Density of finite gap potentials for the Zakharov-Shabat system

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Abstract

For various spaces of potentials we prove that the set of finite gap potentials of the Zakharov-Shabat system is dense. In particular our result holds for Sobolev spaces and for spaces of analytic potentials of a given type.

1 Introduction

This paper is an addendum to our investigation of spectral properties of the Zakharov-Shabat operator

$$L(\psi) := i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & \psi_1 \\ \psi_2 & 0 \end{pmatrix}$$

initiated in [GKM], [GK2] (cf also [GK1]). Its aim is to establish the density of finite gap potentials in weighted Sobolev spaces. Here $\psi = (\psi_1, \psi_2)$ and the components ψ_1 and ψ_2 are 1-periodic functions in the weighted Sobolev space $H^w \equiv H^w_{\mathbb{C}}$ of 2-periodic functions

$$H^w := \{f | f(x) := \sum_{-\infty}^{\infty} \hat{f}(k) e^{i\pi kx} \; ; \; \|f\|_w < \infty\}$$

with

$$\|f\|_w := \left(\sum_{k \in \mathbb{Z}} w(k)^2 |\hat{f}(k)|^2 \right)^{1/2}$$

and $w = (w(k))_{k\in\mathbb{Z}}$ is a weight, i.e. a sequence of positive numbers with $w(k) \geq 1, w(-k) = w(k)$ and $w(k) \leq w(k-j)w(j)$ $\forall j, k \in \mathbb{Z}$. Denote by $(\lambda_n^{\pm}(\psi))_{n\in\mathbb{Z}}$ the periodic eigenvalues of $L(\psi)$ when considered on the interval $[0, 2]$ (cf [GK2] for a complete description of the periodic spectrum). As usual, $(\lambda_n^{\pm}(\psi))_{n\in\mathbb{Z}}$ is arranged in lexicographic ordering. The potential ψ is said to be a *finite gap potential* if $\{n \in \mathbb{Z} | \lambda_n^+(\psi) \neq \lambda_n^-(\psi)\}$ is *finite*. Recall from [GK2] that a weight w is said to be a *δ-weight* ($\delta > 0$) if $w_*(k) := (1 + |k|)^{-\delta} w(k)$ is a weight as well. The potential $\psi = (\psi_1, \psi_2)$ is said to be of real type if $\psi_1 = \overline{\psi_2}$. In such a case the operator $L(\psi)$ is selfadjoint.

Theorem 1.1 *Let w be a δ-weight for some $\delta > 0$. For any $\varepsilon > 0$ and any 1-periodic potential in $\psi = (\psi_1, \psi_2)$ in $H^w \times H^w$ there is a finite gap potential $\psi_\varepsilon = (\psi_{\varepsilon 1}, \psi_{\varepsilon 2})$ in $H^w \times H^w$ at distance at most ε from ψ, i.e. $\sup_{1\leq j\leq 2} \|\psi_{\varepsilon j} - \psi_j\|_w < \varepsilon$ and*

$$\lambda_n^+(\psi_\varepsilon) = \lambda_n^-(\psi_\varepsilon) \quad \forall |n| \geq N_\varepsilon$$

where $N_\varepsilon \equiv N_\varepsilon(\psi) \geq 1$.
If ψ is of real type, i.e. $\psi_1 = \overline{\psi_2}$, then ψ_ε can be chosen to be of real type as well.

Remark 1.2 *The number N_ε in Theorem 1.1 can be chosen uniformly on compact sets of* 1*-periodic potentials in H^w.*

Remark 1.3 *Since $H^w \times H^w$ is dense in $L^2([0,1];\mathbb{C}^2)$ for any δ-weight w Theorem 1.1 implies that finite gap potentials are dense in $L^2([0,1];\mathbb{C}^2)$.*

