Normal form theory for the NLS equation: A preliminary report.

B. Grébert T. Kappeler J. Pöschel

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Chapter 0

Introduction

The (cubic) nonlinear Schrödinger equation (NLS)

$$i\partial_t\psi = -\partial_x^2\psi + 2\kappa|\psi|^2\psi$$

- κ being a real parameter - is an evolution equation in one space dimension. This equation appears as a nonlinear perturbation of the Schrödinger equation for the wave function of a free one dimensional particle of mass $m = 1/2$ - whence the name. However its physical meaning goes far beyond one-particle quantum mechanics.

Actually the NLS equation describes slowly varying wave envelopes in dispersive media and arises in various physical systems such as water waves, plasma physics, solid-state physics and nonlinear optics. One of the most successful applications of the NLS equation is their use for the description of optical solitons in fibers (see for instance [New-Mol92], [Abl-Seg81], [FT], or [Agr-Boy92] and references quoted therein).

The NLS equation was the second (after the KdV equation) evolution equation discovered to be integrable by the inverse scattering approach (cf [ZS]). It turned out that it has the same degree of universality as the KdV equation, both from a mathematical and physical viewpoint. Actually, in many technical respects, the NLS equation is simpler and maybe more fundamental than the KdV equation. For instance the Hamiltonian formalism for the NLS equation is very simple and straightforward (see next section) while the Poisson bracket in the Hamiltonian formalism for KdV is degenerate (cf [Gard] or [KP]).

In this book we only consider the case where $\kappa = 1$ (defocusing case)

$$i\partial_t\psi = -\partial_x^2\psi + 2|\psi|^2\psi$$

with periodic boundary conditions. Our aim is to provide a complete and self-contained study of this evolution equation viewed as a Hamiltonian systems.

Hamiltonian formalism

The NLS equation can be written in Hamiltonian form

$$\frac{\partial\psi}{\partial t} = -i\frac{\partial\mathcal{H}}{\partial\overline{\psi}}$$

with Hamiltonian $\mathcal{H}$ given by

$$\mathcal{H}(\psi,\overline{\psi}) := \int_{S^1}(\partial_x\psi\partial_x\overline{\psi} + \psi^2\overline{\psi}^2)dx$$

where $\frac{\partial H}{\partial\overline{\psi}}$ denotes the L^2-gradient of H considered as a smooth function of ψ and $\overline{\psi}$. Since we are interested in spatially periodic solutions, we take as the underlying phase space the Sobolev space $H^N \equiv H^N(S^1;\mathbb{C})$ of complex valued function with period 1 with $S^1 = \mathbb{R}/\mathbb{Z}$ and $N \geq 1$. For any $N \geq 0$), H^N is given by

$$H^N(S^1;\mathbb{C}) := \left\{ \psi(x) = \sum_{k\in\mathbb{Z}} e^{2i\pi kx}\hat{\psi}(k) \mid \|\psi\|_N < \infty \right\}$$

where

$$\|\psi\|_N := \left(\sum_{k\in\mathbb{Z}} (1+|k|)^{2N} |\hat{\psi}(k)|^2 \right)^{1/2}$$

and $\hat{\psi}(k)$ $(k \in \mathbb{Z})$ denote the Fourier coefficients of ψ.
We endowed $H^N(S^1;\mathbb{C})$ with the standard Poisson bracket

$$\{F,G\}(\psi,\overline{\psi}) := i \int_{S^1} \left(\frac{\partial F}{\partial \psi}\frac{\partial G}{\partial \overline{\psi}} - \frac{\partial F}{\partial \overline{\psi}}\frac{\partial G}{\partial \psi} \right) dx$$

where F, G are functionals on $H^N \times H^N$ of class C^1 with L^2-gradient in L^2. This makes $H^N \times H^N$ a Poisson manifold on which the NLS equation may also be represented in the form

$$\psi_t = \{\mathcal{H}, \psi\} = -i \frac{\partial \mathcal{H}}{\partial \overline{\psi}}$$

familiar from classical mechanics.
Notice that the above Hamiltonian $\mathcal{H}$ is defined only on H^N with $N \geq 1$. However the initial value problem for the NLS equation on the circle S^1 is well posed on any Sobolev space H^N with $N \geq 0$ and thus in particular on $L^2(S^1;\mathbb{C})$ (cf [B2]).
The NLS equation admits infinitely many conserved quantities, or integrals, and there are many ways to construct such integrals (cf [FT], [MV]).
Following P. Lax [L], one obtains a complete set of integrals in a particular elegant way by considering the spectrum of the associated Zakharov-Shabat operator (cf [ZS]). For $\varphi = (\varphi_1, \varphi_2) \in L^2 \equiv L^2(S^1;\mathbb{C}^2)$ consider the differential operator

$$L(\varphi) := i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix} .$$

We will say that φ is of real type and write $\varphi \in L^2_{\mathcal{R}}$ if $\varphi_2 = \overline{\varphi_1}$. In this case $L(\varphi)$ is formally selfadjoint and unitarily equivalent to the AKNS operator ([AKNS]) given by

$$L_{AKNS}(\varphi) := i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} -q & p \\ p & q \end{pmatrix} .$$

where p, q are real valued functions related to φ by

$$\varphi_1 = -q + ip .$$

It is well known (see [GG]) or chapter I) that for $\varphi \in L^2_{\mathcal{R}}$, the spectrum of $L(\varphi)$ considered on the interval $[0,2]$ with periodic boundary conditions is pure point and consists of an unbounded sequence of periodic eigenvalues

$$\ldots < \lambda^-_{-1}(\varphi) \leq \lambda^+_{-1}(\varphi) < \lambda^-_0(\varphi) \leq \lambda^+_0(\varphi) < \lambda^-_1(\varphi) \leq \lambda^+_1(\varphi) < \ldots .$$

The intervals $\left(\lambda^-_k(\varphi), \lambda^+_k(\varphi)\right)$, possibly empty, are called the gaps of the potential φ and

$$\gamma_k(\varphi) := \lambda^+_k(\varphi) - \lambda^-_k(\varphi), \ k \in \mathbb{Z}$$

are the gap lengths.
For $\varphi = \varphi(t,\cdot) \in L^2_{\mathcal{R}}$, depending also on t, define the corresponding operator

$$L(t) = L\left(\varphi(t,\cdot)\right) \ .$$

Then φ_1 is a solution of the NLS equation if and only if

$$\frac{d}{dt} L = [B, L]$$

where $[B, L] = BL - LB$ denotes the commutator of L with

$$B = i \begin{pmatrix} -2\frac{d^2}{dx^2} + |\varphi_1|^2 & -\frac{\partial \varphi_1}{dx} - 2\varphi_1 \frac{d}{dx} \\ \frac{\partial \bar{\varphi}_1}{dx} + 2\bar{\varphi}_1 \frac{d}{dx} & 2\frac{d^2}{dx^2} - |\varphi_1|^2 \end{pmatrix}$$

It follows by an elementary calculation that the flow of

$$\frac{d}{dt} V = BV, \quad V(0) = I$$

defines a family of unitary operators $V(t)$ such that $V^*(t)L(t)V(t) = L(0)$. Consequently, the spectrum of $L(t)$ is independent of t, and so the periodic eigenvalues $\lambda^\pm_k = \lambda^\pm_k(\varphi)$ are conserved quantities under the evolution of the NLS equation. In other words, the flow of the NLS equation defines an isospectral deformation on the space of all potentials in $L^2_{\mathcal{R}}$. From an analytical point of view, however, the periodic eigenvalues are not satisfactory as integrals, as $\lambda^\pm_k$ is not a smooth function φ whenever the corresponding gap collapses. But in section I.6, we prove that the squared gap lengths

$$\gamma^2_k(\varphi), \ k \in \mathbb{Z}$$

are real analytic on all of $L^2_{\mathcal{R}}$. Moreover, Grébert and Guillot [GG] showed that the sequence of gaps lengths determine uniquely the periodic spectrum of a potential in $L^2_{\mathcal{R}}$. Therefore, the sequence of squared gap lengths form

another set of integrals, which is smooth on $L^2_{\mathcal{R}}$ and which is equivalent to the data of the periodic spectrum.
The space $L^2_{\mathcal{R}}$ decomposes into the isospectral sets

$$\mathrm{Iso}(\varphi) = \left\{\psi \in L^2_{\mathcal{R}} \mid \mathrm{spec}(\psi) = \mathrm{spec}(\varphi)\right\},$$

which are invariant under the NLS flow and may also be characterized as

$$\mathrm{Iso}(\varphi) = \left\{\psi \in L^2_{\mathcal{R}} \mid \gamma_k(\psi) = \gamma_k(\varphi),\ k \in \mathbb{Z}\right\}.$$

As shown by Grébert and Guillot [GG] (see also chapter I) these are compact connected tori whose dimension equals the number of positive gap lengths and is infinite generically.
Moreover, as the asymptotic behavior of the gap lengths characterizes the regularity of a potential of real type in the same way as its Fourier coefficients do (see [GK1]), we have

$$\varphi \in H^N_{\mathcal{R}} \Longleftrightarrow \mathrm{Iso}(\varphi) \subset H^N_{\mathcal{R}}$$

for each $N \geq 0$ where

$$H^N_{\mathcal{R}} := \left(H^N \times H^N\right) \cap L^2_{\mathcal{R}} \equiv H^N.$$

Hence also the phase space H^N decomposes into a collection of tori of varying dimension which are invariant under the NLS flow.
All the results about the spectral theory of Zakharov-Shabat operators needed in this book are presented (and proved) in Chapter I.

Normal form and Birkhoff coordinates

In classical mechanics the existence of a foliation of the phase space into Lagrangian invariant tori is tantamount, at least locally, to the existence of action-angle coordinates. This is the content of the Liouville-Arnold-Jost theorem. In an infinite dimensional setting as the one for the NLS equation, however, the existence of such coordinates is far less clear as the dimension of the foliation is nowhere locally constant. Invariant tori of infinite and finite dimension each form dense subsets of the foliation. Nevertheless, action-angle coordinates can be introduced *globally* as we describe now.
To describe the action-angle variables on H^N we introduce the model space $(N \geq 0)$

$$\ell^2_N(\mathbb{Z};\mathbb{R}^2) := \{(x,y) = (x_k, y_k)_{k\in\mathbb{Z}} \mid \|(x,y)\|_N < \infty\}$$

where

$$\|(x,y)\|_N := \left(\sum_{k\in\mathbb{Z}} (1+|k|)^{2N}(x_k^2+y_k^2)\right)^{1/2}.$$

The space $\ell^2_N(\mathbb{Z};\mathbb{R}^2)$ is endowed with the Poisson structure, induced by the canonical symplectic structure $\sum_{k\in\mathbb{Z}} dx_k \wedge dy_k$.
The following theorem was first proven in a quite different form in [BBGK]. A similar version to the one we expand on here was first proven for KdV in [KP] (cf also [BKM2], [KM]).

Theorem 0.1 *There exists a family of diffeomorphisms $\Phi \equiv \Phi^{(N)}, N \geq 0$*

$$\Phi : \ell^2_N(\mathbb{Z},\mathbb{R}^2) \to H^N(S^1,\mathbb{C})$$

with the following properties

(i) *Φ is globally one-to-one, onto, bi-analytic and preserves the Poisson bracket.*

(ii) *The coordinates $(x_k, y_k)_{k\in\mathbb{Z}} = \Phi^{-1}(\varphi)$ are global Birkhoff coordinates for NLS. That is the transformed NLS Hamiltonian $\mathcal{H}\circ\Phi$ depends only on the actions $I_k := \frac{1}{2}(x_k^2+y_k^2)$ $k \in \mathbb{Z}$, with $(x_k, y_k)_{k\in\mathbb{Z}}$, being the (canonical) coordinates on $\ell^2_N(\mathbb{Z};\mathbb{R}^2)$.*

(iii) *For $N > N'$*

$$\Phi^{(N)} = \Phi^{(N')}\big|_{\ell^2_N}.$$

Often it will be convenient to use complex notation

$$x_k + iy_k = \sqrt{2I_k}\, e^{i\theta_k}\ (k \in \mathbb{Z}).$$

The coordinates (I_k, θ_k) are referred to as action-angle coordinates. Note that $\theta_k \in \mathbb{R}/2\pi\mathbb{Z}$ is well defined whenever $I_k \neq 0$.
In the coordinates $(x_k, y_k)_{k\in\mathbb{Z}} \in \ell^2_N(\mathbb{Z};\mathbb{R}^2)$ (with $N \geq 1$ in order $\mathcal{H}$ to be defined) the NLS Hamiltonian $\mathcal{H}$ is a real analytic function of the actions $I = (I_k)_{k\in\mathbb{Z}}$ alone and the NLS equation reads

$$\begin{cases} \dot{x}_k &= \omega_k(I) y_k \\ \dot{y}_k &= -\omega_k(I) x_k \end{cases} \qquad k \in \mathbb{Z}$$

where

$$\omega_k(I) = \frac{\partial \mathcal{H}}{\partial I_k}$$

are the NLS-frequencies which are real analytic functions of I.
It turns out that for $0 \leq N < 1$, the frequencies $\omega_k(I)$ can be defined by continuous extension although NLS Hamiltonian $\mathcal{H}$ itself is not defined.
Theorem 0.1 simultaneously applies to every real analytic Hamiltonian in the Poisson algebra of any of the Hamiltonians which Poisson commute with all action variables I_k, $k \in \mathbb{Z}$. In particular, (I,θ) are action-angle coordinates for every equation in the NLS hierarchy.

Applications

The normal form of NLS stated in Theorem 0.1 gives rise to various applications. First it shows that every solution of the NLS equation is almost periodic in time. Actually in action-angle coordinates, every solution is given by

$$I(t) = I^0, \quad \theta(t) = \theta^0 + \omega(I^0)t,$$

where (I^0, θ^0) corresponds to the initial data $\psi|_{t=0}$ and $\omega(I^0)$ is the (infinite) vector of frequencies associated with I^0. Hence in the model space every solution winds around some underlying invariant torus

$$T_{I^0} = \left\{(x, y) \in \ell^2 \mid x_k^2 + y_k^2 = 2I_k^0,\ k \in \mathbb{Z}\right\}.$$

If the number of non vanishing actions is finite, the torus is finite-dimensional and the solution is quasi-periodic. In this case the solution can also be represented in terms of Riemann theta functions (cf [BBEIM]). These quasi-periodic solutions correspond to finite pap potential: Let $A \subset \mathbb{Z}$ be a finite index set. We introduce the set of A-gap potentials

$$\mathcal{G}_A = \left\{\varphi \in L^2_{\mathcal{R}} \mid \gamma_k(\varphi) > 0 \Longleftrightarrow k \in A\right\}.$$

Actually the set $\mathcal{G}_A$ is analytically diffeomorphic (via the canonical transformation Φ of Theorem 0.1) to

$$h_A = \left\{(x, y) \in \ell^2 \mid x_k^2 + y_k^2 > 0 \Longleftrightarrow k \in A\right\}.$$

The normal form of NLS allows us to consider small perturbations of the NLS equation:

$$\psi_t = -i \left(\frac{\partial \mathcal{H}}{\partial \overline{\psi}} + \varepsilon \frac{\partial K}{\partial \overline{\psi}}\right).$$

In [GK2] (see also [GK3]) we prove that many finite dimensional tori, invariant under NLS-flow, persist under small Hamiltonian perturbations. To obtain this result we follow a procedure developed for the KdV equation in [KP] and use a KAM theorem in infinite dimension due to Kuksin [Ku1].

Chapter I

Spectral theory of ZS operators

In this chapter we present results about the spectral theory of Zakharov-Shabat operators needed to construct and analyze the Birkhoff map. They are elementary and, at least for potentials φ of real type, i.e. $\varphi = (\varphi_1, \overline{\varphi_1})$, well known (cf [G], [GG], [MV]). Throughout this section we use freely techniques and arguments from [GG], [MV], and [PT].
For $\varphi = (\varphi_1, \varphi_2) \in L^2_{\mathbb{C}} \equiv L^2([0,1], \mathbb{C}^2)$ denote by $L(\varphi)$ the Zakharov-Shabat operator

$$L(\varphi) := i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix}.$$

Let $M(\cdot, \lambda) \equiv M(\cdot, \lambda; \varphi)$ be the fundamental 2×2 matrix solution of $L(\varphi)M = \lambda M$, satisfying the initial condition $M(0, \lambda; \varphi) = Id_{2\times 2}$ for any $\lambda \in \mathbb{C}$.

I.1 Basic estimates for $M(x, \lambda)$

Making the ansatz $M(x, \lambda) = E(x, \lambda)N(x, \lambda)$ with $E(x, \lambda)$ given by

$$E(x, \lambda) := \begin{pmatrix} e^{-i\lambda x} & 0 \\ 0 & e^{i\lambda x} \end{pmatrix}$$

one verifies that M satisfies the following integral equation ($x \geq 0$)

$$M(x, \lambda) = E(x, \lambda) + \int_0^x K(x, y, \lambda) M(y, \lambda) dy \tag{I.1}$$

where $K(x, y, \lambda) \equiv K(x, y, \lambda; \varphi)$ is given by

$$K(x, y, \lambda) := i \begin{pmatrix} 0 & e^{-i\lambda(x-y)}\varphi_1(y) \\ -e^{i\lambda(x-y)}\varphi_2(y) & 0 \end{pmatrix}.$$

Formally, the solution of the above integral equation is given by the following power series ($x \geq 0$)

$$M(x, \lambda) = \sum_{k \geq 0} E^{(k)}(x, \lambda) \tag{I.2}$$

where

$$E^{(0)}(x, \lambda) := E(x, \lambda)$$

and, for $k \geq 0$, $E^{(k+1)}(x, \lambda) \equiv E^{(k+1)}(x, \lambda; \varphi)$ is defined by

$$E^{(k+1)}(x, \lambda) = \int_0^x K(x, y, \lambda) E^{(k)}(y, \lambda) dy$$

which leads to the formula ($k \geq 1, x_0 := x$)

$$E^{(k)}(x, \lambda) := \int_0^x dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{k-1}} dx_k \prod_{j=0}^{k-1} K(x_j, x_{j+1}, \lambda) E(x_k, \lambda). \tag{I.3}$$

From (I.3) one deduces ($x \geq 0$)

$$\|E^{(k)}(x,\lambda)\| \leq e^{|Im\lambda|x} \int_{0\leq x_k \leq \ldots \leq x_1 \leq x} \prod_{j=1}^{k} \|\varphi(x_j)\| dx_1 \ldots dx_k.$$

Here $\|A\|$ denotes the usual operator norm of a 2×2 matrix $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, i.e.

$$\|A\| := \max\left(\sqrt{|a|^2+|b|^2}, \sqrt{|c|^2+|d|^2}\right)$$

and, similarly, $\|\varphi(x)\| = \|(\varphi_1(x), \varphi_2(x))\|$ is given by

$$\|\varphi(x)\|^2 = |\varphi_1(x)|^2 + |\varphi_2(x)|^2.$$

Therefore, for any $x \geq 0$,

$$\|E^{(k)}(x,\lambda)\| \leq e^{|Im\lambda|x} \frac{1}{k!} \left(\int_0^x \|\varphi(y)\| dy\right)^k$$

and hence the sum $\sum_{k\geq 0} E^{(k)}(x,\lambda)$ is absolutely convergent, uniformly on bounded subsets of $[0,1]\times\mathbb{C}\times L^2_{\mathbb{C}}$ and

$$\|M(x,\lambda;\varphi)\| \leq \exp(|Im\lambda| + \|\varphi\|_{L^2}) \quad (0 \leq x \leq 1) \tag{I.4}$$

where $\|\varphi\|_{L^2} := \left(\int_0^1 \|\varphi(x)\|^2 dx\right)^{1/2}$.

Lemma I.1 *(1) For any $0 \leq x_0 \leq 1, M(x_0,\cdot,\cdot)$ is an analytic map on $\mathbb{C}\times L^2_{\mathbb{C}}$, depending continuously on x_0.*

(2) M is a weakly continuous map on $[0,1]\times\mathbb{C}\times L^2_{\mathbb{C}}$.

Proof (1) Using induction, one shows that $E^{(k)}(x_0,\cdot;\cdot)$ is an analytic map on $\mathbb{C}\times L^2_{\mathbb{C}}$, depending continuously on x_0. Hence the claimed statement is a consequence of the uniform convergence of $\sum_{k\geq 0} E^{(k)}(x,\lambda;\varphi)$ on bounded subsets of $[0,1]\times\mathbb{C}\times L^2_{\mathbb{C}}$.
(2) In view of the uniform convergence of $\sum_{k\geq 0} E^{(k)}(x,\lambda;\varphi)$, it suffices to prove that for each $k \geq 0$, the map $\varphi \mapsto E^{(k)}(x,\lambda,\varphi)$ is weakly continuous in $L^2_{\mathbb{C}}$, uniformly for (x,λ) in bounded subsets of $[0,1]\times\mathbb{C}$. We argue by induction: clearly, for $E^{(0)}(x,\lambda,\varphi) = E(x,\lambda)$ the above assertion holds as it does not depend on φ. Assume that it holds up to some $k \geq 0$. By definition,

$$E^{(k+1)}(x,\lambda;\varphi) = \int_0^x K(x,y,\lambda;\varphi)E^{(k)}(y,\lambda;\varphi)dy.$$

Let $(\varphi_n)_{(n\geq 1} \subseteq L^2_{\mathbb{C}}$ converge weakly to $\varphi \in L^2_{\mathbb{C}}$, $\varphi_n \rightharpoonup \varphi$. Clearly, for the matrix valued function K, we have $K(x,\cdot,\lambda;\varphi_n) \rightharpoonup K(x,\cdot,\lambda;\varphi)$ uniformly for (x,λ) in bounded subsets of $[0,1]\times\mathbb{C}$. By induction hypothesis, $\lim_{n\to\infty} E^{(k)}(y,\lambda;\varphi_n) = E^{(k)}(y,\lambda;\varphi)$ uniformly for (y,λ) in a bounded subset of $[0,1]\times\mathbb{C}$, one deduces that

$$\begin{aligned}\lim_{n\to\infty} E^{(k+1)}(x,\lambda;\varphi_n) &= \int_0^x K(x,y,\lambda;\varphi)E^{(k)}(y,\lambda;\varphi) \\ &= E^{(k+1)}(x,\lambda;\varphi)\end{aligned}$$

uniformly for (x,λ) in a bounded subset of $[0,1]\times\mathbb{C}$. ■

To obtain asymptotics for $M(x,\lambda,\varphi)$ as $|\lambda| \to \infty$ we need an auxilliary lemma (cf [AG, Proposition A.1]). Denote by $H^1_{\mathbb{C}} \equiv H^1(\mathcal{S}^1;\mathbb{C}^2)$ the set of elements $\varphi = (\varphi_1,\varphi_2) \in L^2_{\mathbb{C}}$ in the Sobolev space $H^1([0,1];\mathbb{C}^2)$ with periodic boundary conditions, $\varphi(0) = \varphi(1)$, and by $\|\varphi\|_{H^1}$ the H^1-norm,

$$\|\varphi\|^2_{H^1} = \|\varphi\|^2_{L^2} + \|\frac{d}{dx}\varphi\|^2_{L^2} \ .$$

Lemma I.2 *Let $\varphi^0 \in L^2_{\mathbb{C}}$, $\varepsilon > 0$ and $r_0 \geq 0$ be given and assume that $\varphi^\varepsilon \in H^1_{\mathbb{C}}$ satisfies $\|\varphi^\varepsilon - \varphi^0\|_2 < \varepsilon$. Then for any $\varphi \in L^2_{\mathbb{C}}$ with $\|\varphi - \varphi^0\|_2 < r_0$ and $(x,\lambda) \in [0,1]\times\mathbb{C}\backslash\{0\}$ one has*

$$\|M(x,\lambda;\varphi) - E(x,\lambda)\| \leq e^{|Im\lambda|x + \|\varphi\|_2}\left(r_0 + \varepsilon + \frac{\|\varphi^\varepsilon\|_{H^1}}{|\lambda|}\right).$$

Proof As $\varphi^\varepsilon \in H^1_{\mathbb{C}}$, one can integrate by parts to get

$$\begin{aligned} E^{(1)}(x,\lambda;\varphi^\varepsilon) &= i\int_0^x \begin{pmatrix} 0 & e^{-i\lambda(x-2y)}\varphi_1^\varepsilon(y) \\ -e^{i\lambda(x-2y)}\varphi_2^\varepsilon(y) & 0 \end{pmatrix} dy \\ &= \frac{1}{2\lambda}\begin{pmatrix} 0 & e^{-i\lambda(x-2y)}\varphi_1^\varepsilon(y) \\ e^{i\lambda(x-2y)}\varphi_2^\varepsilon(y) & 0 \end{pmatrix}\Big|_{y=0}^{x} \\ &- \frac{1}{2\lambda}\int_0^x \begin{pmatrix} 0 & e^{-i\lambda(x-2y)}\partial_y\varphi_1^\varepsilon(y) \\ e^{i\lambda(x-2y)}\partial_y\varphi_2^\varepsilon(y) & 0 \end{pmatrix} dy \ . \end{aligned}$$

In view of the Sobolev inequality $\|\varphi(x)\| \leq \|\varphi\|_{H^1}$ we then obtain, for any $0 \leq x \leq 1$ and $\lambda \neq 0$,

$$\|E^{(1)}(x,\lambda;\varphi^{(\varepsilon)})\| \leq \frac{3}{2}\frac{e^{|Im\lambda|x}}{|\lambda|}\|\varphi^\varepsilon\|_{H^1}.$$

For $\varphi \in L^2_{\mathbb{C}}$ with $\|\varphi - \varphi_0\|_{L^2} < r_0$ we have ($0 \leq x \leq 1$)

$$\|E^{(1)}(x,\lambda;\varphi) - E^{(1)}(x,\lambda;\varphi^\varepsilon)\| \leq (r_0+\varepsilon)e^{|Im\lambda|x}.$$

Combining these two inequalities yields

$$\|E^{(1)}(x,\lambda;\varphi)\| \le (r_0 + \varepsilon + \frac{3}{2|\lambda|}\|\varphi^\varepsilon\|_{H^1})e^{|Im\lambda|x}.$$

As for $k \ge 2$ and $0 \le x \le 1$ (with $x_0 := x$)

$$E^{(k)}(x,\lambda;\varphi) = \int_0^x dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{k-2}} dx_{k-1} \left(\prod_{j=0}^{k-2} K(x_j, x_{j+1})\right) E^{(1)}(x_k,\lambda;\varphi).$$

the above estimate leads to $(0 \le x \le 1, \lambda \ne 0)$

$$\|E^{(k)}(x,\lambda;\varphi)\| \le \left(r_0 + \varepsilon + \frac{3\|\varphi^\varepsilon\|_{H^1}}{2|\lambda|}\right) e^{|Im\lambda|x} \frac{1}{(k-1)!}\left(\int_0^x \|\varphi(y)\|dy\right)^{k-1} \tag{I.5}$$

and hence, by (I.2)

$$\|M(x,\lambda;\varphi) - E(x,\lambda)\| \le e^{|Im\lambda|x + \|\varphi\|_2}\left(r_0 + \varepsilon + \frac{3\|\varphi^\varepsilon\|_{H^1}}{2|\lambda|}\right).$$

■

From Lemma I.2 one obtains the following basic estimates of $M(x,\lambda;\varphi)$ and its derivative $\dot M(x,\lambda;\varphi)$ ($\dot{} = \frac{d}{d\lambda}$).

Proposition I.3 *Let $\varphi \in L^2_{\mathbb{C}}$. Then, uniformly for $0 \le x \le 1$, as $|\lambda| \to \infty$*

(i) $M(x,\lambda;\varphi) = E(x,\lambda) + o(e^{|Im\lambda|x})$,

(ii) $\dot M(x,\lambda;\varphi) = \dot E(x,\lambda) + o(e^{|Im\lambda|x})$.

Proof (i) By Lemma I.2 with $r_0 = 0$, given $\delta > 0$ arbitrary, there exists $\lambda_\delta > 0$ such that

$$|M(x,\lambda;\varphi) - E(x,\lambda)| \le \delta e^{|Im\lambda|x} \quad \forall |\lambda| \ge \lambda_\delta.$$

To prove (ii), derive the integral equation (I.1) with respect to λ,

$$\begin{aligned}\dot M(x,\lambda) = \dot E(x,\lambda) + \int_0^x \dot K(x,y,\lambda)M(y,\lambda)dy + \\ + \int_0^x K(x,y,\lambda)\dot M(y,\lambda)dy.\end{aligned} \tag{I.6}$$

Using (i) and the identity

$$\dot K(x,y,\lambda) = (x-y)\begin{pmatrix} 0 & e^{-i\lambda(x-y)}\varphi_1(y) \\ e^{i\lambda(x-y)}\varphi_2(y) & 0 \end{pmatrix}$$

one sees that

$$\int_0^x \dot K(x,y,\lambda)M(y,\lambda)dy = \int_0^x \dot K(x,y,\lambda)E(y,\lambda)dy + o(e^{|Im\lambda|x}).$$

Approximating φ by an element in $H^1_{\mathbb{C}}$ up to ϵ (cf Lemma I.2) one sees that

$$\int_0^x \dot K(x,y,\lambda)E(y,\lambda)dy = o(e^{|Im\lambda|x}).$$

Hence equation (I.6) is of the form

$$\dot M(x,\lambda) = \widetilde{E}(x,\lambda) + \int_0^x K(x,y,\lambda)\dot M(y,\lambda)dy$$

where

$$\widetilde{E}(y,\lambda) := \dot E(x,\lambda) + \int_0^x \dot K(x,y)M(y,\lambda)dy$$

satisfies

$$\widetilde{E}(y,\lambda) = \dot E(x,\lambda) + o(e^{|Im\lambda|x}).$$

Arguing as in the proof of Lemma I.2 one concludes that

$$\dot M(x,\lambda) = \widetilde{E}(x,\lambda) + o(e^{|Im\lambda|x})$$

and the claimed statement follows. ■

We include in this section a result concerning the Wronskian identity for M. Denote by $W(M(x,\lambda))$ the Wronskian of $M(x,\lambda)$

$$W(M(x,\lambda)) := \det M(x,\lambda).$$

Lemma I.4 *For any $\varphi \in L^2_{\mathbb{C}}, \lambda \in \mathbb{C}$ and $x \in \mathbb{R}$,*

$$W(M(x,\lambda)) = 1.$$

Proof Writing $M(x) \equiv M(x,\lambda,\varphi)$, one has

$$\frac{d}{dx}W(M(x)) = \det(\frac{dM^{(1)}}{dx}(x)M^{(2)}(x)) + \det(M^{(1)}(x)\frac{dM^{(2)}}{dx}(x)) \tag{I.7}$$

where $M^{(1)}(x)$ and $M^{(2)}(x)$ denote the first respectively second column of M. Rewriting $LM^{(j)} = \lambda M^{(j)}$ $(j = 1,2)$ one gets

$$\frac{dM^{(j)}}{dx}(x) = i\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}\left(\lambda M^{(j)}(x) - \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0\end{pmatrix} M^{(j)}(x)\right)$$

which, when substituted into (I.7) leads to $\frac{d}{dx}W(M(x)) = 0$, i.e. $W(M(x))$ is independent of x. As $M(0) = Id_{2\times 2}$, we have $W(M(0)) = 1$ and the claimed statement follows. ■

Later on we will also use the notation $W(F,G)$ for the Wronskian of two functions F, G in $L^2_{\mathbb{C}}$,

$$W(F,G)(x) := F_1(x)G_2(x) - F_2(x)G_1(x). \tag{I.8}$$

I.2 Periodic spectrum

We denote by $spec(\varphi)$ the spectrum of the operator $L(\varphi)$ with domain

$$dom_{per}(L) := \{F \in H^1_{\mathbb{C}} \mid F(1) = \pm F(0)\}.$$

By Floquet theory $spec(\varphi)$ coincides with the spectrum of the operator $L(\varphi)$, considered on $[0,2]$ with periodic boundary conditions.
By the definition of $M(x,\lambda)$, any solution F of $LF = \lambda F$ satisfies $F(1,\lambda) = M(1,\lambda)F(0,\lambda)$. Hence $\lambda \in spec(\varphi)$ iff 1 or -1 is an eigenvalue of $M(1,\lambda)$. As $\det M(1,\lambda) = 1$ (cf Lemma I.4) it follows that $spec(\varphi)$ is the zero set of the entire function $\Delta(\lambda,\varphi)^2 - 4$ where $\Delta(\lambda) \equiv \Delta(\lambda,\varphi)$ is the discriminant

$$\Delta(\lambda) := trM(1,\lambda) = M_{11}(1,\lambda) + M_{22}(1,\lambda). \qquad \text{(I.9)}$$

Lemma I.2 can be used to locate the periodic eigenvalues of $L(\varphi)$. For $\varphi = 0$, $spec(\varphi)$ consists of the eigenvalues $k\pi$ ($k \in \mathbb{Z}$), each eigenvalue having multiplicity two. The discriminant $\Delta(\lambda,0)$ is given by

$$\Delta(\lambda,0) = 2\cos\lambda$$

hence $\Delta(\lambda,0)^2 - 4 = 4(\cos^2\lambda - 1)$ has indeed as zeroes the eigenvalues $k\pi$ ($k \in \mathbb{Z}$) of $L(0)$. To locate the zeroes of $\Delta(\lambda,\varphi)^2 - 4$, we want to use Rouché's Theorem.
By Lemma I.2, given any $\varphi^0 \in L^2_{\mathbb{C}}$, there exist $\varepsilon > 0$ and $N_0 > 0$ so that for any $\varphi \in L^2_{\mathbb{C}}$ with $\|\varphi - \varphi^0\|_2 < \varepsilon, \lambda \in \mathbb{C}$ with $|\lambda| \geq N_0$ and $0 \leq x \leq 1, \|M(x,\lambda,\varphi) - E(x,\lambda)\| \leq \frac{1}{36}e^{|Im\lambda|x}$.
As $\Delta(\lambda,\varphi) = trM(1,\lambda;\varphi)$ and $\Delta(\lambda,0) = 2\cos\lambda$,

$$\begin{aligned}
&|(\Delta(\lambda,\varphi)^2 - 4) - (\Delta(\lambda,0)^2 - 4)| \\
&\qquad = |\Delta(\lambda,\varphi) - 2\cos\lambda||\Delta(\lambda) + 2\cos\lambda| \\
&\qquad \leq |\Delta(\lambda,\varphi) - 2\cos\lambda| \left(|\Delta(\lambda) - 2\cos\lambda| + 4|\cos\lambda|\right) \\
&\qquad \leq 2\|M(1,\lambda;\varphi) - E(1,\lambda)\| \left(2\|M(1,\lambda;\varphi) - E(1,\lambda)\| + 4e^{|Im\lambda|}\right).
\end{aligned}$$

Together with $\Delta(\lambda,0)^2 - 4 = 4(\cos^2\lambda - 1) = (2i\sin\lambda)^2$, we then obtain

$$|(\Delta(\lambda,\varphi)^2 - 4) - (2i\sin\lambda)^2| < \frac{1}{4}e^{2|Im\lambda|}.$$

Notice that for $\lambda \in \mathbb{C}$ with $|\lambda - n\pi| \geq \frac{\pi}{4} \quad \forall n \in \mathbb{Z}$, one has $e^{|Im\lambda|} < 4|\sin\lambda|$ (cf [PT] Lemma 1, p. 27) and hence

$$|(\Delta(\lambda,\varphi)^2 - 4) - (2i\sin\lambda)^2| < 4|\sin\lambda|^2 = |2i\sin\lambda|^2.$$

By Rouché's Theorem applied to the contours $\{\lambda \in \mathbb{C} \mid |\lambda - n\pi| = \pi/4\}$ ($|n| \geq N_0+1$) and $\{\lambda \in \mathbb{C} \mid |\lambda| = N\pi + \pi/4\}$ ($N \geq N_0$) we have proved the following result due to [LM] (see also [GG])

Proposition I.5 *Given any $\varphi^0 \in L^2_{\mathbb{C}}$, there exist $\varepsilon > 0$ and $N_0 > 0$ so that for any $\varphi \in L^2_{\mathbb{C}}$ with $\|\varphi - \varphi^0\|_2 < \varepsilon$, the following statements hold:*

(i) For any $|n| \geq N_0 + 1$ the set $spec(\varphi) \cap \{\lambda \in \mathbb{C} \mid |\lambda - n\pi| < \pi/4\}$ contains precisely one isolated pair λ_n^+, λ_n^- of eigenvalues (counted with multiplicity).

(ii) $spec(\varphi) \backslash \{\lambda_n^\pm \mid |n| \geq N_0 + 1\}$ is contained in $\{\lambda \in \mathbb{C} \mid |\lambda| < N_0\pi + \frac{\pi}{4}\}$ and its cardinality is $4N_0 + 2$ (with multiplicities).

For $|n| \geq N_0 + 1$, the pairs of eigenvalues $\lambda_n^\pm$ are ordered lexicographically, $\lambda_n^- \preccurlyeq \lambda_n^+$, i.e.

$$\begin{aligned}
&Re\lambda_n^- < Re\lambda_n^+ \\
&\text{or} \\
&Re\lambda_n^- = Re\lambda_n^+ \text{ and } Im\lambda_n^- \leq Im\lambda_n^+.
\end{aligned}$$

The $4N_0+2$ eigenvalues of $L(\varphi)$inside the disc $\{|\lambda| < N_0\pi + \pi/4\}$ are denoted by $\lambda_n^\pm$ ($|n| \leq N_0$) so that they are in lexicographic order as well

$$\lambda_{-N_0}^- \preccurlyeq \lambda_{-N_0}^+ \preccurlyeq \lambda_{-N_0+1}^- \preccurlyeq \lambda_{-N_0+1}^+ \preccurlyeq \ldots \preccurlyeq \lambda_{N_0}^- \preccurlyeq \lambda_{N_0}^+.$$

By Proposition I.5, we have

$$spec(\varphi) = \{\lambda_n^\pm \mid n \in \mathbb{Z}\}$$

and $\ldots, \lambda_n^-, \lambda_n^+, \lambda_{n+1}^-, \lambda_{n+1}^+, \ldots$ are ordered lexicographically.
As $\{\lambda_n^+, \lambda_n^-\}$ is an isolated pair of eigenvalues for $|n| \geq N_0 + 1$, one then sees by deforming φ to the zero element that

$$\Delta(\lambda_n^\pm, \varphi) = 2(-1)^n \quad \forall |n| \geq N_0 + 1, \qquad \text{(I.10)}$$

i.e. for $|n| \geq N_0 + 1, \lambda_n^\pm$ are periodic (for n even) or antiperiodic (for n odd) eigenvalues of $L \equiv L(\varphi)$, considered on $dom_{per}(L)$.
Generically, formula (I.10) does not hold for $|n| \leq N_0$ since the eigenvalues, being ordered lexicographically, are not continuous with respect to φ.
However if $\varphi = (\varphi_1, \varphi_2)$ is of real type, i.e. $\varphi_2 = \overline{\varphi_1}$ (cf. section I.4), the eigenvalues $\lambda_n^\pm$ are real and continuous with respect to φ. Hence, in this case (I.10) holds for any $n \in \mathbb{Z}$ and we have $\lambda_n^- \leq \lambda_n^+ < \lambda_{n+1}^- \ \forall n \in \mathbb{Z}$.
To express that a sequence $(a_n)_{n\in\mathbb{Z}}$ in a Banach space $(\mathcal{B}, \|\cdot\|)$ is in $\ell^p(\mathbb{Z}, \mathcal{B})$ it is convenient to write $\ell^p(n)$ for a_n ($n \in \mathbb{Z}$).

Proposition I.6 *Locally uniformly in $\varphi \in L^2_{\mathbb{C}}$,*

(i) $\lambda_n^\pm(\varphi) = n\pi + \ell^2(n)$

(ii) $M(x, \lambda_n^\pm(\varphi); \varphi) = E(x, n\pi) + \alpha_n(x)$ *where* $\left(\sup_{0\leq x\leq 1} \|\alpha_n(x)\|\right)_{n\in\mathbb{Z}} \in \ell^2$.

To prove Proposition I.6 we need the following auxilary result (cf. [AG, Lemma A.1], [Ma] or [Mis]):

Lemma I.7 *Let $\mathcal{E} \subseteq \ell^\infty_{\mathbb{C}}$ and $\mathcal{F} \subseteq L^2(S^1, \mathbb{C})$ be bounded subsets. Then $\left(\int_0^x f(t)e^{i\pi(k+\varepsilon_n)t}dt\right)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{C}}$ uniformly in $0 \le x \le 1, (\varepsilon_n)_{n\in\mathbb{Z}} \in \mathcal{E}$ and $f \in \mathcal{F}$.*

Proof (Proposition I.6) To prove (i) recall that, for $\lambda \in spec(\varphi), trM(1,\lambda;\varphi) = \pm 2$ and

$$trM(1,\lambda;\varphi) = trE(1,\lambda;\varphi) + \sum_{k\ge 1} trE^{(k)}(1,\lambda;\varphi).$$

For λ arbitrary, we have $trE(1,\lambda;\varphi) = 2\cos\lambda$ and $trE^{(1)}(1,\lambda;\varphi) = 0$. To estimate $trE^{(2)}(1,\lambda_n^\pm;\varphi)$, notice that $\lambda_n^\pm = n\pi + 0(1)$ by Proposition I.5. Hence by Lemma I.7, applied to

$$E^{(2)}(x,\lambda_n^\pm;\varphi) = \int_0^x dx_1 \int_0^{x_1} dx_2$$
$$diag\left(e^{-i\lambda_n^\pm(x-2x_1+2x_2)}\varphi_1(x_1)\varphi_2(x_2), e^{i\lambda_n^\pm(x-2x_1+2x_2)}\varphi_1(x_2)\varphi_2(x_1)\right)$$

with respect to the x_2-integration and then to the x_1-integration, one deduces that uniformly, for $0 \le x \le 1$ and locally uniformly in φ

$$E^{(2)}(1,\lambda_n^\pm;\varphi) = \ell^1(n)\ .$$

Arguing as in the proof of Lemma I.2 (cf (I.5)), we conclude that

$$\sum_{k\ge 2} E^{(k)}(x,\lambda_n^\pm;\varphi) = \ell^1(n)$$

uniformly for $0 \le x \le 1$ and locally uniformly in φ. It follows that

$$trM(1,\lambda_n^\pm;\varphi) = 2\cos\lambda_n^\pm + \ell^1(n).$$

Use that $trM(1,\lambda_n^\pm;\varphi) \in \{+2,-2\}$ and

$$\cos\lambda_n^\pm = \cos(n\pi + (\lambda_n^\pm - n\pi)) = (-1)^n\cos(\lambda_n^\pm - n\pi)$$

to conclude that $\cos(\lambda_n^\pm - n\pi) = 1 + \ell^1(n)$. Hence $\lambda_n^\pm - n\pi = o(1)$ as $|n| \to \infty$ locally uniformly in φ and in view of the expansion $\cos x = 1 - \frac{x^2}{2}(1 + O(x^2))$ it then follows that $(\lambda_n^\pm - n\pi)^2 = \ell^1(n)$, i.e. locally uniformly in φ

$$\lambda_n^\pm = n\pi + \ell^2(n)$$

which shows statement (i).
Similarly, one proves that

$$E^{(1)}(x,\lambda_n^\pm;\varphi) = i\int_0^x \begin{pmatrix} 0 & e^{-i\lambda_n^\pm(x-2y)}\varphi_1(y) \\ -e^{i\lambda_n^\pm(x-2y)}\varphi_2(y) & 0 \end{pmatrix} dy$$

satisfies, uniformly in $0 \le x \le 1$ and locally uniformly in φ,

$$E^{(1)}(x,\lambda_n^\pm;\varphi) = \ell^2(n).$$

Again arguing as in the proof of Lemma I.2 it then follows that

$$M(x,\lambda_n^\pm;\varphi) = E(x,\lambda_n^\pm) + \ell^2(n).$$

By (i), we have $E(x,\lambda_n^\pm) = E(x,n\pi) + \ell^1(n)$ uniformly in $0 \le x \le 1$ and locally uniformly in φ. Hence statement (ii) is proved as well. ■

I.3 Dirichlet spectrum

In this section we consider the operator $L(\varphi)$ with Dirichlet boundary conditions.

Definition I.8 *A function $F = (F_1, F_2) \in H^1([0,1], \mathbb{C}^2)$ satisfies Dirichlet boundary conditions if*

$$F_1(0) = F_2(0)\ ;\ F_1(1) = F_2(1) \tag{I.11}$$

When expressed in $\tilde{F} = (\tilde{F}_1, \tilde{F}_2)$ with

$$\tilde{F}_1 := \frac{1}{i\sqrt{2}}(F_1 + F_2)\ ;\ \tilde{F}_2 := \frac{1}{\sqrt{2}}(F_2 - F_1) \tag{I.12}$$

the Dirichlet conditions (I.11) take the more familiar form

$$\tilde{F}_2(0) = 0\ ;\ \tilde{F}_2(1) = 0.$$

the transformation (I.12) is related to the $AKNS$ operator $L_{AKNS}(\varphi)$ given by

$$L_{AKNS}(\varphi) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{d}{dx} + \begin{pmatrix} -q & p \\ p & q \end{pmatrix}$$

with

$$\varphi_1 = -q + ip\ ;\ \varphi_2 = -q - ip.$$

in the following way: if F is a solution of $LF = \lambda F$ then $\tilde{F}$ is a solution of $L_{AKNS}\tilde{F} = \lambda\tilde{F}$, i.e. $L_{AKNS}(\varphi)$ and $L(\varphi)$ are unitarily equivalent.
It follows from the definition of the fundamental matrix $M(x) \equiv M(x,\lambda;\varphi)$ that the vector function $G(x) \equiv G(x,\lambda;\varphi)$ given by

$$G(x) := \begin{pmatrix} M_{11}(x) + M_{12}(x) \\ M_{21}(x) + M_{22}(x) \end{pmatrix}$$

satisfies $L(\varphi)G = \lambda G$ and $G_1(0) - G_2(0) = 0$.

Hence the Dirichlet spectrum $spec_{Dir}(\varphi)$, consisting of all $\lambda \in \mathbb{C}$ for which there exists a solution $F \in H^1([0,1],\mathbb{C}^2)$ of the equation, $LF = \lambda F$, satisfying the Dirichlet boundary conditions (I.11), is the zero set of

$$\delta(\lambda) \equiv \delta(\lambda,\varphi) := (M_{11} + M_{12} - M_{21} - M_{22})\big|_{1,\lambda}. \tag{I.13}$$

By Proposition I.3, we have

$$\delta(\lambda) = -2i \sin\lambda + o\left(e^{|Im\lambda|}\right).$$

Arguing as in the proof of Proposition I.5 one concludes that the zero set of $\delta(\lambda)$ consists of a sequence $(\mu_n)_{n\in\mathbb{Z}}$ with the asymptotics $|\mu_n - n\pi| \le \frac{\pi}{4}$, when listed in such a way that they are ordered lexicographically, i.e. for any $n \in \mathbb{Z}$,

$$Re\mu_n < Re\mu_{n+1} \text{ or } Re\mu_n = Re\mu_{n+1} \text{ and } Im\mu_n \le Im\mu_{n+1}.$$

Following the proof of Proposition I.6 one shows

Proposition I.9 *Locally uniformly on* $L^2_{\mathbb{C}}$,

$$\mu_n(\varphi) = n\pi + \ell^2(n) \tag{I.14}$$

$$M(x,\mu_n(\varphi);\varphi) = E(x,n\pi) + \alpha_n(x) \tag{I.15}$$

where $\left(\sup_{0\le x\le 1}\|\alpha_n(x)\|\right)_{n\in\mathbb{Z}} \in \ell^2$.

As a consequence, we have for the eigenfunction $G_n(x) := G(x,\mu_n)$, corresponding to the Dirichlet eigenvalue μ_n, the asymptotic behaviour

$$G_n(x) = (e^{-in\pi x}, e^{in\pi x}) + \ell^2(n). \tag{I.16}$$

To obtain a convenient formula for the eigenfunctions of $L(\varphi)$ corresponding to simple periodic eigenvalues we need to consider additional boundary conditions for $L(\varphi)$.

Definition I.10 *A function* $F = (F_1, F_2)$ *in* $H^1([0,1],\mathbb{C}^2)$ *satisfies the second Dirichlet boundary conditions if*

$$F_1(0) = -F_2(0);\ F_1(1) = -F_2(1). \tag{I.17}$$

When expressed with respect to the function $\tilde F = (\tilde F_1, \tilde F_2)$ defined in (I.12), the boundary conditions (I.17) take the more familiar form

$$\tilde F_1(0) = 0;\ \tilde F_1(1) = 0.$$

Notice that the vector function $\check G(x) \equiv \check G(x,\lambda,\varphi)$ given by $M^{(1)}(x) - M^{(2)}(x)$, i.e.

$$\check G(x) := (M_{11}(x) - M_{12}(x), M_{21}(x) - M_{22}(x))$$

satisfies $L(\varphi)\check G = \lambda \check G$ and $\check G_1(0) + \check G_2(0) = 0$. Hence the second Dirichlet spectrum is the zero set of $(\check G_1 + \check G_2)\big|_{1,\lambda}$, i.e. of

$$\check\delta(\lambda) \equiv \check\delta(\lambda,\varphi) := (M_{11} - M_{12} + M_{21} - M_{22})\big|_{1,\lambda}. \tag{I.18}$$

Arguing as for $\delta(\lambda)$ one concludes that the zero set of $\check\delta(\lambda)$ consists of a sequence $(\check\mu_n)_{n\in\mathbb{Z}}$ with asymptotics $|\check\mu_n - n\pi| \le \pi/4$ listed in such a way that they are ordered lexicographically and one obtains an analogue of Proposition I.9. As a consequence we have for the eigenfunction $\check G_n$ corresponding to $\check\mu_n$, $\check G_n(x) := \check G(x,\mu_n)$ the asymptotics

$$\check G_n(x) = \left(e^{-in\pi x}, -e^{in\pi x}\right) + \ell^2(n). \tag{I.19}$$

I.4 Spectrum for potentials of real type

We say that a potential $\varphi = (\varphi_1, \varphi_2)$ is of *real type* if both $q := -(\varphi_1+\varphi_2)/2$ and $p := (\varphi_1 - \varphi_2)/2i$ are *real valued*, i.e. $\varphi_2 = \overline{\varphi_1}$. We denote by $L^2_{\mathcal{R}}$ the space of potentials of real type

$$L^2_{\mathcal{R}} := \{(\varphi_1,\varphi_2) \in L^2_{\mathbb{C}} \mid \overline{\varphi_2} = \varphi_1\}.$$

For a potential φ of real type, $L(\varphi)$ and $L_{AKNS}(\varphi)$ are formally selfadjoint. One verifies that L_{per}, i.e. the operator L considered on $[0,2]$ with periodic boundary conditions, and L_{Dir}, i.e. the operator L considered on $[0,1]$ with Dirichlet boundary conditions (cf Definition I.8), are both selfadjoint, hence the periodic and Dirichlet spectrum are real. Moreover, when restricted to potentials of real type the eigenvalues $\lambda_k^\pm(\varphi)$ are continuous in φ. By deforming the zero potential $(0,0)$ continuously to φ, it then follows that $\Delta(\lambda_k^\pm) = 2(-1)^k$ and, in particular,

$$\lambda_k^- \le \lambda_k^+ < \lambda_{k+1}^- \quad \forall k \in \mathbb{Z}. \tag{I.20}$$

For φ of real type, $L(\varphi)$ has an additional symmetry property. Given a vector $F = (F_1,F_2)$ and a 2×2 matrix $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, let

$$A^* = \begin{pmatrix} a_3 & a_4 \\ a_1 & a_2 \end{pmatrix}\ ;\ F^* = (F_2, F_1).$$

One verifies that the fundamental matrix solution satisfies

$$L\overline{M^*(x,\lambda)} = \overline{\lambda}\ \overline{M^*(x,\lambda)}\ .$$

As $\overline{M^*(0,\lambda)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ one then gets $(k = 1, 2)$

$$\overline{M^*_{k1}(x,\lambda)} = M_{k2}(x, \overline{\lambda}) \ ; \ \overline{M^*_{k2}(x,\lambda)} = M_{k1}(x, \overline{\lambda}).$$

In particular, for λ *real* ,

$$\overline{M_{11}(x,\lambda)} = M_{22}(x,\lambda) \ ; \ \overline{M_{12}(x,\lambda)} = M_{21}(x,\lambda). \tag{I.21}$$

As $\delta(\mu_k) = 0$ we have $(M_{11} - M_{22})\big|_{1,\mu_k} = (M_{21} - M_{12})\big|_{1,\mu_k}$ and by the Wronskian identity, $M_{11}M_{22}\big|_{1,\lambda} = 1 + M_{12}M_{21}\big|_{1,\lambda}$ (cf Lemma I.4). One then obtains with $\Delta(\mu_k) = (M_{11} + M_{22})\big|_{1,\mu_k}$

$$\begin{aligned}\Delta(\mu_k)^2 &= (M_{11} - M_{22})^2\big|_{1,\mu_k} + 4M_{11}M_{22}\big|_{1,\mu_k} \\ &= (M_{21} - M_{12})^2\big|_{1,\mu_k} + 4 + 4M_{12}M_{21}\big|_{1,\mu_k} \\ &= 4 + (M_{12} + M_{21})^2\big|_{1,\mu_k}.\end{aligned}$$

By (I.21), $(M_{12} + M_{21})\big|_{1,\mu_k}$ is real, hence $\Delta(\mu_k)^2 \geq 4$. This shows that for any k there exists $n_k \in \mathbb{Z}$ with $\lambda_k^- \leq \mu_{n_k} \leq \lambda_k^+$. By deforming the zero potential continuously to φ and using that $\lambda_k^\pm, \mu_k$ are continuous in φ it follows that $n_k = k$, i.e.

$$\lambda_k^- \leq \mu_k \leq \lambda_k^+ \quad \forall k \in \mathbb{Z}. \tag{I.22}$$

Notice that (I.20) together with (I.22) implies that the Dirichlet eigenvalues are all simple and it then follows that $\mu_k(\varphi)$ is real analytic in q, p with q, p given by $\varphi = (\varphi_1, \varphi_2) = (-q + ip, -q - ip)$.
Finally we present two auxilary results needed later. Note that the second one provides another proof of the simplicity of all Dirichlet eigenvalues.
Recall that we have introduced

$$G(x,\lambda) \equiv G(x,\lambda;\varphi) := (M_{11}(x,\lambda) + M_{12}(x,\lambda), M_{21}(x,\lambda) + M_{22}(x,\lambda)).$$

Lemma I.11 *Let $\varphi \in L^2_{\mathcal{R}}$ and $\lambda \in \mathbb{R}$. Then*

$$G_2(x,\lambda) = \overline{G_1(x,\lambda)} \tag{I.23}$$

and

$$\|G(\cdot,\lambda)\|^2_{L^2} = i\dot{\delta}(\lambda)(M_{11} + M_{12})\big|_{1,\lambda} - i\delta(\lambda)(\dot{M}_{11} + \dot{M}_{12})\big|_{1,\lambda}.$$

In particular, for $\lambda = \mu_k$ $(k \in \mathbb{Z})$

$$\|G_k(\cdot)\|^2 = i\dot{\delta}(\mu_k)(M_{11} + M_{12})\big|_{1,\mu_k}$$

where $G_k(\cdot) = G(\cdot, \mu_k)$.

