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Scattering asymptotics for Riemann surfaces

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1. Introduction and statement of the results

In this article we prove the optimal polynomial lower bound for the number of resonances of a surface with hyperbolic ends. We also give Weyl asymptotics for the relative scattering phase of such a surface. The proofs are based on trace formulæ analogous to those of the Euclidean odd-dimensional scattering. The main technical ingredient is a new proof of the Poisson formula (Theorem 5.7) which is applicable in the Euclidean case as well.

Our lower bound seems to be the first example of an optimal polynomial lower bound for the number of resonances holding for a general class of higher dimensional elliptic operators with no symmetries. The previous general lower bounds or asymptotics were either nonoptimal ([25], [58], [9]), one-dimensional or radial ([65], [67] and [54], [41][1]) or they required some degeneracy of the operator ([59], [63]).

The surfaces we consider generalize arbitrary noncompact finite geometry quotients $\Gamma\backslash\mathbf{H}^2$, where Γ is a discrete isometry group, in the same way finite volume quotients were generalized in [7], [16], [41]. In the case of finite volume and constant curvature (that is, in the case of $\Gamma\backslash\mathbf{H}^2$, Γ a cofinite discrete group), the corresponding results are now classical and more precise — they were established by Selberg in [54] and more recently they were generalized by W. Müller [41] and Parnovski [42].

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[1]From the point of view of scattering theory-finite volume quotients have one-dimensional infinity; see [57, Ex. 1.1] and for a more detailed discussion [2].

For the infinite volume quotients, the study of scattering and spectral properties of the Laplacian was initiated by Patterson in [44]. He defined the scattering matrix, proved its meromorphic continuation and obtained some trace formulæ motivated by the Selberg trace formula. In [45] he introduced the dynamical zeta function which in the constant curvature case is closely related to the scattering matrix (see [17], [45], [48] and for a more detailed and general presentation, [46]). The scattering and spectral theory on infinite volume quotients and their generalizations was then studied in [26], [1], [8], [15], [17], [47], [11], [12], [29], [30].

Through the analogy with the Euclidean and finite volume cases the *resonances* were defined as the poles of the meromorphic continuation of Eisenstein series, the resolvent of the Laplacian or the scattering matrix (see Definition 1.2 below). As in the other situations they constitute a replacement of the discrete spectral data for problems on noncompact domains (see [69] for discussion and references). The analogue of the counting function for eigenvalues is provided by an appropriately defined *scattering phase* (see Definition 1.4 below). Heuristically it measures the averaged phase shift of a wave passing through a compact region containing the perturbation (for $\Gamma\backslash\mathbf{H}^2$, the Fenchel-Nielsen region). The resonances close to the continuous spectrum correspond to the peaks of its derivative which roughly explains the physical origin of the notion of *resonance*. In [18] we proved the optimal upper bound for the number of resonances which we then generalized in a weaker form to a class of higher dimensional asymptotically hyperbolic manifolds in [19]. The methods used here (except for the general aspects of the proof of the Poisson formula) are close to [18] and very specific to two dimensions; we require a very detailed knowledge of the neighbourhood of infinity.

Let (X, g) (see Fig. 1) be a complete two-dimensional (with a compact boundary) connected manifold with a decomposition

$$X = Z \sqcup X_1 \sqcup \cdots \sqcup X_M \sqcup Y_1 \sqcup \cdots \sqcup Y_N\,, \quad N \neq 0\,, \tag{1.1}$$

where Z is a compact manifold with boundary, $\partial Z = \partial X \sqcup \partial X_1 \sqcup \cdots \sqcup \partial X_M \sqcup \partial Y_1 \sqcup \cdots \sqcup \partial Y_N$ and each X_i is isometric to

$$X_i \simeq [a_i, \infty)_r \times (\mathbf{R}/h_i\mathbf{Z})_t\,, \quad g_{|X_i} \simeq dr^2 + e^{-2r}dt^2\,,\ a_i > 0, h_i > 0\,, \tag{1.2}$$

and each Y_j to

$$Y_j \simeq [b_j, \infty)_r \times (\mathbf{R}/\ell_j\mathbf{Z})_t\,, \quad g_{|Y_j} \simeq dr^2 + \cosh^2 r\, dt^2\,,\ b_j > 0, \ell_j > 0\,. \tag{1.3}$$

We will also define the hyperbolic half-cylinders,

$$Y_j^0 = [0, \infty)_r \times (\mathbf{R}/\ell_j\mathbf{Z})_t\,, \quad g_0 = dr^2 + \cosh^2 r\, dt^2\,, \tag{1.4}$$

and consider Y_j as a subset of Y_j^0 with the metric g on Y_j coinciding with the metric g_0 on Y_j^0.

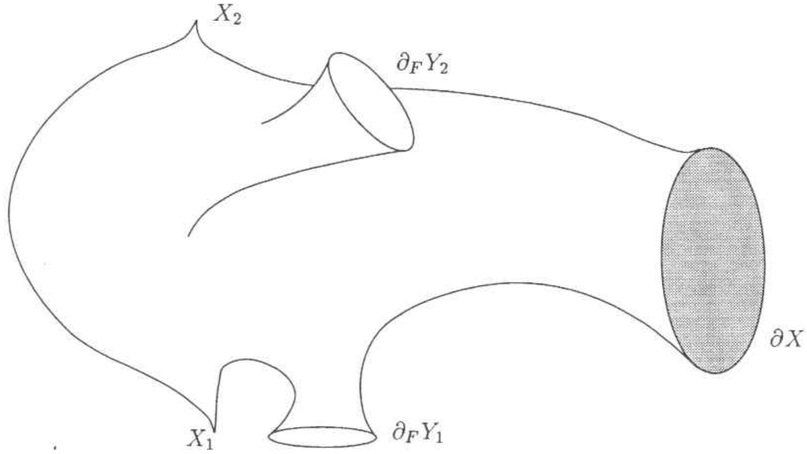

FIGURE 1. A Riemannian surface X

The assumption $N \neq 0$ is made only for expository convenience, in particular to guarantee the absence of embedded eigenvalues [44]. All the methods are applicable in the easier case $N = 0$, but the corresponding results are already included, explicitly or implicitly among those of [54], [41], [44], [49].

The Laplace-Beltrami operator on $L^2(X, d\,\mathrm{vol}_g)$ with some self-adjoint boundary condition on ∂X (Dirichlet, Robin) will be denoted by Δ_g. From [18] we recall the following essentially well-known result:

THEOREM 1.1. *The resolvent*

$$(\Delta_g - s(1-s))^{-1} : L^2(X, d\,\mathrm{vol}_g) \to H^2(X, d\,\mathrm{vol}_g)$$

defined for $\mathrm{Re}\, s > 1/2, s(1-s) \notin \mathrm{spec}_{\mathrm{pp}}(\Delta_g)$ *extends to a meromorphic family of operators*

$$R_X(s): L^2_{\mathrm{comp}}(X, d\,\mathrm{vol}_g) \to H^2_{\mathrm{loc}}(X, d\,\mathrm{vol}_g) \tag{1.5}$$

with poles of finite rank.

The *resonances* are defined as the poles of $R_X(s)$, but the definition of their multiplicities is more problematic. Ideally, the multiplicity should be defined as a trace of a finite rank projection and that definition, through the complex scaling method or the Lax-Phillips theory, is available in the odd dimensional Euclidean case (or for resonances on the first logarithmic sheet in even dimension—see [56], [59]). By a straightforward analogy, in [18] and [19], we put as multiplicity

$$m_{s_0}(R_X) = \mathrm{rank} \int_{\gamma_{s_0,\varepsilon}} R_X(s)(1-2s)ds\,, \quad s_0 \neq \frac{1}{2}\,, \tag{1.6}$$

where $\gamma_{z,\varepsilon}$ will denote the circle $t \in [0, 2\pi] \mapsto z + \varepsilon e^{it}$ around z with radius ε taken sufficiently small to avoid other singularities of the integrand.

Here we will adopt a slightly different point of view and will follow [38] to define the multiplicity.

Definition 1.2. A *complex number* s_0 is called a *resonance* if it is a *pole* of R_X. Its *multiplicity* is given by the *rank of the full polar part of* R_X *at* s_0:

$$
\begin{aligned}
(1.7) \qquad m_{s_0}(R_X) &= \dim \sum_{j=1}^{k} A_j(L^2_{\text{comp}}(X, d\,\text{vol}_g))\,, \\
R_X(s) &= \sum_{j=1}^{k} \frac{A_j}{(s-s_0)^j} + A_0(s)\,,
\end{aligned}
$$

where $A_0(s)$ is holomorphic near s_0.

As we will see this is the same as (1.6) when $\operatorname{Re} s_0 \neq \frac{1}{2}$. For the correctness of the Poisson formula, however we need (1.7). We will denote by $\mathcal{R}_X$ the set of resonances of X included according to the multiplicities given by $m_s(R_X)$. Thus $\mathcal{R}_{Y_j^0}$ is equal to the explicitly computable set of resonances of the hyperbolic half-cylinder (1.4) with the Dirichlet boundary condition (see [18] and §5 below).

A natural definition can also be given by considering the meromorphic continuation of the scattering matrix of X, $S_X(s)$ given by Proposition 2.5 below. Following [14] we define the divisor of S_X, $v_s(S_X)$, by a complicated but natural formula (2.45) which coincides with the definition given in [46]. We show in Section 2 that for $\operatorname{Re} s < 1$ the scattering matrix, S_X can only have a pole when R_X has one and that their multiplicities coincide in the following sense: For $\operatorname{Re} s < 1$,

$$
v_s(S_X) = m_s(R_X) - m_{1-s}(R_X).
$$

The second term on the right-hand side can be nonzero only for $s(1-s) \in \sigma_{\text{pp}}(\Delta_X)$, that is, at finitely many points in $(0,1)$.

We remark also that when $N=0$ *or* when $M=0$ (that is, we only have cusps *or* funnels), the divisor of S_X at its poles in $\operatorname{Re} s < 1$ is given by a simple formula (which for $N=0$ holds everywhere):

$$
v_{s_0}(S_X) = -\frac{1}{2\pi \mathrm{i}} \operatorname{tr} \int_{\gamma_{s_0,\varepsilon}} S_X(s)^{-1} \frac{d}{ds} S_X(s) ds
$$

with ε sufficiently small.

To conclude the general discussion of multiplicities we should mention that the proof in Section 2 is applicable to any conformally compact manifold where the scattering matrix is defined by Proposition 8.2 in [38]. The self-contained discussion of two dimensional surfaces has then to be replaced by the full power of the results of Mazzeo and Melrose [30].

To state our main results we need to introduce the 0-volume of X which comes from a Hadamard type regularization fully explained in Section 5:

$$(1.8) \qquad \text{0-vol}(X) = \text{vol}_g(Z) + \sum_{i=1}^{M} \text{vol}_g(X_i) - \sum_{j=1}^{N} \text{vol}_{g_0}(Y_j^0 \setminus Y_j)\,,$$

where Y_j^0 were given by (1.4). We note that for $X = \Gamma\backslash\mathbf{H}^2$, we have $\text{0-vol}(X) = \text{vol}(N)$ where N is the Fenchel-Nielsen region of Γ, that is, the projection of the convex hull of the limit set of Γ (the set of accumulation points of orbits of hyperbolic elements of Γ); see (5.5) for an explicit expression.

THEOREM 1.3. *If* $\text{0-vol}(X) \neq 0$, *then the counting function of resonances defined as*

$$N_X(r) = \sum_{|s|\leq r} m_s(R_X)$$

satisfies, for some $C > 0$,

$$r^2/C \leq N_X(r) \leq Cr^2\,, \quad r > C\,.$$

The proof is based on a simple Tauberian argument from [59] applied to the Poisson formula (see Theorem 5.7 below) which is the analogue of the Poisson formula of Euclidean scattering [31], [32], [38], [59], [52]:

$$(1.9) \qquad \text{0-tr}\cos t\sqrt{\Delta_g - \frac{1}{4}} = \frac{1}{2}\sum_{s\in\mathbf{C}} \mathrm{e}^{(s-\frac{1}{2})|t|} m_s(R_X)\,, \quad t \neq 0\,,$$

with both sides considered as distribution on $\mathbf{R}\setminus\{0\}$. The *zero-trace*, 0-tr, defined in Section 6, is a regularized trace adapted to the $\mathcal{V}_0$-calculus of [30] in the same way the b-trace is adapted to the $\mathcal{V}_{\mathrm{b}}$-calculus of [37]. We should remark that the assumption that $\text{0-vol}(X) \neq 0$ is a consequence of the method; we do not expect it to play any rôle for the counting function of resonances. The situations for which $\text{0-vol}(X) = 0$ are, however, very special and except for the case of a hyperbolic cylinder, somewhat artificial.

Once we have (1.9), or rather its building components given in Proposition 5.5, we can apply a modification of Melrose's method [35] to obtain the asymptotics of the relative scattering phase. Following [17] we give:

Definition 1.4. The *relative scattering phase*, $\sigma_X(s)$, is defined as

$$(1.10) \qquad \sigma_X(s) = \frac{\mathrm{i}}{2\pi}\log\det\mathcal{S}_X(s)$$

where $\mathcal{S}_X(s)$ is the *relative scattering matrix* given by Definition 2.12:

$$\mathcal{S}_X(s) = S_{M,Y^0}(s)^{-1}S_X(s)$$

where $S_{M,Y^0}(s)$ is the direct sum of the identity operator on $\mathbf{C}$ for each cusp X_i and the scattering matrices for *each* infinite volume end $Y_j^0 \simeq [0,\infty)\times(\mathbf{R}/\ell_j\mathbf{Z})$ with the metric given by (1.3).

The introduction of the relative scattering matrix is necessary since the full scattering matrix is very far from the form $\mathrm{Id}+A$ with A a trace class operator (see §2 for a detailed description of its structure). Heuristically one can say that the phase shift has to be measured relative to some background and that is true in the Euclidean case where we also take the relative scattering matrix as opposed to the absolute one; see [39].

The methods of Sunada, Buser and Bérard (see [4]) apply to the setting of surfaces with infinite volume ends and consequently we can construct pairs of surfaces with conjugate scattering matrices (absolute and relative) and the same scattering phases and resonances; see Remark 2.15 in Section 2.

The next theorem is an exact analogue of the results in the Euclidean, finite volume or cylindrical end settings; see [35], [50], [41], [6], [43] for recent results and references.

THEOREM 1.5. *If the relative scattering phase* $\sigma_X(s)$ *is given by* (1.10) *and the* 0*-volume by* (1.8), *then*

$$\sigma_X(s) = \frac{1}{4\pi}\,0\text{-vol}(X)|s|^2 - \frac{M}{\pi}|s|\log|s| + \mathcal{O}(|s|). \tag{1.11}$$

We point out the analogy with [6], where scattering asymptotics for manifolds with product type b-metrics (that is, with cylindrical ends) are given in terms of the b-volume.

Remark 1.6. It is perhaps unlikely that $N_X(r)$ enjoys a meaningful asymptotic behaviour as $r\to\infty$, that is, a Weyl law with a geometric constant. In the known examples ([65], [63], [41]) the pole dominating the asymptotics were concentrated near the continuous spectrum so their behaviour could be deduced from the microlocal properties of the trace (the left-hand side of (1.9)). The simple Example 6.1 below shows that lacking some concentration near the continuous spectrum, the singularities of the trace cannot give asymptotics for $N_X(r)$. In fact, for exact convex co-compact quotients $\Gamma\setminus\mathbf{H}^2$ (that is, smooth noncompact quotients where Γ has no parabolic elements) the resonances are sparse near the real axis. Following the methods of [55], it is shown in [70] that

$$\sum_{|s|\le r,\,\mathrm{Re}(\frac12-s)<C_1^{-1}|\operatorname{Im}s|^\alpha+C_2} m_s(R_X)\le C_3 r^{1+\delta+\alpha(1-\delta)}$$

where C_1, C_2 and C_3 are large constants, $0\le\alpha\le1$, and $\delta<1$ is the Hausdorff dimension of the limit set of Γ, that is the set of limit points of orbits of the elements of Γ (which are assumed to be hyperbolic). Thus, in spite of high density in discs given by Theorem 2 ($\sim r^2$), the density in any sub-conic neighbourhood of the continuous spectrum is less than $r^{2-\varepsilon}$, $\varepsilon > 0$.

Remark 1.7. As we mentioned above, in the case of constant negative curvature, the scattering matrix is related to the dynamical zeta function which was introduced by Selberg [53] (and studied in higher dimensions by, among others, Ruelle [51], Patterson [45], and Fried [10]). For any finite geometry quotient $\Gamma\backslash\mathbf{H}^2$, the meromorphic continuation of the zeta function, Z_X, was proved in [17] where it was applied to the study of the distribution of lengths of closed geodesics (see also [15]). The recent but long expected results of [46] show that the non-topological part of the divisor of the zeta function (see Theorem 1.2 there) coincides with the divisor of the scattering matrix:

$$v_s(Z_X) = -v_s(S_X)\,, \mathrm{Re}\, s < \frac{1}{2}\,.$$

In other words, the nontrivial zeros of Z_X in $\mathrm{Re}\, s < \frac{1}{2}$ coincide (with multiplicities) with resonances. Thus, Theorem 1.3 gives an optimal polynomial lower bound on the number of zeros of the meromorphic continuation of Z_X. For Γ without parabolic elements but in all even dimensions the identification of the divisors has also been proved in [46].

A much weaker lower bound near the continuous spectrum follows from [58]:

$$\sum_{|\,\mathrm{Im}\, s| \leq r, \mathrm{Re}(\frac{1}{2}-s)<\rho \log\langle s\rangle} \mathrm{e}^{-(d-\varepsilon)(\frac{1}{2}-\mathrm{Re}\, s)} > \left(\frac{1}{2\pi \sinh(d/2)} \sum_{m\ell(\gamma)=d} \ell - o_\varepsilon(1) \right) r\,, \tag{1.12}$$

where $d > 0$ is arbitrary, $\varepsilon \ll 1$, $\rho > \frac{2}{d-\varepsilon^2}$, and $\gamma \in \Gamma$ has a displacement length $\ell(\gamma)$. Hence in logarithmic neighbourhoods of the continuous spectrum, the density of resonances is at least $\sim r$.

The paper is organized as follows. In Section 2 we review, in a slightly modified way, the scattering theory of surfaces with hyperbolic ends. In particular we show that the poles of the scattering matrix, of the resolvent and of the generalized eigenfunctions (Eisenstein series) coincide. In Section 3 we use the characteristic-value estimates similar to those in [33], [66], [62], and [18] (see [69] for more references) to bound the determinant of the relative scattering matrix. Section 4 is devoted to the proof of a Birman-Krein-type trace formula relating the trace of the wave group to the relative scattering phase. The observation that the finite order of the determinant is equivalent to the relative Poisson formula and a detailed analysis of the model problem in Section 5 give us (1.9) (Theorem 5.7) after the introduction of the 0-*trace*.

Finally, in Section 6, the methods of [59] and [35] give Theorems 1.3 and 1.5 respectively.

For notational simplicity we will use C to denote a large, but not necessarily the same, constant. For $s \in \mathbf{C}$ we will denote by $\langle s \rangle$ the shifted modulus $1 + |s|$ and by $\mathbf{N}$ the set of natural numbers $1, 2, \ldots$, with the notation $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

2. Review of scattering theory

We will first modify the geometric description of X and then the Hilbert space on which the Laplacian acts. Each model funnel Y_j^0 given by (1.4) can be compactified at infinity using the natural fundamental domain. We then have two boundary components: ∂Y_j^0 and $\partial_F Y_j^0$, where $\partial_F Y_j^0$ is the boundary at infinity. The same compactification can then be applied to Y_j's, giving the boundaries at infinity, $\partial_F Y_j$, and then to X which gives

$$\partial_F X = \partial_F Y_1 \sqcup \cdots \sqcup \partial_F Y_N \,.$$

We note that, after this partial compactification, X remains noncompact if $M \neq 0$. From now on we shall think of X as a manifold with cusps and boundary $\partial X \sqcup \partial_F X$ and will denote by $\overset{o}{X}$ its interior. In particular $\mathcal{C}^\infty(X)$ denotes functions which are smooth up to $\partial_F X$. Let $x \in \mathcal{C}^\infty(X)$ be a defining function of $\partial_F X$: $x_{|\partial_F X} = 0$, $dx_{|\partial_F X} \neq 0$, $x > 0$ on $\overset{o}{X}$. We demand also that x be bounded from below away from a neighbourhood of $\partial_F X$ (which is a nontrivial requirement if $M \neq 0$). If $\Omega_X^{\frac{1}{2}}$ denotes the line bundle of half densities on X as a manifold with boundary $\partial X \sqcup \partial_F X$, then we define the line bundle of 0-half densities, ${}^0\Omega_X^{\frac{1}{2}}$, by

$$\mathcal{C}^\infty(X; {}^0\Omega_X^{\frac{1}{2}}) = x^{-1} \mathcal{C}^\infty(X; \Omega_X^{\frac{1}{2}}),$$

where the notation is borrowed from [30]; see [36] for an outline of the general philosophy of such bundles.

We note that $\mathcal{C}^\infty(X; {}^0\Omega_X^{\frac{1}{2}})$ can be identified with $\mathcal{C}^\infty(X)$ using the metric and in any local coordinates up to $\partial_F X$ this identification is given by multiplication by a $\mathcal{C}^\infty(X)$ function. If y is a coordinate on a component of $\partial_F X$ then, near that component, an element of $\mathcal{C}^\infty(X; {}^0\Omega_X^{\frac{1}{2}})$ can be written as

$$u \left| \frac{dx}{x} \right|^{\frac{1}{2}} \left| \frac{dy}{x} \right|^{\frac{1}{2}}, \quad u \in \mathcal{C}^\infty(X) \,.$$

The Laplacian, Δ_X, acts on the 0-half densities through the above identification with functions:

$$\Delta_X (u |d \operatorname{vol}_g|^{\frac{1}{2}}) = \Delta_g u \, |d \operatorname{vol}_g|^{\frac{1}{2}} .$$

The resolvent $R_X(s)$ will also act on 0-half densities and Theorem 1.1 recalled from Section 1 can be restated as:

$$R_X(s)\colon\ \mathcal{C}^\infty_{\mathrm{c}}(\overset{o}{X};\ {}^0\Omega^{\frac12}_X) \longrightarrow \mathcal{C}^\infty(\overset{o}{X};\ {}^0\Omega^{\frac12}_X) \tag{2.1}$$

is meromorphic in s with poles of finite rank.