In the special case where ψ is of real type and an element of the Sobolev space H^{w_p} with w_p denoting the Sobolev weight $w_p(k) := (1+|k|)^p (k \in \mathbb{Z})$ and $p \in \mathbb{Z}_{\geq 0}$, Theorem 1.1 has been proved in [Mis] and, independently in [GG], using results established in [MO] and [GT] respectively.
Similar results as the ones presented here for the Zakharov-Shabat operator $L(\psi_1, \psi_2)$ have been obtained previously for the Hill operator $-\frac{d^2}{dx^2} + V$ in [Mit].
To prove Theorem 1.1 (cf [GK2] for a discussion of the spaces H^w) we follow the approach used in [Mit]: as a set-up we take the Fourier bloc decomposition introduced first for the Hill operator in [KM1], [KM2] and worked out subsequently for the Zakharov-Shabat operator in [GKM], [GK2]. Unlike in [Mit] where a contraction mapping argument was used to obtain the density results for the Hill operator, we get a short proof of Theorem 1.1 by applying the inverse function theorem in a straightforward way. As in [Mit] the main feature of the presented proof is that it does not involve any results from inverse spectral theory (cf [CK] for related results for the Hill equation).
Throughout the remainder of this paper we use the notation introduced in [GK2].

2 Proof of Theorem 1.1

Note that for any two weights w, w' with $w \leq w', H^{w'}$ is a dense subspace of H^w. Hence without loss of any generality we may assume that w is a δ-weight with

$$\delta = 1/2.$$

As explained in [GK2, section 2.6], for $M \geq 1$ arbitrary, there exists $N \equiv N_M \geq 1$ so that for any $\psi = (\psi_1, \psi_2)$ with

$$\|\psi\|_w := \sup_{j=1,2} \|\psi_j\|_w \leq M$$

the eigenvalues $\lambda_n^\pm(\psi)$ for $|n| \geq N_M$ are of the form

$$\lambda_n^\pm(\psi) = n\pi + z_n^\pm, \ |z_n^\pm| \leq \pi/2$$

where the complex numbers $z_n^\pm \equiv z_n^\pm(\psi)$ are the two solutions in $|z| < \pi/4$ of the following system

$$z = \alpha(n,z) + \zeta \tag{2.1}$$

$$\zeta^2 - \left(\hat{\psi}_2(2n) + \beta^+(n,z)\right)\left(\hat{\psi}_1(-2n) + \beta^-(n,z)\right) = 0 \tag{2.2}$$

with $\alpha(n,z) \equiv \alpha(n,z,\psi)$ and $\beta^\pm(n,z) \equiv \beta^\pm(n,z,\psi)$ given by (cf [GK2, (2.23) - (2.26), (2.29)])

$$\alpha(n,z) := \langle S^n\hat{\psi}_2, (z-D_n)^{-1}(Id-Q_n)^{-1}S^nJ\hat{\psi}_1\rangle \tag{2.3}$$

$$\beta^+(n,z) := \langle S^n\hat{\psi}_2, (z-D_n)^{-1}R_n^{(1)}(Id-P_n)^{-1}S^n\hat{\psi}_2\rangle \tag{2.4}$$

$$\beta^-(n,z) := \langle S^nJ\hat{\psi}_1, (z-D_n)^{-1}R_n^{(2)}(Id-Q_n)^{-1}S^nJ\hat{\psi}_1\rangle. \tag{2.5}$$

Here S and J denote the shift resp. involution operator given by

$$S : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \ Sa(k) := a(k+1) \quad \forall k \in \mathbb{Z}$$
$$J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \ Ja(k) := a(-k) \quad \forall k \in \mathbb{Z}$$

and, for any $n \in \mathbb{Z}, \langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_{\mathbb{Z}\backslash n}$ denotes the bilinear form (*no* complex conjugation)

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle := \sum_{k \neq n} \left(a(k)c(k) + b(k)d(k)\right).$$

Finally, $D_n, P_n, Q_n, R_n^{(1)}$ and $R_n^{(2)}$ are operators on $\ell^2(\mathbb{Z}\backslash n)$ defined by

$$D_n := \left((k-n)\pi\delta_{kj}\right)_{k,j\in\mathbb{Z}\backslash n}; \ P_n := R_n^{(2)}R_n^{(1)}; Q_n := R_n^{(1)}R_n^{(2)}$$

and

$$R_n^{(1)}a := J\left(\hat{\psi}_1 * (z-D_n)^{-1}\right)a; \ R_n^{(2)}a := \hat{\psi}_2 * J(z-D_n)^{-1}a$$