Proof Formula (I.23) is a consequence of (I.21). To prove the claimed formula for $\|G(\cdot,\lambda)\|^2$ consider first the case where φ is continuous. Then $M(x,\lambda)$ is continuously differentiable in x. Differentiating $LG = \lambda G$ with respect to λ yields $(\cdot = \frac{d}{d\lambda})$.

$$L\dot{G} = G + \lambda\dot{G}$$

and, taking the L^2-inner product with G, one gets

$$\|G\|^2_{L^2} = (L\dot{G}, G) - \lambda(\dot{G}, G).$$

On the other hand, taking the inner product of $LG = \lambda G$ with $\dot{G}$ leads to

$$(\dot{G}, LG) = \lambda(\dot{G}, G).$$

Substituting this identity into the former one one gets

$$\begin{aligned}\|G\|^2_{L^2} &= (L\dot{G}, G) - (\dot{G}, LG) \\ &= i\int_0^1 (\dot{G}_1'\overline{G}_1 - \dot{G}_2'\overline{G}_2 + \dot{G}_1\overline{G}_1' - \dot{G}_2\overline{G}_2') \\ &= i\int_0^1 (\dot{G}_1'G_2 - \dot{G}_2'G_1 + \dot{G}_1G_2' - \dot{G}_2G_1')\end{aligned}$$

where for the last identity we used (I.23). Hence with $W(\dot{G}, G)$ denoting the Wronskian of $\dot{G}$ and G,

$$\begin{aligned}\|G\|^2_{L^2} &= i\int_0^1 \frac{d}{dx}W(\dot{G}, G)dx \\ &= iW(\dot{G}, G)(1,\lambda) \\ &= i\{(\dot{M}_{11} + \dot{M}_{12})(M_{21} + M_{22}) - (\dot{M}_{21} + \dot{M}_{22})(M_{11} + M_{12})\}\big|_{1,\lambda} \\ &= i\{\dot{\delta}(\lambda)(M_{11} + M_{12})\big|_{1,\lambda} - \delta(\lambda)(\dot{M}_{11} + \dot{M}_{12})\big|_{1,\lambda}\}.\end{aligned}$$

As both sides of the last identity are continuous with respect to $\varphi \in L^2_{\mathbb{C}}$, this identity holds for $\varphi \in L^2_{\mathcal{R}}$ as well. ∎

I.5 Spectral properties of potentials near $L^2_{\mathcal{R}}$

For potentials φ near $L^2_{\mathcal{R}}$, the regularity properties and the asymptotics estimates of the periodic eigenvalues (cf Proposition I.6) can be improved. To be more precise we write the eigenvalues $\lambda_n^\pm(\varphi)$ in the form

$$\lambda_n^\pm(\varphi) = \tau_n(\varphi) \pm \gamma_n(\varphi)/2$$

where $\tau_n(\varphi)$ is the arithmetic mean of λ_n^+ and λ_n^-,

$$\tau_n(\varphi) := \frac{1}{2}\left(\lambda_n^+(\varphi) + \lambda_n^-(\varphi)\right)$$

and $\gamma_n(\varphi)$ is the difference,

$$\gamma_n(\varphi) := \lambda_n^+(\varphi) - \lambda_n^-(\varphi).$$

The asymptotics of $\tau_n(\varphi) = n\pi + \ell^2(n)$ obtained from Proposition I.6 can be improved as well as the regularity properties of τ_n and $(\gamma_n)^2$.
First we improve on the localization of the eigenvalues $\lambda_n^\pm(\varphi)$ for φ near $L^2_{\mathcal{R}}$. Recall that for $\varphi_0 \in L^2_{\mathcal{R}}$ (i.e. $\varphi_2 = \overline{\varphi_1}$) the periodic eigenvalues $\lambda_n^\pm = \lambda_n^\pm(\varphi_0)$ are all real and satisfy

$$\ldots < \lambda_n^- \leq \lambda_n^+ < \lambda_{n+1}^- \leq \lambda_{n+1}^+ < \ldots \quad (n \in \mathbb{Z}).$$

Together with the asymptotics $\lambda_n^\pm(\varphi_0) = n\pi + \ell^2(n)$ (cf Proposition I.6) it then follows that $\min_{n\in\mathbb{Z}}(\lambda_{n+1}^-(\varphi_0) - \lambda_n^+(\varphi_0)) > 0$. Set

$$K := \frac{1}{5}\min\{\lambda_{n+1}^-(\varphi_0) - \lambda_n^+(\varphi_0), \frac{\pi}{2} \mid n \in \mathbb{Z}\}. \tag{I.24}$$

For any $n \in \mathbb{Z}$, denote by Γ_n the counterclockwise oriented circle in $\mathbb{C}$ with center $\tau_n(\varphi_0)$ and radius $\frac{1}{2}\gamma_n(\varphi_0)+2K$. Note that the circles Γ_n are pairwise disjoint. By Proposition I.6, $\lambda_n^\pm(\varphi) = \lambda_n^\pm(\varphi_0) + \ell^2(n)$ uniformly for $\varphi \in L^2$ close to φ_0. Thus there exist $N \geq 1$ and a neighborhood $V \equiv V_{\varphi_0} \subseteq L^2$ of φ_0 so that for any $\varphi \in V$

$$\sup_{|n|\geq N} |\lambda_n^\pm(\varphi) - \tau_n(\varphi_0)| \leq K.$$

As $\Delta(\lambda, \varphi)$ is continuous in φ and λ and

$$\sup_\lambda \{|\Delta(\lambda, \varphi_0)| \bigm| |\lambda - \tau_n(\varphi_0)| = \gamma_n(\varphi_0)/2 + K\} > 0$$

there exists a neighborhood $V'_{\varphi_0} \subseteq V$ of φ_0 in L^2 so that for any $\varphi \in V'_{\varphi_0}, -N \leq n \leq N$ and $\lambda \in \mathbb{C}$ with $|\lambda - \tau_n(\varphi_0)| = \gamma_n(\varphi_0)/2 + K$

$$|\Delta(\lambda, \varphi) - \Delta(\lambda, \varphi_0)| \leq |\Delta(\lambda, \varphi_0)|/2.$$

As $\Delta(\lambda, \varphi)$ is analytic in λ it follows then from Rouché's theorem and the lexicographic ordering of $(\lambda_k^\pm(\varphi))_{k\in\mathbb{Z}}$ that for $\varphi \in V'_{\varphi_0}$ and $-N \leq n \leq N$,

$$|\lambda_n^\pm(\varphi) - \tau_n(\varphi_0)| < \gamma_n(\varphi_0)/2 + K.$$

The open set

$$W := \bigcup_{\varphi_0 \in L^2_{\mathcal{R}}} V'_{\varphi_0} \tag{I.25}$$

is then a neighborhood of $L^2_{\mathcal{R}}$ in L^2. We have proved the following

Lemma I.12 *For any φ in W and $n \in \mathbb{Z}$,*

$$\ldots < Re(\lambda_n^-(\varphi)) \leq Re(\lambda_n^+(\varphi)) < Re(\lambda_{n+1}^-(\varphi)) \leq \ldots$$

and

$$Re\lambda_{n+1}^-(\varphi) - Re\lambda_n^+(\varphi) > 3K.$$

Due to the lexicographic ordering, the eigenvalues $\lambda_n^\pm(\varphi)$ are not continuous. However, we will prove that, for any $n \in \mathbb{Z}$, $\tau_n(\varphi)$ and $(\gamma_n(\varphi))^2$ are analytic functions on W. First we need to introduce some more notation. By Proposition I.6, there exists $N \geq 1$ locally uniformly on W so that

$$|\lambda_n^\pm - n\pi| < \pi/4 \quad \forall |n| \geq N. \tag{I.26}$$

For $|n| \geq N$, denote by $S_n(\varphi)$ the counterclockwise oriented circle with center $n\pi$ and radius $\pi/2$ whereas for $|n| < N, S_n \equiv S_n(\varphi)$ is defined to be the circle $\Gamma_n = \Gamma_n(\varphi)$. For any $n \in \mathbb{Z}$, the Riesz projectors $P_n \equiv P_n(\varphi)$ are then well defined,

$$P_n := \frac{1}{2\pi i}\int_{S_n(\varphi)} (\lambda - L(\varphi))^{-1}\, d\lambda$$
$$P_n^0 := \frac{1}{2\pi i}\int_{S_n(0)} (\lambda - L(0))^{-1}\, d\lambda.$$

The Riesz space $E_n \equiv E_n(\varphi)$ corresponding to P_n is defined as the range of P_n,

$$E_n := P_n(L^2);\; E_n^0 := P_n^0(L^2).$$

Both, E_n and E_n^0 are two dimensional subspaces of L^2 and P_n as well as $L(\varphi)P_n$ are bounded operators on L^2 of finite rank depending analytically on φ. Their traces can be computed to be, writing $L \equiv L(\varphi)$,

$$trP_n = 2;\; trLP_n = \lambda_n^+ + \lambda_n^- = 2\tau_n.$$

As $trP_n^0 = 2$ and $trL(0)P_n^0 = 2n\pi$ we then conclude that

$$\begin{aligned} 2\tau_n - 2n\pi &= Tr(LP_n) - Tr\left(L(0)P_n^0\right) \\ &= Tr\left((L - n\pi)P_n\right) - Tr\left((L(0) - n\pi)P_n^0\right) \\ &= TrQ_n \end{aligned}$$

where $Q_n \equiv Q_n(\varphi)$ is given by

$$Q_n := (L - n\pi)P_n - (L(0) - n\pi)P_n^0.$$

Substituting the formulas for P_n and P_n^0 into Q_n one obtains for $|n| \geq N$

$$Q_n = \frac{1}{2\pi i}\int_{S_n} (\lambda - n\pi)\left((\lambda - L)^{-1} - (\lambda - L(0)^{-1}\right) d\lambda.$$

Writing $L \equiv L(\varphi) = L(0) + B$ with $B \equiv B(\varphi) = \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix}$ one has

$$(\lambda - L)^{-1} = (\lambda - L(0))^{-1} + (\lambda - L)^{-1} B (\lambda - L(0))^{-1}.$$

Iterating the latter formula, and substituting into the formula for Q_n, one gets for any integer $N \geq 1$

$$Q_n = \sum_{k=1}^{N} Q_n^{(k)} + \check{Q}_n^{(N+1)}$$

where for any $k \geq 1$,

$$Q_n^{(k)} = \frac{1}{2\pi i} \int_{S_n} (\lambda - n\pi)(\lambda - L(0))^{-1} [B(\lambda - L(0))^{-1}]^k d\lambda$$

and

$$\check{Q}_n^{(k)} = \frac{1}{2\pi i} \int_{S_n} (\lambda - n\pi)(\lambda - L)^{-1} [B(\lambda - L(0))^{-1}]^k d\lambda. \tag{I.27}$$

The sequence e_k^+, e_k^- $(k \in \mathbb{Z})$, defined by

$$e_k^+(x) := \frac{1}{\sqrt{2}}(0,1)e^{ik\pi x} \ ; \ e_k^-(x) := \frac{1}{\sqrt{2}}(1,0)e^{-ik\pi x}$$

is an orthonormal basis of $L^2([0,2], \mathbb{C}^2)$ of eigenfunctions for $L(0)$ where $e_k^\pm$ are associated with the eigenvalue $\lambda_k^+ = \lambda_k^- = k\pi$ of $L(0)$. When expressed with respect to this basis, $(\lambda - L(0))^{-1}$ is a diagonal operator,

$$(\lambda - D)^{-1} e_k^\pm = \frac{1}{\lambda - k\pi} e_k^\pm \quad (k \in \mathbb{Z}).$$

Using the Fourier decomposition of φ_1 and φ_2,

$$\varphi_j(x) = \sum_{k \in \mathbb{Z}} \hat{\varphi}_j(k) e^{ik\pi x}$$

one gets for $k \in \mathbb{Z}$,

$$B(\lambda - L(0))^{-1} e_k^+ = \sum_{j \in \mathbb{Z}} \frac{\hat{\varphi}_1(-k-j)}{\lambda - k\pi} e_j^- \tag{I.28}$$

and

$$B(\lambda - L(0))^{-1} e_k^- = \sum_{j \in \mathbb{Z}} \frac{\hat{\varphi}_2(k+j)}{\lambda - k\pi} e_j^+. \tag{I.29}$$

In particular one deduces that for any $k \in \mathbb{Z}$,

$$tr Q_n^{(2k+1)} = 0.$$

Hence one obtains

$$\tau_n(\varphi) - n\pi = \frac{1}{2}\left(tr Q_n^{(2)} + tr \check{Q}_n^{(4)} \right). \tag{I.30}$$

The trace $tr Q_n^{(2)}$ can be explicitly computed. As

$$Q_n^{(2)} e_k^\pm = \frac{1}{2\pi i} \int_{S_n} (\lambda - n\pi) \sum_{\ell, j} \frac{\hat{\varphi}_1(-k-j)\hat{\varphi}_2(j+\ell)}{(\lambda - k\pi)(\lambda - j\pi)(\lambda - \ell\pi)} e_\ell^\pm$$

one obtains

$$\begin{aligned} tr Q_n^{(2)} &= \frac{2}{2\pi i} \sum_{k,j} \hat{\varphi}_1(-k-j)\hat{\varphi}_2(j+k) \int_{S_n} \frac{\lambda - n\pi}{(\lambda - k\pi)^2(\lambda - j\pi)} d\lambda \\ &= \frac{2}{\pi} \sum_{j \neq n} \hat{\varphi}_1(-n-j)\hat{\varphi}_2(j+n) \frac{1}{n-j}. \end{aligned} \tag{I.31}$$

Proposition I.13 *The map $\varphi \mapsto (\tau_n(\varphi) - n\pi)_{n \in \mathbb{Z}}$ is analytic on W with values in ℓ^p for any $1 < p$.*

Remark For potentials φ with more regularity, $\tau_n(\varphi)$ has an asymptotic expansion of the form $\tau_n = n\pi + \frac{H_1(\varphi)}{n\pi} + o\left(\frac{1}{n}\right)$ where $H_1(\varphi) := \int_0^1 \varphi_1(x)\varphi_2(x)dx$ (cf Lemma I.22).

Proof As $\tau_n(\varphi) - n\pi = \frac{1}{2} tr Q_n(\varphi)$ and $Q_n(\varphi) = (L(\varphi) - n\pi)P_n - (L(0) - n\pi)P_n^0$ is analytic on W with values in the space of operators of finite rank on $L^2([0,2], \mathbb{C}^2), \tau_n(\varphi)$ is analytic on W for any $n \in \mathbb{Z}$. Hence it suffices to prove that $(\tau_n(\varphi) - n\pi)_{n \in \mathbb{Z}} \in \ell^p$ locally uniformly on W. Our starting point is formula (I.30),

$$\tau_n(\varphi) - n\pi = \frac{1}{2} tr Q_n^{(2)} + \frac{1}{2} tr \check{Q}_n^{(4)}.$$

In view of (I.31), introduce

$$\begin{aligned} b(k) &:= |\varphi_1(-k)\varphi_2(k)| \ (k \in \mathbb{Z}) \\ u(k) &:= 1/|k| \ (k \neq 0); \ u(0) = 0 \end{aligned}$$

to obtain the estimate

$$| tr Q_n^{(2)} | \leq \frac{2}{\pi} (b * u)(2n)$$

where $b * u$ denotes the convolution of the two sequences b and u,

$$(b * u)(n) = \sum_{k \in \mathbb{Z}} b(n-k)u(k) \ .$$

As $(b(k))_{k\in\mathbb{Z}} \in \ell^1$ locally uniformly on W and $(u(k))_{k\in\mathbb{Z}} \in \ell^p$ for any $p > 1$, we conclude by Young's inequality that $b * u \in \ell^p$.
By Lemma I.14 below we have $\left(\|\breve{Q}_n^{(4)}\|_{\mathcal{L}(L^2)}\right)_{n\in\mathbb{Z}} \in \ell^p$ locally uniformly on W for any $p > 1$. As the range of $\breve{Q}_n^{(4)}$ is at most of dimension 4, we conclude that $|tr\breve{Q}_n^{(4)}| \leq 4\|\breve{Q}_n^4\|_{\mathcal{L}(L^2)}$. This proves that for any $p > 1$, $(\tau_n(\varphi) - n\pi)_{n\in\mathbb{Z}} \in \ell^p$ locally uniformly on W. ■

Lemma I.14 *For any $p > 1$, $\left(\|\breve{Q}_n^{(4)}\|_{\mathcal{L}(L^2)}\right)_{n\in\mathbb{Z}} \in \ell^p$ locally uniformly on W.*

Proof For any $|n| \geq N$, (cf. (I.26)), the circle S_n is given by $|\lambda - n\pi| = \frac{\pi}{2}$ and $\sup_{\lambda\in S_n} \|(\lambda - L)^{-1}\|_{\mathcal{L}(L^2)}$ is locally uniformly bounded. Hence for $|n| \geq N$, locally uniformly on W,

$$\|\breve{Q}_n^{(4)}\|_{\mathcal{L}(L^2)} \leq C \sup_{\lambda\in S_n} \|B(\lambda - L(0))^{-1}B(\lambda - L(0))^{-1}\|^2_{\mathcal{L}(L_2)}.$$

As (cf (I.28) - (I.29))

$$B(\lambda - L(0))^{-1}B(\lambda - L(0))^{-1}e_k^+ = \sum_{j,\ell} \frac{\hat{\varphi}_1(-k-j)\hat{\varphi}_2(j+\ell)}{(\lambda - k\pi)(\lambda - j\pi)} e_\ell^+$$

$$B(\lambda - L(0))^{-1}B(\lambda - L(0))^{-1}e_k^- = \sum_{j,\ell} \frac{\hat{\varphi}_2(k+j)\hat{\varphi}_1(-j-\ell)}{(\lambda - k\pi)(\lambda - j\pi)} e_\ell^-$$

one has

$$\|B(\lambda - L(0))^{-1}B(\lambda - L(0))^{-1}\|^2_{\mathcal{L}(L^2)} \leq 2\sum_{k,\ell}\left(\sum_j \frac{a(k+j)a(j+\ell)}{\langle k-n\rangle\langle j-n\rangle}\right)^2$$

where $\langle k\rangle = |k| + 1$ and

$$a(k) := \max(|\hat{\varphi}_1(-k)|, |\hat{\varphi}_2(k)|).$$

By the Cauchy-Schwartz inequality one has for any $\varepsilon > 0$,

$$\|B(\lambda - L(0))^{-1}B(\lambda - L(0))^{-1}\|^2_{\mathcal{L}(L^2)} \leq$$
$$\leq 2\sum_{k,\ell}\left(\sum_j \frac{a(j+\ell)^2}{\langle j-n\rangle^{1+\varepsilon}}\right)\left(\sum_j \frac{a(k+j)^2}{\langle k-n\rangle^2\langle j-n\rangle^{1-\varepsilon}}\right).$$

Introduce $r(k) := \langle k\rangle^{-1}$ $(k \in \mathbb{Z})$ and for any sequence $(u(k))_{k\in\mathbb{Z}}$ of nonnegative numbers denote by u^α the sequence $(u(k)^\alpha)_{k\in\mathbb{Z}}$ $(\alpha > 0)$. Hence, for $\lambda \in S_n$,

$$\|B(\lambda - L(0))^{-1}B(\lambda - L(0))^{-1}\|^2_{\mathcal{L}(L^2)} \leq 2\|a\|_{\ell^2}\|r^{1+\varepsilon}\|_{\ell^1}(r^2 * a^2 * r^{1-\varepsilon})(2n).$$

Given any $1 < p$, choose $\varepsilon > 0$ so that $p(1-\varepsilon) > 1$. As $r^2 \in \ell^1, r^{1-\varepsilon} \in \ell^p$ and $a^2 \in \ell^1$ (locally uniformly on W) one concludes by Young's inequality that $\left((r^2 * a^2 * r^{1-\varepsilon})(n)\right)_{n\in\mathbb{Z}} \in \ell^p$ (locally uniformly on W). Hence

$$\left(\sup_{\lambda\in S_n} \|B(\lambda - L(0))^{-1}B(\lambda - L(0))^{-1}\|^2_{\mathcal{L}(L^2)}\right)_{|n|\geq N} \in \ell^p$$

locally uniformly on W. As $\sup_{\lambda\in S_n} \|B(\lambda - L(0))^{-1}B(\lambda - L(0))^{-1}\|^2_{\mathcal{L}(L^2)}$ is bounded for any $|n| \leq N$, the claimed statement follows. ■

The next result concerns the sequence $(\gamma_n(\varphi)^2)_{n\in\mathbb{Z}}$. For $\varphi \in W$, the operator $2(L - \tau_n Id)^2 P_n$ has range E_n and its eigenvalues are $2(\lambda_n^\pm - \tau_n)^2 = \gamma_n^2$ (with multiplicity two) and 0. Hence

$$tr(2(L - \tau_n Id)^2 P_n) = \gamma_n^2.$$

As P_n and τ_n (cf Proposition I.13) are analytic on W, the map

$$W \to \mathcal{L}(L^2), \varphi \mapsto (L - \tau_n Id)^2 P_n$$

and thus $\varphi \mapsto tr(L - \tau_n Id)^2 P_n$ are analytic on W. By Proposition I.6, $\lambda_n^\pm = n\pi + \ell^2(n)$ and thus $\left((\lambda_n^+ - \lambda_n^-)^2\right)_{n\in\mathbb{Z}} \in \ell^1$, locally uniformly on W. We thus have proved

Proposition I.15 *The map $\varphi \mapsto \left((\gamma_n(\varphi))^2\right)_{n\in\mathbb{Z}}$ is analytic on W with values in ℓ^1.*

I.6 Infinite products

The infinite product representations given in this section have been proved in [GG] for $\varphi \in L^2_{\mathcal{R}}$ where $L^2_{\mathcal{R}}$ has been defined by

$$L^2_{\mathcal{R}} := \{\varphi = (\varphi_1, \varphi_2) \in L^2(S^1; \mathbb{C}^2) \mid \varphi_2 = \overline{\varphi_1}\}.$$

The proofs are valid for φ arbitrary. We recall them for the convenience of the reader.
Given a sequence of complex numbers $(a_k)_{k\in K}$ with $K \subseteq \mathbb{Z}$, we say that the product $\prod_{k\in K} a_k$ is convergent if the limit $\lim_{N\to\infty} \prod_{\substack{|k|\leq N\\ k\in K}} a_k$ exists. In such a case we write

$$\prod_{k\in K} a_k := \lim_{N\to\infty}\left(\prod_{\substack{|k|\leq N\\ k\in K}} a_k\right).$$

A sufficient condition for the convergence of $\prod_{k\in\mathbb{Z}} a_k$, with $(a_k)_{k\in\mathbb{Z}}$ being a sequence in $\mathbb{C}$ satisfying $\lim_{k\to\infty} a_k a_{-k} = 1$ is

$$\sum_{k\geq 1} |a_k a_{-k} - 1| < \infty. \tag{I.32}$$

This can easily be seen by applying the logarithm to $\prod_{k\geq k_0} a_k a_{-k}$ for $k_0 > 1$ sufficiently large,

$$|\log\left(\prod_{k\geq k_0} a_k a_{-k}\right)| \leq \sum_{k\geq k_0} |\log(a_k a_{-k})| \leq \sum_{k\geq k_0} |a_k a_{-k} - 1|.$$

Recall the following two fundamental lemmas on product representations:

Lemma I.16 *Assume that $z \equiv (z_m)_{m\in\mathbb{Z}}$ is a sequence in $\mathbb{C}$ with*

$$b \equiv (b_m)_{m\in\mathbb{Z}} := (z_m - m\pi)_{m\in\mathbb{Z}} \in \ell^2 .$$

Then for any $\lambda \in \mathbb{C}$, the infinite product

$$f(\lambda) := -(z_0 - \lambda) \prod_{m\neq 0} \frac{z_m - \lambda}{m\pi}$$

is convergent. The function $f(\lambda)$ is entire, its roots are given by z_m $(m \in \mathbb{Z})$ and it satisfies

$$\sup_{\lambda\in A_N} \Big| \frac{f(\lambda)}{\sin\lambda} - 1 \Big| \leq \left(\frac{\|b\|}{\langle N\rangle^{1/2}} + \left(\sum_{|n|\geq N/2} |b_m|^2 \right)^{1/2} \right) \exp\left(C\|b\|\right) \tag{I.33}$$

where

$$A_N := \cup_{n\geq N}\{(n + \frac{1}{4})\pi \leq |\lambda| \leq (n + \frac{3}{4})\pi\}$$

and $C > 0$ is an absolute constant.

Proof In view of (I.32), the convergence of $\prod_{m\neq 0} \frac{z_m-\lambda}{m\pi}$ follows from

$$\sum_{m\geq 1} \Big| \frac{(z_m - \lambda)(\lambda - z_{-m})}{m^2\pi^2} - 1 \Big| < \infty. \tag{I.34}$$

To verify (I.34), write

$$\begin{aligned} \frac{(z_m - \lambda)(\lambda - z_{-m})}{m^2\pi^2} - 1 &= \frac{(m\pi - \lambda + b_m)(\lambda + m\pi - b_{-m}) - m^2\pi^2}{m^2\pi^2} \\ &= -\frac{b_m b_{-m}}{m^2\pi^2} + \frac{b_m - b_{-m}}{m\pi} + \lambda\frac{b_m + b_{-m}}{m^2\pi^2} - \frac{\lambda^2}{m^2\pi^2}. \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{m\geq 1} \Big| \frac{(z_m - \lambda)(\lambda - z_{-m})}{m^2\pi^2} - 1 \Big| \leq \\ &\sum_{m\geq 1} \frac{|b_m b_{-m}|}{m^2\pi^2} + \sum_{m\geq 1} \frac{|b_m| + |b_{-m}|}{m\pi} \left(1 + \frac{\lambda}{m\pi}\right) + \lambda^2 \sum_{m\geq 1} \frac{1}{m^2\pi^2} \end{aligned}$$

which converges uniformly on bounded subsets of $\mathbb{C}$. This establishes (I.34) and one concludes that $f(\lambda)$ is an entire function. Clearly its roots are z_m $(m \in \mathbb{Z})$. It remains to prove the estimate (I.33). By a straightforward argument one verifies that

$$\begin{aligned} \Big| \prod_m (1 + a_m) - 1 \Big| &\leq \left(\sum_m |a_m|\right) \prod_m (1 + |a_m|) \\ &\leq \left(\sum_m |a_m|\right) \exp\left(\sum_m |a_m|\right) . \end{aligned} \tag{I.35}$$

To apply this estimate recall that $\sin\lambda$ has the product representation

$$\sin\lambda = \lambda \prod_{m\in\mathbb{Z}\setminus 0} \frac{m\pi - \lambda}{m\pi} .$$

Hence, for $\lambda \in \mathbb{C}$ with $(n + \frac{1}{4})\pi \leq |\lambda| \leq \left(n + \frac{3}{4}\right)\pi$

$$\frac{f(\lambda)}{\sin\lambda} = \left(1 - \frac{z_0}{\lambda}\right) \prod_{m\neq 0} \left(1 + \frac{b_m}{m\pi - \lambda}\right)$$

with

$$\Big|\frac{z_0}{\lambda}\Big| \leq 4\frac{b_0}{\pi\langle n\rangle}, \quad \Big|\frac{b_m}{m\pi - \lambda}\Big| \leq \frac{8|b_m|}{\pi\langle |m| - n\rangle}.$$

Hence in view of (I.35) for any $n \geq 0$

$$\sup_{\substack{(n+\frac{1}{4})\pi\leq|\lambda| \\ |\lambda|\leq(n+\frac{3}{4})\pi}} \Big| \frac{f(\lambda)}{\sin\lambda} - 1 \Big| \leq \left(\sum_m \frac{8}{\pi} \frac{|b_m|}{\langle |m| - n\rangle}\right) \exp\left(\sum_m \frac{8}{\pi} \frac{|b_m|}{\langle |m| - n\rangle}\right).$$

By the Cauchy-Schwartz inequality

$$\begin{aligned} \sum_{m\geq 0} \frac{|b_m|}{\langle m - n\rangle} &\leq \|b\| \left(\sum_{0\leq m\leq n/2} \frac{1}{\langle m - n\rangle^2}\right)^{1/2} \\ &\quad + \left(\sum_{m\geq n/2} |b_m|^2\right)^{1/2} \left(\sum_{m\geq n/2} \frac{1}{\langle m - n\rangle^2}\right)^{1/2} \\ &\leq C\left(\frac{\|b\|}{\langle n\rangle^{1/2}} + \left(\sum_{m\geq n/2} |b_m|^2\right)^{1/2}\right) \end{aligned}$$

where $C > 0$ is an absolute constant. Similarly, one has

$$\sum_{m\geq 0} \frac{|b_{-m}|}{\langle m-n\rangle} \leq C\left(\frac{\|b\|}{\langle n\rangle^{1/2}} + \left(\sum_{m\geq n/2} |b_{-m}|^2\right)^{1/2}\right).$$

Hence we conclude that for any $N \geq 0$

$$\Big| \sup_{\lambda\in A_N}\left(\frac{f(\lambda)}{\sin\lambda} - 1\right) \Big| \leq \left(\frac{\|b\|}{\langle N\rangle^{1/2}} + \left(\sum_{|m|\geq N/2} |b_m|^2\right)^{1/2}\right) \exp\left(C\|b\|\right)$$

for some absolute constant $C > 0$. ■

To prove the following Lemma we will use that for any sequence $(a_m)_{m\in\mathbb{Z}}$ in ℓ^1, with $0 \leq a_m \leq 1/2$ one has

$$\begin{aligned}\prod_m (1+a_m) &= \exp\left(\sum_m \log(1+a_m)\right)\\ &= \exp\left(\sum_m a_m\right)\exp\left(\sum_m (\log(1+a_m) - a_m)\right)\end{aligned}$$

and hence

$$\begin{aligned}\prod_m (1+a_m) - 1 &= \exp\left(\sum_m a_m\right) - 1\\ &\quad + \exp\left(\sum_m a_m\right)\left(\exp\left(\sum_m (\log(1+a_m) - a_m)\right) - 1\right).\end{aligned}$$

Using that $|e^x - 1| \leq |x|e^{|x|}$ one then obtains

$$\begin{aligned}&\Big| \prod_m (1+a_m) - 1 \Big| \leq \Big| \sum_m a_m \Big| \exp\left(\Big| \sum a_m \Big|\right)\\ &+ \exp\left(\Big| \sum a_m \Big|\right)\left(\Big| \sum_m (\log(1+a_m) - a_m) \Big| \exp\left(\sum_m \Big| \log(1+a_m) - a_m \Big|\right)\right)\end{aligned}$$

and together with $|\log(1+x) - x| \leq 2|x|^2$ for $|x| \leq 1/2$ one concludes that

$$\begin{aligned}&\Big| \prod_m (1+a_m) - 1 \Big| \leq\\ &\left(\Big| \sum_m a_m \Big| + 2\left(\sum_m \Big| a_m \Big|^2\right)\exp\left(2\sum_m \Big| a_m \Big|^2\right)\right)\exp\left(\Big| \sum a_m \Big|\right).\end{aligned} \tag{I.36}$$

Lemma I.17 *Assume that $z = (z_m)_{m\in\mathbb{Z}}$ is a sequence in $\mathbb{C}$ with*

$$b \equiv (b_m)_{m\in\mathbb{Z}} := (z_m - m\pi)_{m\in\mathbb{Z}} \in \ell^2 .$$

Then for any $k \in \mathbb{Z}\backslash\{0\}$, the infinite product

$$f_k(\lambda) := \frac{\lambda - z_0}{k\pi} \prod_{m\neq 0,k} \frac{z_m - \lambda}{m\pi} \tag{I.37}$$

is convergent for any $\lambda \in \mathbb{C}$ and defines an entire function with

$$\left\| \left(\sup_{|\lambda-k\pi|\leq\pi/4} \Big| \frac{k\pi-\lambda}{\sin\lambda} f_k(\lambda) - 1 \Big| \right)_{k\in\mathbb{Z}} \right\|_{\ell^2} \leq C \tag{I.38}$$

for some constant $C > 0$ which can be chosen uniformly on bounded subsets of sequences $b = (b_m)_{m\in\mathbb{Z}}$ in ℓ^2. As a consequence, given $(r_k)_{k\in\mathbb{Z}} \in \ell^4(\mathbb{Z})$ with $r_k \geq 0$,

$$\left\| \left(\sup_{|\lambda-k\pi|\leq r_k} \Big| f_k(\lambda) + (-1)^k \Big| \right)_{k\in\mathbb{Z}} \right\|_{\ell^2} \leq C \tag{I.39}$$

where $C > 0$ can be chosen uniformly on bounded subsets of sequences $(b_m)_{m\in\mathbb{Z}}$ in ℓ^2 and $(r_k)_{k\in\mathbb{Z}}$ in ℓ^4.

Proof The convergence of the product in (I.37) and the analyticity of f_k is shown as in the proof of Lemma I.16. To prove (I.38), use again the product representation $\frac{\sin\lambda}{\lambda} = \prod_{m\neq 0} \frac{m\pi-\lambda}{m\pi}$ to get for $\lambda \in \mathbb{C}$ with $|\lambda - k\pi| \leq \frac{\pi}{4}$

$$\begin{aligned}\frac{k\pi-\lambda}{\sin\lambda} f_k(\lambda) &= \frac{k\pi}{\lambda}\left(\prod_{m\neq 0,k} \frac{m\pi}{(m\pi-\lambda)}\right)\frac{\lambda - z_0}{k\pi}\left(\prod_{m\neq 0,k} \frac{m\pi-\lambda+b_m}{m\pi}\right)\\ &= \prod_{m\neq k}\left(1 + \frac{b_m}{m\pi-\lambda}\right).\end{aligned}$$

To apply (I.36) to our situation, choose $k_0 \geq 1$ so large that for any $|k| \geq k_0$ and any $\lambda \in \mathbb{C}$ with $|\lambda - k\pi| \leq \pi/4$, $|b_m|/(m\pi - \lambda) \leq 1/2$ for any $m \in \mathbb{Z}\backslash\{k\}$ to obtain

$$\sup_{|\lambda-k\pi|\leq\pi/4} \Big| \frac{k\pi-\lambda}{\sin\lambda} f_k(\lambda) - 1 \Big| \leq \alpha_k \exp(\alpha_k)$$

where

$$\alpha_k := \sup_{|\lambda-k\pi|\leq\pi/4} \Big| \sum_{m\neq k} \frac{b_m}{m\pi-\lambda} \Big| + 2\sum_m \sup_{|\lambda-k\pi|\leq\pi/4} \Big| \frac{b_m}{m\pi-\lambda} \Big|^2. \tag{I.40}$$

The two terms in α_k are estimated separately. Towards the first one, recall that the discrete Hilbert transform is a bounded linear operator from $\ell^2(\mathbb{Z})$ onto $\ell^2(\mathbb{Z})$ (cf. [HLP], Theorem 294), hence for an absolute constant $C > 0$

$$\Big\| \Big(\sum_{m\neq k} \frac{b_m}{(m-k)\pi} \Big)_{k\in\mathbb{Z}} \Big\| \leq C \Big\| (b_m)_{m\in\mathbb{Z}} \Big\|. \tag{I.41}$$

As $|m\pi - \lambda| \geq \frac{1}{2}\pi|m-k|$ for $|\lambda - k\pi| \leq \pi/4$, one also has

$$\begin{aligned} &\Big\| \Big(\sup_{|\lambda-k\pi|\leq\pi/4} \Big| \sum_{m\neq k} \frac{b_m}{m\pi-\lambda} - \sum_{m\neq k} \frac{b_m}{m\pi - k\pi} \Big| \Big)_{k\in\mathbb{Z}} \Big\| \\ &\leq \Big\| \frac{1}{2\pi} \Big(\sum_{m\neq k} \frac{|b_m|}{(m-k)^2} \Big)_{k\in\mathbb{Z}} \Big\| \\ &\leq \frac{1}{2\pi} \|b\| \Big(\sum_k \frac{1}{k^2} \Big) \end{aligned} \tag{I.42}$$

where for the last inequality we used that for $a = (a_k)_{k\in\mathbb{Z}} \in \ell^1$ and $b = (b_k)_{k\in\mathbb{Z}} \in \ell^2$ the convolution $a * b$ is again a sequence in ℓ^2 with $\|a * b\| \leq \|b\| \|a\|_{\ell^1}$.
Combining (I.41) and (I.42) leads to

$$\begin{aligned} &\Big\| \Big(\sup_{|\lambda-k\pi|\leq\pi/4} \Big| \sum_{m\neq k} \frac{b_m}{m\pi-\lambda} \Big| \Big)_{k\in\mathbb{Z}} \Big\| \leq \\ &\Big\| \Big(\sum_{m\neq k} \frac{b_m}{(m-k)\pi} \Big)_{k\in\mathbb{Z}} \Big\| + \Big\| \Big(\sup_{|\lambda-k\pi|\leq\pi/4} \Big| \sum_{m\neq k} \Big(\frac{b_m}{m\pi-\lambda} - \frac{b_m}{(m-k)\pi} \Big) \Big| \Big)_{k\in\mathbb{Z}} \Big\| \\ &\leq C\|b\| \end{aligned} \tag{I.43}$$

Towards the second term in (I.40) use

$$\sup_{|\lambda-k\pi|\leq\pi/4} \frac{1}{|m\pi-\lambda|} \leq \frac{2}{\pi} \frac{1}{|m-k|}$$

to conclude that

$$\begin{aligned} &\sum_k \sum_{m\neq k} \sup_{|\lambda-k\pi|\leq\pi/4} \frac{|b_m|^2}{|m\pi-\lambda|^2} \leq \\ &\leq \sum_k \sum_{m\neq k} \frac{4}{\pi^2} \frac{|b_m|^2}{|m-k|^2} \leq \frac{4}{\pi^2} \Big(\sum_m |b_m|^2 \Big) \Big(\sum_{j\neq 0} \frac{1}{j^2} \Big) \\ &\leq C\|b\|^2 \end{aligned} \tag{I.44}$$

for some absolute constant $C > 0$. Combining (I.43) and (I.44) with (I.40) finally yields

$$\Big\| \Big(\sup_{|\lambda-k\pi|\leq\pi/4} \Big| \frac{k\pi-\lambda}{\sin\lambda} f_k(\lambda) - 1 \Big| \Big)_{|k|\geq k_0} \Big\| \leq C\|b\|$$

where $C > 0$ can be chosen uniformly for bounded subsets of sequences b in ℓ^2. Clearly

$$\Big\| \Big(\sup_{|\lambda-k\pi|\leq\pi/4} \Big| \frac{k\pi-\lambda}{\sin\lambda} f_k(\lambda) - 1 \Big| \Big)_{|k|\leq k_0} \Big\| \leq C$$

where again, $C > 0$ can be chosen uniformly for bounded sets of sequences b in ℓ^2. Hence estimate (I.38) is proved. ∎

In Part II, we will need the following lemma which is a consequence of Lemma I.16 and Lemma I.17.

Lemma I.18 *Let $z := (z_m)_{m\in\mathbb{Z}}$ be a sequence in $\mathbb{C}$ with $(b_m := z_m - m\pi)_{m\in\mathbb{Z}} \in \ell^2$. Then the entire function*

$$f(\lambda) := (\lambda - z_0) \prod_{m\neq 0} \frac{z_m - \lambda}{m\pi}$$

is bounded on $\mathbb{R}$, uniformly for $b \equiv (b_m)_{m\in\mathbb{Z}}$ in bounded subsets of $\ell^2(\mathbb{Z})$.

Proof By Lemma I.16, f is entire, hence bounded on bounded subsets of $\mathbb{C}$ and there exists a constant $C_1 > 0$ so that on $\bigcup_{n\geq 0} \left\{ \left(n + \frac{1}{4}\right)\pi \leq |\lambda| \leq \left(n + \frac{3}{4}\right)\pi \right\}$, one has $|f(\lambda)| \leq C_1$. By Lemma I.17, there exists $C_2 > 0$ so that for any $k \in \mathbb{Z}$ and $|\lambda - k\pi| \leq \pi/4$

$$\Big| \frac{f(\lambda)}{z_k - \lambda} \Big| \leq C_2 \Big| \frac{\sin\lambda}{\lambda - k\pi} \Big|$$

and, as $\left| \frac{\sin\lambda}{\lambda-k\pi} \right| \leq 1$ and $|z_k - \lambda| \leq |b_k| + \pi/4 \leq \|b\| + 1$ for $|\lambda - k\pi| \leq \pi/4$, one then gets $|f(\lambda)| \leq C_2(\|b\| + 1)$. It then follows that

$$|f(\lambda)| \leq (C_1 + C_2)(\|b\| + 1) \quad \forall \lambda \in \mathbb{R}.$$

By Lemma I.16 and Lemma I.17, $C := C_1 + C_2$ can be chosen uniformly on bounded subsets of sequences b in ℓ^2. ∎

Lemma I.16 is used to derive product representations for $\Delta(\lambda)^2 - 4$, $\dot{\Delta}(\lambda)$, $\delta(\lambda)$ and $\check{\delta}(\lambda)$.

In view of the asymptotics $\lambda_n^\pm = n\pi + \ell^2(n)$, (Proposition I.6), the infinite product

$$f(\lambda) := -4(\lambda_0^- - \lambda)(\lambda_0^+ - \lambda)\prod_{k\neq 0}\frac{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}{k^2\pi^2}$$

is convergent.

Lemma I.19 *For $\varphi \in L^2_{\mathbb{C}}$ and $\lambda \in \mathbb{C}$,*

$$\Delta(\lambda,\varphi)^2 - 4 = -4(\lambda_0^- - \lambda)(\lambda_0^+ - \lambda)\prod_{k\neq 0}\frac{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}{k^2\pi^2}.$$

Proof From Proposition I.3 and in view of the estimate

$$e^{|Im\lambda|} < 4|\sin\lambda| \quad \forall\lambda \in \{z\in\mathbb{C} \mid |z-k\pi| \geq \frac{\pi}{4} \quad \forall k\in\mathbb{Z}\}$$

one gets uniformly on $\{|\lambda| = (n+\frac{1}{2})\pi\}$,

$$\Delta(\lambda,\varphi) = 2\cos\lambda + o(\sin\lambda)$$

and thus

$$\Delta(\lambda,\varphi)^2 - 4 = (-4\sin^2\lambda)(1+o(1)).$$

By Lemma I.16,

$$f(\lambda) = -4\sin^2\lambda(1+o(1)).$$

As $f(\lambda)$ and $\Delta(\lambda,\varphi)^2-4$ are both entire functions and have the same roots we conclude that $h(\lambda) := (\Delta(\lambda,\varphi)^2-4)/f(\lambda)$ is also entire and satisfies on $\{|\lambda| = |n+\frac{1}{2}|\pi\}$

$$h(\lambda) = 1 + o(1).$$

The function $h-1$ being harmonic we then conclude by the maximum principle that $h(\lambda)\equiv 1$, i.e. $\Delta(\lambda)^2 - 4 \equiv f(\lambda)$. ■

Next we want to obtain a product representation for $\dot\Delta(\lambda,\varphi)$. First we have to prove asymptotic estimates for its zeroes. They are obtained by arguments similar to the ones used to show the asymptotics of the zeroes $\lambda_n^\pm$ of $\Delta(\lambda)^2-4$. By Proposition I.3, we have for $|\lambda|\to\infty$

$$\dot\Delta(\lambda) = tr\dot M(1,\lambda) = tr\dot E(1,\lambda) + o(e^{|Im\lambda|}).$$

As $tr\dot E(1,\lambda) = -2\sin\lambda$ we conclude that for λ sufficiently large

$$|\dot\Delta(\lambda) + 2\sin\lambda| \leq \frac{1}{4}e^{|Im\lambda|}.$$

By Rouché's Theorem one then argues as in the proof of Proposition I.5 to conclude that $\dot\Delta(\lambda)$ has a sequence of zeroes $(\dot\lambda_n)_{n\in\mathbb{Z}}$ with $\dot\lambda_n = n\pi + o(1)$. Again, we order the $\dot\lambda_n$'s lexicographically,

$$Re\dot\lambda_n < Re\dot\lambda_{n+1} \text{ or } Re\dot\lambda_n = Re\dot\lambda_{n+1} \text{ and } Im\dot\lambda_n \leq Im\dot\lambda_{n+1}.$$

Arguing as in the proof of Proposition I.6 one shows that $\dot\lambda_n = n\pi + \ell^2(n)$. Hence, by Lemma I.16, the following infinite product is absolutely convergent for any $\lambda\in\mathbb{C}$,

$$g(\lambda) := 2(\dot\lambda_0 - \lambda)\prod_{k\neq 0}\frac{\dot\lambda_k - \lambda}{k\pi}$$

and defines an entire function.

Lemma I.20 *For $\varphi \in L^2_{\mathbb{C}}$ and $\lambda \in \mathbb{C}$,*

$$\dot\Delta(\lambda) = 2(\dot\lambda_0 - \lambda)\prod_{k\neq 0}\frac{\dot\lambda_k - \lambda}{k\pi}.$$

Proof Notice that for λ in $\{|\lambda| = |n+\frac{1}{2}|\pi\}$

$$\dot\Delta(\lambda,\varphi) = (-2\sin\lambda)\,(1+o(1))\,.$$

Hence we can argue as in the proof of Lemma I.19 to obtain the claimed result. ■

Recall from section I.3 that $\delta(\lambda)$ and $\breve\delta(\lambda)$ are entire functions with zeroes $(\mu_n)_{n\in\mathbb{Z}}$ and $(\breve\mu_n)_{n\in\mathbb{Z}}$ respectively. Both sequences have asymptotics of the form $n\pi + \ell^2(n)$. Hence, by Lemma I.16 the infinite products

$$2i(\mu_0 - \lambda)\prod_{j\neq 0}\frac{\mu_j - \lambda}{j\pi} \text{ and } 2i(\breve\mu_0 - \lambda)\prod_{j\neq 0}\frac{\breve\mu_j - \lambda}{j\pi}$$

are absolutely convergent for any $\lambda\in\mathbb{C}$. Due to the asymptotics $\delta(\lambda)$ and $\breve\delta(\lambda)$ of the form $-2i\sin\lambda + o(e^{|Im\lambda|})$ one can argue as in the proof of Lemma I.19 to obtain

Lemma I.21 *For $\varphi \in L^2_{\mathbb{C}}$ and $\lambda \in \mathbb{C}$,*

$$\delta(\lambda) = 2i(\mu_0 - \lambda)\prod_{k\neq 0}\frac{\mu_k - \lambda}{k\pi}$$

$$\breve\delta(\lambda) = 2i(\breve\mu_0 - \lambda)\prod_{k\neq 0}\frac{\breve\mu_k - \lambda}{k\pi}.$$

We end this section by providing refined asymptotics for $\dot\lambda_n$ as $|n| \to \infty$. For this purpose introduce the sequences $(\gamma_n)_{n\in\mathbb{Z}}$ and $(\tau_n)_{n\in\mathbb{Z}}$ given by

$$\gamma_n := \lambda_n^+ - \lambda_n^- \ ; \quad \tau_n := (\lambda_n^+ + \lambda_n^-)/2.$$

Lemma I.22 *Locally uniformly for φ in $L^2_{\mathbb{C}}$,*

$$\dot\lambda_n = \tau_n + \gamma_n^{3/2}\ell^2(n). \tag{I.45}$$

Proof In a first step we prove the weaker estimate

$$\dot\lambda_n = \tau_n + \gamma_n\ell^2(n). \tag{I.46}$$

By Lemma I.19,

$$\begin{aligned}&\frac{d}{d\lambda}(\Delta(\lambda)^2-4)\Big|_{\lambda=\lambda_n^+}\\ &= -4(\lambda_0^+-\lambda)(\lambda_0^--\lambda)\frac{\gamma_n}{n^2\pi^2}\prod_{m\neq n,0}\frac{(\lambda_m^+-\lambda)(\lambda_m^--\lambda)}{m^2\pi^2}\Big|_{\lambda=\lambda_n^+}.\end{aligned}$$

On the other hand, as $\Delta(\lambda_n^+) = 2(-1)^n$,

$$\begin{aligned}\frac{d}{d\lambda}(\Delta(\lambda)^2-4)\Big|_{\lambda=\lambda_n^+} &= 2\Delta(\lambda_n^+)\dot\Delta(\lambda_n^+)\\ &= 4(-1)^n\dot\Delta(\lambda_n^+).\end{aligned}$$

Combining the two identities we get

$$\dot\Delta(\lambda_n^+) = (-1)^{n+1}\gamma_n\left(1+0\left(\frac{1}{n}\right)\right)\prod_{m\neq n,0}\frac{(\lambda_m^+-\lambda_n^+)(\lambda_m^--\lambda_n^+)}{m^2\pi^2}.$$

By Lemma I.17,

$$\prod_{m\neq n,0}\frac{\lambda_m^+-\lambda}{m\pi}\ \frac{\lambda_m^--\lambda}{m\pi}\Big|_{\lambda=\lambda_n^+} = 1+\ell^2(n)$$

and hence

$$\dot\Delta(\lambda_n^+) = (-1)^{n+1}\gamma_n(1+\ell^2(n)). \tag{I.47}$$

By the product representation of Lemma I.20,

$$\begin{aligned}\dot\Delta(\lambda_n^+) &= 2(\dot\lambda_0-\lambda_n^+)\frac{\dot\lambda_n-\lambda_n^+}{n\pi}\prod_{m\neq 0,n}\frac{\dot\lambda_m-\lambda_n^+}{m\pi}\\ &= 2(-1)^n(\dot\lambda_n-\lambda_n^+)(1+\ell^2(n))\end{aligned}$$

where we used again Lemma I.17 for the latter identity. In view of (I.47) one then obtains

$$2(\dot\lambda_n-\lambda_n^+) = -\gamma_n(1+\ell^2(n))$$

and substituting $\lambda_n^+ = \tau_n + \frac{\gamma_n}{2}$ into this identity leads to (I.46) .
To obtain the stronger estimate (I.45), write

$$\Delta(\lambda)^2 - 4 = 4(\lambda_n^+-\lambda)(\lambda-\lambda_n^-)f_n(\lambda)$$

where

$$f_n(\lambda) := \frac{(\lambda_0^+-\lambda)(\lambda_0^--\lambda)}{n^2\pi^2}\prod_{m\neq n,0}\frac{(\lambda_m^+-\lambda)(\lambda_m^--\lambda)}{m^2\pi^2}.$$

Clearly for $\gamma_n = 0$, (I.45) trivially holds. Now consider the case $\gamma_n \neq 0$. By Lemma I.17, we have uniformly on $\Gamma_n := \{\lambda \in \mathbb{C} \mid |\lambda - \dot\lambda_n| = |\gamma_n|^{1/2}\}$,

$$f_n = 1 + \ell^2(n).$$

By Cauchy's theorem,

$$\begin{aligned}\left|\frac{df_n}{d\lambda}(\dot\lambda_n)\right| &= \left|\frac{1}{2\pi i}\int_{\Gamma_n}\frac{f_n(\lambda)-1}{(\lambda-\dot\lambda_n)^2}d\lambda\right|\\ &= |\gamma_n|^{-1/2}\ell^2(n).\end{aligned}$$

This is used in

$$\begin{aligned}0 &= \frac{d}{d\lambda}(\Delta(\lambda)^2-4)\Big|_{\lambda=\dot\lambda_n}\\ &= \left(-4(\lambda-\lambda_n^-)+4(\lambda_n^+-\lambda)\right)f_n(\lambda)\Big|_{\lambda=\dot\lambda_n}\\ &\quad + 4(\lambda_n^+-\lambda)(\lambda-\lambda_n^-)\frac{df_n}{d\lambda}(\lambda)\Big|_{\lambda=\dot\lambda_n}\end{aligned}$$

to obtain

$$2(\dot\lambda_n-\tau_n)(1+\ell^2(n)) = (\lambda_n^+-\dot\lambda_n)(\dot\lambda_n-\lambda_n^-)|\gamma_n|^{-1/2}\ell^2(n).$$

In view of (I.46), $\lambda_n^\pm - \dot\lambda_n = 0(\gamma_n)$ and hence $\dot\lambda_n - \tau_n = \gamma_n^{3/2}\ell^2(n)$ as claimed.
■

I.7 Branches of square roots

In the sequel we need to specify various branches of square roots. We denote by $\sqrt[+]{z}$ the principal branch of the square root defined for $z \in \mathbb{C}\backslash(-\infty, 0]$ and determined by $\sqrt[+]{1} = 1$.

Given $a, b \in \mathbb{C}$ with $\Re a < \Re b$ or $\Re a = \Re b$ and $\Im a < \Im b$, we denote by $\sqrt[s]{(a-z)(b-z)}$ the *standard* square root, defined on $\mathbb{C}\backslash[a,b]$ and determined by

$$\sqrt[s]{(a-z)(b-z)}\Big|_{z=b+(b-a)} = -\sqrt[+]{2}\,(b-a)$$

where $[a,b] := \{ta + (1-t)b \mid 0 \leq t \leq 1\}$.
For λ in $\mathbb{C}\backslash\left(\bigcup_{k\in\mathbb{Z}}[\lambda_k^-, \lambda_k^+]\right)$, the *canonical* square root $\sqrt[c]{\Delta(\lambda)^2 - 4}$ is defined by

$$\sqrt[c]{\Delta(\lambda)^2 - 4} := 2i\sqrt[s]{(\lambda_0^- - \lambda)(\lambda_0^+ - \lambda)} \prod_{k\neq 0} \frac{\sqrt[s]{(\lambda_k^- - \lambda)(\lambda_k^+ - \lambda)}}{k\pi}. \tag{I.48}$$

Notice that for $|\lambda| << |k\pi|$, with $\lambda_k^\pm = k\pi + \ell^2(k)$,

$$\sqrt[s]{(\lambda_k^- - \lambda)(\lambda_k^+ - \lambda)} = k\pi \sqrt[+]{\left(1 + 0\left(\frac{1}{k}\right) - \frac{\lambda}{k\pi}\right)\left(1 + 0\left(\frac{1}{k}\right) - \frac{\lambda}{k\pi}\right)}$$

and thus the infinite product on the right side of (I.48) is absolutely convergent.
For $\varphi = (\varphi_1, \varphi_2)$ of real type (i.e. $\varphi_2 = \overline{\varphi_1}$), $\lambda_n^\pm \in \mathbb{R}$ for all n and one gets

$$(-1)^k i \sqrt[c]{\Delta(\lambda)^2 - 4} > 0 \qquad \forall\ \lambda_k^+ < \lambda < \lambda_{k+1}^- \tag{I.49}$$

$$(-1)^k \sqrt[c]{\Delta(\lambda + i0)^2 - 4} > 0 \qquad \forall\ \lambda_k^- < \lambda < \lambda_k^+ \tag{I.50}$$

$$(-1)^{k-1} \sqrt[c]{\Delta(\lambda - i0)^2 - 4} > 0 \quad \forall\ \lambda_k^- < \lambda < \lambda_k^+. \tag{I.51}$$

I.8 Asymptotics for $\Delta(\lambda, \varphi)$

In Part II we need an expansion of $\Delta(\lambda)$ for $|k| \to \infty$ and of $ch^{-1}\left(\frac{\Delta(iy)}{2}\right)$ as $y \to \infty$.
For $\varphi = (\varphi_1, \varphi_2) \in L^2_{\mathbb{C}}$ introduce

$$H_1(\varphi) := \int_0^1 \varphi_1(x)\varphi_2(x)dx.$$

Let $H^N_{\mathbb{C}} \equiv H^N([0,1]; \mathbb{C}^2)$ be the Sobolev space of functions $\varphi : [0,1] \to \mathbb{C}^2$ with $\partial_x^j f \in L^2_{\mathbb{C}}$ for any $0 \leq j \leq N$.

