R} \preceq R^{r_*+2} \ . \tag{4.22}$$

□

Consider now the analytic Hamiltonian

$$H_T = H_0 + P^{(1)} \tag{4.23}$$

and introduce the complex variables (ξ, η) defined by (2.15). Clearly (4.21) holds also in the complex variables.

In the statement of the forthcoming iterative lemma we will use the following notations: For any positive R, define $\delta := R/2r_*$ and $R_r := R - r\delta$.

Proposition 4.20. Iterative Lemma. *Consider the Hamiltonian (4.23), and fix $s \geq 1$. For any $r \leq r_*$ there exists a positive $R_{*r} \ll 1$ and, for any $N > 1$ there exists an analytic canonical transformation*

$$\mathcal{T}^{(r)} : B_s \left(\frac{R_{*r}(2r_* - r)}{2N^\alpha r_*} \right) \to \mathcal{P}_s$$

which puts (4.23) in the form

$$H^{(r)} := H_T \circ \mathcal{T}^{(r)} = H_0 + \mathcal{Z}^{(r)} + f^{(r)} + \mathcal{R}_N^{(r)} + \mathcal{R}_T^{(r)} \ . \tag{4.24}$$

For any $R < R_{*r}/N^\alpha$, the following properties are fulfilled

1) the transformation $\mathcal{T}^{(r)}$ satisfies

$$\sup_{z \in B_s(R_r)} \left\| z - \mathcal{T}^{(r)}(z) \right\|_s \preceq N^\alpha R^2 \ , \tag{4.25}$$

2) $\mathcal{Z}^{(r)}$ is a polynomial of degree at most $r + 2$ and has tame modulus. It is (γ, α, N)–normal form, and has a zero of order 3 at the origin; $f^{(r)}$ is a *polynomial of degree at most r_*+2 and has a zero of order $r+3$ at the origin. Moreover the following estimates hold*

$$\left\langle\left|\mathcal{Z}^{(r)}\right|\right\rangle_{s,R_r} \preceq R^2 \ , \quad \forall r\geq 1 \tag{4.26}$$

$$\left\langle\left|f^{(r)}\right|\right\rangle_{s,R_r} \preceq R^2(RN^\alpha)^r \tag{4.27}$$

3) the remainder terms, $\mathcal{R}_N^{(r)}$ and $\mathcal{R}_T^{(r)}$ have tame modulus and satisfy

$$\left\|X_{\mathcal{R}_T^{(r)}}\right\|_{s,R_r} \preceq (RN^\alpha)^{r_*+2} \tag{4.28}$$

$$\left\|X_{\mathcal{R}_N^{(r)}}\right\|_{s,R_r} \preceq \frac{R^2}{N^{s-1}} \ . \tag{4.29}$$

Proof. We proceed by induction. First remark that the theorem is trivially true when $r=0$ with $\mathcal{T}^{(0)}=I$, $\mathcal{Z}^{(0)}=0$, $f^{(0)}=P^{(1)}$, $\mathcal{R}_N^{(0)}=0$ and $\mathcal{R}_T^{(r)}=0$. Then we split $f^{(r)}$ ($P^{(1)}$ in the case $r=0$) into an effective part and a remainder. Consider the Taylor expansion of $f^{(r)}$, *in the variables $\hat{z}$ only*. Write

$$f^{(r)}=f_0^{(r)}+f_N^{(r)}$$

where $f_0^{(r)}$ is the truncation of such a series at second order (it contains at most terms quadratic in $\hat{z}$) and $f_N^{(r)}$ is the remainder of the expansion. Since both $f_N^{(r)}$ and $f_0^{(r)}$ are truncations of $f^{(r)}$, one has

$$\left\langle\left|f_N^{(r)}\right|\right\rangle_{s,R_r}\leq\left\langle\left|f^{(r)}\right|\right\rangle_{s,R_r} \ , \quad \left\langle\left|f_0^{(r)}\right|\right\rangle_{s,R_r}\leq\left\langle\left|f^{(r)}\right|\right\rangle_{s,R_r} \ .$$

Consider the truncated Hamiltonian

$$H_0+\mathcal{Z}^{(r)}+f_0^{(r)} \ . \tag{4.30}$$

We look for a Lie transform, $\mathcal{T}_r$, eliminating the non normalized part of order $r+3$ in the truncated Hamiltonian. Let χ_r be the Hamiltonian generating $\mathcal{T}_r$. Using the formulae (4.1,4.2) one writes

$$\left(H_0+\mathcal{Z}^{(r)}+f_0^{(r)}\right)\circ\mathcal{T}_r=H_0+\mathcal{Z}^{(r)} \tag{4.31}$$

$$+\{\chi_r;H_0\}+f_0^{(r)} \tag{4.32}$$

$$+\sum_{l\geq1}\mathcal{Z}_l^{(r)}+\sum_{l\geq1}f_{0,l}^{(r)}+\sum_{l\geq2}H_{0,l} \tag{4.33}$$

with $\mathcal{Z}_l^{(r)}$ the l-th term in the expansion of the Lie transform of $\mathcal{Z}^{(r)}$, similarly for the other quantities. Then it easy to see that the right hand side of (4.31) is the already normalized part of the transformed Hamiltonian, (4.32) contains the part of degree $r+3$ from which all the non normalized terms have to be eliminated by a suitable choice of χ_r, (4.33) contains a non normalized remainder having a zero of order $r+4$ at the origin.

We first use lemma 4.17 to determine χ_r as the solution of the equation

$$\{\chi_r; H_0\} + f_0^{(r)} = \mathcal{Z}_r \tag{4.34}$$

with $\mathcal{Z}_r$ in normal form. Then, by (4.16) and by (4.27) one has the estimates

$$\langle|\chi_r|\rangle_{s,R_r} \preceq N^\alpha R^2 \left(N^\alpha R\right)^r , \quad \langle|\mathcal{Z}_r|\rangle_{s,R_r} \preceq R^2 \left(N^\alpha R\right)^r \ . \tag{4.35}$$

In particular, in view of (4.14), estimate (4.25) is proved at rank $r+1$.

Define now $\mathcal{Z}^{(r+1)} := \mathcal{Z}^{(r)} + \mathcal{Z}_r$ and $f_C^{(r+1)}$:=(4.33). From (4.35), remarking that $RN^\alpha < R_{*r}$ the estimate (4.26) holds at rank $r+1$. One has

$$\epsilon := \frac{e}{\delta} \langle|\chi_r|\rangle_{s,R_r} \preceq (N^\alpha R)^{r+1} < \frac{1}{2}$$

provided $N^\alpha R$ (i.e. $R_{*(r+1)}$) is small enough. By lemma 4.14, using (4.26), (4.27) and lemma 4.18 one thus gets

$$\begin{aligned} \left\langle\left|f_C^{(r+1)}\right|\right\rangle_{s,R_r-\delta} &\preceq \sum_{l\geq 1} R^2\epsilon^l + \sum_{l\geq 1} R^2\epsilon^l \left(N^\alpha R\right)^r + \sum_{l\geq 2} R^2\epsilon^{l-1} \left(N^\alpha R\right)^r \\ &\preceq R^2 (N^\alpha R)^{r+1} \end{aligned} \tag{4.36}$$

Write now

$$f_C^{(r+1)} = f^{(r+1)} + \mathcal{R}_{r,T}$$

where $f^{(r+1)}$ is the Taylor polynomial of degree r_*+2 of $f_C^{(r+1)}$ and $\mathcal{R}_{r,T}$ is the remainder which therefore has a zero of order r_*+3 at the origin. Since $f^{(r+1)}$ is a truncation of $f_C^{(r+1)}$ it also fulfills (4.36) and therefore (4.27) at rank $r+1$. Then the remainder is estimated using Lagrange and Cauchy estimates:

$$\left\|X_{\mathcal{R}_{r,T}}\right\|_{s,R_r} \leq \frac{R^{r_*+2}}{(r_*+2)!} \sup_{\|z\|_s \leq R_{*r}/2N^\alpha} \left\|d^{r_*+2} X_{f_C^{(r+1)}}(z)\right\|$$

$$\leq R^{r_*+2} \left(\frac{2N^\alpha}{R_{*r}}\right)^{r_*+2} \sup_{\|z\|_s \leq R_{*r}/N^\alpha} \left\|X_{f_C^{(r+1)}}(z)\right\|_s \preceq (N^\alpha R)^{r_*+2}$$

Define now

$$\mathcal{R}_T^{(r+1)} := \mathcal{R}_T^{(r)} \circ \mathcal{T}_r + \mathcal{R}_{r,T} \ . \tag{4.37}$$

By lemma 4.16 one gets the the estimate (4.28) at rank $r+1$.

