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Polynomial bounds on the number of resonances for some complete spaces of constant negative curvature near infinity

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Abstract

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Let X be a conformally compact n-dimensional manifold with constant negative curvature -1 near infinity. The resolvent $(\Delta - s(n-1-s))^{-1}$, $\Re e\, s > n-1$, of the Laplacian on X extends to a meromorphic family of operators on $\mathbf{C}$ and its poles are called resonances or scattering poles. If $N_X(r)$ is the number of resonances in a disc of radius r we prove the following upper bound: $N_X(r) \leqslant Cr^{n+1} + C$.

1. Introduction and statement of the results

The purpose of this paper is to provide polynomial upper bounds on the number of resonances for the Laplacian on some infinite volume complete manifolds with constant negative curvature near infinity. Referring to Definitions 1 and 2 below we can state the result as

Theorem. *Let X be a conformally compact n-dimensional manifold with constant negative curvature near infinity. If $\mathcal{R}_X$ is the set of resonances of X then there exists a constant C such that*

$$\#\{s \in \mathcal{R}_X\colon\ |s| \leqslant r\} \leqslant Cr^{n+1} + C. \tag{1.1}$$

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The class of manifolds we consider constitutes the simplest general case similar to the one studied in [13], in which neighbourhoods of infinity are locally isometric to neighbourhoods of infinity of the free hyperbolic space $\mathbf{H}^n = \mathrm{SO}(n,1)/\mathrm{SO}(n)$. A large class of examples is provided by the convex co-compact quotients $\Gamma\backslash\mathbf{H}^n$ (see Fig. 1) which were studied in [1, 6, 14, 19–22].

The new upper bound (1.1) generalizes, in a weaker form, the results of [9] to higher dimensions but the method of proof is rather different. In [9] we proceeded by an analogy with the Euclidean case replacing $\mathbf{R}^2$ by $\langle w \to e^{\ell} w\rangle\backslash\mathbf{H}^2$. Through a careful analysis of this model quotient we could apply the methods of [16, 24, 25], once the meromorphic continuation was established as in [23]. By using a more refined approach to meromorphic continuation due to Mazzeo and Melrose [13] we can now proceed more directly. Roughly speaking, we are taking advantage of the regularity of the hyperbolic Laplacian when considered in terms of degenerate elliptic boundary value problems.

Since for $n = 2$ the bound in (1.1) improves to $\mathrm{O}(r^2)$ (see [9]) and since the asymptotics are known for some explicit higher dimensional quotients (see Remark 1), our present result is almost certainly *not* optimal. The expected power on the right hand side of (1.1) is n.

The class of manifolds considered here is given by

Definition 1. A complete Riemannian manifold (X, g) is called conformally compact with constant negative curvature near infinity if and only if

(i) there exists an incomplete metric h on X such that $(\overline{X}, h)$ is a compact manifold with a C^∞ boundary $X(\infty) = \partial\overline{X}$;
(ii) there exists $\rho \in C^\infty(\overline{X})$, $\rho|_{X(\infty)} = 0$, $d\rho|_{X(\infty)} \neq 0$ such that $g = \rho^{-2}h$;
(iii) for some neighbourhood $Y \subset \overline{X}$ of $X(\infty)$ all sectional curvatures of g are equal to -1.

For a manifold X, let us denote the Laplacian by Δ. Then the resolvent $R(s) = (\Delta - s(n-1-s))^{-1}$, $\Re e\, s > n-1$, extends to the complex plane as a meromorphic family of operators from $L^2_{\mathrm{comp}}(X)$ to $H^2_{\mathrm{loc}}(X)$ (see Proposition 3.2). The resonances of the Laplacian Δ are defined

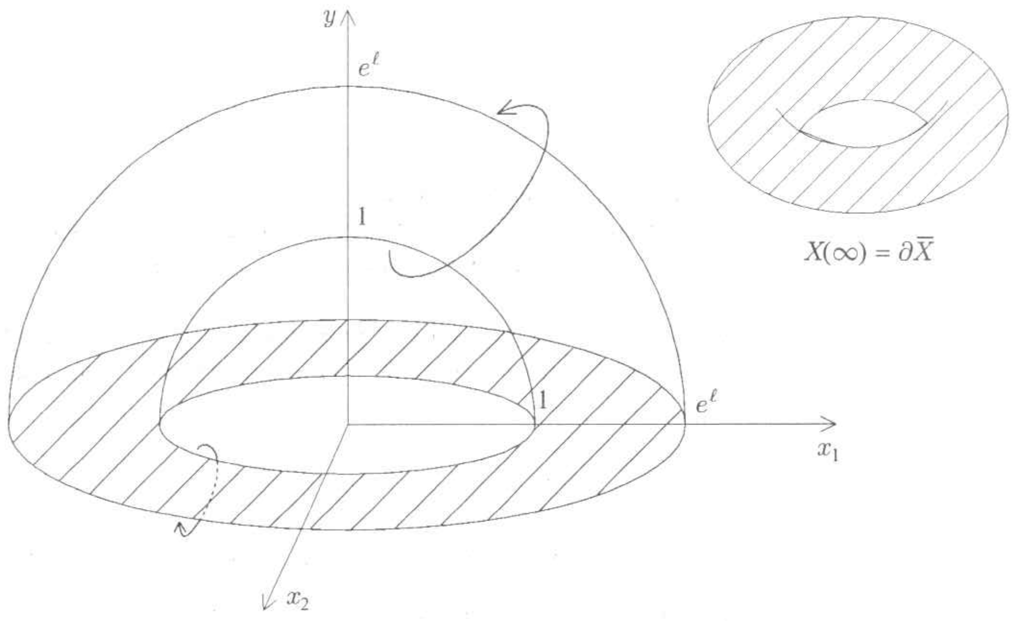

Fig. 1. $X = \langle w \to e^{\ell} w\rangle\backslash\mathbf{H}^3$.

as the poles of that meromorphic continuation. This is motivated by the Euclidean scattering as, for instance, in the Lax–Phillips theory [11, 23]. For hyperbolic quotients, resonances correspond also to the poles of the meromorphic continuation of Eisenstein series [19, 20].

Definition 2. A resonance s_0 of the Laplacian is a pole of R and the multiplicity is given by the rank of

$$\int_{\gamma_{s_0}} (n-1-2s)R(s)\,\mathrm{d}s$$

where γ_{s_0} is sufficiently small closed curve of index 1 relative to s_0.

Remark 1. By using elementary representation theory (see [5]), the resonances for $\mathbf{H}^n$ can be computed: for n odd, there are no resonances, while for n even each negative integer $-k$ is a resonance with multiplicity m_k^n, equal to the multiplicity of the eigenvalue $k(k+n-1)$ of the Laplacian on the standard sphere S^n. Consequently,

$$\#\{s \in \mathcal{R}_{\mathbf{H}^n}\colon\ |s| \leqslant r\} \sim \frac{2}{\Gamma(n+1)} r^n, \quad r \to \infty.$$

Resonances for hyperbolic cylinders $X = \langle w \to e^{\ell} w\rangle \backslash \mathbf{H}^n$ were computed by C.L. Epstein [4] and the first author [8], see also the appendix of [9]:

$$\mathcal{R}_X = -\mathbf{N} + \mathrm{i}\mathbf{Z}2\pi/\ell,$$

where each vertical line $-K + \mathrm{i}\mathbf{Z}2\pi/\ell$ has multiplicity

$$2\sum_{p=0}^{[K/2]} m_{K-2p}^{n-2}.$$

Consequently

$$\#\{s \in \mathcal{R}_X\colon\ |s| \leqslant r\} \sim \frac{\ell}{2^{n-2} n \Gamma\left(\frac{n}{2}\right)^2} r^n, \quad r \to \infty.$$

The preceding computation is based on the spectral analysis of one-dimensional Schrödinger operators with Pöschl–Teller potentials, which can also be used for the hyperbolic space $\mathbf{H}^n$.

Remark 2. In the case of constant negative curvature, Perry proposed to study the distribution of resonances through the meromorphic continuation of the Fried–Ruelle zeta function which is a topic of great interest in its own right. That meromorphic continuation was proved in dimension two by the first author [8] and a meromorphic continuation with an order estimate for higher dimensional convex co-compact quotients was announced in [22]. That would give a bound

$$\#\{s \in \mathcal{R}_{\Gamma\backslash\mathbf{H}^n}\colon\ |s| \leqslant r\} = \mathrm{O}_\varepsilon\left(r^{n+\varepsilon}\right)$$

for all $\varepsilon > 0$.

For notational simplicity we will use below the letter C to denote a large, but not necessarily the same constant. For u a system of coordinates, Δ_u will denote the pull-back through u of the Euclidean Laplacian.