The space on which $R_X(s)$ acts can be defined in several ways and we will follow a modification of the approach of [18] which was based on [7] and [56]. Thus we define

$$\mathcal{H}^a = \mathcal{H}^{a,b}_{\mathrm{int}} \oplus \bigoplus_{i=1}^{M} L^2(X^a_{0i}, \Omega^{\frac12}_{\mathbf{R}_+}) \oplus \bigoplus_{j=1}^{N} L^2(Y^b_j,\ {}^0\Omega^{\frac12}_{Y_j})\,,$$

where

$$\begin{aligned}\mathcal{H}^{a,b}_{\mathrm{int}} = L^2(Z, \Omega^{\frac12}_Z) \quad &\oplus \quad \bigoplus_{i=1}^{M} L^2(X_i \setminus X^a_i, \Omega^{\frac12}_{X_i})\\ &\oplus \quad \bigoplus_{i=1}^{M} {}_0L^2(X^a_i, \Omega^{\frac12}_{X_i})\\ &\oplus \quad \bigoplus_{j=1}^{N} L^2(Y_j \setminus Y^b_j, \Omega^{\frac12}_{Y_j})\end{aligned}$$

with

$$\begin{aligned} X^a_i &\simeq [a,\infty)_r \times (\mathbf{R}/h_i\mathbf{Z})_t\,, \quad X^a_{0i} \simeq [a,\infty)_r\,, \quad a > a_i\,,\\ Y^b_j &\simeq [b,\infty)_r \times (\mathbf{R}/\ell_j\mathbf{Z})_t\,, \quad b > b_j\,.\end{aligned}$$

The space ${}_0L^2(X^a_i, \Omega^{\frac12}_{X_i})$ is the $L^2(X^a_i, \Omega^{\frac12}_{X_i})$ closure of

$${}_0\mathcal{C}^\infty_{\mathrm{c}}(X^a_i, \Omega^{\frac12}_{X_i}) = \{f \in \mathcal{C}^\infty_{\mathrm{c}}(X_i, \Omega^{\frac12}_{X_i}) : \widehat{f}^i_0(r) = 0, r > a\} \tag{2.2}$$

where

$$\widehat{f}^i_0(r) = h_i^{-1}\mathrm{e}^{-r/2}\int_0^{h_i} f(r,t)dt\,. \tag{2.3}$$

We have three orthogonal projections

$$\begin{aligned}\pi_{\mathrm{int}}\colon\ &\mathcal{H}^a \longrightarrow \mathcal{H}^{a,b}_{\mathrm{int}}\,,\\ \pi_a\colon\ &\mathcal{H}^a \longrightarrow \textstyle\bigoplus_{i=1}^M L^2(X^a_{0i}, \Omega^{\frac12}_{\mathbf{R}_+})\,,\\ \pi_b\colon\ &\mathcal{H}^a \longrightarrow \textstyle\bigoplus_{j=1}^N L^2(Y^b_j,\ {}^0\Omega^{\frac12}_{Y_j})\,,\end{aligned}$$

which extend naturally to $\mathcal{H}^a_{\mathrm{loc}}$ once L^2 is replaced by L^2_{loc} in the target spaces. To define the projection

$$\pi_a\colon\ \mathcal{C}^\infty(X,\ {}^0\Omega^{\frac12}_X) \cap L^2(X,\ {}^0\Omega^{\frac12}_X) \longrightarrow \bigoplus_{i=1}^{M} L^2(X^a_{0i}, \Omega^{\frac12}_{\mathbf{R}_+})\,,$$

we use the coordinates on X_i which are implicit in its definition (1.1):

$$e^{-r/2} f(r,t) |dt|^{\frac{1}{2}} |dr|^{\frac{1}{2}} \longmapsto (\widehat{f}_0^i(r))_{i=1}^M |dr|^{\frac{1}{2}} ,$$

where $\widehat{f}_0^i$ is given by (2.3). This shows that we have a natural identification which gives

$$\mathcal{C}^\infty(X, {}^0\Omega_X^{\frac{1}{2}}) \cap L^2(X, {}^0\Omega_X^{\frac{1}{2}}) \longrightarrow \mathcal{H}^a . \tag{2.4}$$

We then define

$$\mathcal{H}^a_{\mathrm{comp},N} = x^N \mathcal{H}^a_{\mathrm{comp},0} , \quad \mathcal{H}^a_{\mathrm{comp},0} = \{ u \in \mathcal{H}^a : \operatorname{supp} \pi_a(u) \subset\subset \sqcup_{i=1}^M X_i^a \} ,$$

and we note that if $M = 0$ then

$$\mathcal{H} = L^2(X, {}^0\Omega_X^{\frac{1}{2}}) , \quad \mathcal{H}_N = x^N L^2(X, {}^0\Omega_X^{\frac{1}{2}}) .$$

As we will see below (and as is implicit in Section 5 of [18]) the resolvent is a meromorphic operator

$$\mathcal{H}^a_{\mathrm{comp},N} \longrightarrow \mathcal{H}^a_{\mathrm{loc},-N} , \quad N > -\operatorname{Re} s , \tag{2.5}$$

where the space $\mathcal{H}^a_{\mathrm{loc},-N}$ is obtained by allowing the image of π_a to lie in L^2_{loc}. This could be refined further by using the complex scaling method in $L^2(X^a_{0i}, \Omega^{\frac{1}{2}}_{\mathbf{R}_+})$ (see Example 1.3 of [57]) but we shall not need this here.

We can instead introduce the space

$$\begin{aligned} \mathcal{H}^a_N \quad = \quad & \{ u \in \mathcal{H}^a_{\mathrm{loc}} : \pi_b u \in x^N \pi_b(\mathcal{H}^a), (\pi_a u)_i \in \mathrm{e}^{(-N+\frac{1}{2})r} L^2(X_0^a, \Omega^{\frac{1}{2}}_{\mathbf{R}_+}) , \\ & 1 \leq i \leq M \} . \end{aligned}$$

As in scattering by compactly supported perturbations on $\mathbf{R}_+$ [65], we can modify the space on which the resolvent acts to

$$R_X(s) \colon \mathcal{H}^a_N \to \mathcal{H}^a_{-N} , \quad N > -\operatorname{Re} s . \tag{2.6}$$

As in [18, §2], we have two model problems: the shifted Dirichlet Laplacian on the Euclidean half line $X_0^0 \simeq \mathbf{R}^+$:

$$\Delta_0^0 = D_r^2 + \frac{1}{4} , \tag{2.7}$$

and the hyperbolic Dirichlet Laplacian on the half cylinder $Y_j^0 \simeq \mathbf{R}^+ \times \mathbf{R}/\ell_j \mathbf{Z}$:

$$\Delta^0_{Y_j} = D_r^2 - i \tanh r D_r + \cosh^{-2} r D_t^2 . \tag{2.8}$$

They coincide with Δ_X on the ends in the following sense: if $f \in \mathcal{C}^\infty(X, {}^0\Omega_X^{\frac{1}{2}}) \cap L^2(X, {}^0\Omega_X^{\frac{1}{2}}) \cap \mathcal{H}^a$ then

$$\pi_a(\Delta_X f) = (\Delta_0^0 \widehat{f}_0^i(r) |dr|^{\frac{1}{2}})_{i=1}^M , \quad \pi_a(f) = (\widehat{f}_0^i(r) |dr|^{\frac{1}{2}})_{i=1}^M ,$$

where $\widehat{f}_0^i$ is as defined in (2.3), and

$$\pi_b(\Delta_X f) = ((\Delta^0_{Y_j} f_j)|d\,\mathrm{vol}_{Y_j^0}|^{\frac{1}{2}})_{j=1}^N\,, \quad \pi_b(f) = (f_j|d\,\mathrm{vol}_{Y_j^0}|^{\frac{1}{2}})_{j=1}^N\,.$$

For the meromorphic continuation we use a smooth decomposition of Δ_X which we recall from [18]. For that we need generalized cut-off functions: Let χ be a smooth function on $\mathbf{R}$ with support in $(-\infty, 2/3]$, $\chi(r) = 1$ for $r \leq 1/3$ and χ_A the translate $\chi(\cdot - A)$. Using the identification (1.1) we define the following linear cut-off operators acting on $f \in \mathcal{H}$:

$$\chi^{X_i}_{a,b} f = \begin{cases} \chi_a f & \text{if } f \in L^2(X^a_{0i}, dr)\,, \\ f & \text{if } f \in \mathcal{H}^{a,b}_{\mathrm{int}} \oplus \bigoplus_{k\neq i} L^2(X^a_{0k}, dr) \oplus \bigoplus_{j=1}^N L^2(Y^b_j, d\mathrm{vol}_g)\,, \end{cases}$$

$$\chi^{Y_j}_{a,b} f = \begin{cases} \chi_b f & \text{if } f \in L^2(Y^b_j, d\mathrm{vol}_g)\,, \\ f & \text{if } f \in \mathcal{H}^{a,b}_{\mathrm{int}} \oplus \bigoplus_{i=1}^M L^2(X^a_{0i}, dr) \oplus \bigoplus_{k\neq j} L^2(Y^b_k, d\mathrm{vol}_g)\,, \end{cases}$$

and

$$\chi^Z_{a,b} = \begin{cases} f & \text{if } f \in \mathcal{H}^{a,b}_{\mathrm{int}}\,, \\ \chi_a f & \text{if } f \in \bigoplus_{i=1}^M L^2(X^a_{0i}, dr)\,, \\ \chi_b f & \text{if } f \in \bigoplus_{j=1}^N L^2(Y^b_j, d\mathrm{vol}_g)\,. \end{cases}$$

We then immediately have

$$\begin{aligned} \Delta = \chi^Z_{a+1,b+1}\Delta_X\chi^Z_{a,b} &+ \sum_{i=1}^M (1-\chi^{X_i}_{a-1,b-1})\Delta^0_0(1-\chi^{X_i}_{a,b}) \\ &+ \sum_{j=1}^N (1-\chi^{Y_j}_{a-1,b-1})\Delta^0_{Y_j}(1-\chi^{Y_j}_{a,b})\,. \end{aligned} \tag{2.9}$$

The abstract space $\mathcal{H}^a$ was introduced because of the presence of cusps but the discussion needs to be more geometric in order to understand the structure of the resolvent, $R_X(s)$, near infinity, $\partial_F X$. To make the exposition clearer we will first discuss the case $M = 0$ (no cusps) and then introduce the necessary modifications. Then the Schwartz kernel of $R_X(s)$ has a nice description in the geometric framework of Mazzeo and Melrose.

Thus we recall from Section 3 of [30] the blown-up product space obtained by resolving the intersection of the diagonal and $\partial_F X \times \partial_F X$ in $X \times X$ (see Fig. 2):

$$\begin{aligned} X \times_0 X &= [X \times X, \partial_F \Delta] \xrightarrow{\beta} X \times X\,, \\ \Delta &= \{(z,z) : z \in X\} \subset X \times X\,, \\ \partial_F \Delta &= \Delta \cap (\partial_F X \times \partial_F X)\,. \end{aligned} \tag{2.10}$$

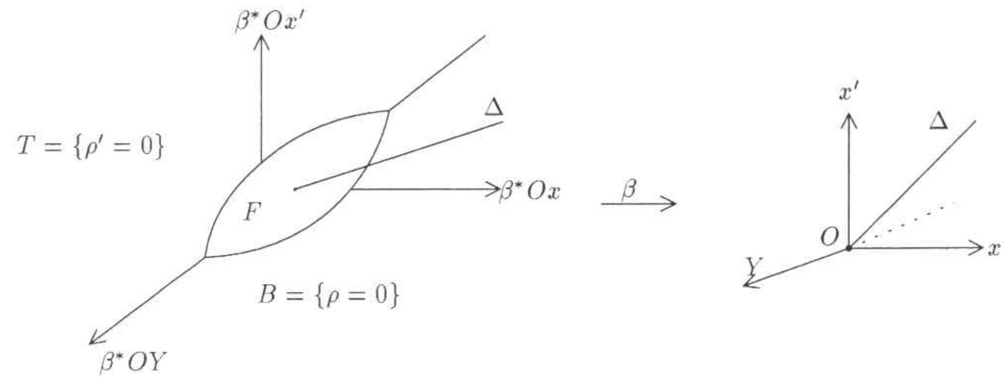

FIGURE 2. The boundary faces of the blown-up space at the funnels

Writing as usual $\beta^* Z = \mathrm{cl}\beta^{-1}(Z \setminus \partial\Delta)$, we define the top (T), bottom (B) and front (F) faces

$$T = \beta^*(\partial_F X \times X)\,, \quad B = \beta^*(X \times \partial_F X)\,, \quad F = \beta^{-1}(\partial_F \Delta)\,.$$

Denoting by $\mathcal{M}(\mathbf{C},\mathcal{U})$ the meromorphic functions with values in a function space $\mathcal{U}$, we first prove the following lemma about the model ends:

LEMMA 2.1. *Let $X = Y_j^0$ and let $\rho,\rho' \in \mathcal{C}^\infty(X \times_0 X)$ be defining functions of T and B respectively. Then for $\psi \in \mathcal{C}^\infty(X \times_0 X)$ with support sufficiently near to $T \cap B$,*

$$\psi\beta^* R_{Y_j}(s) \in \rho^s \rho'^s \mathcal{M}(\mathbf{C},\mathcal{C}^\infty(X \times_0 X, {}^0\Omega^{\frac{1}{2}}(X \times X))).$$

Proof. This is a refinement of Lemma 1.2 of [17], where it was proved that

$$R_{Y_j}(s) \in x^s x'^s \mathcal{M}(\mathbf{C}, \beta^*\mathcal{C}^\infty(X \times X, {}^0\Omega^{\frac{1}{2}}(X \times X)))$$

up to a term locally equal to the resolvent $R_{\mathbf{H}^2}(s)$.

To define coordinates for the blown-up space $\mathbf{H}^2 \times_0 \mathbf{H}^2$, we consider the Poincaré model $\mathbf{H}^2 = \{(u,v); v > 0\}$ of the hyperbolic plane and introduce $\{(r,\rho,\rho',\omega,y) \in \mathbf{R}^3_+ \times \mathbf{R}^2, \rho^2+\rho'^2+\omega^2 = 1\}$. Then the projection $\beta :\to \mathbf{H}^2 \times \mathbf{H}^2$ is expressed by

$$\beta(r,\rho,\rho',\omega,y) = ((y, r\rho), (y + r\omega, r\rho'))\,.$$

The resolvent kernel on $\mathbf{H}^2$ is given by

$$R_{\mathbf{H}^2}(z,z') = \frac{2^{-2s-1}\Gamma(s)}{\sqrt{\pi}\Gamma(s+1/2)} u^s(z,z') F(s,s,2s; u(z,z'))\,,$$

where $u(z,z') = \cosh^{-2}(d(z,z')/2)$. The lemma follows then directly from

$$\beta^* u(r,\rho,\rho',\omega,y) = \frac{4\rho\rho'}{1+2\rho\rho'}\,. \qquad \square$$

LEMMA 2.2. *Let* $\rho, \rho' \in \mathcal{C}^\infty(X \times_0 X)$ *be defining functions of* T *and* B *respectively. Then for* $\psi \in \mathcal{C}^\infty(X \times_0 X)$ *with support sufficiently near to* $T \cap B$,

$$\psi\beta^* R_X(s) \in \rho^s \rho'^s \mathcal{M}(\mathbf{C}, \mathcal{C}^\infty(X \times_0 X, {}^0\Omega^{\frac{1}{2}}(X \times X))) \tag{2.11}$$

with poles of finite rank.

Proof. We will use a direct argument based on Section 5 of [18]. Thus we recall that

$$Q_0(s_0) + Q(s) = R_X(s)(\mathrm{Id} + L(s_0, s)) \tag{2.12}$$

where

$$\begin{aligned} Q_0(s_0) &= \chi^Z_{a+2,b+2} R_X(s_0) \chi^Z_{a+1,b+1}, \\ Q(s) &= \sum_{j=1}^N (1 - \chi^{Y_j}_{a,b}) R_{Y_j^0}(s)(1 - \chi^{Y_j}_{a+1,b+1}), \\ L(s_0, s) &= [\Delta, \chi^Z_{a+2,b+2}] R_X(s_0) \chi^Z_{a+1,b+1} \\ &\quad + (s_0(1 - s_0) - s(1 - s)) Q_0(s_0) \\ &\quad - \sum_{j=1}^N [\Delta_{Y_j^0}, \chi^{Y_j}_{a,b}] R_{Y_j^0}(s)(1 - \chi^{Y_j}_{a+1,b+1}). \end{aligned} \tag{2.13}$$

The kernels of $Q_0(s_0)$ and $L(s_0, s)$ are supported in $\{x > \delta > 0\}$ and (2.13) also shows that $L(s_0, s)\varphi \in x'^s \mathcal{M}(\mathbf{C}, \mathcal{C}^\infty(X \times X, {}^0\Omega^{\frac{1}{2}}(X \times X)))$. Hence $L(s_0, s) : \mathcal{H}^a_N \to \mathcal{H}^a_{\mathrm{comp}} \subset \mathcal{H}^a_N, N > -\mathrm{Re}\, s$, forms a meromorphic family of operators, with $\mathrm{Id} + L(s_0, s)$ invertible if s_0 is chosen so that $\mathrm{Re}\, s_0 \gg \frac{1}{2}$. The meromorphic operator $(\mathrm{Id} + L(s_0, s))^{-1}$ is again of the form $\mathrm{Id} + F(s_0, s)$ and the identities

$$\begin{aligned} F(s_0, s) &= -L(s_0, s) - L(s_0, s) F(s_0, s), \\ F(s_0, s) &= -L(s_0, s) - F(s_0, s) L(s_0, s) \end{aligned}$$

show that its kernel is supported in $\{x > \delta\}$ and that

$$F(s_0, s) \in x'^s \mathcal{M}(\mathbf{C}, \mathcal{C}^\infty(X \times X, {}^0\Omega^{\frac{1}{2}}(X \times X))).$$

Hence $Q(s)F(s_0, s) \in x^s x'^s \mathcal{M}(\mathbf{C}, \mathcal{C}^\infty(X \times X, {}^0\Omega^{\frac{1}{2}}(X \times X)))$ and

$$\begin{aligned} \psi\beta^* R_X(s) \in\; & \rho^s \rho'^s \mathcal{M}(\mathbf{C}, \mathcal{C}^\infty(X \times_0 X, {}^0\Omega^{\frac{1}{2}}(X \times X))) \\ & + x^s x'^s \mathcal{M}(\mathbf{C}, \beta^* \mathcal{C}^\infty(X \times X, {}^0\Omega^{\frac{1}{2}}(X \times X))) \end{aligned} \tag{2.14}$$

since

$$\psi\beta^* R_X(s) = \psi\beta^* \left[Q(s) + Q(s) F(s_0, s)\right]$$

and

$$\psi\beta^* Q(s) \in \rho^s \rho'^s \mathcal{M}(\mathbf{C}, \mathcal{C}^\infty(X \times_0 X, {}^0\Omega^{\frac{1}{2}}(X \times X)))$$

by Lemma 2.1. □

We should remark that the proofs of the two lemmas above give considerably more than (2.11) and will be exploited in the proof of Lemma 2.6 below.

The complete description of $R_X(s)$ is more complicated, both combinatorically (there are many ends) and analytically (the one-dimensional Euclidean ends given by X_{0i}^a with the Laplacian Δ_0^0 are different from the hyperbolic ends).

When $M \neq 0$ (that is, there are cusps), we can still construct $X \times_0 X$ in the same way. Lemma 2.2 then generalizes to:

Proposition 2.3. *Let r be a coordinate on X_0^0 as in (2.7), let $\rho,\rho' \in \mathcal{C}^\infty(X \times_0 X)$ be the defining functions of T and B respectively and let $\psi \in \mathcal{C}^\infty(X \times_0 X)$ be supported sufficiently near $T \cap B$. Then*

$$(2.15) \quad (1-\chi_{a,b}^{X_i})R_X(s)(1-\chi_{a,b}^{X_{i'}}) \in e^{(\frac{1}{2}-s)r}e^{(\frac{1}{2}-s)r'}\mathcal{M}(\mathbf{C}) + \delta_{ii'}e^{(\frac{1}{2}-s)r}e^{-(\frac{1}{2}-s)r'}\mathcal{M}(\mathbf{C}),$$

$$(2.16) \quad (1-\chi_{a,b}^{X_i})R_X(s)(1-\chi_{a,b}^{Y_j}) \in e^{(\frac{1}{2}-s)r}x'^s\mathcal{M}(\mathbf{C},\mathcal{C}^\infty(Y_j^b,{}^0\Omega_{Y_j}^{\frac{1}{2}})),$$

$$(1-\chi_{a,b}^{Y_j})R_X(s)(1-\chi_{a,b}^{X_i}) \in x^s e^{(\frac{1}{2}-s)r'}\mathcal{M}(\mathbf{C},\mathcal{C}^\infty(Y_j^b,{}^0\Omega_{Y_j}^{\frac{1}{2}})),$$

$$\psi\beta^*[(1-\chi_{a,b}^{Y_j})R_X(s)(1-\chi_{a,b}^{Y_i})] \in \delta_{ij}\rho^s\rho'^s\mathcal{M}(\mathbf{C},\beta^*\mathcal{C}^\infty(Y_j^b\times Y_i^b,{}^0\Omega^{\frac{1}{2}}(Y_j^b\times Y_i^b)) + x^s x'^s\mathcal{M}(\mathbf{C},\beta^*\mathcal{C}^\infty(Y_j^b\times Y_i^b,{}^0\Omega^{\frac{1}{2}}(Y_j^b\times Y_i^b)).$$

Proof. This follows from the same analysis as in the proof of Lemma 2.2: in (2.13) we now add the contribution from the cusps to $Q_0(s)$ and $L(s_0,s)$:

$$\begin{aligned} Q(s) &= \sum_{i=1}^{M}(1-\chi_{a,b}^{X_i})R_0^0(s)(1-\chi_{a+1,b+1}^{X_i}) \\ &\quad + \sum_{j=1}^{N}(1-\chi_{a,b}^{Y_j})R_{Y_j^0}(s)(1-\chi_{a+1,b+1}^{Y_j}), \end{aligned}$$

$$(2.17) \quad \begin{aligned} L(s_0,s) &= [\Delta,\chi_{a+2,b+2}^{Z}]R_X(s_0)\chi_{a+1,b+1}^{Z} \\ &\quad + (s_0(1-s_0)-s(1-s))Q_0(s_0) \\ &\quad - \sum_{i=1}^{M}[\Delta_0^0,\chi_{a,b}^{X_i}]R_0^0(s)(1-\chi_{a+1,b+1}^{X_i}) \\ &\quad - \sum_{j=1}^{N}[\Delta_{Y_j}^0,\chi_{a,b}^{Y_j}]R_{Y_j^0}(s)(1-\chi_{a+1,b+1}^{Y_j}), \end{aligned}$$

and (2.15) and (2.16) follow from the form of the free resolvent on X_0^0 (see Lemma 2.2 of [18]):

$$R_0^0(s) = \frac{2\mathrm{i}}{2s-1}\left[\mathrm{e}^{(\frac{1}{2}-s)|r-r'|} - \mathrm{e}^{(\frac{1}{2}-s)|r+r'|}\right]. \tag{2.18}$$

□

The structure of the extended resolvent is described in the next lemma. We should remark that in the Euclidean odd dimensions case a more detailed and accurate description is possible (see [56, p. 744] and [24] for an explicit presentation) and that it probably holds here as well. The lemma below is, however, sufficient to us.

LEMMA 2.4. *If $R_X(s)$ has a pole at $s = s_0 \neq \frac{1}{2}$, then*

$$R_X(s) = \sum_{k=1}^{p} \frac{A_k(s_0)}{(s(1-s) - s_0(1-s_0))^k} + H(s_0,s) \tag{2.19}$$

where $H(s_0,s)$ is holomorphic near s_0,

$$\begin{aligned} A_k(s_0) &= \sum_{\ell,m=1}^{q} a_k^{\ell m}(s_0)\varphi_\ell \otimes \varphi_m\,, \\ \varphi_\ell \otimes \varphi_m(f) &= \varphi_\ell \int f\varphi_m,\ f \in \mathcal{C}_\mathrm{c}^\infty(X;\Omega^{\frac{1}{2}}(X))\,, \end{aligned} \tag{2.20}$$

with $\{\varphi_\ell\}_{\ell=1}^q$ satisfying

$$(1-\chi_{a,b}^{X_i})\widehat{(\varphi_\ell)}_0^{\,i} \in \mathrm{e}^{(\frac{1}{2}-s_0)r}\,\Omega^{\frac{1}{2}}(\mathbf{C})\,, \quad (1-\chi_{a,b}^{Y_j})\varphi_\ell \in x^{s_0}\mathcal{C}^\infty(Y_j, {}^0\Omega_{Y_j}^{\frac{1}{2}})\,. \tag{2.21}$$

If $a_k(s_0)$ denotes the matrix $(a_k^{\ell m}(s_0))_{1\leq \ell,m\leq q}$, then $a_1(s_0)$ is symmetric with rank q, $d(s_0) = a_1(s_0)^{-1}a_2(s_0)$ is nilpotent and $a_k(s_0) = a_1(s_0)d(s_0)^{k-1}, k > 1$.

Proof. Near $s_0 \neq \frac{1}{2}$, $R_X(s)$ is meromorphic in $s(1-s)$ and Lemma 5.1 of [18] showed that the poles are of finite rank; hence (2.19) follows. For $N > \mathrm{Re}(-s_0)$, we have

$$R_X(s)\colon\ \mathcal{H}_N^a \to \mathcal{H}_{-N}^a$$

and consequently $A_1(s_0) = \sum_{\ell=1}^q \varphi_\ell \otimes \psi_\ell$, $\operatorname{rank} A_1(s_0) = q$, $\varphi_\ell \in \mathcal{H}_{-N}^a$ and $\psi_\ell \in (\mathcal{H}_N^a)' = \mathcal{H}_{-N}^a$, where we took the complexification of the real Banach space dual (that is, used the form $\langle u, v\rangle = \int_X uv$). Taking the corresponding transpose

$${}^tA_1(s_0)\colon\ (\mathcal{H}_{-N}^a)' = \mathcal{H}_N^a \longrightarrow (\mathcal{H}_N^a)' = \mathcal{H}_{-N}^a$$

we note that $A_1(s_0) = {}^tA_1(s_0)$. In fact, the symmetry of $R_X(s)$ for $s \in \frac{1}{2} + i\mathbf{R}$ (since $\overline{(\Delta_X - s(1-s))u} = (\Delta_X - s(1-s))\bar{u}$ there) shows that ${}^tR_X(s) = R_X(s)$ for all s and since

$$A_1(s_0) = \frac{1}{2i\pi}\int_{\gamma_{s_0,\varepsilon}} R_X(s)(1-2s)ds\,,$$

$A_1(s_0)$ has the same symmetry. We conclude that $\psi_i = \sum_{j=1}^q a_1^{ij}(s_0)\varphi_j$ where $(a_1^{ij}(s_0))_{1\leq i,j\leq q}$ is a rank q symmetric matrix. The structure of the generalized eigenfunctions (2.21) is an immediate consequence of the structure of the resolvent given by Proposition 2.3.