By (2.1), $z_n^+ = z_n^-$ whenever $\zeta_n^+ = 0$ *and* $\zeta_n^- = 0$. Hence by (2.2), a sufficient condition for $\lambda_n^+(\psi) = \lambda_n^-(\psi)$ is that there exists z_n with $|z_n| < \pi/4$ so that

$$\hat{\psi}_2(2n) + \beta^+(n, z_n, \psi) = 0 \tag{2.6}$$

$$z_n - \alpha(n, z_n, \psi) = 0. \tag{2.7}$$

This suggests a way how to prove Theorem 1.1: Given $\varepsilon > 0$ and a 1-periodic potential $\psi \in H^w \times H^w$ with $\|\psi\|_w \leq M/4$ we want to find $N_\varepsilon \equiv N_\varepsilon(\psi) \geq N_M$ so that there exists a 1-periodic potential ψ_ε in $H^w \times H^w$ with $\|\psi_\varepsilon - \psi\|_w \leq \varepsilon$ and $(z_n)_{|n|\geq N_\varepsilon}$ with $|z_n| < \pi/4$ satisfying the system of equations (2.6) - (2.7). For the appropriate set-up of this system we review some properties of the coefficients $\alpha(n,z)$ and $\beta^+(n,z)$ established in [GK2].
First let us introduce some more notation. Denote by $\mathcal{D}_r \subseteq \mathbb{C}$ the open disc, $\mathcal{D}_r := \{z \in \mathbb{C} \,\big|\, |z| < r\}$ and by B_M^w the open ball,

$$B_M^w := \{\psi = (\psi_1, \psi_2) \,\big|\, \psi_j \in H^w \text{ 1-periodic } ; \ \|\psi_j\|_w < M, \ j = 1,2\}.$$

Concerning the coefficient $\alpha(n,z,\psi)$, recall from [GK2] that for any $|n| \geq N_M$, $\alpha(n,z,\psi)$ is defined for $(z,\psi) \in \mathcal{D}_{\pi/4} \times B_M^w$. From the definition of $\alpha(n,z,\psi)$ it is straightforward to see that it is analytic on $\mathcal{D}_{\pi/4} \times B_M^w$.
Recall the following Lemma 2.7 of [GK2]. (Note that we assume throughout that w is a δ-weight with $\delta = 1/2$.)

Lemma 2.1 *For any $(z,\psi) \in \mathcal{D}_{\pi/4} \times B_M^w$ and any $|n| \geq N_M$,*

$$\big|\, \alpha(n,z,\psi) \,\big| \leq \frac{4M^2}{\langle n \rangle}.$$

By Cauchy's estimate, Lemma 2.1 leads to the following estimates of the differentials $d_\psi \alpha$ and $d_z \alpha$.

Corollary 2.2 *For any $(z,\psi) \in \mathcal{D}_{\pi/8} \times B_{M/2}^w$ and any $|n| \geq N_M$*

$$\|d_\psi \alpha(n,z,\psi)\| \leq \frac{16M}{\langle n \rangle}$$
$$|d_z \alpha(n,z,\psi)| \leq \frac{32M^2}{\pi} \frac{1}{\langle n \rangle}$$

To state the next result denote by

$$\mathcal{D}_{\pi/8}^\infty \equiv \mathcal{D}_{\pi/8}^\infty(\mathbb{Z} \backslash [-N_M, N_M]) \subseteq \ell^\infty(\mathbb{Z} \backslash [-N_M, N_M]; \mathbb{C})$$

the open ball of radius $\pi/8$ centered at 0 in the Banach space $\ell^\infty(\mathbb{Z} \backslash [-N_M, N_M]; \mathbb{C})$.