Concerning the terms at least cubic in $\hat{z}$, define

$$\mathcal{R}_N^{(r+1)} := \left(\mathcal{R}_N^{(r)} + f_N^{(r)}\right) \circ \mathcal{T}_r \ . \tag{4.38}$$

Using lemma 4.11 one estimates $\left\|X_{f_N^{(r)}}\right\|_{s,R_r}$. Adding the iterative estimate of $X_{\mathcal{R}_N^{(r)}}$ and estimating the effects of $\mathcal{T}_r$ by lemma 4.16 one gets (4.29) at rank $r+1$. □

As a corollary we obtain

Corollary 4.21. *For any $r_* \geq 1$ and $s \geq 1$ there exists a constant C such that for any R satisfying*

$$R \leq \frac{C}{N^\alpha}$$

the following holds true:

There exists a canonical transformation $\mathcal{T} : B_s(R) \to \mathcal{P}_s$ fulfilling

$$\|z - \mathcal{T}(z)\|_s \preceq (N^\alpha R)^2 R$$

such that the transformed Hamiltonian has the form

$$H^{(r_*)} = H_0 + \mathcal{Z}^{(r_*)} + \mathcal{R}_N + \mathcal{R}_T + \mathcal{R}_* \circ \mathcal{T} \tag{4.39}$$

where $\mathcal{Z}^{(r_*)}$ is in (γ, α, N)–normal form, and the remainders fulfill the following estimate

$$\|X_{\mathcal{R}_* \circ \mathcal{T}}\|_{s,R} \ , \ \|X_{\mathcal{R}_T}\|_{s,R} \preceq (RN^\alpha)^{r_*+2} \tag{4.40}$$

$$\|X_{\mathcal{R}_N}\|_{s,R} \preceq \frac{R^2}{N^{s-1}} \tag{4.41}$$

End of the proof of theorem 2.13. To conclude the proof we choose N and s in order to obtain that both $\mathcal{R}_T$ and $\mathcal{R}_N$ are small. First take $N = R^{-a}$ with a still undetermined a. Then in order to obtain that $\mathcal{R}_T$ is of order $R^{r_*+3/2}$ choose $a := \frac{1}{2(r_*+2)\alpha}$. Taking $s > 2\alpha r_*(r_* + 2) + 1$ also $\mathcal{R}_N$ is of the same order of magnitude. □

5 Verification of the Nonresonance Properties

5.1 Dirichlet frequencies of the wave equation

The proof follows very closely the proof of theorem 6.5 of [Bam03b] (we just simplify the last step, namely lemma 5.7). We repeat the main steps for completeness and because we will use a variant of this procedure in the next subsection. We use the notations introduced in section 3.2.1. We fix V_0 with zero average and r once for all and denote by C any constant depending only on V_0 and r.

Lemma 5.1. *For any $K \leq N$, consider K indexes $j_1 < ... < j_K \leq N$; consider the determinant*

$$D := \begin{vmatrix} \omega_{j_1} & \omega_{j_2} & \cdots & \omega_{j_K} \\ \frac{d\omega_{j_1}}{dm} & \frac{d\omega_{j_2}}{dm} & \cdots & \frac{d\omega_{j_K}}{dm} \\ \vdots & \vdots & & \vdots \\ \frac{d^{K-1}\omega_{j_1}}{dm^{K-1}} & \frac{d^{K-1}\omega_{j_2}}{dm^{K-1}} & \cdots & \frac{d^{K-1}\omega_{j_K}}{dm^{K-1}} \end{vmatrix} \tag{5.1}$$

One has

$$
\begin{aligned}
D &= \pm\left[\prod_{j=1}^{K-1}\frac{(2j-3)!}{2^{j-2}(j-2)!2^j}\right]\left(\prod_l \omega_{i_l}^{-2K+1}\right)\left(\prod_{1\le l<k\le K}(\lambda_{j_l}-\lambda_{j_k})\right) && (5.2)\\
&\ge \frac{C}{N^{2K^2}}. && (5.3)
\end{aligned}
$$

Proof. First remark that, by explicit computation, one has

$$
\frac{d^j\omega_i}{dm^j}=\frac{(2j-1)!}{2^{j-1}(j-1)!2^j}\frac{(-1)^j}{(\lambda_i+m)^{j-\frac{1}{2}}}\ . \tag{5.4}
$$

Substituting (5.4) in the l.h.s. of (5.1) we get the determinant to be computed. Factorize from the $l-th$ column the term $(\lambda_{j_l}+m)^{1/2}$, and from the $j-th$ row the term $\frac{(2j-3)!}{2^{j-2}(j-2)!2^j}$. The determinant becomes, up to the sign,

$$
\left[\prod_{l=1}^{K}\omega_{j_l}\right]\left[\prod_{j=1}^{K-1}\frac{(2j-3)!}{2^{j-2}(j-2)!2^j}\right]
\times\begin{vmatrix}
1 & 1 & 1 & . & . & . & 1\\
x_{j_1} & x_{j_2} & x_{j_3} & . & . & . & x_{j_K}\\
x_{j_1}^2 & x_{j_2}^2 & x_{j_3}^2 & . & . & . & x_{j_K}^2\\
. & . & . & . & . & . & .\\
. & . & . & . & . & . & .\\
. & . & . & . & . & . & .\\
x_{j_1}^{K-1} & x_{j_2}^{K-1} & x_{j_3}^{K-1} & . & . & . & x_{j_K}^{K-1}
\end{vmatrix} \tag{5.5}
$$

where we denoted by $x_j := (\lambda_j+m)^{-1}\equiv\omega_j^{-2}$. The last determinant is a Vandermond determinant whose value is given by

$$
\prod_{1\le l<k\le K}(x_{j_l}-x_{j_k})=\prod_{1\le l<k\le K}\frac{\lambda_{j_k}-\lambda_{j_l}}{\omega_{j_l}^2\omega_{j_k}^2}=\left(\prod_{1\le l<k\le K}(\lambda_{j_l}-\lambda_{j_k})\right)\prod_{l=1}^{K}\omega_{j_l}^{-2K}\ . \tag{5.6}
$$

Using the asymptotic of the frequencies and the fact that all the eigenvalues are different one gets also the second of (5.3). □

From [BGG85] appendix B we learn

Lemma 5.2. *Let $u^{(1)},...,u^{(K)}$ be K independent vectors with $\left\|u^{(i)}\right\|_{\ell^1}\le 1$. Let $w\in\mathbb{R}^K$ be an arbitrary vector, then there exist $i\in[1,...,K]$, such that*

$$
|u^{(i)}\cdot w|\ge\frac{\|w\|_{\ell^1}\det(u^{(1)},\dots,u^{(K)})}{K^{3/2}}\ .
$$

Combining Lemmas 5.1 and 5.2 we deduce

Corollary 5.3. *Let $w \in \mathbb{R}^\infty$ be a vector with K components different from zero, namely those with index $i_1, ..., i_K$; assume $K \le N$, and $i_1 < ... < i_K \le N$. Then, for any $m \in [m_0, \Delta]$ there exists an index $i \in [0, ..., K-1]$ such that*

$$\left| w \cdot \frac{d^i \omega}{dm^i}(m) \right| \ge C \frac{\|w\|_{\ell^1}}{N^{2K^2+2}} \tag{5.7}$$

where ω is the frequency vector.

Now we need the following lemma from [XYQ97].

Lemma 5.4. *(Lemma 2.1 of [XYQ97]) Suppose that $g(\tau)$ is m times differentiable on an interval $J \subset \mathbb{R}$. Let $J_h := \{\tau \in J \ : \ |g(\tau)| < h\}$, $h > 0$. If on J, $\left|g^{(m)}(\tau)\right| \ge d > 0$, then $|J_h| \le M h^{1/m}$, where*

$$M := 2(2 + 3 + ... + m + d^{-1}) \ .$$

For any $k \in \mathbb{Z}^N$ with $|k| \le r$ and for any $n \in \mathbb{Z}$, define

$$\mathcal{R}_{kn}(\gamma, \alpha) := \left\{ m \in [m_0, \Delta] \ : \ \left| \sum_{j=1}^N k_j \omega_j + n \right| < \frac{\gamma}{N^\alpha} \right\} \tag{5.8}$$

Applying lemma 5.4 to the function $\sum_{j=1}^N k_j \omega_j + n$ and using corollary 5.3 we get as in [Bam99b] lemma 8.4

Corollary 5.5. *Assume $|k| + |n| \ne 0$, then*

$$|\mathcal{R}_{kn}(\gamma, \alpha)| \le C(\Delta - m_0) \frac{\gamma^{1/r}}{N^\varsigma} \tag{5.9}$$

with $\varsigma = \frac{\alpha}{r} - 2r^2 - 2$.