2. The model problem

The purpose of this section is to present a convenient representation of the resolvent of the free hyperbolic Laplacian. In the half-space model we have

$$(\mathbf{H}^n, g_{\mathbf{H}^n}) = \left(\mathbf{R}^{n-1}_{\mathbf{x}} \times (0,\infty)_y, y^{-2}\left(\mathrm{d}\mathbf{x}^2 + \mathrm{d}y^2\right)\right) \tag{2.1}$$

so that

$$\Delta_{\mathbf{H}^n} = y^n D_y\left(y^{2-n} D_y\right) + y^2 \Delta_{\mathbf{x}}. \tag{2.2}$$

We will write $w = (\mathbf{x}, y)$ and denote by $d(w, w')$ the hyperbolic distance between $w, w' \in \mathbf{H}^n$. We recall that

$$\cosh d(w, w') = \frac{|\mathbf{x} - \mathbf{x}'|^2 + y^2 + y'^2}{2yy'}. \tag{2.3}$$

The resolvent kernel $R_0(s)(w, w')$ satisfies

$$\begin{aligned} &\left(\Delta_{\mathbf{H}^n} - s(n-1-s)\right) R_0(s)(w, w') = \delta_{\mathbf{H}^n}(w, w'), \\ &f(w) = \int_{\mathbf{H}^n} \delta_{\mathbf{H}^n}(w, w') f(w') \,\mathrm{d}\,\mathrm{vol}_{\mathbf{H}^n}, \quad f \in C_0^\infty(\mathbf{H}^n), \end{aligned} \tag{2.4}$$

and since the Laplacian is invariant under the isometry group, $R_0(s)(w, w')$ is a function of $d(w, w')$ alone. We have a well-known expression (see, e.g., [19, 21])

$$\begin{aligned} R_0(s)(w, w') = {} & \frac{\pi^{-\frac{n-1}{2}} 2^{-2s-1} \Gamma(s)}{\Gamma\left(s - \frac{n-3}{2}\right)} \cosh^{-2s}\left[\frac{d(w, w')}{2}\right] \\ & \times F\left(s, s - \frac{n}{2} + 1, 2s - n + 2; \cosh^{-2}\left[\frac{d(w, w')}{2}\right]\right), \end{aligned} \tag{2.5}$$

where

$$F(a, b, c; u) = 1 + \frac{a \cdot b}{1 \cdot c} u + \frac{a(a+1)b(b+1)}{1 \cdot 2c(c+1)} u^2 + \ldots$$

is the hypergeometric function.

It will be convenient to use a coordinate system $(\mathbf{x}, z = y^2)$ on $\mathbf{H}^n$, for which the relation

$$\Delta_{\mathbf{H}^n}\left(z^\zeta f\right) = 2\zeta(n - 1 - 2\zeta) z^\zeta f + z^{\zeta+1} Q(\zeta) f \tag{2.6}$$

holds with Q a differential operator with smooth coefficient on $\mathbf{R}^n_{\mathbf{x},z}$. This will be essential in the construction for the parametrix of the resolvent on weighted spaces (see Proposition 3.1). The choice of these coordinates amounts to fixing a differentiable structure $\mathcal{D}$ on $\mathbf{R}^n_{\mathbf{x}} \times [0,\infty)_z$; taking the usual differentiable structure with the function y smooth would not give a relation such as (2.6) with a smooth Q up to the boundary. The expansion of the resolvent kernel adapted to the structure $\mathcal{D}$, where $\cosh d(w, w') = z^{-1/2} z'^{-1/2} q(w, w')/2$ with q smooth, is given by the following lemma.

Lemma 2.1. *The free resolvent is given by*

$$R_0(s)(w,w') = \cosh^{-s} d(w,w') G\big(s, \cosh d(w,w')\big), \tag{2.7}$$

where for $\tau > 1$

$$G(s,\tau) = \pi^{-\frac{n-1}{2}} 2^{-s-1} \sum_{j=0}^{\infty} 2^{-2j} \frac{\Gamma(s+2j)}{\Gamma\big(s - \frac{n-3}{2} + j\big)\Gamma(j+1)} \tau^{-2j}. \tag{2.8}$$

Proof. If $r = d(w,w')$ then $g(s,r) = R_0(s)(w,w')$ solves

$$\left(D_r^2 - \mathrm{i}(n-1)\frac{\cosh r}{\sinh r} D_r - s(n-1-s)\right) g(s,r) = 0,$$

$$g(s,r) \sim c_1(s)\mathrm{e}^{-sr}, \quad r \to \infty,$$

as can be seen from writing (2.2) in polar coordinates and from the requirement that the resolvent $R_0(s)$ is bounded on $L^2(\mathbf{H}^n, \mathrm{d}\,\mathrm{vol}_{\mathbf{H}^n})$ for $\Re e\, s > (n-1)/2$.

Changing the variables $\tau = \cosh r$ we obtain an equation for $\tau^{-s}G(s,\tau) = g(s,r)$:

$$\big((\tau^2-1)D_\tau^2 - \mathrm{i}n\tau D_\tau - s(n-1-s)\big)\big(\tau^{-s}G(s,\tau)\big) = 0, \quad \tau > 1.$$

Substituting

$$G(s,\tau) = \sum_{k=0}^{\infty} a_k(s)\tau^{-k}$$

we obtain $a_1(s) = 0$ and an iterative formula for the coefficients

$$a_{k+2}(s) = \frac{(s+k)(s+k+1)}{(k+2)(2s-n+k+3)} a_k(s)$$

which shows that only even powers appear and that

$$a_{2j}(s) = 2^{-2j} \frac{\Gamma\big(s-\frac{n-3}{2}\big)\Gamma(s+2j)}{\Gamma(s)\Gamma\big(s-\frac{n-3}{2}+j\big)\Gamma(j+1)} a_0(s).$$

A comparison of the leading terms in the expansions in $\cosh d(w,w')$ in $G(s,\tau)$ and (2.5) gives (2.8). □

As the expression (2.5), the expansion (2.8) immediately shows that R_0 is entire for n odd and meromorphic for n even. In the latter case the poles are simple and the residue at $s = -k \in -\mathbf{N}$ is

$$\operatorname*{Res}_{s=-k} R_0(s)(w,w') = \sum_{0 \leqslant 2j \leqslant k} \frac{\pi^{-\frac{n-1}{2}}(-1)^{k+2j}2^{k-2j-1}}{\Gamma(j+1)\Gamma(k-2j+1)\Gamma\big(j-\frac{n-3}{2}-k\big)} \cosh^{k-2j} d(w,w').$$

In view of (2.3) the rank is clearly finite, as we saw in Remark 1. For applications in Sections 3 and 5 we need a more general but less accurate approach.

Lemma 2.2. *If an operator* A: $\mathcal{C}_0^\infty(\mathbf{H}^n) \to \mathcal{C}^\infty(\mathbf{H}^n)$ *has the kernel of the form*

$$A(w, w') = \sum_{0 \leqslant p \leqslant k} a_p(w) \cosh^p d(w, w') b(w')$$

where $a_p, b \in \mathcal{C}^\infty(\mathbf{H}^n)$, $0 \leqslant p \leqslant k$, *then there exists a constant* C_1 *depending only on* n *such that*

$$\operatorname{rank}_{\mathcal{C}_0^\infty(\mathbf{H}^n) \to \mathcal{C}^\infty(\mathbf{H}^n)}(A) \leqslant C_1 \langle 2k \rangle^n.$$

Proof. Using (2.3), we expand the kernel A in a sum of decomposed elements $f \otimes f'$ of the tensor product $\mathcal{C}^\infty(\mathbf{H}^n)^{\otimes 2}$ where f' is of the form

$$f'(w') = y'^{-k} \left(y'^{k-p} x_1'^{\alpha_1} \ldots x_{n-1}'^{\alpha_{n-1}} \left(|\mathbf{x}'|^2 + y'^2 \right)^{\alpha_n} b(w') \right)$$

with $\alpha_1 + \ldots + \alpha_n \leqslant k$. The number of such terms is less than then number of monomials of degree less than or equal to $2k$ in n variables, that is less than or equal to

$$\sum_{0 \leqslant p \leqslant 2k} N(p, n)$$

where $N(i, j) = \binom{i+j-1}{i}$ is the number of integer solutions of $n_1 + \ldots + n_j = i$. The lemma follows. □

3. Meromorphic continuation

As mentioned in the introduction we follow, with additional care, the parametrix construction of [13]. That is greatly simplified by the constant curvature assumption and for that we need an essentially well-known lemma which follows from the proof of Theorem 3.82 of [7] and Section III of [12].

Lemma 3.1. *If* X *is a conformally compact* n*-dimensional manifold with constant negative curvature near infinity* (*Definition* 1) *then there exists a neighbourhood* Y *of* $X(\infty)$, *in* $\overline{X}$ *and an open covering*

$$Y \subset \bigcup_{j=1}^{M} Y_j$$

such that each Y_j *is isometric to the set*

$$U_1 = \left\{ (\mathbf{x}, y) \in \mathbf{H}^n \colon |\mathbf{x}|^2 + y^2 < 1 \right\} \tag{3.1}$$

where we use the upper half-plane model (2.1).

