Perturbations of the defocusing NLS equation

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June 13, 2002

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Abstract: We prove that many finite dimensional tori, invariant under the flow of the defocusing nonlinear Schrödinger equation, persist under small Hamiltonian perturbations.These invariant tori are not necessarily close to the zero solution.

1 Introduction

Consider the *defocusing* nonlinear Schrödinger equation (NLS) with periodic boundary conditions

$$i\partial_t\varphi = -\partial_x^2 + 2|\varphi|^2\varphi; \quad \varphi(x+1,t) = \varphi(x,t) \quad (x \in \mathbb{R}, t \in \mathbb{R}). \tag{1}$$

It is a completely integrable system with phase space $H^N \equiv H^N(S^1;\mathbb{C})$ ($N \in \mathbb{R}_{\geq 1}$) and Hamiltonian $\mathcal{H} = \mathcal{H}(\varphi, \overline{\varphi})$ ([ZS] - see also [FT], [MV]). Here, for $N \geq 0$,

$$H^N(S^1;\mathbb{C}) := \{\varphi(x) = \sum_{k\in\mathbb{Z}} e^{2i\pi kx}\hat{\varphi}(k) \mid ||\varphi||_N < \infty\} \tag{2}$$

where

$$||\varphi||_N := (\sum_{k\in\mathbb{Z}}(1+|k|)^{2N}|\hat{\varphi}(k)|^2)^{1/2}$$

and $\hat{\varphi}(k) \quad (k \in \mathbb{Z})$ denote the Fourier coefficients of φ, viewed as a function of period 1, and, for any $\varphi \in H^1$,

$$\mathcal{H}(\varphi, \overline{\varphi}) := \int_{S^1} \left(\partial_x \varphi \partial_x \overline{\varphi} + \varphi^2 \overline{\varphi}^2 \right) dx \ . \tag{3}$$

The Poisson structure is given by the standard Poisson bracket.

$$\{F, G\}(\varphi, \overline{\varphi}) := i \int_{S^1} \Big(\frac{\partial F}{\partial \varphi} \frac{\partial G}{\partial \bar{\varphi}} - \frac{\partial F}{\partial \bar{\varphi}} \frac{\partial G}{\partial \varphi} \Big) dx \tag{4}$$

where F, G are functionals on $L^2 \equiv L^2(S^1; \mathbb{C})$ of class C^1. When written in Hamiltonian form, NLS becomes

$$\partial_t \varphi = -i \frac{\partial \mathcal{H}}{\partial \overline{\varphi}}.$$

It is well known that NLS admits a Lax pair representation

$$\frac{d}{dt} L(\varphi) = [L(\varphi), A(\varphi)] \tag{5}$$

where L is the Zakharov-Shabat operator (see [ZS])

$$L(\varphi) := i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & \varphi \\ \overline{\varphi} & 0 \end{pmatrix}, \tag{6}$$

and A is a rather complicated operator given in [FT]. As a consequence, the periodic spectrum, $spec_{per} L(\varphi)$, remains invariant under the NLS flow. Here, $spec_{per} L(\varphi)$ denotes the spectrum of $L(\varphi)$ when considered on the interval [0,2] with periodic boundary conditions. The periodic spectrum consists of two interlacing sequences $(\lambda_j^+(\varphi))_{j \in \mathbb{Z}}, (\lambda_j^-(\varphi))_{j \in \mathbb{Z}}$ of real numbers satisfying

$$\ldots < \lambda_j^-(\varphi) \leq \lambda_j^+(\varphi) < \lambda_{j+1}^-(\varphi) \leq \ldots . \tag{7}$$

They are uniquely determined by the sequence of the gap lengths, $\gamma(\varphi) = (\gamma_k(\varphi))_{k \in \mathbb{Z}}$ with $\gamma_k(\varphi) = \lambda_k^+(\varphi) - \lambda_k^-(\varphi)$ (cf e.g. [GG], [GKP], [MV]).

To describe the structure of the phase space $H^N(S^1; \mathbb{C})$ we introduce the model space

$$l_N^2(\mathbb{Z}; \mathbb{R}^2) := \{(x, y) = (x_j, y_j)_{j \in \mathbb{Z}} \mid ||(x, y)||_N < \infty\}$$

where

$$||(x,y)||_N := \left(\sum_{j\in\mathbb{Z}} (1+|j|)^{2N}(x_j^2+y_j^2) \right)^{1/2} < \infty.$$

The space $l_N^2(\mathbb{Z};\mathbb{R}^2)$ is endowed with the Poisson structure induced by the canonical symplectic structure $\sum_{j\in\mathbb{Z}} dx_j \wedge dy_j$. In [GKP] (cf also [BKM], [KMa], [KPö] and [MV]) we have proved the following result:

Theorem 1.1 *There exists a family of diffeomorphisms* $\Phi \equiv \Phi^{(N)}$, $N \in \mathbb{R}_{>0}$,

$$\Phi : l_N^2(\mathbb{Z};\mathbb{R}^2) \to H^N(S^1;\mathbb{C})$$

with the following properties:

(i) Φ *is globally 1-1, onto, bianalytic and preserves the Poisson bracket.*

(ii) The coordinates $(x_j,y_j)_{j\in\mathbb{Z}} = \Phi^{-1}(\varphi)$ *are Birkhoff coordinates for NLS (and its hierarchy). That is the transformed NLS Hamiltonian* $\mathcal{H}\circ\Phi$ *depends only on the actions* $I_j := (x_j^2+y_j^2)/2$, $j\in\mathbb{Z}$, *with* (x_j,y_j) *being canonical coordinates on* $l_N^2(\mathbb{Z};\mathbb{R}^2)$.

(iii) For $N > N'$

$$\Phi^{(N)} = \Phi^{(N')}|_{l_N^2}.$$

Often it will be convenient to use complex notation

$$x_j + y_j = \sqrt{2I_j}\; e^{i\theta_j} \quad (j\in\mathbb{Z}).$$

The coordinates $(I_j,\theta_j)_{j\in\mathbb{Z}}$ are referred to as action-angle coordinates. Note that $\theta_j \in R/2\pi\mathbb{Z}$ is well defined whenever $I_j \neq 0$.
As a consequence of Theorem 1.1, the solution $\varphi(x,t) \equiv \varphi_t(x)$ of the initial value problem for NLS with initial profile $\varphi_0 = \Phi((\sqrt{2I_j}e^{i\theta_j})_{j\in\mathbb{Z}})$ in $H^N(S^1;\mathbb{C})$, with $N \geq 1$, is given by

$$\varphi_t = \Phi((\sqrt{2I_j}\; e^{i(\theta_j+t\omega_j(I))})_{j\in\mathbb{Z}})$$

where $\omega_j(I) = \frac{\partial\mathcal{H}(I)}{\partial I_j}$ $(j\in\mathbb{Z})$ denote the frequencies and $\mathcal{H} = \mathcal{H}(I)$ is the Hamiltonian of NLS, expressed in action variables.

An asymptotic expansion of the frequencies (cf. section 4) shows that $\omega_k \sim \omega_{-k}$ for $|k|$ large. In order to control the effect of these "asymptotic" resonances on perturbed equations we impose symmetry conditions so that the action-angle variables (I_k, θ_k) with $k < 0$ are uniquely determined by $(I_j, \theta_j)_{j\geq 0}$. More precisely, we introduce the space $H^N_\alpha(S^1;\mathbb{C})$ with $\alpha \in \mathbb{R}$,

$$H^N_\alpha(S^1;\mathbb{C}) := \Phi(l^2_{N;\alpha}(\mathbb{Z};\mathbb{R}^2)) \tag{8}$$

where $(\sqrt{2I_j}e^{i\theta_j})_{j\in\mathbb{Z}} \in l^2_{N;\alpha}$ iff $(\sqrt{2I_j}\, e^{i\theta_j})_{j\in\mathbb{Z}} \in l^2_N$,

$$I_{-j} = I_j \quad \forall j \geq 0 \tag{9}$$

and

$$\theta_{-j} \equiv \theta_j + \alpha \ (\mathrm{mod}\ 2\pi) \quad \forall j \geq 0 \text{ with } I_j \neq 0. \tag{10}$$

Notice that for $\alpha \not\equiv 0 (\mathrm{mod}\ 2\pi)$, (10) implies that $I_0(\varphi) = 0$ for all $\varphi \in H^N_\alpha$. It turns out that the subspaces $H^N_\alpha(S^1;\mathbb{C})$ are invariant under the NLS-flow. A way to prove this (cf section 3) is to show that the symmetries of the NLS Hamiltonian $\mathcal{H}$, expressed in action-variables, imply that for any $I = (I_k)_{k\in\mathbb{Z}}$, $\mathcal{H}(I') = \mathcal{H}(I)$ where $I' = (I'_k)_{k\in\mathbb{Z}}$ is given by $I'_k := I_{-k}$ $(k \in \mathbb{Z})$. As a consequence we have the following

Proposition 1.2 *(i)* $\mathcal{H}(I') = \mathcal{H}(I)$;

(ii) if $I' = I$, *then* $\omega_j(I) = \omega_{-j}(I)$ $\forall j \geq 1$.

In section 3 (see also [GK3]) we prove that $H^N_\alpha(S^1;\mathbb{C})$ can be characterized as follows,

$$H^N_\alpha(S^1;\mathbb{C}) = \{\varphi \in H^N(S^1;\mathbb{C}) \mid e^{i\alpha}\check{\varphi} \equiv \varphi\}$$

where $\check{\varphi}(x) = \varphi(-x)$. In particular, $H^N_\pi \cap C^\infty$ (resp. $H^N_0 \cap C^\infty$) is the space of elements φ in $H^N \cap C^\infty$ satisfying generalized Dirichlet (resp. Neumann) conditions, i.e. $\forall k \geq 0$

$$\partial_x^{2k}\varphi(0) = \partial_x^{2k}\varphi(1) = 0 \quad [\ \partial_x^{2k+1}\varphi(0) = \partial_x^{2k+1}\varphi(1) = 0\ ]\ .$$