By comparing terms in the Laurent expansion of

$$(\Delta - s(1-s))R_X(s) = \mathrm{Id} = R_X(s)(\Delta - s(1-s))$$

we now see that (by the density of $\mathcal{H}_N^a$ in $\mathcal{H}_{-N}^a$)

$$\begin{aligned}(2.22)\qquad (\Delta - s_0(1-s_0))A_k(s_0) &= A_{k+1}(s_0)\\ &= A_k(s_0)(\Delta - s_0(1-s_0)), \quad A_{p+1}(s_0) = 0.\end{aligned}$$

Hence Δ preserves $\mathrm{Im}\, A_1(s_0)$ and the $A_k(s_0)$'s can be represented by (2.20) for $k>1$ as well. The equalities (2.22) show that $(\Delta - s_0(1-s_0))_{|\,\mathrm{Im}\, A_1(s_0)}$ is nilpotent and that its matrix for the basis $\{\varphi_\ell\}_{1\leq\ell\leq q}$ of $\mathrm{Im}\, A_1(s_0)$ is given by $d(s_0) = a_1(s_0)^{-1}a_2(s_0)$. We also see from (2.22) that $a_1(s_0)$ commutes with $a_k(s_0)$ and consequently that $a_k(s_0) = a_1(s_0)d(s_0)^{k-1}$. □

We will now follow a method motivated by classical scattering theory (see [1], [45] and [17, Def. 2.1]) and will define the generalized eigenfunctions on X which extend the notion of Eisenstein series in the special case $X = \Gamma\backslash\mathbf{H}^2$. To include the dependence of the defining function of the boundary, we proceed in a way similar to that in [39], rather than by using the conformal structure of the boundary ([45], [48]).

Thus, for an embedded orientable hypersurface $H \subset V$, V a $\mathcal{C}^\infty$ manifold, and L a line bundle over H, we denote by N^*H the conormal bundle and define a line bundle $L\otimes|N^*H|^a$ over H as follows. If H is given by $h=0$, $dh_{|H}\neq 0$, then we write

$$h^a\mathcal{C}^\infty(H,L) = \mathcal{C}^\infty(H, L\otimes|N^*H|^a)\,.$$

For a specific choice of h we will locally write sections of $|N^*H|^a$ as $u|dh|^a$. The new line bundle captures the dependence on the defining function of H. With this new notation we will define the generalized eigenfunction

$$(2.23)\quad E_X(s;z,\xi) \in \mathcal{C}^\infty(\overset{o}{X}, [\mathcal{C}^\infty(\partial_F X; \Omega^{\frac12}_{\partial X}\otimes|N^*\partial_F X|^{s-\frac12})\oplus\mathbf{C}^M]\otimes {}^0\Omega^{\frac12}_X)$$

where we use ξ for $\xi\in\partial_F X$ or for indices of cusps $i=1,\ldots,M$.

For $\xi = i, 1\leq i\leq M$,

$$(2.24)\qquad E_X(s;z,\xi) = \mathrm{e}^{(s-\frac12)r'}(1-\chi^{X_i}_{a,b}(z'))R_X(s)(z',z)_{|r'\gg_z a}\,,$$

where we mean that r', a coordinate given by a specific choice of an isometry in (1.1), is sufficiently large depending on z, and for $\xi\in\partial_F Y_j$ (recall that $\partial_F X = \sqcup_{j=1}^N \partial_F Y_j$),

$$(2.25)\qquad E_X(s;z,\xi) = x'^{-s+\frac12}(1-\chi^{Y_j}_{a,b}(z'))R_X(s)(z',z)_{|x'=0}\,,\quad z'=(x',\xi),$$

where ξ is the variable on the boundary, the term $\frac{1}{2}$ in the exponent of x cancels the x dependence in the the half density factor $|dy/x|^{\frac{1}{2}}$ in $R_X(s)$ (see Proposition 2.3) and we obtain E_X as a section of ${}^0\Omega_X^{\frac{1}{2}} \otimes \Omega_{\partial_F X}^{\frac{1}{2}} \otimes |N^*\partial_F X|^{s-\frac{1}{2}}$ over $X \times \partial_F X$.

We rephrase Proposition 2.1 from [18] in slightly modified language:

$$\begin{aligned} (2.26) \qquad & R_X(s)(z,z') - R_X(1-s)(z,z') \\ &= (1-2s)\left[\sum_{i=1}^{M} E_X(s;z,i)E_X(1-s;z',i) \right. \\ &\quad \left. + \int_{\partial_F X} E_X(s;z,\xi)E_X(1-s;z',\xi)\right], \quad \operatorname{Re} s = \frac{1}{2}, N \neq 0\,. \end{aligned}$$

Since the spectral measure is given by the difference of resolvents at s and $1-s$ (which can be thought of as limits of the usual resolvent $(\Delta_X - \zeta)^{-1}$ at $\zeta = s(1-s) \pm \mathrm{i}0$), we obtain a parametrization of the continuous spectrum of Δ_X. Proposition 2.3 shows that the $E_X(s)$ continue meromorphically in s and, in the precise sense described below by the scattering matrix, their poles coincide with those of the resolvent.

The scattering matrix can be defined as the operator intertwining $E_X(1-s;z)$ and $E_X(s;z)$ and we recall from [17] the following proposition (our convention differs slightly from that of [17] and follows traditional scattering theory [1], [39]):

PROPOSITION 2.5. *There exists a meromorphic operator*

$$\begin{aligned} (2.27) \qquad S_X(s)\colon \mathbf{C}^M \oplus \mathcal{C}^\infty(\partial_F X; \Omega_{\partial_F X}^{\frac{1}{2}} \otimes |N^*\partial_F X|^{-s+\frac{1}{2}}) \\ \longrightarrow \mathbf{C}^M \oplus \mathcal{C}^\infty(\partial_F X; \Omega_{\partial_F X}^{\frac{1}{2}} \otimes |N^*\partial_F X|^{s-\frac{1}{2}}) \end{aligned}$$

which satisfies:

(i) $S_X(s)E_X(1-s;z) = E_X(s;z)\,,$

(ii) $S_X(s)S_X(1-s) = \mathrm{Id}\,, \quad S_X(s)^* = S_X(\bar{s})\,.$

The scattering matrix can also be introduced as the operator intertwining the boundary data of the degenerate elliptic boundary value problem at infinity; see Proposition 8.2 in [38].

We now recall from Section 2 of [17] how the Schwartz kernel of $S_X(s)$, $\operatorname{Re} s = \frac{1}{2}$, appears in the distributional expansion of $E_X(s)$ at the infinities of X. Let $\operatorname{Re} s = \frac{1}{2}, s \neq \frac{1}{2}$. First we have the part of the scattering matrix responsible for the interaction between cusps: If $\xi = i$ and $\eta = i', 1 \leq i, i' \leq M$, then

$$(2.28) \quad (1 - \chi_{a,b}^{X_{i'}})E_X(s;r,\xi) = (2s-1)^{-1}\left[\delta_{ii'}\mathrm{e}^{(s-\frac{1}{2})r} - \mathrm{e}^{(\frac{1}{2}-s)r}S_X(s)(\xi,\eta)\right]|dr|^{\frac{1}{2}}\,.$$

For the interaction between a funnel and a cusp, we have, for $\xi \in \partial Y_j$, $1 \leq j \leq N$ and for $\eta = i, 1 \leq i \leq M$, and when r denotes the coordinates on X^a_{0i},

$$(1-\chi^{X_i}_{a,b})E_X(s;r,\xi) = -(2s-1)^{-1}\mathrm{e}^{(\frac{1}{2}-s)r}S_X(s)(\xi,\eta)|d\xi|^{\frac{1}{2}}|dx|^{s-\frac{1}{2}} \tag{2.29}$$

for $r \gg a$.

We then have a transpose statement: $\eta \in \partial_F Y_j$, $1 \leq j \leq N$ and $\xi = i, 1 \leq i \leq M$:

$$\begin{aligned}(1-\chi^{Y_j}_{a,b})E_X(s;(x,\eta),\xi) &\\ = -(2s-1)^{-1}x^{s-\frac{1}{2}}&\left[S_X(s)(\xi,\eta)|d\eta|^{\frac{1}{2}}|dx|^{s-\frac{1}{2}}\left|\frac{dx}{x}\right|^{\frac{1}{2}}\right.\\ &\left.+\varphi_{\xi,s}(x,\eta)\left|\frac{dx}{x}\right|^{\frac{1}{2}}\right],\end{aligned} \tag{2.30}$$

where $\varphi_{\xi,s}(x,\eta) \to 0$ as $x \to 0$ in $\mathcal{D}'(\partial_F Y_j, \Omega^{\frac{1}{2}}_{\partial_F Y_j})$.

Finally, for $\xi \in \partial_F Y_j$, $1 \leq j \leq N$, and in local coordinates near $\partial_F Y_{j'}$, $1 \leq j' \leq N$,

$$\begin{aligned}(1-\chi^{Y_{j'}}_{a,b})&E_X(s;(x,\eta),\xi)\\ &= (2s-1)^{-1}\left\{x^{-s+\frac{1}{2}}\delta_\xi(\eta)|d\eta|^{\frac{1}{2}}|d\xi|^{\frac{1}{2}}|dx|^{s-\frac{1}{2}}\right.\\ &\quad - x^{s-\frac{1}{2}}[S_X(s)(\xi,\eta)|d\eta|^{\frac{1}{2}}|d\xi|^{\frac{1}{2}}|dx|^{2s-1}]|dx|^{\frac{1}{2}-s}\\ &\quad \left.+\psi_{\xi,s}(x,\eta)\right\}\left|\frac{dx}{x}\right|^{\frac{1}{2}},\end{aligned} \tag{2.31}$$

where $\psi_{\xi,s} \to 0$, as $x \to 0$ in $\mathcal{D}'(\partial_F Y_{j'}; \Omega^{\frac{1}{2}}_{\partial_F X} \otimes |N^*\partial_F Y_{j'}|^{s-\frac{1}{2}})$. We note that the first term in the expansion is nonzero only if $j = j'$.

The expression in local coordinates makes clear the following fact already implicit in (2.27): When operators are identified with their kernels,

$$\begin{aligned}S_X(s) \in\ & \mathbf{C}^M \otimes \mathbf{C}^M \oplus \mathcal{D}'(\partial_F X, \Omega^{\frac{1}{2}}_{\partial_F X} \otimes |N^*\partial_F X|^{s-\frac{1}{2}}) \otimes \mathbf{C}^M\\ &\oplus \mathbf{C}^M \otimes \mathcal{D}'(\partial_F X, \Omega^{\frac{1}{2}}_{\partial_F X} \otimes |N^*\partial_F X|^{s-\frac{1}{2}})\\ &\oplus \mathcal{D}'(\partial_F X \times \partial_F X, \Omega^{\frac{1}{2}}(\partial_F X \times \partial_F X) \otimes |N^*\partial_F X|^{2s-1}).\end{aligned}$$

Consequently it is convenient to think of $S_X(s)$ as an $(M+N) \times (M+N)$ matrix

$$S_X = \begin{pmatrix} S^{CC}_X & S^{FC}_X \\ S^{CF}_X & S^{FF}_X \end{pmatrix}, \tag{2.32}$$

where

$$\begin{aligned}(S_X^{CC}(s))_{ii'} &\in \mathbf{C}, 1 \le i, i' \le M\,,\\ (S_X^{CF}(s))_{ij} &\in \mathcal{D}'(\partial_F Y_j, \Omega^{\frac{1}{2}}_{\partial_F X} \otimes |N^*\partial_F X|^{s-\frac{1}{2}}), \quad 1 \le i \le M, 1 \le j \le N\,,\\ (S_X^{FC}(s))_{ji} &\in \mathcal{D}'(\partial_F Y_j, \Omega^{\frac{1}{2}}_{\partial_F X} \otimes |N^*\partial_F X|^{s-\frac{1}{2}}), \quad 1 \le i \le M, 1 \le j \le N\,,\\ (S_X^{FF}(s))_{jj'} &\in \mathcal{D}'(\partial_F Y_j \times \partial_F Y_{j'}, \Omega^{\frac{1}{2}}_{\partial_F X \times \partial_F X} \otimes |N^*\partial_F X|^{2s-1}), \quad 1 \le j, j' \le N.\end{aligned}$$

The description above can be improved significantly and extended to all $s \in \mathbf{C}$; so far we have considered $\operatorname{Re} s = \frac{1}{2}$ only. For this improvement we apply Proposition 2.3. We start by observing that the blow-down map on B restricts to $T \cap B$ naturally:

$$\text{(2.33)} \qquad \begin{aligned} T \cap B &= [\partial X \times \partial X, \Delta_{\partial X}] \xrightarrow{\beta_\partial} \partial X \times \partial X,\\ \Delta_{\partial X} &= \{(y,y)\colon\, y \in \partial X\} \subset \partial X \times \partial X\,.\end{aligned}$$

Here $[\partial X \times \partial X, \Delta_{\partial X}]$ denotes the space obtained by blowing up the diagonal over the boundary, $\Delta_{\partial X}$, in $\partial X \times \partial X$; see [36, §III] or Section 5 of [39] for brief self-contained discussions. We will also use the notation for conormal sections of line bundles from [21, §18.2]: $I^m_{\mathrm{phg}}(Y, Z; E \otimes \Omega_Y^{1/2})$ denotes distributional sections of $E \otimes \Omega_Y^{1/2}$ over Y classically conormal of order m to the submanifold Z of Y.

LEMMA 2.6. *For $s \notin \mathbf{N} + \frac{1}{2}$, the kernel of $S_X^{FF}(s)$ is the unique conormal distribution in $I^{2\operatorname{Re} s-1}_{\mathrm{phg}}(\partial_F X \times \partial_F X, \Delta_{\partial_F X}\ \Omega^{\frac{1}{2}}_{\partial_F X \times \partial_F X} \otimes |N^*\partial_F X|^{2s-1})$ given by*

$$\text{(2.34)} \qquad \beta_\partial^* S_X^{FF}(s) = (1-2s)\beta^*[x^{-s+\frac{1}{2}} x'^{-s+\frac{1}{2}} R_X(s)]_{|T \cap B}\,.$$

The mixed terms satisfy

$$S_X^{CF}(s) = {}^t(S_X^{FC}(s)) \in \mathcal{M}(\mathbf{C}, \mathcal{C}^\infty(\partial_F X, \Omega^{\frac{1}{2}}_{\partial_F X} \otimes |N^*\partial_F X|^{s-\frac{1}{2}}) \otimes \mathbf{C}^M)$$

and are given by

$$\text{(2.35)} \quad (S_X^{CF}(s))_{ij} = (1-2s)\left[e^{(s-\frac{1}{2})r}(1-\chi^{X_i}_{a,b})R_X(s)(1-\chi^{Y_j}_{a,b})x'^{-s+\frac{1}{2}}\right]_{|x'=0}\,.$$

Finally, $S_X^{CC} \in \mathcal{M}(\mathbf{C}, \mathbf{C}^M \otimes \mathbf{C}^M)$.

At $s \in \mathbf{N} + \frac{1}{2}$, $S_X(s)$ has a pole of infinite rank and (2.34) holds after division by $\Gamma(-s + \frac{1}{2})$.

Proof. The left-hand side of (2.34) is meromorphic in s and hence (2.34) will follow once we have the meromorphy of the right-hand side with the equality of the two sides for $\operatorname{Re} s = \frac{1}{2}$.

To study the right-hand side we shall use Proposition 2.3 (but see Lemma 2.2 for a clearer exposition in the case of no cusps). Let us fix a defining function x of ∂X and let x' denote the same function on the left factor in $X \times X$. Away from F, $(x, x', y, y'), y, y' \in \partial X$ give coordinates for $X \times_0 X$ near $T \cap B$ and we can put $\rho = x, \rho' = x'$. Then (2.11) shows that the right-hand side of (2.34) has a smooth kernel and the same holds for the restriction. Locally a section of ${}^0\Omega^{1/2}(X \times X)$ can be represented by

$$u(x, y, x', y') \left|\frac{dx}{x}\right|^{\frac{1}{2}} \left|\frac{dx'}{x'}\right|^{\frac{1}{2}} \left|\frac{dy}{x}\right|^{\frac{1}{2}} \left|\frac{dy'}{x'}\right|^{\frac{1}{2}}, \quad u \in \mathcal{C}^\infty(X \times X),$$

so that the dependence on s of the restriction comes from the factor $x^{-s+1/2}x'^{-s+1/2}$. The restriction is invariantly defined as

$$u(y, y')|dy|^{1/2}|dy'|^{1/2}|dx|^{2s-1},$$

that is, as a section on $\Omega^{1/2}_{\partial X \times \partial X} \otimes |N^*\partial X|^{2s-1}$. Near F, we choose local coordinates in $\partial X \times \partial X$, $y, y' \in \mathbf{R}$, so that the coordinates in the blow-up space are given by

$$Y = y - y', y, \quad \rho = \frac{x}{|Y|}, \quad \rho' = \frac{x'}{|Y|}$$

where now ρ and ρ' define T and B respectively. Hence, near $T \cap B$ and by the proof of Lemma 2.2,

$$\begin{aligned} (2.36) \qquad & \beta^* \left[x^{-s+1/2}x'^{-s+1/2}R_X(s)(z, z')\right] \\ &= \left[|Y|^{-2s}K_0(s, \rho,\ \rho', Y, y) + K_1(s, \rho; \rho', Y, y)\right] \\ &\quad \times \left|\frac{d\rho}{\rho}\right|^{\frac{1}{2}} \left|\frac{d\rho'}{\rho'}\right|^{\frac{1}{2}} |dy|^{1/2}|dY|^{1/2}|dx|^{2s-1}, \end{aligned}$$

where K_0 and K_1 are meromorphic in s with smooth kernels in the Y, y coordinates and poles of finite rank. The kernel of the scattering matrix is obtained by restriction to $T \cap B = \{\rho = \rho' = 0\}$. The homogenous distribution on $\mathbf{R}$, $(1-2s)|Y|^{-2s}$, is well defined and holomorphic in s for $s \notin \mathbf{N} + 1/2$ (it satisfies $(3.2.25)'$ in [21, §3.2]). The proof of Lemma 2.2 shows that K_0 is completely determined by $R_{Y_i}(s)$, that is, by the *free* cylindrical ends. Hence it contributes to the pole of infinite order produced by the singularities in $(1-2s)|Y|^{-2s}$.

Then (2.36) shows that the right-hand side of (2.34) is meromorphic for all $s \in \mathbf{C}$. The equalities of both sides for $\operatorname{Re} s = 1/2$ follows from the definition of the generalized eigenfunction (2.25) and the definition of S_X^{FF} through their asymptotics (2.31). □

Remark 2.7. On the operator level we could reformulate this proposition as

$$S_X^{FF}(s) \in \Psi^{2\operatorname{Re} s-1}(\partial_F X; |N^*\partial_F X|^{-s+\frac{1}{2}}, |N^*\partial_F X|^{s-\frac{1}{2}}), \quad s \notin \mathcal{R}_X \cup \mathbf{N} + \frac{1}{2}.$$

Then (2.27) shows that $S_X^{FF}(s)$ is an elliptic pseudodifferential operator of order $2\,\mathrm{Re}\,s - 1$.

The following lemma will be useful later.

LEMMA 2.8. *For* $s \in -\mathbf{N} + \frac{1}{2}$,

$$S_X^{FF}(s) \equiv 0\,, \quad S_X^{FC}(s) = {}^t(S_X^{CF}(s)) \equiv 0\,. \tag{2.37}$$

Proof. The explicit formula for $S_{Y_j^0}$ (given in Lemma 3.3 of [18]) shows that $S_{Y_j^0}(s) \equiv 0$ for $s \in -\mathbf{N} + \frac{1}{2}$ which together with the functional equation (2.26) (with X replaced by Y_j^0) shows that $R_{Y_j^0}(s) = R_{Y_j^0}(1-s)$ for $s \in -\mathbf{N}+\frac{1}{2}$. To show (2.37), we will use Lemma 2.6 and the representation of the resolvent given in the proof of Lemma 2.2. We note that for ψ supported in $x \gg 0$,

$$x^{-s+\frac{1}{2}} R_{Y_j^0}(1-s)\psi_{|x=0} = 0\,, \quad \mathrm{Re}\,s < \frac{1}{2}\,, \tag{2.38}$$

which shows that the term of the form $Q(s)F(s_0,s)$ (see (2.12) and (2.13); $\mathrm{Id} + F(s_0,s) = (\mathrm{Id} + L(s_0,s))^{-1}$) contribute zero to the right-hand side of (2.34) at $s \in -\mathbf{N} + \frac{1}{2}$. Since the contribution from $Q(s)$ to $S_X^{FF}(s)$ is the same as in the *free* case of Y_j^0's, we obtain the free scattering matrix which is equal to 0 at $s \in -\mathbf{N} + \frac{1}{2}$ as recalled above. By (2.35) the terms contributing to $S_X^{FC}(s)$ are of the form $Q(s)F(s_0,s)$ and consequently (2.38) shows that they vanish. □

Remark 2.9. The proof of this lemma shows also that if $M = 0$ (that is, there are no cusps) then

$$\mathcal{R}_X \cap -\mathbf{N} + \frac{1}{2} = \emptyset\,.$$

The relationship between the poles of R_X and those of S_X is given by the following:

LEMMA 2.10. *The scattering matrix* $S_X(s)$ *has a pole at* s_0, $\mathrm{Re}\,s_0 < 1, s_0 \neq \frac{1}{2}$ *if and only if* $R_X(s)$ *has a pole at* s_0. *Moreover if* s_0 *is such a pole then*

$$S_X(s) = \sum_{k=1}^{p} \frac{A_k^{\#}(s_0)}{(s(1-s) - s_0(1-s_0))^k} + H^{\#}(s_0,s), \tag{2.39}$$

where $A_k^{\#}(s_0) = {}^t\Phi^{\#} a_k(s_0)\Phi^{\#}$, *with* $\Phi^{\#}: \mathbf{C}^M \oplus L^2(\partial_F X) \to \mathbf{C}^q$ *surjective and* $a_k(s_0)$ *is as in Lemma 2.4 and the function* $H^{\#}(s_0,s)$ *is holomorphic near* s_0.

Proof. We recall (2.21) from Lemma 2.4:

$$(1 - \chi_{a,b}^{X_i})\varphi_\ell = e^{(\frac{1}{2}-s_0)r}\tilde{\varphi}_{\ell i}^{\#}|dr|^{\frac{1}{2}}\,, \quad \tilde{\varphi}_{\ell i}^{\#} \in \mathbf{C}\,, \tag{2.40}$$

$$(1 - \chi_{a,b}^{Y_j})\varphi_\ell = x^{s_0}\varphi_{\ell j}^{\#}\frac{|dxdy|^{\frac{1}{2}}}{x}\,, \quad \varphi_{\ell j}^{\#} \in \mathcal{C}^\infty(Y_j)\,. \tag{2.41}$$

Hence by Lemma 2.2, Definitions (2.25) and (2.24) and the characterization of $S_X(s)$, (2.28)–(2.31), we obtain that

$$\begin{aligned}(S_X^{CC}(s))_{ii'} &= \sum_{k=1}^{p}\sum_{\ell,m=1}^{q} a_k^{\ell m}(s_0)\frac{\widetilde{\varphi}_{\ell i}^{\#}\otimes\widetilde{\varphi}_{mi'}^{\#}}{(s(1-s)-s_0(1-s_0))^k} \\ &\quad + (H^{\#,CC}(s_0,s))_{ii'}\,, \\ (S_X^{FC}(s))_{ij} &= \sum_{k=1}^{p}\sum_{\ell,m=1}^{q} a_k^{\ell m}(s_0)\frac{\widetilde{\varphi}_{\ell i}^{\#}\otimes\varphi_{mj|x=0}^{\#}}{(s(1-s)-s_0(1-s_0))^k} \\ &\quad +(H^{\#,FC}(s_0,s))_{ij}\,,\end{aligned}$$

and finally

$$\begin{aligned}(S_X^{FF}(s))_{jj'} &= \sum_{k=1}^{p}\sum_{\ell,m=1}^{q} a_k^{\ell m}(s_0)\frac{\varphi_{\ell j|x=0}^{\#}\otimes\varphi_{mj'|x=0}^{\#}}{(s(1-s)-s_0(1-s_0))^k} \\ &\quad + (H^{\#,FF}(s_0,s))_{jj'}\,.\end{aligned} \tag{2.42}$$

Hence

$$\begin{aligned}A_k^{\#}(s_0) &= \begin{pmatrix} \left(\sum_{\ell,m} a_k^{\ell m}(s_0)\widetilde{\varphi}_{\ell i}^{\#}\otimes\widetilde{\varphi}_{\ell i'}^{\#}\right)_{ii'} & \left(\sum_{\ell,m} a_k^{\ell m}(s_0)\varphi_{\ell j|x=0}^{\#}\otimes\widetilde{\varphi}_{mi}^{\#}\right)_{ji} \\ \left(\sum_{\ell,m} a_k^{\ell m}(s_0)\widetilde{\varphi}_{\ell i}^{\#}\otimes\varphi_{mj|x=0}^{\#}\right)_{ij} & \left(\sum_{\ell,m} a_k^{\ell m}(s_0)\varphi_{\ell j|x=0}^{\#}\otimes\varphi_{mj'|x=0}^{\#}\right)_{jj'}\end{pmatrix} \\ &= {}^t\Phi^{\#}a_k(s_0)\Phi^{\#},\end{aligned} \tag{2.43}$$

where $a_k(s_0)$ is as defined in Lemma 2.4 and $\Phi^{\#}\colon \mathbf{C}^M\oplus L^2(\partial_F X)\to\mathbf{C}^q$ is given by

$$\Phi^{\#}\colon (v,w)\longrightarrow\left(\sum_{i=1}^{M} v_i\widetilde{\varphi}_{\ell i}^{\#}+\langle\varphi_{\ell}^{\#}{}_{|x=0},w\rangle\right)_{\ell=1,\dots,q},$$

with $\langle v_1,v_2\rangle=(v_1,\overline{v}_2)_{L^2(\partial_F X;\Omega^{\frac12}_{\partial_F X})}$. Since $\operatorname{rank}A_1(s_0)=\operatorname{rank}a_1(s_0)$, we conclude that $\operatorname{rank}A_k^{\#}(s_0)\le\operatorname{rank}A_1(s_0)$.