Lemma 5.6. *Fix $\alpha > 2r^3 + r^2 + 5r$. For any positive γ small enough there exists a set $\mathcal{I}_\gamma \subset [m_0, \Delta]$ such that $\forall m \in \mathcal{I}_\gamma$ one has that for any $N \ge 1$*

$$\left| \sum_{j=1}^N k_j \omega_j + n \right| \ge \frac{\gamma}{N^\alpha} \tag{5.10}$$

for all $k \in \mathbb{Z}^N$ with $0 \ne |k| \le r$ and for all $n \in \mathbb{Z}$. Moreover,

$$|[m_0, \Delta] - \mathcal{I}_\gamma| \le C \gamma^{1/r} \ . \tag{5.11}$$

Proof. Define $\mathcal{I}_\gamma := \bigcup_{nk} \mathcal{R}_{nk}(\gamma, \alpha)$. Remark that, from the asymptotic of the frequencies, the argument of the modulus in (5.10) can be small only if $|n| \le CrN$, By (5.9) one has

$$\left| \bigcup_k \mathcal{R}_{nk}(\gamma, \alpha) \right| \le \sum_k |\mathcal{R}_k(\gamma, \alpha)| < C \frac{N^r (\Delta - m_0) \gamma^{1/r}}{N^\varsigma} \ ,$$

summing over n one gets an extra factor rN. Provided α is chosen according to the statement, one has that the union over N is also bounded and therefore the thesis holds. $\square$

Denote $\omega^{(N)} := (\omega_1, ..., \omega_N)$, then we have

Lemma 5.7. *For any γ positive and small enough, there exist a set $\mathcal{J}_\gamma$ satisfying, $|[m_0, \Delta] - \mathcal{J}_\gamma| \to 0$ when $\gamma \to 0$, and a real number α' such that for any $m \in \mathcal{J}_\gamma$ one has for $N \geq 1$*

$$\left|\omega^{(N)} \cdot k + \epsilon_1 \omega_j + \epsilon_2 \omega_l\right| \geq \frac{\gamma}{N^{\alpha'}} \tag{5.12}$$

for any $k \in \mathbb{Z}^N$, $\epsilon_i = 0, \pm 1$, $j \geq l > N$, and $|k| + |\epsilon_1| + |\epsilon_2| \neq 0$, $|k| \leq r + 2$.

Proof. The case $\epsilon_1 = \epsilon_2 = 0$ reduces to the previous lemma with $n = 0$. Consider the case $\epsilon_1 = \pm 1$ and $\epsilon_2 = 0$. In view of the asymptotic of the frequencies, the argument of the modulus can be small only if $j < 2rN$. Thus to obtain the result one can simply apply lemma 5.6 with $N' := 2rN$ in place of N and $r' := r + 2$ in place of r. This just amounts to a redefinition of the constant C in (5.11). The argument is identical in the case $\epsilon_1 \epsilon_2 = 1$.

Consider now the case $\epsilon_1 \epsilon_2 = -1$. Here the main remark is that

$$\omega_j - \omega_l = j - l + a_{jl} \text{ with } |a_{jl}| \leq \frac{C}{l} \tag{5.13}$$

So the quantity to be estimated reduces to

$$\omega^{(N)} \cdot k \pm n \pm a_{jl} \ , \quad n := j - l$$

If $l > 2CN^\alpha/\gamma$ then the a_{jl} term represents an irrelevant correction and therefore the lemma follows from lemma 5.6. In the case $l \leq 2CN^\alpha/\gamma$ one reapplies the same lemma with $N' := 2CN^\alpha/\gamma$ in place of N and $r' := r + 2$ in place of r. As a consequence one has that the nonresonance condition (5.12) holds provided $\alpha' = \alpha^2 \sim r^6$ and assuming that m is in a set whose complement has its measure estimated by a constant times $\gamma^{\frac{\alpha+1}{r}}$. $\square$

To obtain theorem 3.12 just define $\mathcal{J} := \bigcap_{r \geq 1} \bigcup_{\gamma > 0} \mathcal{J}_\gamma$ and remark that its complement is the union of a numerable infinity of sets of zero measure.

5.2 Periodic frequencies of the wave equation

We use the notations of section 3.2.2. We fix r and σ once for all, and denote by C any constant depending only on r and σ. We introduce $\tau_0 = \omega_0$ and for $j \geq 1$,

$$\gamma_j := \omega_j - \omega_{-j} \ , \quad \tau_j = \frac{\omega_j + \omega_{-j}}{2} \ .$$

We recall that from the Sturm Liouville theory (cf. [KM01]) there exists an absolute constant C_σ such that, for $V \in \mathcal{V}_R$,

$$|\lambda_j - \lambda_{-j}| \leq C_\sigma R e^{-2\sigma j} \ . \tag{5.14}$$

We are going to prove the following result:

Theorem 5.8. *Fix a positive $r \geq 2$, define $\alpha := 200r^3$. There exists a positive R and a positive b with the following property: for any $\gamma > 0$ small enough there exists a set $\mathcal{S}_\gamma \subset \mathcal{V}_R$ and a constant β such that*

i) For all $V \in \mathcal{S}_\gamma$ one has that for all $N \geq 1$ the following inequality, with $J := b \ln \frac{N}{\gamma^\beta}$, holds

$$\left| \sum_{j=1}^{N} \tau_j k_j + \sum_{j=1}^{J} \gamma_j n_j + \sum_{j=J+1}^{N} \gamma_j l_j \right| > \frac{\gamma}{N^\alpha} \tag{5.15}$$

for any $k \in \mathbb{Z}^{N+1}$, $l \in \mathbb{Z}^{N-J}$, $n \in \mathbb{Z}^J$ fulfilling $|k| + |n| \neq 0$, $|k| + |l| + |n| \leq r$.

ii) $|\mathcal{V}_R - \mathcal{S}_\gamma| \leq C\gamma^{(1/4r)}$.

Using arguments similar to those in the proof of lemma 5.7, one easily concludes that theorem 5.8 implies theorem 3.14.

The strategy of the proof of theorem 5.8 is as follow: when $n = 0$ (and then $k \neq 0$) we only move the mass m in order to move the τ_j in such a way that the l.h.s. of (5.15) is not too small. We also use that, for $V \in \mathcal{V}_R$, γ_j is exponentially small with respect to j and thus the third term in the l.h.s. of (5.15) is not relevant. In a second step we consider the case where $n \neq 0$ and we use the Fourier coefficients v_i to move the γ_j for small indexes j.

Fix $k \in \mathbb{Z}^N$, $l \in \mathbb{Z}^{N-J}$, $n \in \mathbb{Z}^J$. For any $\alpha \in \mathbb{R}$ and $\gamma > 0$ define

$$\mathcal{S}_{knl}(\gamma, \alpha) := \left\{ (m, v_k) \in \mathcal{V}_R \ : \ \left| \sum_{j=1}^{N} \tau_j k_j + \sum_{j=1}^{J} \gamma_j n_j + \sum_{j=J+1}^{N} \gamma_j l_j \right| \leq \frac{\gamma}{N^\alpha} \right\} \tag{5.16}$$

Lemma 5.9. *For $0 \neq |k| \leq r$*

$$|\mathcal{S}_{k00}(\gamma_1, \alpha_1)| < C \frac{\gamma_1^{1/r}}{N^\varsigma}$$

with $\varsigma = \frac{\alpha_1}{r} - 2r^2 - 2$.

Proof. In this case the l.h.s. of (5.15) reduces to $\sum_{j=1}^N \tau_j k_j$ and we use m to move it away from zero. To this end we follow closely the argument of subsection 5.1. So, consider the determinant

$$\left| \begin{array}{ccccc} \tau_{j_1} & . & . & . & \tau_{j_K} \\ \frac{d\tau_{j_1}}{dm} & . & . & . & \frac{d\tau_{j_K}}{dm} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{d^{K-1}\tau_{j_1}}{dm^{K-1}} & . & . & . & \frac{d^{K-1}\tau_{j_K}}{dm^{K-1}} \end{array} \right| . \tag{5.17}$$

Using the definition of τ one immediately has that such a determinant is the sum of some determinant of the form (5.1) *all with the same sign*. Therefore the modulus of the determinant (5.17) is estimated from below by a negative power of N (as the determinant (5.1)). To see that all the determinants composing (5.17) have the same sign remark that the sign of (5.1) is determined by

$$\prod_{1\leq l<k\leq K} (\lambda_{j_l} - \lambda_{j_k}) \tag{5.18}$$

Choose now arbitrarily two of the determinants composing (5.17). The corresponding products (5.18) differ only because they involve indexes with the same modulus, but different sign. Provided the potential (i.e. R) is small enough one has that the sign of $\lambda_{\pm j} - \lambda_{\pm k}$ does not depend on the choice of the $\pm$'s. Thus the result on (5.17) follows.