Proposition 2.3 *The map*

$$\mathcal{D}^{\infty}_{\pi/8} \times B^{w}_{M/2} \to \ell^2(\mathbb{Z}\backslash[-N_M, N_M], \mathbb{C})$$
$$\left((z_n)_{|n|>N_M}, \psi\right) \mapsto (\alpha(n, z_n, \psi))_{|n|>N_M}$$

is analytic and satisfies for any $N \geq N_M$

$$\| \left(\alpha(n, z_n, \psi)\right)_{|n|>N} \| \leq 2M^2 N^{-1/2}$$
$$\|d_\psi \left(\alpha(n, z_n, \psi)\right)_{|n|>N} \| \leq 32 M N^{-1/2}$$

and, with $Z := (z_n)_{|n|>N}$

$$\|d_Z \left(\alpha(n, z_n, \psi)\right)_{|n|>N} \| = \| \left(\delta_{nk} \left(\partial_{z_k} \alpha(n, z_n, V)\right)_{|n|,|k|>N}\right) \| \leq \frac{32M^2}{\pi} N^{-1}.$$

Concerning the coefficient $\beta^+(n, z, \psi)$ recall from [GK2] that for any $|n| \geq N_M, \beta^+(n, z, \psi)$ is well defined for $(z, V) \in \mathcal{D}_{\pi/4} \times B^w_M$.
From its definition, it is straightforward to see that $\beta^+(n, z, \psi)$ is analytic on $\mathcal{D}_{\pi/4} \times B^w_M$. Recall the following Proposition 2.10 in [GK2]. (Again note that we have chosen $\delta = 1/2$, hence $\delta_* = 1/2$.)

Lemma 2.4 *There exists* $C \geq 1$ *so that for any* $\psi \in B^w_M$,

$$\sum_{|n|>N_M} \langle n \rangle^2 w(2n)^2 \sup_{|z| \leq \pi/4} |\beta^+(n, z, \psi)|^2 \leq C.$$

Again by Cauchy's estimate we obtain estimates for the derivatives of $\beta^+(n, z, \psi)$. To state them recall that ℓ^2_w denotes the ℓ^2-Hilbert space with weight w and w_1 denotes the weight

$$w(k) := \langle k \rangle w(k) \quad (\forall k \in \mathbb{Z}).$$

Proposition 2.5 *The map*

$$\mathcal{D}^{\infty}_{\pi/8} \times B^{w}_{M/2} \to \ell^2_{w_1}\left(\mathbb{Z}\backslash[-N_M, N_M]; \mathbb{C}\right)$$
$$\left((z_n)_{|n|>N_M}, \psi\right) \mapsto \left(\beta^+(n, z_n, \psi)\right)_{|n|>N_M}$$

is analytic and satisfies for any $N \geq N_M$, *with* $C \geq 1$ *given as in Lemma 2.4,*

(i) $\left\| \left(\beta^+(n, z_n, \psi)\right)_{|n|>N} \right\|_w \leq 2C \; N^{-1/2}$

(ii) $\left\|d_\psi \left(\beta^+(n, z_n, \psi)\right)_{|n|>N} \right\|_w \leq \frac{4C}{M} N^{-1/2}$

and for any $Z := (z_n)_{|n|>N} \in \mathcal{D}^{\infty}_{\pi/8}$

(iii) $\left\|d_Z \left(\beta^+(n, z_n, \psi)\right)_{|n|>N} \right\|_w \leq \frac{16C}{\pi} N^{-1/2}$.

It turns out that a convenient set-up for the system of equations (2.6) - (2.7) is the following one: introduce for $N \geq N_M$

$$E \equiv E_N := \{v = (v_n)_{|n|>N} \mid \|v\|_w < M/4\}$$

where $\|v\|_w = \left(\sum_{|n|>N} w(2n)^2 |v_n|^2 \right)^{1/2}$ and

$$\mathcal{Z} \equiv \mathcal{Z}_N := \{Z = (z_n)_{|n|>N} \mid \|Z\| = \left(\sum_{|n|>N} |z_n|^2 \right)^{1/2} < \pi/8\}.$$