The easiest way to see the equality of the ranks comes from the functional equation (i) in Proposition 2.5 and (2.26): since $E_X(1-s;\xi,z)$ are holomorphic for $\operatorname{Re}s>\frac12$, the rank of the residue of R_X is less than or equal to the rank of the residue of S_X. Hence $\Phi^{\#}$ is surjective and the lemma follows. □

Following [38], we define the divisor of the resolvent, $\{(s,m_s(R_X))\}$, for all s, by putting

$$m_s(R_X)=\dim\operatorname{Im}\Xi_s(R_X)\,, \tag{2.44}$$

where $\Xi_s(R_X)$ denotes the singular part of R_X at the point s and

$$\operatorname{Im}\Xi_s(R_X) = \{\Xi_s(R_X)(s')\varphi(s')_{|s'=s} \in \mathcal{H}^a_{\mathrm{loc}} \text{ with } \varphi,\ \Xi_s(R_X)\varphi : \mathbf{C} \to \mathcal{H}^a_{\mathrm{comp}} \text{ holomorphic}\}.$$

Lemma 2.4 shows that $m_s(R_X) = q$ so that this definition agrees with (1.6) (and of course with (1.7)).

By Lemma 2.6, $S_X(s)$ forms a family of operators with poles of finite rank in $\operatorname{Re} s < 1$. Hence a natural definition of the divisor of S_X, $\{(s, v_s(S_X))\}$, is provided by [14] once we eliminate the zeros of infinite rank:

$$(2.45)\qquad \begin{aligned} v_s(S_X) &= -\frac{1}{2\pi \mathrm{i}} \operatorname{tr} \int_{\gamma_{s_0,\varepsilon}} \widetilde{S}_X(s)^{-1} \frac{d}{ds} \widetilde{S}_X(s) ds\,, \\ \widetilde{S}_X(s) &= \begin{pmatrix} S_X^{CC}(s) & S_X^{FC}(s) \\ \Gamma(s+\frac{1}{2}) S_X^{CF} & \Gamma(s+\frac{1}{2}) S_X^{FF} \end{pmatrix}. \end{aligned}$$

Note that in our convention the poles have positive multiplicities while the corresponding zeros, under the reflection, $s \mapsto 1-s$, have negative multiplicities.

PROPOSITION 2.11. *If the divisors of R_X and S_X are defined by* (2.44) *and* (2.45) *respectively, then for* $\operatorname{Re} s < 1$,

$$(2.46)\qquad v_s(S_X) = m_s(R_X) - m_{1-s}(R_X)\,,$$

and for $s \neq \frac{1}{2}$,

$$(2.47)\qquad m_s(R_X) = \operatorname{rank} \int_{\gamma_{s,\varepsilon}} R_X(s')(1-2s')ds'\,,$$

where the last quantity is the rank of the residue of R_X as a function of $\lambda = s(1-s)$.

Proof. As has been already stated, the second equality (2.47) follows from (2.19) and (2.20) and the invertibility of $a_1(s_0)$. To see the first equality (2.46) we need to recall some facts from [14].

Let $A(\lambda)$ be a meromorphic family of Fredholm operators on a Banach space $\mathcal{B}$, such that $A(\lambda)$ has a pole of finite rank at λ_0, $A(\lambda)$ is holomorphic for $\lambda \neq \lambda_0$ on a small neighbourhood of λ_0 and the index of $[A(\lambda) - \Xi_\lambda(A)]_{|\lambda=\lambda_0}$ is equal to 0 (here $\Xi_\lambda(A)$ denotes the singular part of A at the point λ). We also assume the existence of a meromorphic operator $B(\lambda)$ such that $B(\lambda)A(\lambda) = A(\lambda)B(\lambda) = \mathrm{Id}$. By a root function $\varphi(\lambda)$ of $B(\lambda)$ we mean a holomorphic vector valued function such that $\varphi(\lambda) \not\equiv 0$ and $B(\lambda_0)\varphi(\lambda_0) = 0$. The vector $\varphi_0 = \varphi(\lambda_0)$ is called a null vector of $B(\lambda_0)$ and the maximal order of vanishing of $B(\lambda)\varphi(\lambda)$ at λ_0 for any root vector with $\varphi(\lambda_0) = \varphi_0$ is called the *rank* of φ_0. Under the above assumptions the rank is always finite since $A(\lambda)$ has a finite-rank pole at $\lambda = \lambda_0$. A canonical system of null vectors of $B(\lambda)$ at λ_0,

$\{\varphi_0^{(1)}, \ldots, \varphi_0^{(k)}\}$, is a basis of $\ker B(\lambda_0)$ satisfying the following property: The rank of $\varphi_0^{(1)}$ is the maximal rank corresponding to λ_0 and the rank of $\varphi_0^{(j)}$ is the maximal rank of null vectors in some direct complement of the span of $\{\varphi_0^{(1)}, \ldots, \varphi_0^{(j-1)}\}$ in $\ker B(\lambda_0)$. If $r_j = \operatorname{rank} \varphi_0^{(j)}$ then the set

$$\mathfrak{n}_{\lambda_0}(B) = \{r_1, \ldots, r_k\}$$

(where the same number may appear several times) is determined by $B(\lambda)$ and is called the set of partial null multiplicities of $B(\lambda)$ at λ_0. Following [14] we also put

$$N_{\lambda_0}(B) = \sum_{r \in \mathfrak{n}_{\lambda_0}(B)} r \,.$$

Having the set of partial null multiplicities, $\mathfrak{n}_{\lambda_0}(A) = \{\widetilde{r}_1, \cdots, \widetilde{r}_{\widetilde{k}}\}$, for A and the corresponding $N_{\lambda_0}(A)$, we recall Theorem 2.1 of [14]:

$$\begin{aligned} N_{\lambda_0}(B) - N_{\lambda_0}(A) &= \frac{1}{2\pi \mathrm{i}} \operatorname{tr} \int_{\gamma_{\lambda_0,\varepsilon}} B(\lambda) \frac{d}{d\lambda} B(\lambda) d\lambda \\ &= -\frac{1}{2\pi \mathrm{i}} \operatorname{tr} \int_{\gamma_{\lambda_0,\varepsilon}} A(\lambda) \frac{d}{d\lambda} A(\lambda) d\lambda \,. \end{aligned} \tag{2.48}$$

We will now *assume* that

$$A(\lambda) = E(\lambda) \left[\sum_{\ell=1}^{n} (\lambda - \lambda_0)^{-k_\ell} P_\ell + H(\lambda) \right] D(\lambda) \,, \quad k_1 \geq \cdots \geq k_n > 0 \tag{2.49}$$

where $E(\lambda)$ and $D(\lambda)$ are holomorphic and invertible near λ_0 and P_i are mutually orthogonal one-dimensional projections (that is $P_i P_j = \delta_{ij} P_i$, $\operatorname{tr} P_j = 1$). Following Section 2.2 of [14] we can in fact assume that $A(\lambda)$ is equal to the operator in the square brackets of (2.49). We now claim that

$$\mathfrak{n}_{\lambda_0}(B) = \{k_1, \ldots, k_n\} \,. \tag{2.50}$$

In fact, if $\varphi^{(j)}$ is the root function corresponding to $\varphi_0^{(j)}$ and with the order of vanishing of $B(\lambda)\varphi^{(j)}(\lambda)$ at λ_0 equal to the rank of $\varphi_0^{(j)}$, r_j, then $\psi^{(j)}(\lambda) = (\lambda - \lambda_0)^{-r_j} B(\lambda) \varphi^{(j)}(\lambda)$ satisfies $\psi_0^{(j)} = \psi^{(j)}(\lambda_0) \neq 0$ and $A(\lambda)\psi^{(j)}(\lambda) = (\lambda - \lambda_0)^{-r_j} \varphi^{(j)}(\lambda)$. Hence

$$\varphi^{(j)}(\lambda) = \sum_{\ell=1}^{n} (\lambda - \lambda_0)^{r_j - k_\ell} P_\ell \psi^{(j)}(\lambda) + (\lambda - \lambda_0)^{r_j} H(\lambda) \psi^{(j)}(\lambda)$$

and consequently

$$\varphi_0^{(j)} \in \operatorname{span} \{ P_\ell \mathcal{B} \,:\, r_j \leq k_\ell \} \,.$$

In particular $\dim \operatorname{Ker} B(\lambda_0) \leq n$ and for the elements of $\mathfrak{n}_{\lambda_0}(B)$, $r_j \leq k_\ell$ for some $\ell = \ell(j)$. To see the equality (2.50) we first construct n independent root vectors: Let $\psi_0^{(j)} \neq 0$ be such that $P_\ell \psi_0^{(j)} = \delta_{j\ell}\psi_0^{(j)}$ and put

$$\varphi^{(j)}(\lambda) = \psi_0^{(j)} + (\lambda - \lambda_0)^{k_j} H(\lambda)\psi_0^{(j)} = (\lambda - \lambda_0)^{k_j} A(\lambda)\psi_0^{(j)}, \quad \varphi_0^{(j)} = \varphi^{(j)}(\lambda_0).$$

Then the set $\{\varphi_0^{(1)}, \ldots, \varphi_0^{(n)}\}$ is linearly independent and $B(\lambda)\varphi^{(j)}(\lambda) = (\lambda - \lambda_0)^{k_j}\psi_0^{(j)}$. Hence $\varphi_0^{(j)} \in \ker B(\lambda_0)$ with rank greater than or equal to k_j. We conclude that $k = \dim\ker B(\lambda_0) = n$ and that $r_j \geq k_j, j = 1, \ldots, n$. But since already had the opposite inequalities, $r_j \leq k_\ell$, we obtain (2.50).

If $B(\lambda)$ is also of the form (2.49), with k_j, $j = 1, \ldots, n$, replaced by, say, $\widetilde{k}_j$, $j = 1, \ldots, \widetilde{n}$, then the analogue of (2.50) holds for $\mathfrak{n}_{\lambda_0}(A)$ as well: $\mathfrak{n}_{\lambda_0}(A) = \{\widetilde{k}_1, \ldots, \widetilde{k}_{\widetilde{n}}\}$. Hence (2.48) equals $\sum_{j=1}^{n} k_j - \sum_{j=1}^{\widetilde{n}} \widetilde{k}_j$.

To apply the discussion above to S_X we introduce a family of operators acting on a fixed Banach space. Thus we choose an invertible elliptic operator of order 1: $\Lambda \in \Psi^1(\partial_F X; \Omega^{\frac{1}{2}}_{\partial_F X}, \Omega^{\frac{1}{2}}_{\partial_F X})$ and choose a defining function of $\partial_F X$ to trivialize the bundle $|N^*\partial_F X|^{s-\frac{1}{2}}$. We then put $\lambda = s(1-s)$ $(s \neq \frac{1}{2})$ and

$$\begin{aligned} A(\lambda) &= \begin{pmatrix} S_X^{CC}(s) & S_X^{CF}(s)\Lambda^{-s+\frac{1}{2}} \\ \Gamma(s+\frac{1}{2})\Lambda^{-s+\frac{1}{2}} S_X^{FC}(s) & \Gamma(s+\frac{1}{2})\Lambda^{-s+\frac{1}{2}} S_X^{FF}(s)\Lambda^{-s+\frac{1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \Lambda^{-s+\frac{1}{2}} \end{pmatrix} \tilde{S}_X(s) \begin{pmatrix} 1 & 0 \\ 0 & \Lambda^{-s+\frac{1}{2}} \end{pmatrix}, \\ B(\lambda) &= \begin{pmatrix} S_X^{CC}(1-s) & \Gamma(s+\frac{1}{2})^{-\frac{1}{2}} S_X^{CF}(1-s)\Lambda^{s-\frac{1}{2}} \\ \Lambda^{s-\frac{1}{2}} S_X^{FC}(1-s) & \Gamma(s+\frac{1}{2})^{-\frac{1}{2}}\Lambda^{s-\frac{1}{2}} S_X^{FF}(1-s)\Lambda^{s-\frac{1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \Lambda^{s-\frac{1}{2}} \end{pmatrix} \tilde{S}_X(s)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \Lambda^{s-\frac{1}{2}} \end{pmatrix}, \end{aligned}$$

which by Proposition 2.5, Lemmas 2.6 and 2.8 satisfy the general assumption above with $\mathcal{B} = \mathbf{C}^M \oplus L^2(\partial_F X, \Omega^{\frac{1}{2}}_{\partial_F X})$ and $\lambda_0 = s_0(1-s_0)$. We will now use Lemma 2.10 to obtain (2.49) with

$$m_{s_0}(R_X) = q = \sum_{\ell=1}^{n} k_\ell\,. \tag{2.51}$$

By (2.19) and Lemma 2.10 the singular part of $A(\lambda)$ at λ_0 can be rewritten as (2.52)

$${}^t\Phi^{\#} a_1(s_0)\left[(\lambda-\lambda_0)^{-1} + d(s_0)(\lambda-\lambda_0)^{-2} + \cdots + d(s_0)^{p-1}(\lambda-\lambda_0)^{-p}\right]\Phi^{\#}\,.$$

We can put the matrix $d(s_0)$ into its Jordan normal form, with $k_1 + \cdots + k_n = q$,

$$
Fd(s_0)F^{-1} = \begin{pmatrix}
\left.\begin{matrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & & 1 \\ \mathbf{0} & & & 0 \end{matrix}\right\} k_1 & & & \mathbf{0} \\
& \left.\begin{matrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & & 1 \\ \mathbf{0} & & & 0 \end{matrix}\right\} k_2 & & \\
& & \ddots & \\
\mathbf{0} & & & \left.\begin{matrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & & 1 \\ \mathbf{0} & & & 0 \end{matrix}\right\} k_n
\end{pmatrix} .
$$

Then it is easy to see that there exist matrices $E_\ell^{\#}(\lambda)$ and $F_\ell^{\#}(\lambda)$, holomorphic and invertible near λ_0, such that

$$
\begin{pmatrix}
(\lambda-\lambda_0)^{-1} & (\lambda-\lambda_0)^{-2} & (\lambda-\lambda_0)^{-k_\ell} \\
\ddots & \ddots & (\lambda-\lambda_0)^{-k_\ell+1} \\
& \ddots & \vdots \\
& & (\lambda-\lambda_0)^{-2} \\
\mathbf{0} & & (\lambda-\lambda_0)^{-1}
\end{pmatrix}
$$
$$
= E_\ell^{\#}(\lambda) \begin{pmatrix}
(\lambda-\lambda_0)^{-k_\ell} & & & \mathbf{0} \\
& 1 & & \\
& & \ddots & \\
\mathbf{0} & & & 1
\end{pmatrix} F_\ell^{\#}(\lambda) .
$$

Consequently we can rewrite (2.52) as

$$
E^{\#}(\lambda) \left[\sum_{j=1}^{n} (\lambda-\lambda_0)^{-k_j} P_j + P_0 \right] F^{\#}(\lambda),
$$

$P_i P_j = \delta_{ij} P_i \ \operatorname{tr} P_i = 1, i \neq j$ and $\operatorname{tr} P_0 = q - n$. From this, (2.49), with the exponents satisfying (2.51), easily follows. The same conclusion holds also for $B(\lambda)$, where now $\sum_{j=1}^{\widetilde{n}} \widetilde{k}_j = \widetilde{q} = m_{1-s_0}(R_X)$.

The equality $v_{s_0}(S_X) = m_{s_0}(R_X) - m_{1-s_0}(R_X)$ is now immediate from the definition (2.44) and the identities (2.48) and (2.50):

$$\begin{aligned} v_{s_0}(S_X) &= N_{\lambda_0}(d(s_0)) - N_{\lambda_0}(d(1-s_0)) \\ &= \sum_{j=1}^{n} r_j - \sum_{j=1}^{\widetilde{n}} \widetilde{r}_j = \sum_{j=1}^{n} k_j - \sum_{j=1}^{\widetilde{n}} \widetilde{k}_j \\ &= q - \widetilde{q} = m_{s_0}(R_X) - m_{1-s_0}(R_X)\,. \end{aligned}$$

We assumed so far that $s \neq \frac{1}{2}$. However, as $S_X(s)$ is unitary for $\operatorname{Re} s = \frac{1}{2}$, we have $v_{\frac{1}{2}}(S_X) = 0$ so the formula holds as well ($\frac{1}{2} = 1 - \frac{1}{2}!$). □

Following [17] we now introduce the relative scattering matrix of X, $\mathcal{S}_X(s)$. For $M \geq 0$ and

$$Y^0 \overset{\text{def}}{=} Y_1^0 \sqcup \cdots \sqcup Y_N^0 \tag{2.53}$$

which depends on X through its number of components and the lengths of their necks, we define

$$\begin{aligned} S_{M,Y^0}(s)\colon\ &\mathbf{C}^M \oplus \mathcal{C}^\infty(\partial_F X; \Omega^{\frac{1}{2}}_{\partial_F X} \otimes |N^*\partial_F X|^{-s+\frac{1}{2}}) \\ &\longrightarrow \mathbf{C}^M \oplus \mathcal{C}^\infty(\partial_F X; \Omega^{\frac{1}{2}}_{\partial_F X} \otimes |N^*\partial_F X|^{s-\frac{1}{2}}) \end{aligned} \tag{2.54}$$

as

$$S_{M,Y^0}(s) = \mathrm{Id}_{\mathbf{C}^M} \oplus \bigoplus_{j=1}^{N} S_{Y_j^0}(s)\,,$$

where $S_{Y_j^0}(s)$ is the scattering matrix for Y_j^0 acting on sections of $\Omega^{\frac{1}{2}}_{\partial_F Y_j} \otimes |N^*\partial_F Y_j|^{s-\frac{1}{2}}$ over $\partial_F Y_j$ (we recall that $\partial_F X = \partial_F Y_1 \sqcup \cdots \sqcup \partial_F Y_N$ and the direct sum corresponds to this decomposition).

Definition 2.12. The *relative scattering matrix* of X, $\mathcal{S}_X(s)$, is defined by

$$\mathcal{S}_X(s) = S_{M,Y^0}(s)^{-1} S_X(s)\,. \tag{2.55}$$

We note (see [17, §3]) that $\mathcal{S}_X(s)$ can also be defined using the wave operators, where scattering on Y^0 is considered as a *free* problem. It is not surprising then that $\mathcal{S}_X(s)$ has properties similar to those of scattering matrices in traditional scattering theory:

PROPOSITION 2.13. *The relative scattering matrix $\mathcal{S}_X(s)$ is an operator from $\mathbf{C}^M \oplus \mathcal{C}^\infty(\partial_F X;\Omega^{\frac{1}{2}}_{\partial_F X})$ to itself and it satisfies*

$$\mathcal{S}_X^{FF} = \mathrm{Id} + \mathcal{A}_X^{FF}\,, \quad \mathcal{A}_X^{FF} \in \mathcal{M}(\mathbf{C}, \mathcal{C}^\infty(\partial_F X \times \partial_F X; \Omega^{\frac{1}{2}}(\partial_F X \times \partial_F X)));$$

that is, $\mathcal{A}_X^{FF}$ is a meromorphic function of smoothing operators with poles of finite rank. In particular,

$$\mathcal{S}_X = \mathrm{Id} + \mathcal{A}_X ,$$
$$\mathcal{A}_X \in \mathcal{M}(\mathbf{C}, \mathcal{L}_1(\mathbf{C}^M \otimes L^2(\partial_F X; \Omega^{\frac{1}{2}}(\partial_F X)), \mathbf{C}^M \otimes L^2(\partial_F X; \Omega^{\frac{1}{2}}(\partial_F X)))) ,$$

where $\mathcal{L}_1(\mathcal{H}, \mathcal{H})$ denotes the space of trace class operators on a Hilbert space $\mathcal{H}$.

Proof. The regularity statement is essentially proved in [17] (see §4.4 there) and it also follows from Proposition 2.3 and Lemma 2.6 above. Since $\mathcal{S}_X(s) = S_{M,Y^0}(1-s) S_X(s)$, the meromorphy is clear and we only need to discuss the possible poles of infinite rank at $s \in \mathbf{Z} \setminus \{0\} + \frac{1}{2}$. Since the poles of $S_{M,Y^0}^{FF}(1-s)$ at $s \in \mathbf{Z} \setminus \{0\} + \frac{1}{2}$ are simple (see Lemma 3.3 in [18]), the holomorphy of $\mathcal{S}_X^{FF}(s)$ there follows from Lemma 2.8. Lemma 2.6 shows that the poles of $S_X(s)$ at $\mathbf{N} + \frac{1}{2}$ are also simple and appear only in the S_X^{FF} component. As $S_{M,Y^0}^{FF}(1-s)$ vanishes there, we conclude the holomorphy of $\mathcal{S}_X(s)$ at $s \in \mathbf{N} + \frac{1}{2}$. □

We will now consider the determinant τ_X of the relative scattering matrix which is well defined by Proposition 2.13 and which is used in the definition of the relative scattering phase:

$$\begin{aligned} \tau_X(s) &= \det \mathcal{S}_X(s) , \\ \sigma_X(s) &= \frac{\mathrm{i}}{2\pi} \log \tau_X(s) . \end{aligned} \tag{2.56}$$

The main result of this section is the following:

PROPOSITION 2.14. *If τ_X is defined by* (2.56), *then*

$$\tau_X(s) = \mathrm{e}^{g(s)} \frac{P_X(1-s)}{P_X(s)} \frac{P_{Y^0}(s)}{P_{Y^0}(1-s)} , \tag{2.57}$$

where g is an entire function and $P_V(s) = \prod_{\rho \in \mathcal{R}_V} E(s/\rho, 2)$ are the Weierstrass products over the resonances of V (included according to their multiplicities), $E(z,m) = (1-z)\exp(z + z^2/2 + \cdots + z^m/m)$.