Proof. We recall that the metric on X is of the form $\rho^{-2} h$ where ρ defines $\partial \overline{X}$ and h is a smooth metric on the manifold with boundary $\overline{X}$. If $m_0 \in \partial \overline{X}$, we follow [12] and put

$$N = \left\{ d_h(m, m_0) = \varepsilon \colon \rho(m) > 0 \right\} \quad \text{and} \quad M = \left\{ d_h(m, m_0) < \varepsilon \colon \rho(m) > 0 \right\},$$

with $\varepsilon > 0$ small enough for N to be a complete embedded submanifold of X. If for $p \in N$ we denote by n_p the g-unit normal vector at p pointing into M, then $M \simeq N \times \mathbf{R}_+$, $(p, s) \mapsto \exp_p(sn_p)$. In those coordinates the metric takes the form $g = ds^2 + k(s)$, where $k(s)$ is a family of complete metrics on N. We claim that $k(s)$ is determined by the the metric g near N. In fact, for $w \in T_pN$,

$$k(s)_p(w, w) = g_{\exp_p(sn_p)}\big(J_w(s), J_w(s)\big),$$

where $J_w(s)$ is the Jacobi field on $s \mapsto \exp_p(sn_p)$ with initial conditions $J_w(0) = w$, $J'_w(0) = \nabla^g_w n$ (see Section III of [12]). But since the sectional curvature is equal to -1 for ε sufficiently small, $J''_w - J_w = 0$ and consequently

$$J_w(s) = U_w(s)\cosh s + U_{\nabla^g_w n}(s)\sinh s,$$

where $U_v(s)$ is the parallel transport of v along $s \mapsto \exp_p(sn_p)$. Hence

$$g_{\exp_p(sn_p)}\big(U_v(s), U_v(s)\big) = g_p(v, v)$$

and we have

$$k(s)_p(w, w) = \cosh^2 s\, g_p(w, w) + 2\sinh s\cosh s\, g_p\big(w, \nabla^g_w n\big) + \sinh^2 s\, g_p\big(\nabla^g_w n, \nabla^g_w n\big),$$

and thus $k(s)$ is determined by the metric near N.

Let now $p_0 \in N$ be such that $\exp_{p_0}(sn_{p_0}) \to m_0$ as $s \to \infty$. By using the curvature assumption and Jacobi fields again we conclude that there exists a neighbourhood Ω of p_0 in X where the metric is hyperbolic (see Theorem 3.82 of [7]). Hence $M_0 \simeq (N \cap \Omega) \times \mathbf{R}_+$ is isometric to an open cylinder in $\mathbf{H}^n$ and consequently m_0 has a neighbourhood isometric to U_1 in (3.1). The compactness of $\partial\overline{X}$ now yields the final conclusion. $\square$

Let ι_j denote the isometry from Y_j to U_1. We will denote ι_j^* the induced pull-back operation transforming operators acting on U_1 to operators acting on Y_j and, for a function θ on Y_j, we will denote by $\tilde{\theta}$ its pull-back by ι_j^{-1} on U_1.

Using the covering of Lemma 3.1 (or possibly its refinement) we construct partitions of unity in X (see Fig. 2). Thus let $\varphi_i^j \in \mathcal{C}^\infty(X(\infty))$, $1 \leqslant j \leqslant M$, $i = 1, 2$, satisfy, for some $\varepsilon > 0$,

$$\operatorname{dist}_{h_\partial}\big(\operatorname{supp}\varphi_2^j, \operatorname{supp}\big(1 - \varphi_1^j\big)\big) > \varepsilon, \quad \operatorname{supp}\varphi_i^j \subset \overline{Y}_j \cap X(\infty), \quad \sum_{j=1}^{M} \varphi_2^j = 1. \tag{3.2}$$

Here h_∂ is the metric induced by h on $X(\infty) = \partial\overline{X}$. For $\delta > 0$ sufficiently small and for each j, $1 \leqslant j \leqslant M$, we also define functions $\psi_i^{j\delta} \in \mathcal{C}^\infty(\overline{X})$, $i = 1, 2$, which through the isometry ι_j depend only on the variable y, such that

$$\rho|_{\operatorname{supp}(1-\psi_i^{j\delta})\cap Y_j} > \delta, \quad \rho|_{\operatorname{supp}\psi_i^{j\delta}\cap Y_j} < 2\delta, \tag{3.3}$$

and

$$\psi_2^{j\delta}\psi_1^{j\delta} = \psi_2^{j\delta}.$$

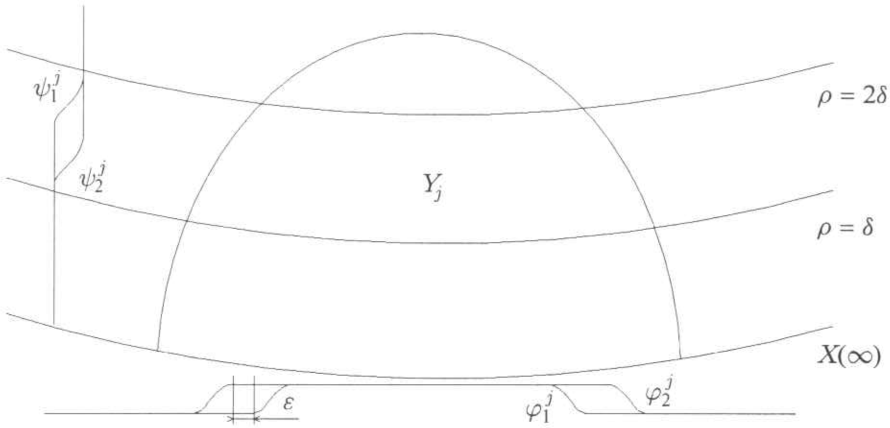

Fig. 2. The open set Y_j and its associated cut-off functions.

Using the identifications given by Lemma 3.1 to extend functions on $X(\infty)$ to functions on X we also assume that $\chi_i^{j\delta} = \varphi_i^j \psi_i^{j\delta}$, $i = 1, 2$, satisfies

$$\operatorname{supp}\left(\chi_i^{j\delta}\right) \subset Y_j. \tag{3.4}$$

Furthermore we introduce $\bar{\rho}$, a modified defining function for the boundary such that

$$\bar{\rho}(w) = \begin{cases} 1 & \text{for } w \in \operatorname{supp}(1 - \psi_i^{j\delta}),\ i = 1, 2, \\ \rho(w) & \text{if } \rho(w) \leqslant \delta/2. \end{cases}$$

Let χ^δ be the function defined by

$$\chi^\delta = \sum_{j=1}^{M} \chi_2^{j\delta}$$

so that it satisfies the two inequalities of (3.3) and $\operatorname{supp} \chi^\delta \subset Y$. We now drop the superscript δ, although it will play a rôle in the proof of Lemma 4.1.

The Mazzeo–Melrose construction will give us the proof of the following

Proposition 3.1. *For every $N \in \mathbf{N}$ there exists a family of operators*

$$E_N(s)\colon\ \bar{\rho}^N L^2(X) \to \bar{\rho}^{-N} H^2(X)$$

meromorphic in the half-plane $\{s\colon \Re e\, s > -N + (n-1)/2\}$ and such that

$$\big(\Delta_X - s(n-1-s)\big) E_N(s) = \chi + K_N(s)$$

where $K_N(s)\colon \bar{\rho}^N L^2(X) \to \bar{\rho}^N L^2(X)$ is meromorphic family of trace class operators. The singularities of $E_N(s)$ and $K_N(s)$ are simple poles at $s \in -\mathbf{N}$, of ranks uniformly bounded by $C_1\langle 2s\rangle^n$. For $s_N = N/C$ we have

$$\left\|K_N(s_N)\right\|_{\bar{\rho}^N L^2(X) \to \bar{\rho}^N L^2(X)} \leqslant 1/4. \tag{3.5}$$

Assuming this we apply the simple argument from [23] (see also [9]) to obtain

Proposition 3.2. *If* (X, g) *is a conformally compact manifold with constant negative curvature near infinity and* Δ *is the Laplacian on* X *then the resolvent*

$$R(s) = \big(\Delta - s(n-1-s)\big)^{-1}\colon\ L^2(X) \to H^2(X),\ \Re e\, s > \frac{n-1}{2},\ s(n-1-s) \notin \operatorname{spec}(\Delta),$$

extends meromorphically to $\mathbf{C}$ *as an operator*

$$R(s)\colon\ L^2_{\mathrm{comp}}(X) \to H^2_{\mathrm{loc}}(X).$$

Proof. Let χ_0 be a function which as χ satisfies (3.3) while $\operatorname{supp}(\chi_0) \subset$ Y, so that $\chi_0\chi = \chi_0$. To continue $R(s)$ to the half-plane $\{s\colon \Re e\, s > -N\}$ we take s_0 with $\Re e\, s_0 \gg (n-1)/2$ and define

$$P_N(s, s_0) = (1 - \chi_0)R(s_0)(1 - \chi) + E_N(s).$$

Hence

$$\big(\Delta - s(n-1-s)\big)P_N(s, s_0) = I + K_N(s, s_0)$$

where by Proposition 3.1

$$\begin{aligned} K_N(s, s_0) = {}& -[\Delta, \chi_0]R(s_0)(1-\chi) \\ & + \big(s_0(n-1-s_0) - s(n-1-s)\big)(1-\chi_0)R(s_0)(1-\chi) + K_N(s). \end{aligned} \tag{3.6}$$