For any $\psi = (\psi_1, \psi_2) \in B^w_{M/4}$ and any $v \in E_N$ define $\psi^v = (\psi^v_1, \psi^v_2)$ by

$$\psi^v_2(x) := \sum_{|n|\leq N} \hat{\psi}_2(2n) e^{i2\pi nx} + \sum_{|n|>N} v_n e^{i2\pi nx} \tag{2.8}$$

and

$$\psi^v_1(x) := \sum_{|n|\leq N} \hat{\psi}_1(2n) e^{i2\pi nx} + \sum_{|n|>N} \overline{v}_{-n} e^{i2\pi nx}. \tag{2.9}$$

The definition of ψ^v_1 has been chosen in such a way that ψ^v is of real type if ψ is of real type. Notice that for $v \in E$ and $\psi \in B^w_{M/4}$, $\|\psi^v_j\|_w \leq M/2$ for $j = 1, 2$ and hence $\psi^v \in B^w_{M/2}$. Finally, introduce for any given $\psi \in B^w_M$ and $N \geq N_M$ a map $\Lambda \equiv \Lambda^{(N)}_\psi$ defined for any $(v, Z) \in E \times \mathcal{Z}$ by $\Lambda(v, Z) := (\Lambda_n(v, Z))_{|n|>N}$ with

$$\Lambda_n(v, Z) := \left(v_n + \beta^+(n, z_n, \psi^v), z_n - \alpha(n, z_n, \psi^v)\right).$$

By Proposition 2.3 and Proposition 2.5, the range of $\lambda^{(N)}$ is contained in $\ell^2_w(\mathbb{Z}\backslash[-N, N]; \mathbb{C}) \times \ell^2(\mathbb{Z}\backslash[-N, N]; \mathbb{C})$ and is analytic. Choose on $\ell^2_w \times \ell^2$ the norm defined for any $(v, Z) \in \ell^2_w \times \ell^2$ by

$$\|(v, Z)\| = \max\left(\|v\|_w, \|Z\|\right).$$

Lemma 2.6 *For any $M \geq 1$ there exists $N \geq N_M$ so that for any $\psi \in B^w_{M/4}$ and any $(v, Z) \in E_N \times \mathcal{Z}_N$,*

$$\|d_{v,Z}\Lambda^{(N)}_\psi - Id^{(N)}\| \leq 1/2$$

where $Id^{(N)}$ denotes the identity map on $E_N \times \mathcal{Z}_N$.

Proof The differential $d_{v,Z}\Lambda^{(N)}$ takes the form

$$d_{v,Z}\Lambda^{(N)} = \begin{pmatrix} \mathrm{Id}_E + d_v\,(\beta_n^+)_{|n|>N} & d_Z\,(\beta_n^+)_{|n|>N} \\ -d_v(\alpha_n)_{|n|>N} & \mathrm{Id}_{\mathcal{Z}} - d_Z(\alpha_n)_{|n|>N} \end{pmatrix} .$$

Hence by Proposition 2.3 and Proposition 2.5 there exists $N \geq N_M$ so that for any $\psi \in B^w_{M/4}$ and $(v, Z) \in E \times \mathcal{Z}$,

$$\begin{aligned} &\|d_{v,Z}\Lambda^{(N)} - \mathrm{Id}^{(N)}\|_w \leq \\ &\quad \|d_v\left(\beta_n^+\right)_{|n|>N}\|_w + \|d_Z\left(\beta_n^+\right)_{|n|>N}\|_w + \\ &\quad +\|d_v(\alpha_n)_{|n|>N}\| + \|d_Z(\alpha_n)_{|n|>N}\| \\ &\quad \leq 1/2. \end{aligned}$$

■

Proof (of Theorem 1.1) Let $0 < \varepsilon \leq 1$ and a 1-periodic potential ψ in H^w. Let $M > 4\|\psi\|_w$ and choose $N_1 \geq N$ with N as in Lemma 2.6 so that

$$\left(\sum_{|n|>N_1} w(2n)|\hat{\psi}_1(2n)|^2\right)^{1/2} < \varepsilon/2 \text{ and } \|Z^{(0)}(\psi)\| < \pi/8 \tag{2.10}$$

where

$$Z^{(0)} \equiv Z^{(0)}(\psi) := \left(\lambda_n^+(\psi) - n\pi\right)_{|n|>N_1} .$$