Proof. Because of the bound on the resonance counting function obtained in [18], the Weierstrass products P_X and P_{Y^0} are convergent on $\mathbf{C}$. Since $N \neq 0$ there are no resonances for $\mathrm{Re}\, s = \frac{1}{2}, s \neq \frac{1}{2}$. At $s = \frac{1}{2}$ the formula (2.57) is obvious as the Weierstrass products cancel, and by the unitarity of the scattering matrix $|\tau_X(s)| = 1$ for $\mathrm{Re}\, s = \frac{1}{2}$. Hence we have to show that near any s_0 with $\mathrm{Re}\, s_0 < \frac{1}{2}$

$$\tau_X(s) = (s - s_0)^{-m_{s_0}(R_X) + m_{1-s_0}(R_X) + m_{s_0}(R_{Y^0}) - m_{1-s_0}(R_{Y^0})} \tau_X^\sharp(s_0, s) ,$$

where $\tau_X^\sharp(s_0, s_0) \neq 0$, and which by Proposition 2.11 is the same as

$$\tau_X(s) = (s - s_0)^{-v_{s_0}(S_X) + v_{1 s_0}(S_{M,Y^0})} \tau_X^\sharp(s_0, s) .$$

That in turn is equivalent to showing that

$$-\frac{1}{2\pi\mathrm{i}}\int_{\gamma_{s_0,\varepsilon}}\tau_X(s)^{-1}\tau'_X(s)ds = v_{s_0}(S_X) - v_{1-s_0}(S_{M,Y^0})\,.$$

Since $\mathcal{S}_X = \widetilde{S}_{M,Y^0}^{-1}\widetilde{S}_X$, where $\widetilde{S}_M$, $M = X, Y^0$, is given by (2.45), the left-hand side above is equal to

$$\begin{aligned}
-\frac{1}{2\pi\mathrm{i}}&\int_{\gamma_{s_0,\varepsilon}}\operatorname{tr}\mathcal{S}_X(s)^{-1}\mathcal{S}'_X(s)ds \\
&= -\frac{1}{2\pi\mathrm{i}}\operatorname{tr}\int_{\gamma_{s_0,\varepsilon}}\Big[\mathcal{S}_X(s)^{-1}(\widetilde{S}_{M,Y^0}(s)^{-1}\widetilde{S}'_X(s) \\
&\quad + (\widetilde{S}_{M,Y^0}(s)^{-1})'\widetilde{S}_X(s))\Big]\,ds \\
&= -\frac{1}{2\pi\mathrm{i}}\left[\operatorname{tr}\int_{\gamma_{s_0,\varepsilon}}\widetilde{S}_X(s)^{-1}\widetilde{S}'_X(s)ds\right. \\
&\quad \left. + \operatorname{tr}\int_{\gamma_{s_0,\varepsilon}}\widetilde{S}_X(s)^{-1}\widetilde{S}_{M,Y^0}(s)(\widetilde{S}_{M,Y^0}(s)^{-1})'\widetilde{S}_X(s)ds\right].
\end{aligned}$$

The contribution from the first integral is precisely $v_{s_0}(S_X)$ while that from the second equals

$$\frac{1}{2\pi\mathrm{i}}\operatorname{tr}\int_{\gamma_{s_0,\varepsilon}}\widetilde{S}_{M,Y^0}(s)(\widetilde{S}_{M,Y^0}(s)^{-1})'ds,$$

that is, to $-v_{1-s_0}(S_{M,Y^0})$. We refer to [14] for justification of the formal trace cyclicity argument. □

Remark 2.15. The transplantation method of Bérard [4] provides pairs of nonisometric surfaces of the form (1.1) (allowing $N \neq 0$), X_K and X_H, and a transplantation map $\mathcal{T}$: $\mathcal{C}^\infty(X_H) \to \mathcal{C}^\infty(X_K)$ which intertwines the Laplacians, $\mathcal{T}\Delta_{X_H} = \Delta_{X_K}\mathcal{T}$. The surfaces decompose as $X_K = \sqcup_k X_K^k$ and $X_K = \sqcup_k X_H^k$, where the components are isometric, $X_K^k \simeq X_H^\ell$, for all k, ℓ. The transplantation map $\mathcal{T}$ is given by

$$\mathcal{T}(f)_{|X_K^k} = \sum_\ell t_{k\ell} f_{|X_H^\ell}$$

with the matrix $(t_{k\ell})$ invertible.

Since $\mathcal{T}$ intertwines the resolvents, definitions (2.24) and (2.25) show that it relates the Eisenstein functions of the two surfaces. Consequently by (2.28)–(2.31), $\mathcal{T}$ induces a conjugation between the scattering matrices S_{X_H} and S_{X_K} as well as the relative scattering matrices $\mathcal{S}_{X_H}$ and $\mathcal{S}_{X_K}$. Hence, by Definition 1.9, the scattering phases σ_{X_H} and σ_{X_K} are equal. By Proposition 2.14 the resonant sets $\mathcal{R}_{X_H}$ and $\mathcal{R}_{X_K}$ are also equal.

For $N = 0$, that is, the surfaces have finite volume, iso-scattering pairs were constructed in [4], [64].

3. Estimates on the relative scattering matrix

The purpose of this section is to prove an upper bound on the determinant, $\tau_X(s) = \det \mathcal{S}_X$, of the relative scattering matrix (given in Definition 2.54) away from its poles. More precisely, we will find an entire function g of order 4 such that $g\tau_X$ is entire of order 4. Starting with a convenient representation of the kernel of $\mathcal{S}_X(s) = S_{M,Y^0}(s)^{-1}S_X(s)$, we recall the following definitions, taken essentially from Lemma 4.3 of [18]:

$$\begin{aligned}(3.1)\qquad &\mathbf{E}_j^{\chi_j}(s) : L^2(X, \Omega_X^{\frac{1}{2}}) \to L^2(\partial_F Y_j, \Omega_{Y_j(\infty)}^{\frac{1}{2}} \otimes |N^*\partial Y_j|^s)\\ &\mathbf{E}_j^{\chi_j}(s)u(\xi) = \int_{Y_j} E_{Y_j^0}(s;\xi,y)\chi_j(y)u(y)\,, \quad \chi_j \in \mathcal{C}_c^\infty(Y_j)\,.\end{aligned}$$

Combining this with the definitions of $E_{Y_j^0}$ and Lemmas 2.2 and 2.6, we obtain a description of the 'funnel-to-funnel' (FF) part of the relative scattering matrix:

PROPOSITION 3.1. *Let $L(s_0,s)$ and $\chi_{a,b}^{Y_j}$, $\chi_{a+3,b+3}^Z$ be as in (2.17). Then the (j,j'), $1 \le j,j' \le N$, component of the relative scattering matrix $\mathcal{S}_X(s)$ is given by*

$$\begin{aligned}(3.2)\qquad \left[(\mathcal{S}_X(s) - \mathrm{Id})^{\mathrm{FF}}\right]_{jj'} &= (1-2s)\mathbf{E}_j^{\chi_j}(1-s)(\mathrm{Id} + K(s_0,s))^{-1}\\ &\qquad \times [\Delta_{Y_{j'}}, \chi_{a,b}^{Y_{j'}}]\,{}^t\mathbf{E}_{j'}^{\widetilde{\chi}_{j'}}(s)\,,\end{aligned}$$

where $\widetilde{\chi}_j' \in \mathcal{C}_c^\infty(Y_j)$, $\chi_j \in \mathcal{C}_c^\infty(Y_j)$ and $K(s_0,s) = L(s_0,s)\chi_{a+3,b+3}^Z$.

Proof. We recall (2.14) and its consequence (2.34): The kernel of $(1-2s)^{-1}S_X^{\mathrm{FF}}(s)$ is given by

$$\begin{aligned}(3.3)\qquad &(\beta_\partial)_*\left[\beta^*(x^{-s+\frac{1}{2}}x'^{-s+\frac{1}{2}}R_X(s))_{|T\cap B}\right]\\ &= (\beta_\partial)_*\left[\beta^*(x^{-s+\frac{1}{2}}x'^{-s+\frac{1}{2}}R_{Y^0}(s))_{|T\cap B}\right.\\ &\qquad \left.+ x^{-s+\frac{1}{2}}x'^{-s+\frac{1}{2}}Q(s)F(s_0,s)_{|x=x'=0}\right]\\ &= (1-2s)^{-1}S_{Y^0}(s) + x^{-s+\frac{1}{2}}x'^{-s+\frac{1}{2}}Q(s)F(s_0,s)_{|x=x'=0}\,,\end{aligned}$$

where we denote the kernels and the operator by the same symbols. If $\chi \in \mathcal{C}_c^\infty(\overset{o}{X})$, then

$$(3.4)\qquad (x^{-s+\frac{1}{2}}Q(s)\chi_{|x=0})_j = \mathbf{E}_j^{\chi_j}(s)\,, \quad \chi_j = (1-\chi_{a+1,b+1}^{Y_j})\chi\,.$$

On the other hand

$$\begin{aligned}(3.5)\qquad F(s_0,s) &= -(\mathrm{Id} + L(s_0,s))^{-1}L(s_0,s)\\ &= -\chi_{a+3,b+3}^Z(\mathrm{Id} + L(s_0,s)\chi_{a+3,b+3}^Z)^{-1}L(s_0,s)\,,\end{aligned}$$

since $\chi^Z_{a+3,b+3}L(s_0,s) = L(s_0,s)$. Thus we can take $\chi = \chi^Z_{a+3,b+3}\psi$ in (3.4), where $\psi \in \mathcal{C}^\infty(X)$ is equal to 1 on Y_j and 0 near all the cusps, $\sqcup_{\ell=1}^M X_\ell$. Similarly we obtain

$$(L(s_0,s)x'^{-s+\frac{1}{2}}{}_{|x'=0})_j = -[\Delta_{Y_j}, \chi^{Y_j}_{a,b}]\,{}^t\mathbf{E}^{\chi_j}_j(s)$$

where $\widetilde{\chi}_{j'} \in \mathcal{C}^\infty_{\rm c}(\overset{o}{X})$ is equal to 1 on the support of the coefficients of $[\Delta_{Y_{j'}}, \chi^{Y_{j'}}_{a,b}]$.

This combined with (3.3) gives

$$\begin{aligned}(S^{\rm FF}_X(s))_{jj'} &= (S^{\rm FF}_{M,Y^0}(s))_{jj'} + (1-2s)\mathbf{E}^{\chi_j}_j(s)(\mathrm{Id}+K(s_0,s))^{-1}\\ &\quad\times[\Delta_{Y_{j'}}, \chi^{Y_{j'}}_{a,b}]\,{}^t\mathbf{E}^{\widetilde{\chi}_{j'}}_{j'}(s)\,,\end{aligned}$$

where $K(s_0,s) = L(s_0,s)\chi^Z_{a+3,b+3}$. To obtain (3.2) we note that $(S^{\rm FF}_{M,Y^0}(s))_{jj'} = S_{Y^0_j}(s)\delta_{jj'}$ and that $S_{Y^0_j}(s)^{-1}\mathbf{E}^{\chi_j}_j(s) = S_{Y^0_j}(1-s)\mathbf{E}^{\chi_j}_j(s) = \mathbf{E}^{\chi_j}_j(1-s)$. □

The mixed terms have a similar description. To simplify the notation we introduce the operator

(3.6)
$$\mathbf{e}^{a,b}_i(s) : \mathcal{H}^a \to \mathbf{C}$$
$$\mathbf{e}^{a,b}_i(s)u = \frac{4}{2s-1}\int_{\mathbf{R}_+}[(1-\chi^{X_i}_{a+1,b+1})\chi^Z_{a+3,b+3}]u(r)\sin\left[\mathrm{i}\left(s-\frac{1}{2}\right)r\right]dr\,.$$

We note that this is an exact analogue of (3.1) as the normalized sine function is the generalized eigenfunction for Δ^0_0.

PROPOSITION 3.2. *With the notation of Proposition* (3.1) *and* (3.6),

$$\begin{aligned}\left(\mathcal{S}^{CF}_X(s)\right)_{ij} &= (1-2s)\mathbf{e}^{a,b}_i(s)(\mathrm{Id}+K(s,s_0))^{-1}[\Delta_{Y_j},\chi^{Y_j}_{a,b}]^t\mathbf{E}^{\widetilde{\chi}_j}_j(s),\\ \left(\mathcal{S}^{FC}_X(s)\right)_{ji} &= (1-2s)\mathbf{E}^{\widetilde{\chi}_j}_j(1-s)^t[\Delta_{Y_j},\chi^{Y_j}_{a,b}]\\ &\quad\times(\mathrm{Id}+{}^tK(s,s_0))^{-1t}\mathbf{e}^{a.b}_i(s).\end{aligned} \tag{3.7}$$

Proof. We proceed as in the proof of Proposition 3.1 using (2.35) and the expression for the resolvent given in the proof of Proposition 2.1 (see also §5 of [18]):

$$\begin{aligned}R_X(s) &= (Q(s)+Q_0(s_0))(\mathrm{Id}+L(s,s_0))^{-1}\\ &= Q(s)+Q_0(s_0)+(Q(s)+Q_0(s_0))F(s,s_0).\end{aligned} \tag{3.8}$$

Hence for large r,

$$\begin{aligned}&\mathrm{e}^{(s-\frac{1}{2})r}(1-\chi^{X_i}_{a,b})R_X(s)(1-\chi^{Y_j}_{a,b})x'^{-s+\frac{1}{2}}_{|x'=0}\\ &= \mathrm{e}^{(s-\frac{1}{2})r}(1-\chi^{X_i}_{a+4,b+4})R^0_0(s)(1-\chi^{X_i}_{a+1,b+1})F(s,s_0)(1-\chi^{Y_j}_{a,b})x'^{-s+\frac{1}{2}}_{|x'=0}\\ &= \mathbf{e}^{a,b}_i(s)(\mathrm{Id}+K(s,s_0))^{-1}[\Delta_{Y_j},\chi^{Y_j}_{a,b}]^t\mathbf{E}^{\widetilde{\chi}_j}_j(s)\,,\end{aligned}$$

where we used (2.18) the facts that $(1-\chi^{X_i}_{a+4,b+4})\chi^{Z}_{a+3,b+3}=0$, $\chi^{Z}_{a+3,b+3}F(s,s_0)=F(s,s_0)$. □

The 'cusp-to-cusp' (CC) term has an equally nice description:

PROPOSITION 3.3. *With the notation of Proposition* 3.2,

$$
\begin{aligned}
(3.9)\qquad \left[(\mathcal{S}_X(s)-\mathrm{Id})^{CC}\right]_{ii'} &= \left(S^{CC}_X(s)-\mathrm{Id}_{\mathbf{C}^M}\right)_{ii'} \\
&= (1-2s)\mathbf{e}^{a,b}_i(s)(\mathrm{Id}+K(s,s_0))^{-1} \\
&\quad \times[\Delta^0_{X_i},\chi^{X_i}_{a,b}]^t\mathbf{e}^{a-2,b-2}_i(s)\,,
\end{aligned}
$$

provided a and b are taken sufficiently large.

Proof. We will again use (3.8), now combined with (2.24) and (2.28):

$$
\begin{aligned}
\left(S^{CC}_X(s)\right)_{ii'} &= (1-2s)\lim_{r'\to\infty} \\
&\quad \times \mathrm{e}^{(s-\frac{1}{2})r'}\left[(1-\chi^{X_{i'}}_{a,b})E_X(s;r',i)+(1-2s)^{-1}\delta_{ii'}\mathrm{e}^{(s-\frac{1}{2})r'}\right],
\end{aligned}
$$

where in fact the kernel on the right-hand side is constant in r'. Then using (3.8) and (2.18) (we are also using the symmetry of the kernel of $R_X(s)$) we obtain that this is equal to

$$
\begin{aligned}
(1-2s)\lim_{r'\to\infty}\mathrm{e}^{(s-\frac{1}{2})r'}&\Big[\delta_{ii'}(1-\chi^{X_{i'}}_{a,b})R^0_0(s)\mathrm{e}^{(s-\frac{1}{2})r}(1-\chi^{X_i}_{a,b})_{|r\gg a} \\
&+(1-2s)^{-1}\delta_{ii'}\mathrm{e}^{(s-\frac{1}{2})r'}+(1-\chi^{X_{i'}}_{a,b})R^0_0(s) \\
&\times(1-\chi^{X_{i'}}_{a+1,b+1})F(s,s_0)(1-\chi^{X_i}_{a,b})\mathrm{e}^{(s-\frac{1}{2})r}_{|r\gg a} \\
&-(1-2s)^{-1}\delta_{ii'}\mathrm{e}^{(s-\frac{1}{2})r'}\Big] \\
=(1-2s)\lim_{r'\to\infty}\mathrm{e}^{(s-\frac{1}{2})r'}&\Big[\mathrm{e}^{(\frac{1}{2}-s)r'}\mathbf{e}^{a,b}_{i'}(s)(\mathrm{Id}+K(s,s_0))^{-1}[\Delta^0_{X_i},\chi^{X_i}_{a,b}] \\
&\times R^0_{X_i}(s)(1-\chi^{X_i}_{a+1,b+1})\mathrm{e}^{(s-\frac{1}{2})r}_{|r\gg a}\Big] \\
=(1-2s)\mathbf{e}^{a,b}_{i'}(s)&(I+K(s,s_0))^{-1}[\Delta^0_{X_i},\chi^{X_i}_{a,b}]^t\mathbf{e}^{a-2,b-2}_i(s)\,.
\end{aligned}
$$

By symmetry we have the same formula with i and i' exchanged and that gives (3.9). □

To estimate $\tau_X(s)$ away from its poles we shall use Weyl inequalities following the method introduced by Melrose [32], [33] and developed for hyperbolic scattering in [18]. We recall that if A is a trace class operator with eigenvalues $\{\lambda_k(A)\}_{1\leq k\leq\infty}$, $|\lambda_1(A)|\geq\dots|\lambda_k(A)|\to 0$ then $\det(\mathrm{Id}+A)=\prod_{k=1}^{\infty}(1+\lambda_k(A))$. The characteristic values of A are defined as the eigenvalues of $|A|=(AA^*)^{\frac{1}{2}}$, $\mu_1(A)\geq\mu_2(A)\geq\dots\mu_k(A)\to 0$, say. Then (see [13]),

$$
(3.10)\qquad |\det(\mathrm{Id}+A)|\leq\det(\mathrm{Id}+|A|)\,.
$$

Hence we need to estimate the characteristic values of the left-hand side of (3.2).

LEMMA 3.4. *Let* $\chi \in \mathcal{C}_c^\infty(Y_j), Q \in \operatorname{Diff}^m(X, {}^0\Omega_X^{\frac{1}{2}})$ *and let* $\mathbf{E}_j^\chi$ *be defined by* (3.1). *Then for* $\operatorname{Re} s > \varepsilon$,

$$\mu_k(\mathbf{E}_j^\chi(s)Q) \leq \exp(C\langle s\rangle - k/C),$$

and for $\operatorname{Re} s < \varepsilon$,

$$\mu_k(\mathbf{E}_j^\chi(s)Q) \leq \begin{cases} d(s,\mathcal{R}_{Y_j^0})^{-2} e^{C\langle s\rangle \log\langle s\rangle}, & k \leq 2 \\ e^{C\langle s\rangle + 2\operatorname{Re}(s-\frac{1}{2})\log\langle s\rangle/k}, & 2 \leq k \leq C\langle s\rangle \\ e^{-k/C}, & k > C\langle s\rangle. \end{cases}$$

Here, $d(s,\mathcal{R})$ *denotes the distance between* $s \in \mathbf{C}$ *and the set* $\mathcal{R} \subset \mathbf{C}$.

Proof. The first part of the lemma follows from Lemma 4.3 of [18] and the analyticity of $E_j(s;\xi,z)$ in z. To see the second part, we write

$$\mu_k(\mathbf{E}_j^\chi(s)Q) = \mu_k(S_{Y_j^0}(s)\mathbf{E}_j^\chi(1-s)Q) \leq \mu_{k_1}(S_{Y_j^0}(s))\mu_{k_2}(\mathbf{E}_j^\chi(1-s)Q),$$

$k = k_1 + k_2 - 1$, and then apply Lemmas 4.2 and 4.3 of [18]. □

As in Section 6 of [18] we define $D(s) = \det(\operatorname{Id} + K(s_0, s)^3)$, where $K(s_0, s)$ is as given after (3.2). From Lemmas 6.2 and 6.3 there we recall:

LEMMA 3.5. *The function* $D(s)$ *is meromorphic of order* 2 *and more precisely*

$$D(s) = \frac{h_1(s)}{h_2(s)} \quad , \quad |h_i(s)| \leq e^{C\langle s\rangle^2}, \quad i = 1,2;$$

that is, $D(s)$ *is of finite type.*

Let us denote by $D(z,r)$ the open disk of center z and radius r. We now have:

LEMMA 3.6. *Let* $\mathcal{L}_D^0$ *be the set of zeros of* $D(s)$. *Then for* $s \notin \cup_{\zeta\in\mathcal{L}_D^0\cup\mathcal{R}_{Y^0}} D(\zeta,\langle\zeta\rangle^{-2-\delta}), \delta > 0$

$$\|(\operatorname{Id} + K(s_0,s))^{-1}\|_{L^2(X)\to L^2(X)} \leq \exp(C_\varepsilon\langle s\rangle^{2+\varepsilon}),$$

for all $\varepsilon > 0$.

Proof. We use the estimate ([13, Chap. V, Th. 5.1]):

$$\|(\operatorname{Id} + T)^{-1}\| \leq (\det(\operatorname{Id} + T^3))^{-1} \det(\operatorname{Id} + |T|^3)$$

with $T = K(s_0, s)$. The second term was estimated in Section 6 of [18]; we put

$$g_{Y^0}(s) = \prod_{\rho\in\mathcal{R}_{Y^0}} E\left(\frac{s}{\rho}, 3\right)$$

so that $|g_{Y^0}(s)| \leq e^{C_\varepsilon \langle s\rangle^{2+\varepsilon}}$ (in fact the estimate is valid with $\varepsilon = 0$ but we do not need it here). Then for some P sufficiently large and any $\varepsilon \geq 0$,

$$|g_{Y^0}(s)|^P \det(\mathrm{Id} + |K(s_0, s|^3) \leq e^{C_\varepsilon \langle s\rangle^{2+\varepsilon}}.$$

By applying the minimum modulus theorem (see [61, 8.71]) to the first term on the left-hand side we see that the second term is bounded by $\exp(C_\varepsilon \langle s\rangle^{2+\varepsilon})$ away from

$$\cup_{\zeta \in \cup \mathcal{L}_D^0 \cup \mathcal{R}_{Y^0}} D(\zeta, \langle \zeta\rangle^{-2-\delta}).$$

Lemma 4.2 and an application of the minimum modulus theorem to h_1 give the same bound for $\det(\mathrm{Id} + K(s_0, s)^3)$ away from $\cup_{\zeta \in \mathcal{L}_D^0} D(\zeta, \langle\zeta\rangle^{-2-\delta})$ and the lemma follows. □

The main result of this section is a refinement of Proposition 2.14:

PROPOSITION 3.7. *The determinant of the relative scattering matrix, $\tau_X(s)$, satisfies*

$$\tau_X(s) = e^{q(s)} \frac{P_X(1-s)}{P_X(s)} \frac{P_{Y^0}(s)}{P_{Y^0}(1-s)},$$

where q is a polynomial of degree less than or equal to 4 and where the P_Y are the Weierstrass products over the resonances of Y, as in (2.57).