Then the thesis of the lemma follows from the procedure of the previous section in the same way that lemma 5.1 leads to corollary 5.5 . □

Corollary 5.10. *For each $\gamma_1 > 0$, $\alpha_1 > 0$, $R > 0$, r, N define*

$$J := \frac{1}{2\sigma} \ln \frac{rRC_\sigma N^{\alpha_1}}{\gamma_1} \tag{5.19}$$

then for any $0 \neq |k| \leq r$ and any $l \in \mathbb{Z}^{N-J}$

$$|\mathcal{S}_{k0l}(\gamma_1/2, \alpha_1)| < C\frac{\gamma_1^{1/r}}{N^\varsigma}$$

with $\varsigma = \frac{\alpha_1}{r} - 2r^2 - 2$.

Proof. From (5.14) one has

$$|\gamma_j| \leq \frac{1}{2j} C_\sigma R e^{-2\sigma j} \tag{5.20}$$

which, using the definition of J leads to

$$\left|\sum_{j=J+1}^{N} \gamma_j l_j\right| \leq \frac{C_{\mathcal{V}} R}{2J} e^{-2\sigma J} r < \frac{\gamma_1}{2N^{\alpha_1}}$$

Then for $(m, v_k) \in \mathcal{S}_{k00}(\gamma_1, \alpha_1)$,

$$\left|\sum_{j=1}^{N} \tau_j k_j + \sum_{j=J+1}^{N} \gamma_j l_j\right| > \frac{\gamma_1}{N^{\alpha_1}} - \frac{\gamma_1}{2N^{\alpha_1}} = \frac{\gamma_1}{2N^{\alpha_1}} \ .$$

□

In the case where $n \neq 0$, we need some informations on the periodic spectrum of a Sturm Liouville operator. As remarked in section 3.2.2, in the case of even potentials the periodic spectrum is the union of the Dirichlet and the Neumann spectrum. Thus the forthcoming lemma 5.13 has the following

Corollary 5.11. *For $V \in \mathcal{V}_R$ one has*

$$\frac{\partial \gamma_j}{\partial v_{2i}} = -\frac{1}{4}\frac{\tau_j}{\sqrt{\omega_j \omega_{-j}}}\delta_{ji} + \mathcal{R}_{ji} \ , \quad \frac{\partial \tau_j}{\partial v_{2i}} = -\frac{1}{8}\frac{\gamma_j}{\sqrt{\omega_j \omega_{-j}}}\delta_{ji} + \mathcal{R}'_{ji} \tag{5.21}$$

with

$$|\mathcal{R}_{ji}| \ , \ |\mathcal{R}'_{ji}| \le C\frac{Re^{-\sigma|j-i|}}{j} \ . \tag{5.22}$$

Lemma 5.12. *With the definition of J given by (5.19), there exists a positive R (independent of J, γ, α) such that for any $V \in \mathcal{V}_R$, if $n \neq 0$, $|k| + |n| + |l| \le r$ then*

$$|\mathcal{S}_{knl}(\gamma_2, \alpha_2)| < C\frac{\gamma_2}{\gamma_1^2}\frac{1}{N^{\alpha_2 - 2\alpha_1}}$$

Proof. Let $1 \le i \le J$ such that $n_i \neq 0$ and consider

$$\frac{\partial}{\partial v_{2i}}\left(\sum_{j=1}^{N}\tau_j k_j + \sum_{j=J+1}^{N}\gamma_j l_j + \sum_{j=1}^{J} n_j \gamma_j\right) \ . \tag{5.23}$$

By corollary 5.11 this quantity is given by

$$\sum_{j=1}^{N}\mathcal{R}'_{ji}k_j + \sum_{j=J+1}^{N}\mathcal{R}_{ji}l_j + \sum_{j=1}^{J} n_j \mathcal{R}_{ji} - \frac{n_i}{4}\frac{\tau_i}{\sqrt{\omega_i \omega_{-i}}} - \frac{k_i}{8}\frac{\gamma_i}{\sqrt{\omega_i \omega_{-i}}} \ . \tag{5.24}$$

From (5.22) one has (noticing that $\frac{e^{-\sigma|i-j|}}{j} \le \frac{C}{i}$ for all $i, j \ge 1$)

$$\left|\mathcal{R}'_{ji}\right| \ , \ |\mathcal{R}_{ji}| < \frac{CR}{i} \ .$$

Combining this estimate with (5.20) one gets that the modulus (5.24) is estimated from below by

$$\frac{1}{i}(\frac{n_i}{8} - CRr) > \frac{1}{16J}$$

taking $R = 1/16Cr$. It follows that excising from the domain of v_{2i} a segment of length $\gamma_2 16J/N^{\alpha_2}$, whose normalized measure is estimated by

$$\frac{\gamma_2 16J}{N^{\alpha_2}}\frac{e^{2J\sigma}}{R} \le \frac{16\gamma_2}{N^{\alpha_2}R}e^{4J\sigma} = C\frac{\gamma_2}{\gamma_1^2}\frac{1}{N^{\alpha_2 - 2\alpha_1}}$$

one fulfills the nonresonance condition. □

Proof of theorem 5.8 Fix α as in the statement and a positive γ. Define $\alpha_1 :=$

$\alpha/4$, $\alpha_2 := \alpha$, $\gamma_1 := \gamma^{1/4}$, $\gamma_2 := \gamma$. Take J as in (5.19). Define

$$\mathcal{S}_{\gamma,N} := \bigcup_{knl} \mathcal{S}_{knl}(\gamma, \alpha) \tag{5.25}$$

$$= \left[\bigcup_{n\neq 0} \mathcal{S}_{knl}(\gamma, \alpha) \right] \bigcup \left[\bigcup_{k\neq 0} \mathcal{S}_{k0l}(\gamma, \alpha) \right] \tag{5.26}$$

$$\subset \left[\bigcup_{n\neq 0} \mathcal{S}_{knl}(\gamma_2, \alpha_2) \right] \bigcup \left[\bigcup_{k\neq 0} \mathcal{S}_{k0l}(\gamma_1, \alpha_1) \right] \tag{5.27}$$

thus one can use corollary 5.10 and lemma 5.12 to get the estimate

$$|\mathcal{S}_{\gamma,N}| \leq CN^r \left(\frac{\gamma_1^{1/r}}{N^{\varsigma}} + \frac{\gamma_2}{\gamma_1^2} \frac{1}{N^{\alpha_2 - 2\alpha_1}} \right) \tag{5.28}$$

Inserting the definitions of the various parameters one has that both $\alpha_2 - 2\alpha_1$ and ς are bigger than $r + 2$ and thus one can define

$$\mathcal{S}_\gamma := \bigcup_N \mathcal{S}_{\gamma,N}$$

and estimate its measure by the sum over N of the r.h.s. of (5.28) getting a convergent series and the result. □

5.3 Frequencies of the 1-d NLS

Denote by λ_j the Dirichlet eigenvalues of $-\partial_{xx} + V$.

Lemma 5.13. *For any j and k and any $V \in \mathcal{V}_R$, with R small enough, one has*

$$\frac{\partial \lambda_j}{\partial v_k}(V) = -\delta_{j,2k}\frac{1}{2} + \mathcal{R}_{jk} \tag{5.29}$$

with

$$|\mathcal{R}_{jk}| \leq CRe^{-\sigma ||j|-2k|} \ .$$

For the Neumann eigenvalues the formula (5.29) without the minus sign holds. Moreover, for any $j > k, l$

$$\left.\frac{\partial^2 \lambda_j}{\partial v_k \partial v_l}\right|_{V=0} = -\delta_{kl}\frac{1}{2}\frac{1}{(2j)^2 - k^2} = -\delta_{kl}\frac{1}{8}\sum_{s\geq 1}\frac{(k^2)^{s-1}}{2^{s-1}j^{2s}} \ . \tag{5.30}$$

Proof. Denote $L_0 := -\partial_{xx} + V$ with Dirichlet boundary conditions (all what follows holds also for the case of Neumann boundary conditions). Fix j and let $\lambda = \lambda(V)$ be the j–th eigenvalue of L_0. As the Dirichlet spectrum is simple, the function $V \mapsto \lambda(V)$ is smooth and admit a Taylor expansion at any order. In particular we are interested in computing such a Taylor expansion up to second order, i.e. in computing $d\lambda$ and $d^2\lambda$. The idea is to fix a potential h and to construct iteratively, by Lyapunof–Schmidt method, the expansion of $\lambda(V+\epsilon h)$. Thus one will get

$$\lambda(V+\epsilon h) = \lambda(V) + \epsilon d\lambda(V)h + \frac{\epsilon^2}{2}\langle d^2\lambda(V); h, h\rangle + ...$$

which obviously allows to compute $d\lambda$, but also $d^2\lambda$ which is the bilinear form associated to the quadratic form $\langle d^2\lambda(V); h, h\rangle$.