Lemma I.23 *Uniformly for φ in a bounded subset of $H^2_{\mathbb{C}}$,*

$$\Delta(\lambda, \varphi) = 2\cos\lambda + \frac{\sin\lambda}{\lambda} H_1(\varphi) + 0\left(\frac{e^{|Im\lambda|}}{|\lambda|^2}\right).$$

Remark The asymptotic holds under weaker regularity assumption on φ.
Proof Recall that $\Delta(\lambda) = trM(1,\lambda)$ and $M(x,\lambda)$ is given by

$$M(x,\lambda) = E(x,\lambda) + \sum_{k\geq 1} E^{(k)}(x,\lambda).$$

We have $trE(1,\lambda) = e^{-i\lambda} + e^{i\lambda} = 2\cos\lambda$ and $trE^{(1)} = 0$. Hence

$$\Delta(\lambda) = 2\cos\lambda + \sum_{k\geq 2} trE^{(k)}(1,\lambda). \tag{I.52}$$

To obtain the asymptotics for $trE^{(k)}(1,\lambda)$ we write

$$E^{(k)}(x,\lambda) = \int_0^x K(x,y)E^{(k-1)}(y,\lambda)dy\ .$$

First consider $E^{(1)}(x,\lambda)$, given by

$$E^{(1)}(x,\lambda) = i\int_0^x \begin{pmatrix} 0 & e^{-i\lambda(x-2y)}\varphi_1(y) \\ -e^{i\lambda(x-2y)}\varphi_2(y) & 0 \end{pmatrix} dy.$$

Integrate by parts to get

$$E^{(1)}(x,\lambda) = \frac{1}{2\lambda}\begin{pmatrix} 0 & e^{-i\lambda(x-2y)}\varphi_1(y) \\ e^{i\lambda(x-2y)}\varphi_2(y) & 0 \end{pmatrix}\Big|_{y=0}^{x} - R^{(1)}(x,\lambda)$$

$$= \frac{1}{2\lambda}\begin{pmatrix} 0 & e^{i\lambda x}\varphi_1(x) \\ e^{-i\lambda x}\varphi_2(x) & 0 \end{pmatrix} - \frac{1}{2\lambda}\begin{pmatrix} 0 & e^{-i\lambda x}\varphi_1(0) \\ e^{i\lambda x}\varphi_2(0) & 0 \end{pmatrix} + 0\left(\frac{e^{|Im\lambda|x}}{\lambda^2}\right)$$

where for the last identity we used that

$$\begin{aligned} R^{(1)}(x,\lambda) :&= \frac{1}{2\lambda}\int_0^x \begin{pmatrix} 0 & e^{i\lambda(2y-x)}\varphi_1'(y) \\ e^{i\lambda(x-2y)}\varphi_2'(y) & 0 \end{pmatrix} dy \\ &= \frac{i}{(2\lambda)^2}\begin{pmatrix} 0 & -e^{i\lambda(2y-x)}\varphi_1'(y) \\ e^{i\lambda(x-2y)}\varphi_2'(y) & 0 \end{pmatrix}\Big|_{y=0}^{x} \\ &- \frac{i}{(2\lambda)^2}\int_0^x \begin{pmatrix} 0 & -e^{i\lambda(2y-x)}\varphi_1''(y) \\ e^{i\lambda(x-2y)}\varphi_2''(y) & 0 \end{pmatrix} dy \\ &= 0\left(\frac{e^{|Im\lambda|x}}{\lambda^2}\right). \end{aligned}$$

To get the asymptotics of $E^{(2)}(x,\lambda)$ write $E^{(2)}(x,\lambda) = \int_0^x K(x,y)E^{(1)}(y,\lambda)dy$ with

$$K(x,y) = i\begin{pmatrix} 0 & e^{-i\lambda(x-y)}\varphi_1(y) \\ -e^{i\lambda(x-y)}\varphi_2(y) & 0 \end{pmatrix}$$

to obtain, with $\Phi(x) := \int_0^x \varphi_1(y)\varphi_2(y)dy$,

$$E^{(2)}(x,\lambda) = \frac{i}{2\lambda}\Phi(x)\begin{pmatrix} e^{-i\lambda x} & 0 \\ 0 & -e^{i\lambda x}\end{pmatrix} + 0\left(\frac{e^{|Im\lambda|x}}{\lambda^2}\right)$$

where we used that

$$\begin{aligned}
&\int_0^x K(x,y)\frac{1}{2\lambda}\begin{pmatrix} 0 & e^{-i\lambda y}\varphi_1(0) \\ e^{i\lambda y}\varphi_2(0) & 0\end{pmatrix} dy \\
&= \frac{i}{2\lambda}\int_0^x diag\left(e^{i\lambda(2y-x)}\varphi_1(y)\varphi_1(0), -e^{i\lambda(x-2y)}\varphi_1(0)\varphi_2(y)\right) dy \\
&= \frac{1}{(2\lambda)^2} diag\left(e^{i\lambda(2y-x)}\varphi_1(y)\varphi_2(0), e^{i\lambda(x-2y)}\varphi_1(0)\varphi_2(y)\right)\Big|_{y=0}^{x} \\
&- \frac{1}{(2\lambda)^2}\int_0^x diag\left(e^{i\lambda(2y-x)}\varphi_1'(y)\varphi_2(0), e^{i\lambda(x-2y)}\varphi_1(0)\varphi_2'(y)\right) dy \\
&= 0\left(\frac{e^{|Im\lambda|x}}{\lambda^2}\right).
\end{aligned}$$

To get the asymptotics of $E^{(3)}(x,\lambda)$ write $E^{(3)}(x,\lambda) = \int_0^x K(x,y)E^{(2)}(y,\lambda)dy$. This leads to

$$\begin{aligned}
E^{(3)}(x,\lambda) = &-\frac{1}{2\lambda}\int_0^x \begin{pmatrix} 0 & -e^{i\lambda(2y-x)}\varphi_1(y) \\ -e^{i\lambda(x-2y)}\varphi_2(y) & 0\end{pmatrix}\Phi(y)dy \\
&+ 0\left(\frac{e^{|Im\lambda|x}}{\lambda^2}\right).
\end{aligned}$$

Integrating by parts once more, one obtains

$$E^{(3)}(x,\lambda) = 0\left(\frac{e^{|Im\lambda|x}}{\lambda^2}\right).$$

Arguing as in the proof of Lemma I.2 we then conclude that

$$\sum_{k\geq 3} E^{(k)}(x,\lambda) = 0\left(\frac{e^{|Im\lambda|x}}{\lambda^2}\right).$$

Thus we get

$$\begin{aligned}
M(x,\lambda) &= E(x,\lambda) + E^{(1)}(x,\lambda) \\
&+ \frac{i}{2\lambda}\Phi(x)\begin{pmatrix} e^{-i\lambda x} & 0 \\ 0 & -e^{i\lambda x}\end{pmatrix} + 0\left(\frac{e^{|Im\lambda|x}}{|\lambda|^2}\right).
\end{aligned}$$

Hence, in view of (I.52)

$$\Delta(\lambda,\varphi) = 2\cos\lambda + \frac{\sin\lambda}{\lambda}H_1(\varphi) + 0\left(\frac{e^{|Im\lambda|}}{|\lambda|^2}\right).$$

■

Denote by ch^{-1} the branch of $arccosh$ defined on $\mathbb{C}\backslash(-\infty,1)$ which is given for $z\in\mathbb{C}\backslash(-\infty,1)$ with $|z|>1$ by

$$ch^{-1}(z) = \log(z + z\sqrt[+]{1-\frac{1}{z^2}})$$

with log denoting the principal branch of the logarithm.
Notice that for y large and positive, $\Delta(iy)$ is closed to chy and hence $\frac{\Delta(iy)}{2}$ is in the domain of definition of $ch^{-1}(z)$.

Lemma I.24 *Uniformly for φ in a bounded subset of $H^2_{\mathbb{C}}$, one has for $\lambda = iy$ with $y\to\infty$*

$$ch^{-1}\left(\frac{\Delta(\lambda,\varphi)}{2}\right) = -i\lambda + \frac{iH_1(\varphi)}{2\lambda} + 0\left(\frac{1}{\lambda^2}\right).$$

Proof To make notation easier, introduce $g(z) := ch^{-1}(z)$. By Lemma I.23

$$\frac{\Delta(iy)}{2} = chy + u(y)\ ; \quad u(y) = \frac{shy}{2y}H_1(\varphi) + 0\left(\frac{e^y}{y^2}\right).$$

By the Taylor expansion of $g(z)$ at $z = ch(y)$,

$$g\left(\frac{\Delta(iy)}{2}\right) = g(chy) + u(y)g'(chy) + \frac{u(y)^2}{2}g''(ch(y)+\theta u(y))$$

for some $0<\theta<1$. As $chy + \theta u(y) \sim \frac{e^y}{2}$, one has

$$g''(chy+\theta u(y)) = 0(e^{-2y})$$

and hence

$$ch^{-1}\left(\frac{\Delta(iy)}{2}\right) = y + \frac{H_1}{2y} + 0\left(\frac{1}{y^2}\right).$$

■

Assuming more regularity on φ one can obtain the subsequent terms in the expansion of $\Delta(\lambda)$ and $ch^{-1}\left(\frac{\Delta(iy)}{2}\right)$. In the following result we give the next two terms in the expansion for $ch^{-1}\left(\frac{\Delta(iy)}{2}\right)$. For this purpose introduce for $\varphi\in H^2_{\mathbb{C}}$

$$\begin{aligned}
H_2(\varphi) &:= i\int_0^1 \varphi_1'\varphi_2 dx = \frac{i}{2}\int_0^1 (\varphi_1'\varphi_2 - \varphi_1\varphi_2' \\
H_3(\varphi) &:= \int_0^1 \left(\varphi_1'\varphi_2' + (\varphi_1\varphi_2)^2\right)dx.
\end{aligned}$$

Notice that H_1, H_2, H_3 are real for φ of real type and that H_3 is the Hamiltonian of the defocusing NLS-equation.

Lemma I.25 *Uniformly for φ in a bounded subset of $H^4_{\mathbb{C}}$, one has for $\lambda = iy$ with $y \to \infty$*

$$ch^{-1}\left(\frac{\Delta(\lambda,\varphi)}{2}\right) = -i\lambda + \frac{iH_1}{2\lambda} + \frac{iH_2}{4\lambda^2} + \frac{iH_3}{8\lambda^3} + 0\left(\frac{1}{\lambda^4}\right).$$

Proof Similar to the one of Lemma I.24, using a refined version of Lemma I.23 (see also [MV]). ∎

I.9 Gradients of the discriminant

Given a complex valued function $F : L^2_{\mathbb{C}} \to \mathbb{C}$ of class C^1 we denote its differential by $d_\varphi F$ and its L^2-gradient by $\nabla_{\varphi(x)} F = \left(\frac{\partial F}{\partial \varphi_1(x)}, \frac{\partial F}{\partial \varphi_2(x)}\right)$. For any $v = (v_1, v_2) \in L^2_{\mathbb{C}}$ one then has

$$d_\varphi F \cdot v = \left(\nabla_{\varphi(x)} F, v\right) = \int_0^1 \left(\frac{\partial F}{\partial \varphi_1(x)} v_1(x) + \frac{\partial F}{\partial \varphi_2(x)} v_2(x)\right) dx$$

where $(\cdot,\cdot)$ denotes the dual pairing between the dual of $L^2_{\mathbb{C}}$ and itself.
To obtain formulas for the gradients of the fundamental solutions $M^{(1)} = (M_{11}, M_{21})$ and $M^{(2)} = (M_{12}, M_{22})$ we need to establish an auxilary result. Let $G \in L^2_{\mathbb{C}}$ and $(a_0, b_0) \in \mathbb{C}^2$ and consider the following initial value problem

$$LF(x) = \lambda F(x) + G(x) \tag{I.53}$$
$$F(0) = (a_0, b_0). \tag{I.54}$$

Lemma I.26 *The initial value problem* (I.53) - (I.54) *admits a unique solution given by*

$$F(x) = \left(a_0 - i\int_0^x (G_1(t)M_{22}(t) + G_2(t)M_{12}(t))\,dt\right) M^{(1)}(x)$$
$$+\left(b_0 + i\int_0^x (G_2(t)M_{11}(t) + G_1(t)M_{21}(t))\,dt\right) M^{(2)}(x).$$

Proof By the method of the variation of constants, $F(x)$ is of the form

$$F(x) = a(x)M^{(1)}(x) + b(x)M^{(2)}(x)$$

with $(a(0), b(0)) = (a_0, b_0)$. Substituted into (I.53) one gets

$$\lambda F + i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\left(a'M^{(1)} + b'M^{(2)}\right) = \lambda F + G$$

which leads to

$$a'M^{(1)} + b'M^{(2)} = i\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} G.$$

In view of the Wronskian identity, $\det(M^{(1)}(x), M^{(2)}(x)) = 1$ one obtains by Cramer's rule

$$a'(x) = i\ \det\left(\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} G(x), M^{(2)}(x)\right)$$
$$= -i\left(G_1(x)M_{22}(x) + G_2(x)M_{12}(x)\right)$$

and, similarly

$$b'(x) = i\ \det\left(M^{(1)}(x), \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} G(x)\right)$$
$$= i\left(M_{11}(x)G_2(x) + M_{21}(x)G_1(x)\right).$$

The claimed formulas are then obtained by integrating $a'(x)$ and $b'(x)$. ∎

Lemma I.27 *For any $\varphi \in L^2_{\mathbb{C}}$, $0 \le x \le 1$, and $\lambda \in \mathbb{C}$*

$$\frac{\partial M^{(1)}(x)}{\partial \varphi_1(t)} = \left(iM_{21}(t)M_{22}(t)M^{(1)}(x) - iM_{21}(t)^2 M^{(2)}(x)\right) 1_{[0,x]}(t)$$
$$\frac{\partial M^{(1)}(x)}{\partial \varphi_2(t)} = \left(iM_{11}(t)M_{12}(t)M^{(1)}(x) - iM_{11}(t)^2 M^{(2)}(x)\right) 1_{[0,x]}(t)$$
$$\frac{\partial M^{(2)}(x)}{\partial \varphi_1(t)} = \left(iM_{22}(t)^2 M^{(1)}(x) - iM_{22}(t)M_{21}(t)M^{(2)}(x)\right) 1_{[0,x]}(t)$$
$$\frac{\partial M^{(2)}(x)}{\partial \varphi_2(t)} = \left(iM_{12}(t)^2 M^{(1)}(x) - iM_{12}(t)M_{11}(t)M^{(2)}(x)\right) 1_{[0,x]}(t)$$

Proof As each term in the above formulas depends continuously on φ it suffices to establish the formulas for continuous potentials φ. Then $M(x)$ is continuously differentiable with respect to x and we may interchange x-differentiation with φ-differentiation. Hence, for $1 \le j \le 2$ and any $v \in L^2_{\mathbb{C}}$,

$$d_{\varphi_j}(LM(x)) \cdot v = (d_{\varphi_j} L \cdot v)\ M(x) + L\ d_{\varphi_j} M(x) \cdot v$$

and it follows that $d_{\varphi_j} M \cdot v$ satisfies the following inhomogeneous equation

$$L\ d_{\varphi_j} M \cdot v = \lambda d_{\varphi_j} M \cdot v - (d_{\varphi_j} L \cdot v)\ M$$

with initial conditions

$$d_{\varphi_j} M(0) \cdot v = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

Note that

$$-d_{\varphi_1} L \cdot v = \begin{pmatrix}0 & -v\\ 0 & 0\end{pmatrix};\ -d_{\varphi_2} L \cdot v = \begin{pmatrix}0 & 0\\ -v & 0\end{pmatrix}$$

and thus, by Lemma I.26, one obtains

$$\frac{\partial M^{(1)}(x)}{\partial \varphi_1(t)} = \Big(iM_{21}(t)M_{22}(t)M^{(1)}(x) - iM_{21}(t)^2 M^{(2)}(x)\Big)\, 1_{[0,x]}(t)$$
$$\frac{\partial M^{(1)}(x)}{\partial \varphi_2(t)} = \Big(iM_{11}(t)M_{12}(t)M^{(1)}(x) - iM_{11}(t)^2 M^{(2)}(x)\Big)\, 1_{[0,x]}(t).$$

Similarly, one computes $\frac{\partial M^{(2)}(x)}{\partial \varphi_j(t)}$. ■

As an application we obtain

Proposition I.28 *For any $\varphi \in L^2_{\mathbb{C}}$ and $\lambda \in \mathbb{C}$*

$$\nabla_{\varphi(x)}\Delta(\lambda,\varphi) = i\,(M_{11} - M_{22})\,\Big|_{1,\lambda} \begin{pmatrix} M_{21}(x,\lambda)M_{22}(x,\lambda) \\ M_{11}(x,\lambda)M_{12}(x,\lambda)\end{pmatrix}$$
$$- iM_{12}(1,\lambda)\begin{pmatrix} M_{21}(x,\lambda)^2 \\ M_{11}(x,\lambda)^2\end{pmatrix}$$
$$+ iM_{21}(1,\lambda)\begin{pmatrix} M_{22}(x,\lambda)^2 \\ M_{12}(x,\lambda)^2\end{pmatrix}$$

I.10 Gradients of eigenvalues

In this section we compute the gradients of the Dirichlet eigenvalues $\mu_k(\varphi)$ and simple periodic eigenvalues $\lambda_k^+(\varphi) \notin \{\mu_k(\varphi), \check{\mu}_k(\varphi)\}$ for $\varphi \in L^2_{\mathcal{R}}$. Recall that the Dirichlet eigenvalues are real analytic on $L^2_{\mathcal{R}}$ and that

$$G_k(x) = M^{(1)}(x,\mu_k) + M^{(2)}(x,\mu_k)$$

is an eigenfunction corresponding to $\mu_k \equiv \mu_k(\varphi)$.

Proposition I.29 *For any $k \in \mathbb{Z}$ and $\varphi \in L^2_{\mathcal{R}}$*

$$\nabla_{\varphi(x)}\mu_k = \frac{1}{\|G_k(\cdot)\|^2}\left((M_{21}+M_{22})^2, (M_{11}+M_{12})^2\right)\Big|_{x,\mu_k}$$

where $\|\cdot\|$ denotes the norm in $L^2([0,1],\mathbb{C}^2)$.

Proof Let $\varphi \in L^2_{\mathcal{R}}$. Arguing as in the proof of Lemma I.27 it suffices to consider potentials φ which are continuous. Thus we may interchange in the expression $d_{\varphi_1}(L(\varphi)G_k(t))$ differentiation with respect to t and φ_1. Hence, for $v \in L^2([0,1],\mathbb{C})$,

$$d_{\varphi_1}(L(\varphi)G_k(t)) \cdot v = (d_{\varphi_1}L(\varphi)\cdot v)G_k(t) + L(\varphi)\; d_{\varphi_1}G_k(t)\cdot v\ . \qquad \text{(I.55)}$$

On the other hand, as $LG_k = \mu_k G_k$,

$$d_{\varphi_1}(L(\varphi)G_k(t)) \cdot v = (d_{\varphi_1}\mu_k \cdot v)G_k(t) + \mu_k d_{\varphi_1}G_k(t)\cdot v\ . \qquad \text{(I.56)}$$

As $d_{\varphi_1}L(\varphi)\cdot v = \begin{pmatrix} 0 & v \\ 0 & 0\end{pmatrix}$, (I.55) - (I.56) leads to

$$(d_{\varphi_1}\mu_k \cdot v)\; G_k(t) = (L(\varphi) - \mu_k)d_{\varphi_1}G_k(t)\cdot v + \begin{pmatrix} 0 & v(t) \\ 0 & 0\end{pmatrix} G_k(t).$$

Then we take the L^2-inner product of both sides of this identity with $G_k(\cdot,\varphi)$. As $L_k(\varphi) - \mu_k$ is selfadjoint, the L^2-inner product of $(L(\varphi) - \mu_k)d_{\varphi_1}G_k(t)\cdot v$ with G_k vanishes and we obtain

$$\|G_k(\cdot)\|^2 d_{\varphi_1}\mu_k \cdot v = \int_0^1 \begin{pmatrix} 0 & v(t) \\ 0 & 0\end{pmatrix} G_k(t)\cdot \overline{G_k(t)}dt. \qquad \text{(I.57)}$$

As φ is of real type, we have, in view of (I.21),

$$\overline{G_k(t)} = G_k(t)^* \qquad \text{(I.58)}$$

where we recall that for a vector $a = (a_1, a_2)$, the vector a^* is given by $a^* = (a_2, a_1)$. Hence the identity (I.57) takes the form

$$d_{\varphi_1}\mu_k \cdot v = \frac{1}{\|G_k(\cdot)\|^2}\int_0^1 v(t)\,(M_{21}(t,\mu_k) + M_{22}(t,\mu_k))^2\, dt.$$

This means that

$$\frac{\partial \mu_k}{\partial \varphi_1(x)} = \frac{1}{\|G_k(\cdot)\|^2}\,(M_{21}(x,\mu_k) + M_{22}(x,\mu_k))^2$$

and $\frac{\partial \mu_k}{\partial \varphi_2(x)}$, evaluated at $\varphi = (\varphi_1, \overline{\varphi_1})$, is given by $\frac{\partial \mu_k}{\partial \varphi_2} = \frac{\partial \mu_k}{\partial \overline{\varphi_1}} = \overline{\frac{\partial \mu_k}{\partial \varphi_1}}$, i.e.

$$\frac{\partial \mu_k}{\partial \varphi_2(x)} = \frac{1}{\|G_k(\cdot)\|^2}\,(M_{12}(x,\mu_k) + M_{11}(x,\mu_k))^2\, .$$

■

Substituting the asymptotics of G_k (cf Proposition I.9) into the formula for $\nabla_{\varphi(x)}\mu_k$ (cf. Proposition I.29) one obtains

Corollary I.30 *Locally uniformly on $L^2_{\mathcal{R}}$,*

$$\nabla_{\varphi(x)}\mu_k = \frac{1}{2}\begin{pmatrix} e^{2ik\pi x} \\ e^{-2ik\pi x}\end{pmatrix} + \ell^2(k)$$

In order to compute $\nabla_{\varphi(x)}\lambda_k^{\pm}$ for a simple periodic eigenvalue $\lambda_k^{\pm} \neq \{\mu_k, \check{\mu}_k\}$ we first have to derive a formula for a L^2-normalized eigenfunction $F_k^{\pm}(x)$ corresponding to $\lambda_k^{\pm}$. In section I.3 we have introduced the entire functions $\delta(\lambda)$ and $\check{\delta}(\lambda)$. They both have an infinite product representation (cf section I.6)

$$\delta(\lambda) = 2i(\mu_0 - \lambda)\prod_{n\neq 0}\frac{\mu_n - \lambda}{n\pi}\ ,\ \check{\delta}(\lambda) = 2i(\check{\mu}_0 - \lambda)\prod_{n\neq 0}\frac{\check{\mu}_n - \lambda}{n\pi}.$$

Further, the derivative $\dot{\Delta}(\lambda)$ of $\Delta(\lambda)$ admits an infinite product representation as well (cf section I.6)

$$\dot{\Delta}(\lambda) = 2(\dot{\lambda}_0 - \lambda) \prod_{n \neq 0} \frac{\dot{\lambda}_n - \lambda}{n\pi}.$$

Hence it follows that for a simple eigenvalue $\lambda_k^\pm$ with $\lambda_k^\pm \notin \{\mu_k, \check{\mu}_k\}$ of a potential $\varphi \in L^2_{\mathcal{R}}$,

$$\frac{-i\delta(\lambda_k^\pm)}{\dot{\Delta}(\lambda_k^\pm)} = 2\frac{\mu_0 - \lambda_k^\pm}{\dot{\lambda}_0 - \lambda_k^\pm} \prod_{n \neq 0} \frac{\mu_n - \lambda_k^\pm}{\dot{\lambda}_n - \lambda_k^\pm} > 0 \tag{I.59}$$

and similarly,

$$-i\check{\delta}(\lambda_k^\pm)/\dot{\Delta}(\lambda_k^\pm) > 0. \tag{I.60}$$

Finally we note that for φ of real type, $\overline{M}_{12} = M_{21}$. Hence $M_{12} + M_{21}$ is real valued. As

$$(M_{12} + M_{21})^2 \Big|_{1,\lambda_k^\pm} = -\check{\delta}(\lambda_k^\pm)\delta(\lambda_k^\pm) \neq 0,$$

the following expression is well defined

$$\varepsilon_k^\pm := sign\left((M_{12} + M_{21}) \Big|_{1,\lambda_k^\pm} / \dot{\Delta}(\lambda_k^\pm) \right). \tag{I.61}$$

In view of (I.59) - (I.61) we can introduce for any $\varphi \in L^2_{\mathcal{R}}$ and any $k \in \mathbb{Z}$ with $\lambda_k^\pm \notin \{\mu_k, \check{\mu}_k\}$ the following functions

$$F_k^\pm(x) := \frac{\varepsilon_k^\pm}{2} \sqrt[+]{\frac{-i\check{\delta}(\lambda_k^\pm)}{\dot{\Delta}(\lambda_k^\pm)}} G(x, \lambda_k^\pm) + \frac{i}{2} \sqrt[+]{\frac{-i\delta(\lambda_k^\pm)}{\dot{\Delta}(\lambda_k^\pm)}} \check{G}(x, \lambda_k^\pm) \tag{I.62}$$

where we recall that

$$G(x,\lambda) = (M^{(1)} + M^{(2)}) \Big|_{x,\lambda}; \quad \check{G}(x,\lambda) = (M^{(1)} - M^{(2)}) \Big|_{x,\lambda}.$$

As φ is of real type, we have $\overline{M}_{11} = M_{22}$ and $\overline{M}_{21} = M_{21}$, hence $\overline{G}_1 = G_2$, $\overline{\check{G}}_1 = -\check{G}_2$ and thus

$$F_{k,2}^\pm(x) = \overline{F_{k,1}^\pm(x)}. \tag{I.63}$$

Using Proposition I.28 one verifies the following

Lemma I.31 *For $\varphi \in L^2_{\mathcal{R}}$ and $k \in \mathbb{Z}$ with $\lambda_k^\pm \notin \{\mu_k, \check{\mu}_k\}$,*

$$\nabla_{\varphi(x)} \Delta(\lambda) \Big|_{\lambda = \lambda_k^\pm} = -\dot{\Delta}(\lambda_k^\pm)(F_{k,2}^\pm(x)^2, F_{k,1}^\pm(x)^2).$$

Proposition I.32 *For $\varphi \in L^2_{\mathcal{R}}$ and $k \in \mathbb{Z}$ with $\lambda_k^\pm \notin \{\mu_k, \check{\mu}_k\}$, $F_k^\pm(x)$ is an eigenfunction corresponding to $\lambda_k^\pm$ satisfying $\int_0^1 \|F_k^\pm(x)\|^2 dx = 1$ and*

$$\nabla_{\varphi(x)} \lambda_k^\pm = (F_{k,2}^\pm(x)^2, F_{k,1}^\pm(x)^2).$$

Proof The condition $\lambda_k^\pm \notin \{\mu_k, \check{\mu}_k\}$ is valid in a sufficiently small neighborhood of φ in $L^2([0,1]; \mathbb{C}^2)$. As $\Delta(\lambda_k^\pm) = 2(-1)^k$ and $\lambda_k^\pm$ is analytic near φ one obtains by implicit differentiation that

$$\nabla_{\varphi(x)} \lambda_k^\pm = -\frac{1}{\dot{\Delta}(\lambda_k^\pm)} \nabla_{\varphi(x)} \Delta(\lambda) \Big|_{\lambda_k^\pm}.$$

By Lemma I.31 we then conclude that

$$\nabla_{\varphi(x)} \lambda_k^\pm = (F_{k,2}^\pm(x)^2, F_{k,1}^\pm(x)^2).$$

Arguing as in the proof of Proposition I.29 one sees that $F_k^\pm(x)$ is an eigenfunction corresponding to $\lambda_k^\pm$ satisfying $\int_0^1 \|F_k^\pm(x)\|^2 dx = 1$. ■

I.11 Poisson brackets

In this section we provide formulas for the Poisson brackets $\{\Delta(\mu), \Delta(\lambda)\}$ and $\{\Delta(\lambda), \mu_k\}\big|_{\lambda=\mu_n}$. Recall that the Poisson bracket of two functionals F, G defined on $L^2_{\mathbb{C}}$ is defined by

$$\{F, G\}(\varphi) = i \int_0^1 \left(\frac{\partial F}{\partial \varphi_1(x)} \frac{\partial G}{\partial \varphi_2(x)} - \frac{\partial F}{\partial \varphi_2(x)} \frac{\partial G}{\partial \varphi_1(x)} \right) dx.$$

To compute $\{\Delta(\mu), \Delta(\lambda)\}$ it is convenient to rewrite the gradient $\nabla_{\varphi(x)} \Delta(\lambda)$, (cf Proposition I.28) in a different form. Denote by $m_\pm(\lambda)$ the Floquet multipliers, i.e. the two eigenvalues of the monodromy matrix $M(1, \lambda)$,

$$m_\pm(\lambda) := \frac{\Delta(\lambda)}{2} \pm \frac{1}{2} \sqrt[c]{\Delta(\lambda)^2 - 4}. \tag{I.64}$$

Notice that $m_+(\lambda) + m_-(\lambda) = \Delta(\lambda)$ and, by the Wronskian identity ,

$$m_+(\lambda) m_-(\lambda) = 1. \tag{I.65}$$

For λ with $M_{12}(1, \lambda) \neq 0$ the eigenvectors corresponding to $m_\pm(\lambda)$ are given by $(1, b_\pm(\lambda))$ where

$$b_\pm(\lambda) := \frac{m_\pm(\lambda) - M_{11}(1, \lambda)}{M_{12}(1, \lambda)}.$$

Further denote by $F^{\pm}(x,\lambda)$ the Baker-Akhiezer functions

$$F^{\pm}(x,\lambda) := M_1(x,\lambda) + b_{\pm}(\lambda)M_2(x,\lambda).$$

Note that $LF^{\pm} = \lambda F^{\pm}$ and

$$F^{\pm}(x+1,\lambda) = m_{\pm}(\lambda)F^{\pm}(x,\lambda).$$

Writing $F^{\pm} = (F_1^{\pm}, F_2^{\pm})$ we get the following formula for the gradient of $\Delta(\lambda)$:

Lemma I.33 *For $\lambda \in \mathbb{C}$ with $M_{12}(1,\lambda) \neq 0$*

$$\nabla_{\varphi(x)}\Delta(\lambda) = -iM_{12}(1,\lambda)\begin{pmatrix} F_2^+(x,\lambda)F_2^-(x,\lambda) \\ F_1^+(x,\lambda)F_1^-(x,\lambda) \end{pmatrix}$$

Proof Each component of the right side of the latter identity is treated separately. For the second component, note that

$$\begin{aligned} F_1^+(x)F_1^-(x) &= (M_{11}(x) + b_+M_{12}(x))\,(M_{11}(x) + b_-M_{12}(x)) \\ &= M_{11}(x)^2 + b_+b_-M_{12}(x)^2 + (b_+ + b_-)M_{11}(x)M_{12}(x). \end{aligned} \tag{I.66}$$

As $m_+ + m_- = \Delta(\lambda)$,

$$\begin{aligned} M_{12}(1)(b_+ + b_-) &= m_+ + m_- - 2M_{11}(1) \\ &= M_{22}(1) - M_{11}(1) \end{aligned}$$

and as $m_+m_- = 1$ (cf (I.65)),

$$\begin{aligned} M_{12}(1)^2 b_+b_- &= (m_+ - M_{11}(1))(m_- - M_{11}(1)) \\ &= m_+m_- - (m_+ + m_-)M_{11}(1) + M_{11}(1)^2 \\ &= 1 - \Delta M_{11}(1) + M_{11}(1)^2 \\ &= 1 - M_{11}(1)M_{22}(1) \\ &= -M_{12}(1)M_{21}(1) \end{aligned}$$

where for the last identity we used the Wronskian identity once more. Hence

$$M_{12}(1)b_+b_- = -M_{21}(1).$$

Substituting the identities above into (I.66) one obtains

$$\begin{aligned} -iM_{12}(1)F_1^+(x)F_1^-(x) = &- iM_{12}(1)M_{11}(x)^2 + iM_{21}(1)M_{12}(x)^2 \\ &+ i(M_{11}(1) - M_{22}(1))M_{11}(x)M_{12}(x). \end{aligned}$$

Hence, by Proposition I.28,

$$-iM_{12}(1)F_1^+(x)F_1^-(x) = \frac{\partial\Delta}{\partial\varphi_2(x)}.$$

Similarly, one proves

$$-iM_{12}(1)F_2^+(x)F_2^-(x) = \frac{\partial\Delta}{\partial\varphi_1(x)}.$$

■

We need two additional auxilary results for the computation of the Poisson bracket $\{\Delta(\mu), \Delta(\lambda)\}$. Assume that $F(x) = (F_1(x), F_2(x))$ and $G(x) = (G_1(x), G_2(x))$ are in $L^2([0,1];\mathbb{C}^2)$ satisfying

$$L(\varphi)F = \lambda F;\; L(\varphi)G = \mu G$$

for given $\lambda, \mu \in \mathbb{C}$ and $\varphi \in L^2(S^1;\mathbb{C}^2)$. Recall that $W(F(x), G(x))$ denotes the Wronskian of F and G (cf (I.8)).

Lemma I.34 *For any $0 \leq x \leq 1$,*

$$\frac{d}{dx}W(F(x), G(x)) = i(\mu - \lambda)(F_1(x)G_2(x) + F_2(x)G_1(x)).$$

Proof Note that $\frac{d}{dx}W(F,G) = W(F',G) + W(F,G')$ and, as $LF = \lambda F$,

$$\begin{aligned} F' &= i\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} LF - \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix} F \\ &= i\lambda\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} F - \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix} F. \end{aligned}$$

Similarly, one has

$$G' = i\mu\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} G - \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix} G.$$

Substituting these formulas for F' and G' into the expression for $\frac{d}{dx}W(F,G)$ one gets

$$\begin{aligned} \frac{d}{dx}W(F,G) &= i\lambda W\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} F, G\right) + i\mu W\left(F, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} G\right) \\ &- W\left(\begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix} F, G\right) - W\left(F, \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix} G\right) \\ &= i(\mu - \lambda)(F_1G_2 + F_2G_1). \end{aligned}$$

■

Given $\lambda \in \mathbb{C}$, introduce the following subset of $L^2_{\mathbb{C}}$,

$$\mathcal{N}_\lambda := \{\varphi \in L^2_{\mathbb{C}} \mid M_{12}(1,\lambda;\varphi) = 0\}.$$

Lemma I.35 *For any $\lambda \in \mathbb{C}$, $\mathcal{N}_\lambda$ is an analytic submanifold of $L^2_{\mathbb{C}}$ of complex codimension* 1.

Proof Recall that $M_{12}(1,\lambda;\varphi)$ is analytic in φ. It remains to prove that its gradient does not vanish on $\mathcal{N}_\lambda$. By Lemma I.27,

$$\frac{\partial M_{12}(1,\lambda)}{\partial \varphi_1(t)} = iM_{22}(t,\lambda)\left(M_{22}(t,\lambda)M_{11}(1,\lambda) - M_{21}(t,\lambda)M_{12}(1,\lambda)\right).$$

which is continuous in t. For $\varphi \in \mathcal{N}_\lambda$, the Wronskian identity becomes $M_{11}(1,\lambda)M_{22}(1,\lambda) = 1$. In particular, $M_{22}(1,\lambda) \neq 0$ and, evaluating $\frac{\partial M_{12}(1,\lambda)}{\partial \varphi_1(t)}$ at $t = 1$,

$$\frac{\partial M_{12}(1,\lambda)}{\partial \varphi_1(1)} = iM_{22}(1,\lambda) \neq 0.$$

Hence $\frac{\partial M_{12}(1,\lambda)}{\partial \varphi(t)} = \left(\frac{\partial M_{12}}{\partial \varphi_1(t)}, \frac{\partial M_{12}}{\partial \varphi_2(t)}\right)\Big|_{1,\lambda}$ does never vanish on $\mathcal{N}_\lambda$. ■

Proposition I.36 *For any $\lambda, \mu \in \mathbb{C}$ and $\varphi \in L^2_{\mathbb{C}}$*

$$\{\Delta(\lambda,\varphi), \Delta(\mu,\varphi)\} = 0.$$

Proof By Lemma I.35, $\mathcal{N}_\lambda$ and $\mathcal{N}_\mu$ are submanifolds in $L^2_{\mathbb{C}}$ of codimension 1, hence $\mathcal{M} := \{(\varphi,\lambda,\mu) \mid \lambda,\mu \in \mathbb{C};\ \varphi \notin \mathcal{N}_\lambda \cup \mathcal{N}_\mu\}$ is dense in $L^2_{\mathbb{C}} \times \mathbb{C} \times \mathbb{C}$. As $\{\Delta(\lambda,\varphi), \Delta(\mu,\varphi)\}$ is continuous on $L^2_{\mathbb{C}} \times \mathbb{C} \times \mathbb{C}$, it suffices to prove that $\{\Delta(\lambda,\varphi), \Delta(\mu,\varphi)\} = 0$ on $\mathcal{M}$. Further the result clearly holds for $\lambda = \mu$. Thus let us consider $(\varphi,\lambda,\mu) \in \mathcal{M}$ with $\lambda \neq \mu$. In view of Lemma I.33,

$$\begin{aligned}\{\Delta(\lambda),\Delta(\mu)\} &= i\int_0^1 \left(\frac{\partial\Delta(\lambda)}{\partial\varphi_1(x)}\frac{\partial\Delta(\mu)}{\partial\varphi_2(x)} - \frac{\partial\Delta(\lambda)}{\partial\varphi_2(x)}\frac{\partial\Delta(\mu)}{\partial\varphi_1(x)}\right)dx \\ &= -iM_{12}(1,\lambda)M_{12}(1,\mu)\int_0^1 \big(F_2^+(x,\lambda)F_2^-(x,\lambda)F_1^+(x,\mu)F_1^-(x,\mu) \\ &\quad - F_1^+(x,\lambda)F_1^-(x,\lambda)F_2^+(x,\mu)F_2^-(x,\mu)\big)dx.\end{aligned}$$

Denote the latter integrand by $f(x,\lambda,\mu)$. Then

$$f(x,\lambda,\mu) = (ac - bd)$$

with

$$\begin{aligned} a &:= F_2^+(x,\lambda)F_1^-(x,\mu), \quad b := F_1^+(x,\lambda)F_2^-(x,\mu), \\ c &:= F_2^-(x,\lambda)F_1^+(x,\mu), \quad d := F_1^-(x,\lambda)F_2^+(x,\mu)\ . \end{aligned}$$

Using the definition of the Wronskian, $W\left((a_1,a_2),(b_1,b_2)\right) = a_1b_2 - a_2b_1$ and the identity

$$2(ac - bd) = (a+b)(c-d) + (a-b)(c+d)$$

one gets

$$\begin{aligned} 2f(x,\lambda,\mu) = &(a+b)W\left(F^+(x,\mu),F^-(x,\lambda)\right) \\ &+ (c+d)W\left(F^-(x,\mu),F^+(x,\lambda)\right). \end{aligned} \tag{I.67}$$

By Lemma I.34, we have

$$\frac{-i}{\lambda-\mu}\,\frac{d}{dx}W\left(F^+(x,\mu),F^-(x,\lambda)\right) = c + d$$

and

$$\frac{-i}{\lambda-\mu}\,\frac{d}{dx}W\left(F^-(x,\mu),F^+(x,\lambda)\right) = a + b.$$

Thus substituting the last two identities into (I.67) one obtains

$$2f(x,\lambda,\mu) = -\frac{1}{(\lambda-\mu)}\frac{d}{dx}\left(W(F^-(x,\mu),F^+(x,\lambda))\ W(F^+(x,\mu),F^-(x,\lambda))\right).$$

Hence

$$\begin{aligned}\{\Delta(\lambda),\Delta(\mu)\} &= -iM_{12}(1,\lambda)M_{12}(1,\mu)\int_0^1 f(x,\lambda,\mu)dx \\ &= \frac{1}{2}\frac{1}{\mu-\lambda}M_{12}(1,\lambda)M_{12}(1,\mu)\cdot \\ &\quad W(F^+(x,\lambda),F^-(x,\mu))W(F^-(x,\lambda),F^+(x,\mu))\Big|_0^1.\end{aligned}$$

As $F^\pm(1,\lambda) = m_\pm(\lambda)F^\pm(0,\lambda)$ and $m_+(\lambda)m_-(\lambda) = 1$ (cf (I.65)) we conclude that $\{\Delta(\lambda),\Delta(\mu)\} = 0$. ■

To state our next result notice that

$$\begin{aligned}(M_{21}+M_{12})^2\Big|_{1,\mu_k} &= \left((M_{21}-M_{12})^2 + 4M_{21}M_{21}\right)\Big|_{1,\mu_k} \\ &= \left((M_{11}-M_{22})^2 + 4(M_{11}M_{22}-1)\right)\Big|_{1,\mu_k} \\ &= (M_{11}+M_{22})^2\Big|_{1,\mu_k} - 4 \\ &= \Delta(\mu_k)^2 - 4.\end{aligned}$$

Denote by $\sqrt[*]{\Delta(\mu_k)^2-4}$ the square root with the sign determined such that

$$\sqrt[*]{\Delta(\mu_k)^2-4} = (M_{21}+M_{12})\Big|_{1,\mu_k}. \tag{I.68}$$

Using the formulas for the gradients $\nabla_{\varphi(x)}\Delta(\lambda)$ and $\nabla_{\varphi(x)}\mu_k$ obtained in Lemma I.33 and Proposition I.29 one then obtains the following

Proposition I.37 *Let $\varphi \in L^2_{\mathcal{R}}$. For any $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$ one has*

$$(\mu_k - \lambda)\{\Delta(\lambda), \mu_k\} = \frac{1}{2}\ \frac{\delta(\lambda)}{\dot\delta(\mu_k)}\ (M_{12} + M_{21})\Big|_{1,\mu_k}. \tag{I.69}$$

In particular for any k and $n \in \mathbb{Z}$ one has

$$\{\Delta(\lambda), \mu_k\}\Big|_{\lambda=\mu_n} = -\frac{1}{2}\sqrt[*]{\Delta(\mu_k)^2 - 4}\ \delta_{nk}. \tag{I.70}$$

Proof Arguing as in Proposition I.36 it suffices to prove (I.69) for $\varphi \in L^2_{\mathcal{R}}\backslash\mathcal{N}_\lambda$. For such a φ, we have by Lemma I.33 and Proposition I.29

$$\{\Delta(\lambda), \mu_k\} = i\int_0^1 \left(\frac{\partial\Delta(\lambda)}{\partial\varphi_1(x)}\frac{\partial\mu_k}{\partial\varphi_2(x)} - \frac{\partial\Delta(\lambda)}{\partial\varphi_2(x)}\frac{\partial\mu_k}{\partial\varphi_1(x)}\right)dx$$
$$= \frac{M_{12}(1,\lambda)}{\|G_k(\cdot)\|^2}\int_0^1 \left(F_2^+(x,\lambda)F_2^-(x,\lambda)G_{k,1}(x)^2 - F_1^+(x,\lambda)F_1^-(x,\lambda)G_{k,2}(x)^2\right)dx$$

where $G_k(x) = (G_{k,1}(x), G_{k,2}(x))$ is given by $M_1(x,\mu_k) + M_2(x,\mu_k)$. We argue as in the proof of Proposition I.36. Denote by $f(x,\lambda)$ the latter integrand. Then

$$f(x,\lambda) = ac - bd$$

with

$$a := F_2^+(x,\lambda)G_{k,1}(x), \quad b := F_1^+(1,\lambda)G_{k,2}(x)\ ,$$
$$c := F_2^-(x,\lambda)G_{k,1}(x), \quad d := F_1^-(x,\lambda)G_{k,2}(x)\ .$$

Using the definition of the Wronskian and the identity

$$2(ac - bd) = (a+b)(c-d) + (a-b)(c+d)$$

one gets

$$\begin{aligned} 2f(x,\lambda) &= (a+b)W\left(G_k(x), F^-(x,\lambda)\right) \\ &\quad + (c+d)W\left(G_k(x), F^+(x,\lambda)\right). \end{aligned} \tag{I.71}$$

By Lemma I.34, we have, for $\lambda \neq \mu_k$,

$$\frac{d}{dx}\left(W(G_k(x), F^-(x,\lambda))\right) = i(\lambda - \mu_k)(c+d)$$

and

$$\frac{d}{dx}\left(W(G_k(x), F^+(x,\lambda))\right) = i(\lambda - \mu_k)(a+b)\ .$$

Thus substituting the last two identities into (I.71)

$$2i(\lambda - \mu_k)f(x,\lambda) = \frac{d}{dx}\left(W(G_k(x), F^-(x,\lambda))W(G_k(x), F^+(x,\lambda))\right).$$

Hence

$$\begin{aligned} (\mu_k - \lambda)\{\Delta(\lambda), \mu_k\} &= \frac{M_{12}(1,\lambda)}{\|G_k(\cdot)\|^2}\int_0^1 (\mu_k - \lambda)f(x,\lambda)dx \\ &= \frac{M_{12}(1,\lambda)}{\|G_k(\cdot)\|^2}\ \frac{i}{2}\ (J(1) - J(0)) \end{aligned} \tag{I.72}$$

where

$$J(x) := W\left(G_k(x), F^-(x,\lambda)\right) W\left(G_k(x), F^+(x,\lambda)\right).$$

Note that $G_k(0) = (1,1)$ and, as G_k satisfies Dirichlet boundary conditions, $G_{k,1}(1) = G_{k,2}(1)$, i.e.

$$G_k(1) = (M_{11}(1,\mu_k) + M_{12}(1,\mu_k))\cdot(1,1).$$

Further

$$F^\pm(0,\lambda) = (1, b_\pm(\lambda))\ ;\ F^\pm(1,\lambda) = m_\pm(\lambda)F^\pm(0,\lambda).$$

Hence

$$J(0) = (b_-(\lambda) - 1)(b_+(\lambda) - 1)$$
$$J(1) = (M_{11} + M_{12})^2\Big|_{1,\mu_k} m_-(\lambda)m_+(\lambda)(b_-(\lambda) - 1)(b_+(\lambda) - 1)$$

and, using that $m_+(\lambda)m_-(\lambda) = 1$ (cf (I.65)),

$$J(1) - J(0) = \left((M_{11} + M_{12})^2\Big|_{1,\mu_k} - 1\right)(b_-(\lambda) - 1)(b_+(\lambda) - 1). \tag{I.73}$$

Recall that $M_{12}(1,\lambda)b_\pm(\lambda) = m_\pm(\lambda) - M_{11}(1,\lambda)$ and

$$\delta(\lambda) = M_{11}(1,\lambda) + M_{12}(1,\lambda) - M_{21}(1,\lambda) - M_{22}(1,\lambda).$$

By a straight forward computation we obtain

$$M_{12}(1,\lambda)(b_-(\lambda) - 1)(b_+(\lambda) - 1) = \delta(\lambda). \tag{I.74}$$

Further, as $\delta(\mu_k) = 0$, we have $(M_{11} + M_{12})\big|_{1,\mu_k} = (M_{21} + M_{22})\big|_{1,\mu_k}$ and thus

$$\begin{aligned} (M_{11} + M_{12})^2\Big|_{1,\mu_k} - 1 &= (M_{11} + M_{12})(M_{22} + M_{21})\Big|_{1,\mu_k} - 1 \\ &= ((M_{11}M_{22} - 1 + M_{12}M_{21}) + M_{11}M_{21} + M_{12}M_{22})\Big|_{1,\mu_k} \\ &= (2M_{21}M_{12} + M_{11}M_{21} + M_{12}M_{22})\Big|_{1,\mu_k} \\ &= (M_{21}(M_{12} + M_{11}) + M_{12}(M_{21} + M_{22}))\Big|_{1,\mu_k} \\ &= (M_{21} + M_{12})(M_{12} + M_{11})\Big|_{1,\mu_k}. \end{aligned} \tag{I.75}$$

Substitute (I.74) and (I.75) into (I.73) then leads to

$$(\mu_k - \lambda)\{\Delta(\lambda), \mu_k\} = \frac{i\delta(\lambda)}{2} \frac{1}{\|G_k(\cdot)\|^2} \left((M_{11} + M_{12})(M_{12} + M_{21})\right)\Big|_{1,\mu_k}.$$

As φ is assumed to be of real type, $\|G_k(\cdot)\|^2$ is given by (cf Lemma I.11)

$$\|G_k(\cdot)\|^2 = i\dot{\delta}(\mu_k)(M_{11} + M_{12})\Big|_{1,\mu_k}.$$

and one finally obtains

$$(\mu_k - \lambda)\{\Delta(\lambda), \mu_k\} = \frac{1}{2} \frac{\delta(\lambda)}{\dot{\delta}(\mu_k)} (M_{12} + M_{21})\Big|_{1,\mu_k}.$$

To prove (I.70) notice that, if $\lambda = \mu_k$ with $n \neq k$, $\delta(\mu_k) = 0$ and thus

$$\{\Delta(\lambda), \mu_k\}\Big|_{\lambda=\mu_n} = 0$$

and, if $\lambda = \mu_n$ with $n = k$,

$$\lim_{\lambda \to \mu_k} \frac{\delta(\lambda)}{\lambda - \mu_k} = \dot{\delta}(\mu_k)$$

and thus

$$\{\Delta(\lambda), \mu_k\}\Big|_{\lambda=\mu_k} = -\frac{1}{2}(M_{12} + M_{21})\Big|_{1,\mu_k}.$$

■

I.12 Isospectral flows

In this section we study auxilary isospectral flows of vector fields on the space of potentials of real type. Recall that $\mathrm{Iso}(\varphi_0)$ with $\varphi_0 \in L^2_{\mathcal{R}}$ denotes the isospectral set of φ_0,

$$\mathrm{Iso}(\varphi_0) := \{\varphi \in L^2_{\mathcal{R}} \mid \lambda_k^{\pm}(\varphi) = \lambda_k^{\pm}(\varphi_0) \quad \forall k \in \mathbb{Z}\}.$$

From the infinite product representation of $\Delta(\lambda)^2 - 4$, one concludes that, for any element $\varphi \in L^2_{\mathbb{C}}$, $\varphi \in \mathrm{Iso}(\varphi_0)$ iff

$$\Delta(\lambda, \varphi)^2 - 4 = \Delta(\lambda, \varphi_0)^2 - 4 \quad \forall \lambda \in \mathbb{C}.$$

By Proposition I.3, $\Delta(\lambda, \varphi) = 2cos\lambda + o(1)$ as $\lambda = Re\lambda \to \infty$, and hence

$$\mathrm{Iso}(\varphi_0) := \{\varphi \in L^2_{\mathcal{R}} \mid \Delta(\lambda, \varphi) = \Delta(\lambda, \varphi_0) \quad \forall \lambda \in \mathbb{C}\}.$$

The first flow we consider is the so called phase flow ($t \in \mathbb{R}, x \in \mathbb{R}$)

$$\chi(x,t) = (\chi_1(x,t), \chi_2(x,t)) = (e^{it}\varphi_1(x), e^{-it}\varphi_2(x))$$

with initial conditions $\chi(\cdot, 0) = \varphi(\cdot) \in L^2_{\mathbb{C}}$. This flow is Hamiltonian

$$\frac{d}{dt}\chi = \left(i\frac{\partial H_1(\chi)}{\partial \varphi_2}, -i\frac{\partial H_1(\chi)}{\partial \varphi_1}\right)$$

where H_1 is given by $H_1(\varphi) = \int_0^1 \varphi_1(x)\varphi_2(x)dx$. Note that for any given solution $F = (F_1, F_2)$ of $L(\varphi)F = \lambda F$, the function $G = (e^{it/2}F_1, e^{-it/2}F_2)$ and hence $(F_1, e^{-it}F_2)$ as well as $(e^{it}F_1, F_2)$ satisfy $L(\chi(\cdot,t))G = \lambda G$. Hence the fundamental matrix $M(x, \lambda, \chi(\cdot,t))$ is related to $M(x, \lambda, \varphi)$ as follows

$$M(x, \lambda, \chi(\cdot,t)) = \begin{pmatrix} M_{11} & e^{it}M_{12} \\ e^{-it}M_{21} & M_{22} \end{pmatrix}\Bigg|_{x,\lambda,\varphi}. \tag{I.76}$$

In particular its trace is invariant,

$$\Delta(\lambda, \chi(\cdot,t)) = \Delta(\lambda, \varphi) \quad \forall \lambda \in \mathbb{C}.$$

This means that the phase flow is isospectral. Further notice that for $\varphi \in L^2_{\mathcal{R}}$, $\chi(\cdot,t)$ stays in $L^2_{\mathcal{R}}$ for any $t \in \mathbb{R}$.
Finally we would like to know how the Dirichlet eigenvalues evolve under the phase flow. Recall that the Dirichlet eigenvalues $(\mu_k(\varphi))_{k\in\mathbb{Z}}$ are given by the zero set of

$$\delta(\lambda, \varphi) = (M_{11} + M_{12} - M_{21} - M_{22})\Big|_{1,\lambda,\varphi}.$$

Let us evaluate $\delta(\lambda, \varphi)$ at $(\mu^s, \chi^t) = (\mu_k(\chi(\cdot,s)), \chi(\cdot,t))$ for $k \in \mathbb{Z}, s, t \in \mathbb{R}$ arbitrary. Using (I.76) one then obtains

$$\begin{aligned} \delta\Big|_{\mu^s,\chi^t} &= (M_{11} - M_{22})\Big|_{1,\mu^s,\chi^s} + (e^{it}M_{12} - e^{-it}M_{21}\Big|_{1,\mu^s,\varphi} \\ &= \delta\Big|_{\mu^s,\chi^s} - (M_{12} - M_{21})\Big|_{1,\mu^s,\chi^s} + (e^{it}M_{12} - e^{-it}M_{21})_{1,\mu^s,\varphi} \\ &= (-e^{is}M_{12} + e^{-is}M_{21} + e^{it}M_{12} - e^{-it}M_{21})_{1,\mu^s,\varphi} \end{aligned}$$

where for the last identity we used that $\delta(\mu^s, \chi^s) = 0$. Hence we have

$$\delta\Big|_{\mu^s,\chi^t} = (e^{it} - e^{is})M_{12}\Big|_{1,\mu^s,\varphi} - (e^{-it} - e^{-is})M_{21}\Big|_{1,\mu^s,\varphi}. \tag{I.77}$$

Formula (I.77) can be used to get some information on the location of the Dirichlet spectrum of $\chi(\cdot,t)$. Recall that for $\varphi \in L^2_{\mathcal{R}}$, all the eigenvalues $\lambda_k^{\pm}, \mu_k$ are real and satisfy $\lambda_k^- \leq \mu_k \leq \lambda_k^+$ ($k \in \mathbb{Z}$).