Proof. From (2.57) it is known that $H(s) = \tau_X(s) P_X(s) P_{Y^0}(1-s)$ is entire and so by Hadamard's factorization theorem (see [61, 8.24]) it suffices to show that it is of order 4. Since the Weierstrass products satisfy the estimates $\exp(C_\varepsilon \langle s\rangle^{2+\varepsilon})$, it is in turn sufficient to show that $|\tau_X(s)|$ is bounded by $\exp(C_\varepsilon \langle s\rangle^{4+\varepsilon})$ for

$$s \notin \mathcal{B} = \bigcup_{\zeta \in \mathcal{L}_D^0 \cup \mathcal{R}_{Y^0} \cup 1 - \mathcal{R}_{Y^0}} D(\zeta, \langle\zeta\rangle^{-2-\delta}), \quad \delta > 0.$$

For that we use (3.10):

$$|\tau_X(s)| \leq \prod_{k=1}^{\infty} (1 + \mu_k(\mathcal{S}_X(s) - \mathrm{Id})). \tag{3.11}$$

To estimate the characteristic values we first note the crude estimate

$$\begin{aligned}\mu_k(\mathcal{S}_X(s) - \mathrm{Id}) \quad &\leq \quad \mu_{[\frac{k}{4}]}((\mathcal{S}_X(s) - \mathrm{Id})^{FF}) + \mu_{[\frac{k}{4}]}(\mathcal{S}_X^{FC}(s)) \\ &\quad + \mu_{[\frac{k}{4}]}(\mathcal{S}_X^{CF}(s)) + \mu_{[\frac{k}{4}]}(\mathcal{S}_X^{CC}(s) - \mathrm{Id}_{\mathbf{C}^M}).\end{aligned}$$

The operators appearing in the last three terms are of rank M and hence have at most M nonzero characteristic values. Propositions 3.2, 3.3 and Lemmas 3.6, 3.4 show that the nonzero characteristic values are bounded by $\exp(C_\varepsilon \langle s\rangle^{2+\varepsilon})$

for $s \notin \mathcal{B}$. For instance,

$$\begin{aligned}\mu_\ell(\mathcal{S}_X^{CF}(s)) &\leq \|\mathcal{S}_X^{FC}(s)\|_{\mathbf{C}^M \to L^2(\partial_F X)} \\ &\leq C_{M,N} \max_{i,j} |1-2s| \|\mathbf{E}_i^{a,b}(s)\|_{L^2(X)\to \mathbf{C}} \\ &\quad \times \|(\operatorname{Id} + K(s,s_0))^{-1}\|_{L^2(X)\to L^2(X)} \\ &\quad \times \|\mathbf{E}_j^{\widetilde{\chi}_j}(s)^t [\Delta_{Y_j}, \chi_{a,b}^{Y_j}]\|_{L^2(\partial_F X)\to L^2(X)} \\ &\leq \langle s\rangle e^{C\langle s\rangle} e^{C\langle s\rangle^{2+\varepsilon}} \langle s\rangle^{2+\delta} e^{C\langle s\rangle \log\langle s\rangle} \leq e^{C\langle s\rangle^{2+\varepsilon}}.\end{aligned}$$

The terms of the 'funnel-to-funnel' (FF) type are more interesting. By Proposition 3.1,

$$\begin{aligned}&\mu_k(S_{M,Y^0}(s)^{-1} S_X(s) - \operatorname{Id}) \qquad (3.12)\\ &\leq \sum_{1\leq j,j'\leq N} \mu_{\left[\frac{k}{N^2}\right]}(\mathbf{E}_j^{\chi_j}(1-s)(\operatorname{Id}+K(s_0,s))^{-1}[\Delta_{Y_{j'}}, \chi_{a,b}^{Y_{j'}}]\, {}^t\mathbf{E}_{j'}^{\widetilde{\chi}_{j'}}(s)).\end{aligned}$$

We now apply Lemmas 3.4 and 3.6 to estimate the characteristic values away from the *bad* set $\mathcal{B}$:

$$\begin{aligned}&\mu_k\left(\mathbf{E}_j^{\chi_j}(1-s)(\operatorname{Id}+K(s_0,s))^{-1}[\Delta_{Y_{j'}}, \chi_{a,b}^{Y_{j'}}]^t\mathbf{E}_{j'}^{\widetilde{\chi}_{j'}}(s)\right) \\ &\leq \min\{\mu_k(\mathbf{E}_j^{\chi_j}(1-s)), \mu_k(\mathbf{E}_{j'}^{\widetilde{\chi}_{j'}}(s)^t[\Delta_{Y_{j'}}, \chi_{a,b}^{Y_{j'}}])\} \\ &\quad \times \max\{\|\mathbf{E}_j^{\chi_j}(1-s)\|_{L^2(X)\to L^2(\partial_F X)}, \\ &\quad \|\mathbf{E}_{j'}^{\widetilde{\chi}_{j'}}(s)^t[\Delta_{Y_{j'}}, \chi_{a,b}^{Y_{j'}}]\|_{L^2(X)\to L^2(\partial_F X)}\} \\ &\quad \times \|(\operatorname{Id}+K(s_0,s))^{-1}\|_{L^2(X)\to L^2(X)} \\ &\leq e^{C\langle s\rangle - k/C}.|s|^{(2+\delta)2} e^{C\langle s\rangle \log\langle s\rangle} e^{C_\varepsilon \langle s\rangle^{2+\varepsilon}} \leq e^{\widetilde{C}_\varepsilon \langle s\rangle^{2+\varepsilon} - k/C}.\end{aligned}$$

Summing up all the terms, we conclude that outside of $\mathcal{B}$,

$$\mu_k(\mathcal{S}_X(s) - \operatorname{Id}) \leq e^{\widetilde{C}_\varepsilon \langle s\rangle^{2+\varepsilon} - k/C}.$$

Hence, by (3.11)

$$|\tau_X(s)| \leq e^{C_\varepsilon \langle s\rangle^{4+\varepsilon}}, \quad s \notin \mathcal{B},$$

for any $\varepsilon > 0$ and consequently $H(s)$ is an entire function of order 4. □

4. Trace formula involving the relative scattering matrix

In this section we will present a trace formula of the Birman-Krein type; see [38, §4.1]. It is a refinement of the formula in [17] and is obtained by using the Maaß-Selberg type relation (in the infinite volume case this technique was used in [44], [48]).

Before proceeding with the proof we have to discuss the singularities of the resolvent which contribute to the trace formula. The only pole of the resolvent and of the generalized eigenfunctions for $\operatorname{Re} s = \frac{1}{2}$ can occur at $s = \frac{1}{2}$ (we recall the assumption that $N \neq 0$ which guarantees the absence of embedded eigenvalues). The following two lemmas describe the precise structure of $R_X(s)$ there.

LEMMA 4.1. *For s near $\frac{1}{2}$, the resolvent satisfies*

$$R_X(s) = \frac{A}{(2s-1)^2} + \frac{B}{2s-1} + C(s)\,,$$

where $C(s)$ is holomorphic near $\frac{1}{2}$, A is a surjection of $\mathcal{H}^a$ onto the eigenspace of Δ_X corresponding to the eigenvalue $\frac{1}{4}$, and $B = \sum_{\ell=1}^{k} \psi_\ell \otimes \psi_\ell$ where

$$\begin{aligned}(4.1)\quad \psi_\ell \in \Big\{ u \in \mathcal{H}^a_{\mathrm{loc}}\colon \Big(\Delta_X - \frac{1}{4}\Big)u = 0\,,\ (1-\chi^{X_i}_{a+3,b+3})u(r) = C_i|dr|^{\frac{1}{2}}, \\ r \gg a_i,\ (1-\chi^{Y_j}_{a+3,b+3})u = R_{Y_j^0}(1/2)h_j\,,\ h_j \in \mathcal{C}^\infty_{\mathrm{c}}(Y_j^0; {}^0\Omega_{Y_j^0})\Big\}.\end{aligned}$$

Proof. For the convenience of the reader, we recall an essentially standard argument (see [38, §2.10] or [23]). We first note that, for $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\overset{o}{X}; {}^0\Omega^{\frac{1}{2}}_X)$,

$$\begin{aligned}(4.2)\qquad \left|\int_X \bar\phi(\Delta_X - s(1-s))\phi\right| &\geq \left|\operatorname{Im}\int_X \bar\phi(\Delta_X - s(1-s))\phi\right| \\ &= 2\left|\operatorname{Re} s - \frac{1}{2}\right| |\operatorname{Im} s|\, \|\phi\|^2\,,\end{aligned}$$

so that for $\operatorname{Re} s > 1/2$, $\|R_X(s)\|_{L^2\to L^2} \leq (2|\operatorname{Re} s - 1/2||\operatorname{Im} s|)^{-1}$ and, consequently, the pole at $s = 1/2$ is at most double. The equation $(\Delta_X - s(1-s))R_X(s) = \mathrm{Id}$ shows that the elements, u, of the ranges of A and B satisfy the equation $(\Delta_X - 1/4)u = 0$. The bound (4.2) shows then that $A : \mathcal{H}^a_{\mathrm{comp}} \to L^2(X; {}^0\Omega^{\frac{1}{2}}_X)$. To see that all L^2 eigenfunctions are in the range of A we observe that, for $\operatorname{Re} s > 1/2$, and $u \in L^2(X; {}^0\Omega^{1/2}_0)$,

$$(4.3)\quad (\Delta_X - s(1-s))u = \Big(\frac{1}{4} - s(1-s)\Big)u \quad\Longrightarrow\quad u = \Big(s - \frac{1}{2}\Big)^2 R_X(s)u\,,$$

and thus u is the range of A. That the range of B is contained in (4.1) follows from the representation of the resolvent in the proofs of Lemma 2.2 and Proposition 2.1 (see also Section 5 of [18]): if $M = Y_j$ or $M = X_i$ then

$$(4.4)\qquad (1-\chi^M_{a+3,b+3})R_X(s) = (1-\chi^M_{a+3,b+3})Q(s)(\mathrm{Id} + F(s_0,s))$$

with the kernel of $F(s_0, s)$ meromorphic in s and compactly supported on the left.

Finally, to see that $B = \sum_{\ell=1}^{k} \psi_\ell \otimes \psi_\ell$ we note that $R_X(s)$ is self-adjoint for $s \in \mathbf{R}$, $s > 1/2$ so that

$$R_X(s) = \frac{A}{(2s-1)^2} + \frac{\sum_{\ell=1}^{k} \psi_\ell(s) \otimes \psi_\ell(s)}{2s-1} + \widetilde{C}(s)\,,$$

where $\psi_\ell(s) \in L^2(X;\, {}^0\Omega_X^{\frac{1}{2}})$ for $s > 1/2$ and $\psi_\ell(1/2) = \psi_\ell$. □

Only the term B will contribute to the generalized eigenfunctions as can be seen from the next result:

LEMMA 4.2. *If* $u \in L^2(X; {}^0\Omega_X^{\frac{1}{2}})$ *satisfies* $(\Delta_X - \frac{1}{4})u = 0$ *then* $u \in \dot{\mathcal{C}}^\infty(X; {}^0\Omega^{\frac{1}{2}})$ *and in particular,*

$$u = \widetilde{u} \left|\frac{dx}{x}\right|^{\frac{1}{2}} \left|\frac{dy}{x}\right|^{\frac{1}{2}}, \quad \widetilde{u}_{|\partial_F X} = 0\,.$$

Proof. From (4.4) and (4.3) we see that the behaviour of $u \in L^2(X;\, {}^0\Omega_0^{\frac{1}{2}})$ satisfying $(\Delta_X - 1/4)u = 0$ is given, modulo $\mathcal{O}(x^\infty)$, by the behaviour of $R_{Y_j}(1/2)$ and $R_{X_j^0}(1/2)$. Either from the explicit analysis of the model resolvents (see Appendix A to [18]) or from the general regular singular point analysis (see [37]), the L^2 requirement gives the vanishing of all the terms in the expansion of u at $x = 0$ and of the zero mode at the cusp (that is of $(1 - \chi_{a,b}^{X_i})u$). Hence u has to vanish to infinite order at $x = 0$ and consequently $u \in \dot{\mathcal{C}}^\infty(X;\, {}^0\Omega_X^{\frac{1}{2}})$. □

These lemmas have the following consequence for the scattering matrix:

LEMMA 4.3. *For s close to $\frac{1}{2}$,*

$$\mathcal{S}_X(s) = I + C_X + (2s-1)T_X(s)\,, \tag{4.5}$$

where $T_X(s)$ is holomorphic near $s = \frac{1}{2}$, C_X is a finite rank operator and

$$\frac{1}{2}\operatorname{tr} C_X = \dim\left\{u \in L^2(X; {}^0\Omega_X^{\frac{1}{2}}) : \left(\Delta_X - \frac{1}{4}\right)u = 0\right\} - m_{\frac{1}{2}}(R_X)\,. \tag{4.6}$$

Proof. We first note that the line bundle $|N^*\partial_F X|^{s-\frac{1}{2}}$ is trivial at $s = \frac{1}{2}$ and that $S_{Y^0,M}(1/2) = \mathrm{Id}$ on $\mathbf{C}^M \oplus L^2(\partial_F Y^0; \Omega_{\partial_F Y^0}^{\frac{1}{2}})$. Hence from (ii) of Proposition 2.5 we conclude that $\mathcal{S}_X(1/2) = \mathcal{S}_X(1/2)^{-1} = \mathcal{S}_X(1/2)^*$. Thus, if $C_X = \mathcal{S}_X(1/2) - \mathrm{Id}$ then $C_X(C_X + 2) = 0$, $C_X^* = C_X$ and, by Proposition 2.13, C_X is compact. Consequently,

$$C_X = -2P\,, \quad P^2 = P = P^*\,, \quad \operatorname{rank} P < \infty\,. \tag{4.7}$$

We now want to show that

$$\operatorname{rank} P = m_{\frac{1}{2}}(R_X) - \operatorname{rank} A\,, \tag{4.8}$$

where A is as in Lemma 4.1. From the definitions of E_X, (2.24), (2.25), and Lemma 4.1 we have that

$$E_X(s;z,\xi) = \frac{E_X^\sharp(z,\xi)}{2s-1} + E_X^\flat(s;z,\xi)\,,$$

where $E_X^\flat$ is holomorphic in s near $1/2$ and $E_X^\sharp(z,\xi) = \sum_{\ell=1}^k \psi_\ell^\sharp(\xi)\psi_\ell(z)$ with

$$\psi_\ell^\sharp(\xi) = \begin{cases} (1-\chi_{a,b}^{X_i})\psi_{\ell|r\gg a}\,, & \xi = i\,, \\ (1-\chi_{a,b}^{Y_j})\psi_\ell(z)_{|x=0}\,, & z=(x,\xi)\,, \xi\in\partial_F Y^j\,. \end{cases}$$

By Lemma 4.2 and its proof, the dimension of $\operatorname{span}\{\psi_\ell^\sharp\} \subset \mathbf{C}^M \oplus L^2(\partial_F X;\Omega^{\frac{1}{2}}_{\partial_F X})$ is equal to $m_{\frac{1}{2}}(R_X) - \operatorname{rank} A$. On the other hand the asymptotic expansions (2.28)– (2.31) (and the fact that $S_{Y^0,M} = \mathrm{Id}$) show that $C_X = \sum_{\ell=1}^k \psi_\ell^\sharp \otimes \psi_\ell^\sharp$. Hence

$$\operatorname{rank} C_X = \dim \operatorname{span}\{\psi_\ell^\sharp\} = m_{\frac{1}{2}}(R_X) - \operatorname{rank} A\,.$$

This proves (4.8) and by (4.7) also proves (4.6). □

We start the discussion of the trace formula by normalizing the trace near the cusps. Let us first assume (here only) that $N=0$ and $M>0$ (that is, X is a finite volume surface). For a one-form $\omega \in \mathcal{C}^\infty(X,\Omega_X^1)$, thought of as a one-density, we define

$$ {}^b\!\!\int_X \omega = \mathrm{FP}_{\varepsilon\to 0}\left[\sum_{i=1}^M \int_{r\le \log\varepsilon^{-1}} (1-\chi_a^{X_i})\omega + \int_X \chi_a^Z \omega\right]\,,$$

for a large, and where FP denotes the finite part (if it exists); this is a regularized integral in the sense of Hadamard. We note that, for ω bounded, $\chi_a^Z\omega \in \mathcal{C}^\infty(X,\Omega_X^1)\cap L^1(X,\Omega_X^1)$ and $(1-\chi_a^{X_i})\omega \in \mathcal{C}^\infty(X_i,\Omega^1_{X_i}) \simeq \mathcal{C}^\infty(\mathbf{R}_+,\Omega^1_{\mathbf{R}^+})$. The definition of the b-integral, ${}^b\!\!\int$, depends on the choice of the variable r. It coincides with the b-integral of [37], since the Euclidean metric on $\mathbf{R}_+$ is a b-metric after a logarithmic change of variables and the dependence on the 1-jet of the defining function of infinity can be seen as in that case.

For an operator acting on half-densities and with a smooth Schwartz kernel, the b-trace is defined by taking the b-integral of the Schwartz kernel on the diagonal. Since for $\rho\in\mathcal{C}_c^\infty(\mathbf{R})$, $\langle \cos\bullet\sqrt{\Delta_X - \frac{1}{4}}, \rho(\bullet)\rangle$ is such an operator, we define the regularized distributional trace as

$$\langle \widetilde{\operatorname{tr}} \cos\bullet\sqrt{\Delta_X - \frac{1}{4}}, \rho(\bullet)\rangle = \text{b-tr}\langle \cos\bullet\sqrt{\Delta_X - \frac{1}{4}}, \rho(\bullet)\rangle\,.$$

It is quite easy to check (see for instance [59, §3]) that

$$\cos t\sqrt{\Delta_X - \frac{1}{4}} - \pi_a^* \bigoplus_{i=1}^{M} \cos t\sqrt{\Delta^0_{X_{i0}} - \frac{1}{4}}\pi_a \in \mathcal{D}'(\mathbf{R}; \mathcal{L}^1(\mathcal{H}_a, \mathcal{H}_a)),$$

so that

$$\begin{aligned}\operatorname{tr}&\left(\cos t\sqrt{\Delta_X - \frac{1}{4}} - \pi_a^* \bigoplus_{i=1}^{M} \cos t\sqrt{\Delta^0_{X_{i0}} - \frac{1}{4}}\pi_a\right)\\ &= \widetilde{\operatorname{tr}} \cos t\sqrt{\Delta_X - \frac{1}{4}} - \sum_{i=1}^{M} \widetilde{\operatorname{tr}}\left(\mathbf{1}_{r>a}\cos t\sqrt{\Delta^0_0 - \frac{1}{4}}\right)\\ &= \widetilde{\operatorname{tr}} \cos t\sqrt{\Delta_X - \frac{1}{4}} + \frac{M}{2}\mathbf{1}_{|t|>2a},\end{aligned}$$

which relates $\widetilde{\operatorname{tr}}$ to more standard regularizations, such as those given in Section 2 of [41] for instance. We recall that Δ^0_0 given by (2.7) is the shifted Dirichlet Laplacian on $\mathbf{R}_+$. Consequently

$$\cos t\sqrt{\Delta^0_0 - \frac{1}{4}} = \frac{1}{2}[\delta(t-(x-y)) + \delta(t+(x-y)) - \delta(t-(x+y)) - \delta(t+(x+y))].$$

When $N \neq 0$ and K is an operator acting on 0-half-densities, $\mathcal{C}^\infty(X, \Omega_0^{\frac{1}{2}})$, and such that

$$(1-\pi_a)K \in \mathcal{L}^1(\mathcal{H}^a, \mathcal{H}^a),$$

then we define

$$\widetilde{\operatorname{tr}}\, K = \operatorname{tr}(1-\pi_a)K + \operatorname{b-tr}\pi_a K, \tag{4.9}$$

where the b-trace is defined as above, using the variable r appearing in (1.2). To normalize the trace of the wave group near the funnels we shall first use the same procedure as in the Euclidean case (see [38] and [59, Lemma 3.1]). For that we need a reference problem, which should also be explicitly computable. For $j = 1, \dots, N$, let U_j be a small collar neighbourhood of ∂Y_j in X, with a smooth boundary ∂U_j and with $U_j, U_{j'}$ disjoint for j, j' distinct. We can now use any smooth identification of $U_j \cup Y_j$ with the hyperbolic half-cylinder Y_j^0 given by (1.4); see Figure 3.

Hence we can consider the union Y^0 of the model hyperbolic ends, defined in (2.53), as a subset of X, with a smooth boundary of N components where the hyperbolic metric on Y^0 coincides with the metric on X when restricted to $\bigcup_{j=1}^N Y_j$. The wave group of the Dirichlet Laplacian on the disjoint union Y^0 of the half-cylinders is then our *free* wave group (replacing $U_0(t)$ of [38] and [59]).

LEMMA 4.4. *If $Y^0 \subset X$ is given by (2.53) then for $\varphi \in \mathcal{C}^\infty_{\rm c}(\mathbf{R})$,*

$$(1-\pi_a)\left\langle \cos \bullet\sqrt{\Delta_X - \frac{1}{4}} - {}^t\mathbf{1}_{Y^0}\cos\bullet\sqrt{\Delta^0_{Y^0} - \frac{1}{4}}\mathbf{1}_{Y^0}, \varphi(\bullet)\right\rangle \in \mathcal{L}^1(\mathcal{H}^a, \mathcal{H}^a),$$

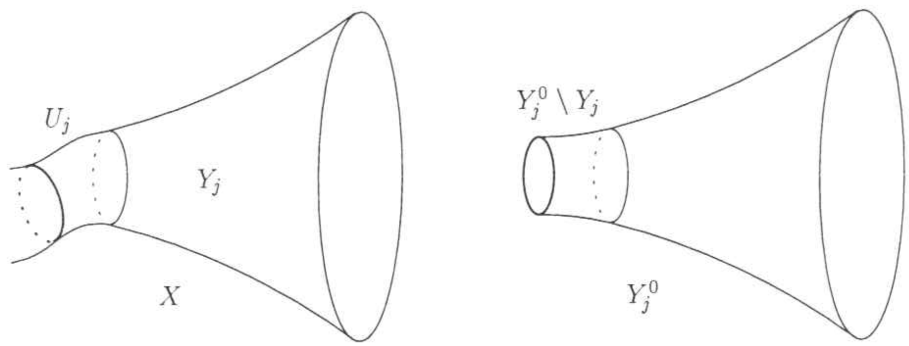

FIGURE 3. The funnel Y_j in X:$U_j \simeq Y_j^0 \setminus Y_j$ as a subset of the model half-cylinder Y_j^0

where the $\Delta^0_{Y^0}$ denote Dirichlet Laplacians on the union Y^0 of hyperbolic half-cylinders.

Proof. The argument is the same as in the Euclidean case (see for instance §3 of [56]); the smoothness of the kernel follows from the propagation of singularities and the trace class property from the finite speed of propagation. □

Using Lemma 4.4 and the definition (4.9) of the regularized trace, $\widetilde{\mathrm{tr}}$, we can now define the regularized distributional wave trace of X with $N \neq 0$:

$$u_X(t) = \widetilde{\mathrm{tr}} \left(\cos t \sqrt{\Delta_X - \frac{1}{4}} - {}^t\mathbf{1}_{Y^0} \cos t \sqrt{\Delta^0_{Y^0} - \frac{1}{4}} \mathbf{1}_{Y^0} \right) . \tag{4.10}$$

The main result of this section relates this wave trace to the relative scattering phase:

PROPOSITION 4.5. *Let $u_X(t)$ be the regularized wave trace given by* (4.10), *let $\sigma_X(s)$ be the relative scattering phase defined in* (2.56) *and put*

$$\widetilde{\sigma}_X(\lambda) = \sigma_X(1/2 + \mathrm{i}\lambda) .$$

Then

$$u_X(t) = \frac{1}{2} \widehat{\frac{d\widetilde{\sigma}_X}{d\lambda}}(t) + \sum_{\mathrm{Re}\, s \geq \frac{1}{2}} m_s(R_X) \cosh t \left(s - \frac{1}{2} \right) , \tag{4.11}$$

in the sense of tempered distributions on $\mathbf{R}$, *where the divisor $m_s(R_X)$ is defined by* (2.44).

Proof. By Lemma 4.1, the term $\sum_{\mathrm{Re}\, s > \frac{1}{2}} m_s(R_X) \cosh t(s - 1/2) + \mathrm{rank}(A)$ in (4.11) corresponds to the discrete spectrum. We need to evaluate the contribution of the continuous spectrum given by

$$u_X^{\mathrm{c}}(t) = \widetilde{\mathrm{tr}}\left(P_X^{\mathrm{c}} \cos t\sqrt{\Delta_X - \frac{1}{4}} - {}^t\mathbf{1}_{Y^0} \cos t\sqrt{\Delta_{Y^0}^0 - \frac{1}{4}}\mathbf{1}_{Y^0}\right), \tag{4.12}$$

where P_X^{c} denotes the spectral projection on the continuous part of Δ_X.

Let us recall that through functional calculus we have, for $\varphi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbf{R})$ and $M = X, Y^0$,

$$\begin{aligned} &\int \varphi(t) P_M^{\mathrm{c}} \cos t\sqrt{\Delta_M - \frac{1}{4}}dt \\ &\quad = -\frac{1}{4\mathrm{i}\pi}\int_{\frac{1}{2}+\mathrm{i}\mathbf{R}} \widehat{\varphi}\left(\mathrm{i}\left(\frac{1}{2} - s\right)\right)(2s-1)[R_M(s) - R_M(1-s)]ds. \end{aligned} \tag{4.13}$$

The operators $R_M(s) - R_M(1-s)$, $M = X, Y^0$, have smooth kernels and the proof of Lemma 2.2 shows that

$$\mathcal{R}_X(s) = R_X(s) - R_X(1-s) - {}^t\mathbf{1}_{Y^0}[R_{Y^0}(s) - R_{Y^0}(1-s)]\mathbf{1}_{Y^0}$$

is a trace class operator for $\mathrm{Re}\, s = 1/2$ and $s \neq 1/2$. We will first show that

$$\begin{aligned} (2s-1)\,\widetilde{\mathrm{tr}}\,\mathcal{R}_X(s) &= \partial_s \log\det \mathcal{S}_X(s) \\ &\quad + 4\pi\mathrm{i}\left(m_{\frac{1}{2}}(R_X) - \mathrm{rank}\, A\right)\delta_0(s) \end{aligned} \tag{4.14}$$

in the sense of distributions on $\frac{1}{2} + \mathrm{i}\mathbf{R}$. Thus we will consider, for $\psi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbf{R})$,

$$T_X^{\mathrm{c}}(\psi) = \frac{1}{4\mathrm{i}\pi}\left[\int_{\frac{1}{2}+\mathrm{i}\mathbf{R}} \psi\left(\mathrm{i}\left(\frac{1}{2} - s\right)\right)(2s-1)\,\widetilde{\mathrm{tr}}\,\mathcal{R}_X(s)ds\right]. \tag{4.15}$$

The formula (4.13) is the same as $\langle u_X^{\mathrm{c}}, \varphi\rangle = T_X^{\mathrm{c}}(\widehat{\varphi})$.