By a slight abuse of notation, the restriction of Φ to $l^2_{N;\alpha}(\mathbb{Z};\mathbb{R}^2)$ is again denoted by Φ.
Given $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, a finite subset $A \subseteq \mathbb{Z}_{\geq 0}$ (with $0 \notin A$ if $\alpha \not\equiv 0 \bmod 2\pi$) and $I_A \in (\mathbb{R}_{>0})^{|A|}$ we denote by $T^\alpha_{I_A}$ the $|A|$ dimensional torus of the model space $l^2(\mathbb{Z};\mathbb{R}^2)$, defined by

$$\begin{aligned} T^\alpha_{I_A} \ :=\ & \{(\sqrt{2J_j}e^{i\theta_j})_{j\in\mathbb{Z}} | J_j = J_{-j} = I_j \quad \forall j \in A; \\ & J_j = J_{-j} = 0 \quad \forall j \notin A;\ \theta_j = \theta_{-j} + \alpha \quad \forall j \in A\} \end{aligned} \tag{11}$$

and by $\mathcal{T}^\alpha_{I_A}$ the $|A|$ dimensional, NLS invariant torus in H^N_α

$$\mathcal{T}^\alpha_{I_A} := \Phi(T^\alpha_{I_A}). \tag{12}$$

A potential $\varphi \in H^N(S^1;\mathbb{C})$ is *symmetric* if $I_{-j} = I_j$ for any $j \geq 1$ and is said to be a *finite gap* [K-gap with $K \in \mathbb{Z}_{\geq 1}$] potential if there exists a finite subset $B \subset \mathbb{Z}$ [of cardinality $|B| = K$] so that

$$I_j(\varphi) = 0 \text{ iff } j \in \mathbb{Z} \setminus B\ .$$

Hence any element in $\mathcal{T}^\alpha_{I_A}$ is a symmetric $2|A|$ gap potential (if $0 \notin A$) or $2|A|-1$ gap potential (if $0 \in A$) and thus in particular smooth by Theorem 1.1 (cf also [GK2]).
For $\Gamma \subseteq (\mathbb{R}_0)^{|A|}$ compact and of positive Lebesgue measure introduce the following union of tori

$$\mathcal{T}^\alpha_\Gamma := \cup_{I_A \in \Gamma} \mathcal{T}^\alpha_{I_A}. \tag{13}$$

We will consider Hamiltonian perturbations, $\mathcal{H}_\varepsilon = \mathcal{H} + \varepsilon K$ on $H^N_\alpha(S^1;\mathbb{C})$ with the following properties:

(P1) K is real analytic on some symmetric neighborhood U_Γ of $\{(\varphi,\overline{\varphi}) \mid \varphi \in \mathcal{T}^\alpha_\Gamma\}$ in $(H^N(S^1;\mathbb{C}))^2$. [1]

(P2) $\frac{\partial K}{\partial \varphi}$, $\frac{\partial K}{\partial \psi}$ are bounded as functions from U_Γ into $H^N(S^1;\mathbb{C})$ and verify the normalization condition

$$\sup\left\{ ||\frac{\partial K}{\partial \varphi}(\varphi,\psi)||_N + ||\frac{\partial K}{\partial \psi}(\varphi,\psi)||_N \mid (\varphi,\psi) \in U_\Gamma \right\} \leq 1.$$

(P3) K satisfies the symmetry condition, $((\varphi,\psi) \in U_\Gamma)$

$$K(\varphi,\psi) = K(e^{i\alpha}\check{\varphi}, e^{-i\alpha}\check{\psi}).$$

Notice that, together with Proposition 1.2, condition (P3) insures that solutions of $\frac{\partial \varphi}{\partial t} = i\frac{\partial \mathcal{H}_\varepsilon}{\partial \varphi}$ for initial data in $H^N_\alpha(S^1;\mathbb{C})$ evolve in the same space $H^N_\alpha(S^1;\mathbb{C})$.

Our KAM Theorem states that, for ε small enough, many of the NLS-invariant tori $\mathcal{T}^\alpha_{I_A}$ persist under perturbation of the NLS Hamiltonian by

[1] U_Γ is said to be symmetric iff $(e^{i\alpha}\check{\varphi}, e^{-i\alpha}\check{\psi}) \in U_\Gamma$ for any $(\varphi,\psi) \in U_\Gamma$.

εK with K satisfying (P1), (P2), and (P3). Moreover these tori and their linear flows are only slightly perturbed.
Denote by T^n the $n-$dimensional torus $(\mathbb{R}/\mathbb{Z})^n$.

Theorem 1.3 *Let $N \geq 1, A, \Gamma, \alpha, U_\Gamma$ be given as above. Then, for K satisfying (P1), (P2) and (P3), there exists ε_0 so that for any ε with $|\varepsilon| \leq \varepsilon_0$*

(i) there exists a Cantor set $\Gamma_\varepsilon \subset \Gamma$ with meas $(\Gamma \setminus \Gamma_\varepsilon) \overset{\varepsilon \to 0}{\to} 0$,

(ii) there exists a Lipschitz family of real analytic torus embeddings

$$\Psi : T^{|A|} \times \Gamma_\varepsilon \to U_\Gamma \cap \{(\varphi, \overline{\varphi}) | \varphi \in H^N_\alpha\}$$

and

(iii) there exists a Lipschitz map $f : \Gamma_\varepsilon \to \mathbb{R}^{|A|}$

such that for $I_A \in \Gamma_\varepsilon$ and $\theta_A \in T^{|A|}, \Psi(\theta_A + tf(I_A), I_A)$ is a quasiperiodic solution of $\partial_t \varphi = i\frac{\partial \mathcal{H}}{\partial \overline{\varphi}} + i\varepsilon \frac{\partial K}{\partial \overline{\varphi}}$. The deformed invariant tori $\Psi(T^{|A|} \times \{I_A\})$ are linearly stable.

Remarks: Theorem 1.3 generalizes results due to Kuksin-Pöschel [KP] which concern the special case where $\Gamma \subseteq \mathbb{R}^{|A|}_+$ is contained in a sufficiently small neighborhood of $0 \in \mathbb{R}^{|A|}$ and the phase space consists of elements satisfying generalized Dirichlet boundary conditions. In this situation, action-angle variables are not needed as the Fourier coefficients $(\hat{\varphi}(k))_{k \in \mathbb{Z}}$ are a sufficiently good approximation of the Birkhoff coordinates close to the origin.

Similarly, the results of [CW], [C] and their generalization by [B1], [B2] while not directly comparable with our Theorem 1.3, concern only small perturbations of NLS around $\varphi = 0$.

Our results and methods continue the investigation in [KPö] on the Korteweg-de Vries equation. The purpose of this paper is to document similar features of the defocusing NLS equation.

Theorem 1.3 has been announced in [GK1].

This paper is organized as follows: In section 2 we present a historical background on the topic of this paper. In section 3 we review elements of the normal form theory of NLS (cf [GKP]) needed for our KAM result and in section 4 we express the various symmetries of NLS in action-angle coordinates (cf [GK3]). In section 5 a detailed treatment of the NLS frequencies $(\omega_k)_{k\in\mathbb{Z}}$ are given. In particular we provide new closed formulas for ω_k (cf Theorem 5.6) - of independent interest - which can be used to derive asymptotics of ω_k for $|k| \to \infty$. Finally, in section 6, we prove our KAM result.

2 Background

In this article we consider the periodic NLS equation as an infinite dimensional integrable Hamiltonian system and subject it to small Hamiltonian perturbations. To this end many concepts - in particular KAM theory - are extended from the theory of finite dimensional Hamiltonian systems to systems of infinite dimension.

It is the purpose of this section to provide a brief summary of the perturbation theory of integrable systems of finite dimension - see section "classical background" in the forthcoming book [KPö] for a more detailed account.

Integrable systems Let (M, σ) be a symplectic manifold M being a smooth, connected manifold without boundary and σ a closed, nondegenerated 2-form on M. Given a smooth function $H : M \to \mathbb{R}$, denote by X_H its Hamiltonian vector field, i.e. the unique vector field satisfying $\sigma(X_H, \cdot) = dH$, and by X_H^t or Φ^t its flow, referred to as the Hamiltonian flow of H on the phase space M. $X_H^t(x)$ solves the initial value problem $\dot{x}(t) = X_H(x(t))$ with the initial data $x(0) = x$ in M. Clearly, the Hamiltonian H is constant along the flow X_H^t, $\frac{d}{dt} H \circ X_H^t = 0$. More generally, we say that a smooth function $F : M \to R$ is an integral of the Hamiltonian system (M, σ, H) if F is constant along the flow, $\frac{d}{dt} F \circ X_H^t = 0$. Given a symplectic structure one defines the Poisson bracket of two smooth functions $F, G : M \to R$ is

$$\{F, G\} := \sigma(X_F, X_G).$$

It is straight forward to verify that F is an integral for (M, σ, H) iff $\{F, H\} \equiv 0$. To preserve the Hamiltonian nature of vector fields, a diffeomorphism $\Phi : M \to M$ of a symplectic manifold (M, σ) has to preserve σ or, equivalently, the Poisson bracket, i.e. for any $F, G \in C^\infty(M)$,

$$\{F, G\} \circ \Phi = \{F \circ \Phi, G \circ \Phi\}.$$

Such transformations are called symplectic or canonical. Integrable systems are particular Hamiltonian systems that can be solved for any initial data by quadratures - whence the name:

Definition 2.1 *A Hamiltonian system* (M, σ, H) *with* $\dim M = 2n$ *is called integrable if its Hamiltonian* H *admits* n *independent integrals* $F_1, \cdots, F_n$ *in involution. That is, everywhere on* M*,*

(i) $\{F_i, H\} = 0 \quad \forall 1 \leq i \leq n$*;*

(ii) $\{F_i, F_j\} = 0 \quad \forall 1 \leq i, j \leq n$*;*

(iii) $dF_1 \wedge \ldots \wedge dF_n \neq 0$*.*

Given an integrable system with $F_1, \ldots, F_n$ being independent integrals in involution, denote by $M^c = F^{-1}(c)$ the level sets of $F = (F_1, \ldots, F_n)$. They define a foliation of M into Lagrangian submanifolds, meaning that the restriction of σ to M^c vanishes.

Theorem 2.2 *(Liouville-Arnold-Jost; see [HZ]) Assume that* (M, σ, H) *is an integrable system. If one of the level sets* M^c *is compact, then there exists a neighborhood* U *of* M^c *which is foliated into invariant* n*-dimensional Lagrangian tori and one can introduce angle-action coordinates* $(\theta, I) = \Psi^{-1}(x)$ *where* $\Psi : (\mathbb{T}^n \times D, \sigma_0) \to (U, \sigma)$ *is a canonical transformation, such that the transformed Hamiltonian* $H \circ \Psi$ *is a function of the actions* I *alone. Here* $D \subseteq \mathbb{R}^n, \mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ *and* σ_0 *is the standard symplectic form* $\sigma = \sum_{j=1}^{0} d\theta_j \wedge dI_j$ *with respect to angle-action coordinates.*

With respect to action-angle coordinates, the equations of motion are $(1 \leq i \leq n)$

$$\dot{\theta}_i \equiv \omega_i(I); \quad \dot{I}_i = 0$$

where $\omega_i(I) = \frac{\partial H}{\partial I_i}(I)$ are called frequencies. These equations are easily integrated whence the name integrable system,

$$\theta(t) = \theta^0 + \omega(I^0)t, \; I(t) = I^0.$$

Note that the frequency vector $\omega(I) = (\omega_1(I), \cdots, \omega_n(I))$ completely determines the dynamics on the torus $\mathbb{T}^n \times \{I\}$. On the phase space M, these solutions are the form $\Psi\left(\theta^0 + \omega(I^0)t, I^0\right)$ and hence quasiperiodic in t.