Lemma I.38 *Let $\varphi \in L^2_{\mathcal{R}}$ and $k \in \mathbb{Z}$ with $\gamma_k(\varphi) \neq 0$. Then $\mu_k(t) \neq \lambda_k^{\pm}$ for all $-\pi \le t < \pi$ except possibly two.*

Proof Assume that $-\pi \le t_0 < \pi$ satisfies $\mu_k(\chi(\cdot, t_0)) = \lambda_k^+(\varphi)\ \big(\equiv \lambda_k^+(\chi(\cdot, t))\big)$. By (I.77) one has

$$\delta\Big|_{\lambda_k^+, \chi^t} = (e^{it} - e^{it_0}) M_{12}\Big|_{1, \lambda_k^+, \varphi} - (e^{-it} - e^{-it_0}) M_{21}\Big|_{1, \lambda_k^+, \varphi}.$$

Hence, for any $-\pi \le t < \pi$ with $t \neq t_0$ and $\mu_k(\chi(\cdot, t)) = \lambda_k^+$,

$$(e^{it} - e^{it_0}) M_{12}\Big|_{1, \lambda_k^+, \varphi} = (e^{-it} - e^{-it_0}) M_{21}\Big|_{1, \lambda_k^+, \varphi}$$

or, as $e^{it} - e^{it_0} = -e^{i(t+t_0)}(e^{-it} - e^{-it_0})$,

$$e^{i(t+t_0)} M_{12}\Big|_{1, \lambda_k^+, \varphi} = -M_{21}\Big|_{1, \lambda_k^+, \varphi}. \tag{I.78}$$

As $\gamma_k \neq 0$, the eigenvalue λ_k^+ is simple and thus the Floquet matrix $M(1, \lambda_k^+, \varphi)$ satisfies $M(1, \lambda_k^+, \varphi) \neq (-1)^k \mathrm{Id}$. As $\det M(1, \lambda_k^+, \varphi) = 1$ and $\Delta(\lambda_k^+, \varphi) = 2(-1)^k$ we conclude that

$$(M_{12}, M_{21})\Big|_{1, \lambda_k^+, \varphi} \neq (0, 0)$$

hence the identity (I.78) can hold for at most one value of t in $[-\pi, \pi)\backslash\{t_0\}$. The result for $\mu_k(t) = \lambda_k^-$ can be proved in the same fashion. ■

From Lemma I.38, it follows that for $\varphi \in L^2_{\mathcal{R}}$ the set

$$T(\varphi) := \big\{ t \in [-\pi, \pi) \mid \exists k \in \mathbb{Z} \text{ with } \mu_k(t) \in \{\lambda_k^+, \lambda_k^-\} \big\}$$

is *countable* and we obtain the following

Corollary I.39 *For any $\varphi \in L^2_{\mathcal{R}}$ there exists a sequence $(\varphi_n)_{n \ge 1}$ in $L^2_{\mathcal{R}}$ with the following properties:*

(i) $\varphi = \lim_{n\to\infty} \varphi_n$ *in* $L^2_{\mathbb{C}} \times L^2_{\mathbb{C}}$;

(ii) $\varphi_n \in Iso(\varphi)$ *and* $\|\varphi_n\| = \|\varphi\|\ \forall n \ge 1$;

(iii) for any $n \ge 1$ and any $k \in \mathbb{Z}$ with $\gamma_k(\varphi) \neq 0$,

$$\lambda_k^-(\varphi) = \lambda_k^-(\varphi_n) < \mu_k(\varphi_n) < \lambda_k^+(\varphi_n) = \lambda_k^+(\varphi).$$

Proof Choose a sequence $(t_n)_{n\ge1} \subseteq [-\pi, \pi)\backslash T(\varphi)$ so that $\lim_{n\to\infty} t_n = 0$. This is possible as $T(\varphi)$ is countable. Then $(\varphi_n := \chi(t_n, \varphi))_{n \ge 1}$ is a sequence with the claimed properties. ■

For any given k, we now define an isospectral flow on $L^2_{\mathcal{R}}$ which leaves all Dirichlet eigenvalues except the k'th invariant. For any $n \in \mathbb{Z}$, denote by $X_n(\varphi)$ the vector field on $L^2_{\mathbb{C}}$ given by

$$X_n(\varphi)(x) := \left(i \frac{\partial \Delta(\lambda, \varphi)}{\partial \varphi_2(x)}, -i \frac{\partial \Delta(\lambda, \varphi)}{\partial \varphi_1(x)} \right) \Big|_{\lambda = \mu_n(\varphi)}.$$

Notice that $X_n(\varphi)$ is analytic in φ and $X_n(\varphi) \in L^2_{\mathcal{R}}$ for $\varphi \in L^2_{\mathcal{R}}$ as $\Delta(\lambda, \varphi)$ and $\mu_n(\varphi)$ are real valued for $(\lambda, \varphi) \in \mathbb{R} \times L^2_{\mathcal{R}}$. Hence the differential equation

$$\frac{d}{dt} \eta^t = X_n(\eta^t)\ ; \quad \eta^t \Big|_{t=0} = \varphi \tag{I.79}$$

is locally (in time) integrable in $L^2_{\mathcal{R}}$ for any initial data $\varphi \in L^2_{\mathcal{R}}$. Denote by $I_{max}(\varphi)$ the maximal interval of existence of the solution $\eta(\cdot, t)$ of (I.79). To see that (I.79) is globally integrable, i.e. $I_{max}(\varphi) = \mathbb{R}$ we prove that $\|\eta^t\|_{L^2}$ remains bounded for $t \in I_{max}(\varphi)$. To this end, note that

$$\begin{aligned} \frac{d}{dt} \Delta(\lambda, \eta^t) &= \int_0^1 \nabla_{\varphi(x)} \Delta(\lambda, \eta^t) \cdot \frac{d}{dt} \eta^t(x) dx \\ &= \int_0^1 \nabla_{\varphi(x)} \Delta(\lambda, \eta^t) \cdot \nabla_{\varphi(x)} \Delta(\mu, \eta^t)\Big|_{\mu = \mu_n(\eta^t)} dx \\ &= \Big\{ \Delta(\lambda, \eta^t), \Delta(\mu, \eta^t) \Big\}\Big|_{\mu = \mu_n(\eta^t)}. \end{aligned}$$

By Proposition I.36, $\{\Delta(\lambda), \Delta(\mu)\} = 0$ and thus

$$\frac{d}{dt} \Delta(\lambda, \eta^t) = 0$$

which shows that the flow η^t is isospectral. From the asymptotic expansion of $\Delta(\lambda)$ given by Lemma I.23 it follows that for $\varphi \in H^2([0,1]; \mathbb{C}^2)$ the Hamiltonian $H_1(\varphi) = \int_0^1 \varphi_1(x) \varphi_2(x) dx$ is a spectral invariant. More precisely, for any $u, v \in H^2([0,1]; \mathbb{C}^2)$ with $u \in \mathrm{Iso}(v)$ one has $H_1(u) = H_1(v)$. Consider first the case where the initial data φ in (I.79) is in $H^2_{\mathcal{R}} := H^2([0,1]; \mathbb{C}^2) \cap L^2_{\mathcal{R}}$. Using the equation $L(\varphi) M(x, \lambda, \varphi) = \lambda M(x, \lambda, \varphi)$ one sees that the entries of the fundamental matrix solution $M(x, \lambda, \varphi)$ belong to $H^2([0,1]; \mathbb{C})$ (in fact, they belong to $H^3([0,1]; \mathbb{C})$. Hence, by Proposition I.28, the vectorfield $X_n(\varphi)$ is an element in $H^2_{\mathcal{R}}$ and the initial value problem (I.79) is locally integrable in $H^2_{\mathcal{R}}$. As the flow η^t is isospectral we then conclude that $\eta^t \in \mathrm{Iso}(\varphi)$ and therefore

$$\begin{aligned} \|\eta^t\|^2_{L^2} = \|\eta_1^t\|^2_{L^2} + \|\eta_2^t\|^2_{L^2} = 2 \int_0^1 \eta_1^t(x) \overline{\eta_1^t(x)} dx = 2 H_1(\eta^t) \\ = \|\varphi\|^2_{L^2}. \end{aligned}$$

It follows that $I_{max}(\varphi) = \mathbb{R}$ and $\eta^t \in H^2([0,1];\mathbb{C}^2)$ for any $t \in \mathbb{R}$. Now let us consider the general case $\varphi \in L^2_{\mathcal{R}}$. Approximate φ by a sequence $(\varphi_k)_{k\geq 1}$ in $H^2_{\mathcal{R}}$ so that $\lim_{k\to\infty} \|\varphi_k - \varphi\|_{L^2} = 0$. As the solution $\eta^t(\varphi_k)$ of (I.79) with initial data φ_k depends continuously on the initial data and $I_{max}(\eta^t_k) = \mathbb{R}$ for all k, we conclude that locally uniformly in $I_{max}(\varphi)$,

$$\lim_{k\to\infty} \|\eta^t(\varphi_k) - \eta^t(\varphi)\|_{L^2} = 0.$$

Furthermore, X_k is a continuous map on $L^2_{\mathcal{R}}$ (cf Proposition I.28 and Lemma I.4) it follows that

$$\lim_{k\to\infty} \|\frac{d}{dt}\eta^t(\varphi_k) - \frac{d}{dt}\eta^t(\varphi)\|_{L^2} = 0.$$

Hence

$$\begin{aligned}\frac{d}{dt}H_1(\eta^t(\varphi)) &= 2Re\int_0^1 \eta^t_1(x,\varphi)\frac{d}{dt}\overline{\eta^t_1(x,\varphi)}dx \\ &= \lim_{k\to\infty} 2Re\int_0^1 \eta^t_1(x,\varphi_k)\frac{d}{dt}\overline{\eta^t_1(x,\varphi_k)}dx \\ &= \lim_{k\to\infty}\frac{d}{dt}H_1(\eta^t(\varphi_k)) \\ &= 0.\end{aligned}$$

Thus $H_1(\eta^t(\varphi)) = \frac{1}{2}\|\eta^t(\varphi)\|^2_{L^2}$ is constant along the flow $\eta^t(\varphi)$ and we conclude that for any $\varphi \in L^2_{\mathcal{R}}$, $I_{max}(\varphi) = \mathbb{R}$.
Along the flow $\eta^t(\cdot) \equiv \eta^t(\cdot,\varphi)$, the Dirichlet eigenvalues $\mu_k(t) := \mu_k(\eta^t)$ evolve according to the following equation,

$$\begin{aligned}\frac{d}{dt}\mu_k(t) &= \int_0^1 \nabla_{\varphi(x)}\mu_k(t)\cdot\frac{d}{dt}\eta^t(x)dt \\ &= \int_0^1 \nabla_{\varphi(x)}\mu_k(t)\cdot\nabla_{\varphi(x)}\Delta(\lambda)dx\Big|_{\lambda=\mu_n(t)} = \{\mu_k,\Delta(\lambda)\}\Big|_{\lambda=\mu_n(t)}.\end{aligned}$$

Hence, using Proposition I.37, we get

$$\frac{d}{dt}\mu_k(t) = \frac{1}{2}\sqrt[*]{\Delta(\lambda)^2-4}\Big|_{\lambda=\mu_n(t)}\delta_{nk}. \tag{I.80}$$

As an application we obtain the following

Proposition I.40 *Let $\varphi_0 \in L^2_{\mathcal{R}}$, $(\nu_k)_{k\in\mathbb{Z}}$ a sequence of real numbers satisfying $\lambda^-_k(\varphi_0) \leq \nu_k \leq \lambda^+_k(\varphi_0)$ $(\forall k \in \mathbb{Z})$ and $(\epsilon_k)_{k\in\mathbb{Z}}$ a sequence with $\epsilon_k \in \{1,-1\}$. Then there exists a potential $\varphi \in Iso(\varphi_0) \cap L^2_{\mathcal{R}}$ such that $\|\varphi\| \leq \|\varphi_0\|$,*

$$\mu_k(\varphi) = \nu_k \quad \forall k \in \mathbb{Z}$$

and, for any $k \in \mathbb{Z}$ with $\lambda^-_k < \nu_k < \lambda^+_k$,

$$sign\sqrt[*]{\Delta(\mu_k)^2-4} = \varepsilon_k.$$

Remark One can show, using the injectivity of the Birkhoff map (cf section III.8), that the map

$$\varphi \mapsto \left(\mu_k - \tau_k,\ \sqrt[+]{(\gamma_k/2)^2 - (\mu_k-\tau_k)^2}, sign\sqrt[*]{\Delta(\mu_k)^2-4}\right)_{k\in\mathcal{O}}$$

is a bijection of $\mathrm{Iso}(\varphi_0) \cap L^2_{\mathcal{R}}$ onto the torus

$$\{(\xi_k,\eta_k)_{k\in\mathcal{O}} \mid \xi^2_k + \eta^2_k = (\gamma_k/2)^2\ \forall k \in \mathcal{O}\}$$

of genus $\sharp\mathcal{O}$ where $\mathcal{O} \equiv \mathcal{O}(\varphi_0) := \{k \in \mathbb{Z} \mid \gamma_k(\varphi_0) \neq 0\}$.
Proof In view of Corollary I.39 one can choose $\psi_0 \in \mathrm{Iso}(\varphi_0) \cap L^2_{\mathcal{R}}$ so that

$$\lambda^-_k(\varphi_0) < \mu_k(\psi_0) < \lambda^+_k(\varphi_0) \quad \forall k \in \{n \in \mathbb{Z} \mid \gamma_n(\varphi_0) \neq 0\}$$

and $\|\psi_0\| = \|\varphi_0\|$. For any $n \in \mathbb{Z}$ with $\gamma_n \neq 0$, consider the orbit $\mu_k(t) := \mu_k(\eta^t)$ $(t \in \mathbb{R},\ k \in \mathbb{Z})$ where η^t is the solution of (I.79) with initial data ψ_0. By (I.80), $\mu_k(t)$ remains constant for any $k \in \mathbb{Z}\backslash\{n\}$. As $\sqrt{\Delta(\mu_n)^2-4} = 0$ iff $\mu_n \in \{\lambda^-_n, \lambda^+_n\}$, there exist $t_1 < 0 < t_2$ so that $\mu_n(t)$ is monotone in $t_1 < t < t_2$ and $\lim_{t\to t_\pm}\mu_n(t) = \lambda^\pm_n(\varphi_0)$ with t_+, t_- agreeing with t_1, t_2 up to permutation. As the flow η^t is isospectral, $\Delta(\lambda,\eta^t)$ is independent of t and using formula (I.80) one obtains

$$\begin{aligned}\frac{d^2}{dt^2}\mu_n(t) &= \frac{1}{2}\frac{d}{dt}\sqrt[*]{\Delta(\mu_n(t))^2-4} \\ &= \frac{1}{4}\Delta(\mu_n(t))\dot{\Delta}(\mu_n(t)).\end{aligned}$$

Therefore

$$\frac{d^2}{dt^2}\mu_n(t_\pm) = \lim_{t\to t_\pm}\frac{d^2}{dt^2}\mu_n(t) = \frac{1}{4}\Delta(\lambda^\pm_n)\dot{\Delta}(\lambda^\pm_n) \neq 0$$

and it follows that $\frac{d}{dt}\mu_n$ and hence $\sqrt[*]{\Delta(\mu_n(t))^2-4}$ changes sign at these points. Using the same arguments once more one concludes that $\mu(t)$ moves back and forth between λ^-_n and λ^+_n without stopping. By composing the flows corresponding to finitely many of the vector fields X_n, one sees that for any $N \geq 1$, there exists $\psi_N \in L^2_{\mathcal{R}} \cap \mathrm{Iso}(\psi_0)$ so that for any $-N \leq k \leq N$

$$\mu_k(\psi_N) = \nu_k \text{ and, if } \lambda^-_k < \nu_k < \lambda^+_k, \quad sign\sqrt[*]{\Delta(\mu_k)^2-4} = \varepsilon_k.$$

As each of the flows corresponding to the vector fields X_n preserves the L^2-norm we have $\|\psi_N\| = \|\psi_0\|(= \|\varphi_0\|)$. Therefore there exists a subsequence of $(\psi_N)_{N\geq 1}$, again denoted by $(\psi_N)_{N\geq 1}$, which converges weakly to an element $\varphi \in L^2_{\mathcal{R}}$ and hence $\|\varphi\| \leq \|\varphi_0\|$. By Lemma I.4 $\Delta(\lambda,\cdot)$ is a continuous map with respect to the weak topology in $L^2_{\mathbb{C}}$. Hence $\Delta(\lambda,\psi_N) \underset{N\to\infty}{\to} \Delta(\lambda,\varphi)$, and, as $\psi_N \in \mathrm{Iso}(\varphi_0)$, we have $\Delta(\lambda,\psi_N) = \Delta(\lambda,\varphi_0)$ for any $N \geq 1$ and thus $\Delta(\lambda,\varphi) = \Delta(\lambda,\varphi_0)$, or $\varphi \in \mathrm{Iso}(\varphi_0)$.

On the other hand, for any $k \in \mathbb{Z}$ and $N \geq k$, $\mu_k(\psi_N) = \nu_k$. By Lemma I.4, the map δ (cf (I.13)) is weakly continuous on $\mathbb{C} \times L^2([0,1], \mathbb{C}^2)$ and hence, as $\delta(\nu_k, \psi_N) = 0 \;\; \forall N \geq k$

$$0 = \lim_{N \to \infty} \delta(\nu_k, \psi_N) = \delta(\nu_k, \psi),$$

i.e. ν_k is a Dirichlet eigenvalue of ψ. As $\lambda_k^-(\varphi) \leq \nu_k \leq \lambda_k^+(\varphi)$ it follows that $\nu_k = \mu_k(\varphi)$.
Recall that $\sqrt[*]{\Delta(\mu_k)^2 - 4} = (M_{12} + M_{21})\big|_{1,\mu_k}$ and use that $(\lambda, \varphi) \to (M_{12} + M_{21})\big|_{1,\lambda,\varphi}$ is compact on $\mathbb{C} \times L^2([0,1], \mathbb{C}^2)$ to conclude by the same reasoning that $sign \sqrt[*]{\Delta(\mu_k)^2 - 4} = \varepsilon_k$ for any $k \in \mathbb{Z}$ with $\lambda_k^- < \nu_k < \lambda_k^+$. ■

As a second application of the flows introduced above one obtains a density result that will be used in Part II. Let

$$D_n := \{\varphi \in W \mid \gamma_n(\varphi) = 0\}$$

and

$$B_n := \{\varphi \in L^2_{\mathcal{R}} \mid \gamma_n \neq 0;\; \mu_n = \tau_n;\; sign \sqrt[*]{\Delta(\mu_n)^2 - 4} = (-1)^{n+1}\}.$$

Proposition I.41 *For any $n \in \mathbb{Z}$, $D_n \cap L^2_{\mathcal{R}}$ is contained in the L_2-closure of B_n.*

First we prove the following auxilary result

Lemma I.42 *Let $n \in \mathbb{Z}$, $\varphi \in L^2_{\mathcal{R}}$ and $\nu \in \mathbb{R}$ with $\gamma_n^- < \nu < \lambda_n^+$ be given. If $\lambda_n^- < \mu_n < \lambda_n^+$ then there exists $\psi \in Iso(\varphi)$ with the following properties*

(i) $\mu_n(\psi) = \nu$; $sign \sqrt[*]{\Delta(\mu_n(\psi))^2 - 4} = (-1)^{n-1}$;

(ii) $\mu_k(\psi) = \mu_k(\varphi) \;\forall k \in \mathbb{Z} \backslash \{n\}$

(iii) $\|\psi - \varphi\| \leq C(\gamma_n + \gamma_n^{1/2})$

where the constant C is locally uniform in φ.

Proof (Lemma I.42) Let η^t denote the flow satisfying (I.79) with initial data given by φ. By Proposition I.40 there exists $t_* > 0$ so that $\psi := \eta^{t_*}$ satisfies (i) - (ii). It remains to estimate $\|\psi - \varphi\| = \| \int_0^{t_*} X_n(\eta^t) dt \|$.
By (I.80), $\mu_k(t) := \mu_k(\eta^t)$ satisfies

$$\frac{d}{dt} \mu_k(t) = \frac{1}{2} \sqrt[*]{\Delta(\mu_n(t))^2 - 4} \;\; \delta_{nk}.$$

Notice that $0 < t_* < T$ where T denotes the period of $t \to \mu_n^*(t)$ and for $\psi \in L^2_{\mathcal{R}}$, $\mu_n^*(\psi)$ is defined by

$$\mu_n^*(\psi) := (\mu_n(\psi),\; sign \sqrt[*]{\Delta(\mu_n(\psi))^2 - 4}).$$

From the proof of Proposition I.40 one learns that $\frac{d}{dt}\mu_n(t)$ vanishes precisely at $\lambda_n^\pm$. Using the product representation of $\sqrt{\Delta(\mu)^2 - 4}$ together with the asymptotic estimates given by Lemma I.17 one then obtains

$$\begin{aligned} t_* \leq T &= 2 \int_{\lambda_n^-}^{\lambda_n^+} \frac{2 d\mu}{|\sqrt{\Delta(\mu)^2 - 4}|} \\ &\leq C \int_{\lambda_n^-}^{\lambda_n^+} \frac{d\mu}{\sqrt[+]{(\lambda_n^+ - \mu)(\mu - \lambda_n^-)}} \\ &= \pi C \end{aligned}$$

where $C > 0$ is independent of n and can be chosen locally uniformly in φ. Hence

$$\|\psi - \varphi\| \leq \int_0^{t_*} \|X_n(\eta^t)\| dt \leq \pi C \sup_{t \in \mathbb{R}} \|X_n(\eta^t)\|.$$

As the flow η^t is isospectral we have $\Delta(\lambda, \eta^t) = \Delta(\lambda, \varphi)$ and therefore

$$\|X_n(\eta^t)\| \leq \sup_{\lambda_n^- \leq \lambda \leq \lambda_n^+} \|\nabla_{\varphi(x)} \Delta(\lambda)\Big|_{\varphi = \eta^t}\|$$

By Proposition I.28,

$$\begin{aligned} \nabla_{\varphi(x)} \Delta(\lambda) = i(M_{11} - M_{22})\Big|_{1,\lambda} &\begin{pmatrix} M_{21}(x,\lambda) M_{22}(x,\lambda) \\ M_{11}(x,\lambda) M_{12}(x,\lambda) \end{pmatrix} \\ - i M_{12}(1,\lambda) &\begin{pmatrix} M_{21}(x,\lambda)^2 \\ M_{11}(x,\lambda)^2 \end{pmatrix} \\ + i M_{21}(1,\lambda) &\begin{pmatrix} M_{22}(x,\lambda)^2 \\ M_{12}(x,\lambda)^2 \end{pmatrix}. \end{aligned}$$

By (I.4), $\|M(x, \lambda, \varphi)\| \leq e^{\|\varphi\|}$ for $0 \leq x \leq 1, \lambda \in \mathbb{R}$ and hence

$$\|X_n(\eta^t)\| \leq 2 e^{2\|\varphi\|} \sup_{\substack{\lambda_n^- \leq \lambda \leq \lambda_n^+ \\ t \in \mathbb{R}}} (|M_{11} - M_{22}| + |M_{12}| + |M_{21}|)\Big|_{1,\lambda,\eta^t}. \tag{I.81}$$

Note that

$$(M_{11} + M_{22})\Big|_{1,\lambda_k^-} = 2(-1)^k;\; \delta(\mu_k) = 0;\; \check{\delta}(\check{\mu}_k) = 0.$$

As $M(1,\lambda)$ and thus $\dot{M}(1,\lambda)$ is analytic, hence locally bounded near λ_n^- it follows that for $t \in \mathbb{R}$ and $\lambda_n^- \leq \lambda \leq \lambda_n^+$,

$$\begin{aligned} (M_{11} + M_{22})\Big|_{1,x,\eta^t} &= 2(-1)^n + 0(\gamma_n) \\ \delta(\lambda, \eta^t) = 0(\gamma_n);\;& \check{\delta}(\lambda, \eta^t) = 0(\gamma_n). \end{aligned} \tag{I.82}$$

Together with the identities $2(M_{11} - M_{22})\big|_{1,\lambda} = (\delta + \check{\delta})(\lambda)$ and $2(M_{12} - M_{21})\big|_{1,\lambda} = (\delta - \check{\delta})(\lambda)$ one then gets

$$(M_{11} - M_{22})\Big|_{1,\lambda} = 0(\gamma_n);\ (M_{12} - M_{21})\Big|_{1,\lambda} = 0(\gamma_n). \qquad \text{(I.83)}$$

Combining (I.82) and (I.83) one sees that

$$M_{11}(1,\lambda,\eta^t) = (-1)^n + 0(\gamma_n),\ M_{22}(1,\lambda,\eta^t) = (-1)^n + 0(\gamma_n).$$

By (I.21), $M_{12}\big|_{1,\lambda} = \overline{M_{21}}\big|_{1,\lambda}$. Hence the Wronskian identity $(M_{11}M_{22} - M_{12}M_{21})\big|_{1,\lambda} = 1$ leads to $|M_{12}|^2 = 0(\gamma_n)$. Substituting these estimates into (I.81) one sees that

$$\|X_n(\eta^t)\| = 0(\gamma_n + \gamma_n^{1/2}).$$

■

Proof (Proposition I.41). Notice that

$$D_n \cap L^2_{\mathcal{R}} = \{\varphi \in L^2_{\mathcal{R}} \mid \Delta(\dot{\lambda}_n) = 2(-1)^n\}$$

is a submanifold of real codimension 2 in $L^2_{\mathcal{R}}$. Therefore given $\varphi_0 \in D_n \cap L^2_{\mathcal{R}}$ and $\varepsilon > 0$, there exists $\varphi_\varepsilon \in L^2_{\mathcal{R}} \backslash D_n$ with $\|\varphi_\varepsilon - \varphi_0\| < \varepsilon$. As $\varphi \mapsto \gamma_n(\varphi)$ is continuous on $L^2_{\mathcal{R}}$ and hence $\lim_{\varepsilon \to 0} \gamma_n(\varphi_\varepsilon) = 0$, we can choose φ_ε so that $0 < \gamma_n(\varphi_\varepsilon) < \varepsilon$ and in view of Corollary I.39, by changing φ_ε within $\mathrm{Iso}(\varphi_\varepsilon)$ if necessary, we may assume that for any ε,

$$\lambda_n^-(\varphi_\varepsilon) < \mu_n(\varphi_\varepsilon) < \lambda_n^+(\varphi_\varepsilon)\ .$$

Hence by Lemma I.42 it follows that there exists $\psi_\varepsilon \in \mathrm{Iso}(\varphi_\varepsilon)$ satisfying

$$\begin{aligned}
&\mu_n(\psi_\varepsilon) = \tau_n(\psi_\varepsilon)(= \tau_n(\varphi_\varepsilon)),\\
&sign \sqrt[*]{\Delta(\mu_n(\psi_\varepsilon))^2 - 4} = (-1)^{n-1},\\
&\|\psi_\varepsilon - \varphi_\varepsilon\| \le C(\gamma_n + \gamma_n^{1/2})\Big|_{\varphi_\varepsilon}.
\end{aligned}$$

It follows that $\psi_\varepsilon \in B_n$ and

$$\begin{aligned}
\|\psi_\varepsilon - \varphi_0\| &\le \|\psi_\varepsilon - \varphi_\varepsilon\| + \|\varphi_\varepsilon - \varphi_0\|\\
&\le C(\varepsilon + \varepsilon^{1/2}) + \varepsilon.
\end{aligned}$$

As C can be chosen locally uniformly in φ and thus independent of ε for ε sufficiently small one concludes that $\|\psi_\varepsilon - \varphi_0\| \to 0$ for $\varepsilon \to 0$. ■

We end this section with an approximation result that will be used in Part II.

Proposition I.43 *Let $\varphi \in L^2_{\mathcal{R}}$ and $n \in \mathbb{Z}$. Assume $\lambda_n^- = \mu_n$. Then there exists a sequence $(\psi_j)_{j\ge1} \subseteq L^2_{\mathcal{R}}$ with $\lim_{j\to\infty} \psi_j = \varphi$ satisfying*

(i) $\lambda_n^-(\psi_j) < \mu_n(\psi_j) < \lambda_n^+(\psi_j) \quad \forall j \in \mathbb{Z}$

(ii) $\frac{\mu_n(\psi_j) - \lambda_n^-(\psi_j)}{\lambda_n^+(\psi_j) - \mu_n(\psi_j)} \underset{j\to\infty}{\longrightarrow} 0.$

Proof If $\lambda_n^-(\varphi) < \mu_n(\varphi) \le \lambda_n^+(\varphi)$ use the isospectral flow η^t satisfying (I.79) with initial condition $\psi \in \mathrm{Iso}(\varphi)$ satisfying $\mu_k^*(\psi) = \mu_k^*(\varphi)\ \forall k \in \mathbb{Z}\backslash\{n\}$ and $\mu_n(\psi) = \tau_n(\varphi)$. As $\lim_{t\to t_-} \mu_n(\eta^t) = \lambda_n^-$ (cf proof of Proposition I.40) the claimed statement follows if one chooses $\psi_j = \eta^{\tau_j}$ with $(\tau_j)_{j\ge1}$ being a monotone sequence satisfying $\lim_{j\to\infty} \tau_j = t_-$.
If $\lambda_n^-(\varphi) = \lambda_n^-(\varphi)$, then by Proposition I.41 we can choose a sequence $(\varphi_j)_{j\ge1}$ in B_n such that $\lim_{j\to\infty} \varphi_j = \varphi$. By Lemma I.42 there exists for any $j \ge 1$ $\psi_j \in \mathrm{Iso}(\varphi_j)$ such that

$$\mu_n(\psi_j) = \lambda_n^-(\varphi_j) + \frac{1}{j}\gamma_n(\varphi_j)$$

and

$$\|\psi_j - \varphi_j\| \le C\left(\gamma_n(\varphi_j) + \gamma_n(\varphi_j)^{1/2}\right)$$

where $C > 0$ can be choosen independently of $j \ge 1$. As $\lim_{j\to\infty} \gamma_n(\varphi_j) = 0$ we then conclude that $\lim_{j\to\infty} \psi_j = \varphi$ and (i) and (ii) are satisfied. ■

Chapter II

Holomorphic 1-forms

Figure II.1: a-cycles

II.1 Introduction

Consider the Floquet matrix

$$M(1,\lambda) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} (1,\lambda)$$

associated with the Zakharov-Shabat equation $L(\varphi)F = \lambda F$ and its discriminant $\Delta(\lambda) = TrM(1,\lambda)$. The periodic spectrum of φ is precisely the zero set of the entire function $\Delta^2(\lambda;\varphi)-4$ and we have the product representation (cf section I.6)

$$\Delta^2(\lambda) - 4 = -4(\lambda_0^- - \lambda)(\lambda_0^+ - \lambda) \prod_{k\neq 0} \frac{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}{k^2\pi^2} \ .$$

The square root of $\Delta^2(\lambda)-4$ is defined on the hyperelliptic Riemann surface

$$\Sigma_\varphi = \left\{(\lambda, y) \in \mathbb{C}^2 \mid y^2 = \Delta^2(\lambda) - 4\right\}$$

whose genus is precisely the number of open gaps of φ minus 1. The Riemann surface is a spectral invariant associated with φ. It may be viewed as two copies of the complex plane slit open along each open gap and then glued together crosswise along the slits.

The aim of this chapter is to construct a normalized basis of holomorphic differential on Σ_φ. To make this statement more precise, we introduce the complex Hilbert space $\mathcal{I}_0$ of entire function of order ≤ 1 and type ≤ 1, quadratically integrable on $\mathbb{R}$ (see section II.2). For $f \in \mathcal{I}_0$, the differential

$$\omega = \frac{f(\lambda)d\lambda}{\sqrt{\Delta^2(\lambda) - 4}}$$

is holomorphic except eventually at the two infinities of Σ_φ. Denote by a_k, $k \in \mathbb{Z}$, the cycles on the canonical sheet

$$\Sigma_\varphi^c := \left\{(\lambda, y) \in \Sigma_\varphi \mid y = \sqrt[c]{\Delta^2(\lambda) - 4}\right\}$$

described by figure 1. Here $\sqrt[c]{\Delta^2(\lambda) - 4}$ denotes the canonical root defined in I.I.7

Theorem II.1 below states that there exists a family of differentials

$$\sigma_j = \frac{\psi_j(\lambda)d\lambda}{\sqrt{\Delta^2(\lambda) - 4}} \ ,$$

with $\psi_j \in \mathcal{I}_0$, such that for any $j,k \in \mathbb{Z}$

$$\int_{a_k} \sigma_j = 2\pi\delta_{jk} \ .$$

Further the zeroes of ψ_j are located near the center of each gap. From the infinite product representation (II.2), we can see that, in fact, σ_j have poles at infinities with non vanishing residues. Thus $(\sigma_j)_{j\in\mathbb{Z}}$ is a family of Abelian differentials of the third kind. This family will play a crucial rule in the construction of angles for the NLS equation (see chapter III).
The existence of the σ_j's was first proven for φ of real type in McKean-Varinsky [MV] (see [MT2] for a similar construction for the Hill's equation). The extension of σ_j to a complex neighborhood of $L^2_{\mathcal{R}}$ is not straight forward. Actually, when φ is not of real type Σ_φ is a more complicated object since the periodic eigenvalues are no more real. We construct the ψ_j using the implicit function theorem.
Let us mention that the above arguments have been used in [KP] to obtain a similar result for Hill's equation.
To give a precise statement, let us recall some notations. Let

$$W = \cup_{\varphi_0 \in L^2_{\mathcal{R}}} V'_{\varphi_0}$$

be the neighborhood of $L^2_{\mathcal{R}}$ in L^2 given by (I.25) where V'_{φ_0} is the neighborhood of $\varphi_0 \in L^2_{\mathcal{R}}$ in L^2 constructed in section I. I.5. There, for any $\varphi \in W$, we have chosen $\varphi_0 \in L^2_{\mathcal{R}}$ with $\varphi \in V'_{\varphi_0}$ and denoted by $\Gamma_n(\varphi)$ the counter clockwise oriented circle $\Gamma_n(\varphi_0)$ of center $\tau_n(\varphi_0)$ and radius $\frac{1}{2}\gamma_n(\varphi_0) + 2K$ with $K > 0$ given as in (I.24). We recall that the circles $\Gamma_n(\varphi)$ are pairwise disjoint and that there are exactly two periodic eigenvalues of $L(\varphi)$, namely $\lambda_n^+(\varphi)$ and $\lambda_n^-(\varphi)$, inside $\Gamma_n(\varphi)$.

Theorem II.1 *There exists a neighborhood U of $L^2_{\mathcal{R}}$ in L^2 with $U \subseteq W$ so that for any $\varphi \in U$ one can find a sequence of entire functions $\psi_j(\lambda) \equiv \psi_j(\lambda,\varphi)$ $(j \in \mathbb{Z})$ such that*

$$\int_{\Gamma_k(\varphi)} \frac{\psi_j(\lambda,\varphi)}{\sqrt[c]{\Delta(\lambda,\varphi)^2-4}}\, d\lambda = 2\pi\delta_{jk} \quad (j,k \in \mathbb{Z}). \tag{II.1}$$

The functions $\psi_j(\lambda,\varphi)$ are analytic in λ,φ and admit a product representation, for $j \neq 0$,

$$\psi_j(\lambda) = -2\frac{\nu_0^j - \lambda}{j\pi} \prod_{k\neq 0,j} \frac{\nu_k^j - \lambda}{k\pi} \tag{II.2}$$

and, given for $j = 0$ by,

$$\psi_0(\lambda) = -2\prod_{k\neq 0} \frac{\nu_k^0 - \lambda}{k\pi} \tag{II.3}$$

where the zeroes $\nu_k^j \equiv \nu_k^j(\varphi)$ $(j,k \in \mathbb{Z}, k \neq j)$ depend analytically on $\varphi \in U$ and are real for φ of real type. They satisfy the estimate

$$\sup_{j\neq k} |\nu_k^j(\varphi) - \tau_k(\varphi)| \leq |\gamma_k(\varphi)|^2 a_k \tag{II.4}$$

where $(a_k)_{k\in\mathbb{Z}} \in \ell^2$ can be chosen locally independently of φ.
For $\varphi = 0$, the zeroes ν_k^j are given by $\nu_k^j = k\pi$ $(j,k \in \mathbb{Z}$ with $j \neq k)$.

We prove this theorem with the help of the implicit function theorem. To this end we reformulate the statement in terms of a functional equation.
For $\alpha \in \ell^2 \equiv \ell^2(\mathbb{Z},\mathbb{C})$, introduce

$$\bar\alpha_k := k\pi + \alpha_k \quad (k \in \mathbb{Z})$$

and define for $j \in \mathbb{Z}$ the entire function

$$\chi_j(\lambda,\alpha) := \frac{\bar\alpha_0 - \lambda}{j\pi} \prod_{k\neq 0,j} \frac{\bar\alpha_k - \lambda}{k\pi} \quad (j \neq 0)$$

and

$$\chi_0(\lambda,\alpha) := \prod_{k\neq 0} \frac{\bar\alpha_k - \lambda}{k\pi}.$$

For $\varphi \in W$ and $k \in \mathbb{Z}$, denote by $A_k \equiv A_k(\varphi)$ the linear functional defined on the space of entire functions by

$$A_n \cdot f := \int_{\Gamma_n(\varphi)} \frac{f(\lambda)d\lambda}{\sqrt[c]{\Delta(\lambda,\varphi)^2-4}}.$$

For each $j \in \mathbb{Z}$ we then consider the functional

$$F^j : \ell^2 \times W \to \mathbb{C}^{\mathbb{Z}} \ ; \ (\alpha,\varphi) \mapsto \left(F_k^j(\alpha,\varphi)\right)_{k\in\mathbb{Z}}$$

where for $k \neq j$

$$F_k^j(\alpha,\varphi) := (k-j)\ A_k(\varphi) \cdot \chi_j(\cdot,\alpha)$$

and

$$F_j^j(\alpha,\varphi) := \alpha_j + j\pi - \tau_j(\varphi)\ .$$

The proof of Theorem II.1 will be presented in the subsequent sections.
In section II.2 we analyse the case $\varphi = 0$ and then use this case to define a complex Hilbert space of entire functions $\mathcal{I}_0$. In section II.3 we prove the analyticity of the maps $\varphi \mapsto A_k(\varphi)$, $\alpha \mapsto \chi_j(.,\alpha)$ and $(\alpha,\varphi) \mapsto F^j(\alpha,\varphi)$, and in section II.4 we apply the implicit function theorem to prove that for each $j \in \mathbb{Z}$ the functional equation $F^j(\alpha,\varphi) = 0$ has a unique solution

$\alpha = \alpha^j(\varphi) \in \ell^2$ which is defined and analytic on some complex neighborhood U of $L^2_{\mathcal{R}}$ which can be chosen to be independent of j.
In section II.5 we verify that

$$A_j(\varphi) \cdot \chi_j(\cdot, \alpha^j(\varphi)) = -\pi \ .$$

Finally in section II.6 we prove that the zeroes of $\chi_j(\cdot, \alpha^j(\varphi))$ satisfy estimate (II.4) and thus the entire functions

$$\psi_j(, \varphi) := -2\chi_j(\cdot, \alpha^j(\varphi))$$

have all the required properties.

II.2 The zero potential

Proposition II.2 *For $\varphi = 0$, the functions $\psi_j(\lambda) \equiv \psi_j(\lambda, 0)$ of Theorem II.1 are given by*

$$\psi_j(\lambda) = \frac{2\lambda}{j\pi} \prod_{k\neq 0,j} \frac{k\pi - \lambda}{k\pi} \ (j \neq 0) \ \textit{and} \ \psi_0(\lambda) = -2 \prod_{k\neq 0} \frac{k\pi - \lambda}{k\pi}.$$

Proof By (I.9), $\Delta(\lambda, 0) = 2\cos\lambda$ and hence

$$\sqrt[c]{\Delta(\lambda, 0)^2 - 4} = -2i \sin\lambda.$$

Hence by Cauchy's theorem (II.1) is satisfied for $\psi_j(\lambda, 0) := 2\frac{\sin\lambda}{j\pi - \lambda}$. From the product representation $\sin\lambda = \lambda \prod_{k\neq 0} \frac{k\pi - \lambda}{k\pi}$ one then obtains

$$\psi_j(\lambda, 0) = \frac{2\lambda}{j\pi} \prod_{k\neq 0,j} \frac{k\pi - \lambda}{k\pi} \quad (j \neq 0)$$

and $\psi_0(\lambda, 0) = -2\prod_{k\neq 0} \frac{k\pi - \lambda}{k\pi}$ as claimed. ■

The sequence $(\psi_j(\cdot, 0))_{j\in\mathbb{Z}}$ has some additional properties which will be discussed in the remainder of this section. First we need to introduce some notation. Denote by $\mathcal{I}_0$ the complex Hilbert space of entire functions $f : \mathbb{C} \to \mathbb{C}$ satisfying

$$\int_{-\infty}^{\infty} |f(\lambda)|^2 d\lambda < \infty \ ; \ |f(\lambda)| \leq Ce^{|\lambda|} \quad \forall \lambda \in \mathbb{C}.$$

The inner product in $\mathcal{I}_0$ is given by

$$\langle f, g\rangle := \int_{-\infty}^{\infty} f(\lambda)\overline{g(\lambda)} d\lambda$$

and $\|f\|$ denotes the corresponding norm, $\|f\| := \langle f, f\rangle^{1/2}$. By the Paley-Wiener Theorem, the Fourier transform

$$\hat{f}(x) \equiv \mathcal{F}(f)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\lambda} f(\lambda) d\lambda$$

of a function$f \in \mathcal{I}_0$ satisifes

$$\hat{f}(x) = 0 \text{ for } x \in \mathbb{R} \ , \ |x| \geq 1.$$

Hence

$$\int_{-1}^{1} |\hat{f}(x)| dx \leq \sqrt{2} \left(\int_{-1}^{1} |\hat{f}(x)|^2 dx \right)^{1/2} = \sqrt{2} \left(\int_{-\infty}^{\infty} |f(\lambda)|^2 d\lambda \right)^{1/2}$$

and as $f(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i\lambda x} \hat{f}(x) dx$ it then follows that for any $\lambda \in \mathbb{C}$,

$$\begin{aligned} |f(\lambda)| &\leq \frac{1}{\sqrt{2\pi}} \ e^{|Im\lambda|} \int_{-1}^{1} |\hat{f}(x)| dx \\ &\leq \frac{1}{\sqrt{\pi}} \ e^{|Im\lambda|} \|f\|. \end{aligned} \tag{II.5}$$

In fact, the Fourier transform is an isometry of $\mathcal{I}_0$ onto

$$L^2_{PW} := \{ f \in L^2_{\mathbb{C}}(\mathbb{R}) \mid \text{supp}(f) \subseteq [-1, 1] \}.$$

Clearly, $u_n(x) := \frac{1}{\sqrt{2}} e^{-i\pi nx} 1_{[-1,1]}(x)$ $(n \in \mathbb{Z})$ is an orthonormal basis of L^2_{PW} where $1_{[-1,1]}(x)$ denotes the characteristic function of the interval $[-1, 1]$. Therefore $v_n := \mathcal{F}^{-1}(u_n)$ $(n \in \mathbb{Z})$ is an orthonormal basis of $\mathcal{I}_0$. The v_n's can be computed to be

$$v_n(\lambda) = \frac{(-1)^{n+1}}{\sqrt{\pi}} \frac{\sin\lambda}{n\pi - \lambda}$$

and we conclude that $\frac{1}{2\sqrt{\pi}} \psi_n(\lambda, 0) = (-1)^{n+1} v_n(\lambda)$ $(n \in \mathbb{Z})$ is an orthonormal basis for $\mathcal{I}_0$. For $f \in \mathcal{I}_0$, apply to

$$\mathcal{F}(f) = \sum_{n\in\mathbb{Z}} \langle \mathcal{F}(f), u_n\rangle u_n = \sum_{n\in\mathbb{Z}} \left(\int_{-1}^{1} \mathcal{F}(f)(x) \frac{1}{\sqrt{2}} e^{i\pi nx} dx \right) \frac{1}{\sqrt{2}} e^{-in\pi x}$$

the inverse Fourier transform $\mathcal{F}^{-1}$ to obtain

$$f(\lambda) = \sum_{n\in\mathbb{Z}} \sqrt{\pi} f(\pi n) v_n(\lambda) \tag{II.6}$$

where we used that supp$\mathcal{F}(f) \subseteq [-1, -1]$ and thus

$$\int_{-1}^{1} \mathcal{F}(f)(x) \frac{1}{\sqrt{2}} e^{i\pi nx} dx = \sqrt{\pi} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(f)(x) e^{i\pi nx} dx = \sqrt{\pi} f(\pi n).$$

Formula (II.6) is known as Kotelnikov's theorem. As a consequence we have that for any entire function f with $|f(\lambda)| \leq Ce^{|\lambda|}$, one has

$$f \in \mathcal{I}_0 \text{ iff } \left(f(n\pi)\right)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}). \tag{II.7}$$

For $\varphi \in W$ and $n \in \mathbb{Z}$, denote by $A_n \equiv A_n(\varphi)$ the linear functional $A_n : \mathcal{I}_0 \to \mathbb{C}$,

$$A_n \cdot f := \int_{\Gamma_n(\varphi)} \frac{f(\lambda)d\lambda}{\sqrt[c]{\Delta(\lambda,\varphi)^2 - 4}}.$$

In view of (II.5), $A_n(\varphi)$ is continuous and thus an element in the dual space $\mathcal{I}_0^*$ of $\mathcal{I}_0$. Hence, by (II.1) and Proposition II.2, $\left(\frac{1}{2\pi}A_n(0)\right)_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{I}_0^*$ which is biorthogonal to the *orthogonal* basis $\left(\psi_j(\cdot,0)\right)_{j\in\mathbb{Z}}$ and therefore an *orthogonal* basis of $\mathcal{I}_0^*$. Notice that

$$\psi_j(n\pi,0) = 2\sqrt{\pi}(-1)^{j+1}v_j(n\pi) = 2(-1)^{j+1}\delta_{jn}$$

and hence one concludes from (II.1) that $\frac{(-1)^{n+1}}{\pi} \cdot A_n(0)$ is the Dirac measure $m_{n\pi}$ at $n\pi$,

$$\frac{1}{\pi}A_n(0) = (-1)^{n+1}m_{n\pi} \ .$$

Further, as $(-1)^{n+1}v_n(\lambda) = \frac{1}{2\sqrt{\pi}}\psi(\lambda,0)$ $(n \in \mathbb{Z})$ is an *orthonormal* basis of $\mathcal{I}_0$, $\frac{1}{\sqrt{\pi}}A_n(0)$ $(n \in \mathbb{Z})$ is an *orthonormal* basis of $\mathcal{I}_0^*$.
We end this section with two simple observations that will be used in the subsequent sections.

Lemma II.3 *Let φ be an element of $L^2_{\mathcal{R}}$ and f an entire function which is real on the real line. If $A_k(\varphi) \cdot f = 0$ for some $k \in \mathbb{Z}$ then f has a root in $G_k(\varphi) := [\lambda_k^-, \lambda_k^+]$.*

Proof By assumption,

$$\int_{\Gamma_k} \frac{f(\lambda)d\lambda}{\sqrt[c]{\Delta(\lambda)^2 - 4}} = 0$$

where $\gamma_k = \gamma_k(\varphi)$ and $\Delta(\lambda) = \Delta(\lambda,\varphi)$. If $\gamma_k \neq 0$, we can shrink the contour Γ_k to the interval $[\lambda_k^-, \lambda_k^+]$ to obtain

$$\int_{\lambda_k^-}^{\lambda_k^+} \frac{f(\lambda)d\lambda}{\sqrt[c]{\Delta(\lambda - i0)^2 - 4}} = 0 \ .$$

Notice that as φ_0 is of real type, $\lambda_k^\pm$ are real, $\sqrt[c]{\Delta(\lambda - i0)^2 - 4}$ is real valued for $\lambda \in [\lambda_k^-, \lambda_k^+]$ and does not vanish for $\lambda_k^- < \lambda < \lambda_k^+$. Therefore f has to change sign in $G_k(\varphi_0)$.
If $\gamma_k = 0$, we may extract the factor $(\lambda - \tau_k)^2$ from the product representation of $\Delta(\lambda)^2 - 4$ and note that the contour integral above turns into a Cauchy integral arround τ_k, which then gives $f(\tau_k) = 0$. ■

Clearly Lemma II.3 is the motivation why we look for the entire functions ψ_j of Theorem II.1 of the form (II.2), (II.3).
The next lemma says that a function in $\mathcal{I}_0$ cannot have too many zeroes.

Lemma II.4 *Assume that $f \in \mathcal{I}_0$ and $(z_k)_{k\in\mathbb{Z}}$ is a sequence of complex numbers with $z_k = k\pi + \ell^2(k)$. If $f(z_k) = 0$ for any $k \in \mathbb{Z}$, then $f \equiv 0$.*

Proof By Lemma I.17, the infinite product representation

$$g(\lambda) := -(z_0 - \lambda)\prod_{j\neq 0} \frac{z_j - \lambda}{j\pi}$$

is convergent and defines an entire function satisfying $g(\lambda) = (1 + 0(1))\sin\lambda$ uniformly for $(n + 1/4)\pi \leq |\lambda| \leq (n + 3/4)\pi$ and $n \in N$. Hence $h(\lambda) := f(\lambda)/g(\lambda)$ is an entire function. As $f \in \mathcal{I}_0$ it follows that there exists $C > 0$ so that for any $\lambda \in \mathbb{C}$ with $|\lambda| = (n + \frac{1}{2})\pi$ $(n \geq 1)$,

$$|h(\lambda)| \leq Ce^{|Im\lambda|}/|\sin\lambda| \leq 4C$$

where for the latter inequality we used that

$$e^{|Im\lambda|} \leq 4|\sin\lambda| \quad \forall\lambda \in \cap_{k\in\mathbb{Z}}\{|\lambda - k\pi| \geq \pi/4\}$$

(cf [PT, Chapter 2]). Hence by the maximum principle, h is bounded on $\mathbb{C}$ and thus constant by Liouville's theorem. It follows that for some $c \in \mathbb{C}$,

$$f(\lambda) = cg(\lambda) = c\,(1 + 0(1))\sin\lambda$$

uniformly on $\cup_{n\geq 1}\{(n+1/4)\pi < |\lambda| < (n+3/4)\pi\}$. By assumption, $f \in \mathcal{I}_0$, hence $\|f\| < \infty$ and one concludes that $c = 0$, hence $f \equiv 0$. ■

II.3 Analyticity properties

In this section we analyze the maps $\varphi \mapsto A_k(\varphi), \alpha \mapsto \chi_j(\cdot,\alpha)$ and $(\alpha,\varphi) \mapsto F^j(\alpha,\varphi)$ defined in section II.1. As already pointed out in section II.2, $A_n(\varphi) \in \mathcal{I}_0^*$ for any $\varphi \in W$ and $n \in \mathbb{Z}$.

Lemma II.5 *For given $\varphi \in W$,*

$$\sup_{n\in\mathbb{Z}} \|A_n(\varphi)\|_{\mathcal{I}_0^*} \leq C \tag{II.8}$$

where $C > 0$ can be chosen locally uniformly in φ.