Proposition 3.1 also shows that $K_N(s_0, s_0)\colon\ \bar\rho^N L^2(X) \to \bar\rho^N L^2(X)$ is meromorphic and compact and that

$$\big\|K_N(s_0, s_0)\big\|_{\bar\rho^N L^2(X) \to \bar\rho^N L^2(X)} \to 0$$

as $|s_0| \to \infty$, $|\Im m\, s_0| < C\Re e\, s_0$. Then by the analytic Fredholm theory

$$\big(I + K_N(s, s_0)\big)^{-1}\colon\ \bar\rho^N L^2(X) \to \bar\rho^N L^2(X)$$

is meromorphic for $\Re e\, s > -N + (n-1)/2$ and consequently

$$R(s) = P_N(s, s_0)\big(I + K_N(s, s_0)\big)^{-1}\colon\ L^2_{\mathrm{comp}}(X) \subset \bar\rho^N L^2(X) \to \bar\rho^{-N} H^2(X) \subset H^2_{\mathrm{loc}}(X)$$

enjoys a meromorphic continuation in the same half-plane. Since N can be taken to be arbitrarily large this concludes the proof. □

To motivate the rest of this section we now state the simple lemma which is the basis of Section 5.

Lemma 3.2. *Let* $K_N(s, s_0)$ *be the operator defined in the proof of Proposition* 3.2. *There exists a constant* C *such that the set of resonances of the Laplacian in* $\{s\colon \Re e\, s > -N + (n-1)/2\}$ *is included in the set of zeros of*

$$D_N(s) = \det{}_{\bar\rho^N L^2(X)}\big(I - \big(-K_N(s, s_0)\big)^n\big)$$

with multiplicities and $-\mathbf{N}$ *with multiplicity of* $k \in -\mathbf{N}$ *bounded by* $v_k(D_N) + C\langle 2k\rangle^n$ *where* $v_z(f)$ *denotes the valuation of the meromorphic function* f *at* z.

Proof. The proof is based on two general lemmas we present in the appendix at the end of the paper. Let us write

$$\begin{aligned} R(s) =& \left[P_N(s,s_0)\big(1-K_N(s,s_0)+\ldots+\big(-K_N(s,s_0)\big)^{n-1}\big)\right] \\ &\times\big(1-(-K_N(s,s_0))^n\big)^{-1}. \end{aligned} \tag{3.7}$$

Outside of $-\mathbf{N}$, resonances are in the set of the poles of $(1-(-K_N(s,s_0))^n)^{-1}$. Their multiplicities are at most the orders of zeros of determinant D_N. This is because of Lemma A.2, where we take B_1, B_2 holomorphic.

For $k \in -\mathbf{N}$, by Lemma 2.2, we have, for $m \leqslant n$,

$$\operatorname{rank}_{\mathrm{p}k}\big(1-\big(-K_N(\cdot,s_0)\big)^m\big) \leqslant C\langle 2k\rangle^n,$$

hence, by Lemma A.2, the multiplicity of k as resonance is at most $v_k(D_N)+C\langle 2k\rangle^n$. □

We will now prove Proposition 3.1 noting that in Sections 4 and 5 we will need some more specific aspects of the construction. All functions of $w, z, \ldots$, except of course the powers z^ζ, $\zeta \notin \mathbf{N}$, will be, if not otherwise stated, smooth with respect to the differentiable structure $\mathcal{D}$ introduced before Lemma 2.1.

We proceed by iteration and start by defining

$$E_N^0(s) = \sum_{j=1}^M \chi_1^j \iota_j^*\big(R_0(s)\big)\chi_2^j. \tag{3.8}$$

So that

$$\begin{aligned} \big(\Delta - s(n-1-s)\big)E_N^0(s) &= \chi + \sum_{j=1}^M \iota_j^*\big(\big[\Delta_{\mathbf{H}^n}, \tilde\chi_1^j\big]R_0(s)\tilde\chi_2^j\big) \\ &= \chi + K_N^0(s) + L_N^0(s), \end{aligned} \tag{3.9}$$

where

$$K_N^0(s) = \sum_{j=1}^M \iota_j^*\big(\tilde\psi_1^j\big[\Delta_{\mathbf{H}^n}, \tilde\varphi_1^j\big]R_0(s)\tilde\chi_2^j\big),$$

$$L_N^0(s) = \sum_{j=1}^M \iota_j^*\big(\tilde\varphi_1^j\big[\Delta_{\mathbf{H}^n}, \tilde\psi_1^j\big]R_0(s)\tilde\chi_2^j\big). \tag{3.10}$$

To simplify the notation we will now drop the sums and the pull-back operators ι_j^* and write $\varphi_1 \equiv \tilde\varphi_1^j$, $\psi_1 \equiv \tilde\psi_1^j$, $\chi_2 \equiv \tilde\chi_2^j$.

Since $\chi_2[\Delta_{\mathbf{H}^n}, \psi_1] = 0$ and $[\Delta_{\mathbf{H}^n}, \psi_1]$ has compactly supported coefficients we immediately see that

$$L_N^0(s)\colon\ \bar{\rho}^N L^2(X) \to \mathcal{C}_0^\infty(X),$$

so that we have the desired mapping property for this term (see Proposition 3.1).

To study $K_N^0(s)$ we use Lemma 2.1 and (2.3) and write

$$K_N^0(s) = \psi_1 K_N^{00}(s) + \ldots + \psi_1 K_N^{0N}(s) + K_N^{\#}(s) \tag{3.11}$$

where, for $0 \leqslant j \leqslant N$,

$$\begin{aligned} K_N^{0j}(s)(w,w') &= z^{s/2+j+1} a_{2j}(s)[\Delta_x, \varphi_1] q(w,w')^{-s-2j} z'^{s/2+j} \chi_2(w'),\\ a_{2j}(s) &= \frac{\pi^{-\frac{n-1}{2}} 2^{-1} \Gamma(s+2j)}{\Gamma\big(s - \frac{n-3}{2} + j\big)\Gamma(j+1)},\\ q(w,w') &= |\mathbf{x} - \mathbf{x}'|^2 + z + z'. \end{aligned}$$

The iteration is now a refinement of that from [13] and is based on

$$\begin{aligned} \Delta_{\mathbf{H}^n}\big(z^\zeta f(\mathbf{x},z)\big) &= 2\zeta(n-1-2\zeta) z^\zeta f(\mathbf{x},z) + z^{\zeta+1} Q(\zeta) f(\mathbf{x},z),\\ Q(\zeta) &= 2(n-3-4\zeta)\mathrm{i}D_z + 4zD_z^2 + \Delta_{\mathbf{x}}. \end{aligned} \tag{3.12}$$

At the first step we set

$$E_N^1(s) = E_N^0(s) + \big[2(2s-n+3)\big]^{-1} \psi_1 K_N^{00}(s)$$

so that

$$\big(\Delta - s(n-1-s)\big) E_N^1(s) = \chi + K_N^1(s) + L_N^1(s)$$

with

$$\begin{aligned} K_N^1(s) &= \psi_1 K_N^{11}(s) + \psi_1 K_N^{12}(s) + \ldots + \psi_1 K_N^{1N}(s) + K_N^{\#}(s),\\ K_N^{11}(s) &= K_N^{01}(s) + \big[2(2s-n+3)\big]^{-1} Q(s/2+1) K_N^{00}(s),\\ K_N^{1k}(s) &= K_N^{0k}(s), \quad 2 \leqslant k \leqslant N,\\ L_N^1(s) &= L_N^0(s) + \big[2(2s-n+3)\big]^{-1} [\Delta_{\mathbf{H}^n}, \psi_1] K_N^{00}(s). \end{aligned} \tag{3.13}$$