So consider the eigenvalue equation

$$(L_0 + \epsilon h)\varphi = \lambda\varphi \tag{5.31}$$

and formally expand φ and λ in power series:

$$\varphi = \sum_{l\geq 0} \epsilon^l \varphi_l \ , \quad \lambda = \sum_{l\geq 0} \epsilon^l \lambda_l \ ,$$

inserting in (5.31) and equating terms of equal order one gets

$$L_0\varphi_0 \ = \ \lambda_0\varphi_0 \tag{5.32}$$

$$(L_0 - \lambda_0)\varphi_l \ = \ \psi_l + \lambda_l\varphi_0 \tag{5.33}$$

where

$$\psi_l := -h\varphi_{l-1} + \sum_{l_0=1}^{l} \lambda_{l_0}\varphi_{l-l_0} \ .$$

Decompose L^2 =span$\varphi_0\oplus$(span$\varphi_0)^\perp$=Ker$(L_0-\lambda_0)\oplus$Range $(L_0-\lambda_0)$, and let P be the projector on the orthogonal to spanφ_0. Taking φ_0 normalized in L^2, eq. (5.33) turns out to be equivalent to

$$\lambda_l := -\langle\psi_l; \varphi_0\rangle_{L^2} = \langle\varphi_{l-1}; \varphi_0\rangle_{L^2} \tag{5.34}$$

$$\varphi_l := (L_0 - \lambda_0)^{-1} P\psi_l \ . \tag{5.35}$$

By taking $V \in \mathcal{V}_R$, $h = \cos(kx)$ and using (S1) one gets (5.29). By taking $V = 0$ and $h = \mu_1\cos(kx) + \mu_2\cos(lx)$ (and developing the computations) one gets (5.30) □

From [Mar86] (theorem 1.5.1 p. 71) we learn that for each $\rho > 0$, the Dirichlet spectrum of $-\partial_{xx} + V$ admits an asymptotic expansion of the form

$$\omega_j = j^2 + c_0(V) + c_1(V)j^{-2} + \ldots + c_\rho(V)j^{-2\rho} + C_\rho(V)j^{-2\rho-2} \tag{5.36}$$

where $c_0(V) = \int_0^{2\pi} V(x)dx$ and the $c_s(V)$ are certain multilinear expressions in V which depend smoothly on each of its Fourier coefficients (this is the only property we need). Moreover one has $|C_\rho(V)| < C(\rho)R$ for all potentials in $\mathcal{V}_R$ (here $C(\rho)$ denote a constant depending on ρ).

Corollary 5.14.

$$\left.\frac{\partial^2 c_s}{\partial v_k \partial v_l}\right|_{V=0} = \delta_{kl}\frac{1}{4}\frac{k^{2(s-1)}}{2^s}. \tag{5.37}$$

Proof. Just take the second derivative of equation (5.36) and compare with (5.30). □

It follows that the 'constants' c_s have an expansion at the origin of the form

$$c_s(v_1, v_2, ...) = \sum_{k\geq 1} A_{sk} v_k^2 + w_s(v_1, v_2, ...) \tag{5.38}$$

where the matrix A_{sk} is given by the r.h.s of (5.37) (without the Kronecker symbol) and the functions w_s have a zero of third order at the origin. In particular there exists for each $k \geq 1$ a constant $C(k)$ such that

$$|w_k(v_1, v_2, ...)| \leq C(k) R^3 \ . \tag{5.39}$$

Remark 5.15. For any ρ, the matrix

$$A := (A_{sk})_{k=1,\dots,\rho}^{s=1,\dots,\rho} \ ,$$

is invertible and there exists a constant C that depends only on ρ such that

$$\left\|A^{-1}\right\| \leq C \ . \tag{5.40}$$

(Indeed remark that the determinant of A is a non vanishing Vandermonde determinant.)

In the remainder part of this section we fix once for all $r \geq 1$. We begin with the following simple lemma:

Lemma 5.16. *Fix $\rho \geq l \geq 1$ two integers. For any $J \geq 2$ and any positive μ consider an arbitrary collection of l indexes*

$$J \leq j_1 < j_2 < ... < j_l \leq J^{\mu} \ , \tag{5.41}$$

and define the matrix

$$B = (B_{is})_{i=1,\dots,l}^{s=1,\dots,\rho} \ , \quad B_{is} := \frac{1}{j_i^{2s}} \ . \tag{5.42}$$

There exists an $l \times l$ submatrix B_l which is invertible and fulfills

$$\left\|B_l^{-1}\right\| \leq C J^{\beta(\mu,l)} \tag{5.43}$$

with $\beta(\mu, l) := \frac{3}{2}\mu l(l-1)$ and $C \equiv C(l)$ is independent of J and of the choice of the indexes.

Proof. Define the submatrix $B_l = (B_{is})_{i=1,\dots,l}^{s=1,\dots,l}$, notice that all the coefficients of the comatrix of B are smaller than l^2 and therefore

$$\left\|B_l^{-1}\right\|_{\infty} \leq l^2 (\det\, B_l)^{-1} \ .$$

On the other hand, the determinant of B_l is a Vandermonde determinant whose value is given by

$$\prod_{1\leq i<k\leq l} \left(\frac{1}{j_i^2} - \frac{1}{j_k^2}\right) \prod_{1\leq k\leq l} \frac{1}{j_k^2} \ .$$

By the limitation (5.41) each term of the first product is estimated from below by $J^{-3\mu}$. Since the number of pairs is $l(l-2)/4$ one has the result. □

Lemma 5.17. *Given $J \geq 2$ and $\mu > 1$ consider an arbitrary collection of indexes*

$$J \leq j_1 < j_2 < ... < j_l \leq J^\mu \ , \ l \leq r \tag{5.44}$$

Let $(a_{j_i})_{i=1,...,l}$ be a vector with components in $\mathbb{Z}$ fulfilling

$$a \neq 0 \quad , \sum_i |a_{j_i}| \leq r \ , \ \sum_i j_i^2 a_{j_i} = 0 \ . \tag{5.45}$$

For any positive Γ and $\alpha > 2\beta(\mu, l)$ define

$$\mathcal{R} := \left\{ V \in \mathcal{V}_R \ : \ \left| \sum_i \lambda_{j_i} a_{j_i} \right| \leq \frac{\Gamma}{J^\alpha} \right\} \ . \tag{5.46}$$

For R small enough, there exists two constants $C_1(r, \alpha)$ and $C_2(r, \alpha)$ such that, provided

$$J > \frac{C_1}{\Gamma^{1/2}} \tag{5.47}$$

one has

$$|\mathcal{R}| \leq C_2 \frac{\Gamma^{1/2}}{J^{\frac{\alpha}{2} - \beta}}$$

where $\beta \equiv \beta(\mu, l)$ is defined in the preceding lemma.