Proof By Lemma I.19, $\Delta(\lambda)^2 - 4$ admits an infinite product representation

$$\Delta(\lambda,\varphi)^2 - 4 = -4(\lambda_0^+ - \lambda)(\lambda_0^- - \lambda)\prod_{k\neq 0}\frac{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}{k^2\pi^2}.$$

Recall that for any $\varphi \in W$ there exists $\varphi_0 \in L^2_{\mathcal{R}}$ so that $\varphi \in V'(\varphi_0)$ and $\Gamma_n(\varphi) = \Gamma_n(\varphi_0)$ is the circle of center $\tau_n(\varphi_0)$ and radius $\frac{1}{2}\gamma_n(\varphi_0)+2K$ where K is given by

$$K := \frac{1}{5}\min\left\{\left(\lambda_{n+1}^-(\varphi) - \lambda_n^+(\varphi_0)\right), \frac{\pi}{2} \mid n \in \mathbb{Z}\right\} \leq \frac{\pi}{10}.$$

By Lemma I.17, for any $\lambda \in \Gamma_n(\varphi_0)$ $(n \in \mathbb{Z})$

$$\Delta(\lambda,\varphi)^2 - 4 = -4(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)\left(\frac{\sin\lambda}{\lambda - n\pi}\right)^2(1+\alpha_n(\lambda))$$

where $(\alpha_n(\lambda))_{n\in\mathbb{Z}}$ satisfies $\sum_n \sup_{\lambda\in\Gamma_n(\varphi_0)} |\alpha_n(\lambda)|^2 < \infty$ locally uniformly in φ. Choose $n_0 \geq 1$ so that for $|n| \geq n_0, \gamma_n(\varphi_0) \leq \frac{1}{10}, |\tau_n(\varphi_0) - n\pi| \leq \frac{1}{20}$ and $\sup_{\lambda\in\Gamma_n(\varphi_0)} |\alpha_n(\lambda)| \leq \frac{1}{2}$. It then follows that for $\lambda \in \Gamma_n(\varphi_0)$,

$$|\lambda - n\pi| \leq |\lambda - \tau_n(\varphi_0)| + |\tau_n(\varphi_0) - n\pi| < \pi/4$$

and hence

$$\sup_{\substack{|n|\geq n_0\\ \lambda\in\Gamma_n(\varphi_0)}} \left| \left(\frac{\sin\lambda}{\lambda - n\pi}\right)^2 - 1 \right| < 1.$$

This shows that there exists $C_1 > 0$ such that for any $\lambda \in \cup_{|n|\geq n_0}\Gamma_n(\varphi_0)$

$$|\Delta(\lambda,\varphi)^2 - 4| \geq 1/C_1^2 \tag{II.9}$$

and thus, for $C_2 := C_1\pi^2/2$ and $|n| \geq n_0$,

$$\begin{aligned}|A_n(\varphi)\cdot f| &\leq \text{ length } (\Gamma_n)C_1 \sup_{\lambda\in\Gamma_n}|f(\lambda)|\\ &\leq C_2\|f\|\end{aligned}$$

where for the last inequality we have used (II.5). By the continuity of the A_n's, (II.8) then follows. As indicated, the constant C in (II.8) can be chosen locally uniformly. ■

Lemma II.6 *For any $n \in \mathbb{Z}$, the map $A_n : W \to \mathcal{I}_0^*, \varphi \mapsto A_n(\varphi)$ is analytic.*

Proof By Lemma II.5, A_n is locally bounded. Moreover, for any $f \in \mathcal{I}_0$,

$$\varphi \mapsto A_n(\varphi)\cdot f = \int_{\Gamma_n(\varphi)} \frac{f(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}}d\lambda$$

is analytic, hence A_n is weakly analytic and the claimed statement follows (cf e.g. [PT, Appendix A]). ■

Lemma II.7 *Let $\alpha \in \ell^2$. Then $\chi_j(\cdot,\alpha) \in \mathcal{I}_0$ for any $j \in \mathbb{Z}$. Furthermore*

$$\sup_{j\in\mathbb{Z}}\|\chi_j(\cdot,\alpha)\| \leq C$$

where $C > 0$ is a constant which can be chosen locally uniformly for α in ℓ^2.

Proof Let $\alpha \in \ell^2$ and $j \in \mathbb{Z}$ be given. By Lemma I.17, $\chi_j(\cdot,\alpha)$ is entire. Hence $f(\lambda) := (j\pi - \lambda)\chi_j(\lambda,\alpha)$ is entire as well and satisfies by Lemma I.16,

$$f(\lambda) = (1 + 0(1))\sin\lambda$$

uniformly on $\cup_{n\geq 0}\{(n+\frac{1}{4})\pi \leq |\lambda| \leq (n+\frac{3}{4})\pi\}$. By the maximum principle it then follows that for some constant $C > 0$

$$|f(\lambda)| \leq Ce^{|\lambda|} \quad \forall\lambda \in \mathbb{C}.$$

As $\chi_j(\cdot,\alpha)$ is entire one then concludes that

$$|\chi_j(\lambda,\alpha)| \leq C'e^{|\lambda|}$$

for some constant $C' > 0$. It remains to show that

$$\|\chi_j(\cdot,\alpha)\| := \left(\int_{-\infty}^{\infty} |\chi_j(\lambda,\alpha)|^2 d\lambda\right)^{1/2}$$

can be bounded as claimed.
By Lemma I.19, there exists a constant $C_1 > 0$ so that

$$\left|(j\pi - \lambda)\chi_j(\lambda,\alpha)\right| \leq C_1 \quad \forall\lambda \in \mathbb{R}, \quad \forall j \in \mathbb{Z}.$$

By Lemma I.17, there exists $C_2 > 0$ so that for any $j \in \mathbb{Z}$

$$|\chi_j(\lambda,\alpha)| \leq C_2 \quad \forall\lambda \in \mathbb{C} \text{ with } |\lambda - j\pi| \leq \pi/4.$$

Combining the last two estimates it follows that for any $j \in \mathbb{Z}$,

$$|\chi_j(\lambda,\alpha)| \leq \frac{C}{1 + |j\pi - \lambda|} \quad \forall\lambda \in \mathbb{R}.$$

As C_1, C_2 can be chosen uniformly on bounded subsets of α's in ℓ^2, so can the constant $C > 0$. The above estimate then leads to the claimed result

$$\|\chi_j(\cdot,\alpha)\| \leq C\left(\int_{\mathbb{R}} \frac{dx}{1+x^2}\right)^{1/2} = \pi C.$$

■

Lemma II.8 *For any $j \in \mathbb{Z}, \ell^2 \to \mathcal{I}_0,\ \alpha \mapsto \chi_j(\cdot,\alpha)$ is analytic.*

Proof By Lemma II.7, the map $\alpha \mapsto \chi_j(\cdot,\alpha)$ is locally bounded, hence it suffices to prove that the map $\alpha \mapsto \chi_j(\cdot,\alpha)$ is weakly analytic. As $\mathcal{I}_0$ is a Hilbert space and in view of Kotelnikov's theorem (cf (II.6)) it is to prove that for any $n \in \mathbb{Z}, \alpha \mapsto A_n(0)\cdot\chi_j(\cdot,\alpha) = (-1)^{n+1}\pi\chi_j(n\pi,\alpha)$ is weakly analytic on ℓ^2 (cf [PT, Appendix A, Theorem 3]). By definition, one has for $j \neq 0$,

$$\chi_j(n\pi,\alpha) = \frac{\alpha_0 - n\pi}{j\pi}\prod_{k\neq j,0}\frac{\alpha_k + (k-n)\pi}{k\pi}$$

and for $j = 0$,

$$\chi_0(n\pi,\alpha) = \prod_{k\neq 0}\frac{\alpha_k + k\pi}{k\pi}.$$

Hence for any $\alpha,\beta \in \ell^2$, the analyticity of $\chi_j(n\pi,\alpha+z\beta)$ in $z \in \mathbb{C}$ is easily established. ∎

To study the properties of the maps F^j, we first need to establish additional estimates for $\chi_j(\lambda,\alpha)$ and $A_n(\varphi)$.

Lemma II.9 *For any given $\alpha \in \ell^2$ there exists $C = C(\alpha) > 0$ so that for any $n, j \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$*

$$|\chi_j(\lambda,\alpha)| \leq \frac{C}{|j\pi - \lambda|}\ |\bar{\alpha}_n - \lambda|e^{|Im\lambda|}$$

where $\bar{\alpha}_n := n\pi + \alpha_n$ $(n \in \mathbb{Z})$. The constant C can be chosen uniformly on bounded subsets of α's in ℓ^2.

Proof Introduce for any $n, j \in \mathbb{Z}$

$$f_n^j(\lambda,\alpha) := \frac{j\pi - \lambda}{\bar{\alpha}_n - \lambda}\chi_j(\lambda,\alpha).$$

Notice that

$$f_n^j(\lambda,\alpha) = \chi_n(\lambda,\alpha^{j,n})$$

where $\alpha^{j,n} \in \ell^2$ is defined by

$$\begin{aligned} &\alpha_k^{j,n} := \alpha_k \quad \forall k \in \mathbb{Z}\backslash\{n\} \\ &\alpha_j^{j,n} := 0\ ; \quad \alpha_n^{j,n} := 0\ . \end{aligned} \tag{II.10}$$

As $\|\alpha^{j,n}\| \leq \|\alpha\|$ for any $n, j \in \mathbb{Z}$, one concludes from Lemma II.7 that $f_n^j(\cdot,\alpha) \in \mathcal{I}_0$ and

$$\sup_{j,n}\|f_n^j(\cdot,\alpha)\| \leq C$$

where the constant $C > 0$ can be chosen uniformly on bounded subsets of α's in ℓ^2. Hence, by (II.5), one gets

$$\sup_{n,j}|f_n^j(\lambda,\alpha)| \leq \frac{C}{\sqrt{\pi}}e^{|Im\lambda|}.$$

∎

Lemma II.10 *Let $\varphi \in W$. Then there exits $C \equiv C(\varphi) > 0$ so that for any $f \in \mathcal{I}_0$ and $n \in \mathbb{Z}$*

$$|A_n(\varphi)\cdot f| \leq C\sup\{|f(\lambda)|\ \ |\ \ |\lambda - \tau_n(\varphi)| \leq |\gamma_n|\}\ .$$

The constant C can be chosen locally uniformly in $\varphi \in W$.

Proof Recall that $A_n \equiv A_n(\varphi)$ is given by

$$A_n\cdot f = \int_{\Gamma_n}\frac{f(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}}d\lambda.$$

By the product representation of $\Delta(\lambda)^2 - 4$ and Lemma I.17,

$$\Delta(\lambda)^2 - 4 = -4(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)\,(1 + \varrho_n(\lambda))$$

where

$$\left(\sup_{|\lambda-\tau_n|\leq|\gamma_n|}|\varrho_n(\lambda)|\right)_{n\in\mathbb{Z}} \in \ell^2.$$

Choose n_0 so that for any $|n| \geq n_0$, $\sup_{|\lambda-\tau_n|\leq|\gamma_n|}|\varrho_n(\lambda)| \leq 1/2$. It then follows that there exists $C_1 > 0$ so that

$$\inf_{|\lambda-\tau_n|\leq|\gamma_n|}|1 + \varrho_n(\lambda)|^{1/2} \geq 1/C_1 \quad \forall n \in \mathbb{Z}. \tag{II.11}$$

Also recall that, with $\sqrt[s]{\ }$ denoting the standard root (cf Section I. Ss:Branches of square roots)

$$\sqrt[c]{\Delta(\lambda)^2 - 4} = 2i\sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}\sqrt{1 + \varrho_n(\lambda)}$$

with the appropriate choice of the sign of the root $\sqrt{1 + \varrho_n(\lambda)}$. For $n \in \mathbb{Z}$ with $\gamma_n = 0$ one gets by Cauchy's theorem

$$|A_n\cdot f| = 2\pi\left|\frac{f(\tau_n)}{\sqrt{1 + \varrho_n(\tau_n)}}\right| \leq 2\pi C_1|f(\tau_n)|.$$

If $\gamma_n \neq 0$, deform the contour Γ_n to the straight line

$$\lambda(t) = \tau_n + t\cdot\gamma_n/2\ (-1 \leq t \leq 1)$$

traversed in both directions, but with different choices of the sign of the root. In this way one obtains

$$\begin{aligned}|A_n \cdot f| &= 2\Big|\int_{-1}^{1} \frac{f(\lambda(t))}{2i\frac{\gamma_n}{2}\sqrt{t^2-1}\sqrt{1+\varrho_n(\lambda(t))}}\frac{\gamma_n}{2}dt\Big| \\ &\leq \pi C_1 \sup_{|\lambda-\tau_n|\leq|\gamma_n|}|f(\lambda)|.\end{aligned}$$

It is easy to see that C_1 in (II.11) can be chosen locally uniformly on W. ■

We can now state the main result of this section:

Proposition II.11 *For each $j \in \mathbb{Z}$ the map F^j is analytic from $\ell^2 \times W$ into ℓ^2. Furthermore the F^j are locally bounded uniformly in $j \in \mathbb{Z}$.*

Proof Recall that for any $(\alpha, \varphi) \in \ell^2 \times W$ and $k, j \in \mathbb{Z}$ with $k \neq j$,

$$F^j_k(\alpha, \varphi) = (k-j)\ A_k(\varphi) \cdot \chi_j(\cdot, \alpha).$$

By Lemma II.6 and Lemma II.8, each component F^j_k, with $k \neq j$, of F^j is analytic and $F^j_j(\alpha,\varphi) = \bar{\alpha}_j - \tau_j(\varphi)$. Thus F^j is analytic if it is locally bounded as a map from $\ell^2 \times W$ into ℓ^2 (cf [KP, Appendix A, Theorem A.3]). To prove this notice that by Lemma II.10, there exists $C(\varphi) > 0$ so that

$$|F^j_k(\alpha,\varphi)| \leq C(\varphi) \sup_{|\lambda-\tau_k|\leq|\gamma_k|} |k-j|\ |\chi_j(\lambda,\alpha)|.$$

By Lemma II.9, for any $k, j \in \mathbb{Z}$ with $k \neq j$ and $\lambda \in \mathbb{C}$

$$|\chi_j(\lambda,\alpha)| \leq C(\alpha)\Big|\frac{\bar{\alpha}_k - \lambda}{j\pi - \lambda}\Big| e^{|Im\lambda|}.$$

Notice that for $|\lambda - \tau_k| \leq |\gamma_k|$ one has

$$\begin{aligned}|\bar{\alpha}_k - \lambda| &\leq |\alpha_k| + |\tau_k - k\pi| + |\gamma_n|\ , \\ |j\pi - \lambda| &\geq C_1\langle j-k\rangle\end{aligned}$$

and

$$|Im\lambda| \leq |\lambda - k\pi| \leq |\tau_k - k\pi| + |\gamma_k|\ .$$

Furthermore $(\tau_k - k\pi)_{k\in\mathbb{Z}}$ is a sequence in ℓ^2 which can be bounded locally uniformly on W. Hence one concludes that

$$\sup_{j\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|F^j_k(\alpha,\varphi)|^2\right)^{1/2} \leq C$$

where the constant $C \equiv C(\alpha,\varphi)$ can be chosen uniformly on bounded subsets of α's and locally uniformly in $\varphi \in W$. ■

II.4 Implicit Function Theorem

In order to apply the implicit function theorem, we have to analyze the Jacobian of F^j with respect to α. At any point (α,φ) of $\ell^2 \times L^2_{\mathcal{R}}$ this Jacobian is a bounded linear operator $B^j \equiv B^j(\alpha,\varphi)$ on ℓ^2 which is represented by an infinite matrix $B^j = (b^j_{k,n})$ with elements

$$b^j_{k,n} \equiv b^j_{k,n}(\alpha,\varphi) := \frac{\partial}{\partial\alpha_n}F^j_k(\alpha,\varphi).$$

One easily computes that for $k \neq j$ and $n \neq j$

$$b^j_{k,n} = (k-j)\ A_k(\varphi) \cdot \frac{\chi_j(\lambda,\alpha)}{\bar{\alpha}_n - \lambda} \tag{II.12}$$

whereas

$$b^j_{k,n} = \delta_{kn} \text{ for } k = j \text{ or } n = j\ . \tag{II.13}$$

Notice that by Proposition II.2 one has

$$\bar{\alpha}^j_k(0) = k\pi \quad j,k \in \mathbb{Z}\ ,$$

hence $\alpha^j(0) \equiv \left(\alpha^j_k(0)\right) = 0$ and for $k \neq j$ and $n \neq j$, $b^j_{k,n}$ at $\alpha = \alpha^j(0)$, $\varphi = 0$ is given by

$$b^j_{k,n} = \frac{k-j}{2i}\int_{\Gamma_k(0)} \frac{d\lambda}{(n\pi-\lambda)(\lambda - j\pi)} = \delta_{kn}\ .$$

Together with (II.12) we then conclude that B^j at $\alpha = \alpha^j(0)$, $\varphi = 0$ is the identity on ℓ^2.
For $\alpha \in \ell^2$ and $\varphi \in L^2_{\mathcal{R}}$ arbitrary we decompose B^j into its diagonal part, $D^j \equiv (d^j_{k,n})$, and its off-diagonal part Q^j,

$$Q^j := B^j - D^j\ .$$

Denote by $\ell^2_{\mathbb{R}}$ the set of real valued sequences in ℓ^2.

Lemma II.12 *For any $j \in \mathbb{Z}$ and any $(\alpha,\varphi) \in \ell^2_{\mathbb{R}} \times L^2_{\mathcal{R}}$ the diagonal operator $D^j \equiv D^j(\alpha,\varphi)$ is boundedly invertible on $\ell^2_{\mathbb{R}}$.*
Further there exists a constant $C > 0$ such that for all $j \in \mathbb{Z}$

$$\|D^j\|_{\mathcal{L}(\ell^2)} \leq C \ \textit{and}\ \|(D^j)^{-1}\|_{\mathcal{L}(\ell^2)} \leq C$$

where C can be chosen locally uniformly in $(\alpha,\varphi_0) \in \ell^2_{\mathbb{R}} \times L^2_{\mathcal{R}}$.

Proof For any $j,k \in \mathbb{Z}$, let

$$d_k^j := (k-j)A_k \cdot \left(\frac{\chi_j(\lambda)}{\bar{\alpha}_k - \lambda}\right).$$

We prove that locally uniformly on $\ell^2_{\mathbb{R}} \times L^2_{\mathcal{R}}$

$$0 < \inf_{\substack{k,j\in\mathbb{Z}\\ j\neq k}} |d_k^j| < +\infty \ .$$

Notice that $\chi_j(\lambda)/(\bar{\alpha}_k - \lambda)$ is real valued on $\mathbb{R}$ and does not change sign for $\lambda_k^- \leq \lambda \leq \lambda_k^+$. Further, $(-1)^{k+1} \sqrt[c]{\Delta(\lambda - i0)^2 - 4} > 0$ for $\lambda_k^- < \lambda < \lambda_k^+$ (cf (I.50)). It follows that

$$\left(\min_{\lambda_k^- \leq \lambda \leq \lambda_k^+} \left| \frac{(k-j)\chi_j(\lambda)}{\bar{\alpha}_k - \lambda} \right| \right) a_k \leq |d_k^j| \leq \left(\max_{\lambda_k^- \leq \lambda \leq \lambda_k^+} \left| \frac{(k-j)\chi_j(\lambda)}{\bar{\alpha}_k - \lambda} \right| \right) a_k$$

where for k with $\gamma_k = 0$, $a_k := \pi$ and for k with $\gamma_k \neq 0$

$$a_k := 2 \int_{\lambda_k^-}^{\lambda_k^+} \frac{d\lambda}{\sqrt[+]{\Delta(\lambda)^2 - 4}} \quad .$$

Hence $a_k > 0$ for any $k \in \mathbb{Z}$ and by the infinite product representation of $\sqrt[+]{\Delta(\lambda)^2 - 4}$ and Lemma I.17,

$$\begin{aligned} a_k &= \int_{\lambda_k^-}^{\lambda_k^+} \frac{d\lambda}{\sqrt[+]{(\lambda_k^+ - \lambda)(\lambda - \lambda_k^-)}} \left(1 + \ell^2(k)\right) d\lambda \\ &= \pi \left(1 + \ell^2(k)\right) . \end{aligned}$$

It follows that $\inf_k a_k > 0$ and $\sup_k a_k < +\infty$ locally uniformly for $\varphi \in L^2_{\mathcal{R}}$ and it remains to show that

$$\sup_{\substack{k,j\in\mathbb{Z}\\ j\neq k}} \max_{\lambda_k^- \leq \lambda \leq \lambda_k^+} |\chi_j(\lambda)(j\pi - \lambda)/(\bar{\alpha}_k - \lambda)| < +\infty \tag{II.14}$$

and

$$\inf_{\substack{k,j\in\mathbb{Z}\\ j\neq k}} \min_{\lambda_k^- \leq \lambda \leq \lambda_k^+} |\chi_j(\lambda)(j\pi - \lambda)/(\bar{\alpha}_k - \lambda)| > 0 \ . \tag{II.15}$$

Estimate (II.14) is a consequence of Lemma II.9. To prove (II.15) notice that

$$\chi_j(\lambda)(j\pi - \lambda)/(\bar{\alpha}_k - \lambda) = \chi_k(\lambda, \alpha^{j,k})$$

where $\alpha^{j,k} \in \ell^2$ is given by (II.10). Hence (II.15) is equivalent to

$$\inf_{\substack{k,j\in\mathbb{Z}\\ j\neq k}} \min_{\lambda_k^- \leq \lambda \leq \lambda_k^+} |\chi_k(\lambda, \alpha^{j,k})| > 0 \ . \tag{II.16}$$

Clearly, for any $k,j \in \mathbb{Z}$ with $j \neq k$, $\min_{\lambda_k^- \leq \lambda \leq \lambda_k^+} |\chi_k(\lambda, \alpha^{j,k})| > 0$ and by Lemma I.17, one has

$$|\chi_k(\lambda, \alpha^{j,k})| = 1 + \ell^2(k)$$

uniformly for $|\lambda - \tau_k| \leq \gamma_k$ and locally uniformly for $\alpha \in \ell^2$ as $\{\alpha^{j,k} \mid \ k,j \in \mathbb{Z},\ j \neq k\}$ is relatively compact in ℓ^2. Hence there exists $K \geq 1$ so that for any $|k| \geq K+1$, and $j \in \mathbb{Z}$ with $j \neq k$

$$\min_{\lambda_k^- \leq \lambda \leq \lambda_k^+} |\chi_k(\lambda, \alpha^{j,k})| \geq \frac{1}{2}$$

which leads to (II.16). ■

The Jacobian B^j can be written as

$$B^j = D^j(Id + T^j) \text{ and } T^j = (D^j)^{-1} Q^j \ .$$

By construction, $T^j = \left(T^j_{k,n}\right)_{k,n\in\mathbb{Z}}$ is given by

$$T^j_{k,n} = (d_k^j)^{-1}(k-j)A_k \cdot \left(\frac{\chi_j(\cdot)}{\bar{\alpha}_n - \lambda}\right) \quad k \neq n,\ j \neq n \tag{II.17}$$

and

$$T^j_{k,n} = 0 \quad k \neq n \text{ or } j \neq n \ . \tag{II.18}$$

Lemma II.13 *For any $j \in \mathbb{Z}$ and any $(\alpha, \varphi) \in \ell^2_{\mathbb{R}} \times L^2_{\mathcal{R}}$ such that $F^j(\alpha, \varphi) = 0$, the operator $T^j \equiv T^j(\alpha, \varphi)$ is a compact on ℓ^2 .*

Proof Use that $(k-j)A_k \cdot \chi_j = 0$ to get

$$\begin{aligned} (k-j)A_k \cdot \left(\frac{\chi_j}{\bar{\alpha}_n - \lambda}\right) &= (k-j)A_k \cdot \left(\frac{\chi_j(\cdot)}{\bar{\alpha}_n - \lambda} - \frac{\chi_j(\cdot)}{\bar{\alpha}_n - \tau_k}\right) \\ &= \frac{(k-j)}{\tau_k - \bar{\alpha}_n} 2 \int_{\lambda_k^-}^{\lambda_k^+} \frac{\chi_j(\lambda)(\tau_k - \lambda)}{(\bar{\alpha}_n - \lambda)\sqrt[c]{\Delta(\lambda)^2 - 4}} d\lambda. \end{aligned} \tag{II.19}$$

By Lemma II.9

$$\left| \frac{\chi_j(\lambda)}{\bar{\alpha}_n - \lambda} \right| \leq \frac{C_1}{|j\pi - \lambda|} \quad \forall \lambda \in \mathbb{R} \tag{II.20}$$

where $C_1 > 0$ can be chosen independently of $j, n \in \mathbb{Z}$. By Lemma I.17

$$|\sqrt[c]{\Delta(\lambda)^2 - 4}| \geq \sqrt[+]{(\lambda_k^+ - \lambda)(\lambda - \lambda_k^-)}/C_2 \quad \forall \lambda_k^- \leq \lambda \leq \lambda_k^+ \tag{II.21}$$

for some constant $C_2 > 0$ independent of k.

For k with $\gamma_k \neq 0$, parametrize the path of integration in (II.19) by $\lambda(t) = \tau_k + t\gamma_k/2, -1 \leq t \leq 1$, to obtain from (II.20) - (II.21) for $k, j, n \in \mathbb{Z}$ with $k \neq n$, $j \neq n$

$$\begin{aligned}
&\Big| (k-j)A_k \cdot \frac{\chi_j(\lambda)}{\bar{\alpha}_n - \lambda} \Big| \\
&\leq \frac{1}{|\tau_k - \bar{\alpha}_n|} 2 \int_{-1}^{1} \Big| \frac{(j-k)\chi_j(\lambda)}{\bar{\alpha}_n - \lambda} \Big| \frac{|t|\gamma_k/2}{|\sqrt[c]{\Delta(\lambda)^2 - 4}|} \frac{\gamma_k}{2} dt \\
&\leq \frac{\gamma_k}{|\tau_k - \bar{\alpha}_n|} \int_{-1}^{1} \Big| \frac{j-k}{j\pi - \lambda(t)} \Big| C_1 \frac{1}{\sqrt{1-t^2}} C_2 dt \ .
\end{aligned}$$

Hence in view of Lemma II.12 and (II.17) (II.18)

$$|T^j_{k,n}| \leq C\gamma_k / \langle k - n \rangle \tag{II.22}$$

where $C > 0$ is a constant independent of n, j, k and locally uniformly in α. This inequality continues to hold for k with $\gamma_k = 0$ since in this case $\lambda_k^+ = \lambda_k^- = \bar{\alpha}_k$ and thus $(k-j)A_k \cdot \frac{\chi_j(\lambda)}{\bar{\alpha}_n - \lambda} = 0$.
Therefore

$$\sum_{k,n} |T^j_{k,n}|^2 < \infty$$

i.e. T^j is Hilbert-Schmidt and hence compact. ■

Notice that by Lemma II.3 if $(\alpha, \varphi) \in \ell^2_{\mathbb{R}} \times L^2_{\mathcal{R}}$, satisfies $F^j(\alpha, \varphi) = 0$ then for any $k \in \mathbb{Z}$ the entire function $\chi_j(\alpha)$ has a root in $G_k(\varphi) := [\lambda_k^-, \lambda_k^+]$. It therefore make sense to restrict ourselves to the open domain $V \subset \ell^2_{\mathbb{R}} \times L^2_{\mathcal{R}}$ characterized by

$$\frac{\lambda^+_{k-1} + \lambda^-_k}{2} < \bar{\alpha}_k < \frac{\lambda^+_k + \lambda^-_{k+1}}{2} \ .$$

As a consequence, any solution $(\alpha, \varphi) \in V$ of $F^j(\alpha, \varphi) = 0$ leads to a monotone sequence $(\bar{\alpha}^j_k)_{k \in \mathbb{Z}}$, which in turn makes α unique.

Lemma II.14 *Let $j \in \mathbb{Z}$. For any $(\alpha, \varphi) \in V$ such that $F^j(\alpha, \varphi) = 0$ the operator $B^j \equiv B(\alpha, \varphi)$ is a linear isomorphism on ℓ^2.*

Proof As φ is of real type and α is real, the matrix elements of B^j are real. To prove that B^j is 1-1 it suffices to show that for any $\beta \in \ell^2_{\mathbb{R}}$ with $B^j \cdot \beta = 0$ one has $\beta = 0$. By the definition of B^j, $B^j \cdot \beta = 0$ implies that $\beta_j = 0$ and

$$\sum_n \beta_n (k-j) A_k \cdot (\chi_j(\lambda)/(\bar{\alpha}_n - \lambda)) = 0 \quad \forall k \in \mathbb{Z} \tag{II.23}$$

where $\chi_j(\lambda) \equiv \chi_j(\lambda, \alpha)$, $A_k \equiv A_k(\varphi)$ and $\bar{\alpha}_n \equiv n\pi + \alpha_n$.
Introduce

$$f_j(\lambda) := \sum_n \beta_n \chi_j(\lambda)/(\bar{\alpha}_n - \lambda) \ .$$

By Lemmas II.7 and II.4, $f_j \in \mathcal{I}_0$ and the identities (II.23) then read

$$A_k \cdot f_j = 0 \quad \forall k \in \mathbb{Z} \backslash \{j\}.$$

On the other hand, as $F(\alpha, \varphi) = 0$, one has

$$A_k \cdot \chi_j = 0 \quad \forall k \in \mathbb{Z} \backslash \{j\}.$$

As $\chi_j \neq 0$ one deduces by Lemma II.4 and Lemma II.3 that $A_j \cdot \chi_j \neq 0$.
Thus one can define

$$g_j(\lambda) := f_j(\lambda) - \frac{A_j \cdot f_j}{A_j \cdot \chi_j} \chi_j(\lambda)$$

and one has

$$A_k g_j = 0 \quad \forall k \in \mathbb{Z}.$$

The entire function $g_j(\lambda)$ is real valued for $\lambda \in \mathbb{R}$ and as both χ_j and f_j are in $\mathcal{I}_0$, g_j is in $\mathcal{I}_0$ as well. Hence in view of Lemma II.3 there exists for any $k \in \mathbb{Z}$ a real number $\lambda_k^- \leq \eta_k \leq \lambda_k^+$ so that $g_j(\eta_k) = 0$. By Lemma II.4 it then follows that, $g_j \equiv 0$, i.e. $f_j = c_j \ \chi_j$ with $c_j = \frac{A_j \cdot f_j}{A_j \cdot \chi_j}$ or

$$\sum_{n \neq j} \chi_j(\lambda) \frac{\beta_n}{\bar{\alpha}_n - \lambda} = c_j \chi_j(\lambda)$$

or for any $\lambda \in \mathbb{C} \setminus \{\bar{\alpha}_n \mid n \in \mathbb{Z}\}$

$$\pi \sum_{n \neq j} \beta_n / (\bar{\alpha}_n - \lambda) = c_j \ .$$

As $(\alpha, \varphi) \in V$, the zeroes $\bar{\alpha}_n$ are pairwise distinct and one concludes that $\beta_n = 0 \ \forall n \in \mathbb{Z}$. This shows that B_j is one to one.
By Lemmas II.12, II.13 and the Fredholm alternative, B_j is thus a linear isomorphism. ■

Lemma II.14 and Proposition II.11 allow to apply the implicit function theorem to any particular solution of $F^j(\alpha, \varphi) = 0$ in V. The upshot is the following result.

Proposition II.15 *For any $j \in \mathbb{Z}$ there exists a unique real analytic map*

$$\alpha^j : L^2_{\mathcal{R}} \to \ell^2_{\mathbb{R}}$$

with graph in V such that

$$F^j(\alpha^j(\varphi), \varphi) = 0 \ .$$

Further, for any $k \in \mathbb{Z}$ and any $\varphi \in L^2_{\mathcal{R}}$, $\bar{\alpha}^j_k(\varphi) \in G_k(\varphi)$.

Remark To be precise, uniqueness holds within the class of all such real analytic maps with graph in V.
Proof Let $j \in \mathbb{Z}$ and define $\mathcal{E} \equiv \mathcal{E}^j$ by

$$\mathcal{E} := \{\varphi \in L^2_{\mathcal{R}} \mid \exists \alpha \in \ell^2_{\mathbb{R}} \text{ such that } (\alpha, \varphi) \in V \text{ and } F^j(\alpha, \varphi) = 0\} \ .$$

Note that the zero potential is in $\mathcal{E}$ (cf Proposition II.2) and $L^2_{\mathcal{R}}$ is connected. To prove the existence of the map α^j it thus suffices to show that $\mathcal{E}$ is open and closed in $L^2_{\mathcal{R}}$.
Applying the implicit function theorem at any solution $(\alpha, \varphi) \in V$ of $F^j(\alpha, \varphi) = 0$ we conclude from Lemma II.14 that $\mathcal{E}$ is open. Further we claim that for any solution $(\alpha, \varphi) \in V$ of $F^j(\alpha, \varphi) = 0$ one has

$$\bar{\alpha}_k \in G_k(\varphi) \quad \forall k \in \mathbb{Z} \ .$$

For $k = j$, this holds by definition. For $k \neq j$, the fact that $A_k \cdot \chi_j(\alpha) = 0$ together with Lemma II.3 imply that χ_j has at least one root in G_k. As $(\alpha, \varphi) \in V$, it follows that $\bar{\alpha}_k \in G_k \ \forall k \in \mathbb{Z}$.
This claim allows to prove that $\mathcal{E}$ is closed. Let $(\varphi_n)_{n \geq 1}$ be a sequence in $\mathcal{E}$ converging to φ_0 in $L^2_{\mathcal{R}}$. For each $n \geq 1$ let $\alpha(n)$ be an element of $\ell^2_{\mathbb{R}}$ such that $F^j(\alpha(n), \varphi_n) = 0$. Then $\bar{\alpha}_n := \alpha_k(n) + k\pi$ is in $G_k(\varphi_n)$. As the periodic eigenvalues are locally bounded, $\cup_{n \geq 1} G_k(\varphi_n)$ is a bounded set of $\mathbb{R}$ for each $k \in \mathbb{Z}$. Therefore, there exists a subsequence, again denoted by $(\varphi_n)_{n \geq 1}$ such that $\alpha_k(n) \to \alpha_k$ for all $k \in \mathbb{Z}$. It follows that $\bar{\alpha}_k := \alpha_k + k\pi$ is in G_k for any $k \in \mathbb{Z}$ and in particular $(\alpha, \varphi) \in V$ and $F^j(\alpha, \varphi_0) = 0$. This shows that $\mathcal{E}$ is closed.
To prove tha claimed uniqueness, assume that $F^j(\alpha, \varphi) = F^j(\beta, \varphi) = 0$ for some (α, φ) and (β, φ) in V. Then for all $k \neq j$, $A_k \cdot \chi_j(\alpha) = A_k \cdot \chi_j(\beta) = 0$ and by Lemma II.4, $A_j \cdot \chi_j(\alpha) \neq 0$ and $A_j \cdot \chi_j(\beta) \neq 0$. Thus the entire function $g := \chi_j(\alpha) - \frac{A_j \cdot \chi_j(\alpha)}{A_j \cdot \chi_j(\beta)} \chi_j(\beta)$ satisfies $A_k \cdot g = 0$ for all $k \in \mathbb{Z}$. By Lemma II.7 g is in $\mathcal{I}_0$ and we conclude from Lemma II.3 and Lemma II.4 that $g \equiv 0$. As (α, φ) and (β, φ) are in V, it then follows that $\alpha = \beta$.
As a consequence the map $\alpha^j : L^2_{\mathcal{R}} \to \ell^2_{\mathbb{R}}$ is well defined and has is graph in V. By the implicit function theorem α^j is analytic on a complex neighbourhood of $L^2_{\mathcal{R}}$. ■

Lemma II.16 *For any φ in $L^2_{\mathcal{R}}$ and $k \in \mathbb{Z}$,*

$$\bar{\alpha}^j_k(\varphi) \to \dot{\lambda}_k(\varphi) \quad \text{as } j \to \pm\infty$$

where $\dot{\lambda}_k(\varphi)$ are the roots of $\dot{\Delta}(\cdot, \varphi) = \frac{d}{d\lambda}\Delta(\cdot, \varphi)$.

Proof We focus on the limit $j \to +\infty$ as the limit $j \to -\infty$ is calculated in the same way. For each k the sequence $(\bar{\alpha}^j_k)_{j \in \mathbb{N}}$ lies in G_k. As G_k is compact,

there exists a subsequence, again denoted by $(\bar{\alpha}^j_k)_{j \in \mathbb{N}}$ which converges to some element $\bar{\alpha}^\infty_k \in G_k$. Let $\alpha^\infty = (\alpha^\infty_k)_{k \in \mathbb{Z}}$ where $\alpha^\infty_k := \bar{\alpha}^\infty_k - k\pi$.
For each sequence $\beta \in \ell^2$, introduce in addition to the functions χ_j the entire function

$$\chi(\lambda, \beta) := (\beta_0 - \lambda) \prod_{k \neq 0} \frac{\beta_k + k\pi - \lambda}{k\pi} \ .$$

Notice that uniformly on the contours Γ_k one has as $j \to +\infty$,

$$(k - j)\chi_j(\lambda, \alpha^j) = \frac{k - j}{j\pi - \lambda} \chi(\lambda, \alpha^j) \to -\pi \chi(\lambda, \alpha^\infty) \qquad \text{(II.24)}$$

and hence

$$A_k \cdot \chi(\cdot, \alpha^\infty) = 0 \quad \text{ for all } k \in \mathbb{Z} \ .$$

On the other hand, by Lemma I.20, $\dot{\Delta}(\cdot) = 2\chi(\lambda, \beta)$ with $\beta = (\dot{\lambda}_k - k\pi)_{k \in \mathbb{Z}}$ and as $\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} d\lambda$ is an exact differential on Γ_k, one has

$$A_k \cdot \chi(\cdot, \beta) = 0 \quad \text{ for all } k \in \mathbb{Z} \ .$$

Let $f := \chi(\beta) - \chi(\alpha^\infty)$. By Lemma II.3 there exists $\xi \equiv (\xi_k)_{k \in \mathbb{Z}} \in \ell^2$ such that $f(\xi_k + k\pi) = 0$ for all $k \in \mathbb{Z}$. Therefore the function $\lambda \mapsto \frac{f(\lambda)}{\chi(\lambda, \xi)}$ is entire.
In view of Lemma I.17

$$\chi(\lambda, \beta) = \sin\lambda(1 + o(1)), \ \ \chi(\lambda, \alpha^\infty) = \sin\lambda(1 + o(1)), \ \ \chi(\lambda, \xi) = \sin\lambda(1 + o(1))$$

uniformly on $\{(n + 1/4)\pi \leq |\lambda| \leq (n + 3/4)\pi\}$ and hence as $n \to +\infty$

$$\frac{f(\lambda)}{\chi(\lambda, \xi)} = o(1)$$

uniformly on $\{(n + 1/4)\pi \leq |\lambda| \leq (n + 3/4)\pi\}$. Hence by the maximum principle, $f \equiv 0$, i.e. $\alpha^\infty = \beta$. ■

Each map α^j extends to a complex neighbourhood of $L^2_{\mathcal{R}}$ which might depend on j. The following proposition asserts that the complex extensions can be done uniformly in $j \in \mathbb{Z}$.

Proposition II.17 *The real analytic maps $\alpha^j : L^2_{\mathcal{R}} \to \ell^2_{\mathbb{R}}$ $(j \in \mathbb{Z})$ of Proposition II.15 extend to a common complex neighbourhood U of $L^2_{\mathcal{R}}$.*

Proof By the implicit function theorem it suffices to show that the inverses of the Jacobians

$$B^j(\alpha^j(\varphi), \varphi) = \frac{\partial F^j}{\partial \alpha}(\alpha^j(\varphi), \varphi)$$

are bounded uniformly in j and locally uniformly in $\varphi \in L^2_{\mathcal{R}}$.
We begin by proving that at *each* point φ_0 of $L^2_{\mathcal{R}}$

$$\|(B^j(\alpha^j(\varphi_0), \varphi_0))^{-1}\|_{\mathcal{L}(\ell^2)} < \infty \tag{II.25}$$

uniformly for all $j \in \mathbb{Z}$. By Lemma II.16 and in particular (II.24), one has

$$(k-j)\chi_j(\lambda, \alpha^j) \to -\frac{1}{2\pi}\dot{\Delta}(\lambda) \quad \text{for } j \to \pm\infty$$

and hence for $b^j_{k,n} \equiv (B^j(\alpha^j(\varphi_0), \varphi_0))_{k,n}$ given in (II.12)

$$b^j_{k,n} \to A_k \cdot \frac{-1}{2\pi}\frac{\dot{\Delta}(\lambda)}{\dot{\lambda}_n - \lambda} =: b^\infty_{k,n} \text{ as } j \to \pm\infty$$

uniformly for k and n in $\mathbb{Z}$. Let $B^\infty := (b^\infty_{k,n})_{k,n\in\mathbb{Z}}$ and denote by D^∞ the diagonal part of B^∞. The sequence $(\alpha^j)_{j\in\mathbb{Z}}$ is relatively compact in ℓ^2. Hence by Lemma II.12, $D^j \to D^\infty$ in operator norm and D^∞ is boundedly invertible. Write $B^\infty = D^\infty(Id+T^\infty)$ with $T^\infty := (D^\infty)^{-1}(B^\infty - D^\infty)$. By (II.22) and the compacity of $(\alpha^j)_{j\in\mathbb{Z}}$, one sees that

$$|T^j_{k,n}(\alpha^j(\varphi_0), \varphi_0)| \leq C\gamma_k/\langle k-n\rangle$$

Hence the same estimate holds for T^∞ and thus $T^j \to T^\infty$ in operator norm and T^∞ is compact.
We claim that B^∞ is one to one. Assume that $B^\infty \cdot \beta = 0$ for some element $\beta \in \ell^2_{\mathbb{R}}$. Following the proof of Lemma II.14 one gets $A_k \cdot f = 0$ for all $k \in \mathbb{Z}$ where $f := \sum_n \beta_n \frac{\dot{\Delta}(\lambda)}{\dot{\lambda}_n - \lambda} \in \mathcal{I}_0$ and w proves that $f \equiv 0$, i.e. $\beta = 0$.
In view of the Fredholm alternative, B^∞ is boundedly invertible. As $B^j \to B^\infty$ in operator norm and for each j the linear operator $B^j(\alpha^j(\varphi_0), \varphi_0)$ is boundedly invertible, one concludes that (II.25) holds uniformly for $j \in \mathbb{Z}$ i.e.

$$\sup_{j\in\mathbb{Z}} \|B^j(\alpha^j(\varphi_0), \varphi_0)^{-1}\| < \infty \ .$$

By Propostion II.11, the maps F^j are analytic and locally uniformly bounded on W uniformly in j. Thus by Cauchy's estimates (II.25) remains valid uniformly on a complex neighbourhood of each φ_0 in $L^2_{\mathcal{R}}$ and uniformly for $j \in \mathbb{Z}$. ■

II.5 Normalization

Consider the entire functions $\tilde{\psi}_j := \chi_j(\alpha^j)$. By construction, $A_k \cdot \tilde{\psi}_j = 0$ for all $k \neq j$ and by Lemma II.3 $A_j \cdot \tilde{\psi}_j \neq 0$. Hence we can normalize $\tilde{\psi}_j$

$$\psi_j := \frac{1}{A_j \cdot \tilde{\psi}_j}\ \tilde{\psi}_j$$

which then sastisfy (II.1). It turns out that the constants $A_j \cdot \tilde{\psi}_j$ can be explicitely computed.

Proposition II.18 *For any $j \in \mathbb{Z}$ and any $\varphi \in L^2_{\mathcal{R}}$,*

$$A_j \cdot \chi_j(\alpha^j) = -\pi \ .$$

Proof Let $\tilde{\psi}_j := \chi_j(\alpha^j)$. By Lemma I.16 one has as $n \to +\infty$

$$\tilde{\psi}_j(\lambda) = \frac{-1}{j\pi - \lambda}\sin\lambda\ (1+o(1))$$

and

$$\Delta(\lambda)^2 - 4 = -4\sin^2\lambda\ (1+o(1))$$

uniformly on $\{(n+\frac{1}{4})\pi \leq |\lambda| \leq (n+\frac{3}{4})\pi\}$. As in view of (I.49), $\sqrt[c]{\Delta(\lambda)^2-4} = -2i\sin\lambda(1+o(1))$, one then has for $n \to \infty$

$$\frac{\tilde{\psi}_j(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}} = \frac{-1}{2i\lambda}\ (1+o(1)) \tag{II.26}$$

uniformly on $\{(n+\frac{1}{4})\pi \leq |\lambda| \leq (n+\frac{3}{4})\pi\}$.
Due to the asymptotics of the eigenvalues $\lambda^\pm_k$ one can choose $N \geq 1$ so large that $|\lambda^\pm_n - n\pi| < \pi/2$ for $|n| \geq N$. By Cauchy's theorem it follows that for $n > \max(N, j)$,

$$\int_{C(0,n\pi+\pi/2)} \frac{\tilde{\psi}_j(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}d\lambda = A_j \cdot \tilde{\psi}_j \tag{II.27}$$

where $C(0, n\pi+\pi/2)$ denotes the counterclockwise oriented circle of radius $n\pi + \pi/2$ centered at 0. Comparing (II.26) and (II.27) one concludes that $A_j \cdot \tilde{\psi}_j = -\pi$. ■

Remark One can explain this computation in aa easier way for finite gap potentails. It turns out that $\eta_j := \frac{\tilde{\psi}_j}{\sqrt{\Delta^2-4}}d\lambda$ is an Abelian differential on the Riemann surface Σ, given by $y^2 = \Delta(\lambda)^2 - 4$. These Abelian differentials are holomorphic except at the two points at infinity where they have a simple pole. Let a_k be the cycle on the canonical sheet Σ^c of Σ determined by the canonical root $\sqrt[c]{\Delta(\lambda)^2-4}$ which correspond to the contours Γ_k on $\mathbb{C}$. Then by Cauchy's theorem, the conditions $\int_{\Gamma_k}\eta_j = 0$ for all $k \neq j$ lead to a relation between $\int_{\Gamma_j}\eta_j$ and the residue of the pole of η_j at infinity on Σ^c.

II.6 Estimates for the zeroes

In this section we prove the refined estimates for the zeroes $(\dot{\alpha}^j_k)_{k\in\mathbb{Z}\setminus\{j\}}$ of $\chi_j(\alpha^j)$ as stated in Theorem II.1.
Let U be a neighbourhood of $L^2_{\mathcal{R}}$ as in Proposition PropositionII.23.

Proposition II.19 *For any $\varphi \in U$,*

$$\sup_{j \neq k} |\nu_k^j(\varphi) - \tau_k(\varphi)| \leq C|\gamma_k(\varphi)|^2 a_k \tag{II.28}$$

where $(a_k)_{k\in\mathbb{Z}} \in \ell^2$ can be chosen locally independently of φ.

Proof Denote by $\Gamma_k' \equiv \Gamma_k'(\varphi)$ the counterclockwise oriented circle with center τ_j and radius $\left|\frac{\gamma_j}{2}\right| + \min\left(K, \frac{1}{j}\right)$. By Lemma I.17 one has uniformly for $\lambda \in \Gamma_k'$ and $k \neq j$

$$\chi_j\left(\lambda, \alpha^j\right) = \frac{\bar{\alpha}_k^j - \lambda}{\tau_j - \lambda}\left(1 + \ell^2(k)\right)$$

and

$$\sqrt[c]{\Delta(\lambda)^2 - 4} = 2i\sqrt[s]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}\left(1 + \ell^2(k)\right)$$

where the error terms are uniform with respect to j and $\lambda \in \Gamma_k'$ and, locally with respect to $\varphi \in U$. Therefore

$$\frac{\chi_j\left(\lambda, \alpha^j\right)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} = \frac{\bar{\alpha}_k^j - \lambda}{2i(\tau_j - \lambda)\sqrt[s]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} f_{jk}(\lambda)$$

where uniformly for $\lambda \in \Gamma_k'$ and $k \neq j$

$$f_{jk}(\lambda) = 1 + \ell^2(k) \ .$$

As $A_k \cdot \chi_j\left(\alpha^j\right) = 0$ for $k \neq j$, it follows that

$$\int_{\lambda_k^-}^{\lambda_k^+} \frac{\bar{\alpha}_k^j - \lambda}{(\tau_j - \lambda)\sqrt[s]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} f_{jk}(\lambda) d\lambda = 0.$$

Using an integration variable $\lambda(t) = \tau_k + t\frac{\gamma_k}{2}$ and writing $\bar{\alpha}_k^j = \tau_k + \xi_k^j$ one gets

$$\int_{-1}^{1} \frac{\xi_k^j - t\ \gamma_k/2}{(\tau_j - \tau_k - t\gamma_k/2)\sqrt[+]{1 - t^2}} f_{jk}(\lambda(t)) dt = 0.$$

Denoting

$$A_{jk}(\lambda) := \frac{\tau_j - \tau_k}{\tau_j - \lambda} f_{jk}(\lambda)$$

the last equation can be written as

$$\xi_k^{(j)} \int_{-1}^{1} \frac{A_{jk}(\lambda(t))}{\sqrt[+]{1 - t^2}} dt = \frac{\gamma_k}{2} \int_{-1}^{1} t \frac{A_{jk}(\lambda(t))}{\sqrt[+]{1 - t^2}} dt \ .$$

Making the change of variable of integration $t \to -t$ and adding the expression to the one above one gets

$$\begin{aligned} \xi_k^{(j)} \int_{-1}^{1} \frac{dt}{\sqrt[+]{1 - t^2}} \left(A_{jk}(\lambda(t)) + A_{jk}(\lambda(-t))\right) &= \\ \frac{\gamma_k}{2} \int_{-1}^{1} \frac{t dt}{\sqrt[+]{1 - t^2}} \left(A_{jk}(\lambda(t)) - A_{jk}(\lambda(-t))\right) . \end{aligned} \tag{II.29}$$

Observe that uniformly for $t \in (-1, 1)$

$$A_{jk}(\lambda(t)) = 1 + \ell^2(k) \ . \tag{II.30}$$

To estimate the difference $A_{jk}(\lambda(t)) - A_{jk}(\lambda(-t))$, we consider k large enough (locally uniformly in φ) such that $|\lambda_k^+ - k\pi| < \pi/4$ and $|\lambda_k^- - k\pi| < \pi/4$ (cf. Proposition I.5). We denote by D_k the circle of center $k\pi$ and radius $\pi/2$. Express $A_{jk}(\lambda) - 1$ for λ in the interior of D_k by the Cauchy formula

$$A_{jk}(\lambda) - 1 = \frac{1}{2i\pi} \int_{D_k} \frac{A_{jk}(z) - 1}{z - \lambda} dz$$

to get

$$A_{jk}(\lambda(t)) - A_{jk}(\lambda(-t)) = \frac{1}{2i\pi} \int_{D_k} (A_{jk}(z) - 1) \frac{\lambda(t) - \lambda(-t)}{(z - \lambda(t))(z - \lambda(-t))} dz \ .$$

Since $\lambda(t) - \lambda(-t) = t\gamma_k$, one then obtains taking into account (II.30) and the size of D_k,

$$A_{jk}(\lambda(t)) - A_{jk}(\lambda(-t)) = \gamma_k \ell^2(k) \ .$$

Substituting this expression into (II.29), we get

$$\xi_k^{(j)} \int_{-1}^{1} \frac{dt}{\sqrt[+]{1 - t^2}} \left(A_{jk}(\lambda(t)) + A_{jk}(\lambda(-t))\right) = |\gamma_k|^2 \ell^2(k) \ .$$

Using again (II.30) we conclude $|\xi_k^j| \leq |\gamma_k|^2 \ell^2(k)$ as claimed. ■

Chapter III

Birkhoff coordinates

III.1 Introduction

In this chapter we prove that NLS can be brought into normal form as stated in Theorem 0.1.

For a finite dimensional integrable system there is a well known procedure to define actions and angles. Actually Flaschka and McLaughlin, extending this procedure, defined in ([FM]) actions for the KdV equation. Angular variables which linearize the KdV flow were introduced by a number of authors (Dubrovin, Its, Krichever, Matveev, Novikov [D], [DKN1], [DKN2], [DMN], [DN], [IM], McKean and van Moerbecke [MM] and McKean and Trubowitz [MT1], [MT2]). In [BBEIM] and [MV] a similar algebro-geometric construction is carried out for the NLS equation.

The proof of Theroem 0.1 uses the definitions of actions I_k and angular variables θ_k given by the algebro-geometric approach as an ansatz. We then prove that the associated Birkhoff coordinates

$$x_k = \sqrt{2I_k}\cos\theta_k,\ \ y_k = \sqrt{2I_k}\sin\theta_k$$

can be extended to real analytic functions defined on all of $L^2_{\mathcal{R}}$, are canonical and, thus, give rise to a global canonical coordinate system.

Action-angle construction

Before going on into the details of the proof we would like to review the formal construction of actions and angles. To this end we recall some notation of chapter I and introduce some more concepts.