We observe that $E_N^1(s)$, $K_N^1(s)$, $L_N^1(s)$ are meromorphic with simple poles at

$$s \in -\mathbf{N}_0 \Big\backslash \Big(-\mathbf{N}_0 + \frac{n-5}{2} \Big)$$

of rank $\mathrm{O}(\langle s\rangle^n)$, since

$$\frac{a_0(s)}{s-\frac{n-3}{2}} = \frac{\pi^{-\frac{n-1}{2}} 2^{-s-1}\Gamma(s)}{\Gamma\bigl(s-\frac{n-5}{2}\bigr)}$$

and we apply Lemma 2.2 to estimate the rank. We also observe that the kernels of $K_N^{1k}(s)$, $1 \leqslant k \leqslant N$, are of the form

$$z^{s/2+k+1} f_N^{1k}(s)(w,w') z'^{s/2}$$

while $K_N^{\#}(s)$ is of the form

$$z^{s/2+N+2} f_N^{\#}(s)(w,w') z'^{s/2}.$$

We now go on to obtain

$$E_N(s) = E_N^N(s) = E_N^{N-1}(s) + \bigl[2N(2s+2N-n+1)\bigr]^{-1}\psi_1 K_N^{N-1,N-1}(s) \tag{3.15}$$

with

$$\bigl(\Delta - s(n-1-s)\bigr)E_N(s) = \chi + K_N^N(s) + L_N^N(s) = \chi + K_N(s)$$

where

$$K_N^N(s) = \psi_1 K_N^{NN}(s) + K_N^{\#}(s),$$

$$\begin{aligned} K_N^{NN}(s)(w,w') = {} & \frac{\pi^{-\frac{n-1}{2}} 2^{-2N-1}}{\Gamma\bigl(s-\frac{n-3}{2}+N\bigr)\Gamma(N+1)} z^{s/2+N+1} \sum_{j=0}^{N-1} \Gamma(s+2j) 2^{2j} \\ & \times \left[\prod_{m=1}^{N-j} Q(s/2+j+m)_w \bigl[[\Delta_x,\varphi_1] q(w,w')^{-s-2j}\bigr] z'^{s/2+j}\chi_2(w')\right], \end{aligned} \tag{3.16}$$

$$L_N^N(s) = L_N^{N-1}(s) + \bigl[2N(2s+2N-n+1)\bigr]^{-1}[\Delta_{\mathbf{H}^n},\psi_1] K_N^{N-1,N-1}(s). \tag{3.17}$$

To conclude the proof of Proposition 3.1 we recall the following easy

Lemma 3.3. *Let X be a conformally compact n-dimensional manifold with constant negative curvature near infinity and ρ the associated function of Definition* 1. *If A: $C_0^\infty(X) \to \mathcal{D}'(X)$ has a kernel of the form $\rho^r(m)F(m,m')\rho^s(m')$, with $r, s \in \mathbf{C}$ and $F \in L^\infty(X\times X)$ then*

$$p_1 > -\Re e\, s - \frac{n+1}{2},\ p_2 < \Re e\, r - \frac{n-1}{2} \Rightarrow A \in \mathcal{L}\bigl(\rho^{p_1}L^2(X), \rho^{p_2}L^2(X)\bigr).$$

If $F \in C^\infty(\overline{X}\times\overline{X})$ then

$$p_1 \geqslant -\Re e\, s - \frac{n}{2},\ p_2 \leqslant \Re e\, r - \frac{n}{2} \Rightarrow A \in \mathcal{L}_1\bigl(\rho^{p_1}L^2(X), \rho^{p_2}L^2(X)\bigr),$$

where $L^2(X) \equiv L^2(X, \mathrm{d}\,\mathrm{vol}_g)$, and $\mathcal{L}$ and $\mathcal{L}_1$ denote the spaces of bounded and trace class operators, respectively.

From this the desired boundedness of $E_N(s) \equiv E_N^N(s)$ and the trace class property of $K_N(s) \equiv K_N^N(s) + L_N^N(s)$ easily follow. Except for (3.5) which will be proved in Section 4, the proof of Proposition 3.1 is now completed, with the finite rank statement following from (3.16) and Lemma 2.2.

Remark 3. To obtain finite rank of the poles we use detailed information about the model resolvent (see Lemma 2.1 above). As was pointed out to us by Melrose [17] that can be avoided by using the method of Section 5.19 of [18]. But since the continuation proceeds then through narrow strips, it is not as useful for estimating the number of poles.

4. Estimates on the characteristic values

To apply Lemma 3.2 we need to estimate the characteristic values of $K_N(s, s_0)$ away from its poles. That amounts in estimating the characteristic values of $K_N^N(s)$ and $L_N^N(s)$ acting on $\bar{\rho}^N L^2(X)$. Thus we denote by $\mu_k(A)$, $k \geqslant 1$, the eigenvalues of $[A^*A]^{1/2}$: $\bar{\rho}^N L^2(X) \to \bar{\rho}^N L^2(X)$ where A: $\bar{\rho}^N L^2(X) \to \bar{\rho}^N L^2(X)$ is a compact operator. We recall that the weight functions $\bar{\rho}$ and consequently the characteristic values depend on the small parameter δ.

For each integer N, to construct the operator E_N in Section 3, we use functions φ_1 (where following the notation adopted there we drop the superscript j) which satisfy:

$$\text{for } C \text{ independent of } N,\ \left\|D^\alpha \varphi_1\right\|_\infty \leqslant C^{|\alpha|} N^{|\alpha|}, \quad |\alpha| \leqslant 4N. \tag{4.1}$$

The existence of functions satisfying (4.1) and the conditions required in Section 3 is guaranteed by Theorem 1.4.2 of [10].

Lemma 4.1. *For* $|s| \leqslant N/C$, $d(s, -\mathbf{N}) > \eta > 0$, δ *small enough and* $0 \leqslant j \leqslant N-1$

$$\mu_k\big(A_j^N(s)\big) \leqslant \mathrm{e}^{-N/C} k^{-2}, \tag{4.2}$$

where the operator $A_j^N(s)$ *has the kernel*

$$\begin{aligned} A_j^N(s)(w, w') = {} & \frac{\pi^{-\frac{n-1}{2}} 2^{-2N+2j-1} \Gamma(s+2j)}{\Gamma\big(s - \frac{n-3}{2} + N\big)\Gamma(N+1)} \psi_1(z) z^{s/2+N+1} \\ & \times \prod_{m=1}^{N-j} Q(s/2+j+m)_w \big[[\Delta_{\mathbf{x}}, \varphi_1] q(w, w')^{-s-2j}\big] z'^{s/2+j} \chi_2(w'). \end{aligned}$$

Proof. We start with an elementary estimate: for $|s| \leqslant N/C$, C large, $d(s, -\mathbf{N}) > \eta > 0$, $0 \leqslant j \leqslant N-1$

$$\left| \frac{2^{-2N+2j-1} \Gamma(s+2j)}{\Gamma\big(s - \frac{n-3}{2} + N\big)\Gamma(N+1)} \right| \leqslant \mathrm{e}^{CN} N^{-2(N-j)}, \tag{4.3}$$

which follows from the Stirling formula applied after using, when $\Re e\,(s+2j) \leqslant 0$, the complement formula $\Gamma(u) = \pi \sin^{-1}(\pi u) \Gamma^{-1}(1-u)$.

The decay on the right of (4.3) will be used to cancel the growth in the next estimate: for $|s| \leqslant N/C$, $\mathbf{x} \in \operatorname{supp} \nabla\varphi_1$, $\mathbf{x}' \in \operatorname{supp}\varphi_2$, $j \leqslant N$, any fixed $p \in \mathbf{N}$, and if N is large enough

$$\begin{aligned}&\left| \max_{|\alpha|+k\leqslant 2p} D_{\mathbf{x}}^{\alpha} D_z^k \prod_{m=1}^{N-j} Q(s/2+j+m)_w \big[[\Delta_{\mathbf{x}}, \varphi_1] q(w,w')^{-s-2j}\big](w,w')\right| \\ &\quad\leqslant \mathrm{e}^{CN} N^{2(N-j)}.\end{aligned} \tag{4.4}$$

To obtain this we use the assumptions on w, w' to note that

$$C \geqslant \left|q(w,w')\right| \geqslant \varepsilon^2$$

and that $q(w,w')^{-s-2j}$ is holomorphic as a function of $\mathbf{x}, z, \mathbf{x}', z'$ in some complex neighbourhood of

$$\big\{(\mathbf{x}, z, \mathbf{x}', z')\colon\ \mathbf{x}' \in \operatorname{supp}\varphi_2,\ \mathbf{x} \in \operatorname{supp}\nabla\varphi_1,\ 0 < z < \delta^2,\ 0 < z' < \delta^2\big\},$$

depending on ε but not on δ and is bounded by e^{CN} there. We obtain from this

$$\left|D_{\mathbf{x}}^{\alpha} D_z^k q(w,w')^{-s/2-j}\right| \leqslant C^{|\alpha|+k}\big(|\alpha|+k\big)!\,\mathrm{e}^{CN}. \tag{4.5}$$

On the other hand, the quasi-analytic estimates on φ_1, (4.1), give that

$$\left|D_{\mathbf{x}}^{\beta} \nabla\varphi_1\right| \leqslant C^{|\beta|} N^{|\beta|}, \quad |\beta| \leqslant 2N. \tag{4.6}$$