The properties of the flow $t \mapsto \theta^0 + \omega(I^0)t$ on $\mathbb{T}^n$ differ sharply depending on arithmetical properties of its frequencies ω. If $\omega \equiv \omega(I^0)$ is *nonresonant*, i.e. $\langle \omega, k\rangle \neq 0 \ \forall k \in \mathbb{Z}^n \backslash \{0\}$ then the orbit $t \mapsto \theta^0 + \omega t$ is dense on $\mathbb{T}^n$. Otherwise ω is called *resonant*. The torus $\Psi\left(\mathbb{T}^n \times \{I^0\}\right)$ is referred to as a nonresonant or resonant torus respectively.
Another type of integrable system arises in the study of equilibrium of Hamiltonian systems, the so called Birkhoff integrable systems. On the one hand, this type is more special than the Liouville type as it suffices to look at a neighborhood of a simple point. On the other hand it is more general, as such a neighborhood is foliated into invariant tori of any dimension between 0 and n.
It has been shown that around a nonresonant elliptic fixed point a Birkhoff integrable system is modeled by coupled harmonic oscillators - see [Ve], [It]. Theorem 1.1 is an infinite dimensional of this result for the NLS equation.

Classical KAM theory Integrable systems are the exception, not the rule. On the other hand, many Hamiltonian systems occuring in applications may be viewed as small perturbations of an integrable system - the planetary system we live in being such an example. The KAM theory addresses the question what happens to a foliation of invariant tori and their associated quasi-periodic notions under small perturbations of the Hamiltonian. Consider a Hamiltonian in angle-action coordinates (θ, I) of the form

$$H = H_0(I) + \varepsilon H_1(\theta, I)$$

where H_0 is the unperturbed integrable Hamiltonian and $\varepsilon H_1(\theta, I)$ a perturbation of size ε. The unperturbed system is assumed to be nondegenerate meaning that the frequency map

$$I \mapsto \omega(I) = \frac{\partial H_0}{\partial I}(I)$$

is a local diffeomorphism everywhere.
The first result goes back to Poincaré and is of a negative nature. He observed that the resonant tori are in general destroyed by an arbitrarily small perturbation. So in a nondegenerate system a *dense* set of tori is typically destroyed. Hence there seems to be no hope for the other tori to survive. Indeed, until the 1950's it was a common belief that arbitrarily small perturbations can turn an integrable system into an ergodic one on each energy surface. But in 1954 Kolmogorov [Ko] observed that the converse is true -

the majority of tori survives. More precisely, he showed that the so called *strongly nonresonant* tori persist, meaning that the corresponding frequency ω satisfies

$$\big|\langle k,\omega\rangle\big| \geq \frac{\alpha}{|k|^\tau}\ , \quad \forall k \in \mathbb{Z}^n\backslash 0\ .$$

Such a condition is called a small divisor condition, as the expression $\langle k,\omega\rangle$ enter into the denominators of formal series expansion of quasi-periodic solutions of the perturbed system. The existence of such frequencies of a nondegenerate integrable system is easy to verify. Fixing τ, let Δ_α denote the set of all $\omega \in \mathbb{R}^n$ satisfying these infinitely many conditions with given $\alpha > 0$. If $\tau > n-1$, then for any bounded subset $\Omega \subseteq \mathbb{R}^n$ one has

$$\text{meas}\big(\Omega\backslash\Delta_\alpha\big) = 0(\alpha).$$

Note however that the parameter α in the small divisor condition limits the size of the perturbation through the condition $\varepsilon << \alpha^2$. To state the KAM theorem we therefore simple out from a bounded domain $\Omega \subseteq \mathbb{R}^n$ for any given $\alpha > 0$ the subset $\Omega_\alpha \subseteq \Omega$ of frequencies belonging to Δ_α and also having at least distance α to the boundary. These, like Δ_α, are Cantor sets: they are closed, nowhere dense and have no isolated points, hence of first Baire category. But they also have large Lebesgue measure:

$$\text{meas}\big(\Omega\backslash\Omega_\alpha\big) = 0(\alpha).$$

The main theorem of Kolmogorov, Arnold and Moser can now be stated as follows

Theorem 2.3 *([Ko], [Ar], [Mo]). Suppose the Hamiltonian $H = H_0 + \varepsilon H_1$ is real analytic on the closure of $\mathbb{T}^n \times D$ where D is a bounded domain in $\mathbb{R}^n$. If the integrable Hamiltonian H_0 is nondegenerate, and its frequency map is a diffeomorphism $D \to \Omega$, then there exists a constant $\delta > 0$ such that for $|\varepsilon| < \delta\alpha^1$ all invariant tori $\mathbb{T}^n \times I$ of the unperturbed system with $\omega(I) \in \Omega_\alpha$ persist, being only slightly deformed. Moreover, they depend in a Lipschitz continuous way on ω and fill the phase space $\mathbb{T}^n \times D$ up to a set of measure $0(\alpha)$.*

Since its conception the KAM theorem has been generalized and extended in numerous ways, and all its assumptions have been relaxed - see [Bo], [BHS], [Laz] and the references therein.

The perturbation theory for lower dimensional elliptic tori in its general form is due to Eliasson [E] - see Theorem 6.1, where a version of this result is extended to NLS, viewed as an infinite dimensional integrable system (cf also [Ku3], [Pö]). Recent significant extensions of Eliasson's work are due to Rüssmann [Rü] and Xu, You and Qiu [XYQ].

There has been a great interest in extending the classical KAM theory to infinite dimensional systems in order to apply it, in particular, to certain nonlinear partial differential equations in Hamiltonian form. The first results in this direction are due to Kuksin [Ku0] and Wayne [Wa]. In subsequent developments Kuksin applied then to various integrable PDE's [Ku3].

Another approach was taken by Craig and Wayne [CW]. They extended Lyapunov's classical result about the existence of families of periodic solutions near an elliptic equilibrium and constructed Cantor discs of periodic solutions for a nonlinear wave equation with periodic boundary conditions. Their approach was considerably extended by Bourgain, who not only obtained quasi-periodic solutions for Schrödinger equations in this way, but also periodic and quasi-periodic solutions for some two-dimensional Schrödinger and wave equations - see [B2]. All these results concern the existence of quasi-periodic solutions filling finite dimensional invariant tori in an infinite dimensional phase space. Almost nothing is known, however, about the existence of almost-periodic solutions for nonlinear PDE's. It seems that the nonlinearities affect strongly the long range coupling which is beyond the control of the current techniques.

3 Birkhoff normal form

By Theorem 1.1, we already know that $\mathcal{H}$, the Hamiltonian of NLS, admits a Birkhoff normal form in a neighborhood $U \subset (H^N(S^1;\mathbb{C}))^2$ of $(0,0)$. In [KP], S. Kuksin and J. Pöschel have computed explicitely the first few coefficients of the Birkhoff normal form of $\mathcal{H}$ when restricted to the reduced phase space formed by functions satisfying generalized Dirichlet boundary conditions.

Using the fact that NLS admits Birkhoff coordinates near $\varphi = 0$, the coefficients of the Birkhoff normal form of $\mathcal{H}$ up to order 2 (included) can be read off by considering $\mathcal{H}(\varphi, \overline{\varphi})$ as a function of the Fourier coefficients $\hat{\varphi}(k)$ $(k \in \mathbb{Z})$ of $\varphi \in H^N(S^1;\mathbb{C})$.

By a slight abuse of notation, we also denote by $\mathcal{H}$ the pull back $\Phi^*\mathcal{H} := \mathcal{H}\cdot\Phi$ of $\mathcal{H}$ by the Birkhoff map Φ. Of course, $\mathcal{H}$ is a function of the actions alone, $\mathcal{H} = \mathcal{H}((I_j)_{j\in\mathbb{Z}})$.

Proposition 3.1 *(cf [KP]) In a neighborhood of $I = 0$,*

$$\mathcal{H} = \Lambda_2 + \Lambda_4 + 0_6(I) \tag{14}$$

where

$$\Lambda_2(I) := \sum_{j\in\mathbb{Z}} (2\pi j)^2 I_j \tag{15}$$

$$\Lambda_4(I) := 2\sum_{j\neq k} I_k I_j + \sum_{j\in\mathbb{Z}} I_j^2 \tag{16}$$

and where $0_6(I)$ is a remainder term of order 6, i.e. $0_6(I) = \sum_{|\alpha|\geq 3} a_\alpha I^\alpha$, the action variables being counted of order 2.

Proof: Substitute $\varphi(x) = \sum_{k\in\mathbb{Z}} \hat{\varphi}(k) e^{2i\pi kx}$ into (3),

$$\mathcal{H}(\varphi,\overline{\varphi}) = \sum_{k\in\mathbb{Z}} (2\pi k)^2 \hat{\varphi}(k)\overline{\hat{\varphi}(k)} + G(\varphi,\overline{\varphi}) \tag{17}$$

where

$$G(\varphi,\overline{\varphi}) = \sum_{k+l=j+m} \hat{\varphi}(k)\hat{\varphi}(l)\overline{\hat{\varphi}(j)}\ \overline{\hat{\varphi}(m)}. \tag{18}$$

Notice that (17) is already in Birkhoff normal form up to order 2. Recall that the Poisson bracket with respect to $\hat{\varphi}(k), \overline{\hat{\varphi}(k)}$ $(k \in \mathbb{Z})$ is given by

$$\{F,G\} = i\sum_{k\in\mathbb{Z}} \Big(\frac{\partial F}{\partial\hat{\varphi}(k)} \frac{\partial G}{\partial\overline{\hat{\varphi}(k)}} - \frac{\partial F}{\partial\overline{\hat{\varphi}(k)}} \frac{\partial G}{\partial\hat{\varphi}(k)} \Big)$$

and that near $\varphi = 0$, the action I_j is close to $|\hat{\varphi}(j)|^2 (j \in \mathbb{Z})$. Using the fact that the coefficients in the Birkhoff normal form are unique, we then conclude that $\Lambda_1(I)$ can be read off from (17)

$$\Lambda_1(I) = \sum_{k\in\mathbb{Z}} 4\pi^2 k^2 I_k.$$

Similarly, we can read off the coefficients of the 4'th order from (17). In fact we only have to consider

$$S := \sum_{\{k,l\}=\{j,m\}} \hat{\varphi}(k)\hat{\varphi}(l)\overline{\hat{\varphi}(j)\hat{\varphi}(m)}.$$

This sum is split into two parts, $S = S_1 + S_2$, where $S_1 := \sum_{k\in\mathbb{Z}} |\hat{\varphi}(k)|^4$ includes the terms in S with $k = l$ and $S_2 := 2\sum_{k\neq l} |\hat{\varphi}(k)|^2|\hat{\varphi}(l)|^2$. Arguing as above, we obtain (16). ■

From Proposition 3.1, we obtain an expansion of the NLS-frequencies $\omega_j(I) = \frac{\partial \mathcal{H}}{\partial I_j}$ $(j \in \mathbb{Z})$ at $I = 0$:

Corollary 3.2 *In a neighborhood of* $I = 0$,

$$\omega_j(I) = (2\pi j)^2 + 2(2\sum_{k\in\mathbb{Z}} I_k - I_j) + 0_4(I); \tag{19}$$

$$\frac{\partial \omega_j}{\partial I_k} = 2(2 - \delta_{jk}) + 0_2(I); \tag{20}$$

As a consequence, for any finite set $A \subseteq \mathbb{Z}$ *with* $|A| \geq 2$,

$$\det\left(\left(\frac{\partial \omega_j}{\partial I_k}\right)_{j,k\in A}\right)|_{I=0} = -(-1)^{|A|}(2(|A| - 2) + 3) \neq 0. \tag{21}$$

Proof: The asymptotics (19) and (20) follow immediately from Proposition 3.1. Towards (21), notice that at $I = 0$

$$\frac{1}{2}(\frac{\partial \omega_j}{\partial I_k})_{j,k\in A} = 2E_{|A|} - Id_{|A|},$$

where $E_{|A|}$ is the $|A| \times |A|$ matrix whose entries are all equal to 1 and where $Id_{|A|}$ is the $|A| \times |A|$ identity matrix. By substracting the first column from the other ones and then expanding with respect to the last column, we obtain, with $d_n := \det(2E_n - I_n)$,

$$d_n = -d_{n-1} + (-1)^{n+1}2.$$

Hence, for $n \geq 2$, we get $d_n = (-1)^{n+1}(2(n - 2) + 3)$. ■

4 Symmetries of NLS

In this section, we recall results of [GK3] where symmetries of NLS are expressed in action-angle coordinates and prove Proposition 1.2 stated in the introduction. We briefly describe the steps needed to prove that the reduced phase space $H^N_\alpha(S^1;\mathbb{C})$ (defined in (8)) is equal to $\{\varphi \in H^N(S^1;\mathbb{C}) | e^{i\alpha}\check{\varphi} = \varphi\}$.

Introduce the symmetry operator acting on L^2, $S_\alpha(\varphi) := e^{i\alpha}\check{\varphi}$ where, as above, $\check{\varphi}$ is defined by $\check{\varphi}(x) = \varphi(-x)$ and $\alpha \in \mathbb{R}$.

In view of (1), the NLS flow commutes with S_α and from (3) it follows that the Hamiltonian $\mathcal{H}(\varphi) \equiv \mathcal{H}(\varphi, \overline{\varphi})$ is invariant under S_α,

$$\mathcal{H}(S_\alpha \varphi) = \mathcal{H}(\varphi). \tag{22}$$

Concerning the periodic spectrum we have (cf [GK3]) the following

Lemma 4.1 *For $\varphi \in L^2$ and $\alpha \in \mathbb{R}$,*

$$\lambda^{\pm}_k(S_\alpha(\varphi)) = -\lambda^{\mp}_{-k}(\varphi) \ (k \in \mathbb{Z}).$$

In particular, $spec_{per}(L(S_\alpha(\varphi))) = -spec_{per}(\varphi)$.

Using this lemma, the definition of the action and angle variables (see [GKP]) we derived in [GK3] the following

Lemma 4.2 *For $\varphi \in L^2$ and $\alpha \in \mathbb{R}$,*

(i) $I_k(S_\alpha(\varphi)) = I_{-k}(\varphi) \quad (k \in \mathbb{Z})$;

(ii) $\theta_k(S_\alpha(\varphi)) \equiv \theta_{-k}(\varphi) + \alpha \pmod{2\pi} \quad \forall \mathrm{k} \in \mathbb{Z}$ *with* $I_k(S_\alpha(\varphi)) \neq 0$.

As an application of Lemma 4.2 we have

Proposition 4.3 *For all $\alpha \in \mathbb{R}$,*

$$H^N_\alpha(S^1;\mathbb{C}) = \{\varphi \in H^N(S^1;\mathbb{C}) | e^{i\alpha}\check{\varphi} = \varphi\}.$$

Proof: Introduce $G^N_\alpha := \{\varphi \in H^N(S^1;\mathbb{C}) | e^{i\alpha}\check{\varphi} = \varphi\}$. Using Lemma 4.2 and Theorem 1.1, we get $G^N_\alpha \subset H^N_\alpha$. Conversely, if $\varphi \in H^N_\alpha$, it follows from Lemma 4.2 that $I_k(S_\alpha(\varphi)) = I_k(\varphi)$ $(k \in \mathbb{Z})$ and $\theta_k(S_\alpha(\varphi)) \equiv \theta_k(\varphi) \pmod{2\pi}$

($\forall$k $\in \mathbb{Z}$ with $I_k(\varphi) \neq 0$). Since the Birkhoff map Φ is one to one (cf Theorem 1.1), $S_\alpha(\varphi) = \varphi$, i.e. $\varphi \in G_\alpha^N$. ∎

Lemma 3.2 can also be used to prove Proposition 1.2, which is stated in the introduction.

Proof (of Propostion 1.2). As in the introduction, denote by $I'(\varphi) = (I'_k(\varphi))_{k\in\mathbb{Z}}$ the sequence $I'_k(\varphi) := I'_{-k}(\varphi)$ $(k \in \mathbb{Z})$. The formula in Lemma 4.2 (i) then reads

$$I(S_\alpha(\varphi)) = I'(\varphi). \tag{23}$$

On the other hand, when expressing the Hamiltonian with respect to action coordinates, formula (22) reads

$$\mathcal{H}(I(S_\alpha(\varphi)) = \mathcal{H}(I) \tag{24}$$

Combining (23) and (24) leads to the claimed identity $\mathcal{H}(I') = \mathcal{H}(I)$. This proves (i). Statement (ii) follows immediately from (i). ∎

5 Frequencies of NLS

In this section we obtain new formulas for the NLS-frequencies

$$\omega_n := \frac{\partial \mathcal{H}}{\partial I_n} \quad (n \in \mathbb{Z})$$

in terms of differentials defined on the Riemann surface associated to the spectral data of a potential φ. These formulas then lead to the asymptotics of ω_n for $n \to \pm\infty$ which will be needed for the KAM result.

5.1 Differential forms

Let $\varphi \in L^2(S^1;\mathbb{C})$ and $(\lambda_k^\pm)_{k\in\mathbb{Z}} \equiv spec\,(L(\varphi))$. The periodic eigenvalues can be used to define the discriminant $\Delta(\lambda)$ by the following product representation (cf e.g. [GG] or [GKP])

$$\Delta(\lambda) - 2 = -2(\lambda_0^+ - \lambda)(\lambda_0^- - \lambda) \prod_{\substack{k\in 2\mathbb{Z}\\ k\neq 0}} \frac{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}{k^2\pi^2}$$

and

$$\Delta(\lambda) + 2 = 2 \prod_{k \in 2\mathbb{Z}+1} \frac{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}{k^2\pi^2}.$$

These infinite products converge [2] as the eigenvalues $\lambda_k^\pm$ satisfy the asymptotic estimate $\lambda_k^\pm = k\pi + \ell^2(k)$, meaning that

$$(\lambda_k^\pm - k\pi)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}). \tag{25}$$

Denote by Σ_φ the hyperelliptic Riemann surface

$$\Sigma_\varphi := \{(\lambda, y) \in \mathbb{C}^2 \mid y^2 = \Delta(\lambda)^2 - 4\}.$$

Let Σ_φ^c be the canonical sheet of Σ_φ determined by the normalization

$$i\sqrt{\Delta(\lambda)^2 - 4} > 0 \quad \text{ for any } \lambda_0^+ < \lambda < \lambda_1^-. \tag{26}$$

On Σ_φ^c, $\sqrt[c]{\Delta(\lambda)^2 - 4}$ is then given by

$$\sqrt[c]{\Delta(\lambda)^2 - 4} = 2i\sqrt[s]{(\lambda_0^- - \lambda)(\lambda_0^+ - \lambda)} \prod_{k\neq 0} \frac{\sqrt[s]{(\lambda_k^- - \lambda)(\lambda_k^+ - \lambda)}}{k\pi}$$

where for $a, b \in \mathbb{R}$ with $a \leq b$, $\sqrt[s]{(a - \lambda)(b - \lambda)}$ denotes the *standard* square root, defined on $\mathbb{C}\backslash[a, b]$ and determined by

$$\sqrt[s]{(a - \lambda)(b - \lambda)} < 0 \quad \text{ for } b < \lambda < \infty.$$

For any $k \in \mathbb{Z}$, let a_k be a counterclockwise oriented cycle on Σ_φ^c around the gap $[\lambda_k^-, \lambda_k^+]$ and denote by β_n $(n \in \mathbb{Z})$ the differential on Σ_φ (cf [GKP], [MV])

$$\beta_n := \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}}\, d\lambda$$

where $\psi_n(\lambda)$ is an entire function given by, for $n \neq 0$,

$$\psi_n(\lambda) := -2\, \frac{\nu_0^n - \lambda}{\tau_n - \lambda} \prod_{k\neq 0} \frac{\nu_k^n - \lambda}{k\pi}$$

[2] Given a sequence $(a_k)_{k\in\mathbb{Z}}$, we say that the infinite product $\prod_{k\in\mathbb{Z}} a_k$ is convergent if $\lim_{N\to\infty} \prod_{|k|\leq N} a_k$ exists.

whereas for $n = 0$,

$$\psi_0(\lambda) := -2\prod_{k\neq 0} \frac{\nu_k^0 - \lambda}{k\pi}.$$

The zeroes ν_k^n $(k \neq n)$ of β_n are uniquely determined by the normalization conditions

$$\int_{a_k} \beta_n = 2\pi\delta_{nk}$$

and $\nu_n^n := \tau_n$ with $\tau_n = (\lambda_n^+ + \lambda_n^-)/2$. For any $k \in \mathbb{Z}$,

$$\lambda_k^- \leq \nu_k^n \leq \lambda_k^+.$$

From the asymptotics (25) of the eigenvalues it follows that

$$\nu_k^n = k\pi + \ell^2(k)$$

insuring the convergence of the infinite products defining the β_n.

The one-forms β_n are holomorphic except at the points at infinity $\infty^\pm$ on each of the two sheets $\Sigma_\varphi^\pm$ of Σ_φ. In the case where φ is a finite gap potential, the β_n have a simple pole at infinity and thus are Abelian differentials of the third kind.

To compute the asymptotic expansion of β_n at ∞ on Σ_φ^c we need first to establish an auxilary result for *finite gap* potentials: Let $(x_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers with $\lambda_k^- \leq x_k \leq \lambda_k^+$ and define

$$g(\lambda) := 2(x_0 - \lambda)\prod_{k\neq 0} \frac{x_k - \lambda}{k\pi}.$$

As $x_k = k\pi + \ell^2(k)$, the infinite product above converges and g defines an entire function (cf [GKP] or [GG]).