Consider the Floquet matrix

$$M(1,\lambda) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} (1,\lambda)$$

associated with the equation $L(\varphi)F = \lambda F$ and its discriminant $\Delta(\lambda) = TrM(1,\lambda)$. The periodic spectrum of φ is given by the zeroes of the entire function $\Delta^2(\lambda;\varphi) - 4$ (with multiplicities) and we have the product representation (cf section I.6)

$$\Delta^2(\lambda) - 4 = -4(\lambda_0^- - \lambda)(\lambda_0^+ - \lambda)\prod_{k\neq 0} \frac{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}{k^2\pi^2}\ .$$

Hence this function is a spectral invariant. The square root of $\Delta^2(\lambda) - 4$ is defined on the hyperelliptic Riemann surface

$$\Sigma_\varphi = \left\{(\lambda, y) \in \mathbb{C}^2 \mid y^2 = \Delta^2(\lambda) - 4\right\}$$

whose genus is precisely the number of open gaps of φ minus 1. It may be viewed as two copies of the complex plane slit open along each open gap and

then glued together crosswise along the slits. Clearly, the Riemann surface is a spectral invariant associated with φ.
To define actions and angles we also need to consider the Dirichlet spectrum $(\mu_k(\varphi))_{k\in\mathbb{Z}}$ which satisfies (cf sections I.3 and I.4)

$$\ldots < \mu_{-1}(\varphi) < \mu_0(\varphi) < \mu_1(\varphi) < \ldots$$

and

$$\lambda_k^-(\varphi) \le \mu_k(\varphi) \le \lambda_k^+(\varphi), \quad k \in \mathbb{Z}.$$

Using the Wronskian identity one proves (cf (I.68))

$$\Delta^2(\mu_k) - 4 = (M_{21} + M_{12})^2 \big|_{1,\mu_k}.$$

Therefore with any Dirichlet eigenvalue μ_k one can uniquely and analytically associate a sign of the root $\sqrt{\Delta^2(\mu_k) - 4}$ by defining (cf (I.68))

$$\sqrt[*]{\Delta^2(\mu_k) - 4} = (M_{21} + M_{12}) \big|_{1,\mu_k}.$$

This in turn defines the Dirichlet divisor

$$\mu_k^* = \left(\mu_k, \sqrt[*]{\Delta^2(\mu_n) - 4}\right)$$

on the Riemann surface Σ_φ.
The Dirichlet eigenvalues can be complemented to a symplectic coordinate system on $L^2_{\mathcal{R}}$ by introducing the quantities (cf [GG])

$$\mathcal{K}_k(\varphi) = 2 \log \frac{(-1)^k}{2} \ (M_{11} + M_{12} + M_{21} + M_{22}) \big|_{1,\mu_k}.$$

Then

$$\varphi \mapsto (\hat{\mu}_k(\varphi), \ \mathcal{K}_k(\varphi))_{k\in\mathbb{Z}}$$

where $\hat{\mu}_k = \mu_k - k\pi$, defines a real analytic diffeomorphism from $L^2_{\mathcal{R}}$ into a subset of a suitable Hilbert space of sequences; furthermore

$$\begin{aligned} \{\mu_k, \mu_\ell\} &= 0 \\ \{\mathcal{K}_k, \mu_\ell\} &= \delta_{k\ell} \\ \{\mathcal{K}_k, \mathcal{K}_\ell\} &= 0 \end{aligned}$$

for all $k, \ell \in \mathbb{Z}$ (cf [GG] and also [G]). Hence the new variables are canonical and the induced symplectic 1-form is given by

$$\alpha = \sum_{j\in\mathbb{Z}} \mathcal{K}_j d\mu_j.$$

We may now define actions by Arnold's formula,

$$I_k = \frac{1}{2\pi} \int_{c_k} \alpha = \frac{1}{2\pi} \sum_{j\in\mathbb{Z}} \int_{c_k} \mathcal{K}_j d\mu_j \ ,$$

Figure III.1: a-cycles

where c_k is a cycle on the invariant torus $\mathrm{Iso}(\varphi)$ corresponding to μ_k. As $d\mu_j = 0$ along c_k for $j \ne k$,

$$I_k = \frac{1}{2\pi} \int_{c_k} \mathcal{K}_k d\mu_k = \frac{1}{\pi} \int_{c_k} \mu_k \ \frac{(\dot{M}_{11} + \dot{M}_{12} + \dot{M}_{21} + \dot{M}_{22})}{(M_{11} + M_{12} + M_{21} + M_{22})} \Big|_{1,\mu_k} \ d\mu_k$$

by partial integration.
Noticing that

$$(M_{11} + M_{12} + M_{21} + M_{22}) \big|_{1,\mu_k} = \Delta(\mu_k) + \sqrt[*]{\Delta^2(\mu_k) - 4}$$

a short calculation gives

$$I_k = \frac{1}{\pi} \int_{c_k} \mu \frac{\dot{\Delta}(u)}{\sqrt{\Delta^2(\mu) - 4}} d\mu.$$

In particular the actions only depend on the periodic spectrum. Finally by analytic continuation the latter integral may be interpreted as a contour integral on Σ_φ, with contour given by the cycle a_k on the canonical sheet Σ_φ^c around the lift of $[\lambda_k^-, \lambda_k^+]$ as indicated in figure III.1. This formula was first established in [FM] for KdV and in [MV] for NLS.
Assume that the actions I_k admit canonically conjugate angles θ_k . Then the 1-form α reads

$$\alpha = \sum_{j\in\mathbb{Z}} I_j d\theta_j + dS$$

where dS is some exact 1-form. A priori there is no reason for the 1-form dS to be identically zero. But it turns out that with the choice $dS = 0$ the corresponding angles give rise to the canonical relations $\{I_k, \theta_\ell\} = \delta_{k\ell}$ (see section III.6).
Assuming that $\alpha = \sum_{j\in\mathbb{Z}} I_j d\theta_j$, we obtain, at least formally

$$d\theta_k = \frac{\partial \alpha}{\partial I_k} := \alpha_k.$$

Integrating along any path on $\mathrm{Iso}(\varphi)$ from some fixed point φ_0 we then get

$$\theta_k = \int_{\varphi_0}^{\varphi} \alpha_k.$$

This integral is independent of the path chosen since $d\alpha = 0$ on $\mathrm{Iso}(\varphi)$.
Recall from [G] (see also Proposition I.40) that the isospectral torus can be parametrized by $\left(\mu_j, \mathrm{sign} \sqrt[*]{\Delta^2(\mu_j) - 4}\right)_{j\in\mathbb{Z}}$. We then take φ_0 to be the unique element of $\mathrm{Iso}(\varphi)$ with

$$\mu_j(\varphi_0) = \lambda_j^-(\varphi_0) \quad \forall j \in \mathbb{Z}$$

and choose the path from φ_0 to φ obtain by moving successively μ_j from λ_j^- to $\mu_j^*(\varphi)$ for $j = 0, 1, -1, 2, \ldots$. This way we obtain

$$\theta_k = \sum_{j\in\mathbb{Z}} \int_{\lambda_j^-}^{\mu_j^*} \alpha_k.$$

It remains to identify the 1-forms α_k. From $\alpha = \sum_{j\in\mathbb{Z}} I_j d\theta_j$ we have

$$I_j = \frac{1}{2\pi}\int_{c_j} \alpha = \frac{1}{2\pi}\int_{a_j} \alpha,$$

hence

$$\frac{1}{2\pi}\int_{a_j} \alpha_k = \frac{\partial I_j}{\partial I_k} = \delta_{jk}$$

for all $j, k \in \mathbb{Z}$. But these properties uniquely characterize a holomorphic 1-form on $\Sigma_\varphi \setminus \{\infty\pm\}$. Actually α_k coincides with the 1-form constructed in chapter II, i.e. (cf Theorem II.II.1)

$$\alpha_k = \frac{\psi_k(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}}\, d\lambda\,.$$

Then we define

$$\theta_k = \sum_{j\in\mathbb{Z}} \int_{\lambda_j^-}^{\mu_j^*} \frac{\psi_k(\lambda)}{\sqrt{\Delta^2(\lambda)-4}} d\lambda \mod 2\pi$$

for each open gap $(\lambda_k^-, \lambda_k^+)$. By a slight abuse of terminology, we may refer to the map $\varphi \mapsto \theta(\varphi) \equiv (\theta_k(\varphi))_{k\in\mathbb{Z}}$ as the Abel map.

Birkhoff coordinates

The actions I_k are real analytic on $L^2_{\mathcal{R}}$ (see section III.2) and each angle θ_k is real analytic modulo 2π on the dense open domain $L^2_{\mathcal{R}} \backslash D_k$ with

$$D_k = \{\varphi \in L^2_{\mathcal{R}} \mid \gamma_k(\varphi) = 0\}$$

(see section III.3). In section III.4 we show that the associated Birkhoff coordinates ($k \in \mathbb{Z}$)

$$x_k = \sqrt{2I_k}\cos\theta_k, \quad y_k = \sqrt{2I_k}\sin\theta_k$$

extend real analytically to a complex neighborhood W of $L^2_{\mathcal{R}}$.
To extend x_k and y_k to all of $L^2_{\mathcal{R}}$ we prove that the blow up of θ_k when γ_k collapses is compensated by the rate at which I_k vanishes in the process. For complex potential, i.e. $\varphi \in W\backslash L^2_{\mathcal{R}}$, the situation is more complicated since the associated Zakharov-Shabat operator is no more selfadjoint. In particular, it may happen that $\lambda_k^- = \lambda_k^+$ but $\mu_k \neq \lambda_k^-$. In such a case, although I_k vanishes, x_k and y_k will not vanish.
The canonical relations $\{x_k, x_\ell\} = \{y_k, y_\ell\} = 0$ and $\{x_k, y_\ell\} = \delta_{k\ell}, k, \ell \in \mathbb{Z}$, are proved in section III.6. In section III.7 we use these canonical relations (cf [KM]) to show that the map

$$\Omega \;:\; L^2_{\mathcal{R}} \ni \varphi \mapsto (x_k, y_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{R}^2)$$

is a local diffeomorphism at every point $\varphi \in L^2_{\mathcal{R}}$. Finally in section III.8 we prove, using the property of the action map, that Ω is a global diffeomorphism and hence a canonical transformation.

III.2 Actions

In this section we define the action variables I_n as introduced by McKean-Vaninsky [MV] (cf also [FM]) and prove their analyticity as well as asymptotic estimates.
Choose a connected neighborhood W of $L^2_{\mathcal{R}} := \{(\varphi_1, \overline{\varphi_1}) \mid \varphi_1 \in L^2_{\mathbb{C}}\}$ in $L^2_{\mathbb{C}} \times L^2_{\mathbb{C}}$, as given by Lemma I.12. For $\varphi \in W$ and $n \in \mathbb{Z}$, one then has $Re\lambda_n^+ < Re\lambda_{n+1}^-$. In particular,

$$[\lambda_n^-, \lambda_n^+] := \{(1-t)\lambda_n^- + t\lambda_n^+ \mid 0 \le t \le 1\} \subseteq \mathbb{C}$$

are pairwise disjoint intervals. They admit mutually disjoint discs $Disc_n \subseteq \mathbb{C}$ which can be chosen locally independently of φ.

Definition III.1 *The neighborhoods $Disc_n$ are called isolating neighborhoods for the intervals $[\lambda_n^-, \lambda_n^+]$.*

Arguing as in [FM] (cf [MV]) one defines the action variables,

$$I_n := \frac{1}{\pi}\int_{\Gamma_n} \lambda \frac{\dot\Delta(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}}\, d\lambda \quad (n \in \mathbb{Z})$$

where Γ_n is a circuit around $[\lambda_n^-, \lambda_n^+]$ inside $Disc_n$ with counterclockwise orientation, and the dot in $\dot\Delta(\lambda)$ denotes differentiation with respect to λ. The root $\sqrt[c]{\Delta(\lambda)^2 - 4}$ is defined in section I. I.7 . By Cauchy's theorem the definition of I_n does not depend on the choice of Γ_n as long as it stays inside $Disc_n$. In particular, Γ_n can be chosen to be locally independent of φ.

Theorem III.2 *For any $n \in \mathbb{Z}$, the function I_n is analytic on W with L^2-gradient*

$$\nabla_{\varphi(x)} I_n = -\frac{1}{\pi}\int_{\Gamma_n} \nabla_{\varphi(x)}\Delta(\lambda)\frac{1}{\sqrt[c]{\Delta(\lambda)^2-4}}\, d\lambda. \qquad \text{(III.1)}$$

Moreover, for $\varphi \in L^2_{\mathcal{R}}$, each function I_n is real, nonnegative and vanishes iff $\lambda_n^+ = \lambda_n^-$.

Proof Locally on W the contours of integration Γ_n can be chosen independently of φ. As Δ is an analytic function of λ and φ and $\sqrt[c]{\Delta(\lambda)^2-4}$ is analytic in a neighborhood of Γ, the function I_n is clearly analytic on W.
To obtain its gradient we observe that for $\varphi \in V, (-1)^k\Delta(\lambda) \geq 2$ and hence $(-1)^n\Delta(\lambda) \pm \sqrt{\Delta(\lambda)^2-4} > 0$ on the interval $[\lambda_n^-, \lambda_n^+]$. Therefore, on a sufficiently small neighborhood $W_n \subseteq W$ of $L^2_{\mathcal{R}}$ and a circuit Γ_n sufficiently close to $[\lambda_n^-, \lambda_n^+]$, the principle branch of the logarithm

$$h(\lambda) = log(-1)^n\left(\Delta(\lambda) - \sqrt[c]{\Delta(\lambda)^2-4}\right)$$

is well defined along Γ_n. Since $\dot{h}(\lambda) = -\dot{\Delta}(h)/\sqrt[c]{\Delta(\lambda)^2-4}$, partial integration gives

$$I_n = \frac{1}{\pi}\int_{\Gamma_n} log(-1)^n\left(\Delta(\lambda) - \sqrt[c]{\Delta(\lambda)^2-4}\right)d\lambda.$$

Again keeping Γ_n fixed and taking the gradient with respect to $\varphi \in W_n$ one obtains the above formula for $\frac{\partial I_n}{\partial \varphi_j}$ $(1 \leq j \leq 2)$ on W_n. As both sides of (III.1) are analytic on W and W is assumed to be connected, (III.1) holds on all of W.
To prove the last statement of the theorem we observe that

$$\int_{\Gamma_n} \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}\,d\lambda = 0 \qquad \text{(III.2)}$$

in view of the existence of a primitive. With $\dot{\lambda}_n$ denoting the root of $\dot{\Delta}$ near λ_n^- and λ_n^+ we can therefore also write

$$I_n = \frac{1}{\pi}\int_{\Gamma_n}(\lambda - \dot{\lambda}_n)\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}\,d\lambda. \qquad \text{(III.3)}$$

For $\varphi \in L^2_{\mathcal{R}}$, we then obtain

$$I_n = \frac{2}{\pi}\int_{\lambda_n^-}^{\lambda_n^+}(-1)^{n-1}(\lambda - \dot{\lambda}_n)\frac{\dot{\Delta}(\lambda)}{\sqrt[+]{\Delta(\lambda)^2-4}}\,d\lambda$$

by shrinking the contour of integration to the real interval $[\lambda_n^-, \lambda_n^+]$ and taking into account the definition of the canonical root $\sqrt[c]{\Delta(\lambda)^2-4}$. Since $sign\left((\lambda - \dot{\lambda}_n)\dot{\Delta}(\lambda)\right) = (-1)^{n-1}$ on $[\lambda_n^-, \lambda_n^+]$, the integrand is nonnegative and the result follows. ■

Let

$$D_n := \{\varphi \in W \mid \gamma_n = 0\}$$

be the subvariety of potentials in W with collapsed n'th gap. Here γ_n denotes the gap length $\gamma_n := \lambda_n^+ - \lambda_n^- \in \mathbb{C}$. As I_n and γ_n^2 are both analytic on W, their quotient is analytic on $W \backslash D_n$. We show that I_n/γ_n^2 extends analytically to all of W to a nonvanishing function.

Theorem III.3 *The quotient I_n/γ_n^2 extends analytically to W and satisfies*

$$4\frac{I_n}{\gamma_n^2} = 1 + \ell^2(n) \quad (n \in \mathbb{Z})$$

locally uniformly on W. Moreover, the real part of $\left(4\frac{I_n}{\gamma_n^2}\right)$ is locally uniformly bounded away from zero in a sufficiently small neighborhood $W' \subseteq W$ of $L^2_{\mathcal{R}}$. As a consequence

$$\xi_n := \sqrt[+]{4I_n/\gamma_n^2}$$

is a real analytic, nonvanishing function on W' with

$$\xi_n = 1 + \ell^2(n)$$

locally uniformly on W'.

Remark In the sequel, we will assume that the neighborhood W has been chosen so that W' can be chosen to be W.

Proof We show that I_n/γ_n^2 extends continuously to all of W and is weakly analytic when restricted to D_n. By standard arguments it then follows that I_n/γ_n^2 is analytic on all of W. Recall the product expansion (cf Lemma I.19 and Lemma I.20),

$$\Delta(\lambda)^2 - 4 = -4(\lambda_0^+ - \lambda)(\lambda_0^- - \lambda)\prod_{k\neq 0}\frac{(\lambda_k^- - \lambda)(\lambda_k^+ - \lambda)}{k^2\pi^2}$$

$$\dot{\Delta}(\lambda) = 2(\dot{\lambda}_0 - \lambda)\prod_{k\neq 0}\frac{(\dot{\lambda}_k - \lambda)}{k\pi}$$

where $\dot{\lambda}_k$ are the roots of $\dot{\Delta}(\lambda)$. Along the circuit Γ_n we then can write

$$\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}} = \frac{\lambda - \dot{\lambda}_n}{\sqrt[s]{(\lambda_n^- - \lambda)(\lambda_n^- - \lambda)}}\,\chi_n(\lambda)$$

with (cf (I.48))

$$\begin{aligned}\chi_n(\lambda) :&= \frac{\dot{\Delta}(\lambda)}{\lambda - \dot{\lambda}_n}\frac{\sqrt[s]{(\lambda_n^- - \lambda)(\lambda_n^- - \lambda)}}{\sqrt[c]{\Delta(\lambda)^2-4}}\\ &= i\prod_{k\neq n}\frac{\dot{\lambda}_k - \lambda}{\sqrt[s]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}}.\end{aligned}$$

Notice that $\chi_n(\lambda)$ is analytic in λ on $Disc_n$. With the formula (III.3) for I_n we then get for $\varphi \in W \backslash D_n$,

$$\begin{aligned} I_n &= \frac{1}{\pi}\int_{\Gamma_n} (\lambda - \dot\lambda_n)\frac{\dot\Delta(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}\, d\lambda \\ &= \frac{1}{\pi}\int_{\Gamma_n} \frac{(\lambda-\dot\lambda_n)^2}{\sqrt[s]{(\lambda_n^+-\lambda)(\lambda_n^--\lambda)}}\,\chi_n(\lambda)d\lambda \\ &= \frac{\gamma_n^2}{4\pi}\int_{\Gamma_n'}\frac{(z-\delta_n)^2}{\sqrt[s]{z^2-1}}\,\chi_n\left(\tau_n + z\frac{\gamma_n}{2}\right)dz \end{aligned}$$

upon the substitution $\lambda = \tau_n + z\frac{\gamma_n}{2}$, where Γ_n' is a circuit around $[-1,1]$, $\tau_n = \frac{1}{2}(\lambda_n^+ + \lambda_n^-)$ and $\delta_n = 2(\dot\lambda_n - \zeta_n)/\gamma_n$ satisfies by Lemma I.22

$$\delta_n = |\gamma_n|^{1/2}\ell^2(n) \tag{III.4}$$

Thus, on $W\backslash D_n$,

$$\frac{4I_n}{\gamma_n^2} = \frac{1}{\pi}\int_{\Gamma_n'}\frac{(z-\delta_n)^2}{\sqrt[s]{z^2-1}}\,\chi_n\left(\tau_n + z\frac{\gamma_n}{2}\right)dz.$$

The right side of the last identity is continuous on all of W including D_n, since in view of (III.4), when γ_n tends to 0, it tends to

$$\begin{aligned} \frac{1}{\pi}\int_{\Gamma_n'}\frac{z^2}{\sqrt[s]{z^2-1}}\,\chi_n(\tau_n)dz &= \chi_n(\tau_n)\frac{2}{\pi}\int_{-1}^{1}\frac{x^2}{i\sqrt[+]{1-x^2}}\,dx \\ &= -i\chi_n(\tau_n). \end{aligned}$$

where we used that $\sqrt[s]{z^2-1}\;|_{z=x-i0} = i$ by section I.7. But χ_n and τ_n are analytic on W and hence $\frac{4I_n}{\gamma_n^2}$ restricted to the analytic subvariety D_n is analytic. By standard arguments we then conclude that I_n/γ_n^2 extends analytically to all of W. Moreover, by the estimates Lemma I.17 and the definitions of the standard and the canonical square roots, $\chi_n(\lambda) = i + \ell^2(n)$ for λ near $[\lambda_n^-, \lambda_n^-]$ locally uniformly on W. Together with the last two identities and in view of the asymptotics of δ_n (cf (III.4))we then conclude that

$$4\frac{I_n}{\gamma_n^2} = 1 + \ell^2(n)$$

locally uniformly on W.
Finally on $L^2_{\mathcal{R}}$

$$\begin{aligned} 0 < 4\frac{I_n}{\gamma_n^2} &= \frac{2}{\pi}\int_{-1}^{1}\chi_n\left(\tau_n + \frac{s\gamma_n}{2}\right)\frac{(s-\delta_n)^2}{\sqrt[+]{1-s^2}}\,ds \\ &\underset{n\to\infty}{\to} \lim_{n\to\infty}\chi_n(\tau_n) = 1 \end{aligned}$$

locally uniformly. Therefore, by choosing the complex neighborhood $W' \subseteq W$ of $L^2_{\mathcal{R}}$ sufficiently small we can assure that, for any $n \in \mathbb{Z}$, $Re(4I_n/\gamma_n^2)$ is positive and on W' locally uniformly bounded away from zero. ■

In section III.3 we show that the map Ω from $L^2_{\mathbb{C}}$ into the space of Birkhoff coordinates is proper - that is, the preimages of compact sets are compact. This requires an apriori estimate of $\int_0^1 \varphi_1\varphi_2 dx$ in terms of the actions I_n. It stems from an identity, relating the actions (given by contour integrals) and the asymptotic expansion of $\int_{\lambda_0}^{\lambda}\lambda\frac{\dot\Delta(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}d\lambda$ at $\lambda = \infty$.

Proposition III.4 *For $\varphi \in L^2_{\mathcal{R}}$,*

$$\sum_{k\in\mathbb{Z}} I_k(\varphi) = \int_0^1 \varphi_1\varphi_2 dx. \tag{III.5}$$

Proof As both sides of (III.5) are real analytic on $L^2_{\mathcal{R}}$ it suffices to prove the identity near $\varphi = 0$. Moreover, as finite gap potentials are dense it suffices to show the identity for finite gap potentials near zero. Given a finite gap potential $\varphi \in L^2_{\mathcal{R}}$ (near the zero potential) there exists $K \geq 1$ such that

$$\sum_{k\in\mathbb{Z}} I_k = \sum_{|k|\leq K} I_k = \sum_{|k|\leq K}\frac{1}{\pi}\int_{\Gamma_k}\frac{\lambda\dot\Delta(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}\,d\lambda.$$

By Cauchy's theorem, we have for $R = \pi(K' + \frac{1}{2})$ with $K' \geq K$ sufficiently large

$$\begin{aligned} \sum_{|k|\leq K} I_k &= \frac{1}{\pi}\int_{|\lambda|=R}\lambda\frac{\dot\Delta(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}\,d\lambda \\ &= -\frac{1}{\pi}\int_{|\lambda|=R} f(\lambda)d\lambda \end{aligned}$$

by partial integration with

$$f(\lambda) := \int_R^{\lambda}\frac{\dot\Delta(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}\,d\lambda.$$

As $\int_{\Gamma_n}\frac{\dot\Delta(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}d\lambda = 0$ for any $n \in \mathbb{Z}$ (cf (III.2)), $f(\lambda)$ is a well defined function which is analytic on $\mathbb{C}\backslash \cup_{|k|\leq K}[\lambda_k^-, \lambda_k^+]$. It remains to show that

$$-\frac{1}{\pi}\int_{|\lambda|=R} f(\lambda)d\lambda = \int_0^1 \varphi_1\varphi_2 dx. \tag{III.6}$$

Notice that $z \mapsto f\left(\frac{1}{z}\right)$ is analytic on the punctured disc

$$\{z \in \mathbb{C} \mid 0 < |z| < \frac{1}{R}\}\ .$$

As $f(\lambda) = 0(\lambda)$ near infinity (cf Proposition I.3) and in view of the assumption that φ is a finite gap potential it follows that $z \mapsto f\left(\frac{1}{z}\right)$ is meromorphic in a neighborhood of $z = 0$. Therefore

$$\begin{aligned}\oint_{|\lambda|=R} f(\lambda)d\lambda &= \oint_{|z|=1/R} f(1/z)\frac{dz}{z^2} \\ &= 2i\pi Res\left(f(1/z)z^{-2}\right)\end{aligned} \tag{III.7}$$

where $Res\left(f(1/z)z^{-2}\right)$ denotes the residue of $z \mapsto f(1/z)z^{-2}$ at 0.
Denote by ch^{-1} the principal branch of the inverse function of ch. Its domain is given by $\mathbb{C}\backslash(-\infty, 1]$ and the branch is characterized by $ch^{-1}(ch2) = 2$. As (cf Lemma I.23)

$$\frac{\Delta(iy)}{2} \sim chy \text{ for } y \to \infty \tag{III.8}$$

$g(y) := ch^{-1}\left(\frac{\Delta(iy)}{2}\right)$ is well defined for y real with $y \to +\infty$. Furthermore, $g'(y) = \frac{i\dot{\Delta}(iy)}{\sqrt[+]{\Delta(iy)^2-4}}$ and hence

$$g'(y) = sign\left(Re\sqrt[c]{\Delta(iy)^2 - 4}\right) if'(iy). \tag{III.9}$$

As the asymptotic estimates (III.8) hold locally uniformly,

$$sign\left(Re\sqrt[c]{\Delta(iy)^2 - 4}\right)$$

is constant for y large and φ sufficiently close to the zero potential. For $\varphi = 0$ one has $\Delta(\lambda) = 2cos\lambda$ and, by the definition of the canonical root (cf (I.48))

$$\sqrt[c]{\Delta(iy)^2 - 4} = -2i\sin i\lambda = shy > 0.$$

We then conclude from (III.9) that we have $g'(y) = if'(iy)$ and $g(y) = f(iy)$. As finite gap potentials are smooth one has, by the expansion in Lemma I.24,

$$Res\left(f(1/2)z^{-2}\right) = \frac{i}{2}H_1(\varphi). \tag{III.10}$$

substituting (III.10) into (III.7) one gets

$$-\frac{1}{\pi}\int_{|\lambda|=R} f(\lambda)d\lambda = -2i\left(\frac{i}{2}H_1(\varphi)\right) = H_1(\varphi)$$

and the claimed identity (III.6) follows. ■

Remark that (III.5) is a trace formula relating the actions and the Hamiltonian

$$H_1(\varphi) := \int_0^1 \varphi_1\varphi_2 dx$$

which corresponds to the phase flow, $(\varphi_1, \varphi_2)(x,t) = (e^{-it}\varphi_1(x), e^{it}\varphi_2(x))$. In particular we recover from (III.5) that the frequencies of H_1 are all 1.
To obtain estimates for the NLS-frequencies, a trace formula involving the NLS-Hamiltonian will be useful. To this end we introduce, for $\varphi \in L^2_{\mathbb{C}}$ and $k \in \mathbb{Z}$ (cf [MV], actions at the third level)

$$J_k(\varphi) := \frac{1}{\pi}\int_{\Gamma_k} \lambda^3 \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}}\, d\lambda.$$

Expanding $\lambda^3 = \left((\lambda - \dot{\lambda}_k) + \dot{\lambda}_k\right)^3$ and using that $\int_{\Gamma_k} \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}} d\lambda = 0$ one gets

$$\begin{aligned}J_k &= 3\dot{\lambda}_k^2 I_k + 3\dot{\lambda}_k \frac{1}{\pi}\int_{\Gamma_k} \frac{(\lambda - \dot{\lambda}_k)^2\dot{\Delta}(\lambda)d\lambda}{\sqrt[c]{\Delta(\lambda)^2 - 4}} \\ &+ \frac{1}{\pi}\int_{\Gamma_k} \frac{(\lambda - \dot{\lambda}_k)^3\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} d\lambda.\end{aligned}$$

For potentials of real type we obtain the following estimate

Lemma III.5 *For $\varphi \in L^2_{\mathcal{R}}$,*

$$|J_k| \le (3\dot{\lambda}_k^2 + 3|\dot{\lambda}_k|\gamma_k + \gamma_k^2)I_k.$$

By [GK1], we have $(\gamma_k)_{k\in\mathbb{Z}} \in \ell^2_1$ for $\varphi \in H^1 \equiv H^1(S^1; \mathbb{C}^2)$. Together with Theorem III.3 and Lemma III.5 one then sees that for $\varphi \in H^1$,

$$(I_k)_{k\in\mathbb{Z}} \in \ell^1_2\ ;\ (J_k)_{k\in\mathbb{Z}} \in \ell^2.$$

Recall that for $\varphi \in H^1$ we have introduced the NLS-Hamiltonian

$$H_3(\varphi) := \int_0^1 \left(\varphi_1'\varphi_2' + (\varphi_1\varphi_2)^2\right) dx.$$

Introduce $H^1_{\mathcal{R}} := H^1 \cap L^2_{\mathcal{R}}$. Then the J_k's satisfy the following trace formula.

Proposition III.6 *For $\varphi \in W \cap H^1_{\mathcal{R}}$,*

$$\sum_{k\in\mathbb{Z}} J_k(\varphi) = \frac{3}{4}H_3(\varphi).$$

Remark Notice that in view of Lemma III.5,

$$\frac{3}{4}\frac{\partial H_3}{\partial I_k}\Big|_{I=0} = \sum_k \frac{\partial J_k}{\partial I_k}\Big|_{I=0} = 3\dot{\lambda}_k^2$$

Hence at $I=0$, the k'th frequency $\omega_k := \frac{\partial H_3}{\partial I_k}$ is given by

$$\omega_k \Big|_{I=0} = 4\dot{\lambda}_k^2 \Big|_{I=0} = 4k^2\pi^2.$$

Proof (Proposition III.6) As in the proof of Proposition III.4 it suffices to establish the identity for finite gap potentials of real type near zero. Following the line of arguments of the proof of Proposition III.4 one has

$$\sum_{k\in\mathbb{Z}} J_k = -2iRes\left(\frac{3f(1/z)}{z^4}, 0\right)$$

where we recall that $f(\lambda) := \int_R^\lambda \frac{\dot{\Delta}(\mu)}{\sqrt[s]{\Delta(\mu)^2-4}} d\mu$. For y positive and sufficiently large we have (cf Proposition III.4) $f(iy) = ch^{-1}\left(\frac{\Delta(iy)}{2}\right)$ and hence from Lemma I.25 we get

$$Res\left(\frac{f(1/z)}{z^4}, 0\right) = \frac{i}{8}H_3.$$

Substituting this formula into the above expression for $\sum_{k\in\mathbb{Z}} J_k$ one obtains

$$\sum_{k\in\mathbb{Z}} J_k = -2i\cdot 3\cdot \frac{i}{8}H_3 = \frac{3}{4}H_3.$$

■

III.3 Angles

Next we define the angular coordinates Θ_n for $\varphi \in W$ where, to simplificate the notations, we denote again by W the neighbourhood of $L^2_{\mathcal{R}}$ in $L^2_{\mathbb{C}} \times L^2_{\mathbb{C}}$ on which Theorem II.1 holds. More precisely the n'th angle Θ_n is defined for $\varphi \in W\backslash D_n$ (with $D_n := \{\varphi \in W \mid \gamma_n = 0\}$) by

$$\Theta_n(\varphi) = \eta_n(\varphi) + \sum_{k\neq n} \beta_k^{(n)}(\varphi)$$

where

$$\eta_n(\varphi) = \int_{\lambda_n^-}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2-4}}\, d\lambda \pmod{2\pi}$$

and

$$\beta_k^{(n)}(\varphi) = \int_{\lambda_k^-}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2-4}}\, d\lambda. \tag{III.11}$$

Here $\sqrt{\Delta(\lambda)^2-4}$ is a function defined on the Riemann surface

$$\sum_\varphi := \{(\lambda, y) \in \mathbb{C}^2 \mid y^2 = \Delta(\lambda)^2 - 4\}\ ,$$

for any $n \in \mathbb{Z}$, the Dirichlet divisor μ_n^* is the point $(\mu_n, (M_{12}+M_{21})\mid_{1,\mu_n})$ on $\sum_\varphi$ (cf formula (I.68)) and the entire functions $(\psi_n)_{n\in\mathbb{Z}}$ have been constructed in Chapter II. The paths of integration can be chosen arbitrarily on $\sum_\varphi$ as long as they stay inside an isolating neighborhood (cf Definition III.1) of the corresponding interval $[\lambda_n^-, \lambda_n^+]$. We call such paths admissible.
Note that, since by construction (cf. (II.1)) $\int_{\Gamma_n} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2-4}}\, d\lambda = 2\pi$, the function η_n is considered as a function on $W\backslash D_n$ taking values in the cylinder $\mathbb{C}/2\pi\mathbb{Z}$ rather than $\mathbb{C}$, whereas the β_k^n can be considered as functions taking values in $\mathbb{C}$.
We begin by showing that these functions are well defined in the sense that they are independent of the path of integration. In fact the $\beta_k^{(n)}$ are well defined on all of W.

Lemma III.7 *(1) The functions $\beta_k^{(n)} (k\neq n)$ are well defined on all of W. (2) The functions $\eta_n (n\in\mathbb{Z})$ are well defined on $W\backslash D_n$.*

Proof Consider $\beta_k^{(n)}$ for $k \neq n$. By the product expansions for $\Delta(\lambda)^2 - 4$ (cf Lemma I.19) and $\psi_n(\lambda)$ (cf Theorem II.1) we have

$$\frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2-4}} = \frac{\nu_k^{(n)} - \lambda}{\sqrt{(\lambda_k^+ - \lambda)(\lambda - \lambda_k^-)}}\ \zeta_k^{(n)}(\lambda) \tag{III.12}$$

where $\sqrt{\Delta(\lambda)^2-4}$ and hence $\sqrt{(\lambda_k^+ - \lambda)(\lambda - \lambda_k^-)}$ are understood as functions on a neighborhood around $[\lambda_k^-, \lambda_k^+]$ on the Riemann surface $\sum_\varphi$. (The sign of $\sqrt{(\lambda_k^+ - \lambda)(\lambda - \lambda_k^-)}$ is determined by the one of $\sqrt{\Delta(\lambda)^2-4}$.) The functions $\zeta_k^{(n)}(\lambda)$ for $k\neq n$ are defined as follows ($\nu_n^{(n)} := \tau_n$)

$$\zeta_k^{(n)}(\lambda) := -\frac{1}{\tau_n - \lambda} \prod_{j\neq k} \frac{\nu_j^{(n)} - \lambda}{\sqrt[s]{(\lambda_j^- - \lambda)(\lambda_j^+ - \lambda)}}. \tag{III.13}$$

Clearly, $\zeta_k^{(n)}(\lambda)$ is analytic in λ (near $[\lambda_k^-, \lambda_k^+]$) and $\psi \in W$. If $\gamma_k \neq 0$, the factor $\frac{\nu_k^{(n)}-\lambda}{\sqrt{(\lambda_k^+-\lambda)(\lambda-\lambda_k^-)}}$ is integrable on any admissible path. If $\gamma_k = 0$, then $\lambda_k^- = \nu_k^{(n)} = \lambda_k^+$, $\frac{\nu_k^{(n)}-\lambda}{\sqrt{(\lambda_k^+-\lambda)(\lambda-\lambda_k^-)}} = \pm i$ and the integrand of $\beta_k^{(n)}$ is an analytic function near $[\lambda_k^-, \lambda_k^+]$ on each sheet of $\sum_\varphi$. Hence in both cases, $\beta_k^{(n)}$ is well defined.
The integral is independent of any admissible path of integration, since

$$\int_{\lambda_k^+}^{\lambda_k^-} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2-4}}\, d\lambda = 0 \quad k\neq n.$$

This proves (i). As to η_n, the integral exists along an admissible path as long as $\lambda_n^- \neq \lambda_n^+$. It is well defined modulo 2π. This shows claim (ii). ■

Next we prove the analyticity of the coefficients $\beta_k^{(n)}$ $(k \neq n)$.

Lemma III.8 *(i) The functions $\beta_k^{(n)}$ $(k \neq n)$ are real analytic on W.*
(ii) The functions $\eta_n (n \in \mathbb{Z})$ are real analytic on $W \backslash D_n$ if taken $\mod \pi$.

Remark The values of η_n have to be taken modulo π due to the discontinuities of the periodic eigenvalues as functions of φ when φ is not of real type (lexicographic ordering).

Proof In W consider the two subsets

$$\begin{aligned} D_k :=& \{\varphi \in W \mid \gamma_k(\varphi) = 0\} \\ E_k :=& \left\{\varphi \in W \mid \mu_k(\varphi) \in \{\lambda_k^+(\varphi), \lambda_k^-(\varphi)\}\right\}. \end{aligned}$$

Taking into account that $\tau_k(\varphi), \mu_k(\varphi)$ are analytic on W and $\Delta(\lambda, \varphi)$ is analytic on $\mathbb{C} \times W, D_k$ and W_k are in fact analytic subvarieties of W,

$$\begin{aligned} D_k &= \{\varphi \in W \mid \Delta(\tau_k) = 2(-1)^k; \dot{\Delta}(\tau_k) = 0\} \\ E_k &= \{\varphi \in W \mid \Delta(\mu_k) = 2(-1)^k\}. \end{aligned}$$

Our plan is to prove that $\beta_k^{(n)}$ is analytic on $W \backslash (D_k \cup E_k)$ as well as continuous on all of W and that $\beta_k^{(n)} |_{E_k}, \beta_k^{(n)} |_{D_k \backslash E_k}$ are weakly analytic. By standard arguments it then follows that $\beta_k^{(n)}$ is analytic on W. To prove that $\beta_k^{(n)}$ is analytic on $W \backslash (D_k \cup E_k)$ notice that outside of D_k, λ_k^+ and λ_k^- are simple eigenvalues and *locally* there exist analytic functions $\tilde{\lambda}_k^+$ and $\tilde{\lambda}_k^-$ such that the sets $\{\tilde{\lambda}_k^+, \tilde{\lambda}_k^-\}$ and $\{\lambda_k^+, \lambda_k^-\}$ are equal. In view of the normalizing condition $\int_{\lambda_k^-}^{\lambda_k^+} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2-4}} d\lambda = 0$ for $k \neq 0$, we obtain upon the substitution $\lambda = \tilde{\lambda}_k^- + z$

$$\beta_k^{(n)} = \int_{\tilde{\lambda}_k^-}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}}\, d\lambda = \int_0^{\mu_k^* - \tilde{\lambda}_k^-} \frac{\psi_n(\tilde{\lambda}_k^- + z)}{\sqrt{z}\sqrt{D(z)}}\, dz$$

where $D(z) := \frac{\Delta^2(\tilde{\lambda}_k^- + z) - 4}{z}$ is analytic near $z = 0$ and satisfies $D(0) \neq 0$. As path of integration we choose an admissible path which does *not* go through $\tilde{\lambda}_k^+$. (This is possible as $\varphi \notin E_k$.) Then $D(z) \neq 0$ along the path and hence $\frac{\psi_n(\tilde{\lambda}_k^- + z)}{\sqrt{D(z)}}$ is smooth along the path and locally analytic on $W \backslash (D_k \cup E_k)$. As $\mu_k^* = (\mu_k, (M_{21} + M_{12}) |_{1,\mu_k})$ is analytic on W and $\tilde{\lambda}_k^-$ is analytic we then conclude, in view of Leibniz's rule, that $\int_0^{\mu_k^* - \tilde{\lambda}_k^-} \frac{\psi_n(\tilde{\lambda}_k^- + z)}{\sqrt{z}\sqrt{D(z)}} dz$

is analytic. Hence $\beta_k^{(n)}$ is analytic on $W \backslash (D_k \cup E_k)$. Next we show that $\beta_k^{(n)} |_{D_k}$ and $\beta_k^{(n)} |_{E_k}$ are weakly analytic. In view of the normalization $\int_{\lambda_k^-}^{\lambda_k^+} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2-4}} d\lambda = 0$ we have that $\beta_k^{(n)} |_{E_k} = 0$ and hence it is analytic.
On D_k we have

$$\lambda_k^- = \lambda_k^+ = \tau_k = \nu_k^{(n)},$$

and hence with (III.12), one can write $(\psi \in D_k)$

$$\beta_k^{(n)} = \int_{\tau_k}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda = \varepsilon \int_{\tau_k}^{\mu_k^*} \zeta_k^{(n)}(\lambda) d\lambda$$

where ε is the sign ± 1 determined by μ_k^*. As $\mu_k^* = (\mu_k, (M_{12} + M_{21}) |_{1,\mu_k})$ is analytic, $\beta_k^{(n)} |_{D_k}$ is analytic. Alltogether, we conclude that $\beta_k^{(n)} |_{D_k}$ and $\beta_k^{(n)} |_{E_k}$ are weakly analytic.
It remains to prove that $\beta_k^{(n)}$ is continuous on all of W. Clearly $\beta_k^{(n)}$ is continuous on $W \backslash (D_k \cup E_k)$. One shows easily that it is continuous in points of $E_k \backslash D_k$ and $D_k \backslash E_k$. The continuity in points of $D_k \cap E_k$ follows from (III.12) and the estimate $\nu_k^{(n)} - \tau_k = 0(\gamma_k^2)$ (cf Theorem II.1). This establishes the analyticity of $\beta_k^{(n)}$ on W.
The proof for η_n is analogous and even simpler, since we only need to consider the domain $W \backslash D_n$. In view of the normalization condition (II.1) we have

$$\int_{\lambda_n^-}^{\lambda_n^+} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} = \pm \pi \tag{III.14}$$

for the straight line integral. So as above we can write

$$\eta_n = \int_{\tilde{\lambda}_n^-}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda \ (\text{mod } \pi).$$

We conclude that modulo π, the function η_n is analytic on $W \backslash (D_n \cup E_n)$ and continuous on $W \backslash D_n$. Since $\eta_n |_{E_n} = 0$ $(mod \pi), \eta_n |_{E_n}$ is weakly analytic and the statement is proved. ■

Lemma III.9 *For $k \neq n$,*

$$\beta_k^{(n)} = 0\left(\frac{|\gamma_k| + |\mu_k - \tau_k|}{|n-k|}\right)$$

locally uniformly on W.

Proof By (II.1),

$$\beta_k^{(n)} = \int_{\lambda_k^-}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}}\, d\lambda = \int_{\lambda_k^+}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}}\, d\lambda\ .$$

The following argument is not affected if one interchanges the roles of λ_k^+ and λ_k^-. Therefore we assume that $|\mu_k - \lambda_k^-| \leq |\mu_k - \lambda_k^+|$. For λ near $[\lambda_k^-, \lambda_k^+]$ we have

$$\frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} = \frac{\nu_k^{(n)} - \lambda}{\sqrt{(\lambda_k^+ - \lambda)(\lambda - \lambda_k^-)}} \zeta_k^{(n)}(\lambda)$$

by equation (III.12). In view of formula (III.13) and the asymptotics of $\nu_m^{(n)}$ and $\lambda_m^{\pm}, \zeta_k^{(n)}(\lambda)$ satisfies

$$\zeta_k^{(n)}(\lambda) = 0\left(\frac{1}{\langle n-k \rangle}\right)$$

for λ near $[\lambda_k^-, \lambda_k^+]$. Moreover if we integrate along a straight line ℓ from λ_k^- to μ_k on the sheet of Σ_φ determined by μ_k^* we have

$$\sqrt{\frac{\nu_k^{(n)} - \lambda}{\lambda_k^+ - \lambda}} = 0(1)$$

since $|\mu_k - \lambda_k^-| \leq |\mu_k - \lambda_k^+|$ and $\nu_k^{(n)} = \tau_k + 0(\gamma_k^2)$. Thus it remains to show that

$$\int_{\lambda_k^-}^{\mu_k^*} \sqrt[+]{\left|\frac{\lambda - \nu_k^{(n)}}{\lambda - \lambda_k^-}\right|} d\lambda = 0(|\gamma_k| + |\mu_k - \tau_k|)$$

when integrating along the straight line ℓ.
This follows with the substitution $\lambda = \lambda_k^- + t(\mu_k - \lambda_k^-)$

$$\begin{aligned}
\Big| \int_{\lambda_k^-}^{\mu_k^*} \sqrt[+]{\left|\frac{\lambda - \nu_k^{(n)}}{\lambda - \lambda_k^-}\right|}\, d\lambda \Big| &\leq \int_0^1 \frac{\sqrt[+]{|\lambda_k^- - \nu_k^{(n)}| + t|\mu_k - \lambda_k^-|}}{\sqrt{t}|\mu_k - \lambda_k^-|^{1/2}} |\mu_k - \lambda_k^-| dt \\
&\leq \sqrt[+]{|\lambda_k^- - \nu_k^{(n)}| + |\mu_k - \lambda_k^-|}\ |\mu_k - \lambda_k^-|^{1/2} \int_0^1 \frac{1}{\sqrt{t}} dt \\
&\leq \left(|\lambda_k^- - \nu_k^{(n)}| + |\mu_k - \lambda_k^-|\right) + |\mu_k - \lambda_k^-| \\
&\leq 0(|\mu_k - \tau_k| + |\gamma_k|)
\end{aligned}$$

where we used that $2ab \leq a^2 + b^2$ for any real numbers $a,\ b$. ■

By Young's inequality it follows from Lemma III.9 that for any $p > 2$, $\left(\sum_{k \neq n} \beta_k^{(n)}\right)_{n \in \mathbb{Z}} \in \ell^p$ locally uniformly on W. In particular, $\beta_n := \sum_{k \neq n} \beta_k^{(n)}$ are analytic functions on W with the property that $(\beta_n)_{n \in \mathbb{Z}} \in \ell^p$, and hence $\lim_{n \to \infty} \beta_n = 0$, locally uniformly on W. We summarize our results of this section in the following

Theorem III.10 *The function $\varphi \mapsto (\beta_n(\varphi))_{n \in \mathbb{Z}}$ is real analytic from W with values in ℓ^p for any $p > 2$. The angle function*

$$\Theta_n = \eta_n + \beta_n = \eta_n + \sum_{k \neq n} \beta_k^{(n)}$$

is a smooth real valued function defined on $L^2_{\mathcal{R}}$ and extends to a real analytic function on $W \backslash D_n$ when taken modulo π.

III.4 Cartesian coordinates

In section III.2 and III.3 we have defined actions $I_n = \xi_n^2(\gamma_n/2)^2$ for $\varphi \in L^2_{\mathcal{R}}$ and angles $\Theta_n = \eta_n + \beta_n$ for potentials φ in $L^2_{\mathcal{R}} \backslash D_n$ and showed that there exists a neighborhood W of $L^2_{\mathcal{R}}$ in $L^2_{\mathbb{C}} \times L^2_{\mathbb{C}}$ so that for any $n \in \mathbb{Z}$, I_n is real analytic on W and Θ_n is real analytic on $W \backslash D_n$ when taken modulo π.
In this section we introduce the associated Cartesian coordinates. For $\varphi \in L^2_{\mathcal{R}} \backslash D_n$, they are defined as

$$x_n = \sqrt{2I_n} cos\Theta_n,\ y_n = \sqrt{2I_n} sin\Theta_n.$$

With this choice we have

$$dx_n \wedge dy_n = dI_n \wedge d\Theta_n = d(I_n d\Theta_n).$$

This definition extends to the complex domain $W \backslash D_n$,

$$x_n = \sqrt{2}\xi_n \frac{\gamma_n}{2} cos\Theta_n,\ y_n = \sqrt{2}\xi_n \frac{\gamma_n}{2} sin\Theta_n. \tag{III.15}$$

In this section we prove that the functions x_n, y_n ($n \in \mathbb{Z}$) are in fact real analytic functions on W.
Recall that we have already proved that ξ_n and $\beta_n := \Theta_n - \eta_n$ are real analytic on W and thus it remains to analyze

$$z_n^{\pm} := \gamma_n e^{\pm i\eta_n}. \tag{III.16}$$

The functions $z_n^{\pm}$ are defined on $W \backslash D_n$. Although γ_n is not continuous on $W \backslash D_n$ and η_n is analytic only modulo π, the functions $z_n^{\pm}$ are analytic on $W \backslash D_n$ as can be seen from the following argument.

Lemma III.11 *The functions $z_n^{\pm} = \gamma_n e^{\pm i\eta_n}$ are analytic on $W \backslash D_n$.*

Proof Locally around every point in $W \backslash D_n$ there exist analytic functions $\tilde{\lambda}_n^+$ and $\tilde{\lambda}_n^-$ such that as sets $\{\tilde{\lambda}_n^-(\varphi), \tilde{\lambda}_n^+(\varphi)\} = \{\lambda_n^-(\varphi), \lambda_n^+(\varphi)\}$. Let

$$\tilde{\gamma}_n = \tilde{\lambda}_n^+ - \tilde{\lambda}_n^-,\ \tilde{\eta}_n := \int_{\tilde{\lambda}_n^+}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda\ .$$

Depending on whether $\tilde\lambda_n^- = \lambda_n^-$ or $\tilde\lambda_n^- = \lambda_n^+$, respectively, we then have, in view of (III.14),

$$\gamma_n = \tilde\gamma_n \text{ and } \eta_n = \tilde\eta_n \quad \text{or} \quad \gamma_n = -\tilde\gamma_n \text{ and } \eta_n = \tilde\eta_n + \pi \ (mod 2\pi).$$

In both cases, we thus obtain

$$\gamma_n e^{\pm i\eta_n} = \tilde\gamma_n e^{\pm i\tilde\eta_n}.$$

As the right side of the latter identity is analytic, the claimed statement follows. ■

In the case $k = n$ we write instead of the representation III.12 for λ near $[\lambda_n^-, \lambda_n^+]$,

$$\frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} = \frac{1}{i\sqrt{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}}\zeta_n(\lambda) \qquad \text{(III.17)}$$

where $\zeta_n(\lambda)$ is given by

$$\zeta_n(\lambda) = -\frac{\nu_0^{(n)} - \lambda}{\sqrt[s]{(\lambda_0^- - \lambda)(\lambda_0^+ - \lambda)}} \prod_{j\neq 0,n} \frac{\nu_j^{(n)} - \lambda}{\sqrt[s]{(\lambda_j^- - \lambda)(\lambda_j^+ - \lambda)}} \quad (n \neq 0) \qquad \text{(III.18)}$$

$$\zeta_0(\lambda) = -\prod_{j\neq 0} \frac{\nu_j^{(0)} - \lambda}{\sqrt[s]{(\lambda_j^- - \lambda)(\lambda_j^+ - \lambda)}} \quad (n = 0) \qquad \text{(III.19)}$$

and the two roots in (III.17) are understood as functions on $\sum_\varphi$ related to each other by this identity.

Lemma III.12 *For $\mu \in [\lambda_n^-, \lambda_n^-]$,*

$$\zeta_n(\mu) = 1 + 0(\gamma_n)$$

locally uniformly for φ in W and uniformly in $n \in \mathbb{Z}$.

Proof First let $\varphi \in L^2_{\mathcal{R}}$ with $\gamma_n > 0$. In view of Theorem II.1 and the definition of the canonical root $\sqrt[c]{\Delta(\lambda)^2 - 4}$ (cf Section I.7) we have

$$(-1)^{n+1} \int_{\lambda_n^-}^{\lambda_n^+} \frac{\psi_n(\lambda)}{\sqrt[+]{\Delta(\lambda)^2 - 4}}\, d\lambda = \pi$$

for the straight line integral from λ_n^- to λ_n^+. On this line, ζ_n is positive while the sign of $\psi_n(\lambda)$ is $(-1)^{n+1}$. With (III.17), we then obtain

$$\begin{aligned} \pi &= \int_{\lambda_n^-}^{\lambda_n^+} \frac{\zeta_n(\lambda)}{\sqrt[+]{(\lambda_n^+ - \lambda)(\lambda - \lambda_n^-)}}\, d\lambda \\ &= \int_{\lambda_n^-}^{\lambda_n^+} \frac{\zeta_n(\mu)}{\sqrt[+]{(\lambda_n^+ - \lambda)(\lambda - \lambda_n^-)}}\, d\lambda + \int_{\lambda_n^-}^{\lambda_n^+} \frac{\zeta_n(\lambda) - \zeta_n(\mu)}{\sqrt[+]{(\lambda_n^+ - \lambda)(\lambda - \lambda_n^-)}}\, d\lambda \\ &= \pi\zeta_n(\mu) + 0\left(\sup_{\lambda_n^- \leq \lambda \leq \lambda_n^+} \left|\, \zeta_n(\lambda) - \zeta_n(\mu) \,\right|\right) \end{aligned}$$

for any $\lambda_n^- \leq \mu \leq \lambda_n^+$. Hence

$$\zeta_n(\mu) = 1 + 0\left(\sup_{\lambda_n^- \leq \lambda \leq \lambda_n^+} \left|\, \zeta_n(\lambda) - \zeta_n(\mu) \,\right|\right).$$

By Lemma I.17, $\psi_n(\lambda) = -2\frac{sin\lambda}{\lambda - n\pi}\,(1 + \ell^2(n))$ (cf Theorem II.1) and

$$\frac{\Delta(\lambda)^2 - 4}{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)} = -4\left(\frac{sin\lambda}{\lambda - n\pi}\right)^2 (1 + \ell^2(n))$$

(cf Lemma I.19), uniformly for $|\lambda - n\pi| \leq \frac{\pi}{4}$. Hence $\zeta_n(\lambda)$ is bounded uniformly for λ with $|\lambda - n\pi| \leq \pi/4$, $n \in \mathbb{Z}$, and locally uniformly on W. In view of the asymptotics $\lambda_n^\pm = n\pi + \ell^2(n)$ we then obtain by Cauchy's estimate that $\frac{d}{d\lambda}\zeta_n(\lambda)$ is bounded and hence we have

$$\left|\, \zeta_n(\lambda) - \zeta_n(\mu) \,\right| \leq c \left|\, \lambda - \mu \,\right| \leq c \left|\, \gamma_n \,\right| \qquad \text{(III.20)}$$

for any $\lambda, \mu \in [\lambda_n^-, \lambda_n^+]$ with a uniform constant c. This proves the claim for $\varphi \in L^2_{\mathcal{R}}$.
For $\varphi \in W$ one has

$$\int_{\lambda_n^-}^{\lambda_n^+} \frac{\psi_n(\lambda)}{\sqrt[c]{\Delta(\lambda + i0)^2 - 4}}\, d\lambda = \pi$$

and thus

$$\pm\pi = \int_{\lambda_n^-}^{\lambda_n^+} \frac{\zeta_n(\lambda)}{\sqrt[+]{(\lambda_n^+ - \lambda)(\lambda - \lambda_n^-)}}\, d\lambda\ .$$

As the estimate (III.20) holds for $\varphi \in W$, we can use the same arguments as in the real type case to conclude that for $\varphi \in W$,

$$\zeta_n(\mu) = \pm 1 + 0(\gamma_n)$$

As $\zeta_n(\lambda)$ is continuous in λ and φ the claimed statement follows. ■

We now have to investigate the limiting behavior of $z_n^\pm$ as φ approaches a point ψ in D_n. This limit is different from zero when ψ is in the open set

$$X_n := \{\varphi \in W \mid \mu_n(\varphi) \notin [\lambda_n^-(\varphi), \lambda_n^+(\varphi)]\}.$$

Notice that $X_n \cap L^2_{\mathcal{R}} = \emptyset$. Let

$$\chi_n(\varphi) := \int_{\tau_n}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\tau_n)}{\lambda - \tau_n}\, d\lambda. \tag{III.21}$$

This integral exists due to the analyticity of ζ_n in λ and is analytic in φ. To facilitate the statement of the following result define $\varepsilon_n = \pm 1$ for potentials φ in X_n in such a way that (cf. (III.17))

$$-i\varepsilon_n \sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}\,\Big|_{\lambda = \mu_n} = \sqrt[*]{\Delta(\mu_n)^2 - 4}\; \frac{\zeta_n(\mu_n)}{\psi_n(\mu_n)} \tag{III.22}$$

where the root $\sqrt[*]{\Delta(\mu)^2 - 4}$ has been defined in (I.68). Notice that $\mu_n \notin [\lambda_n^-, \lambda_n^+]$ for $\varphi \in X_n$, hence ε_n is well defined.

Lemma III.13 *As $\varphi \in W \setminus D_n$ tends to $\psi \in D_n \cap X_n$,*

$$\gamma_n e^{\pm i\eta_n} \to -2(1 \pm \varepsilon_n)(\mu_n - \tau_n) e^{\pm\varepsilon_n} \chi_n$$

where ε_n is given by (III.22).