Let us introduce the integral

$$\begin{aligned} I_X(s,\varepsilon) &= (1-2s)\int_{X_\varepsilon}\left[\sum_{i=1}^M E_X(s;z,i)E_X(1-s;z,i)\right. \\ &\qquad \left. + \int_{\partial_F X} E_X(s;z,\xi)E_X(1-s;z,\xi)\right], \end{aligned}$$

where the compact manifold X_ε is defined by

$$X_\varepsilon = \{r_i \leq \log\varepsilon^{-1}, i = 1,\dots,M; x_j \geq \varepsilon, j = 1,\dots,N\}.$$

Then, by (2.26) and the definition of the regularized trace $\widetilde{\mathrm{tr}}$, we have

$$T_X^{\mathrm{c}}(\psi) = \mathrm{FP}_{\varepsilon\to 0^+}\frac{1}{4\mathrm{i}\pi}\int_{\frac{1}{2}+\mathrm{i}\mathbf{R}} \psi\left(\mathrm{i}\left(\frac{1}{2} - s\right)\right)(2s-1)\left[I_X(s,\varepsilon) - I_{Y^0}(s,\varepsilon)\right]ds. \tag{4.16}$$

As far as the cusp contribution in the integral $I_X(s,\varepsilon)$ is concerned, it is transformed using the method of Maaß-Selberg almost directly (and in fact might be best thought of in terms of one-dimensional scattering); the *funnel part* is studied by essentially the same method which in higher dimensional scattering appeared in the work of Buslaev [5]. Denoting $\omega(s,t) = (s+t)(1-s-t) - s(1-s)$, we then have:

$$\begin{aligned}
I_X(s,\varepsilon) &= \lim_{t\to 0^+}(1-2s)\int_{X_\varepsilon}\Big[\sum_{i=1}^{M} E_X(s+t;z,i)E_X(1-s;z,i) \\
&\quad + \int_{\partial_F X} E_X(s+t;z,\xi)E_X(1-s;z,\xi)\Big] \\
&= \lim_{t\to 0^+}\frac{1-2s}{\omega(s,t)}\int_{X_\varepsilon}\Big[\sum_{i=1}^{M}\Delta E_X(s+t;z,i)E_X(1-s;z,i) \\
&\quad + \int_{\partial_F X}\Delta E_X(s+t;z,\xi)E_X(1-s;z,\xi) \\
&\quad - \sum_{i=1}^{M} E_X(s+t;z,i)\Delta E_X(1-s;z,i) \\
&\quad - \int_{\partial_F X} E_X(s+t;z,\xi)\Delta E_X(1-s;z,\xi)\Big] \\
&= \lim_{t\to 0^+}\frac{1-2s}{\omega(s,t)}\int_{\partial X_\varepsilon}\Big[\sum_{i=1}^{M}\partial_\nu E_X(s+t;z,i)E_X(1-s;z,i) \\
&\quad + \int_{\partial_F X}\partial_\nu E_X(s+t;z,\xi)E_X(1-s;z,\xi) \\
&\quad - \sum_{i=1}^{M} E_X(s+t;z,i)\partial_\nu E_X(1-s;z,i) \\
&\quad - \int_{\partial_F X} E_X(s+t;z,\xi)\partial_\nu E_X(1-s;z,\xi)\Big] \\
&= \int_{\partial X_\varepsilon}\Big[\sum_{i=1}^{M}\partial_\nu\partial_s E_X(s;z,i)E_X(1-s;z,i) \\
&\quad + \int_{\partial_F X}\partial_\nu\partial_s E_X(s;z,\xi)E_X(1-s;z,\xi) \\
&\quad - \sum_{i=1}^{M}\partial_s E_X(s;z,i)\partial_\nu E_X(1-s;z,i) \\
&\quad - \int_{\partial_F X}\partial_s E_X(s;z,\xi)\partial_\nu E_X(1-s;z,\xi)\Big]
\end{aligned}$$

where ∂_ν is the Riemannian normal derivative. In particular, near $\partial_F X$, the defining function of infinity, x, can be chosen so that $\partial_\nu = x\partial_x$.

We will now use the asymptotics of the Eisenstein forms $E_X(s;z,\xi)$ described after Proposition 2.5 to study the asymptotics in small ε of $I_X(s,\varepsilon) - I_{Y^0}(s,\varepsilon)$. Let us first assume for notational simplicity that $M = 0$ and $N = 1$.

We then have the following lemma which can be easily modified to the general case:

LEMMA 4.6. *Let us assume that* $M = 0$ *and* $N = 1$. *Then, in the notation of* (2.31) *and Proposition* (2.5), *for* $Y = Y_1$, $\operatorname{Re} s = \frac{1}{2}$,

$$
\begin{aligned}
(4.17)\qquad (1-\chi^Y_{a,b})E_X(s;(x,\eta),\xi) &= (1-\chi^Y_{a,b})E_{Y^0}(s;(x,\eta),\xi) \\
&- \frac{x^{s-\frac{1}{2}}}{2s-1} S_{Y^0}(s)\mathcal{A}^{FF}(s)(\xi,\eta)|d\xi|^{\frac{1}{2}}|d\eta|^{\frac{1}{2}}|dx|^{\frac{1}{2}-s}\left|\frac{dx}{x}\right|^{\frac{1}{2}} \\
&+ \frac{1}{2s-1} R(s;(x,\eta),\xi)\,,
\end{aligned}
$$

where R *is holomorphic in* s *in a neighbourhood of* $\operatorname{Re} s = \frac{1}{2}$ *and*

$$
(4.18)\qquad R \in x^{s+\frac{1}{2}}\mathcal{C}^\infty(Y^0 \times \partial_F Y^0; {}^0\Omega^{\frac{1}{2}}_{Y^0\times\partial_F Y^0} \otimes |N^*\partial_F Y|^{s-\frac{1}{2}})\,.
$$

Proof. As in the proof of Lemma 4.2 this follows from the regular singular nature of the equation satisfied by E_X at $x = 0$. In dimension two, rather than to resort to the general methods of [30] or [37], we can proceed directly by expanding in Fourier modes of $\partial_F Y \simeq S^1$. Since for $\operatorname{Re} s = \frac{1}{2}$, $s \neq \frac{1}{2}$, the roots of the indicial equation do *not* differ by an integer, the solution $E_X - E_{Y^0}$ in Y has an expansion at $x = 0$ and its leading term determines that expansion up to terms in $\dot{\mathcal{C}}^\infty$. For $s = \frac{1}{2}$,

$$
E_X(s;(x,\eta),\xi) = \frac{E^\sharp_X((x,\eta),\xi)}{2s-1} + E^\flat_X(s;(x,\eta),\xi)
$$

where now

$$
E^\sharp_X((x,\eta),\xi) = -C_X(\xi,\eta)|d\xi|^{\frac{1}{2}}|d\eta|^{\frac{1}{2}}\left|\frac{dx}{x}\right|^{\frac{1}{2}} + R^\sharp((x,\eta),\xi)\left|\frac{dx}{x}\right|^{\frac{1}{2}},
$$
$$
\left(\Delta_X - \frac{1}{4}\right)E_X(\bullet,\xi) = 0\,,
$$

and

$$
\begin{aligned}
E^\flat_X(s;(x,\eta),\xi) &= E_{Y^0}(s;(x,\eta),\xi) - x^{s-\frac{1}{2}}S_{Y^0}(s)T_X(s)(\xi,\eta)|d\eta|^{\frac{1}{2}}|dx|^{\frac{1}{2}-s}\left|\tfrac{dx}{x}\right|^{\frac{1}{2}} \\
&\quad + R^\flat(s;(x,\eta),\xi)|d\eta|^{\frac{1}{2}}\left|\tfrac{dx}{x}\right|^{\frac{1}{2}},
\end{aligned}
$$
$$
\left(\Delta_X - \frac{1}{4}\right)E^\flat_X(1/2;\bullet,\xi) = 0\,,
$$

where $T_X(s)$ is as in (4.5). From the characterization (2.31) we obtain the fact that $R^\sharp$ and $R^\flat(1/2)$ tend to 0 as x tends to 0. Hence they contribute to lower order terms in the asymptotic expansion of the solution at $x = 0$. □

Still in the setting where $N = 1$ and $M = 0$, we will use Lemma 4.6 to simplify $I_X(s,\varepsilon) - I_{Y^0}(s,\varepsilon)$. We first observe that the integrals

$$\frac{1}{2s-1}\int_{\partial X_\varepsilon}\int_{\partial_F X} \partial_\nu \partial_s E_{Y^0}(s;z,\xi) R(s;z,\xi)\,,$$
$$\frac{1}{2s-1}\int_{\partial X_\varepsilon}\int_{\partial_F X} \partial_s E_{Y^0}(s;z,\xi) \partial_\nu R(s;z,\xi)\,,$$
$$\frac{1}{2s-1}\int_{\partial X_\varepsilon}\int_{\partial_F X} \partial_\nu \partial_s R(s;z,\xi) E_{Y^0}(s;z,\xi)\,,$$
$$\frac{1}{2s-1}\int_{\partial X_\varepsilon}\int_{\partial_F X} \partial_s R(s;z,\xi) \partial_\nu E_{Y^0}(s;z,\xi)\,,$$
$$\frac{1}{(2s-1)^2}\int_{\partial X_\varepsilon}\int_{\partial_F X} \partial_\nu \partial_s R(s;z,\xi) R(s;z,\xi)\,,$$
$$\frac{1}{(2s-1)^2}\int_{\partial X_\varepsilon}\int_{\partial_F X} \partial_s R(s;z,\xi) \partial_\nu R(s;z,\xi)\,,$$
$$\frac{1}{(2s-1)^2}\int_{\partial X_\varepsilon}\int_{\partial_F X} \partial_\nu R(s;z,\xi) E_{Y^0}(s;z,\xi)\,,$$
$$\frac{1}{(2s-1)^2}\int_{\partial X_\varepsilon}\int_{\partial_F X} R(s;z,\xi) \partial_\nu E_{Y^0}(s;z,\xi)\,,$$

do not contribute when the limit (4.16) is taken (when $M = 0$, we can drop the FP there). In view of (4.18), this is clear for the first four terms. For the last four, we note that the integral in (4.16) can be replaced by the principal value integral at $s = 1/2$ and again we have no contribution as $\varepsilon \to 0$. Hence, the last term in (4.17) can be neglected for the asymptotic evaluation of $I_X(s,\varepsilon) - I_{Y^0}(s,\varepsilon)$. All the terms involving E_{Y^0} only can also be dropped as they cancel with the terms in $I_{Y^0}(s,\varepsilon)$. Finally, we note that in the only surviving terms, that is involving *both* E_{Y^0} and $(2s-1)^{-1}S_{Y^0}(s)\mathcal{A}^{FF}(s)$, we can replace E_{Y^0} by the leading terms of its expansion (2.31): $(2s-1)^{-1} \cdot (x^{-s+\frac{1}{2}}\delta_\xi(\eta) - x^{\frac{1}{2}-s}S_{Y^0}(s)(\xi,\eta))$, where we dropped the density factors. In fact, since the remainder in (2.31) tends to 0 as $\varepsilon \to 0$ in the distributional sense on $\partial_F Y$, the smoothness of the kernel of $S_{Y^0}(s)\mathcal{A}^{FF}(s)$ shows that the terms involving $S_{Y^0}\mathcal{A}^{FF}$ and $\psi_{\xi,s}(x,\eta)$ disappear in the limit $\varepsilon \to 0$. In the arguments above, we use the compactness of the support of ψ as we did not discuss the uniform dependence on s of the remainder terms.

Thus, in the analysis of $I_X(s,\varepsilon) - I_{Y^0}(s,\varepsilon)$, we can use the formal approximation

$$\begin{aligned} E_M(s;(x,\eta),\xi) \quad \equiv \quad & (2s-1)^{-1}\left\{x^{-s+\frac{1}{2}}\delta_\xi(\eta)|d\eta|^{\frac{1}{2}}|d\xi|^{\frac{1}{2}}|dx|^{s-\frac{1}{2}}\right. \\ & \left. - x^{s-\frac{1}{2}}\left[S_X(s)(\xi,\eta)|d\eta|^{\frac{1}{2}}|d\xi|^{\frac{1}{2}}|dx|^{2s-1}\right]|dx|^{\frac{1}{2}-s}\right\}\left|\frac{dx}{x}\right|^{\frac{1}{2}}, \end{aligned}$$

with $M = X, Y^0$. To simplify the notation we will now put

$$F_M = (2s-1)E_M(s)\,,\ F_M^- = F_M(1-s)\,,\ S_M = S_M(s)\,,\ S_M^- = S_M(1-s)$$

so that

$$I_M(s,\varepsilon) = \frac{1}{(2s-1)^2}\int_{\partial M_\varepsilon}\int_{\partial_F M} \partial_\nu \partial_s F_M F_M^- - \partial_s F_M \partial_\nu F_M^-\,,$$

and, dropping the density factors,

$$\begin{aligned} F_M &\equiv \delta_\xi(\eta)x^{-s+\frac{1}{2}} - S_M(\xi,\eta)x^{s-\frac{1}{2}},\\ F_M^- &\equiv \delta_\xi(\eta)x^{s-\frac{1}{2}} - S_M^-(\xi,\eta)x^{-s+\frac{1}{2}}\,. \end{aligned}$$

The integrand in $I_X(s,\varepsilon) - I_{Y^0}(s,\varepsilon)$ can be rewritten as

$$\begin{aligned} &\partial_\nu\partial_s F_X F_X^- - \partial_s F_X \partial_\nu F_X^- - (\partial_\nu\partial_s F_Y F_Y^- - \partial_s F_Y \partial_\nu F_Y^-)\\ &\quad = \partial_\nu\partial_s F_X(F_X^- - F_Y^-) + \partial_\nu\partial_s[F_X - F_Y]F_Y^-\\ &\qquad + \partial_s[F_Y - F_X]\partial_\nu F_X^- + \partial_s F_Y \partial_\nu[F_Y^- - F_X^-] \end{aligned}$$

which when we use the approximation above and the identification of ∂X_ε with ∂Y_ε^0 and of $\partial_F X$ with $\partial_F Y^0$, gives (when $\partial_\nu = x\partial_x$):

$$\begin{aligned} I_X(s,\varepsilon) - I_{Y^0}(s,\varepsilon) \ &\equiv\ \frac{1}{(2s-1)^2}\int_{\partial X_\varepsilon}\int_{\partial_F X}\Big[(1-2s)\Big(\partial_s S_X(\xi,\eta)S_X^-(\xi,\eta)\\ &\qquad - \partial_s S_{Y^0}(\xi,\eta)S_{Y^0}^-(\xi,\eta)\Big)\\ &\qquad - \varepsilon^{1-2s}\delta_\xi(\eta)\Big(S_{Y^0}^-(\xi,\eta) - S_X^-(\xi,\eta)\Big)\\ &\qquad - \varepsilon^{2s-1}\delta_\xi(\eta)\left(S_X(\xi,\eta) - S_{Y^0}(\xi,\eta)\right)\Big]\\ &=\ \frac{1}{2s-1}\Big[\operatorname{tr}\Big[\partial_s S_X S_X^- - \partial_s S_{Y^0} S_{Y^0}^-\Big]\\ &\qquad - \frac{1}{1-2s}\Big[\varepsilon^{2s-1}\operatorname{tr}(S_X - S_{Y^0}) - \varepsilon^{1-2s}\operatorname{tr}(S_X^- - S_{Y^0}^-)\Big]\Big]. \end{aligned}$$

Here we used the symmetry of the kernel of S_M, $M = X, Y^0$, and Proposition 2.13 to obtain the needed trace class properties.

Going back to (4.16) we obtain

$$\begin{aligned} T_X^{\mathrm{c}}(\psi) \ &=\ \frac{1}{4\pi \mathrm{i}}\int_{\frac{1}{2}+\mathrm{i}\mathbf{R}} \psi\left(\mathrm{i}\left(\frac{1}{2}-s\right)\right)\operatorname{tr}\\ &\qquad \times\Big[\partial_s S_X(s)S_X(1-s) - \partial_s S_{Y^0}(s)S_{Y^0}(1-s)\Big]ds\\ &\qquad - \frac{1}{2}\operatorname{tr}\left(S_X\left(\frac{1}{2}\right) - \mathrm{Id}\right)\widehat{\psi}(0), \end{aligned} \tag{4.19}$$

where the last term comes from writing

$$\operatorname{tr}(S_X(s) - S_Y(s)) = \operatorname{tr}\left(S_X\left(\frac{1}{2}\right) - \mathrm{Id}\right) + (2s-1)G(s)\,. \tag{4.20}$$

Since $\psi(\mathrm{i}(1/2-s))G(s) \in \mathcal{C}_{\mathrm{c}}^{\infty}(1/2+\mathrm{i}\mathbf{R})$, the integration against $\varepsilon^{\pm(2s-1)}$ produces terms which are rapidly vanishing as $\varepsilon \to 0$. The first term in (4.20) then contributes to the principal value integral in s giving the last term on the right-hand side of (4.19). The argument presented above generalizes to the case $N > 1$, $M = 0$, with the formula (4.19) standing unchanged.

When $M > 0$ and $N = 0$, we can apply the same procedure as above (which is then the standard Maaß-Selberg analysis) for $F = (2s-1)(1-\chi_{a,b}^{X_i})E_X(s)$ in each cusp; we note that any terms involving higher Fourier modes in the cusp, that is, $\chi_{a,b}^{X_i}$ restricted to the cusp X_i, will vanish rapidly as $r \to \infty$ (see for instance [40]).

In the general case of arbitrary M and N, we also get the contributions from the cusp-to-funnel and funnel-to-cusp terms. We can again restrict to the zeroth Fourier mode in the cusp and to the leading terms in the expansion in the funnel (see (2.29) and (2.30)). We then obtain

$$
\begin{aligned}
(4.21)\qquad T_X^{\mathrm{c}}(\psi) \;=\; & \frac{1}{4\pi\mathrm{i}}\int_{\frac{1}{2}+\mathrm{i}\mathbf{R}} \psi\left(\mathrm{i}\left(\frac{1}{2}-s\right)\right)\Big[\operatorname{tr}\partial_s S_X^{CC}(s)S_X^{CC}(1-s) \\
& + \operatorname{tr}\partial_s S_X^{CF}(s)S_X^{FC}(1-s) + \operatorname{tr}\partial_s S_X^{FC}(s)S_X^{CF}(1-s) \\
& + \operatorname{tr}[\partial_s S_X^{FF}(s)S_X^{FF}(1-s) - \partial_s S_{Y^0}^{FF}(s)S_{Y^0}^{FF}(1-s)]\Big]\,ds \\
& - \frac{1}{2}\operatorname{tr}\left(S_X\left(\frac{1}{2}\right)-\mathrm{Id}\right)\psi(0)\,.
\end{aligned}
$$

To finish the proof, we have to show the decomposition of the derivative of the scattering phase induced by the matricial expression (2.32) of the scattering matrices. As $\partial_s \log\det A = \operatorname{tr}\partial_s A A^{-1}$, we have, for $\mathcal{S}_X(s) = S_{M,Y^0}(s)^{-1}S_X(s) = S_{M,Y^0}(1-s)S_X(s)$,

$$
\begin{aligned}
&\partial_s \log\det \mathcal{S}_X(s) \\
&= \operatorname{tr}\Big[(S_{M,Y^0}(1-s)\partial_s S_X(s) - \partial_s S_{M,Y^0}(1-s)S_X(s))S_X(1-s)S_{M,Y^0}(s)\Big] \\
&= \operatorname{tr}\Big[S_{M,Y^0}(1-s)\partial_s S_X(s)S_X(1-s)S_{M,Y^0}(s) - S_{M,Y^0}(1-s)\partial_s S_{M,Y^0}(s)\Big] \\
&= \operatorname{tr}\Big[\partial_s S_X(s)S_X(1-s) - \partial_s S_{M,Y^0}(s)S_{M,Y^0}(1-s)\Big]
\end{aligned}
$$

where we use the functional equation of Proposition 2.5 and its derivative.

Using the matrix form (2.32) related to the two types of boundary at infinity, we get that $\partial_s \log\det \mathcal{S}_X(s)$ is equal to the second factor in the integrand of (4.21). Hence using Lemma 4.3 to rewrite the trace of the scattering matrix at $1/2$ we obtain

$$
\begin{aligned}
T_X^{\mathrm{c}}(\psi) \;=\; & \frac{1}{4\pi\mathrm{i}}\int_{\frac{1}{2}+\mathrm{i}\mathbf{R}} \psi\left(\mathrm{i}\left(\frac{1}{2}-s\right)\right)\partial_s \log\det \mathcal{S}_X(s)\,ds \\
& + \Big(m_{\frac{1}{2}}(R_X) - \operatorname{rank} A\Big)\psi(0)\,,
\end{aligned}
$$

which is (4.14).

We now need the following lemma which will be refined in Section 6 to give Theorem 1.5:

LEMMA 4.7. *The relative scattering phase satisfies*

$$|\log\det \mathcal{S}_X(s)| \leq C\langle s\rangle^4, \quad \operatorname{Re} s = \frac{1}{2},$$

and consequently, $\partial_s \log\det \mathcal{S}_X \in \mathcal{S}'(1/2 + i\mathbf{R})$.

Proof. We will use the results of Section 3 and the method of [35] in a way that will be developed further in Sections 5 and 6 below. We recall the notation $\sigma_X(s) = (2\pi)^{-1} i \log\det \mathcal{S}_X(s)$ and, following [35], we put for $s \in \frac{1}{2} + i\mathbf{R}$,

$$2\pi \frac{d\sigma_X^\sharp}{ds}(s) = \sum_{\zeta\in\mathcal{R}_X} \psi\left(\frac{|\zeta|}{\operatorname{Im} s}\right) \frac{1-2\operatorname{Re}\zeta}{|\zeta - s|^2} - \sum_{\zeta\in\mathcal{R}_{Y^0}} \psi\left(\frac{|\zeta|}{\operatorname{Im} s}\right) \frac{1-2\operatorname{Re}\zeta}{|\zeta - s|^2},$$

$\sigma_X^\sharp(0) = 0$, where $\psi \in \mathcal{C}_c^\infty((-3,3);[0,1])$ satisfies $\psi(\tau) = 1$ for $|\tau| \leq 2$. The polynomial bounds on the counting functions of $\mathcal{R}_X$ and $\mathcal{R}_{Y^0}$ (Theorem 2 of [18]) show that $|\sigma_X^\sharp(s)| \leq C\langle s\rangle^2$; see [35] or the proof of Lemma 6.4 below. For the difference of σ_X and $\sigma_X^\sharp$, Proposition 3.7 shows that (see the computations in the proof of Proposition 5.1 below)

$$\begin{aligned} 2\pi &\left(\frac{d}{ds}\right)^{k+1} (\sigma_X - \sigma_X^\sharp)(s) \\ &= \left(\frac{d}{ds}\right)^{k+1} q(s) + \sum_{\zeta\in\mathcal{R}_X} \left(\frac{d}{ds}\right)^k \left(\left(1 - \psi\left(\frac{|\zeta|}{\operatorname{Im} s}\right)\right) \frac{1-2\operatorname{Re}\zeta}{|\zeta - s|^2}\right) \\ &\quad - \sum_{\zeta\in\mathcal{R}_{Y^0}} \left(\frac{d}{ds}\right)^k \left(\left(1 - \psi\left(\frac{|\zeta|}{\operatorname{Im} s}\right)\right) \frac{1-2\operatorname{Re}\zeta}{|\zeta - s|^2}\right), \end{aligned}$$

for $k \geq 2$. Since q is a polynomial of degree less than or equal to four it follows that for $k \geq 3$

$$\left|\left(\frac{d}{ds}\right)^{k+1} (\sigma_X - \sigma_X^\sharp)(s)\right| \leq C,$$

(a finer estimate terms to term will be provided by Lemma 6.5 below). Hence,

$$|\sigma_X(s)| \leq C\langle s\rangle^4 + |\sigma_X^\sharp(s)| \leq C'\langle s\rangle^4,$$

proving the lemma. □

In view of (4.15), the lemma shows that T_X^c is a tempered distribution. Then (4.13) shows that u_X^c is also tempered. Combining its contribution with the contribution of the discrete spectrum gives the trace formula of the proposition.