Lemma 5.1 *For any finite gap potential φ and any entire function g as above, $i\frac{g(\lambda)}{\sqrt{\Delta(\lambda)^2-4}}$ admits on Σ_φ^c an asymptotic expansion for $|\lambda| \to \infty$ of the form*

$$i\frac{g(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} = 1 + a\lambda^{-1} + \frac{1}{2}\left(a^2 + \sum_k(\tau_k^2 - x_k^2) + \frac{1}{4}\sum_k \gamma_k^2\right)\lambda^{-2} + 0\left(\lambda^{-3}\right) \tag{27}$$

where

$$a := \sum_{k\in\mathbb{Z}} (\tau_k - x_k) \qquad (28)$$

Remark Note that the sums in (27) and (28) are finite as φ is assumed to be a finite gap potential.

Proof Using the definition of $\sqrt[c]{\Delta(\lambda)^2 - 4}$ one gets

$$\frac{g(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} = -i \prod_{k\in\mathbb{Z}} \frac{x_k - \lambda}{\sqrt[s]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}}.$$

With $z = -\lambda^{-1}$ as a local parameter near ∞ on Σ_φ^c,

$$\frac{g(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} = -i \prod_{k\in\mathbb{Z}} \frac{1 + z x_k}{\sqrt[+]{(1 + z\lambda_k^+)(1 + z\lambda_k^-)}}$$

where, in view of the definition of the standard root $\sqrt[s]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}$,

$$\sqrt[s]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)} = -\lambda \sqrt[+]{(1 - \lambda_k^+/\lambda)(1 - \lambda_k^-/\lambda)}$$

for $|\lambda|$ large enough. In particular, for any k with $\lambda_k^+ = \lambda_k^-$ one gets

$$\sqrt[s]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)} = \lambda_k^+ - \lambda.$$

Hence $f(z) := i \frac{g(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} \Big|_{\lambda = -z^{-1}}$ is given by

$$f(z) = \prod_{k\in\mathbb{Z}} f_k(z)$$

with

$$f_k(z) = \frac{1 + x_k z}{\sqrt[+]{(1 + \lambda_k^+ z)(1 + \lambda_k^- z)}}$$

and $f_k(z) \equiv 1$ for $|k|$ sufficiently large. Therefore $f(z)$ admits a Taylor expansion at $z = 0$,

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2} z^2 + 0(z^3)$$

where $f'(0)$ and $f''(0)$ are given by

$$f'(0) = \sum_{k\in\mathbb{Z}} x_k - \tau_k \quad (=: -a)$$

and

$$f''(0) = a^2 + \sum_{k\in\mathbb{Z}} \left(\tau_k^2 + \gamma_k^2/4 - x_k^2\right).$$

The sums in the expressions for $f'(0)$ and $f''(0)$ are finite as φ is assumed to be a finite gap potential. ■

Lemma 5.1 is now applied to complete the asymptotic expansion of the one-form β_n.

Corollary 5.2 *For any finite gap potential φ, $\frac{\psi_n(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}$ admits for $|\lambda| \to \infty$ an asymptotic expansion of the form*

$$\begin{aligned} i\frac{\psi_n}{\sqrt[c]{\Delta(\lambda)^2-4}} =& \frac{1}{\lambda} + (\tau_n+\alpha_n)\frac{1}{\lambda^2} + \Big(\tau_n^2 + \alpha_n\tau_n + \frac{1}{2}\alpha_n^2 + \frac{1}{2}\sum_k \left(\tau_k - (\nu_k^n)^2\right) \\ &+ \frac{1}{8}\sum_k \gamma_k^2\Big)\frac{1}{\lambda^3} + 0\Big(\frac{1}{\lambda^4}\Big) \end{aligned}$$

where $\alpha_n := \sum_k (\tau_k - \nu_k^n)$.

Proof Let φ be a finite gap potential. By definition,

$$\psi_n(\lambda) \;=\; \frac{1}{\lambda-\tau_n}\, 2(\nu_0^n - \lambda)\prod_{k\neq 0}\frac{\nu_k^n - \lambda}{k\pi}.$$

Thus combining Lemma 5.1 with

$$\frac{1}{\lambda-\tau_n} = \frac{1}{\lambda}\sum_{k\geq 0}(\tau_n/\lambda)^k$$

the claimed formula follows. ■

5.2 NLS-frequencies

To obtain formulas for the NLS-frequencies we use that according to [GKP] (see also [MV]), the NLS-Hamiltonian appears in the expansion at infinity of the discriminant $\Delta(\lambda)$, or more conveniently of $ch^{-1}\left(\frac{\Delta(\lambda)}{2}\right)$ where ch^{-1} denotes the branch of *arccosh* defined on $\mathbb{C}\backslash(-\infty,1)$ which for $|z|>1$ is given by

$$ch^{-1}(z)=\log\left(z+z\sqrt[+]{1-\frac{1}{z^2}}\right)$$

with log denoting the principal branch of the logarithm.
Denote by H_j $(j\geq 1)$ the NLS-hierarchy. Recall that H_1,H_2 are given by

$$H_1\equiv H_1(\varphi):=\int_0^1\varphi\overline{\varphi}\,dx\qquad\text{for }\varphi\in H^0$$
$$H_2\equiv H_2(\varphi):=i\int_0^{\prime}\varphi'\overline{\varphi}\,dx\qquad\text{for }\varphi\in H^1$$

with $\varphi'(x)=\frac{d}{dx}\varphi(x)$, whereas H_3 is the NLS-Hamiltonian (denoted by $\mathcal{H}$ in the introduction)

$$H_3\equiv H_3(\varphi):=\int_0^1\left(|\varphi'|^2+|\varphi|^4\right)dx\qquad\text{for }\varphi\in H^1\ .$$

According to [MV] (cf also [GKP]) one has

Lemma 5.3 *At any finite gap potential,*

$$ch^{-1}\left(\frac{\Delta(\lambda)}{2}\right)=-i\lambda+\frac{iH_1}{2\lambda}+\frac{iH_2}{4\lambda^2}+\frac{iH_3}{8\lambda^3}+0\left(\frac{1}{\lambda^4}\right)$$

for λ near ∞.

Recall that for any $N>0$, $(\lambda,\varphi)\mapsto\Delta(\lambda,\varphi)$ is analytic on $\mathbb{C}\times H^N(S^1;\mathbb{C})$. As the Birkhoff map $\varphi\mapsto(x_k(\varphi),y_k(\varphi))_{k\in\mathbb{Z}}$ is a canonical diffeomorphism from $H^N(S^1;\mathbb{C})$ to $\ell^2_N(\mathbb{Z};\mathbb{R}^2)$ and $\Delta(\lambda,\varphi)$ is a spectral invariant, $\Delta(\lambda)$ is a function of the actions alone and $(\lambda,(I_k)_{k\in\mathbb{Z}})\mapsto\Delta(\lambda)$ is analytic on $\mathbb{C}\times\ell^1(\mathbb{Z};\mathbb{R})$. Hence, for any $n\in\mathbb{Z}$,

$$\lambda\mapsto\frac{\partial\Delta(\lambda)}{\partial I_n}$$

is an entire function in λ.
For $K \geq 1$, denote by G_K the following set of finite gap potentials

$$G_K := \{\varphi \in L^2(S^1;\mathbb{C}) \mid \lambda_k^+ = \lambda_k^- \text{ iff } |k| > K\} .$$

For $\varphi \in G_K$ and $|n| \leq K$, $\lambda \mapsto \frac{\partial \Delta(\lambda)}{\partial I_n}$ vanishes at each λ_k^+ with $|k| > K$ and thus

$$\eta_n := -2 \frac{\frac{\partial \Delta(\lambda)}{\partial I_n}}{\sqrt{\Delta(\lambda)^2 - 4}}\, d\lambda$$

is an Abelian differential on Σ_φ which is holomorphic except possibly at $\infty^\pm$.
In [GKP] (cf also [MV]) it is proved that H_1 can be expressed in terms of the action variables $(I_n)_{n\in\mathbb{Z}}$ as follows

$$H_1 = \sum_{k\in\mathbb{Z}} I_k.$$

Hence for any $n \in \mathbb{Z}$,

$$\frac{\partial H_1}{\partial I_n} = 1.$$

Lemma 5.3 then leads to an expansion of η_n at infinity on Σ_φ^c. Introduce, for any $n \in \mathbb{Z}$,

$$w_n := \frac{\partial H_2}{\partial I_n} \text{ and } \omega_n := \frac{\partial H_3}{\partial I_n}.$$

Corollary 5.4 *Let φ be a finite gap potential in G_K for some $K \geq 1$. Then for any $|n| \leq K$, η_n is a holomorphic one-form on Σ_φ except at $\infty^\pm$ where it has a simple pole. At infinity ∞^c of the canonical sheet Σ_φ^c, η_n admits an asymptotic expansion of the form*

$$\eta_n = \frac{1}{i}\left(\frac{1}{\lambda} + \frac{w_n}{2}\frac{1}{\lambda^2} + \frac{\omega_n}{4}\frac{1}{\lambda^3} + 0\left(\frac{1}{\lambda^4}\right)\right)\, d\lambda \tag{29}$$

Proof We already know that η_n is holomorphic on $\Sigma_\varphi \backslash \{\infty^\pm\}$. Next, let us prove the expansion (29). By a straightforward computation one sees that on Σ^c, for $|\lambda|$ sufficiently large

$$\eta_n = \frac{\partial}{\partial I_n}\left(-2h(\lambda)\right)\, d\lambda \tag{30}$$

where h is given by

$$h(\lambda) := \log\left(\Delta(\lambda) - \sqrt[c]{\Delta(\lambda)^2 - 4}\right)$$

with log denoting the principal branch of the logarithm. Recall that (cf [GG], [GKP]) as $|\lambda| \to \infty$,

$$\Delta(\lambda) = 2\cos\lambda + o(1).$$

Hence $\Delta(iy) = 2chy + o(1)$ and

$$\Re\left(\Delta(iy)\right) > 0 \quad \text{and} \quad |\Delta(iy)| > 4$$

for y large enough. By the definition of the canonical root it then follows that

$$\sqrt[c]{\Delta(iy)^2 - 4} = -\Delta(iy)\sqrt[+]{1 - 4/\Delta(iy)^2}$$

and thus

$$h(y) = ch^{-1}\left(\frac{\Delta(iy)}{2}\right)$$

for y sufficiently large.

In view of Lemma 5.3, it then follows that

$$-\eta_n = i\left(\frac{1}{\lambda} + \frac{w_n}{2\lambda^2} + \frac{\omega_n}{4\lambda^3} + 0\left(\frac{1}{\lambda^4}\right)\right) d\lambda$$

as claimed. In the same fashion one shows that η_n admits an expansion at infinity on the second sheet of Σ_φ of the same type as (29). This implies that η_n has simple poles at $\infty^\pm$. ■

It turns out that the one forms η_n can be identified with the one-forms β_n introduced in subsection 5.1.