Proof As X_n is open and $\psi \in X_n \cap D_n$ we have $\varphi \in X_n$ for φ sufficiently close to ψ. By assumption, we have $\varphi \in W \setminus D_n$, hence we can write, modulo 2π,

$$\begin{aligned} \eta_n &= \int_{\lambda_n^-}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}}\, d\lambda \\ &= \int_{\lambda_n^-}^{\mu_n} i \frac{\zeta_n(\lambda)}{\varepsilon_n \sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}}\, d\lambda \\ &= \eta_n' + \eta_n'' \end{aligned}$$

where

$$\eta_n' := i\varepsilon_n \int_{\lambda_n^-}^{\mu_n} \frac{\zeta_n(\lambda_n^-)}{\sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}}\, d\lambda$$

and

$$\eta_n'' := i\varepsilon_n \int_{\lambda_n^-}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\lambda_n^-)}{\sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}}\, d\lambda.$$

Let us first analyze the limiting behaviour of η_n''. If $\varphi \to \psi$, then $\gamma_n \to 0$ and $\sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)} \to -(\lambda - \tau_n)$ by the definition of the standard root $\sqrt[s]{\ }$. Hence

$$i\eta_n'' \to \varepsilon_n \int_{\tau_n}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\tau_n)}{\lambda - \tau_n}\, d\lambda = \varepsilon_n \chi_n((\psi)\ (\mathrm{mod} 2\pi)\ .$$

Consequently, $e^{\pm i\eta_n''} \to e^{\pm\varepsilon_n \chi_n(\psi)}$.
Now consider η_n'. The substitution $\lambda = \tau_n + z\gamma_n/2$ leads to

$$\varepsilon_n \int_{\lambda_n^-}^{\mu_n} \frac{d\lambda}{\sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}} = f(\varrho_n)$$

with

$$\varrho_n := \frac{\mu_n - \tau_n}{\gamma_n/2}\ ; \quad f(z) := \varepsilon_n \int_{-1}^{z} \frac{dz}{\sqrt[s]{z^2 - 1}}.$$

It follows that

$$e^{\pm i f(z)} = -z \mp \varepsilon_n \sqrt[s]{z^2 - 1}$$

as both sides are analytic univalent functions on $\mathbb{C} \setminus [-1, 1]$ which have the same limit at -1 and satisfy the same differential equation

$$\frac{q'(z)}{q(z)} = \pm \frac{\varepsilon_n}{\sqrt[s]{z^2 - 1}}.$$

Hence we obtain

$$e^{\pm i\eta_n'} = e^{\mp f(z_n)\zeta_n(\lambda_n^-)} = \left(-\varrho_n \pm \varepsilon_n \sqrt[s]{\varrho_n^2 - 1}\right)^{\zeta_n(\lambda_n^-)}.$$

Now let φ converge to ψ. Then $\lim_{\varphi\to\psi}(\mu_n - \tau_n) \neq 0$. Hence $\lim_{\varphi\to\psi} \varrho_n^{-1} = 0$ and we claim that

$$\left(-\varrho_n \pm \varepsilon_n \sqrt[s]{\varrho_n^2 - 1}\right)^{\gamma_n} \to 1. \tag{III.23}$$

To see it note that $\sqrt[s]{z^2 - 1} = -z \sqrt[+]{1 - 1/z^2}$ for $|z| > 1$ and thus

$$-\varrho_n \pm \varepsilon_n \sqrt[s]{\varrho_n^2 - 1} = -\varrho_n \left(1 \pm \varepsilon_n \sqrt[+]{1 - 1/\varrho_n^2}\right) \tag{III.24}$$

and, taking any branch of the logarithm,

$$\begin{aligned} &\log\left(-\varrho_n \pm \varepsilon_n \sqrt[s]{\varrho_n^2 - 1}\right)^{\gamma_n} \\ &= -\gamma_n \log\left(\gamma_n \frac{-2}{\mu_n - \tau_n}\ (1 \pm \varepsilon_n \sqrt[+]{1 - 1/\varrho_n^2})^{-1}\right) \\ &\to 0 \end{aligned}$$

which establishes (III.23). By Lemma III.12, $\zeta_n(\lambda_n^-) = 1 + 0(\gamma_n)$ and hence in order to prove convergence of $\gamma_n e^{\pm i\eta_n'}$ for $\varphi \to \psi$ it remains to show that $\gamma_n(-\varrho_n \pm \varepsilon_n \sqrt[s]{\varrho_n^2 - 1})$ converges. By (III.24) we have, as $\varphi \to \psi$,

$$\gamma_n \left(-\varrho_n \pm \varepsilon_n \sqrt[s]{\varrho_n^2 - 1}\right) = -2(\mu_n - \tau_n)\left(1 \pm \varepsilon_n \sqrt[+]{1 - 1/\varrho_n^2}\right)$$
$$\to -2(\mu_n - \tau_n)(1 \pm \varepsilon_n).$$

Combined with the limit for $e^{\pm i\eta_n''}$ we conclude that

$$\lim_{\varphi \to \psi} \gamma_n e^{\pm i\eta_n} = -2(\mu_n - \tau_n)(1 \pm \varepsilon_n) e^{\pm \varepsilon_n \chi_n} \big|_\psi$$

as claimed. ■

In view of the preceding result it is natural to extend the functions $z_n^\pm$ to D_n by defining

$$z_n^\pm := \begin{cases} -2(1 \pm \varepsilon_n)(\mu_n - \tau_n) e^{\pm \varepsilon_n \chi_n} & \text{on } D_n \cap X_n \\ 0 & \text{on } D_n \backslash X_n. \end{cases}$$

Proposition III.14 *The functions $\gamma_n e^{\pm i\eta_n'}, e^{\pm i\eta_n''}$ and hence $z_n^\pm = \gamma_n e^{\pm i\eta_n}$, extended as above, are analytic on W. Moreover*

$$z_n^\pm = 0(|\gamma_n| + |\mu_n - \tau_n|)$$

locally uniformly on W.

Proof To show that $z_n^\pm$ are analytic on W, notice that D_n is an analytic variety. Hence it suffices to prove that $z_n^\pm$ are continuous on W, analytic on $W \backslash D_n$ and that the restriction of $z_n^\pm$ to D_n is (weakly) analytic.
By Lemma III.11, $z_n^\pm$ are analytic on $W \backslash D_n$ and, by inspection of the formula for $z_n^\pm$, one sees that $z_n^\pm$ are continuous on W. To see that $z_n^\pm \big|_{D_n}$ are weakly analytic notice that, from its definition, $z_n^\pm \big|_{D_n \cap X_n}$ is weakly analytic and $z_n^\pm \big|_{D_n \backslash X_n} \equiv 0$. As $D_n \backslash X_n$ is an analytic variety it then follows that $z_n^\pm \big|_{D_n}$ is weakly analytic. We thus have shown that $z_n^\pm$ are analytic on W.
To prove the claimed estimate, we recall from the proof of Lemma III.13 that on $W \backslash D_n$

$$\gamma_n e^{\pm i\eta_n} = \gamma_n \left(-\varrho_n \pm \varepsilon_n \sqrt[s]{\varrho_n^2 - 1}\right)^{\zeta_n(\lambda_n^-)} e^{\pm i\eta_n''}$$

with $\varrho_n = 2(\mu_n - \tau_n)/\gamma_n$ and

$$\eta_n'' = i\varepsilon_n \int_{\lambda_n^-}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\lambda_n^-)}{\sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}}\, d\lambda.$$

In view of the analyticity and local uniform boundedness of ζ_n (cf. Lemma III.12) we have $e^{\pm i\eta_n''} = 0(1)$. By (III.23),

$$\left(-\varrho_n \pm \varepsilon_n \sqrt[s]{\varrho_n^2 - 1}\right)^{\gamma_n} = 0(1)\ .$$

As $\zeta_n(\lambda_n^-) = 1 + 0(\gamma_n)$ it then suffices to bound $\gamma_n(-\varrho_n \pm \varepsilon_n \sqrt[s]{\varrho_n^2 - 1})$. For $|\varrho_n| \le 1$ we have

$$|\gamma_n(-\varrho_n \pm \varepsilon_n \sqrt[s]{\varrho_n^2 - 1})| \le 2|\gamma_n|$$

whereas for $|\varrho_n| > 1$, we have

$$\big|\, \gamma_n(-\varrho_n \pm \varepsilon_n \sqrt[s]{\varrho_n^2 - 1}) \,\big| = \big|\, 2(\mu_n - \tau_n) \,\big| \,\big|\, 1 \pm \varepsilon_n \sqrt[+]{1 - \varrho_n^{-2}} \,\big| \le 4 \,\big|\, \mu_n - \tau_n \,\big|\ .$$

It follows that the claimed estimate for $\gamma_n e^{\pm i\eta_n}$ holds locally uniformly on $W \backslash D_n$ and uniformly in n. By the continuity of $z_n^\pm$ on W, these estimates hold locally uniformly on all of W. ■

In view of Proposition III.14 it is possible to define for φ in W

$$\Omega(\varphi) := (x_n(\varphi), y_n(\varphi))_{n \in \mathbb{Z}},$$

where $x_n(\varphi), y_n(\varphi)$ are analytic functions on W,

$$\begin{aligned} x_n(\varphi) &= \sqrt{2}\frac{\xi_n}{4}(z_n^+ e^{i\beta_n} + z_n^- e^{-i\beta_n}) \\ y_n(\varphi) &= \sqrt{2}\frac{\xi_n}{4i}(z_n^+ e^{i\beta_n} - z_n^- e^{-i\beta_n}. \end{aligned} \tag{III.25}$$

In view of the asymptotics of $\mu_n - \tau_n$, γ_n, ξ_n and β_n, it follows from Proposition III.14 that Ω is a continuous, locally bounded map with values in $\ell^2(\mathbb{Z}, \mathbb{C}^2)$. As the components $x_n(\varphi)$ and $y_n(\varphi)(n \in \mathbb{Z})$ are analytic on W we have established the following main result of this section

Theorem III.15 *The map*

$$\Omega : L^2_{\mathcal{R}} \to \ell^2(\mathbb{Z}, \mathbb{C}^2)$$

extends to an analytic map on W with values in $\ell^2(\mathbb{Z}, \mathbb{C}^2)$.

III.5 Gradients

In this section we compute the gradients of $z_n^\pm$ on $L^2_{\mathcal{R}} \cap D_n$ and prove asymptotic estimates on finite gap potentials which will be needed later.
To compute $\nabla_{\psi(x)} z_n^\pm$ on $L^2_{\mathcal{R}} \cap D_n$ it is convenient to approximate $\psi \in L^2_{\mathcal{R}} \cap D_n$ (cf. Proposition I.41) by potentials φ in

$$B_n := \{\varphi \in L^2_{\mathcal{R}} \backslash D_n \mid \mu_n = \tau_n\ ; \quad sign \sqrt[*]{\Delta(\mu_n)^2 - 4} = (-1)^{n-1}\}\ .$$

For $\varphi \in L^2_{\mathcal{R}}$ denote by $H_n = (H_{n1}, H_{n2})$ the L^2-normalized eigenfunction for the Dirichlet eigenvalue μ_n

$$H_n = \frac{1}{\|G_n\|} G_n$$

where G_n defined as in section I.I.3 and by $K_n = (K_{n1}, K_{n2})$ the L^2-normalized solution of $LF = \mu_n F$ which is L^2-orthogonal to H_n and satisfies the normalization condition $\frac{1}{i}(K_{n1}(0) - K_{n2}(0)) > 0$. Notice that $K_{n1}(0) - K_{n2}(0) \neq 0$ since, otherwise, K_n would be proportional to H_n. Recall that $\varphi \in L^2_{\mathcal{R}}$ is a finite gap potential if the set $A := \{n \in \mathbb{Z} \mid \lambda_n^+ \neq \lambda_n^-\}$ is finite.

Lemma III.16 *At $\psi \in L^2_{\mathcal{R}} \cap D_n$,*

$$\nabla_{\psi(x)} z_n^{\pm} = \left((K_{n2} \pm iH_{n2})^2 \ , \ (K_{n1} \pm iH_{n1})^2\right) \ .$$

Moreover, for finite gap potentials,

$$\nabla_{\psi(x)} z_n^+ = -2(0, e^{-2\pi i n x}) + \ell^2(n)$$
$$\nabla_{\psi(x)} z_n^- = -2(e^{2\pi i n x}, 0) + \ell^2(n) \ .$$

At $\psi = 0$, the above identities hold without error term.

Proof To compute the gradient $\nabla_\psi z_n^\pm$ at $\psi \in L^2_{\mathcal{R}} \cap D_n$ we approximate ψ by elements φ in $B_n \cap L^2_{\mathcal{R}}$. Recall that $\mathrm{sign}(\psi_n(\mu_n)) = (-1)^{n-1}$, $\mathrm{sign}(\zeta_n(\mu_n)) = 1$, and, as $\varphi \in B_n \cap L^2_{\mathcal{R}}$

$$\sqrt[*]{\Delta(\mu_n)^2 - 4} = (-1)^{n-1}$$

From the definition of the s-root,

$$sign\left(i \sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}\right) \Big|_{\lambda = \mu_n - i0} = 1,$$

it then follows that

$$1 = sign\left(i \sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)} \Big|_{\lambda = \mu_n - i0}\right) = sign\left(\frac{\sqrt[*]{\Delta(\mu_n)^2 - 4}}{\psi_n(\mu_n)} \zeta_n(\mu_n)\right).$$

Going through the calculations in the proof of Lemma III.13 with $\varepsilon_n = 1$ and $\lambda - i0 \in [\lambda_n^-, \lambda_n^+]$, one verifies that they remain valid for $\varphi \in B_n$. In particular $\eta_n = \eta_n' + \eta_n''$ where

$$\eta_n''(\varphi) = \int_{\lambda_n^-}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\lambda_n^-)}{i \sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}} d\lambda.$$

As $\psi \in L^2_{\mathcal{R}} \cap D_n$, $\chi_n(\psi) = 0$ by the definition (III.21) of χ_n and hence $\eta_n''(\psi) = 0$. Thus from

$$z_n^\pm = \gamma_n e^{\pm i \eta_n} = \gamma_n e^{\pm i \eta_n'} e^{\pm i \eta_n''}$$

and the fact that η_n'' is analytic we conclude that at $\psi \in L^2_{\mathcal{R}} \cap D_n$,

$$\nabla_{\psi(x)} z_n^\pm = \lim_{B_n \ni \varphi \to \psi} \nabla_{\varphi(x)} (\gamma_n e^{\pm i \eta_n'}).$$

Moreover, as $\varrho_n = \frac{\mu_n - \tau_n}{\gamma_n / 2}$ satisfies $-1 < \varrho_n < 1$,

$$\gamma_n e^{\pm i \eta_n'} = \gamma_n (-\varrho_n \pm \sqrt[s]{\varrho_n^2 - 1})^{\zeta_n(\lambda_n^-)}$$

can be written as

$$\gamma_n e^{\pm i \eta_n'} = \left(-2(\mu_n - \tau_n) \pm i \gamma_n \sqrt[+]{1 - \varrho_n^2}\right) \left(-\varrho_n \pm i \sqrt[+]{1 - \varrho_n^2}\right)^{\hat{\zeta}_n} \qquad \text{(III.26)}$$

where $\hat{\zeta}_n := \zeta_n(\lambda_n^-) - 1$ satisfies $\hat{\zeta}_n = 0(\gamma_n)$ by Lemma III.12 and therefore

$$\lim_{B_n \ni \varphi \to \psi} \left(-\varrho_n \pm i \sqrt[+]{1 - \varrho_n^2}\right)^{\hat{\zeta}_n} = \lim_{B_n \ni \varphi \to \psi} (\pm i)^{\hat{\zeta}_n} = 1.$$

As $\varphi \in B_n$, one has $\mu_n = \tau_n$ as well as $\varrho_n = 0$. Further, in the limit, the first factor on the right side of (III.26) vanishes. As the gradients of both factors on the right side of (III.26) have a limit on $\varphi \to \psi$, the product rule can be applied and we conclude from the considerations above that

$$\begin{aligned} \nabla_\psi z_n^\pm &= \lim_{B_n \ni \varphi \to \psi} \nabla_\varphi (\gamma_n e^{\pm i \eta_n'}) \\ &= 2(\nabla_\psi \tau_n - \nabla_\psi \mu_n) \pm i \lim_{B_n \ni \varphi \to \psi} \nabla_\varphi \gamma_n. \end{aligned}$$

In particular this shows that $\lim_{B_n \ni \varphi \to \psi} \nabla_\varphi \gamma_n$ exists. Recall that the gradient of a *simple* periodic eigenvalue $\lambda_n^\pm$ is given by (cf. Proposition I.32)

$$\nabla_{\varphi(x)} \lambda_n^\pm = \left(F_{n2}^\pm(x)^2, F_{n1}^\pm(x)^2\right)$$

where $F_n^\pm = (F_{n1}^\pm, F_{n2}^\pm)$ is a normalized eigenfunction of $\lambda_n^\pm(\varphi)$. Hence

$$\lim_{B_n \ni \varphi \to \psi} \nabla_{\varphi(x)} \gamma_n = \lim_{B_n \ni \varphi \to \psi} \left((F_{n2}^+)^2 - (F_{n2}^-)^2, (F_{n1}^+)^2 - (F_{n1}^-)^2\right)$$

and, by the analyticity of τ_n,

$$2 \nabla_{\psi(x)} \tau_n = \lim_{B_n \ni \varphi \to \psi} \left((F_{n2}^+)^2 + (F_{n2}^-)^2, (F_{n1}^+)^2 + (F_{n1}^-)^2\right).$$

Combining the two limits we conclude that the limit of $F_{n2}^\pm(x)^2$ and the limit of $F_{n1}^\pm(x)^2$ exist in L^2 as $\varphi \to \psi$. Actually by Lemma III.17 below,

$$F_n^\pm(\cdot, \psi) = \lim_{B_n \ni \varphi \to \psi} F_n^\pm(\cdot, \varphi)$$

exists in H^1 and, $F_n^+(\cdot,\psi)$ and $F_n^-(\cdot,\psi)$ are orthogonal, of norm 1 and satisfy

$$\frac{1}{i}\left(F_{n1}^{\pm}(0)-F_{n2}^{\pm}(0)\right)>0.$$

It follows that

$$F_n^+ = \alpha K_n + \beta H_n \tag{III.27}$$

$$F_n^- = |\beta| K_n - \alpha\frac{\beta}{|\beta|}H_n \tag{III.28}$$

where $\alpha>0,\beta\in\mathbb{C}\backslash\{0\}$ and $\alpha^2+|\beta|^2=1$. By Lemma III.18 below, $\alpha=\frac{1}{\sqrt{2}}$ and by Lemma III.19 below, $\beta=\frac{1}{\sqrt{2}}$. Thus

$$\begin{aligned}\lim_{B_n\ni\varphi\to\psi}\nabla_{\varphi(x)}\gamma_n &= \left((F_{n2}^+)^2,(F_{n1}^+)^2\right)-\left((F_{n2}^-)^2,(F_{n1}^-)^2\right)\\ &= 2\left(H_{n2}K_{n2},H_{n1}K_{n1}\right)\end{aligned}$$

and

$$\begin{aligned}2\nabla_{\psi(x)}\tau_n &= \left((F_{n2}^+)^2,(F_{n1}^+)^2\right)+\left((F_{n2}^-)^2,(F_{n1}^-)^2\right)\\ &= \left(H_{n2}^2+K_{n2}^2,H_{n1}^2+K_{n1}^2\right).\end{aligned}$$

Further, by Proposition I.29

$$\nabla_{\psi(x)}\mu_n=(H_{n2}^2,H_{n1}^2).$$

It then follows that

$$\begin{aligned}\nabla_{\psi(x)}z_n^{\pm} &= 2\left(\nabla_{\psi(x)}\tau_n-\nabla_{\psi(x)}\mu_n\right)\pm i\lim_{B_n\ni\varphi\to\psi}\nabla_{\varphi(x)}\gamma_n\\ &= \left(K_{n2}^2-H_{n2}^2,K_{n1}^2-H_{n1}^2\right)\pm 2i\left(H_{n2}K_{n2},H_{n1}K_{n1}\right).\end{aligned}$$

To prove the second part of Lemma III.16, let ψ be a finite gap potential and $N\geq 0$ be an integer such that $\lambda_n^+(\psi)=\lambda_n^-(\psi)$ for $|n|\geq N$. Then $\psi\in D_n$ and the above formula for $\nabla_{\psi(x)}z_n^{\pm}$ holds for any $|n|\geq N$. Recall that H_n admits an asymptotic estimate (cf (I.16))

$$H_n=\frac{1}{\sqrt{2}}\left(e^{-in\pi x},e^{in\pi x}\right)+\ell^2(n).$$

As K_n is a linear combination of $M^{(1)}(x,\mu_n)$ and $M^{(2)}(x,\mu_n)$, one deduces from Proposition I.6 (page 20) and the orthogonality and normalization conditions

$$K_n=\frac{i}{\sqrt{2}}\left(e^{-in\pi x},-e^{in\pi x}\right)+\ell^2(n).$$

Inserting these asymptotic estimates into the formula for $\nabla_{\psi(x)}z_n^{\pm}$ leads to the stated asymptotics.

Clearly, the above formula for $\nabla_{\psi(x)}z_n^{\pm}$, evaluated for $\psi=0$, leads to the claimed formulas for $\nabla_0 z_n^{\pm}$. ■

It remains to prove the three Lemmas used in the proof of Lemma III.16.

Lemma III.17 *For $\psi\in L^2_{\mathcal{R}}\cap D_n$, $\lim_{B_n\ni\varphi\to\psi}F_n^{\pm}(\cdot,\varphi)$ exist in H^1. The limiting functions, denoted by $F_n^{\pm}(\cdot,\psi)$, are eigenfunctions of $L(\psi)$ corresponding to the eigenvalue $\lambda_n^+(\psi)=\lambda_n^-(\psi)$, they are orthogonal, of L^2-norm 1 and satisfy $\frac{1}{i}\left(F_{n1}^{\pm}(0)-F_{n2}^{\pm}(0)\right)>0$.*

Proof For $\varphi\in B_n$ write

$$F_n^{\pm}=ia^{\pm}\check{G}(x,\lambda_n^{\pm},\varphi)+b^{\pm}G(x,\lambda_n^{\pm},\varphi)$$

where we recall that $\check{G}=M^{(1)}-M^{(2)}$ and $G=M^{(1)}+M^{(2)}$ and, in view of (I.62), $a_{\pm}$ and $b_{\pm}$ are in $\mathbb{R}$ with $a^{\pm}>0$. It remains to study the convergence of $a^{\pm}$ and $b^{\pm}$ as $\varphi\to\psi$. We already observed in the proof of Lemma III.16 that $(F_{n2}^{\pm})^2$ and $(F_{n1}^{\pm})^2$ converge in L^2. One has for $j=1,2$

$$\begin{aligned}(F_{nj}^{\pm})^2 &= (ia^{\pm}\check{G}_j+b^{\pm}G_j)^2\\ &= -(a^{\pm})^2\check{G}_j^2+2ia^{\pm}b^{\pm}\check{G}_jG_j+(b^{\pm})^2G_j^2\,.\end{aligned}$$

In a straightforward way one verifies that

$$X^{(1)}:=(\check{G}_2^2,\check{G}_1^2)\ ;\quad X^{(2)}:=(\check{G}_2G_2,\check{G}_1G_1)\ ;\quad X^{(3)}:=(G_2^2,G_1^2)$$

are linearly independent. Denote by $Y^{(1)},Y^{(2)},Y^{(3)}$ the biorthogonal basis to $X^{(1)},X^{(2)},X^{(3)}$, i.e. $Y^{(1)},Y^{(2)},Y^{(3)}$ are elements in $\mathrm{span}(X^{(1)},X^{(2)},X^{(3)})$ satisfying

$$\langle Y^{(j)},X^{(k)}\rangle=\delta_{jk}.$$

As $X^{(1)},X^{(2)},X^{(3)}$ are continuous functions of λ,φ with values in L^2 so are $Y^{(1)},Y^{(2)},Y^{(3)}$, hence the L^2-convergence of $\left((F_{n2}^{\pm})^2,(F_{n1}^{\pm})^2\right)$ imply that

$$\begin{pmatrix}(F_{n2}^{\pm})^2\\(F_{n1}^{\pm})^2\end{pmatrix}\cdot Y^{(1)}=-(a^{\pm})^2\overset{\varphi\to\psi}{\longrightarrow}-A^{\pm}$$

$$\begin{pmatrix}(F_{n2}^{\pm})^2\\(F_{n1}^{\pm})^2\end{pmatrix}\cdot Y^{(2)}=2ia^{\pm}b^{\pm}\overset{\varphi\to\psi}{\longrightarrow}B^{\pm}$$

$$\begin{pmatrix}(F_{n2}^{\pm})^2\\(F_{n1}^{\pm})^2\end{pmatrix}\cdot Y^{(3)}=(b^{\pm})^2\overset{\varphi\to\psi}{\longrightarrow}C^{\pm}$$

where the dot denote the dual pairing between $L^2\times L^2$ and itself, i.e. $F\cdot G=\int_0^1(F_1G_1+F_2G_2)dx$ (no complex conjugation).

As $a^{\pm}>0$, we have

$$a^{\pm}\to a_{\infty}^{\pm}:=\sqrt[+]{A^{\pm}}$$

and

$$|b^{\pm}|\to\sqrt[+]{C^{\pm}}.$$

Hence

$$|a^{\pm}b^{\pm}|=a^{\pm}|b^{\pm}|\to a_{\infty}^{\pm}\sqrt[+]{C^{\pm}}.$$

We claim that

$$\lim_{B_n\ni\varphi\to\psi}\frac{a^+}{a^-}=1. \tag{III.29}$$

To see this identity holds recall that (cf. (I.62))

$$F_n^{\pm}(0)=\frac{i}{2}\sqrt[+]{\frac{-i\delta(\lambda_n^{\pm})}{\dot\Delta(\lambda_n^{\pm})}}\check G(0,\lambda_n^{\pm})+\frac{\varepsilon_n^{\pm}}{2}\sqrt[+]{\frac{-i\check\delta(\lambda_n^{\pm})}{\dot\Delta(\lambda_n^{\pm})}}G(0,\lambda_n^{\pm}).$$

By Lemma I.21

$$\delta(\lambda)=2i(\mu_0-\lambda)\prod_{k\neq 0}\frac{\mu_k-\lambda}{k\pi}$$

and by Lemma I.20

$$\dot\Delta(\lambda)=2(\dot\lambda_0-\lambda)\prod_{k\neq 0}\frac{\dot\lambda_k-\lambda}{k\pi}.$$

Hence

$$\begin{aligned}\frac{a^+}{a^-}&=\frac{1}{2}\sqrt[+]{\frac{-i\delta(\lambda_n^+)}{\dot\Delta(\lambda_n^+)}}\Big/\frac{1}{2}\sqrt[+]{\frac{-i\delta(\lambda_n^-)}{\dot\Delta(\lambda_n^-)}}\\&=\sqrt[+]{\frac{\dot\Delta(\lambda_n^-)}{\dot\Delta(\lambda_n^+)}\frac{\delta(\lambda_n^+)}{\delta(\lambda_n^-)}}.\end{aligned}$$

As $\dot\lambda_n=\tau_n+0(\gamma_n^{3/2})$ (cf. Lemma I.22) we conclude that

$$\lim_{B_n\ni\varphi\to\psi}\frac{\dot\Delta(\lambda_n^-)}{\dot\Delta(\lambda_n^+)}=\lim_{B_n\ni\varphi\to\psi}\frac{\dot\lambda_n-\lambda_n^-}{\dot\lambda_n-\lambda_n^+}=-1.$$

As $\mu_n=\tau_n$ for $\varphi\in B_n$ we have

$$\lim_{B_n\ni\varphi\to\psi}\frac{\delta(\lambda_n^+)}{\delta(\lambda_n^-)}=\lim_{B_n\ni\varphi\to\psi}\frac{\tau_n-\lambda_n^+}{\tau_n-\lambda_n^-}=-1$$

hence (III.29) follows. Further we claim that

$$a_\infty^+>0\ . \tag{III.30}$$

To see this, it suffices to show that $a_\infty^+\neq 0$. Notice that for $\varphi\in B_n$,

$$\begin{aligned}0=\langle F_n^+,F_n^-\rangle=&a^+a^-\langle\check G(\cdot,\lambda_n^+),\check G(\cdot,\lambda_n^-)\rangle+\\&+ia^+b^-\langle\check G(\cdot,\lambda_n^+),G(\cdot,\lambda_n^-)\rangle-b^+ia^-\langle G(\cdot,\lambda_n^+),\check G(\cdot,\lambda_n^-)\rangle\\&+b^+b^-\langle G(\cdot,\lambda_n^+),G(\cdot,\lambda_n^-)\rangle\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denote the inner product in L^2. Assume that $a_\infty^+=0$. Then, by (III.29), $a_\infty^-=0$ and as $|b^\pm|\to\sqrt[+]{C^\pm}$ it then follows that as $\varphi\to\psi$

$$0=b^+b^-\langle G(\cdot,\lambda_n^+),G(\cdot,\lambda_n^-)\rangle+o(1)$$

or

$$b^+b^-\overset{\varphi\to\psi}{\longrightarrow}0\text{ and }(b^+b^-)^2\to C^+C^-=0.$$

Hence either $C^+=0$ or $C^-=0$. Without loss of generality assume that $C^+=0$. Together with $A^+=B^+=0$ it then follows that $\left((F_{n1}^+)^2,(F_{n1}^+)^2\right)\to 0$ which contradicts $\|F_n^+\|=1$ hence claim (III.30) follows.
Combining $\lim_{B_n\ni\varphi\to\psi}2ia^\pm b^\pm=B^\pm$ and $\lim_{B_n\ni\varphi\to\psi}a^\pm=a_\infty^\pm>0$ one concludes

$$b^\pm=\frac{2ia^\pm b^\pm}{2ia^\pm}\overset{\varphi\to\psi}{\longrightarrow}\frac{B^\pm}{2ia_\infty^\pm}=:b_\infty^\pm$$

and hence, as $G,\check G$ are continuous in λ,φ, with values in H^1,

$$F_n^\pm=ia^\pm\check G(\cdot,\lambda_n^\pm,\varphi)+b^\pm G(\cdot,\lambda_n^\pm,\varphi)\to ia_\infty^\pm\check G(\cdot,\lambda_*,\psi)+b_\infty^\pm G(\cdot,\lambda_*,\psi),$$

in H^1 with $\lambda_*:=\lambda_n^+(\psi)$. As a consequence

$$F_n^\pm(\cdot,\psi):=ia_\infty^\pm\check G(\cdot,\lambda_*,\psi)+b_\infty^\pm G(\cdot,\lambda_*,\psi)$$

are orthogonal, $\langle F_n^+(\cdot,\psi),F_n^-(\cdot,\psi)\rangle=0$, of norm 1, $\|F_n^\pm(\cdot,\psi)\|=1$, and satisfy the normalization condition $\frac{1}{i}\left(F_{n1}^\pm(0,\psi)-F_{n2}^\pm(0,\psi)\right)=2a_\infty^\pm>0$. ■

Next we compute the values of α and β in (III.27) and (III.28).

Lemma III.18 $\alpha=\frac{1}{\sqrt2}$.

Proof For $\varphi\in L^2_{\mathcal R}$,

$$(\lambda_n^+-\mu_n)\langle F_n^+,H_n\rangle=\langle i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{d}{dx}F_n^+,H_n\rangle-\langle F_n^+,i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{d}{dx}H_n\rangle.$$

As $i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{d}{dx}$ is formally selfadjoint, we have, integrating by parts

$$\langle i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{d}{dx}F_n^+,H_n\rangle=\left(i\begin{pmatrix}1&0\\0&-1\end{pmatrix}F_n^+\right)\cdot\bar H_n\Big|_0^1+\langle F_n^+,i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{d}{dx}H_n\rangle.$$

Hence, with

$$F_{nj}^\pm(1)=(-1)^nF_{nj}^\pm(0)\ ;\quad H_{n1}(1)=H_{n2}(1)\ ;\quad H_{n1}(0)=H_{n2}(0)$$

we get

$$(\lambda_n^+-\mu_n)\langle F_n^+,H_n\rangle=i\left(F_{n1}^+(0)-F_{n2}^+(0)\right)\left((-1)^n\overline{H_{n1}(1)}-\overline{H_{n1}(0)}\right).$$

Similarly one shows

$$(\lambda_n^- - \mu_n)\langle F_n^-, H_n\rangle = i\left(F_{n1}^-(0) - F_{n2}^-(0)\right)\left((-1)^n\overline{H_{n1}(1)} - \overline{H_{n1}(0)}\right).$$

For $\varphi \in B_n$, we have $\mu_n = \tau_n$ and $\gamma_n > 0$. Thus $\lambda_n^+ - \mu_n = -(\lambda_n^- - \mu_n) = \gamma_n/2 > 0$. Hence $H_{n1}(1) - (-1)^n H_{n1}(0) \neq 0$ and then, with the identities above,

$$\frac{\langle F_n^+, H_n\rangle}{\langle F_n^-, H_n\rangle} = -\frac{F_{n1}^+(0) - F_{n2}^+(0)}{F_{n1}^-(0) - F_{n2}^-(0)}. \tag{III.31}$$

The limits of both sides of (III.31) are computed separately,

$$\lim_{B_n \ni \varphi \to \psi} \frac{\langle F_n^+, H_n\rangle}{\langle F_n^-, H_n\rangle} = \frac{\lim_{B_n \ni \varphi \to \psi}\langle F_n^+, H_n\rangle}{\lim_{B_n \ni \varphi \to \psi}\langle F_n^-, H_n\rangle} = \frac{\beta}{-\alpha\frac{\beta}{|\beta|}} = -\frac{|\beta|}{\alpha},$$

$$\lim_{B_n \ni \varphi \to \psi} -\frac{F_{n11}^+(0) - F_{n2}^+(0)}{F_{n1}^-(0) - F_{n2}^-(0)} = -\frac{\alpha\left(K_{n1}(0) - K_{n2}(0)\right) + \beta\left(H_{n1}(0) - H_{n2}(0)\right)}{|\beta|\left(K_{n1}(0) - K_{n2}(0)\right) - \alpha\frac{\beta}{|\beta|}\left(H_{n1}(0) - H_{n2}(0)\right)} = -\frac{\alpha}{|\beta|}$$

where we used convergence in H^1 of $F_n^{\pm}$ and the normalization condition $K_{n1}(0) - K_{n2}(0) \neq 0$. Hence together with (III.31) we get $-\frac{|\beta|}{\alpha} = -\frac{\alpha}{|\beta|}$ or $|\beta|^2 = \alpha^2$. As $\alpha^2 + |\beta|^2 = 1$ and $\alpha > 0$ it follows that $\alpha = 1/\sqrt{2}$. ■

Lemma III.19 $\beta = \frac{1}{\sqrt{2}}$.

Proof As $\alpha^2 + |\beta|^2 = 1$ and $\alpha = 1/\sqrt{2}$ (Lemma III.18) it follows that $|\beta| = 1/\sqrt{2}$. It remains to prove that $\beta \geq 0$. By Lemma III.17 and the definition of β we have

$$\beta = \lim_{B_n \ni \varphi \to \psi}\langle F_n^+, H_n\rangle. \tag{III.32}$$

As in the proof of Lemma III.18 write

$$(\lambda_n^+ - \mu_n)\langle F_n^+, H_n\rangle = i\left(F_{n1}^+(0) - F_{n2}^+(0)\right)\left((-1)^n\overline{H_{n1}(1)} - \overline{H_{n1}(0)}\right). \tag{III.33}$$

By Lemma III.17, one has in the limit $\varphi \to \psi$

$$i\left(F_{n1}^+(0) - F_{n2}^+(0)\right) < 0\,.$$

Further, with $H_{n1}(x) = \frac{G_{n1}(x)}{\|G_n\|}$ and $G_{n1}(0) = 1$,

$$(-1)^n H_{n1}(1) - H_{n1}(0) = \frac{1}{\|G_n\|}\left((-1)^n G_{n1}(1) - 1\right).$$

To compute the limit of $\frac{(-1)^n G_1(1,\mu_n)-1}{\lambda_n^+ - \mu_n}$ for $\varphi \to \psi$ write, for $\varphi \in B_n$,

$$\begin{aligned} G_1(1,\mu_n) &= \frac{1}{2}\left(G_1(1,\mu_n) + G_2(1,\mu_n)\right) \\ &= \frac{1}{2}\left(M_{11}(1,\mu_n) + M_{22}(1,\mu_n)\right) \\ &+ \frac{1}{2}\left(M_{21}(1,\mu_n) + M_{12}(1,\mu_n)\right) \\ &= \frac{1}{2}\Delta(\mu_n) + \frac{1}{2}\sqrt[*]{\Delta(\mu_n)^2 - 4} \end{aligned}$$

where we used that, by definition (I.68),

$$\sqrt[*]{\Delta(\mu_n)^2 - 4} = M_{21}(1,\mu_n) + M_{12}(1,\mu_n).$$

As $\Delta(\lambda_n^-) = 2(-1)^n$ and $\lambda_n^+ - \mu_n = \gamma_n/2$ we then conclude

$$\frac{(-1)^n G_1(1,\mu_n) - 1}{\lambda_n^+ - \mu_n} = \frac{(-1)^n}{\gamma_n}\left(\Delta(\mu_n) - \Delta(\lambda_n^-)\right) + \frac{(-1)^n}{\gamma_n}\sqrt[*]{\Delta(\mu_n)^2 - 4}.$$

By Taylor's remainder formula, with $\mu_n = \tau_n$

$$\Delta(\mu_n) - \Delta(\lambda_n^-) = \frac{\gamma_n}{2}\int_0^1 \dot{\Delta}(\lambda(t))dt$$

where $\lambda(t) = \lambda_n^- + t\gamma_n/2$. Hence, with $\lambda_* := \lambda_n^-(\psi) = \lambda_n^+(\psi)$,

$$\lim_{B_n \ni \varphi \to \psi} \frac{\Delta(\mu_n) - \Delta(\lambda_n^-)}{\gamma_n/2} = \dot{\Delta}(\lambda_*) = 0$$

As, for $\varphi \in B_n$, $(-1)^{n-1}\sqrt[*]{\Delta(\mu_n)^2 - 4} = \sqrt[+]{\Delta(\mu_n)^2 - 4}$ we then have

$$\lim_{B_n \ni \varphi \to \psi} \frac{(-1)^n G_1(1,\mu_n) - 1}{\gamma_n/2} = \lim_{B_n \ni \varphi \to \psi} \frac{(-1)^n}{\gamma_n}\sqrt[*]{\Delta(\mu_n)^2 - 4} \leq 0.$$

Combining (III.32) (III.33) with the results above, one concludes

$$\begin{aligned} \beta &= i\left(F_{n1}^+(0) - F_{n2}^+(0)\right)\frac{\left((-1)^n\overline{H_{n1}(1)} - \overline{H_{n1}(0)}\right)}{\gamma_n/2} \\ &= i\left(F_{n1}^+(0) - F_{n2}^+(0)\right)\lim_{B_n \ni \varphi \to \psi}\left(\frac{(-1)^n G_1(1,\mu_n) - 1}{\gamma_n/2}\right) \\ &\geq 0. \end{aligned}$$

■

Lemma III.16 allows us to obtain asymptotic estimates of $\nabla_{\varphi(x)} x_n$ and $\nabla_{\varphi(x)} y_n$ where φ is a finite gap potential.

Proposition III.20 *For any finite gap potential φ,*

$$\nabla_{\varphi(x)} x_n = -\frac{1}{\sqrt{2}} \left(e^{i2\pi nx}, e^{-i2\pi nx}\right) + \ell^2(n)$$
$$\nabla_{\varphi(x)} y_n = \frac{i}{\sqrt{2}} \left(-e^{i2\pi nx}, e^{-i2\pi nx}\right) + \ell^2(n) \ .$$

At $\varphi = 0$ the latter identities hold without error term.

Proof Let φ be a finite gap potential and let $N \geq 0$ be such that $\lambda_n^+ = \lambda_n^- \quad \forall |n| \geq N+1$. By (III.15) and (III.16)

$$x_n = \sqrt{2}\frac{\xi_n}{4} \left(z_n^+ e^{i\beta_n} + z_n^- e^{-i\beta_n}\right)$$

and

$$y_n = \sqrt{2}\frac{\xi_n}{4i} \left(z_n^+ e^{i\beta_n} - z_n^- e^{-i\beta_n}\right).$$

Recall that $\xi_n = 1+\ell^2(n)$ (cf Theorem III.3) and $\beta_n = 0(\frac{1}{n})$ (cf Lemma III.9, using that $\gamma_k = 0$ and $\mu_k - \tau_k = 0 \ \forall |k| \geq N+1$). As $z_n^\pm = 0$ for $|n| \geq N+1$ one obtains

$$\nabla_{\varphi(x)} x_n = \sqrt{2}\frac{1}{4} \left(\nabla_{\varphi(x)} z_n^+ + \nabla_{\varphi(x)} z_n^-\right) + \ell^2(n)$$

and

$$\nabla_{\varphi(x)} y_n = \sqrt{2}\frac{1}{4i} \left(\nabla_{\varphi(x)} z_n^+ - \nabla_{\varphi(x)} z_n^-\right) + \ell^2(n).$$

The claimed asymptotics then follow from Lemma III.16. ■

III.6 Canonical relations

In this section we prove that the map $\Omega : L^2_{\mathcal{R}} \to \ell^2(\mathbb{Z}, \mathbb{R}^2)$ is Poisson, i.e. that the push forward of the Poisson structure by Ω is the canonical Poisson structure on $\ell^2(\mathbb{Z}, \mathbb{R}^2)$.
First, following Mc Kean-Vaninsky [MV] we establish canonical relations for the action and angle variables.

Proposition III.21 *For any $\varphi \in L^2_{\mathcal{R}}$, and $k, n \in \mathbb{Z}$,*

$$\{I_k(\varphi), I_n(\varphi)\} = 0.$$

Proof By Theorem III.2, for $k \in \mathbb{Z}$ and $j = 1, 2$,

$$\frac{\partial I_k}{\partial \varphi_j(x)} = -\frac{1}{\pi} \int_{\Gamma_k} \frac{\partial \Delta(\lambda)}{\partial \varphi_j(x)} \frac{1}{\sqrt[c]{\Delta(\lambda)^2 - 4}} \, d\lambda$$

and hence

$$\{I_k, I_n\} = \frac{1}{\pi^2} \int_{\Gamma_k} \int_{\Gamma_n} \frac{\{\Delta(\lambda), \Delta(\mu)\}}{\sqrt[c]{\Delta(\lambda)^2 - 4}\sqrt[c]{\Delta(\mu)^2 - 4}} \, d\lambda d\mu.$$

As $\{\Delta(\lambda), \Delta(\mu)\} = 0$ for any $\lambda, \mu \in \mathbb{C}$ (Proposition I.36 page 55) it follows that $\{I_k, I_n\} = 0$. ■

Next we want to show that for $\varphi \in L^2_{\mathcal{R}}$ with $\gamma_k \neq 0$, $\{I_n, \theta_k\} = \delta_{nk}$. Recall that $\theta_k = \eta_k + \sum_{j \neq k} \beta_j^{(k)}$ is a real analytic function on $W \backslash D_k$ with values in $\mathbb{R}/\pi\mathbb{Z}$ (cf Theorem III.10). First we compute $\{\beta_j^{(k)}, I_n\}$ for any $j, n \in \mathbb{Z}$) where, for convenience, $\beta_k^{(k)} := \eta_k$. Introduce

$$g_j^{(k)}(\lambda) := \frac{\delta(\lambda)}{\lambda - \mu_j} \frac{\psi_k(\mu_j)}{\dot{\delta}(\mu_j)}.$$

Lemma III.22 *Let $\varphi \in L^2_{\mathcal{R}}$ and $k \in \mathbb{Z}$ with $\gamma_k(\varphi) \neq 0$. Then, for any $j, n \in \mathbb{Z}$,*

(i) $\{\beta_j^{(k)}, \Delta(\lambda)\} = \frac{1}{2} g_j^{(k)}(\lambda)$.

(ii) $\{\beta_j^{(k)}, I_n\} = -\frac{1}{2\pi} \int_{\Gamma_n} \frac{g_j^{(k)}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} \, d\lambda$.

Proof As the cases $j = k$ and $j \neq k$ are proved in the same way, let us concentrate on $j = k$. Recall that

$$\eta_k \equiv \beta_k^{(k)} = \int_{\lambda_k^-}^{\mu_k^*} \frac{\psi_k(\lambda)}{\sqrt{\Delta(\lambda^2) - 4}} \, d\lambda$$

where $\mu_k^* = (\mu_k, y_k)$ is a point on the Riemann surface Σ_φ with

$$y_k = \sqrt[*]{\Delta(\mu_k)^2 - 4} = (M_{21} + M_{12}) \big|_{1,\mu_k}.$$

Further, the gradient of the action variable I_n is given by (cf Theorem III.2)

$$\nabla_{\varphi(x)} I_n = -\frac{1}{\pi} \int_{\Gamma_n} \nabla_{\varphi(x)} \Delta(\lambda) \frac{1}{\sqrt[c]{\Delta(\lambda)^2 - 4}} \, d\lambda.$$

Hence

$$\{\eta_k, I_n\} = -\frac{1}{\pi} \int_{\Gamma_n} \frac{1}{\sqrt[c]{\Delta(\lambda)^2 - 4}} \{\eta_k, \Delta(\lambda)\} d\lambda. \tag{III.34}$$

By Corollary I.39 there exists a sequence $(\varphi_j)_{j \geq 1} \subseteq Iso(\varphi)$ with $\lim_{j \to \infty} \varphi_j = \varphi$ in $L^2_{\mathcal{R}}$ so that

$$\lambda_k^- < \mu_k(\varphi_j) < \lambda_k^+ \quad \forall j \in \mathbb{Z}.$$

As Δ, λ_k^- and ψ_k are isospectral invariants it then follows that for any φ_j

$$\{\eta_k, \Delta(\lambda)\} = \frac{\psi_k(\mu_k)}{\sqrt[*]{\Delta(\mu_k)^2 - 4}} \{\mu_k, \Delta(\lambda)\}.$$

By Proposition I.37

$$\{\mu_k, \Delta(\lambda)\} = \frac{1}{2} \frac{\delta(\lambda)}{\dot{\delta}(\mu_k)} \frac{1}{\lambda - \mu_k} \sqrt[*]{\Delta(\mu_k)^2 - 4}$$

and thus, for any φ_j,

$$\{\eta_k, \Delta(\lambda)\} = \frac{1}{2} \frac{\delta(\lambda)}{\lambda - \mu_k} \frac{\psi_k(\mu_k)}{\dot{\delta}(\mu_k)} = \frac{1}{2} g_k^{(k)}(\lambda).$$

As both sides of the latter identity are continuous in φ it remains valid for $\varphi = \lim_{j\to\infty} \varphi_j$ and (i) is proved. Substituting this formula into (III.34) then leads to

$$\{\eta_k, I_n\} = -\frac{1}{2\pi} \int_{\Gamma_n} \frac{g_k^{(k)}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}}.$$

■

Lemma III.23 *For any $\varphi \in L^2_{\mathcal{R}}$, the infinite sum $g_k(\lambda) = \sum_j g_j^{(k)}(\lambda)$ is absolutely summable locally uniformly in λ and one has*

$$g_k = \psi_k.$$

Proof Recall that

$$g_j^{(k)}(\lambda) = \frac{\delta(\lambda)}{\lambda - \mu_j} \frac{\psi_k(\mu_j)}{\dot{\delta}(\mu_j)}.$$

Using the product representation of ψ_k and the estimate of the zeroes $\nu_j^{(k)}$ of ψ_k, $\mu_j - \nu_j^{(k)} = \ell^2(j)$ $(j \neq k)$ (cf Theorem II.1) as well as the estimate on infinite products given in Lemma I.17 one sees that $\psi_k(\mu_j) = \frac{\ell^2(j)}{\langle k-j \rangle}$ for any $j \in \mathbb{Z}$. By the infinite product representation for $\dot{\delta}(\lambda)$ and again Lemma I.17 it follows that $1/\dot{\delta}(\mu_j) = 0(1)$ for $j \to \infty$. Finally, as $|\delta(\lambda)| \leq Ce^{|Im\lambda|}$ uniformly for $\lambda \in \mathbb{C}$ one then concludes that the series $\Sigma_j g_j^{(k)}$ converges uniformly in any strip $|Im\lambda| \leq a$ and is an entire function which we denote by $g_k(\lambda)$ satisfying

$$|g_k(\lambda)| \leq C' e^{|Im\lambda|} \quad (\lambda \in \mathbb{C}).$$

Next we prove that g_k is an element in $\mathcal{I}_0$. As $\left| \frac{\delta(\lambda)}{\lambda - \mu_j} \right| \leq C \frac{1}{\langle \lambda - \mu_j \rangle}$ where C can be chosen independently of j it follows from the asymptotics of $\psi_k(\mu_j)/\dot{\delta}(\mu_j)$ that $\int_{-\infty}^{\infty} |g_k(\lambda)|^2 d\lambda < \infty$ and therefore $g_k \in \mathcal{I}_0$.

From the definition of $g_j^{(k)}$ one sees that $g_j^{(k)}(\mu_\ell) = \psi_k(\mu_\ell)\delta_{j\ell}$ $(j, \ell \in \mathbb{Z})$ and thus $g_k(\mu_\ell) = \psi_k(\mu_\ell)$ $\forall \ell \in \mathbb{Z}$. As ψ_k is also an element of $\mathcal{I}_0$, Lemma II.4 implies that $g_k \equiv \psi_k$. ■

Proposition III.24 *Let $k, n \in \mathbb{Z}$. Then for any $\varphi \in L^2_{\mathcal{R}} \backslash D_k$*

$$\{I_n, \theta_k\} = \delta_{nk}.$$

Proof By Theorem III.10, $\beta_k = \Sigma_{j\neq k} \beta_j^{(k)}$ converges locally uniformly for φ in $W \backslash D_k$. Hence by Cauchy's theorem, this holds for the gradient $\bigtriangledown_{\varphi(x)} \beta_k$ as well. Combining Lemma III.22 and Lemma III.23 then yields

$$\{I_n, \theta_k\} = \frac{1}{2\pi} \int_{\Gamma_n} \psi_k(\lambda) \frac{d\lambda}{\sqrt[c]{\Delta(\lambda)^2 - 4}} = \delta_{nk}$$

where for the latter identity we use the normalization condition of ψ_k (cf Theorem II.1). ■

It remains to prove that the variables Θ_k commute with each other.