Since

$$\left|Q(s/2+j+m)v\right| \leqslant C \max_{|\alpha|+k\leqslant 2} \left|D_{\mathbf{x}}^{\alpha} D_z^k v\right| + C|s+2j+2m|\,|D_z v|$$

and since

$$\prod_{m=1}^{N-j} \big[|s+2j+2m| N^{N-j}\big] \leqslant \mathrm{e}^{CN} N^{2(N-j)}, \quad N^{|\beta|}\big(2N-2j-|\beta|\big)! \leqslant \mathrm{e}^{CN} N^{2N-2j},$$

the estimate (4.4) follows from (4.5) and (4.6). Thus we have the following estimate on the reduced kernel

$$\begin{aligned}\left|\Delta_{\mathbf{x},z}^p\big(\bar\rho^{-N} A_j^N(s)\bar\rho^N\big)\right| &\leqslant C\delta^{\Re e\, s-2p} \sup_{\operatorname{supp}\psi_1} z^{\Re e\, s/2+N/2+1-p}\mathrm{e}^{CN} \\ &\leqslant C\delta^{N(1-2/C)-4p}\mathrm{e}^{CN}.\end{aligned} \tag{4.7}$$

By taking δ small enough the left hand side is bounded by $\mathrm{e}^{-N/C}$. A comparison with the eigenvalues of the Dirichlet Laplacian on a domain containing U_1 (see the proof of Lemma 4.1 in [9]) gives (4.2) by using (4.7) with $p \geqslant n$. □

To estimate the characteristic values of $L_N^N(s)$ we follow the argument of [15] and [16] since the exponential growth is no longer cancelled by the powers of y obtained in the iteration. However at this stage we do not achieve the dimension reduction [24, 25] and consequently the estimates are *not* optimal.

Lemma 4.2. *For $|s| \leqslant N/C$, $d(s, -\mathbf{N}) > \eta$ and $0 \leqslant j \leqslant p \leqslant N$, a constant C independent of δ and a constant C_δ depending on δ,*

$$\mu_k\big(B_{jp}^N(s)\big) \leqslant \begin{cases} \mathrm{e}^{CN}\delta^{2(\Re e\, s-1)}, \\ \mathrm{e}^{-N/C}k^{-2}, \quad k > C_\delta N^n, \end{cases} \tag{4.8}$$

where the operator $B_{jp}^N(s)$ has the kernel

$$B_{jp}^N(s)(w,w') = \frac{\pi^{-\frac{n-1}{2}}2^{-1}\Gamma(s+2j)}{\Gamma\big(s-\frac{n-3}{2}+p\big)\Gamma(p+1)}[\Delta_{\mathbf{H}^n},\psi_1]z^{s/2+p}$$
$$\times \prod_{m=1}^{p-j} Q(s/2+j+m)_w\big[[\Delta_{\mathbf{x}},\varphi_1]q(w,w')^{-s-2j}\big]z'^{s/2+j}\chi_2(w'),$$

where the product $\prod_{m=1}^{0} Q(s/2+j+m)$ is reduced to 1 if $j=p$.

Proof. Since on the left of the support of the kernel of $B_{jp}^N(s)$ we have $\bar\rho = 1$, the reduced kernel is equivalent to $B_{jp}^N(s)\bar\rho^N$. We now let $\psi^\# \in C_0^\infty(\mathbf{R})$ satisfy an analogue of (4.1):

$$\big\|D^k\psi^\#\big\|_\infty \leqslant C^kN^k, \quad k \leqslant 2N, \tag{4.9}$$

and $\psi^\#(y) = 0$ for $|y| > 2c$, $\psi^\#(y) = 1$ for $|y| < c$, where c is fixed so that $\operatorname{supp}\varphi_1 \times \operatorname{supp}\psi^\# \subset U_1$. Hence for δ small enough $[\Delta_{\mathbf{H}^n},\psi_1]\psi^\# = [\Delta_{\mathbf{H}^n},\psi_1]$. For $\ell \leqslant N$, the proof of Lemma 4.1 shows, thanks to (4.9), that

$$\Bigg|\Delta_{\mathbf{x},z}^{\ell+n}\Bigg[\frac{\pi^{-\frac{n-1}{2}}2^{-1}\Gamma(s+2j)}{\Gamma\big(s-\frac{n-3}{2}+p\big)\Gamma(p+1)}\psi^\#$$
$$\times \prod_{m=1}^{p-j} Q(s/2+j+m)_w\big[[\Delta_{\mathbf{x}},\varphi_1]q(w,w')^{-s-2j}\big]z'^{s/2+j}\chi_2(w')\bar\rho^N(w')\Bigg]\Bigg|$$
$$\leqslant C^\ell N^{2\ell}\mathrm{e}^{CN}\delta^{\Re e\, s}.$$

Hence a comparison with the eigenvalues of the Dirichlet Laplacian, Δ_Ω, on a bounded domain Ω, $\overline{U}_1 \subset \Omega \subset \mathbf{R}^n$, shows that

$$\mu_k\big(B_{jp}^N(s)\big) \leqslant \big\|[\Delta_{\mathbf{H}^n},\psi_1]z^{s/2}\Delta_\Omega^{-1/2}\big\|\delta^{\Re e\, s}k^{-2}k^{-2\ell/n}C_1^\ell N^{2\ell}\mathrm{e}^{C_1N}$$
$$\leqslant \delta^{2(\Re e\, s-1)}k^{-2}k^{-2\ell/n}C_1^\ell N^{2\ell}\mathrm{e}^{C_1N}.$$

Taking $\ell = N$, as we may, we get

$$k^{-2\ell/n} N^{2N} \leqslant \mathrm{e}^{-(2/n \log B)N} \quad \text{if } k > BN^n. \tag{4.10}$$

Choosing B sufficiently large depending on δ and C_1 we obtain the lemma. □

The arguments in the proof immediately give

Proposition 4.1. *If $K_N = K_N^N + L_N^N$ is given by (3.16) and (3.17), then for $|s| \leqslant N/C$, $d(s, -\mathbf{N}) > \eta$ and δ small enough*

$$\mu_k\big(K_N(s)\big) \leqslant \begin{cases} \mathrm{e}^{CN} \delta^{2(\Re e\, s-1)}, \\ \mathrm{e}^{-N/C} k^{-2}, \quad k > C_\delta N^n. \end{cases} \tag{4.11}$$

Proof. The first term, K_N^N is a sum of N terms of the form A_j^N and the term $K_N^{\#}$. Thus the proof of Lemma 4.1 applies with the analyticity of the kernel $R_0(s)(w,w')$ away from the diagonal used for the last term in place of that of $q(w,w')^{-s-2j}$. In fact

$$\begin{aligned} K_N^{\#}(s)(w,w') &= \psi_1(w) z^{s/2+N+2} [\Delta_{\mathrm{x}}, \varphi_1] \\ &\quad \times \big(q(w,w')^{-s-2N-2} G_N^{\#}\big(s, \cosh d(w,w')\big)\big) z'^{s/2+N} \chi_2(w'), \end{aligned}$$

where

$$\begin{aligned} G_N^{\#}(s,\tau) &= \pi^{-\frac{n-1}{2}} 2^{-s-1} \sum_{j=0}^{\infty} 2^{-2j-2N-2} \\ &\quad \times \frac{\Gamma(s+2j+2N+2)}{\Gamma\big(s - \frac{n-3}{2} + j + N + 1\big)\Gamma(j+2+N)} \tau^{-2j}. \end{aligned}$$

The coefficients are holomorphic in s for $|s| < N/C$ and uniformly bounded by $C\mathrm{e}^{CN}$ there and thus we get a uniform bounds on $q(w,w')^{-s-2N-2} G_N^{\#}$ in the same complex neighbourhood as in the proof of Lemma 4.1.

The L_N^N part is a sum of terms B_{jp}^N, $0 \leqslant j \leqslant p \leqslant N$, and L_N^0 given by (3.10). The proof of Lemma 4.2 applies to the sum of B_{jp}^N's while for L_n^0 we use the same argument as for $K_N^{\#}$ with uniform bounds away from the poles of R_0. □

A simple consequence of the estimates on characteristic values is a norm estimate which for $\Re e\, s$ large and positive completes the proof of Proposition 3.1.

5. Proof of the main theorem

The meromorphic continuation and the characteristic values estimate allow us to follow the same method as in [9]. Thus we start by defining

$$g_P(s) = s^{P2^n} \prod_{\omega \in U_{2(n+1)} \cdot \mathrm{N}_0} \left(E\left(\frac{s}{\omega}, n+1\right)\right)^{P(|2\omega|)^n}, \tag{5.1}$$

where U_m denotes the set of the mth roots of the unity and as usual

$$E(z,p) = (1-z)\exp\left(z + \ldots + p^{-1}z^p\right).$$

As in [9] we use Lindelöf's theorem but now we also need some lower bounds.