Proposition 5.5 *Let φ be a finite gap potential in G_K for some $K \geq 1$. Then for any $|n| \leq K$*

$$\beta_n = \eta_n$$

Proof Let $\varphi \in G_K$. By Corollary 5.4, η_n ($|n| \leq K$) is a meromorphic one-form on Σ_φ which is holomorphic except at $\infty^\pm$ where it has simple poles. To compute $\int_{a_k} \eta_n$, first remark that

$$\int_{a_k} \eta_n = -2\int_{\Gamma_k} \frac{\frac{\partial \Delta}{\partial I_n}}{\sqrt[c]{\Delta(\lambda)^2 - 4}}\, d\lambda$$

where Γ_k is a counterclockwise oriented circuit around $[\lambda_k^-, \lambda_k^+]$ in $\mathbb{C}$. As on Γ_k,

$$\frac{\partial}{\partial \lambda} ch^{-1}\left(\frac{\Delta(\lambda)}{2}\right) = \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} \quad \text{and} \quad \frac{\partial}{\partial I_n} ch^{-1}\left(\frac{\Delta(\lambda)}{2}\right) = \frac{\frac{\partial \Delta}{\partial I_n}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}}$$

one gets, integrating by parts,

$$\begin{aligned}
\int_{a_k} \eta_n &= 2 \int_{\Gamma_k} \lambda \frac{\partial}{\partial \lambda}\left(\frac{\frac{\partial \Delta}{\partial I_n}}{\sqrt[c]{\Delta(\lambda)^2 - 4}}\right) d\lambda \\
&= 2 \frac{\partial}{\partial I_n} \int_{\Gamma_k} \frac{\lambda \dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} d\lambda \\
&= \frac{\partial}{\partial I_n}(2\pi\, I_k) = 2\pi \delta_{nk}
\end{aligned}$$

where we used that by definition

$$I_k = \frac{1}{\pi} \int_{\Gamma_k} \frac{\lambda\, \dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} d\lambda.$$

By the construction of the functions ψ_n,

$$\int_{a_k} \beta_n = 2\pi \delta_{nk}$$

hence for any $k \in \mathbb{Z}$

$$\int_{a_k} \beta_n = \int_{a_k} \eta_n.$$

Further η_n and β_n are holomorphic differentials except at $\infty^\pm$ where they admit simple poles. By Cauchy's theorem,

$$\begin{aligned}
\lim_{j \to \infty} \int_{C(0,j)} \beta_n &= \sum_{k \in \mathbb{Z}} \int_{a_k} \beta_n = \sum_{k \in \mathbb{Z}} 2\pi \delta_{nk} \\
&= \sum_{k \in \mathbb{Z}} \int_{a_k} \eta_n = \lim_{j \to \infty} \int_{C(0,j)} \eta_n
\end{aligned}$$

with $C(0,j)$ denoting the counterclockwise oriented circle of radius j centered at 0 on Σ_φ^c. Hence β_n and η_n have the same residue at ∞^+ and, arguing in the same way, at ∞^-. Thus $\beta_n - \eta_n$ is a holomorphic differential on Σ satisfying

$$\int_{a_k} (\beta_n - \eta_n) = 0 \quad \forall k \in \mathbb{Z}.$$

This implies that $\beta_n = \eta_n$. ■

Determining the expansion of β_n on Σ_φ^c for $|\lambda| \to \infty$ and comparing it with η_n leads to a formula for the frequency ω_n.

Theorem 5.6 *The frequencies of the 2^{nd} and 3^{rd} Hamiltonian H_2 resp. H_3 in the NLS-hierarchy defined for potentials in the appropriate Sobolev spaces are*

$$w_n := \frac{\partial H_2}{\partial I_n} = 2(\tau_n + \alpha_n) \tag{31}$$

$$\omega_n := \frac{\partial H_3}{\partial I_n} = \left((2\tau_n)^2 + 4\alpha_n\tau_n + 2\alpha_n^2 + 2\sum_k (\tau_k^2 - (\nu_k^n)^2) + \frac{1}{2}\sum_k \gamma_k^2 \right) \tag{32}$$

where

$$\alpha_n := \sum_{k\in\mathbb{Z}} (\tau_k - \nu_k^n)$$

Proof Let us fix $n \in \mathbb{Z}$. It has been shown in [GKP] that the ν_k^n as well as $\tau_k := (\lambda_k^+ + \lambda_k^-)/2$ and $\gamma_k^2 := (\lambda_k^+ - \lambda_k^-)^2$ are real analytic functions of φ. The ν_k^n's satisfy the asymptotics

$$\nu_k^n - \tau_k = \gamma_k \ell^2(k)$$

and for any $\varphi \in L^2(S^1;\mathbb{C})$ one has (cf [GK2])

$$(\gamma_k)_{k\in\mathbb{Z}} \in \ell_N^2(\mathbb{Z};\mathbb{R}) \iff \varphi \in H^N.$$

Hence left and right hand side of the identities (31) and (32) are well defined on appropriate Sobolev spaces and continuous. As the set of finite gap potentials is dense in any Sobolev space H^N (cf [GK4]), it suffices to prove these identities for the dense set $\cup_{K\geq 1} G_K$ of finite gap potentials (cf Theorem 1.1). Assume that $\varphi \in G_K$ with $|n| \leq K$. By Corollary 5.2, for $|\lambda|$ sufficiently large

$$\begin{aligned} i\frac{\psi_n}{\sqrt[c]{\Delta(\lambda)^2 - 4}} =& \frac{1}{\lambda} + (\tau_n + \alpha_n)\frac{1}{\lambda^2} + \Big(\tau_n^2 + \alpha_n\tau_n + \frac{1}{2}\alpha_n^2 \\ &+ \frac{1}{2}\sum_k \Big(\tau_k^2 - (\nu_k^n)^2\Big) + \frac{1}{8}\sum_k \gamma_k^2\Big)\frac{1}{\lambda^3} + 0\Big(\frac{1}{\lambda^4}\Big) . \end{aligned}$$

Comparing this with the expansion of η_n given in Corollary 5.4,

$$i\eta_n = \left(\frac{1}{\lambda} + \frac{w_n}{2}\frac{1}{\lambda^2} + \frac{\omega_n}{4}\frac{1}{\lambda^3} + 0\left(\frac{1}{\lambda^4}\right)\right)\ d\lambda$$

leads in view of Proposition 5.5 to the claimed identities. ■

Formula (32) can be simplified by the following observation.

Lemma 5.7 *For any $\varphi \in L^2(S^1; \mathbb{C})$ and $n \in \mathbb{Z}$,*

$$\alpha_n = n\pi - \tau_n.$$

Proof The frequencies w_n of H_2 are the frequencies of the translation flow $T_t\varphi := \varphi(t + \cdot)$ which can be computed to be (cf [GK3])

$$w_n = 2\pi n.$$

Combined with formula (31), $w_n = 2(\tau_n + \alpha_n)$, we obtain the claimed identity. ■

Remark Lemma 5.1 can be applied to obtain an asymptotic expansion of $\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}\ d\lambda$ in view of the product representation of $\dot{\Delta}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$,

$$\dot{\Delta}(\lambda) = 2(\dot{\lambda}_0 - \lambda)\prod_{k\neq 0}\frac{\dot{\lambda}_k - \lambda}{k\pi}.$$

Noticing that for any finite gap potential φ and $|\lambda|$ large,

$$\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}}\ d\lambda = d\left(ch^{-1}\left(\frac{\Delta(\lambda)}{2}\right)\right)$$

one can use the asymptotic expansion of $ch^{-1}\left(\frac{\Delta(\lambda)}{2}\right)$ established in Lemma 5.3 to see that for a finite gap potential,

$$i\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}}\ d\lambda = \Big(1 + \frac{H_1}{2}\frac{1}{\lambda^2} + \frac{H_2}{2}\frac{1}{\lambda^3} + \frac{3}{8}H_3\frac{1}{\lambda^4} + 0\big(\frac{1}{\lambda^5}\big)\Big)\ d\lambda.$$

Comparing the two asymptotic expansions then leads as in the proof of Theorem 5.6 to formulas expressing the Hamiltonians H_j in terms of the spectral data $(\lambda_k^{\pm})_{k\in\mathbb{Z}}$ and $(\dot{\lambda}_k)_{k\in\mathbb{Z}}$. For $H_1 = \frac{1}{2}\|\varphi\|^2$ one obtains in this way

$$\|\varphi\|^2 = \sum_k \left((\lambda_k^+)^2 + (\lambda_k^-)^2 - 2\dot{\lambda}_k^2\right).$$

5.3 Asymptotics of the NLS-frequencies

Theorem 5.6 leads to asymptotics for the NLS-frequencies needed for the KAM result.

Theorem 5.8 *For* φ *in* $H^1(S^1;\mathbb{C})$,

$$\omega_n = (2\pi n)^2 + 0(1)$$

locally uniformly on $H^1(S^1;\mathbb{C})$.

Proof By Theorem 5.6 and Lemma 5.7

$$\omega_n = 4\tau_n\pi n + 2(\pi n - \tau_n)^2 + 2\sum_k(\tau_k - \nu_k^n)(\tau_k + \nu_k^n) + \frac{1}{2}\sum_k \gamma_k^2.$$

By [GKP]

$$\nu_k^n - \tau_k = \gamma_k \ell^2(k)$$

locally uniformly on $L^2(S^1;\mathbb{C})$ and by [GG] (cf also [Ma], [GK2])

$$\tau_k = k\pi + 0\left(\frac{1}{k}\right) \quad \text{and} \quad (k\gamma_k)_{k\in\mathbb{Z}} \in \ell^2$$

locally uniformly on $H^1(S^1;\mathbb{C})$. Hence

$$4\tau_n\pi n = (2\pi n)^2 + 0(1) ,$$

$$\sum_{\tau_k}(\tau_k - \nu_k^n)(\tau_k + \nu_k^n) = \sum_k k\gamma_k\ell^2(k) = 0(1)$$

and

$$\sum_k \gamma_k^2 = 0(1)$$

locally uniformly on $H^1(S^1;\mathbb{C})$. Combining these estimates leads to the claimed asymptotics. ■

6 KAM Theorem for NLS

The KAM Theorem for NLS presented in this paper (Theorem 1.3), is derived from an abstract KAM-Theorem with parameters in infinite dimension due to Kuksin [Ku1] (cf also [Ku2] and [Ku3]) and, in a refined form, to Pöschel [Pö]. For the convenience of the reader we present Pöschel's version in the following paragraph (cf also [KPö] and notation established there).

Unlike as in the case of the Korteweg-de Vries equation, this abstract KAM theorem is well suited for NLS once the equations under consideration are expressed in Birkhoff coordinates (cf Theorem 1.1). In paragraph 5.2 we verify that the assumptions of the abstract KAM theorem are satisfied in the case of NLS by using the results of sections 2 and 4.

6.1 Abstract KAM Theorem

For $n \geq 1$ and $N \in \mathbb{R}_{>0}$ fixed, introduce the real phase space

$$\mathcal{S}^N := \mathbb{T}^n \times \mathbb{R}^n \times l^2_N(\mathbb{N};\mathbb{R}) \times l^2_N(\mathbb{N};\mathbb{R})$$

and denote by $\mathcal{S}^N_{\mathbb{C}}$ its complexification.