Proof. In order to fix ideas take $l = r$ the other case being similar. Consider

$$\sum_i \lambda_{j_i} a_{j_i} = \sum_{i=1}^r a_{j_i} \sum_{s=1}^\rho \frac{c_s}{j_i^{2s}} + \sum_{i=1}^r a_{j_i} \frac{C_\rho(V)}{j_i^{2\rho+2}}$$

The second term at r.h.s. is bounded by $\frac{RrC(\rho)}{J^{2\rho+2}}$, thus if $R \leq 1$, $\rho = \alpha/2$ and (5.47) is satisfied, it can be neglected and it remains to estimate the measure of the set of potentials such that

$$\left| \sum_{i=1}^r a_{j_i} \sum_{s=1}^\rho \frac{c_s}{j_i^{2s}} \right| < \frac{\Gamma}{2J^\alpha} \ . \tag{5.48}$$

We will show that there exists $\bar{k} \leq \rho$ such that the second derivative of (5.48) with respect to $v_{\bar{k}}$ is bounded away from zero, and we will apply lemma 5.4. One has

$$\sum_{i=1}^r a_{j_i} \sum_{s=1}^\rho \frac{c_s}{j_i^{2s}} = \sum_{i=1}^r a_{j_i} \sum_{s=1}^\rho \sum_{k=1}^\rho \frac{A_{sk} v_k^2}{j_i^{2s}} + \sum_{i=1}^r a_{j_i} \sum_{s=1}^\rho \frac{w_s}{j_i^{2s}} + b \tag{5.49}$$

where

$$b := \sum_{i=1}^r a_{j_i} \sum_{s=1}^\rho \sum_{k > \rho} \frac{A_{sk} v_k^2}{j_i^{2s}}$$

is independent of v_k, $k = 1, ..., \rho$. Define $\tilde{w}_k$ by

$$w_s = \sum_{k=1}^{\rho} A_{ks} \tilde{w}_k \ ,$$

so that

$$|\tilde{w}_k| \leq CR^3$$

with a constant depending only on ρ (cf. (5.39) and (5.40)). So, (5.49) can be written as

$$\sum_{k=1}^{\rho} (v_k^2 + \tilde{w}_k) \sum_{s=1}^{\rho} A_{ks} \sum_{i=1}^{r} B_{ij} a_{j_i} + b = \sum_{k=1}^{\rho} (v_k^2 + \tilde{w}_k) f_k + b \tag{5.50}$$

where B is given by (5.42) and f_k is defined as the coefficient of the bracket at l.h.s. Clearly one has $f = ABa$. Now, the image of $\mathbb{R}^r$ under AB is an r dimensional space (due to remark 5.15 and lemma 5.16) and AB is an isomorphism between such r dimensional spaces. By remark 5.15 and lemma 5.16 its inverse $(AB)^{-1}$ is bounded by

$$\left\| (AB)^{-1} \right\| \leq CJ^{\beta} \ ,$$

(with C independent of J), and therefore one has

$$\|f\| \geq \frac{1}{CJ^{\beta}} \|a\|$$

It follows that there exists $\bar{k}$ with $|f_{\bar{k}}| \geq \frac{1}{CJ^{\beta}}$.

Consider now the second derivative of (5.50) with respect to $v_{\bar{k}}$, it is given by

$$f_{\bar{k}} \left(1 + \frac{\partial^2 \tilde{w}_k}{\partial v_{\bar{k}}^2} \right) \ . \tag{5.51}$$

Provided

$$\left| \frac{\partial^2 \tilde{w}_k}{\partial v_{\bar{k}}^2} \right| < \frac{1}{2}$$

which is a smallness assumption on R (independent of J), (5.51) is larger than $\frac{1}{CJ^{\beta}}$. Applying lemma 5.4 one gets the thesis. □

In the next lemme, we relax the hypothesis (5.44) of the previous lemma.

Lemma 5.18. *There exists R and a positive constant C with the following property: for any positive Γ there exists a set $\mathcal{S}_1 \subset \mathcal{V}_R$ such that*

i) denote $\alpha := 4^r (r!)^2$, and let J be an integer fulfilling

$$J > \frac{C}{\Gamma^{1/2}} \tag{5.52}$$

then for any $V \in \mathcal{S}_1$ one has for $N \geq J$

$$\left| \sum_{i=1}^{r} \lambda_{j_i} a_{j_i} \right| \geq \frac{\Gamma}{N^\alpha} \tag{5.53}$$

for any choice of integers $J \leq j_1, \ldots, j_r \leq N$ and of relative integers $\{a_{j_i}\}_{i=1}^r$ fulfilling

$$0 \neq \sum_i |a_{j_i}| \leq r \ .$$

ii) $\mathcal{S}_1$ has large measure, namely

$$|\mathcal{V}_R - \mathcal{S}_1| \leq C\Gamma^{1/2} \tag{5.54}$$

Proof. We prove the lemma by induction on r. More precisely we prove that for any $l \leq r$ there exists a set $\mathcal{N}_l$ and two constants $\gamma_l, \alpha(l)$ such that

$$\left| \sum_{i=1}^{l} \lambda_{j_i} a_{j_i} \right| \geq \frac{\gamma_l}{J^{\alpha(l)}} \tag{5.55}$$

for any $\mathcal{V} \in \mathcal{N}_l$ and moreover the measure estimate holds. The statement is true for $l = 1$. Suppose it is true for $l-1$, we prove it for l. Assume that all the integers $a_{j_1}, ..., a_{j_l}$ are different from zero, otherwise the result follows by the inductive assumption. Order the indexes in increasing order. First remark that if $\sum_{i=1}^{l} j_i^2 a_{j_i} \neq 0$ then $\left|\sum_{i=1}^{l} j_i^2 a_{j_i}\right| \geq 1$. Hence, since the potential is small (for R small enough) the l.h.s. of (5.55) is larger than $1/2$ and the result holds in this particular case. Thus we consider only the case where $\sum_{i=1}^{l} j_i^2 a_{j_i} = 0$.

Now, since

$$\left|\lambda_{j_i} - j_i^2\right| \leq \frac{C}{j_i^2} \ ,$$

if we assume that $j_l^2 > \frac{2rCJ^{\alpha(l-1)}}{\gamma_{l-1}}$, then we deduce from the inductive assumption that

$$\left| \sum_{i=1}^{l} \lambda_{j_i} a_{j_i} \right| \geq \left| \sum_{i=1}^{l-1} \lambda_{j_i} a_{j_i} \right| - \frac{rC}{j_l^2} \geq \frac{\gamma_{l-1}}{2J^{\alpha(l-1)}} \ .$$

and thus, provided $\gamma_l \leq \frac{\gamma_{l-1}}{2}$ and $\alpha_l \geq \alpha_{l-1}$, the lemma holds. So, assuming

$$J > \frac{C(r)}{\gamma_{l-1}^{1/2}} \ , \tag{5.56}$$

we have only to consider the case $j_l < J^{\mu_l}$ where $\mu_l = [\alpha(l-1)+2]/2$.

We now apply lemma 5.17. The number of possible choices for the integers $a_{j_1}, \ldots, a_{j_l}$ is smaller than $J^{l\mu_l}$.Therefore, the measure of the set of potentials that we have to exclude at the l-th step can be estimated by

$$\frac{C\gamma_l^{1/2} J^{l\mu_l}}{J^{\frac{\alpha(l)}{2} - \beta(\mu_l, l)}}$$

Thus in order to obtain the result it suffices to choose α_l in such way that

$$\alpha(l) > \alpha(l-1)(3l^2 + l)$$

which amounts to the requirement of a sufficiently fast growth of $\alpha(l)$. The choice $\alpha(l) = 4^l (l!)^2$ fulfills this requirement. On the other hand, choosing $\gamma_r = \Gamma$, $\gamma_l = \gamma_{l-1}/2$ for $l = 2, \ldots, r$ and assuming (5.52) with a suitable constant C, (5.56) is satisfied for each $l = 2, \ldots, r$. This gives the statement of the lemma with a suitable constant in (5.54) and with J in place of N in (5.53). As $J \leq N$, the lemma is proved. □

We now take into account the small indexes ($j < J$):

Lemma 5.19. *There exists $R > 0$ with the following properties: for any $\gamma > 0$ and $J > 0$ fulfilling*

$$\frac{\gamma e^{2\sigma J}}{R} < 1 \tag{5.57}$$

there exists a set $\mathcal{S}_J \subset \mathcal{V}_{R_*}$ such that

i) $\forall V \in \mathcal{S}_J$ *one has*

$$\left| \sum_{j=1}^{J} \lambda_j k_j + \sum_{j=J+1}^{N} \lambda_j a_j \right| \geq \frac{\gamma}{N^{r+2}} \ , \ \forall N \geq J \tag{5.58}$$

and for all $k \in \mathbb{Z}^J$, $a \in Z^{N-J}$ fulfilling $|k| + |a| \leq r$, $k \neq 0$.

ii) $|\mathcal{V}_{R_*} - \mathcal{S}_J| \leq \dfrac{\gamma e^{2\sigma J}}{R}$.