Lemma 5.1. *The function $g_P(s)$ given by* (5.1) *satisfies* $|g_P(s)| \leqslant \mathrm{e}^{C\langle s\rangle^{n+1}}$. *For each $\varepsilon > 0$, there exists a constant C_ε such that for s with $d(s, U_{2(n+1)}\cdot \mathbf{N}) > \varepsilon$,*

$$\left|g_P(s)\right| > \exp\left(-C_\varepsilon\langle s\rangle^{n+1}\right). \tag{5.2}$$

Proof. The first part follows immediately from Lindelöf's theorem (see, for instance, [2, Theorem 2.10.1]) since for each $k \in \mathbf{N}_0$

$$\sum_{\omega\in U_{2(n+1)}k} \omega^{-(n+1)} = 0, \tag{5.3}$$

and the second part from a slight modification of the standard proof of the minimum modulus theorem:

$$\begin{aligned}
\log\left|g_P(s)\right| &\geqslant P2^n\log|s| + P2^n\sum_{|\omega|\leqslant 2|s|}|\omega|^n\log\left|1-\frac{s}{\omega}\right| \\
&\quad + P2^n\sum_{|\omega|\leqslant 2|s|}|\omega|^n\Re e\left[\frac{s}{\omega}+\cdots+(n+1)^{-1}\left(\frac{s}{\omega}\right)^{n+1}\right] \\
&\quad + P2^n\sum_{|\omega|>2|s|}\log\left|E\left(\frac{s}{\omega}, n+1\right)\right| \\
&\geqslant P2^n\log|s| + P2^n\sum_{|\omega|\leqslant 2|s|}|\omega|^n\log\left|1-\frac{s}{\omega}\right| \\
&\quad - C\sum_{|\omega|\leqslant 2|s|}|s|^n - C\sum_{|\omega|>2|s|}|\omega|^n\left(\frac{|s|}{|\omega|}\right)^{n+2} \\
&\geqslant P2^n\log|s| + P2^n\sum_{|\omega|\leqslant 2|s|}|\omega|^n\log\left|1-\frac{s}{\omega}\right| - \mathrm{O}\left(|s|^{n+1}\right).
\end{aligned}$$

Here the ω's are taken in $U_{2(n+1)}\cdot\mathbf{N}_0$ and we used (5.3) to obtain the second inequality. Since s is assumed to lie away from the zeros of g_P, the lower bound (5.2) follows. □

Lemma 5.2. *Let D_N be given by Lemma* 3.2. *Then assuming* (4.1) *and for* $|s| < N/C$, $d(s, -\mathbf{N}) > \eta$

$$\left|D_N(s)\right| \leqslant \mathrm{e}^{CN^{n+1}}. \tag{5.4}$$

Proof. The estimate is immediate from Proposition 4.1 above and Lemma 6.1 of [9]. □

To remove the poles we will use $g_P(s)$ and Lemma A.1 to get the crucial

Lemma 5.3. *If P is large enough (independently of N), the function $g_P D_N$ is holomorphic on $\{s\colon \Re e\, s > -N + (n-1)/2\}$ and satisfies, for $|s| < N/C$,*

$$\left|g_P(s)D_N(s)\right| \leqslant \mathrm{e}^{CN^{n+1}},$$

with the constant C independent of N.

Proof. By Proposition 3.1 and Lemma A.1, the function D_N is meromorphic with poles at $s \in -\mathbf{N}_0$ of multiplicities bounded by $C\langle 2s\rangle^n$. Thus by taking $P \geqslant C$ we get the holomorphy while the estimate follows from Lemmas 5.1 and 5.2 and the maximum principle. The terms coming from $R(s_0)$ are estimated as in the proof of Lemma 6.2 in [9] using again Lemma 6.1 from that paper. Using the resolvent identity we see that the estimates on the characteristic values of $(1-\chi)R(s_0)(1-\chi)$ are independent of s_0 once $\Re e\, s_0 > C \gg (n-1)/2$, $|\Im m\, s_0| < \Re e\, s_0/C$. □

To apply Jensen's inequality we still need another lemma.

Lemma 5.4. *If in the construction of $K_N(s, s_0)$, $\delta > 0$ is small enough and we put $s_0 = s_N = 1/2 + [N/(4C)]$ with C large enough independently of N, then for some other constant C independent of N we have*

$$\left|D_N(s_N)\right| > 1/C,$$

and consequently

$$\left|g_P(s_N)D_N(s_N)\right| > \exp(-CN^{n+1}). \tag{5.5}$$

Proof. We easily see from Proposition 4.1 that with the s_0 given in the lemma, $\|K_N(s_N, s_N)\| \leqslant 1/2$ so that $I - (-K_N(s_N, s_N))^{n+1}$ is invertible and

$$\left|D_N(s_N)\right| = \left|\det\left[\left(I - (-K_N(s_N, s_N))^{n+1}\right)^{-1}\right]\right|^{-1} \geqslant \det\left(I + 2|K_N(s_N, s_N)^{n+1}|\right)^{-1}.$$

The conclusion now follows from estimates of Section 4 and the proof of Lemma 5.3 while (5.5) from Lemma 5.1. □

The proof of the theorem of Section 1 is now immediate. We use Jensen's formula for the disc $D(s_N, N/2C)$, with the estimates provided by Lemmas 5.3 and 5.4: let $n_N(t)$ be the number of zeros of $g_P D_N$ in $D(s_N, t)$. Then

$$\int_0^{N/2C} \frac{n_N(t)}{t}\,\mathrm{d}t \leqslant \max_{|s-s_N|\leqslant N/2C} \log\left|(g_P D_N)(s)\right| - \log\left|(g_P D_N)(s_N)\right| \leqslant CN^{n+1}.$$

Thus a uniform bound CN^{n+1} is given for the number of resonances in $D(0, N/5C)$ (see Fig. 3).

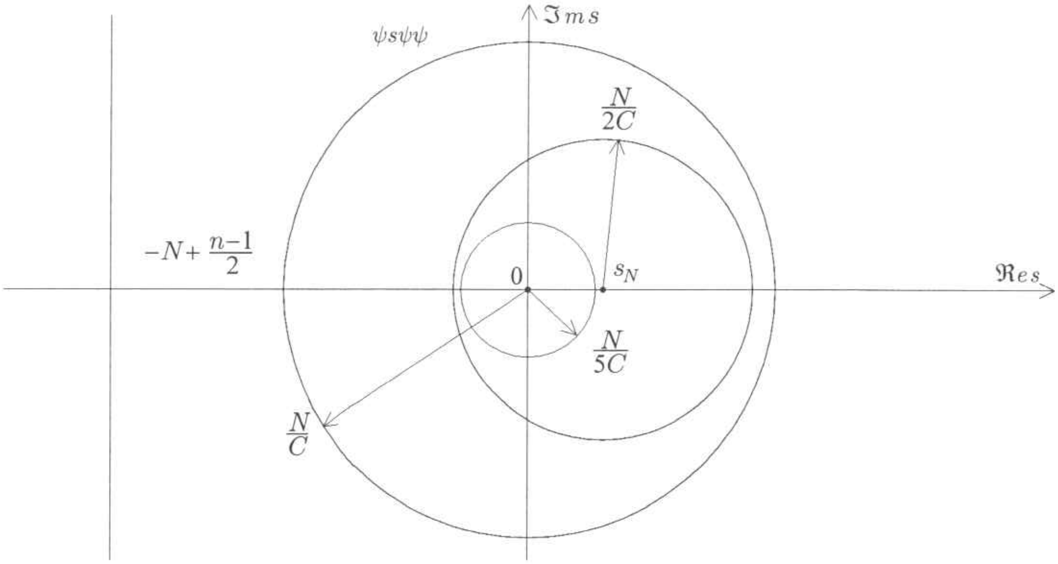

Fig. 3. The meromorphic continuation to $\Re e\, s > -N + (n-1)/2$, a uniform estimate in $D(0, N/C)$, Jensen's formula in $D(s_N, N/(2C))$ and the final estimate in $D(0, N/(5C))$.

Appendix: analytic properties of determinants

Let us consider germs at $u = 0$ of meromorphic functions with values in the space of bounded linear operators on a Hilbert space $\mathcal{H}$, with polar parts of finite rank. For such A we will denote by $v_0(A)$ its valuation at $u = 0$, so that its Laurent expansion takes the form

$$\sum_{p \geqslant v_0(A)} A_p u^p$$

with $A_{v_0(A)} \neq 0$ and all the A_p, $p < 0$, of finite rank. We will denote by $\mathrm{rank}_{\mathrm{p}0}(A)$ the dimension of the range of the polar part

$$A_-(u) = \sum_{v_0(A) \leqslant p < 0} A_p u^p$$

of A: there exists a projection π with a range of dimension $\mathrm{rank}_{\mathrm{p}0}(A)$ such that $\pi A_- = A_-$. Moreover, the *total rank of the polar part* of A is defined as

$$\mathrm{Rank}_{\mathrm{p}0}(A) = \sum_{v_0(A) \leqslant p < 0} \mathrm{rank}\, A_p$$

and the inequality $\mathrm{rank}_{\mathrm{p}0}(A) \leqslant \mathrm{Rank}_{\mathrm{p}0}(A)$ holds trivially. Let us write $m_- = \sup(-m, 0)$ for an integer m.