Further define $v^{(0)} \equiv v^{(0)}(\psi) := \left(\hat{\psi}_2(2n)\right)_{|n|>N_1}$. As $\|v^{(0)}\|_w \leq \|\psi\|_w \leq M/4$ it follows that $(v^{(0)}, Z^{(0)}) \in E_{N_1} \times \mathcal{Z}_{N_1}$. Hence, by Lemma 2.6, $d_{v^{(0)},Z^{(0)}}\Lambda_\psi^{(N_1)}$ is invertible. By the inverse function theorem there is $\eta > 0$ so that one can find an open neighborhood U_1 of $\left(v^{(0)}, Z^{(0)}\right) \equiv \left(v^{(0)}(\psi), Z^{(0)}(\psi)\right)$ contained in the open ball $B_{\varepsilon/2}\left(v^{(0)}, Z^{(0)}\right)$ in $E_{N_1} \times \mathcal{Z}_{N_1}$ so that $\Lambda_\psi^{(N_1)}$ is a diffeomorphism from U_1 onto the ball $B_\eta\left(\Lambda_\psi^{(N_1)}(v^{(0)}, Z^{(0)})\right)$ in $\ell^2_w\left(\mathbb{Z}\backslash[-N_1, N_1]; \mathbb{C}\right) \times \ell^2\left(\mathbb{Z} \times [-N_1, N_1]; \mathbb{C}\right)$. Choose $N_2 \equiv N_2(\psi)$ with $N_2 \geq N_1$ so that

$$\|\Lambda_\psi^{(N_2)}(v^{(0)}, Z^{(0)})\| < \eta.$$

Then

$$\varrho := \left(\left(\Lambda_n(v^{(0)}, Z^{(0)}\right)_{N_1<|n|\leq N_2},\ (0,0)_{|n|>N_2}\right)$$

is an element in $B_\eta\left(\Lambda_\psi^{(N_1)}(v^{(0)}, Z^{(0)})\right)$. Hence there exists $(v, Z) \in U_1$ so that

$$\varrho = \Lambda_\psi^{(N_1)}(v, Z).$$

In particular, for any $|n| > N_2$

$$\begin{aligned} v_n + \beta^+(n, z_n, \psi^v) &= 0 \\ z_n - \alpha(n, z_n, \psi^v) &= 0 \end{aligned}$$

where ψ^v is defined as in (2.8) - (2.9) with N given by N_1. This means that equations (2.6) - (2.7) are satisfied for ψ^v and $(z_n)_{|n|>N_2}$, hence ψ^v is a finite gap potential, or more precisely,

$$\lambda_n^+(\psi^v) = \lambda_n^-(\psi^v) \quad \forall |n| > N_2.$$

Furthermore, as $(v, Z) \in U_1 \subset B_\varepsilon\left(v^{(0)}, Z^{(0)}\right)$ we have

$$\|\psi_2 - \psi_2^v\|_w = \left(\sum_{|n|>N_1} w(2n)^2 |v_n|^2\right)^{1/2} < \varepsilon/2$$

and, by the definition (2.9) of ψ_1^v and the estimate (2.10),

$$\|\psi_1 - \psi_1^v\|_w \le \left(\sum_{|n|>N_1} w(2n)^2 |\hat{\psi}_1(2n)|^2\right)^{1/2} + \left(\sum_{|n|>N_1} w(2n)^2 |v_n|^2\right)^{1/2} < \varepsilon$$

hence

$$\|\psi - \psi^v\|_w = \sup_{1 \le j \le 2} \|\psi_j - \psi_j^v\| < \varepsilon.$$

This shows the claimed result with $\psi_\varepsilon := \psi^v$ and $N_\varepsilon := N_2$ for an arbitrary potential $\psi \in H^w$. If ψ is of real type $\psi_\varepsilon = \psi^v$ is of real type as well by the definition (2.8) - (2.9) of ψ_v. ■

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