Using standard coordinates (x, y, u, v) for $\mathcal{S}^N$ the canonical symplectic structure takes the form

$$\sum_{j=1}^{n} dx_j \wedge dy_j + \sum_{j=1}^{\infty} du_j \wedge dv_j.$$

We perturb a family of infinite dimensional integrable Hamiltonians, parametrized by $\zeta \in \Pi$,

$$\mathcal{N} \equiv \mathcal{N}(x, y, u, v; \zeta) := \sum_{j=1}^{n} \omega_j(\zeta) y_j + \frac{1}{2} \sum_{j=1}^{\infty} \Omega_j(\zeta)(u_j^2 + v_j^2). \qquad (33)$$

Here $\Pi \subseteq \mathbb{R}^n$ is a compact parameter set of positive Lebesgue measure and $\omega_j(\cdot)$ $(1 \leq j \leq n)$, $\Omega_j(\cdot)$ $(j \in \mathbb{N})$ are real functions defined on Π.

Notice that for every value $\zeta \in \Pi$,

$$\mathcal{T}_0 := \mathbb{T}^n \times \{0\} \times \{0\} \times \{0\}$$

is an invariant torus for $\mathcal{N}$ and the Hamiltonian flow on $\mathcal{T}_0$, induced by $\mathcal{N}$, is rotational with frequency vector $\omega(\zeta) := (\omega_j(\zeta))_{1\le j\le n}$. Our aim is to prove the persistence of $\mathcal{T}_0$ under small Hamiltonian perturbations, $H = \mathcal{N} + P$ of $\mathcal{N}$ for many parameter values ζ.

We need to make the following three assumptions:

Assumption A: Asymptotics of the exterior frequencies $(\Omega_j(\xi))_{j\ge 1}$
The frequencies $\Omega_j(\zeta)$ are real valued functions of ζ of the form, $\Omega_j(\zeta) = \overline{\Omega}_j + \tilde{\Omega}_j(\zeta)$, $\overline{\Omega}_j$ being independent of ζ, with the following properties:
There exist two real numbers $d > 1$ and $\delta < d - 1$ so that $\overline{\Omega}_j$ admits a finite expansion in j of order d

$$\overline{\Omega}_j = cj^d + \dots$$

where $c \neq 0$ and the dots stand for an expansion in lower order terms in j and, $(j^{-\delta}\tilde{\Omega}_j(\zeta))_{j\ge 1}$ is uniformly Lipschitz when considered as a map

$$\Pi \to l^\infty(\mathbb{N}),\ \zeta \mapsto (j^{-\delta}\tilde{\Omega}_j(\zeta))_{j\ge 1}.$$

Assumption B: Kolmogorov condition and Melnikov condition
The map, $\zeta \to \omega(\zeta)$, between Π and its image is a homeomorphism which is Lipschitz continuous in both directions. Moreover for every $k \in \mathbb{Z}^n$ and $l \in \mathbb{Z}^{\mathbb{N}}$ with $1 \le |l| := \sum_{j=1}^{\infty} |l_j| \le 2$, the resonance set

$$\mathcal{R}_{k,l} := \{\zeta \in \Pi \mid \langle k, \omega(\zeta)\rangle + \langle l, \Omega(\zeta)\rangle = 0\}$$

has Lebesgue measure zero.

The third assumption concerns the perturbation given by the Hamiltonian P. We assume that P is defined on a neighborhood U of $\mathcal{T}_0$ in $\mathcal{S}^N_{\mathbb{C}}$. The corresponding Hamiltonian vectorfield is denoted as follows

$$X_P = (\partial_y P, -\partial_x P, \partial_v P, -\partial_u P).$$

Assumption $\mathbf{C}_{U,N}$: Regularity of the perturbation

In the neighborhood U of $\mathcal{T}_0$ in $\mathcal{S}^N_{\mathbb{C}}$, the Hamiltonian vectorfield X_P has the following properties:

(C.1) X_P *takes values in* $\mathcal{S}^N_{\mathbb{C}}$, $X_p : U \times \Pi \to \mathcal{S}^N_{\mathbb{C}}$, *and* $X_p(\cdot;\zeta)$ *is real analytic for any given* $\zeta \in \Pi$;

(C.2) $X_P(x,y,u,v;\cdot)$ *is uniformly Lipschitz on* Π *for any given* $(x,y,u,v) \in U$.

To state the abstract KAM Theorem we need to introduce various domains and norms. For $s > 0$ and $r > 0$ denote by $D(s,r)$ the complex neighborhoods of $\mathcal{T}_0$,

$$D(s,r) := \{|Imx| < s\} \times \{|y| < r^2\} \times \{||u||_N + ||v||_N < r\}$$

$$\subseteq \mathcal{S}^N_{\mathbb{C}} := \mathbb{C}^n \times \mathbb{C}^n \times l^2_{N,\mathbb{C}} \times l^2_{N,\mathbb{C}}$$

where, in $\mathbb{C}^n$, we use the norm $|z| = max_{1 \le j \le n}|z_j|$ and $l^2_{N,\mathbb{C}} := l^2_N(\mathbb{N};\mathbb{C})$. On $\mathcal{S}^N_{\mathbb{C}}$ we introduce the weighted norm $||W||_{r,N}$ of $W = (X,Y,U,V)$,

$$||W||_{r,N} = |X| + \frac{1}{r^2}|Y| + \frac{1}{r}||U||_N + \frac{1}{r}||V||_N$$

For a map $W : U \times \Pi \to \mathcal{S}^N_{\mathbb{C}}$ with U a neighborhood of the form $D(s,r)$ we introduce the norms

$$\begin{aligned} ||W||^{sup}_{r,N;U\times\Pi} &:= sup_{(w,\xi)\in U\times\Pi}||W(w,\xi)||_{r,N} \\ ||W||^{lip}_{r,N;U\times\Pi} &:= sup_{\substack{\xi,\zeta\in\Pi \\ \xi\neq\zeta}} \frac{||\Delta_{\xi\zeta}W||^{sup}_{r,N;U}}{|\xi-\zeta|}. \end{aligned}$$

where $(\Delta_{\xi\zeta}W)(w) = W(w,\xi) - W(w,\zeta)$. Similarly we define

$$|\tilde{\Omega}|^{lip}_{-\delta;\Pi} := sup\left\{\frac{j^{-\delta}\Delta_{\xi\zeta}\tilde{\Omega}_j}{|\xi-\zeta|} \mid \xi,\zeta \in \Pi;\ \xi \neq \zeta;\ j \ge 1\right\}.$$

By Assumptions A and B, there exist constants $M < \infty$, $L < \infty$ such that

$$|\omega|^{lip}_{\Pi} + |\Omega|^{lip}_{-\delta;\Pi} \le M; \quad |\omega^{-1}|^{lip}_{\omega(\Pi)} \le L < \infty.$$

The following result is due to Kuksin [Ku2] and is stated here in a refined version, due to [Pö].

Theorem 6.1 *Suppose* $\mathcal{N}$ *is a family of Hamiltonians of the form (6.1) defined on* $\mathcal{S}^N \times \Pi$ *so that Assumptions A and B are satisfied. Given a* *neighborhood* $D(s,r)$ *of* $\mathcal{T}_0$ *with* $s>0, r>0$ *there exists a positive constant* γ, *depending only on* n, d, *the frequencies* ω *and* Ω, *and* s, *and such that for every perturbation* $H = \mathcal{N} + P$ *of* $\mathcal{N}$, *defined on* $D(s,r) \times \Pi$ *with* P *satisfying Assumption* $C_{D(s,r),N}$ *and the smallness condition*

$$\varepsilon := ||X_P||^{sup}_{r,N;D(s,r)\times\Pi} + \frac{\beta}{M} ||X_P||^{lip}_{r,N;D(s,r)\times\Pi} \le \beta\gamma \tag{34}$$

for some $0 < \beta < 1$, *the following holds:*
There exist

(i) *a Cantor set* $\Pi_\beta \subset \Pi$, *depending on the perturbation* P, *with* $lim_{\beta\to 0}$ *meas* $(\Pi \setminus \Pi_\beta) = 0$ *uniformly for* P *satisfying Assumption* $C_{D(s,r),N}$ *and* (34)

(ii) *a Lipschitz family of real analytic torus embeddings* $\Psi : \mathbb{T}^n \times \Pi_\beta \to \mathcal{S}^N$

(iii) *a Lipschitz map* $f : \Pi_\beta \to \mathbb{R}^n$

such that for each $\xi \in \Pi_\beta$, *the restriction* $\Psi|_{\mathbb{T}^n \times \{\xi\}}$ *is a real analytic embedding of a rotational torus with frequencies* $f(\xi)$ *for the Hamiltonian* $H = \mathcal{N} + P$ *at* ξ. *In other words,* $t \to \Psi(\theta + tf(\xi), \xi)$ *is a real analytic, quasi-periodic solution for the Hamiltonian* $H(\cdot, \xi)$ *for every* $\theta \in \mathbb{T}^n$ *and* $\xi \in \Pi_\beta$.

Moreover, each embedding is analytic on $D(s/2) := \{|Im\ x\ | < s/2\}$ *and*

$$||\Psi - \Psi_0||^{sup}_{r,N;D(s/2)\times\Pi_\beta} + \frac{\beta}{M} ||\Psi - \Psi_0||^{lip}_{r,p;D(s/2)\times\Pi_\beta} \le \frac{c\varepsilon}{\beta}$$

$$|f - \omega|^{sup}_{\Pi_\beta} + \frac{\beta}{M} |f - w|^{lip}_{\Pi_P} \le c\varepsilon$$

where $\Psi_0 : \mathbb{T}^n \times \Pi \to \mathcal{T}_0, (x,\xi) \mapsto (x,0,0,0)$ *is the trivial embedding and* $c > 0$ *is a constant which depends on the same parameters as* γ.

6.2 Proof of KAM Theorem for NLS

In this paragraph we prove Theorem 1.2 as given in the introduction, by applying the abstract KAM Theorem stated in paragraph 5.1 (Theorem 6.1).

We follow closely the line of arguments presented in [KPö] for the KdV equation.

To simplify notations we consider in this paragraph the phase space $l^2_{N;\alpha}$ $(\mathbb{Z};\mathbb{R}^2)$ with $\alpha = 0$. In this case $l^2_{N;0}(\mathbb{Z};\mathbb{R}^2)$ may be identifyed with $l^2_N(\mathbb{Z}_{\geq 0};\mathbb{R}^2)$. (In the case $\alpha \neq 0$, $l^2_{N;\alpha}(\mathbb{Z};\mathbb{R}^2) \simeq l^2_N(\mathbb{Z}_{\geq 1};\mathbb{R}^2)$).