Proof. To start with fix k and a. We estimate the measure of the set such that (5.58) is violated. Let $i \leq J$ such that $k_i \neq 0$. Take the derivative of the argument of the modulus in (5.58) with respect to v_{2i}. By (5.29) this is given by $\frac{-1}{2} k_i + \mathcal{R}$ with $|\mathcal{R}| < CR < 1/4$ provided R is small enough. It has to be remarked that, here, C does not depend on J, a, γ but only on r. Thus the measure of the set such that (5.58) is violated for the given choice of (k, a) is estimated by $\frac{\gamma}{4N^{r+2}} \frac{e^{2\sigma J}}{R}$. Summing over all the possible choices of k and a (namely N^r) and summing over N, one gets the thesis. □

Combining the last two lemmas one gets

Theorem 5.20. *Fix $r \geq 1$ and $\alpha = 4^r (r!)^2$. There exists R with the following property: $\forall \gamma > 0$ there exists a set $\mathcal{S} \subset \mathcal{V}_R$ such that*

i) for any $V \in \mathcal{S}$ one has for $N \geq 1$

$$\left| \sum_{j=1}^{N} \lambda_j k_j \right| \geq \frac{\gamma}{N^\alpha} \ , \tag{5.59}$$

for all $k \in \mathbb{Z}^N$ with $0 \neq |k| \leq r$.

ii) $|\mathcal{V}_R - \mathcal{S}| \leq \left[C \ln\left(\frac{R^2}{\gamma}\right)\right]^{-1/2}$.

Proof. With a slight change of notation write (5.59) in the form

$$\left|\sum_{j=1}^{J} \lambda_j k_j + \sum_{j=J+1}^{N} \lambda_j a_j\right| \geq \frac{\gamma}{N^\alpha} \tag{5.60}$$

with a still undetermined J. To begin with consider the case $k = 0$ and apply lemma 5.18 with a still undetermined Γ. In order to fulfill (5.52) we choose now $J := \frac{2C}{\Gamma^{1/2}}$. It follows that (5.60) holds for any $\gamma \leq \Gamma$ provided V belongs to a set $\mathcal{S}_1$ whose measure is estimated by $C\Gamma^{1/2}$. If $k \neq 0$ then assuming

$$\frac{\gamma e^{2\sigma J}}{R} = \frac{\gamma e^{\frac{2\sigma C}{\Gamma^{1/2}}}}{R} < 1$$

with $R \equiv R(r)$ chosen as in lemma 5.19, one can apply lemma 5.19 and (5.60) is still satisfied.

Now we choose Γ in such a way that

$$\frac{\gamma e^{\frac{2\sigma C}{\Gamma^{1/2}}}}{R} = \gamma^{1/2}$$

namely

$$\Gamma^{1/2} = \frac{4\sigma C}{\ln(\frac{R^2}{\gamma})} \ . \tag{5.61}$$

It follows that $\gamma < \Gamma$ and therefore (5.60) is true in $\mathcal{S}_J \cap \mathcal{S}_1$, whose complement is estimated by twice the r.h.s. of (5.54) with Γ given by (5.61). □

Finally, to deduce theorem 3.18 from theorem 5.20 just adapt the proof of lemma 5.7. To this end, instead of (5.13) use that, due to the asymptotic of the frequencies, $\omega_j - \omega_l > l/2$.

A Technical Lemmas

A.1 Proof of lemma 4.11.

Introduce the projector $\bar{\Pi}$ on the modes with index smaller than N and the projector $\hat{\Pi}$ on the modes with larger index. Expand f in Taylor series (in all the variables), namely write

$$f = \sum_{j \leq r_* + 2} f_j$$

with f_j homogeneous of degree j. Consider the vector field of f_j and decompose it into the component on $\bar{z}$ and the component on $\hat{z}$. One has:

$$\bar{\Pi} X_{f_j} = J_{\bar{z}} \nabla_{\bar{z}} f_j \ , \tag{A.1}$$

$$\hat{\Pi} X_{f_j} = J_{\hat{z}} \nabla_{\hat{z}} f_j \ , \tag{A.2}$$

where we denoted by $J_{\bar{z}}$ and $J_{\hat{z}}$ the two components of the Poisson tensor. From(A.1,A.2) one immediately realises that $\bar{\Pi} X_{f_j}$ has a zero of order three as a function of $\hat{z}$ and $\hat{\Pi} X_{f_j}$ has a zero of order two as a function of $\hat{z}$. Consider $\hat{\Pi} X_{f_j}$, write $z = \bar{z} + \hat{z}$, one has

$$\hat{\Pi} X_{f_j}(\bar{z} + \hat{z}) = \hat{\Pi} \widetilde{X}_{f_j}(\underbrace{\bar{z} + \hat{z}, ..., \bar{z} + \hat{z}}_{(j-1)\text{–times}}) \tag{A.3}$$

$$= \sum_{l=2}^{j-1} \binom{j-1}{l} \hat{\Pi} \widetilde{X}_{f_j}(\underbrace{\hat{z}, ..., \hat{z}}_{l\text{–times}}, \underbrace{\bar{z}, ..., \bar{z}}_{(j-l-1)\text{–times}})$$

where the sum starts from 2 since $\hat{\Pi} X_{f_j}$ has a zero of order two as a function of $\hat{z}$. We estimate now a single term of the sum. By the tame property we have

$$\left\| \hat{\Pi} \widetilde{X}_{f_j}(\underbrace{\hat{z}, ..., \hat{z}}_{l\text{–times}}, \underbrace{\bar{z}, ..., \bar{z}}_{(j-l-1)\text{–times}}) \right\|_s \tag{A.4}$$

$$\leq |X_{f_j}|_s^T \frac{1}{j-1} \left[\sum_{i=1}^{l} \|\hat{z}\|_1^{l-1} \|\hat{z}\|_s \|\bar{z}\|_1^{j-1-l} + \sum_{i=l+1}^{j-1} \|\hat{z}\|_1^{l} \|\bar{z}\|_s \|\bar{z}\|_1^{j-2-l} \right]$$

Using the inequalities

$$\begin{aligned} \|\hat{z}\|_1 &\leq \frac{\|\hat{z}\|_s}{N^{s-1}} , \\ \|\hat{z}\|_s &\leq \|z\|_s , \quad \|\bar{z}\|_s \leq \|z\|_s \\ \|z\|_1 &\leq \|z\|_s \end{aligned}$$

which immediately follow from the definition of the norms one can estimate the quantity (A.4) by

$$\frac{1}{N^{(s-1)(l-1)}} \|z\|_s^{j-1}$$

Inserting into (A.3) one gets for $z \in B_s(R)$

$$\left\| \hat{\Pi} X_{f_j}(z) \right\|_s \leq 2^{j-1} \frac{R^{j-1}}{N^{s-1}} |X_{f_j}|_s^T .$$

A similar estimate with $N^{2(s-1)}$ instead of N^{s-1} holds for $\bar{\Pi} X_{f_j}(z)$. Therefore

$$\left\| X_{f_j}(z) \right\|_s \leq 2^j \frac{R^{j-1}}{N^{s-1}} |X_{f_j}|_s^T$$

and summing over j one gets the thesis. □

A.2 Proof of lemma 4.12

One has

$$X_{\{f,g\}} = [X_f, X_g] = dX_f X_g - dX_g X_f$$

Denote $X := X_f$, $Y := X_g$. Write

$$X(z) = \sum_{k,l_1,...l_n} X_k^{l_1,...,l_n} \mathbf{e}_k z_{l_1}...z_{l_n}$$

and similarly for Y, then one has

$$\lfloor [X,Y] \rceil (z)$$

$$= \sum_{\substack{k,l_1,...l_n \\ j_1,...j_m}} \left| nX_k^{l_1,...,l_n} Y_{l_n}^{j_1,...,j_m} - mX_{l_n}^{l_1,...,l_{n-1},j_m} Y_k^{j_1,...,j_{m-1},l_n} \right| \mathbf{e}_k z_{l_1}...z_{l_{n-1}} z_{j_1}...z_{j_m}$$

After symmetrization with respect to the indexes $l_1, ..., l_{n-1}, j_1, ..., j_m$, the quantities in the above modulus are the components of a multilinear symmetric form $\widetilde{\lfloor [X,Y] \rceil}$. Consider also the multilinear form with components obtained by symmetrizing the quantities

$$\sum_{l_n} \left(\left| nX_k^{l_1,...,l_n} Y_{l_n}^{j_1,...,j_m} \right| + \left| mX_{l_n}^{l_1,...,l_{n-1},j_m} Y_k^{j_1,...,j_{m-1},l_n} \right| \right) \tag{A.5}$$

namely the form $d\,\widetilde{\lfloor X \rceil \lfloor Y \rceil} + d\,\widetilde{\lfloor Y \rceil \lfloor X \rceil}$. Let $w = (z^{(1)}, ..., z^{(n+m-1)})$ be a multivector with all the vectors $z^{(j)}$ chosen in the positive octant (see remark 4.5). Then the value of $\widetilde{\lfloor [X,Y] \rceil}$ on w is bounded by the value of $d\,\widetilde{\lfloor X \rceil \lfloor Y \rceil} + d\,\widetilde{\lfloor Y \rceil \lfloor X \rceil}$ on the same multivector and finally it suffices to estimate $d\,\widetilde{\lfloor X \rceil \lfloor Y \rceil}$.