Lemma A.1. *Let T be a meromorphic germ at 0 valued in trace class operators, of valuation $v_0(T)$ and with polar part of finite rank* $\mathrm{rank}_{\mathrm{p}0}(T)$. *Then, if not zero,* $\det(1+T)$ *defines a germ of a meromorphic function with valuation greater than or equal to* $-\mathrm{rank}_{\mathrm{p}0}(T)\,(v_0(T))_-$.

Proof. For some projection π of rank $\operatorname{rank}_{\mathrm{p}0}(T)$ and in a small disk $D(0,\varepsilon)$ centered at $u=0$ we can write $T(u)=\pi\widetilde{T}_-(u)u^{v_0(T)}+T_0(u)$ with $\widetilde{T}_-(u)$ and $T_0(u)$ holomorphic in $D(0,\varepsilon)$. Then

$$1+T(u)=\left(1+\pi u^{v_0(T)}\right)\left[1-\frac{\pi}{1+u^{-v_0(T)}}+\frac{\pi\widetilde{T}_-(u)}{1+u^{-v_0(T)}}+\left(1-\frac{\pi}{1+u^{-v_0(T)}}\right)T_0(u)\right]$$

where the second factor on the right hand side is holomorphic in $D(0,\varepsilon)$ if $v_0(T)<0$. The lemma follows immediately. □

Before giving another lemma, we need to state a useful normal form for some operator valued meromorphic function. Let $\mathcal{O}_1$ be the local ring of germs of holomorphic scalar functions in one variable, $\mathcal{M}_1$ its fraction field (the field of germs of the meromorphic functions) and A in the matrix ring $M(d,\mathcal{M}_1)$. By the theory of invariant factors for matrices over principal rings (see VII, 4.5 in [3]), there exist U_1,U_2 in $\mathrm{GL}(d,\mathcal{O}_1)$, a resolution of the identity $(\pi_p)_{p\in\mathbf{Z}\cup\{\infty\}}$ (i.e., $\sum_{p\in\mathbf{Z}}\pi_p=1$) with projections π_p, non-zero for a finite number of integers p only and with $\pi_p\pi_q=0$, $p\neq q$, such that, for u in some neighbourhood of $u=0$,

$$A(u)=U_1(u)\left[\sum_{p\in\mathbf{Z}}\pi_p u^p\right]U_2(u). \tag{A.1}$$

The projection π_∞ is zero if and only if the germ of meromorphic function $\det A$ is not zero and in this case the valuation of $\det A$ at $u=0$ is given by

$$v_0(\det A)=w_0^+(A)-w_0^-(A) \tag{A.2}$$

with the weights $w_0^\pm(A)$ defined by

$$w_0^\pm(A)=\sum_{\pm p>0}|p|\operatorname{rank}\pi_p.$$

Using the Taylor expansions of U_1 and U_2, we have, for $p<0$

$$A_p=U_{10}\pi_pU_{20}+\sum_{\substack{i+q+j=p\\ i,j\geqslant 0,\ p<q<0}}U_{1i}\pi_qU_{2j}$$

hence

$$\operatorname{rank}\pi_p\leqslant\operatorname{rank}A_p+\sum_{\substack{i+q+j=p\\ i,j\geqslant 0,\ p<q<0}}\operatorname{rank}\pi_q$$

and there exists a constant C_k depending only on the integer k such that $w_0^-(A)\leqslant C_{|v_0(A)|}\operatorname{Rank}_{\mathrm{p}0}(A)$.

The preceding discussion applies to compact perturbations of the identity in any Hilbert space. Let T be a germ of meromorphic function valued in the space of compact operators on a Hilbert space $\mathcal{H}$ and with polar parts of finite rank. There exists a projection π of finite rank, a holomorphic germ H and a meromorphic germ M such that

$$1+T=1+\pi M+H$$

with $1 + H(0)$ invertible. Then there exist π', H', M' with the same properties as π, H, M and such that

$$\begin{aligned} 1+T &= \big(1+\pi M(1+H)^{-1}\big)(1+H) \\ &= \big(1+\pi' M'\pi' + \pi' H'(1-\pi')\big)(1+H) \\ &= \big(1+\pi' H'(1-\pi')\big)(1+\pi' M'\pi')(1+H). \end{aligned}$$

Hence, by applying the preceding finite dimensional result to the central term in the last product, we can assert the existence of germs U_1, U_2 in $\mathcal{O}_1(\mathrm{GL}(\mathcal{H}))$, a resolution $(\pi_p)_{p\in \mathbf{Z}\cup\{\infty\}}$ with a finite number of non zero π_p and all of finite rank except possibly π_0 such that (A.1) is valid for $A = 1 + T$. If T is trace class, the U_1, U_2 are trace class perturbations of the identity and $\det(A)$ has valuation at $u = 0$ given by (A.2) with $w_0^{\pm}(A)$ defined as in the finite dimensional case.

Lemma A.2. *Let Ω be a complex connected neighbourhood of* 0, *B_1, B_2 and T meromorphic functions on Ω with values in the space of bounded operators on a Hilbert space $\mathcal{H}$. If $1 + T(u_0)$ is invertible for one u_0 in Ω and the function T is trace class operator valued, then we have for the rank of the residue of $B_1(1+T)^{-1}B_2$ at* 0

$$\operatorname{rank}\operatorname{res}_0 B_1(1+T)^{-1}B_2 \leqslant v_0\big(\det(1+T)\big) + C_{|v_0(A)|}\mathrm{Rank}_{\mathrm{p}0}(1+T) + \sum_{i=1}^{2}\big(v_0(B_i)\big)_{-}\mathrm{Rank}_{\mathrm{p}0}(B_i).$$

Proof. Let us introduce the resolution $(\pi_p)_{p\in\mathbf{Z}\cup\{\infty\}}$, the germs of holomorphic functions U_i, $i = 1, 2$, with $U_i(0)$ invertible such that $A = 1 + T$ is reduced to the normal form (A.1). Because $\det(1 + T(u_0))$ is non-zero, the projection π_∞ is zero and $(1 + T)^{-1}$ is invertible as a germ of meromorphic function with polar parts of finite rank. Let us write $\widetilde{B}_1 = B_1 U_2^{-1}$ and $\widetilde{B}_2 = U_1^{-1} B_2$ with Laurent expansions

$$\widetilde{B}_i(u) = \sum_{p\geqslant v_0(B_i)} \widetilde{B}_{ip} u^p.$$

Then

$$\begin{aligned} \operatorname{res}_0 B_1(1+T)^{-1}B_2 &= \sum_{p_1-p+p_2=-1} \widetilde{B}_{1p_1}\pi_p\widetilde{B}_{2p_2} \\ &= \sum_{v_0(B_1)\leqslant p_1<0} \widetilde{B}_{1p_1}\Big(\sum_p \pi_p \widetilde{B}_{2,p-p_1-1}\Big) \\ &\quad + \sum_{p_1\geqslant 0,\ p_2\geqslant 0} \widetilde{B}_{1p_1}\pi_{1+p_1+p_2}\widetilde{B}_{2p_2} \\ &\quad + \sum_{v_0(B_2)\leqslant p_2<0} \Big(\sum_p \widetilde{B}_{1,p-p_2-1}\pi_p\Big)\widetilde{B}_{2p_2}, \end{aligned}$$

hence

$$\operatorname{rank}\operatorname{res}_0 B_1(1+T)^{-1}B_2 \leqslant \sum_{v_0(B_1)\leqslant p_1<0} \operatorname{rank} \widetilde{B}_{1p_1} + \sum_{p>0} p \operatorname{rank} \pi_p + \sum_{v_0(B_2)\leqslant p_2<0} \operatorname{rank} \widetilde{B}_{2p_2}.$$

The middle term in the right side is exactly $v_0(\det(1+T)) + w_0^-(1+T)$ while the extreme terms are $\mathrm{Rank}_{\mathrm{p}0}\widetilde{B}_1$ and $\mathrm{Rank}_{\mathrm{p}0}\widetilde{B}_2$. The lemma follows from the upper bounds

$$w_0^-(1+T) \leqslant C_{|v_0(\det(1+T))|}\mathrm{Rank}_{\mathrm{p}0}(1+T), \quad \mathrm{Rank}_{\mathrm{p}0}\widetilde{B}_i \leqslant \big(v_0(B_i)\big)_-\mathrm{Rank}_{\mathrm{p}0}B_i.$$

□

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