As a first step, apply the Birkhoff map Φ of Theorem 1.1, restricted to $l^2_N(\mathbb{Z}_{\geq 0};\mathbb{R}^2)$,

$$\tilde{\Phi} : l^2_N(\mathbb{Z}_{\geq 0};\mathbb{R}^2) \to H^N_0(S^1;\mathbb{C}).$$

Again, to simplify notation, let us assume that $A = \{0, 1, \ldots, n-1\}$. Let $\Gamma \subseteq (\mathbb{R}_+)^A$ be a set of positive Lebesgue measure and set

$$T_\Gamma := \bigcup_{I_A \in \Gamma} \{(\sqrt{2\mathcal{J}_j} e^{i\theta_j})_{j \geq 0} \mid \mathcal{J}_j = I_j, j \in A;\ \mathcal{J}_j = 0, j \notin A\}.$$

Since $\tilde{\Phi}$ is analytic, there exists a neighborhood V_Γ of T_Γ in $l^2_N(\mathbb{Z}_{\geq 0};\mathbb{C}^2)$ which is mapped bianalytically onto a neighborhood $U := \tilde{\Phi}(V_\Gamma)$ of $\{(\varphi, \overline{\varphi}), \varphi \in \mathcal{T}^0_\Gamma\}$ in $H^N(S^1;\mathbb{C}) \times H^N(S^1;\mathbb{C})$.

Choosing V_Γ and/or U_Γ (defined in the Introduction) smaller, if necessary, we may assume that $U = U_\Gamma$. Consider the pull back $\tilde{\mathcal{H}} = \mathcal{H} \circ \tilde{\Phi}$ of $\mathcal{H}$ on U. Notice that $\tilde{\mathcal{H}}$ depends only on the actions $(I_j)_{j \geq 0}$. Using Taylor's formula and the definition of the frequencies, $\omega_j(I) := \frac{\partial \tilde{\mathcal{H}}}{\partial I_j}(I)$, we obtain

$$\tilde{\mathcal{H}}(I + I_0) = \mathcal{H}(I_0) + \sum_{j \geq 0} \omega_j(I_0) I_j + Q \tag{35}$$

where $Q := \sum_{i,j \geq 0} Q_{ij}(I_0, I) I_i I_j$ with

$$Q_{ij}(I_0, I) := \int_0^1 (1-t) \frac{\partial^2 \mathcal{H}}{\partial I_i \partial I_j}(I_0 + tI) dt. \tag{36}$$

As a second step we introduce symplectic polar coordinates near the tori in the family T_Γ : Near T_Γ, introduce new coordinates $(x, y, u, v) = \Psi^{-1}(c, b)$ $((c, b) \in l^2_N(\mathbb{Z}_{\geq 0};\mathbb{R}^2))$ depending on the parameter $\xi = (\xi_0, \ldots, \xi_{n-1}) \in \Gamma$, by setting $(0 \leq j \leq n-1)$

$$\sqrt{\xi_j + y_j}\ e^{-ix_j} := c_j + ib_j,\ \sqrt{\xi_j + y_j}\ e^{ix_j} = c_j - ib_j$$

and $(j \geq 1)$

$$u_j := c_{n-1+j};\quad v_j := b_{n-1+j}\ (j \geq 1).$$

For each $\xi \in \Gamma$, this transformation is real analytic and symplectic on $D(s,r)$ for $s > 0$ and $r > 0$ arbitrary but small enough so that $D(s,r) \subset \Psi(V_\Gamma)$. Actually, ξ parametrizes the invariant n-tori and $y = (y_j)_{0 \le j \le n-1}$ are shifted action variables contained in a (small) neighborhood of 0. Using the expansion of $\tilde{\mathcal{H}}$ in (35) and setting $I_0 := (\xi, 0)$, $\quad \tilde{H} = \tilde{\mathcal{H}} \circ \Psi$ is, up to a constant depending only on ξ, given on $D(s,r)$ by

$$\tilde{H} = \mathcal{N}(y,u,v;\xi) + Q(y,u,v;\xi)$$

where

$$\mathcal{N} := \sum_{j=0}^{n-1} \omega_j(\zeta) y_j + \frac{1}{2}\sum_{j=1}^{\infty} \Omega_j(\zeta)(u_j^2 + v_j^2)$$

with $\Omega_j(\xi) := \omega_{n-1+j}(\xi)$ $(j \ge 1)$ and Q denotes the higher order rest.
As the notation indicates, $\mathcal{N}$ will play the role of the integrable normal form, which is perturbed by $P := Q + \varepsilon R$ with $R := K \circ \tilde{\Phi} \circ \Psi$. We now verify the assumptions A, B and C of Theorem 6.1

Write $\omega_j(\xi) = \overline{\Omega}_j + \tilde{\Omega}_j(\xi)$ with

$$\overline{\Omega}_j := 4\pi^2 (n-1+j)^2$$

and

$$\tilde{\Omega}_j(\xi) := \Omega_j(\xi) - \overline{\Omega}_j = \frac{\partial \mathcal{H}}{\partial I_{n-1+j}}(\xi, 0) - 4\pi^2(n-1+j)^2.$$

By Theorem 5.8, $\tilde{\Omega} : \xi \mapsto (\tilde{\Omega}_j(\xi))_{j \ge 1}$ maps Γ into $l^\infty(\mathbb{Z}_{\ge 1}; \mathbb{R})$ and is analytic on a neighborhood of Γ. Thus $\tilde{\Omega}$ is real analytic on some complex neighborhood of Γ (cf [KPö], Appendix D, Analyticity Lemma).
Hence the map is also Lipschitz by Cauchy's estimate. In all, we conclude that assumption A is satisfiied with $d = 2$ and $\delta = 0$.

To verify Assumption B, recall from Corollary 2.2 that on Γ

$$\det\left(\left(\frac{\partial \omega_j}{\partial \xi_k}\right)_{0 \le j,k \le n-1}\right) \not\equiv 0. \tag{37}$$

In particular, for any given $\eta > 0$ we may excise from Γ a relative open subset Γ_η of Lebesgue measure $< \eta$ so that on $\Gamma \setminus \Gamma_\eta$, the determinant (37) in bounded and uniformly bounded away from 0. Moreover, we may cover $\Gamma \setminus \Gamma_\eta$ by finitely many closed subsets Γ_i so that on each such subset, the map

$\xi \mapsto \omega(\xi)$ is a bianalytic homeomorphism on its image in $\mathbb{R}^n$. We consider each of these parameter sets Γ_i separately.
On each such a set we have, by Corollary 2.2,

$$k \cdot \omega(\xi) + \ell \cdot \Omega(\xi) \not\equiv 0$$

for every $k \in \mathbb{Z}^n$ and $l \in \mathbb{Z}^\infty$ with $1 \leq |l| \leq 2$. Since each such expression is analytic in ξ, its zero set is a set of measure zero. Thus, Assumption B is satisfied for each subset Γ_i.

Moreover, we have for some $M < \infty, L < \infty$

$$|\omega|_{\Pi}^{lip} + |\Omega|_{\Pi}^{lip} \leq M < \infty$$

and

$$|\omega^{-1}|_{\omega(\Pi_i \setminus \Pi_\eta)}^{lip} \leq L < \infty.$$

It remains to check Assumption C for the perturbation $P := Q + \varepsilon R$.
Following the same line of arguments as in [KPö] we prove for any $\beta \leq M$, $s > 0$, $r > 0$ small enough and $\eta > 0$,

$$||X_P||_{r,N;D(s,r)\times\Gamma\setminus\Gamma_\eta}^{sup} + \frac{\beta}{M}||X_P||_{r,N;D(s,r)\times\Gamma\setminus\Gamma_\eta}^{lip} \leq C(r^2 + \frac{\varepsilon}{r^2}). \tag{38}$$

In particular, (38) says that

$$X_P : U \times \Gamma \setminus \Gamma_\eta \to \mathcal{S}_{\mathbb{C}}^N$$

with $U \subseteq D(s,r)$. Thus X_p has the required regularity properties on $\Gamma \setminus \Gamma_\eta$ for each $\eta > 0$. To satisfy the smallness condition (34) of Theorem 6.1 for the perturbation P, choose $r^2 := \sqrt{\varepsilon}, \beta = \frac{2c}{\gamma}\sqrt{\varepsilon}$ with c being the constant from (37) and γ chosen as in Theorem 6.1. We then obtain

$$||X_P||_{r,N;D(s,r)\times\Gamma\setminus\Gamma_\eta}^{sup} + \frac{\beta}{M}||X_P||_{r,p;D(s,r)\times\Gamma\setminus\Gamma_\eta}^{lip} \leq \gamma\beta$$

for all ε as required.

In view of the considerations above, Theorem 6.1 can be applied and Theorem 1.3 is proved. ■

Remark 1: Theorem 1.3 remains true for a larger class of phase spaces (cf [GK5]),

$$G_{\eta,\pm}^N(S^1;\mathbb{C}) := \Phi(\{(x,y) \in l_N^2 \mid S_\eta^\pm(x,y) = (x,y)\}$$

where $\eta : l^1_{2N}(\mathbb{Z};\mathbb{R}) \to l^\infty(\mathbb{Z}_{\geq 1},\mathbb{R})$, $I := (I_k)_{k\in\mathbb{Z}} \mapsto \eta(I)$ is an arbitrary real analytic map and $S^\pm_\eta$ is the real analytic isomorphism, $S^\pm_\eta : l^2_N(\mathbb{Z};\mathbb{R}^2) \to l^2_N(\mathbb{Z};\mathbb{R}^2)$, given in action-angle coordinates by

$$\begin{array}{lcl} I'_k &:=& I_{-k} \ \ (k \in \mathbb{Z}\setminus\{0\}); \quad I'_0 := \pm I_0 \\ \theta'_k &:=& \theta_{-k} + \eta_k(I) \ \ (\text{for } k \geq 1 \text{ with } I_k \neq 0); \\ \theta'_k &:=& \theta_{-k} - \eta_{-k}(I) \ \ (\text{for } k \leq -1 \text{ with } I_k \neq 0); \\ \theta'_0 &:=& \theta_0 \ \ (\text{if } I_0 \neq 0). \end{array}$$

As for $\varphi \in G^N_{\eta,\pm}(S^1;\mathbb{C})$, one has $I_k = I_{-k} \quad (k \geq 1)$, the frequencies satisfy $\omega_k(I) = \omega_{-k}(I)$ $(k \geq 1)$ by Proposition 1.2. Therefore, mutatis mutandis, Theorem 1.3 holds for $G^N_\eta(S^1;\mathbb{C})$ as stated in [GK5].
However, we have succeeded in characterizing G^N_η explicitly in terms of the potential φ only in the case where η is a constant, $\eta_j(I) \equiv \alpha$ with $\alpha \in \mathbb{R}$ independent of $j \geq 1$.

Remark 2: Theorem 1.1, and in turn Theorem 1.3, hold for more general phase spaces. However the version of Theorem 1.1 for Abel-Sobolev spaces $H^{N,a}(S^1;\mathbb{C})$ with $a > 0$ in the form announced in [GK5] is not true in view of the fact that nontrivial finite gap potentials are not entire functions.

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