The value of $d\,\widetilde{\lfloor X \rceil \lfloor Y \rceil}$ on the multivector w is given by

$$n \sum_\sigma \frac{1}{(n+m-1)!} \widetilde{\lfloor X \rceil} \left(z^{(\sigma(1))}, ..., z^{(\sigma(n-1))}, \widetilde{\lfloor Y \rceil}(z^{(\sigma(n))}, ..., z^{(\sigma(n+m-1))}) \right) \tag{A.6}$$

where σ are here all the permutations of the first $n+m-1$ integers.

Consider one of the terms in the last sum, say the one corresponding to the identical permutation. One has

$$\left\| \widetilde{\lfloor X \rceil} \left(z^{(1)}, ..., z^{(n-1)}, \widetilde{\lfloor Y \rceil}(z^{(n)}, ..., z^{(n+m-1)}) \right) \right\|_s \leq | \lfloor X \rceil |_s^T | \lfloor Y \rceil |_s^T \sum_{l=1}^{n+m-1} c_l y_l \ , \tag{A.7}$$

where

$$y_l := \left\| z^{(1)} \right\|_1 ... \left\| z^{(l-1)} \right\|_1 \left\| z^{(l)} \right\|_s \left\| z^{(l+1)} \right\|_1 ... \left\| z^{(n+m-1)} \right\|_1$$

and

$$\begin{aligned} c_l &:= \frac{1}{n} \ , \qquad \text{if} \quad l = 1, ..., n-1 \\ c_l &:= \frac{1}{mn} \ , \quad \text{if} \quad l = n, ..., n+m-1 \ . \end{aligned}$$

Now the important property is that $\sum_l c_l = 1$. Consider now the norm of the sum in (A.6), it is clear that it is again estimated by an expression of the form of the r.h.s. of (A.7) with suitable constants c'_l. To compute the constants c'_l remark that, due to the symmetry of the expression all the coefficients must be equals. Moreover, the symmetrization does not change the property that the sum of the coefficients is 1, and therefore one has $c'_l = 1/(n+m-1)$ for all l's. So, in conclusion one has

$$\left\| d\, \widetilde{\lfloor X \rceil \lfloor Y \rceil}(w) \right\|_s \leq n |\lfloor X \rceil|_s^T |\lfloor Y \rceil|_s^T \|w\|_{s,1} \ ,$$

from which one gets the thesis. □

B On the verification of the tame modulus property

It is clear that the linear combination of polynomials with tame modulus still has tame modulus. The tame modulus property is also stable by composition:

Lemma B.1. *Let $X : \mathcal{P}_s \to \mathcal{P}_s$ and $Y : \mathcal{P}_s \to \mathcal{P}_s$ be two polynomial vector fields with s-tame modulus, then also their composition $X \circ Y$ has s-tame modulus.*

Proof. Just remark that for any multivector w with all components in the positive octant one has

$$\left\| \widetilde{\lfloor X \circ Y \rceil}(w) \right\|_s \leq \left\| \widetilde{\lfloor X \rceil \circ \lfloor Y \rceil}(w) \right\|_s$$

and that the composition of tame maps is still a tame map, so that

$$\left\| \widetilde{\lfloor X \circ Y \rceil}(w) \right\|_s \leq C_s \|w\|_{1,s}$$

with a suitable C_s. □

Now the idea is that if one is able to show that some elementary maps have tame modulus(think of examples 2.9, 2.10), then the same is true for the maps obtained by composing them.

We have already seen (cf example 2.4) that bounded linear maps are tame. Proposition B.2 below gives a simple condition to ensure that a linear map has tame modulus for all $s \geq 1$. Such a condition will be very useful in studying the behaviour of the T_M property under change of basis in $\mathcal{P}_s$. In the following proposition, given a linear operator A, we will consider its matrix defined by $Az = \sum_{kl} A_{kl} z_l \mathbf{e}_k$. We will simply write $A = (A_{kl})$.

Proposition B.2. *Let a be an injective and surjective map from $\bar{\mathbb{Z}}$ to $\bar{\mathbb{Z}}$ with the property that there exists $C > 0$ such that*

$$\frac{|l|}{C} \leq |a(l)| \leq C|l| \qquad \text{(B.1)}$$

and let $A = (A_{kl})$ be a linear operator. If for any n there exists a constant C_n such that

$$|A_{kl}| \leq \frac{C_n}{(1 + |k - a(l)|)^n} \tag{B.2}$$

then A has tame modulus.

Proof. First we get rid of the map a by reordering the basis in the space $\mathcal{P}_s$ constituting the domain of A. Namely we take as l-th element of the basis $\mathbf{e}_{a(l)}$. This has the consequence of changing the norm in the domain; due to (B.1) the new norm is equivalent to the old one. So we consider only the case where a is the identity.

Fix $s \geq 1$, to prove that A has s-tame modulus it suffices to prove that $|A_{kl}|$ are the matrix elements of an operator which is bounded as an operator from $\mathcal{P}_s$ into itself. This is clearly equivalent to the fact that $D_{kl} := |A_{kl}||k|^s|l|^{-s}$ is bounded from $\mathcal{P}_0$ into itself. Remark that, since

$$\frac{|k|}{(1 + |k - l|)|l|} \leq 1 \ ,$$

in view of (B.2), for any $s \geq 1$ there still exists a constant C'_s such that

$$D_{kl} \leq \frac{C'_s}{(1 + |k - l|)^2} \ . \tag{B.3}$$

Thus, using Schwartz inequality, one has

$$\begin{aligned} \|Dz\|_0^2 = \sum_k \left(\sum_l D_{kl} z_l \right)^2 &\leq \sum_k \left[\left(\sum_l D_{kl} \right) \left(\sum_l D_{kl} z_l^2 \right) \right] \\ &= K_s \sum_l |z_l|^2 \left(\sum_k D_{kl} \right) = K_s^2 \, \|z\|_0^2 \end{aligned}$$

where we used the inequality (B.3) and we defined

$$K_s := \sum_k \frac{C'_s}{(1 + |k - l|)^2} \ .$$

□

In the particular case of linear *canonical* maps one has

Proposition B.3. *Let $A^{(1)} ..., A^{(m)}$ be linear operators fulfilling that for any n there exists a constant C_n such that*

$$|A^{(j)}_{kl}| \leq \frac{C_n}{(1 + |k - a_j(l)|)^n} \tag{B.4}$$

where the maps a_j have the property (B.1). *Let $A := A^{(1)} + ... + A^{(m)}$ be a canonical map (which by the proposition B.2 has tame modulus) then also A^{-1} has tame modulus.*

Proof. Denote by J the Poisson tensor, namely the operator $z \equiv (p_k, q_k) \mapsto J(p_k, q_k) := (-q_k, p_k)$, which clearly has tame modulus. Then, by canonicity one has

$$A^{-1} = -JA^T J = (-JA^{(1)T} J - ... - JA^{(m)T} J) \ .$$

But $A^{(j)T}$ still fulfills (B.4) with the roles of k and l exchanged, therefore it has tame modulus. It follows that the above expression has tame modulus. □

Finally concerning changes of coordinates one has the following

Corollary B.4. *Let $z = Aw$ be a linear canonical transformation with the same structure as in proposition B.3. If $f \in T^s_M$ is a polynomial function with s-tame modulus, then also the transformed function $f \circ A$ has s-tame modulus.*

Proof. Just remark that $X_{f\circ A} = A^{-1} X_f \circ A$ which is the composition of maps with s-tame modulus. □

Proof of theorem 3.6. First use the isomorphism (3.4) to identify $\mathcal{B}_s$ with $\mathcal{P}_s$. By hypothesis P has tame modulus in this basis. Then one has to pass to the basis of the normal modes, namely to introduce the coordinates relative to the basis φ_j. The matrix realizing the change of coordinates is canonical and has the form

$$\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

where the matrix elements of A are given by $B_{k,l} = \varphi^k_l$. Therefore if (S1) is satisfied,

$$|B_{k,l}| \leq \frac{C_n}{(1 + |k - l|)^n} + \frac{C_n}{(1 + |k + l|)^n} \ .$$

Thus, as a consequence of corollary B.2, one gets the thesis. □

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