Proposition III.25 *Let $\varphi \in L^2_{\mathcal{R}} \backslash (D_k \cup D_n)$ with n, k arbitrary. Then*

$$\{\Theta_k, \Theta_n\} = 0.$$

The proof of this Proposition requires the following three auxilary Lemmas:

Lemma III.26 *For any $j \neq \ell$ and $\varphi \in L^2_{\mathcal{R}} \backslash D_\ell$*

$$\{\mu_j, \lambda_\ell^-\} = \frac{1}{2} \sqrt[*]{\Delta(\mu_j)^2 - 1} \frac{\delta(\lambda_\ell^-)}{\dot{\delta}(\mu_j)} \frac{1}{\lambda_\ell^- - \mu_j} \frac{1}{\dot{\Delta}(\lambda_\ell^-)}.$$

Proof By Proposition I.29

$$\nabla_{\varphi(x)} \mu_j = \frac{1}{\|G_j(\cdot)\|^2} \left(G_{j,2}(x)^2, G_{j,1}(x)^2 \right)$$

where $G_j(x) = M^{(1)}(x, \mu_j) + M^{(2)}(x, \mu_j)$ is an eigenfunction corresponding to μ_j and by Proposition I.32,

$$\nabla_{\varphi(x)} \lambda_\ell^- = \left(F_{\ell,2}^-(x)^2, F_{\ell,1}^-(x)^2 \right)$$

where (cf (I.62))

$$F_\ell^-(x) = \frac{\varepsilon_\ell^-}{2} \sqrt[+]{\frac{-i\check{\delta}(\lambda_\ell^-)}{\dot{\Delta}(\lambda_\ell^-)}} G(x, \lambda_\ell^-) + \frac{i}{2} \sqrt[+]{\frac{-i\delta(\lambda_\ell^-)}{\dot{\Delta}(\lambda_\ell^-)}} \check{G}(x, \lambda_\ell^-) \qquad \text{(III.35)}$$

is an eigenfunction corresponding to λ_ℓ^-. Hence

$$\{\mu_j, \lambda_\ell^-\} = \frac{i}{\|G_j\|^2}\int_0^1 \Big(G_{j,2}(x)F_{\ell,1}^-(x)\Big)^2 - \Big(G_{j,1}(x)F_{\ell,2}^-(x)\Big)^2 dx$$
$$= \frac{i}{\|G_j\|^2}\int_0^1 \Big(G_{j,2}F_{\ell,1}^- + G_{j,1}F_{\ell,2}^-\Big)\Big(G_{j,2}F_{\ell,1}^- - G_{j,1}F_{\ell,2}^-\Big) dx.$$

By the definition of the Wronskian,

$$W[F_\ell^-, G_j] = F_{\ell,1}^- G_{j,2} - F_{\ell,2}^- G_{j,1}$$

and by Lemma I.34

$$\frac{-i}{\lambda_\ell^- - \mu_j}\frac{d}{dx}W[F_\ell^-, G_j] = F_{\ell,1}^- G_{j,2} + F_{\ell,2}^- G_{j,1}$$

and thus

$$\{\mu_j, \lambda_\ell^-\} = \frac{1}{\lambda_\ell^- - \mu_j}\frac{1}{\|G_j\|^2}\frac{1}{2}\left(W[G_j, F_\ell^-](x)\right)^2 \Big|_0^1. \tag{III.36}$$

As $G_{j,1}(1) = G_{j,2}(1)$ as well as $G_{j,1}(0) = 1 = G_{j,2}(0)$ and F_ℓ^- is periodic or antiperiodic one computes

$$\left(W[G_j, F_\ell^-](x)\right)^2 \Big|_0^1 = \Big(G_{j,1}(1)^2 - 1)(F_{\ell,2}^-(0) - F_{\ell,1}^-(0)\Big)^2. \tag{III.37}$$

The two terms on the right side of (III.37) can be simplified further. Using that G_j satisfies Dirichlet boundary conditions, i.e.

$$(M_{11} + M_{12})\Big|_{1,\mu_j} = (M_{21} + M_{22})\Big|_{1,\mu_j}$$

together with the Wronskian identity $1 = M_{11}M_{22} - M_{12}M_{21}$ one has

$$\begin{aligned} G_{j,1}(1)^2 - 1 &= (M_{11} + M_{12})^2 - 1 \\ &= (M_{11} + M_{12})(M_{21} + M_{22}) - (M_{11}M_{22} - M_{12}M_{21}) \\ &= (M_{11} + M_{12})M_{21} + M_{12}(M_{22} + M_{21}) \\ &= (M_{11} + M_{12})(M_{21} + M_{12}) \end{aligned}$$

where for the last identity we made use of the Dirichlet boundary conditions once more. As

$$\sqrt[*]{\Delta(\mu_j)^2 - 4} = (M_{12} + M_{21})\Big|_{1,\mu_j}$$

and

$$\|G_j\|^2 = i\dot{\delta}(\mu_j)(M_{11} + M_{12})\Big|_{1,\mu_j}$$

one thus obtains

$$G_{j,1}(1)^2 - 1 = \frac{-i}{\dot{\delta}(\mu_j)}\sqrt[*]{\Delta(\mu_j)^2 - 4}\|G_j\|^2. \tag{III.38}$$

Concerning $(F_{\ell,2}^-(0) - F_{\ell,1}^-(0))^2$, use that $G(0,\lambda) = (1,1)$ and $\breve{G}(0,\lambda) = (1,-1)$ to conclude from (III.35) that

$$\Big(F_{\ell,2}^-(0) - F_{\ell,1}^-(0)\Big)^2 = i\delta(\lambda_\ell^-)/\dot{\Delta}(\lambda_\ell^-). \tag{III.39}$$

Substituting (III.37) - (III.39) into (III.36) yields the claimed formula. ■

Lemma III.26 allows us to prove that $\{\Theta_k, \Theta_n\} = 0$ for certain potentials $\varphi \in L^2_{\mathcal{R}}$:

Lemma III.27 *Let $\varphi \in L^2_{\mathcal{R}}\backslash(D_k \cup D_n)$ (with $n, k \in \mathbb{Z}$ arbitrary) with $\mu_j(\varphi) = \lambda_j^-(\varphi)$ for any $j \in \mathbb{Z}$. Then $\{\Theta_k, \Theta_n\} = 0$.*

Proof In view of the definition $\Theta_k = \sum_{j\in\mathbb{Z}} \beta_j^{(k)}$ (with $\beta_k^{(k)} := \eta_k$) the claimed result follows if we prove that for any given $j, \ell \in \mathbb{Z}$, $\{\beta_j^{(k)}, \beta_\ell^{(n)}\} = 0$. As the Poisson bracket is skew symmetric it suffices to consider the case $j \neq \ell$. By Proposition I.43, the potential φ can be approximated in $L^2_{\mathcal{R}}$ by a sequence $(\varphi_\alpha)_{\alpha\geq 1}$ so that, for any $\alpha \geq 1$

$$\begin{aligned} &\lambda_j^-(\varphi_\alpha) < \mu_j(\varphi_\alpha) < \lambda_j^+(\varphi_\alpha), \\ &\lambda_\ell^-(\varphi_\alpha) < \mu_\ell(\varphi_\alpha) < \lambda_\ell^+(\varphi_\alpha), \\ &\lim_{\alpha\to\infty} \frac{\mu_j - \lambda_j^-}{\lambda_j^+ - \mu_j}\Big|_{\varphi_\alpha} = 0 \text{ and } \lim_{\alpha\to\infty} \frac{\mu_\ell - \lambda_\ell^-}{\lambda_\ell^+ - \mu_\ell}\Big|_{\varphi_\alpha} = 0. \end{aligned} \tag{III.40}$$

From the definition of $\beta_j^{(k)}$ (cf (III.11)) and the product representations of ψ_k and $\Delta^2 - 4$ one gets

$$\begin{aligned} \nabla_{\varphi_\alpha(x)}\beta_j^{(k)} = &\frac{\psi_k(\mu_j)}{\sqrt[*]{\Delta(\mu_j)^2 - 4}}\Big(\nabla_{\varphi_\alpha(x)}\mu_j - \nabla_{\varphi_\alpha(x)}\lambda_j^-\Big) \\ &+ 0\left(\sqrt{\frac{\mu_j - \lambda_j^-}{\lambda_j^+ - \mu_j}\Big|_{\varphi_\alpha}}\right) \end{aligned} \tag{III.41}$$

where the error term is uniform in $0 \leq x \leq 1$ and $\alpha \geq 1$. This leads to the following decomposition

$$\{\beta_j^{(k)}, \beta_\ell^{(n)}\} = A + R \tag{III.42}$$

where

$$A := \frac{\psi_k(\mu_j)}{\sqrt[*]{\Delta(\mu_j)^2-4}} \frac{\psi_n(\mu_\ell)}{\sqrt[*]{\Delta(\mu_\ell)^2-4}} \left(-\{\mu_j, \lambda_\ell^-\} - \{\lambda_j^-, \mu_\ell\} \right)$$

and R is the remainder term defined by the identity (III.42). First we want to prove that $\lim_{\alpha\to\infty} R(\varphi_\alpha) = 0$. Recall that (cf Proposition I.29)

$$\nabla_{\varphi_\alpha(x)}\mu_j = \frac{1}{\|G_j(\cdot)\|^2}\left(G_{j,2}(x)^2, G_{j,1}(x)^2\right) \tag{III.43}$$

and (cf Proposition I.32),

$$\nabla_{\varphi_\alpha(x)}\lambda_j^- = \left(F_{j,2}^-(x)^2, F_{j,1}^-(x)^2\right) \tag{III.44}$$

where (cf (I.62))

$$F_j^-(x) = \frac{\varepsilon_j^-}{2}\sqrt[+]{\frac{-i\check{\delta}(\lambda_j^-)}{\dot{\Delta}(\lambda_j^-)}}G(x,\lambda_j^-) + \frac{i}{2}\sqrt[+]{\frac{-i\delta(\lambda_j^-)}{\dot{\Delta}(\lambda_j^-)}}\check{G}(x,\lambda_j). \tag{III.45}$$

is a normalized eigenfunction corresponding to $\lambda_j^- \equiv \lambda_j^-(\varphi_\alpha)$. Using the product representation for $\delta(\lambda), \check{\delta}(\lambda)$ (cf Lemma I.21) $\dot{\Delta}(\lambda)$ (cf Lemma I.20) and the definition $G_j(x) = G(x,\mu_j)$ as well as (III.40) it follows that

$$\lim_{\alpha\to\infty} F_j^-(x,\varphi_\alpha) = \frac{G_j(x,\varphi)}{\|G_j(\cdot,\varphi)\|}.$$

Therefore

$$\lim_{\alpha\to\infty} \|\nabla_{\varphi_\alpha(x)}(\mu_j - \lambda_j^-)\| = 0. \tag{III.46}$$

Furthermore, by the product representations of the quantities involved, the sequence $\left(\psi_k(\mu_j)\sqrt[+]{\frac{\mu_j-\lambda_j^-}{\lambda_j^+-\mu_j}}\frac{1}{\sqrt[*]{\Delta(\mu_j)^2-4}}\Big|_{\varphi_\alpha}\right)_{\alpha\geq 1}$ is bounded. Hence in view of (III.40) - (III.42) and (III.46), $\lim_{\alpha\to\infty} R(\varphi_\alpha) = 0$ and thus

$$\begin{aligned}\left\{\beta_j^{(k)}, \beta_\ell^{(n)}\right\}(\varphi) &= \lim_{\alpha\to\infty}\left\{\beta_j^{(k)}, \beta_\ell^{(n)}\right\}(\varphi_\alpha)\\ &= \lim_{\alpha\to\infty}\frac{\psi_k(\mu_j)\psi_n(\mu_\ell)}{\sqrt[*]{\Delta(\mu_j)^2-4}\sqrt[*]{\Delta(\mu_\ell)^2-4}}\left(\{\lambda_\ell^-, \mu_j\} + \left\{\mu_\ell, \lambda_j^-\right\}\right)\end{aligned}$$

where we also used that $\{\mu_j, \mu_\ell\} = \left\{\lambda_\ell^-, \lambda_j^-\right\} = 0$. The terms on the right side of the equation above are treated separately and in the same way. Let us consider the first term only. By Lemma III.26 we have

$$\begin{aligned}&\lim_{\alpha\to\infty}\frac{\psi_k(\mu_j)\psi_n(\mu_\ell)}{\sqrt[*]{\Delta(\mu_j)^2-4}\sqrt[*]{\Delta(\mu_\ell)^2-4}}\{\mu_j, \lambda_\ell^-\}\\ &= \lim_{\alpha\to\infty}\frac{1}{2}\frac{\psi_k(\mu_j)\psi_n(\mu_\ell)}{\sqrt[*]{\Delta(\mu_\ell)^2-4}}\frac{\delta(\lambda_\ell^-)}{\dot{\delta}(\mu_j)}\frac{1}{\dot{\Delta}(\lambda_\ell^-)}\frac{1}{\lambda_\ell^- - \mu_j}.\end{aligned}$$

In view of (III.40), it follows from the product representation of $\delta(\lambda)$ and $\Delta(\lambda)^2 - 4$ that

$$\lim_{\alpha\to\infty}\frac{\delta(\lambda_\ell^-)}{\sqrt[*]{\Delta(\mu_\ell)^2-4}} = 0.$$

Since $\gamma_\ell \neq 0$ we have $\lim_{\alpha\to\infty}\dot{\Delta}(\lambda_\ell^-) = \dot{\Delta}(\lambda_\ell^-) \neq 0$. As $\dot{\delta}(\mu_j)$ is uniformly bounded away from 0 and $\lim_{\alpha\to\infty}(\lambda_\ell^- - \mu_j) = \lambda_\ell^- - \lambda_j^- \neq 0$ we thus conclude that

$$\lim_{\alpha\to\infty}\frac{\psi_k(\mu_j)\psi_n(\mu_\ell)}{\sqrt[*]{\Delta(\mu_\ell)^2-4}}\frac{\delta(\lambda_\ell^-)}{\dot{\delta}(\mu_j)}\frac{1}{\dot{\Delta}(\lambda_\ell^-)}\frac{1}{\lambda_\ell^- - \mu_j} = 0$$

and Lemma III.27 is proved. ■

Lemma III.28 *Let $\varphi \in L^2_{\mathcal{R}}\backslash(D_k \cup D_n)$ with $n, k \in \mathbb{Z}$ arbitrary. Then for any $\lambda \in \mathbb{C}$*

$$\{\{\Theta_k, \Theta_n\}, \Delta(\lambda)\} = 0.$$

Proof Introduce $h(\lambda) := \{\{\Theta_k, \Theta_n\}, \Delta(\lambda)\}$. First we want to prove that $h \in \mathcal{I}_0$. By the Jacobi identity,

$$h(\lambda) = \{\Theta_k, \{\Theta_n, \Delta(\lambda)\}\} - \{\Theta_n, \{\Theta_k, \Delta(\lambda)\}\}.$$

By Lemma III.22 (i) one gets

$$\{\Theta_k, \Delta(\lambda)\} = \frac{1}{2}\sum_{j\in\mathbb{Z}} g_j^{(k)}(\lambda) = \frac{1}{2}\psi_k(\lambda)$$

where the latter identity holds by Lemma III.23. Therefore

$$h(\lambda) = \frac{1}{2}\{\Theta_k, \psi_n(\lambda)\} - \frac{1}{2}\{\Theta_n, \psi_k(\lambda)\}.$$

Recall that $\psi_n : W \to \mathcal{I}_0, \varphi \mapsto \psi_n(\cdot, \varphi)$ is analytic. Hence, for

$$\varphi \in L^2_{\mathcal{R}}\backslash(D_k \cup D_n),$$

h is in $\mathcal{I}_0$. Moreover, in view of the formula

$$\nabla_{\varphi(x)} I_j = -\frac{1}{\pi}\int_{\Gamma_j}\frac{\nabla_{\varphi(x)}\Delta(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}d\lambda$$

one has

$$\{\{\Theta_k, \Theta_n\}, I_j\} = -\frac{1}{\pi}\int_{\Gamma_j}\frac{h(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}\,d\lambda\ .$$

Using the Jacobi identity for $\{\{\Theta_k, \Theta_n\}, I_j\}$ once more one then concludes from the canonical relations $\{\Theta_k, I_j\} = -\delta_{kj}$ (cf Proposition III.24) that

$$A_j(h) \equiv \int_{\Gamma_j}\frac{h(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}\,d\lambda = 0 \quad \forall j \in \mathbb{Z}.$$

By Lemma II.3 and Lemma II.4 it then follows that $h = 0$. ■

We are now in position to prove Proposition III.25:

Proof (of Proposition III.25) As finite gap potentials are dense in $L^2_{\mathcal{R}} \backslash (D_k \cup D_n)$ and $\{\Theta_k, \Theta_n\}$ is continuous it suffices to show that $\{\Theta_k, \Theta_n\} = 0$ for any finite gap potential φ in $L^2_{\mathcal{R}} \backslash (D_k \cup D_n)$. Using the isospectral flows η^t introduced in (I.79) together with $\{\{\Theta_k, \Theta_n\}, \Delta(\lambda)\} = 0$ (cf Lemma III.28) one sees that

$$\begin{aligned}\frac{d}{dt}\{\Theta_k, \Theta_n\}(\eta^t) &= \int_0^1 \nabla_{\varphi(x)}\{\Theta_k, \Theta_n\} \cdot \frac{d}{dt}\eta^t dx \\ &= \{\{\Theta_k, \Theta_n\}, \Delta(\lambda)\} \Big|_{\lambda=\mu_j(\eta^t)} = 0.\end{aligned}$$

This leads to the identity

$$\{\Theta_k, \Theta_n\}(\varphi_0) = \{\Theta_k, \Theta_n\}(\varphi)$$

with $\varphi_0 \in \mathrm{Iso}(\varphi)$ satisfying

$$\mu_j(\varphi_0) = \lambda_j^-(\varphi) \quad \forall j \in \mathbb{Z}.$$

But by Lemma III.27, $\{\Theta_k, \Theta_n\}(\varphi_0) = 0$ and hence $\{\Theta_k, \Theta_n\}(\varphi) = 0$. ■

The results obtained in this section allow us to prove the following

Theorem III.29 *For any $\varphi \in L^2_{\mathcal{R}}$ and $k, n \in \mathbb{Z}, \{x_k, x_n\} = \{y_k, y_n\} = 0$ and $\{x_k, y_n\} = \delta_{kn}$.*

Proof Let $k, n \in \mathbb{Z}$ be fixed. As $L^2_{\mathcal{R}} \backslash (D_k \cup D_n)$ is dense in $L^2_{\mathcal{R}}$ and the coordinates as well as their Poisson brackets are continuous it suffices to prove the canonical relations for φ in $L^2_{\mathcal{R}} \backslash (D_k \cup D_n)$. Then x_j and y_j $(j = k, n)$ are given by $x_j = \sqrt{2I_j} cos \Theta_j$ and $y_j = \sqrt{2I_j} sin \Theta_j$. Using Propositions III.21, III.24, and III.25 one sees that $\{x_k, x_n\} = 0$ and $\{y_k, y_n\} = 0$. Moreover one computes $\{x_k, y_n\}$ as follows,

$$\begin{aligned}\{x_k, y_n\} &= \frac{1}{\sqrt{2I_k}} \cos\Theta_k \sqrt{2I_n} \cos\Theta_n \{I_k, \Theta_n\} \\ &- \sqrt{2I_k} \sin\Theta_k \frac{1}{\sqrt{2I_n}} \sin\Theta_n \{\Theta_k, I_n\} \\ &= (\sin^2\Theta_n + \cos^2\Theta_n)\{I_k, \Theta_n\} \\ &= \delta_{kn}.\end{aligned}$$

■

III.7 Local diffeomorphism

For any $N \geq 0$, let $H^N \equiv H^N(S^1, \mathbb{C}^2)$ be the Sobolev space of periodic functions $g : \mathbb{R} \to \mathbb{C}^2$ of period 1 of the form $g(x) = \sum_k \hat{g}(k) e^{i2k\pi x}$ satisfying $\|g\|_N < \infty$ where

$$\|g\|_N := \left(\sum_{k \in \mathbb{Z}} \langle k \rangle^{2N} |\hat{g}(k)|^2 \right)^{1/2}$$

and denote by $H^N_{\mathcal{R}}$ the space of potentials of real type in H^N

$$H^N_{\mathcal{R}} := L^2_{\mathcal{R}} \cap H^N(S^1, \mathbb{C}^2)$$

and by $\Omega^{(N)}$ the restriction of $\Omega \equiv \Omega^{(0)}$ to $H^N_{\mathcal{R}}$. From [GK1] we learn that $\gamma(\varphi) := (\gamma_n(\varphi))_{n \in \mathbb{Z}}$ is in the sequence space ℓ^2_N for any $\varphi \in H^N_{\mathcal{R}}$. Here $\ell^2_N \equiv \ell^2_N(\mathbb{Z}, \mathbb{C}^2)$ is the Hilbert space

$$\ell^2_N := \{a = (a_k)_{k \in \mathbb{Z}} \in \ell^2 \,\Big|\, \|a\|_N < \infty\}$$

with

$$\|a\|_N := \left(\sum_{k \in \mathbb{Z}} \langle k \rangle^{2N} |a_k|^2 \right)^{1/2}.$$

Hence $\Omega^{(N)}$ maps $H^N_{\mathcal{R}}$ into ℓ^2_N and, in the sequel, will be viewed as a map

$$\Omega^{(N)} \ : \ H^N_{\mathcal{R}} \to \ell^2_N.$$

In this section we prove that, for any $N \geq 0$, $\Omega^{(N)}$ is real analytic and, at each point in $H^N_{\mathcal{R}}$, a local diffeomorphism. We begin by showing that $\Omega^{(0)}$ is a local diffeomorphism at each point in $L^2_{\mathcal{R}}$.

First we need to introduce some more notation. We introduce the following orthonormal basis of $L^2([0,1], \mathbb{C}^2)$ only used in this section, $(k \in \mathbb{Z})$

$$e_k^+(x) := -\frac{1}{\sqrt{2}} \left(e^{i2\pi kx}, , e^{-i2\pi kx} \right) \ ; \quad e_k^- := \frac{i}{\sqrt{2}} \left(-e^{i2\pi kx}, e^{-i2\pi kx} \right).$$

Further we define for any $\varphi \in L^2_{\mathcal{R}}$ $(k \in \mathbb{Z})$

$$d_k^+(x, \varphi) := \nabla_{\varphi(x)} x_k \ ; \quad d_k^-(x, \varphi) := \nabla_{\varphi(x)} y_k$$

and note that by Proposition III.20, $d_k^\pm(x, 0) = e_k^\pm(x)$ $(k \in \mathbb{Z})$.

Lemma III.30 *For any $\varphi \in L^2_{\mathcal{R}}$, the differential $d_\varphi \Omega$ is a linear isomorphism from $L^2([0,1], \mathbb{C}^2)$ onto $\ell^2(\mathbb{Z}, \mathbb{C}^2)$.*

Proof Let $\varphi \in L^2_{\mathcal{R}}$. For any $F \in L^2([0,1],\mathbb{C}^2)$ one has

$$d_\varphi \Omega(F) = \left(d_k^+(\varphi) \cdot F, d_k^-(\varphi) \cdot F \right)_{k\in\mathbb{Z}}$$

where the dot denotes the dual pairing between the dual of $L^2([0,1],\mathbb{C}^2)$ and itself (no complex conjugation).
Noticing that the sequence $(d_k^\pm(\varphi))_{k\in\mathbb{Z}}$ is bounded in $L^2([0,1],\mathbb{C}^2)$, we introduce $A \equiv A(\varphi)$ the bounded linear operator on $L^2([0,1],\mathbb{C}^2)$ given by

$$A(\varphi) : F \mapsto \sum_{k\in\mathbb{Z}} \langle F, e_k^+ \rangle d_k^+(\varphi) + \langle F, e_k^- \rangle d_k^-(\varphi) \tag{III.47}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2([0,1],\mathbb{C}^2)$ (i.e. $\langle F, G \rangle = F \cdot \bar{G}$).
One has for any $k \in \mathbb{Z}$

$$Ae_k^\pm = d_k^\pm$$

and thus

$$d_k^\pm \cdot F = Ae_k^\pm \cdot F = e_k^\pm \cdot A^* F \ .$$

Therefore

$$d_\varphi \Omega(F) = \left(e_k^+ \cdot A(\varphi)^* F, e_k^- \cdot A(\varphi)^* F \right)_{k\in\mathbb{Z}}$$

and thus it remains to prove that $A \equiv A(\varphi)$ is a linear isomorphism of $L^2([0,1],\mathbb{C}^2)$.
As

$$\sum_{k\in\mathbb{Z}} \|(A-I)(e_k^+)\|^2 + \|(A-I)(e_k^-)\|^2 = \sum_{k\in\mathbb{Z}} \|d_k^+ - e_k^+\|^2 + \|d_k^- - e_k^-\|^2$$

one deduces by Proposition III.20 that, if $\varphi \in L^2_{\mathcal{R}}$ is a finite gap potential, then $A(\varphi) - I$ is a Hilbert-Schmidt operator and thus a compact operator. Given $\varphi \in L^2_{\mathcal{R}}$, choose a sequence of finite gap potentials in $L^2_{\mathcal{R}}$ with $\varphi = \lim_{n\to\infty} \varphi_n$. As $\varphi \mapsto \Omega(\varphi)$ is analytic, $\varphi \mapsto A(\varphi)$ is continuous and thus $A(\varphi) - I = \lim_{n\to\infty}(A(\varphi_n) - I)$ is compact as well.
Further we claim that A is 1-1. Noticing that

$$\{H, K\}(\varphi) = \nabla_{\varphi(x)} H \cdot J \nabla_{\varphi(x)} K$$

where $J = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ one gets, using the canonical relations established in Theorem III.29, that for any $k, n \in \mathbb{Z}$

$$d_k^\pm \cdot J d_n^\mp = \pm \delta_{kn} \quad \text{and} \quad d_k^\pm \cdot J d_n^\pm = 0 \ .$$

Therefore if $AF = 0$ then for any $n \in \mathbb{Z}$

$$0 = AF \cdot J d_n^\mp = \pm \langle F, e_n^\pm \rangle$$

i.e. $F = 0$. Hence A is 1-1 and by the Fredholm alternative we conclude that A is an isomorphism. ■

To prove $\Omega^{(N)}$ is a local diffeomorphism we need the following auxilary result:

Lemma III.31 *Let $N \geq 0$ and φ be any finite gap potential in $L^2_{\mathcal{R}}$. Then for any $F \in H^N(S^1, \mathbb{C}^2)$,*

$$\langle \nabla_{\varphi(x)} x_n, F \rangle = \langle e_n^+, F \rangle + \ell_N^2(n)$$
$$\langle \nabla_{\varphi(x)} y_n, F \rangle = \langle e_n^-, F \rangle + \ell_N^2(n)$$

where the error terms $\ell_N^2(n)$ are uniformly bounded in ℓ_N^2 on bounded sets of functions F in $H^N(S^1, \mathbb{C}^2)$.

Proof Both asymptotic estimates are proven in the same way, so let us concentrate on the first one. We argue by induction. For $N = 0$, the statement follows from Proposition III.20. Next assume that the statement holds up to some integer N. From the definition (III.25) we see that for any n with $\gamma_n = 0$,

$$\nabla_{\varphi(x)} x_n = \sqrt{2} \frac{\xi_n}{4} (e^{i\beta_n} \nabla z_n^+ + e^{-i\beta_n} \nabla z_n^-). \tag{III.48}$$

By Lemma III.16 for $|n|$ sufficiently large,

$$\nabla_{\varphi(x)} z_n^\pm = 2 \left(\nabla_{\varphi(x)} \tau_n - \nabla_{\varphi(x)} \mu_n \right) \pm i \lim_{B_n \ni \psi \to \varphi} \nabla_{\psi(x)} \gamma_n \tag{III.49}$$

where $\lim_{B_n \ni \psi \to \varphi} \nabla_{\psi(x)} \gamma_n$ is of the form

$$\left(\tilde{F}_{n,2}^+(x)^2, \tilde{F}_{n,1}^+(x)^2 \right) - \left(\tilde{F}_{n,2}^-(x)^2, \tilde{F}_{n,1}^-(x)^2 \right)$$

and $\tilde{F}_n^\pm = \left(\tilde{F}_{n,1}^\pm, \tilde{F}_{n,1}^\pm \right)$ being both solutions of the equation $LF = \lambda_n^+ F$.
It turns out that $\nabla_{\varphi(x)} z_n^\pm$ satisfies a nonlocal equation of first order. Introduce the operator $\tilde{L} \equiv \tilde{L}(\varphi)$

$$\tilde{L}(\varphi) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx} + 2 \begin{pmatrix} -\varphi_2 D^{-1} \varphi_1 & \varphi_2 D^{-1} \varphi_2 \\ -\varphi_1 D^{-1} \varphi_1 & \varphi_1 D^{-1} \varphi_2 \end{pmatrix}$$

where D^{-1} denotes the inverse of the restriction of $\frac{d}{dx}$ to $H_0^1(S^1, \mathbb{C})$,

$$D^{-1} : L^2(S^1, \mathbb{C}) \to H_0^1(S^1, \mathbb{C})$$

and

$$H_0^1(S^1, \mathbb{C}) := \{ f \in H^1(S^1, \mathbb{C}) \Big| \int_0^1 f(x) dx = 0 \}.$$

One verifies that for any function $F = (F_1, F_2) \in H^1(S^1, \mathbb{C}^2)$ satisfying $LF = \lambda F$ one has

$$\tilde{L} \left((F_2)^2, (F_1)^2 \right) = 2i\lambda \left((F_2)^2, (F_1)^2 \right) - 2i \left(\int_0^1 F_1 F_2 \ dx \right) (\varphi_2, \varphi_1).$$

In particular, if $F_1(x) = \overline{F_2(x)}$ and $\|F\|^2 = 1$, then

$$\int_0^1 F_1 F_2 dx = 1/2\|F\|^2 = 1/2$$

and the formula above reads, with

$$\varphi^* := (\varphi_2, \varphi_1) \ ,$$

$$\tilde{L}\left((F_2)^2, (F_1)^2\right) = 2i\lambda\left((F_2)^2, (F_1)^2\right) - i\varphi^*.$$

Applying this formula to each term on the right side of the expression (III.49) separately, one concludes from (I.21) (page 25) Proposition I.29 and Proposition I.32

$$\tilde{L}\left(\nabla_{\varphi(x)} z_n^{\pm}\right) = 2i\lambda_n^{+} \nabla_{\varphi(x)} z_n^{\pm}.$$

By (III.48) it then follows that for any $F \in H^{N+1}(S^1, \mathbb{C}^2)$,

$$\begin{aligned}\langle \nabla_{\varphi(x)} x_n, F\rangle_{L^2} &= \frac{1}{2i\lambda_n^+}\langle \tilde{L}(\varphi)\nabla_{\varphi(x)} x_n, F\rangle_{L^2}\\ &= \frac{1}{2i\lambda_n^+}\langle \nabla_{\varphi(x)} x_n, -\tilde{L}(\overline{\varphi^*})F\rangle_{L^2}\end{aligned}$$

where for the latter identity we used that the adjoint of $\tilde{L}(\varphi)$ is given by $-\tilde{L}(\overline{\varphi^*})$. Clearly, $L(\overline{\varphi^*})F \in H^N(S^1, \mathbb{C}^2)$ and thus, by the induction hypothesis applied to it one has

$$\langle \nabla_{\varphi(x)} x_n, F\rangle = \frac{1}{2i\lambda_n^+}\left(\langle e_n^+, -\tilde{L}(\overline{\varphi^*})F\rangle_{L^2} + \ell^2_N(n)\right)$$

where the error term is uniformly bounded on bounded sets in $H^{N+1}(S^1, \mathbb{C}^2)$ since the linear map $\tilde{L}(\overline{\varphi^*}) : H^{N+1}(S^1, \mathbb{C}^2) \to H^N(S^1, \mathbb{C}^2)$ is continuous for $N \geq 0$. Notice that $B := -L(\overline{\varphi^*}) + \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\frac{d}{dx}$ is a bounded operator on H^{N+1} and

$$\begin{aligned}\langle e_n^+, -\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\frac{d}{dx}F\rangle &= \langle\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\frac{d}{dx}e_n^+, F\rangle\\ &= 2i\pi n\langle e_n^+, F\rangle.\end{aligned}$$

Together with the asymptotics $\lambda_n^{\pm} = n\pi + \ell^2(n)$ (cf Proposition I.6) one then obtains

$$\begin{aligned}\langle \nabla_{\varphi(x)} x_n, F\rangle &= \frac{\pi n}{\lambda_n^+}\langle e_n^+, F\rangle + \frac{1}{\lambda_n^+}\langle e_n^+, BF\rangle + \ell^2_{N+1}(n)\\ &= \left(1 + O(\frac{1}{n})\right)\langle e_n^+, F\rangle + \frac{1}{\lambda_n^+}\langle e_n^+, BF\rangle + \ell^2_{N+1}(n).\end{aligned}$$

Integrating by parts we have

$$|\langle e_n^+, F\rangle| \leq \frac{\|F\|_{N+1}}{\langle n\rangle^{N+1}}$$

and

$$|\langle e_n^+, BF\rangle| \leq C(\varphi)\frac{\|F\|_{N+1}}{\langle n\rangle^{N+1}}$$

where $C(\varphi)$ is a constant depending only on φ and its derivatives. Hence it follows that

$$\langle \nabla_{\varphi(x)} x_n, F\rangle = \langle e_n^+, F\rangle + \ell^2_{N+1}(n)$$

where the error term is uniformly bounded on bounded sets of functions F in $H^{N+1}(S^1, \mathbb{C}^2)$. This proves the induction step. ■

Lemma III.31 will now be used to prove

Theorem III.32 *For any $N \geq 0, \Omega^{(N)}$ is real analytic and a local diffeomorphism near any point in $H^N_{\mathcal{R}}$.*

Proof To see that $\Omega^{(N)} : W \cap H^N \to \ell^2_N$ is real analytic (with W as in Theorem III.15) it suffices to show that $\Omega^{(N)}$ is locally bounded and that each component $\Omega_j^{(N)} := (x_j, y_j)$ is real analytic on $W \cap H^N$. The latter statement clearly follows from the fact that $\Omega : W \to \ell^2$ is real analytic. The local boundedness of Ω follows from the asymptotic estimate

$$|x_n| + |y_n| = 0(|\mu_n - \tau_n| + |\gamma_n|)$$

(cf Proposition III.14, Theorem III.3 and Lemma III.9) and the asymptotic estimates for γ_n and $\mu_n - \tau_n$ established in [GK1] (see also [Ma]). These estimates imply that for any $N \geq 1$, the maps $\varphi \mapsto (\gamma_n(\varphi))_{n\in\mathbb{Z}}$ and $\varphi \mapsto (\mu_n(\varphi) - \tau_n(\varphi))_{n\in\mathbb{Z}}$ from $H^N(S^1, \mathbb{C}^2)$ to ℓ^2_N are bounded.
To see that $\Omega^{(N)}$ is a local diffeomorphism on $H^N_{\mathcal{R}}$, one has to prove that, at each point $\varphi \in H^N_{\mathcal{R}}$, $d_\varphi\Omega^{(N)}$ is a linear isomorphism from $H^N(S^1, \mathbb{C}^2)$ onto ℓ^2_N. Clearly, $d_\varphi\Omega^{(N)}$ is the restriction to $H^N(S^1, \mathbb{C}^2)$ of $d_\varphi\Omega$ and thus by Lemma III.30, $d_\varphi\Omega^{(N)}$ is 1-1. Further, following the notation established in this Lemma, on has for any $F \in H^N([0,1], \mathbb{C}^2)$

$$d_\varphi\Omega^{(N)}(F) = \left(d_k^+(\varphi)\cdot F, d_k^-(\varphi)\cdot F\right)_{k\in\mathbb{Z}}$$

and, with $F = (F_1, F_2)$,

$$\begin{aligned}d_0\Omega^{(N)}(F) &= \left(e_k^+\cdot F, e_k^-\cdot F\right)_{k\in\mathbb{Z}}\\ &= \left(\frac{1}{\sqrt{2}}\left(\hat{F}_1(-k), \hat{F}_2(k)\right)\begin{pmatrix} -1 & -i \\ 1 & 1\end{pmatrix}\right)_{k\in\mathbb{Z}}.\end{aligned}$$

As the Fourier transform realises a linear isomorphism from $H^N([0,1],\mathbb{C})$ onto $\ell^2_N(\mathbb{Z},\mathbb{C})$, we deduce that $d_0\Omega^{(N)}$ is a linear isomorphism from $H^N(S^1,\mathbb{C}^2)$ onto ℓ^2_N.
By the Fredholm alternative it thus remains to show that

$$R \equiv R(\varphi) := d_\varphi\Omega^{(N)} - d_0\Omega^{(N)}$$

is a compact operator from $H^N(S^1,\mathbb{C}^2)$ into ℓ^2_N.
Denote by R_k, $k\in\mathbb{Z}$, the k-th component of R, i.e. the operator from H^N into $\mathbb{C}^2$ given by

$$R_kF = \left((d_k^+ - e_k^+)\cdot F, (d_k^- - e_k^-)\cdot F\right)$$

We introduce for each $K\geq 1$ the finite rank operator R^K defined by

$$R^KF \equiv \left(R_k^KF\right)_{k\in\mathbb{Z}}$$

where for $|k|\leq K$, $R_k^KF = R_kF$ and for $|k|>K$, $R_k^KF=0$. Denoting for any $k\in\mathbb{Z}$

$$a_k := \sup\{\|R_k(\varphi)F\| \mid F\in H^N,\ \|F\|_N = 1\}\ .$$

one has

$$\|R-R^K\|_{\mathcal{L}(H^N,\ell^2_N)} \leq \sum_{|k|>K}\langle k\rangle^N a_k^2\ .$$

Therefore, as by Lemma III.31 the sequence $(a_k)_{k\in\mathbb{Z}}$ is in ℓ^2_N, we deduce that, for each finite gap potential φ, $R(\varphi)$ is the uniform limit of finite rank operators and thus a compact operator.
Approximating an arbitrary $\varphi\in H^N_{\mathcal{R}}$ by a sequence of finite gap potentials $(\varphi_n)_{n\geq 1}$ in $H^N_{\mathcal{R}}$ and using that $R(\varphi)$ is real analytic, hence in particular continuous in φ, one sees that $\lim_{n\to\infty}R(\varphi_n) = R(\varphi)$ is compact as well. ■

III.8 Global properties

The purpose of this section is to prove that $\Omega^{(N)}$ is $1-1$ and onto for any $N\geq 0$. To this end we show that the action map

$$I: L^2_{\mathcal{R}}\to\ell^1,\ \varphi\mapsto(I_k(\varphi))_{k\in\mathbb{Z}}\,.$$

is proper. We begin with establishing auxilary results concerning various asymptotic estimates.

Lemma III.33 *Let $\varphi\in L^2_{\mathcal{R}}, \lambda\in\mathbb{R}$ and $F\in H^1([0,1],\mathbb{C}^2)$ with $\|F\|_{L^2}=1$ satisfying $L(\varphi)F=\lambda F$. Then*

$$\|F\|_{L^\infty}\leq\sqrt{2}\exp(\|\varphi_1\|_{L^2}).$$

Proof Introduce $\tilde F(x) := E(x,\lambda)^{-1}F(x)$ where (cf page 14)

$$E(x,\lambda) = diag(e^{-i\lambda x}, e^{i\lambda x}).$$

Substitute $F(x) = E(x,\lambda)\tilde F(x)$ into the equation $LF=\lambda F$ to obtain

$$\frac{d}{dx}\tilde F(x) = i\begin{pmatrix} 0 & e^{2i\lambda x}\varphi_1(x)\\ -e^{-2i\lambda x}\overline{\varphi_1(x)} & 0\end{pmatrix}\tilde F(x).$$

Denoting by $Q(x)$ the matrix

$$Q(x) := i\begin{pmatrix} 0 & e^{2i\lambda x}\varphi_1(x)\\ -e^{-2i\lambda x}\overline{\varphi_1(x)} & 0\end{pmatrix}$$

one gets

$$\frac{d}{dx}\|\tilde F(x)\|^2 = \tilde F\cdot Q\tilde F + \tilde F\cdot\overline{Q}\overline{\tilde F}$$

and thus, for any $0\leq s\leq x\leq 1$,

$$\|\tilde F(x)\|^2 = \|\tilde F(s)\|^2 + \int_s^x\left(\overline{\tilde F}\cdot Q\tilde F + \tilde F\cdot\overline{Q}\overline{\tilde F}\right)dt. \tag{III.50}$$

As $\lambda\in\mathbb{R}$, we have $|\tilde F(x)| = |F(x)|$ for any x and thus $\int_0^x\|\tilde F(s)\|^2ds\leq 1$ for any $0\leq x\leq 1$. Integrating both sides of (III.50) with respect to s from 0 to x then leads to

$$\begin{aligned}x\|\tilde F(x)\|^2 &\leq 1 + \int_0^x ds\int_s^x dt\left|\overline{\tilde F}\cdot Q\tilde F + \tilde F\cdot\overline{Q}\overline{\tilde F}\right|\\ &\leq 1 + 2\int_0^x ds\int_s^x dt\left|\tilde F(t)\right|^2\left|\varphi_1(t)\right|\\ &= 1+2\int_0^x t\left|\tilde F(t)\right|^2\left|\varphi_1(t)\right|dt.\end{aligned}$$

By Gronwall's lemma one gets for $0\leq x\leq 1$,

$$x\|\tilde F(x)\|^2\leq\exp\left(2\int_0^x\left|\varphi_1(t)\right|dt\right)\leq\exp\left(2\|\varphi\|_{L^2}\right).$$

In particular, for any $\frac{1}{2}\leq x\leq 1$,

$$\|\tilde F(x)\|\leq\sqrt{2}\exp\left(\|\varphi\|_{L^2}\right).$$

Reversing the orientation on the interval $[0,1]$ one obtains the same inequality for $x\in[0,1/2]$. ■

Corollary III.34 *For any $\varphi\in L^2_{\mathcal{R}}$ and $n\in\mathbb{Z}$,*
(i) $|\mu_n(\varphi)-n\pi|\leq 2\sqrt{2}\|\varphi\|e^{\sqrt{2}\|\varphi\|}$,
(ii) $|\lambda_n^\pm(\varphi)-n\pi|\leq 2\sqrt{2}\|\varphi\|e^{\sqrt{2}\|\varphi\|}$.

Proof (i) As $\mu_n(0) = n\pi$, one has by Taylor's formula

$$\mu_n(\varphi) - n\pi = \int_0^1 \frac{d}{dt}(\mu_n(t\varphi))\,dt = \int_0^1 \langle \nabla_{t\varphi(\cdot)}\mu_n, \varphi(\cdot)\rangle dt.$$

By Proposition I.29,

$$\nabla_{\varphi(x)}\mu_n = \frac{1}{\|G_n(\cdot)\|^2}\left(G_{n,2}(x)^2, G_{n,1}(x)^2\right)$$

where we recall that

$$G_n(x) = G(x,\mu_n) = M^{(1)}(x,\mu_n) + M^{(2)}(x,\mu_n)\ .$$

Hence by Lemma III.33,

$$\|G\|_{L^\infty}/\|G\|_{L^2} \le \sqrt{2}\exp\left(\|\varphi_1\|_{L^2}\right)$$

and therefore, with $\|\varphi\|_{L^2} = \sqrt{2}\|\varphi_1\|_{L^2}$

$$\left|\langle \nabla_{t\varphi(\cdot)}\mu_n, \varphi(\cdot)\rangle\right| \le 2\exp\left(2\|\varphi_1\|_{L^2}\right)\left(\|\varphi_1\|_{L^2} + \|\varphi_2\|_{L^2}\right) \le 2\sqrt{2}\exp\left(\sqrt{2}\|\varphi\|_{L^2}\right)\|\varphi\|_{L^2}.$$

To prove (ii) recall from Proposition I.40 that for any given $n \in \mathbb{Z}$ there exists $\varphi^{\pm} \in \mathrm{Iso}(\varphi)$ such that $\mu_n(\varphi^{\pm}) = \lambda_n^{\pm}(\varphi)$ and $\|\varphi^{\pm}\| \le \|\varphi\|$. Hence (ii) is a consequence of (i). ■

Corollary Corollary7.2 can be used to prove the following properness result for the actions.

Lemma III.35 *The map* $I : L^2_{\mathcal{R}} \to \ell^1, \varphi \mapsto (I_k(\varphi))_{k\in\mathbb{Z}}$ *is proper.*

Remark With some effort one can prove that (cf [MV]) $\sum |\mu_n(\varphi) - n\pi|^2 \le C(\|\varphi\|)$. This uniform estimate can be used to prove that the map $L^2_{\mathcal{R}} \ni \varphi \mapsto (I_(\varphi))_{k\in\mathbb{Z}}$ is proper. Nevertheless our proof of Lemma III.35 use only the weaker estimates established in Corollary Corollary7.2.

Proof It is to prove that any sequence $(\varphi_j)_{j\ge1}$ in $L^2_{\mathcal{R}}$ with the property that $(I(\varphi_j))_{j\ge1}$ converges in $\ell^1(\mathbb{Z})$ to a sequence $\mathcal{J} := (\mathcal{J}_k)_{k\in\mathbb{Z}}$ admits a convergent subsequence. By Proposition III.4, we have for any $j \ge 1$, $\sum_{k\in\mathbb{Z}} I_k(\varphi_j) = \|\varphi\|^2$ and hence

$$\lim_{j\to\infty}\|\varphi_j\|^2 = \sum_{k\in\mathbb{Z}}\mathcal{J}_k\ .$$

In particular, $(\varphi_j)_{j\ge1}$ admits a weakly convergent subsequence in $L^2_{\mathcal{R}}$ which we again denote by $(\varphi_j)_{j\ge1}$. Denote its limit by φ. By Corollary III.34, the sequences $(\lambda_n^{\pm}(\varphi_j))_{j\ge1}$ are bounded for any $n \in \mathbb{Z}$. Hence without loss of generality we may assume that each of these sequences converges,

$$\xi_n^{\pm} := \lim_{j\to\infty}\lambda_n^{\pm}(\varphi_j).$$

By Lemma I.1, $\Delta(\lambda,\varphi)$ is a weakly continuous map on $\mathbb{C}\times L^2_{\mathcal{R}}$, hence

$$2(-1)^n = \Delta(\lambda_n^{\pm}(\varphi_j),\varphi_j) \to \Delta(\xi_n^{\pm},\varphi).$$

It follows that $\xi_n^{\pm} \in spec(\varphi)$. Since for any j, $(\lambda_n^{\pm}(\varphi_j))_{n\in\mathbb{Z}}$ is a nondecreasing sequence, $(\xi_n^{\pm})_{n\in\mathbb{Z}}$ is nondecreasing as well, more precisely

$$\ldots \le \xi_{n-1}^{+} \le \xi_n^{-} \le \xi_n^{+} \le \ldots .$$

Further, as

$$\Delta(\xi_n^{\pm},\varphi) = (-1)^n 2 \tag{III.51}$$

it is *impossible* that for any $n, k \in \mathbb{Z}$

$$\xi_n^{+} = \lambda_k^{-} \text{ and } \xi_{n+1}^{-} = \lambda_k^{+}$$

or

$$\xi_n^{-} = \lambda_k^{+} \text{ and } \xi_n^{+} = \lambda_{k+1}^{-}.$$

Hence one can choose an increasing sequence $(k_n)_{n\in\mathbb{Z}}$ in $\mathbb{Z}$ so that

$$\xi_n^{\pm} \in \{\lambda_k^{\pm} \mid k_n \le k < k_{n+1}\}.$$

Further choose mutually disjoint open discs in $\mathbb{C}$, $(B_n)_{n\in\mathbb{Z}}$, so that for any $n \in \mathbb{Z}$,

$$B_n \cap spec(\varphi) = \{\lambda_k^{\pm} \mid k_n \le k < k_{n+1}\}$$

and

$$\partial B_n \cap spec(\varphi) = \emptyset$$

where ∂B_n denotes the boundary of B_n with counterclockwise orientation. For any $n \in \mathbb{Z}$, there exists $j_n \ge 1$ so that for any $j \ge j_n$,

$$\lambda_n^{\pm}(\varphi_j) \in B_n$$

and

$$\lambda_k^{\pm}(\varphi_j) \notin \overline{B_n} \quad \forall k \in \mathbb{Z}\backslash\{n\}.$$

Hence we have for any $j \ge j_n$

$$I_n(\varphi_j) = \frac{1}{\pi}\int_{\partial B_n}\frac{\lambda\dot{\Delta}(\lambda,\varphi_j)}{\sqrt[c]{\Delta(\lambda,\varphi_j)^2 - 4}}\,d\lambda.$$

Again using that Δ is weakly continuous it then follows that

$$\mathcal{J}_n = \lim_{j\to\infty} I_n(\varphi_j) = \frac{1}{\pi}\int_{\partial B_n} \frac{\lambda\dot{\Delta}(\lambda,\varphi)}{\sqrt[c]{\Delta(\lambda,\varphi)^2-4}}\, d\lambda.$$

Further, by Cauchy's theorem,

$$\frac{1}{\pi}\int_{\partial B_n} \frac{\lambda\dot{\Delta}(\lambda,\varphi)}{\sqrt[c]{\Delta(\lambda,\varphi)^2-4}}\, d\lambda = \sum_{k_n\le k<k_{n+1}} I_k(\varphi).$$

Hence

$$\mathcal{J}_n = \sum_{k_n\le k<k_{n+1}} I_k(\varphi).$$

Combining the results above one concludes that

$$\begin{aligned}\lim_{j\to\infty}\|\varphi_j\|^2 &= \sum_{n\in\mathbb{Z}}\mathcal{J}_n = \sum_{n\in\mathbb{Z}}\sum_{k_n\le k<k_{n+1}} I_k(\varphi) = \sum_{n\in\mathbb{Z}} I_k(\varphi)\\ &= \|\varphi\|^2\end{aligned}$$

where for the latter equality we used again Proposition III.4. Together with the weak convergence of $(\varphi_j)_{j\ge1}$ it then follows that $\varphi=\lim_{j\to\infty}\varphi_j$ strongly in $L^2_{\mathcal{R}}$. ■

The properness of the action map I allows to prove that Ω is bijective. Denote by $\ell^2_{\mathbb{R}}$ the sequence space $\ell^2(\mathbb{Z};\mathbb{R}^2)$.

Proposition III.36 *The map $\Omega : L^2_{\mathcal{R}} \to \ell^2_{\mathbb{R}}$ is $1-1$ and onto.*

Proof It is convenient to denote by $\Omega_{\mathcal{R}}$ the restriction of Ω to $L^2_{\mathcal{R}}$,

$$\Omega_{\mathcal{R}} : L^2_{\mathcal{R}} \to \ell^2_{\mathbb{R}}\ .$$

For

$$A := \{z\in\ell^2_{\mathbb{R}} \mid \sharp\Omega_{\mathcal{R}}^{-1}(z) = 1\}$$

it is then to show that $A=\ell^2_{\mathbb{R}}$. As $\ell^2_{\mathbb{R}}$ is connected this amounts to show that A is open, closed and nonempty.

First notice that for any $\varphi\in L^2_{\mathcal{R}}$ with $\Omega(\varphi)=0$ we have $I_k(\varphi)=0\ \forall k\in\mathbb{Z}$ and hence, by Proposition III.4,

$$\|\varphi\|^2 = \sum_{k\in\mathbb{Z}} I_k(\varphi) = 0$$

and thus $\varphi=0$. This shows that $0\in A$, hence $A\neq\emptyset$.

Next we show that A is open. Let $z_0 := \Omega(\varphi_0)$ be in A. As $\Omega_{\mathcal{R}}$ is a local diffeomorphism there exist an open neighborhood U of φ_0 in $L^2_{\mathcal{R}}$ and an open neighborhood V of z_0 in $\ell^2_{\mathbb{R}}$ so that $\Omega_{\mathcal{R}}\big|_V : U\to V$ is a diffeomorphism. Thus for any $z\in V$, $\sharp\Omega_{\mathcal{R}}^{-1}(z)\ge1$.

Introduce for any $n\ge1$, the neighborhoods $U_n := U\cap B_n$ of φ_0 and $V_n := \Omega(U_n)$ of z_0 where B_n denotes the ball in $L^2_{\mathcal{R}}$ with center φ_0 and radius $\frac{1}{n}$. We claim that there exists $n\ge1$ so that $V_n\subseteq A$.
Arguing by contradiction, assume that no such n exists. Then one can find a sequence $(z_n)_{n\ge1}$ with $z_n\in V_n$ and two sequences $(\varphi_n)_{n\ge1},(\psi_n)_{n\ge1}$ with $\varphi_n\in U_n$, $\psi_n\in L^2_{\mathcal{R}}\backslash U$ such that $\Omega_{\mathcal{R}}(\varphi_n)=z_n=\Omega_{\mathcal{R}}(\psi_n)$. Notice that $\lim_{n\to\infty} z_n = z_0$ and $\lim_{n\to\infty}\varphi_n=\varphi_0$. By Lemma III.35, $\Omega_{\mathcal{R}} : L^2_{\mathcal{R}}\to\ell^2$ is proper. Therefore there exists a subsequence, again denoted by $(\psi_n)_{n\ge1}$ which L^2 converges to an element ψ in the closed set $L^2_{\mathcal{R}}\backslash U$. By the continuity of $\Omega, z_0=\Omega(\psi)$ which contradicts the assumption $z_0\in A$.

It remains to prove that A is closed. Let $(z_n)_{n\ge1}$ be a sequence in A which converges to z in ℓ^2. Let $\varphi_n=\Omega_{\mathcal{R}}^{-1}(z_n)$. As $\Omega_{\mathcal{R}}$ is proper we may assume that $(\varphi_n)_{n\ge1}$ converges in $L^2_{\mathcal{R}}$. Denote by φ its limit. By continuity of $\Omega,\Omega(\varphi)=z$. Assume that $z\notin A$. Then there exists $\psi\in L^2_{\mathcal{R}}$ with $\psi\neq\varphi$ such that $\Omega(\psi)=\varphi$. As Ω is a local diffeomorphism there exits n with $\varphi_n\notin A$ which contradicts our assumption that the sequence $z_n=\Omega(\varphi_n)$ be in A. ■

To prove that for any $N\ge1$ the restriction $\Omega^{(N)}$ is 1-1 and onto as well we need a result corresponding to Lemma III.35 for $H^1_{\mathcal{R}}\ (:= H^1\cap L^2_{\mathcal{R}})$.
Denote by $I^{(1)}$ the restriction of the action map $I : L^2_{\mathcal{R}}\to\ell^1$ to $H^1_{\mathcal{R}}$ and by ℓ^1_k the weighted Banach space

$$\ell^1_k := \{(a_j)_{j\in\mathbb{Z}} \mid \sum_j \langle j\rangle^k |a_j| < \infty\}\ .$$

Recall from [GK1] that $(\gamma_k(\varphi))_{k\in\mathbb{Z}}\in\ell^2_1$ for $\varphi\in H^1$. As $I_k=\xi_k^2(\gamma_k/2)^2$ and $\xi_k=1+\ell^2(k)$ (Theorem III.3) it then follows that $(I_k)_{k\in\mathbb{Z}}\in\ell^1_2$. Further, $\xi_k>0$ and $\xi_k=1+\ell^2(k)$ locally uniformly on $L^2_{\mathcal{R}}$ (cf Theorem III.3). Hence on any compact subset K of $L^2_{\mathcal{R}}$,

$$\sup_{\substack{k\in\mathbb{Z}\\ \varphi\in K}} 1/\xi_k < \infty \tag{III.52}$$

and, by the continuity of $\gamma : L^2_{\mathcal{R}}\to\ell^2$, $\varphi\mapsto(\gamma_k(\varphi))_{k\in\mathbb{Z}}$

$$\gamma(K) := \big\{(\gamma_k(\varphi))_{k\in\mathbb{Z}} \mid \varphi\in K\big\} \subseteq \ell^2 \text{ is compact}\ . \tag{III.53}$$

Lemma III.37 *The map $I^{(1)} : H^1_{\mathcal{R}}\to\ell^1_2$ is proper.*

Proof Let $(\varphi_n)_{n\ge1}$ be a sequence in $H^1_{\mathcal{R}}$ such that $(I(\varphi_n))_{n\ge1}$ converges in ℓ^1_2 to an element in ℓ^1_2. It is to prove that $(\varphi_n)_{n\ge1}$ admits a convergent subsequence in $H^1_{\mathcal{R}}$. By Lemma III.35 we may assume that $(\varphi_n)_{n\geq 1}$ converges in $L^2_{\mathcal{R}}$. Denote its limit by φ. In section III.2, we have introduced the functionals defined on $L^2_{\mathcal{R}}$

$$J_j = \frac{1}{\pi}\int_{\Gamma_j} \lambda^3 \frac{\dot\Delta(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}\, d\lambda.$$

By Lemma III.5 and (III.53), the convergence of $(I(\varphi_n))_{n\geq 1}$ in ℓ^1_2 implies that $(J_j(\varphi_n))_{j\in\mathbb{Z}}$ converges in ℓ^1.
By Proposition III.6, using that $\varphi_n = (\varphi_{n1}, \varphi_{n2})$ is of real type (i.e. $\overline{\varphi_{n1}} = \varphi_{n2}$),

$$\int_0^1 \left(|\varphi'_{n1}|^2 + |\varphi^2_{n1}|^2\right) dx = \frac{3}{4}\sum_{k\in\mathbb{Z}} J_k(\varphi_n). \tag{III.54}$$

Hence $(\|\varphi'_n\|)_{n\geq 1}$ is bounded and we may assume without loss of generality that φ_n converges weakly in $H^1_{\mathcal{R}}$ and, as a consequence, $\varphi \in H^1_{\mathcal{R}}$ and $\varphi_n \rightharpoonup \varphi$. As $\varphi \mapsto J_k(\varphi)$ is continuous on $L^2_{\mathcal{R}}$ for any $k \in \mathbb{Z}$ one has $\lim_{n\to\infty} J_k(\varphi_n) = J_k(\varphi)$ and therefore

$$\lim_{n\to\infty} (J_k(\varphi_n))_{k\in\mathbb{Z}} = (J_k(\varphi))_{k\in\mathbb{Z}}$$

in ℓ^1. In view of (III.54) and the compactness of the Sobolev embedding $H^1_{\mathcal{R}} \hookrightarrow C(S^1, \mathbb{C}^2)$ it then follows that

$$\lim_{n\to\infty}\int_0^1 \|\varphi'_n(x)\|^2 dx = \int_0^1 \|\varphi'(x)\|^2 dx$$

and hence $\varphi_n \to \varphi$ in $H^1_{\mathcal{R}}$. ■

Arguing as in the proof of Proposition III.36 and using Lemma III.37 one sees that $\Omega^{(1)} : H^1_{\mathcal{R}} \to \ell^2_1$ is a real analytic diffeomorphism. To prove the corresponding result for $\Omega^{(N)}$ with $N \geq 1$ arbitrary we use a spectral characterization of the regularity of a potential, established in [GK1] ($N \geq 1$ arbitrary),

$$\varphi \in H^N_{\mathcal{R}} \Longleftrightarrow \varphi \in H^1_{\mathcal{R}} \text{ and } (\gamma_n(\varphi))_{n\in\mathbb{Z}} \in \ell^2_N.$$

As $\Omega^{(1)}$ is onto this characterization implies that $\Omega^{(N)} : H^N_{\mathcal{R}} \to \ell^2_N$ is onto for any $N \geq 1$.
Summarizing the results of this section one has

Theorem III.38 *For any $N \geq 0$, the map $\Omega^{(N)} : H^N_{\mathcal{R}} \to \ell^2_N$ is a real analytic diffeomorphism.*

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