5. The Poisson formula and the 0-trace

We will now combine the results of Sections 3 and 4 to obtain a Poisson formula relating the regularized wave trace to the resonances. It is an analogue of the Poisson formula of the odd-dimensional Euclidean scattering which was obtained successively by Lax-Phillips [26], Bardos-Guillot-Ralston [3] (for large times), by Melrose [31] (for all nonzero times), Sjöstrand and the second author [57] (in an abstract setting) and by Sá Barreto and the second author [52] (for noncompactly supported super-exponentially decaying perturbations). The arguments used in the works cited above relied on the Lax-Phillips theory which in fact provided a clear motivation for the formula. Since the strong Huyghens principle was a necessary component of that approach, it could not be used in the case of infinite volume surfaces (though it could have been used for the finite volume case, where, by a different method, W. Müller obtained a Poisson formula[2]; see [41, §5]). The results of Section 4 are well known in Euclidean scattering (see [38, Prop. 4.1][3]) and those of Section 3 could be obtained by methods similar to those here. Hence, our approach is applicable there and it provides a more direct argument. However, in even dimensions an appropriately modified Poisson formula is not yet available in the Euclidean case and we cannot offer any help there (essentially because the scattering matrix is not a meromorphic operator on $\mathbf{C}$ but on the logarithmic plane).

PROPOSITION 5.1. *If the regularized wave trace of* X, $u_X(t)$ *is defined by* (4.10) *then, in the distributional sense on* $\mathbf{R}$

$$t^4 u_X(t) = t^4 \left(\frac{1}{2} \sum_{s \in \mathbf{C}} \mathrm{e}^{(s-\frac{1}{2})|t|} m_s(R_X) - \frac{1}{2} \sum_{s \in \mathbf{C}} \mathrm{e}^{(s-\frac{1}{2})|t|} m_s(R_{Y^0}) \right), \tag{5.1}$$

where the divisor of R_M, $M = X, Y^0$, *is defined by* (2.44), Y^0 *is the model infinite volume end given by* (2.53).

Proof. It is convenient to work with the λ variable, $s = \frac{1}{2} + i\lambda$, and denote by $\widetilde{\mathcal{R}}$ the set $i(\frac{1}{2} - \mathcal{R})$. Proposition 3.7 then becomes

$$\widetilde{\tau}_X(\lambda) = \mathrm{e}^{\widetilde{q}(\lambda)} \frac{\widetilde{P}_X(-\lambda)}{\widetilde{P}_X(\lambda)} \frac{\widetilde{P}_{Y^0}(\lambda)}{\widetilde{P}_{Y^0}(-\lambda)},$$

$$\widetilde{P}_M(\lambda) = \prod_{\zeta \in \widetilde{\mathcal{R}}_M} E\left(\frac{\lambda}{\zeta}, 2\right), \quad E(z,2) = (1-z)\mathrm{e}^{z + \frac{1}{2}z^2},$$

[2] In fact, a stronger formula valid through $t = 0$; see Remark 5.2 below.

[3] A small correction is needed there so that half-bound states are included as in Proposition 4.5 above.

where $\widetilde{q}(\lambda)$ is a polynomial of degree at most 4. Hence, for $m \geq 0$,

$$
\begin{aligned}
&\left(\frac{d}{d\lambda}\right)^{m+1} \log \widetilde{\tau}_X(\lambda) \\
&= \left(\frac{d}{d\lambda}\right)^{m+1} \widetilde{q}(\lambda) \\
&\quad + \sum_{\zeta \in \widetilde{\mathcal{R}}_X} \left[\left(\frac{d}{d\lambda}\right)^{m+1} \log E\left(-\frac{\lambda}{\zeta}, 2\right) - \left(\frac{d}{d\lambda}\right)^{m+1} \log E\left(\frac{\lambda}{\zeta}, 2\right) \right] \\
&\quad - \sum_{\zeta \in \widetilde{\mathcal{R}}_{Y^0}} \left[\left(\frac{d}{d\lambda}\right)^{m+1} \log E\left(-\frac{\lambda}{\zeta}, 2\right) - \left(\frac{d}{d\lambda}\right)^{m+1} \log E\left(\frac{\lambda}{\zeta}, 2\right) \right],
\end{aligned}
$$

which for $m \geq 4$ is equal to

$$
\sum_{\zeta \in \widetilde{\mathcal{R}}_X} \left[\frac{(-1)^m m!}{(\lambda+\zeta)^{m+1}} - \frac{(-1)^m m!}{(\lambda-\zeta)^{m+1}} \right] - \sum_{\zeta \in \widetilde{\mathcal{R}}_{Y^0}} \left[\frac{(-1)^m m!}{(\lambda+\zeta)^{m+1}} - \frac{(-1)^m m!}{(\lambda-\zeta)^{m+1}} \right].
$$

Thus, with the notation of Definition 1.2 and Proposition 4.5 and for $\ell \geq 2$,

$$
\begin{aligned}
(5.2) \quad \left(\frac{d}{d\lambda}\right)^{2\ell} \frac{d\widetilde{\sigma}_X}{d\lambda}(\lambda) \; = \; \frac{\mathrm{i}(2\ell)!}{2\pi} \Bigg[&\sum_{\zeta \in \widetilde{\mathcal{R}}_X} \left[\frac{1}{(\lambda+\zeta)^{2\ell+1}} - \frac{1}{(\lambda-\zeta)^{2\ell+1}} \right] \\
&- \sum_{\zeta \in \widetilde{\mathcal{R}}_{Y^0}} \left[\frac{1}{(\lambda+\zeta)^{2\ell+1}} - \frac{1}{(\lambda-\zeta)^{2\ell+1}} \right] \Bigg].
\end{aligned}
$$

Denoting by $\mathcal{F}$ the Fourier transform on $\mathbf{R}$ and by $\mathbf{C}_-$ the open lower complex half-plane we now consider, for $\zeta \in \mathbf{C}_-$,

$$
\begin{aligned}
\mathcal{F}^{-1}\left(t^{2\ell} \mathrm{e}^{-\mathrm{i}\zeta|t|}\right) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{2\ell} \mathrm{e}^{-\mathrm{i}\zeta|t| + \mathrm{i}\lambda t} dt \\
&= \frac{1}{2\pi} \left(\frac{1}{\mathrm{i}} \frac{d}{d\lambda}\right)^{2\ell} \left[\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\zeta|t| + \mathrm{i}\lambda t} dt \right] \\
&= \frac{1}{2\pi} \left(\frac{1}{\mathrm{i}} \frac{d}{d\lambda}\right)^{2\ell} \left[\frac{\mathrm{i}}{\lambda - \zeta} - \frac{\mathrm{i}}{\lambda + \zeta} \right] \\
&= \frac{(-1)^{\ell+1}}{2\mathrm{i}\pi} (2\ell)! \left[\frac{1}{(\lambda+\zeta)^{2\ell+1}} - \frac{1}{(\lambda-\zeta)^{2\ell+1}} \right],
\end{aligned}
$$

which, by continuity in $\mathcal{S}'$, is also valid for $\zeta \in \mathbf{R}$ and, with ζ replaced by $-\zeta$, for $\zeta \in \mathbf{C} \setminus \mathbf{C}_-$. Hence, combining this with (5.2) we obtain

$$\begin{aligned} t^{2\ell}\mathcal{F}\left(\frac{d\widetilde{\sigma}_X}{d\lambda}\right)(t) &= \mathcal{F}\left(\left(\frac{1}{\mathrm{i}}\frac{d}{d\lambda}\right)^{2\ell}\frac{d\widetilde{\sigma}_X}{d\lambda}\right)(t) \\ &= t^{2\ell}\left[\sum_{\zeta\in\widetilde{\mathcal{R}}_X\setminus\{0\}} \mathrm{e}^{\mathrm{i}\zeta|t|} - \sum_{\zeta\in\widetilde{\mathcal{R}}_X\cap\mathbf{C}_-} e^{-\mathrm{i}\zeta|t|} - \sum_{\zeta\in\widetilde{\mathcal{R}}_{Y^0}} \mathrm{e}^{\mathrm{i}\zeta|t|}\right]. \end{aligned}$$

A comparison with Proposition 4.5 proves (5.1) where we put $\ell = 2$. □

Remark 5.2. It is not clear what the optimal value of the power of t in (5.1) should be and in fact it is possible, though unlikely, that the formula is true without a factor of t at all. That would have to include a way of summing over the resonances.

In the infinite volume case, this question is related to the exact value of the order and type of the determinant of the relative scattering matrix. A plausible strengthening of Proposition 3.7 would be that the order is two and the type is finite. That would imply that we could take two rather than four as the power of t in (5.1). When $X = \Gamma\backslash\mathbf{H}^2$ and Γ has no parabolic elements, that is in fact the case. Proposition C of [17] and Proposition 4.5 (or rather its proof) show that

$$\tau_X(s) = e^{2\pi i s m_x}\frac{Z_X(1-s)}{Z_X(s)}\prod_{j=1}^{N}\frac{Z_{Y_j^0}(s)}{Z_{Y_j^0}(1-s)}F_{g,N}(s), \tag{5.3}$$

where $m_X = m_{\frac{1}{2}}(R_X) - \dim\ker_{L^2}(\Delta_X - \frac{1}{4})$ and $F_{g,N}(s)$ is an explicitly given meromorphic function of order 2. Then Theorem 1.1 of [46] shows that the order is two.

In the finite volume case, that is when $N = 0$, it follows from Proposition 4.1 and Theorem 5.11 of [41] that

$$u_X(t) = \text{p.v.}\sum_{s\in\mathbf{C}} \mathrm{e}^{(s-\frac{1}{2})|t|}m_s(R_X) - \log q\,\delta_0(t) \tag{5.4}$$

where $q^{s-\frac{1}{2}}$ is the factor in the Blaschke product representation of the scattering phase (see (0.3) in [41]) and

$$\text{p.v.}\sum_{s\in\mathbf{C}} f(s) = \lim_{R\to\infty}\sum_{|s|\leq R} f(s)\,,$$

here in the sense of distribution in t. In [41], Müller considered only even functions of t, but the proof of Theorem 5.7 above and (6.4) in [41] show that (5.4) holds in general. For potentials in dimension one the trace formula also needs a modification to be valid through $t = 0$ ([68]).

We will now obtain an absolute Poisson formula which will involve the resonances of X only. For that we need to regularize the trace in a more intrinsic way.

For a form $\omega \in \mathcal{C}^\infty(X;\, {}^0\Omega^1_X)$ (thought of as a zero-one-density), we define its 0-integral as a regularized integral in the sense of Hadamard:

Definition 5.3. Let x be a defining function for $\partial_F X$, bounded outside of a neighbourhood of the $\partial_F X$; that is,

$$x_{|\partial_F X} = 0\,, \quad dx_{|\partial_F X} \neq 0\,, \quad x > 0 \text{ on } | \overset{o}{X}\,.$$

The 0-integral of a form $\omega \in \mathcal{C}^\infty(X;\, {}^0\Omega^1_X)$ is defined by

$$ {}^0\!\!\int_X \omega = \mathrm{FP}_{\varepsilon \to 0} \int_{x \geq \varepsilon} \omega\,,$$

where FP denotes the finite part in the asymptotic expansion in powers of ε.

The 0-integral depends on the 2-jet on $\partial_F X$ of the defining function x. A surface of type (1.1) with $N > 0$, although of infinite riemannian volume, is of finite 0-volume (defined by taking the 0-integral of the volume form $d\,\mathrm{vol}_X$). There is a natural choice of the defining function, namely that which gives a null 0-volume to the hyperbolic half-cylinder:

LEMMA 5.4. *For the model infinite volume end Y_j^0 given by (1.4), as the defining function of its boundary at infinity, put $\partial_F Y_j$, $x_j(z){=}(\sinh d(z,\partial Y_j^0))^{-1}$, $z \in Y_j^0$, where ∂Y_j^0 denotes its usual boundary. Then*

$$0\text{-vol}(Y_j^0) = {}^0\!\!\int_{Y_j^0} d\,\mathrm{vol}_{Y_j^0} = 0\,.$$

Proof. In Fermi coordinates with respect to the boundary ∂Y_j, we have

$$\int_{x_j \geq \varepsilon} d\mathrm{vol}_{Y_j} = \int_{0 \leq \sinh r \leq \varepsilon^{-1}} \int_0^{\ell_j} \cosh r dr dt = \ell_j/\varepsilon\,,$$

and hence the finite part equals 0. □

From the functions introduced in Lemma 5.4 we can, by partition of unity, construct a defining function of $\partial_F X$ and that specific defining function will be implicit in the sequel. We can then check that the 0-volume of the surface X is given by the 3-term formula in (1.8), which may be nonpositive:

$$0\text{-vol}(X) = \mathrm{vol}_g(Z) + \sum_{i=1}^{M} \mathrm{vol}_g(X_i) - \sum_{j=1}^{N} \mathrm{vol}_{g_0}(Y_j^0 \setminus Y_j)\,.$$

As mentioned in Section 1, the 0-volume of a hyperbolic surface is the riemannian volume of its Fenchel-Nielsen region which can be computed through the

Gauß-Bonnet formula: if X is a genus g surface with curvature -1 and with M cuspidal and N cylindrical ends, then

$$0\text{-vol}(X) = 2\pi(2g + M + N - 2)\,. \tag{5.5}$$

When $M = 0$ then the 0-trace of a smoothing operator with kernel K in $\mathcal{C}^\infty(X \times X; {}^0\Omega^{\frac{1}{2}}_{X\times X})$ is defined by taking the 0-integral of the kernel form restricted to the diagonal. When $M \neq 0$ we define the 0-trace by combining the above regularization near the infinite volume ends with the regularization near the cusps described in the beginning of Section 4 (but the smoothness of the kernel is not enough for the existence of the regularized trace and the integral may not exist).

We can now state the absolute Poisson formula for the model ends:

PROPOSITION 5.5. *Let Y_j^0 be given by (1.4), that is, be the half-cylinder with closed geodesic of length ℓ_j. If $\Delta^0_{Y_j^0}$ is the Dirichlet Laplacian on Y_j^0 and $\mathcal{R}_{Y_j^0}$ the set of its resonances (included according to their multiplicities) then*

$$0\text{-tr}\cos\left(t\sqrt{\Delta^0_{Y_j^0} - \frac{1}{4}}\right) = \frac{1}{2}\sum_{\rho\in\mathcal{R}_{Y_j^0}} \mathrm{e}^{|t|(\rho-\frac{1}{2})}\,, \quad t \neq 0.$$

Proof. Let us recall from [18, Lemma 3.4], that

$$\mathcal{R}_{Y_j^0} = i\frac{2\pi}{\ell_j}\mathbf{Z} - 2\mathbf{N}_0 - 1\,,$$

and each resonance has multiplicity 2. Hence for $t \neq 0$,

$$\begin{aligned}\frac{1}{2}\sum_{\rho\in\mathcal{R}_{Y_j^0}} e^{|t|(\rho-\frac{1}{2})} &= \sum_{\rho\in \mathrm{i}\frac{2\pi}{\ell_j}\mathbf{Z}-2\mathbf{N}_0-1} e^{|t|(\rho-\frac{1}{2})} = \frac{\mathrm{e}^{-|t|/2}}{2\sinh|t|}\sum_{m\in\mathbf{Z}} \mathrm{e}^{\frac{2i\pi m t}{\ell_j}} \\ &= \ell_j \sum_{k\in\mathbf{N}_0} \frac{\mathrm{e}^{-k\ell_j/2}}{2\sinh k\ell_j}\delta(|t| - k\ell_j)\,,\end{aligned}$$

where we used the (standard) Poisson formula.

The wave operator $W_{\mathbf{H}^2}$ on $\mathbf{H}^2$ has its kernel given by (see [60]),

$$\begin{aligned}W_{\mathbf{H}^2}(t)(w,w') &= \frac{\sin t\sqrt{\Delta_{\mathbf{H}^2} - \frac{1}{4}}}{\sqrt{\Delta_{\mathbf{H}^2} - \frac{1}{4}}}(w,w') \\ &= \frac{1}{2\pi\sqrt{2}}[\cosh t - \cosh d(w,w')]_+^{-\frac{1}{2}}\,, \quad t > 0\,.\end{aligned}$$

Let γ_{ℓ_j} be the hyperbolic isometry of displacement length ℓ_j and axis a_{ℓ_j}, I_a the symmetry with respect to the geodesic a. Then the wave kernel on Y_j^0

for the Dirichlet Laplacian is given by

$$\frac{\sin t\sqrt{\Delta^0_{Y^0_j}-\frac{1}{4}}}{\sqrt{\Delta^0_{Y^0_j}-\frac{1}{4}}}(z,z') = \sum_{k\in\mathbf{Z}}\left[W_{\mathbf{H}^2}(t)(w,\gamma^k_{\ell_j}w') - W_{\mathbf{H}^2}(t)(w,I_{a_{\ell_j}}\gamma^k_{\ell_j}w')\right],$$

with z, z' in $Y^0_j = \langle\gamma_{\ell_j}, I_{a_{\ell_j}}\rangle\backslash\mathbf{H}^2$, projections of w, w' in $\mathbf{H}^2$.

Let us take the set $\mathcal{F}_{\ell_j} = \{(u,v) : u \geq 0, v \in [1, e^{\ell_j}]\}$ as fundamental domain for the action of $\langle\gamma_{\ell_j}, I_{a_{\ell_j}}\rangle$ on $\mathbf{H}^2$. We need to calculate the integrals

$$^0\!\!\int_{\mathcal{F}_{\ell_j}} W_{\mathbf{H}^2}(t)(w,\gamma^k_{\ell_j}w)d\mathrm{vol}_{g_0}, \quad ^0\!\!\int_{\mathcal{F}_{\ell_j}} W_{\mathbf{H}^2}(t)(w,I_{a_{\ell_j}}\gamma^k_{\ell_j}w)d\mathrm{vol}_{g_0}, \quad t>0.$$

First, the integral $^0\!\int_{\mathcal{F}_{\ell_j}} W_{\mathbf{H}^2}(t)(w,w)d\mathrm{vol}_{g_0}$ is zero, as the 0-volume of Y^0_j is null. The other integrals are standard ones; to calculate them, we use the formula

$$\cosh d(w,w') = \frac{(u-u')^2+v^{\ 2}+v'^2}{2vv'}.$$

For $k \neq 0$,

$$\begin{aligned}
&\int_{\mathcal{F}_{\ell_j}} W_{\mathbf{H}^2}(t)(w,\gamma^k_{\ell_j}w)d\mathrm{vol}_{g_0}\\
&= \frac{1}{2\pi\sqrt{2}}\int_1^{e^{\ell_j}}\frac{dv}{v^2}\int_0^\infty du\left[\cosh t - \frac{(1-e^{k\ell_j})^2u^2+(1+e^{2k\ell_j})v^2}{2e^{2k\ell_j}v^2}\right]_+^{-\frac{1}{2}}\\
&= \frac{1}{2\pi\sqrt{2}}\int_1^{e^{\ell_j}}\frac{dv}{v}\int_0^\infty d\widetilde{u}\left[\cosh t-(2\sinh(k\ell_j/2)^2\widetilde{u}^2+\cosh k\ell_j)\right]_+^{-\frac{1}{2}}\\
&= \ell_j\frac{[t-|k|\ell_j]^0_+}{8\sinh(|k|\ell_j/2)},
\end{aligned}$$

where we use $\int_0^\infty[a^2-b^2x^2]_+^{-\frac{1}{2}}dx = \pi/(2b)$. Similarly,

$$\int_{\mathcal{F}_{\ell_j}} W_{\mathbf{H}^2}(t)(w,I_{a_{\ell_j}}\gamma^k_{\ell_j}w)d\mathrm{vol}_{g_0} = \ell_j\frac{[t-|k|\ell_j]^0_+}{8\cosh(k\ell_j/2)}, \quad k\in\mathbf{Z}, \quad t>0.$$

Summing up, we conclude that

$$0\text{-tr}\,\frac{\sin t\sqrt{\Delta^0_{Y^0_j}-\frac{1}{4}}}{\sqrt{\Delta^0_{Y^0_j}-\frac{1}{4}}} = \ell_j\sum_{k\in\mathbf{N}_0}\frac{e^{-k\ell_j/2}}{2\sinh k\ell_j}[t-k\ell_j]^0_+ - \frac{\ell_j}{8}, \quad t>0.$$

The proposition follows by taking the derivative of the last expression. □

Remark 5.6. The same formula holds for the Neumann boundary condition or for the entire hyperbolic cylinder, $\langle z\mapsto e^{\ell}z\rangle\backslash\mathbf{H}^2$. That can be seen directly by the same method as above using the explicit form of resonances given in the appendix to [18] or by using the general theorem below.

For operators (satisfying $(1-\pi_a)K \in \mathcal{L}^1(\mathcal{H}^a, \mathcal{H}^a)$) with smooth kernels, we clearly have $\widetilde{\operatorname{tr}}\, K = 0\text{-tr}\, K$. Hence the definition of the regularized trace $u_X(t)$, given by (4.10), shows that

$$u_X(t) = 0\text{-tr} \cos t\sqrt{\Delta_X - \frac{1}{4}} - 0\text{-tr} \cos t\sqrt{\Delta^0_{Y^0} - \frac{1}{4}}\,.$$

Combining this with Propositions 5.1 and 5.5, we immediately obtain the main result of this section.

THEOREM 5.7. *If the divisor, $m_s(R_X)$, of the meromorphic continuation of the resolvent of Δ_X, R_X is given by (2.44), then*

$$0\text{-tr} \cos t\sqrt{\Delta_X - \frac{1}{4}} = \frac{1}{2} \sum_{s \in \mathbf{C}} m_s(R_X) e^{(s-\frac{1}{2})|t|}\,, \quad t \neq 0\,,$$

in the sense of distributions on $\mathbf{R} \setminus \{0\}$.

6. Scattering asymptotics

We will now use Theorem 5.1 to derive Theorems 1.2 and 1.3 presented in the introduction, and as we stated there, this will be done using modifications of methods of Sjöstrand and the second author [59] and of Melrose [35]. The crucial component of both methods, now standard in spectral theory, is the use of the small-time behaviour of the wave trace.

In the proof of Proposition 5.5, we have already seen that for $\chi \in \mathcal{C}^\infty(Y^0_j)$,

$$(6.1) \quad 0\text{-tr}\, \chi \cos t\sqrt{\Delta^0_{Y^0_j} - \frac{1}{4}} = -\frac{1}{4\pi} {}^0\!\!\int_{Y^0_j} \chi d\mathrm{vol}_{g_0}\, \partial_t \left(\frac{1}{\sinh(t/2)} \right)\,, \quad |t| < \ell_j\,,$$

where $\chi d\mathrm{vol}_{g_0}$ can be considered as an element of $\mathcal{C}^\infty(Y^0_j, {}^0\Omega^{\frac{1}{2}})$. Hence, the small-time behaviour is described by the 0-vol. To describe the behaviour of the regularized wave trace of the cuspidal ends and then that of the surface X, it is convenient to recall the definition of distributions *classically conormal* to 0, $I^m_{\mathrm{phg}}(\mathbf{R}; \{0\})$:

$$u \in I^m_{\mathrm{phg}}(\mathbf{R}; \{0\}) \iff \hat{u} \in \mathcal{C}^\infty(\mathbf{R}),\ \hat{u}(\lambda) \sim \sum_{k=0}^{\infty} a^{\pm}_k \lambda^{m-k-\frac{1}{4}}, \quad \lambda \to \pm\infty\,.$$

With that notation we have:[4]

[4] As kindly pointed out to us by L. Parnovski, a similar result was obtained by Levitan [28].

LEMMA 6.1. *Let $\psi \in \mathcal{C}_c^\infty(\mathbf{R})$ have support close to 0 with $\psi(0) = 1$. Then for $\chi \in \mathcal{C}^\infty(X_i^a)$ which is equal to 1 outside a compact set, the regularized trace of the wave equation of a cusp end X_i^a satisfies*

$$\psi(t)\,\widetilde{\operatorname{tr}}\,\chi\left(\cos t\sqrt{\Delta_{X_i^a} - \frac{1}{4}}\right) - \frac{1}{2}\int \chi d\operatorname{vol}_g |D_t|\delta_0(t) - \frac{1}{2}|t|^{\dot{-1}} \in I^{\frac{1}{4}}_{\mathrm{phg}}(\mathbf{R};\{0\}),$$

where the distribution $|t|^{\dot{-1}}$ is defined by $|t|^{\dot{-1}} = t_+^{-1} + t_-^{-1}$ with $t_\pm^{-1}$ as given in (3.2.5) *of* [20].

Proof. It is convenient to introduce some reference operators. Thus we put

$$X_i^\sharp \simeq \mathbf{R} \times \mathbf{R}/h_i\mathbf{Z} \simeq \langle w \mapsto w + h_i\rangle \backslash \mathbf{H}^2, \quad \widetilde{X}_i^{a,b} \simeq [a,b] \times \mathbf{R}/h_i\mathbf{Z},$$

where we have the same metric as in (1.2). Then by the finite speed of propagation we have for small t and $b > 2a$

$$\begin{aligned}\operatorname{tr}\chi&\left(\cos t\sqrt{\Delta^0_{X_i^a} - \frac{1}{4}} - {}^t\mathbf{1}_{r>a}\cos t\sqrt{\Delta_{X_i^\sharp} - \frac{1}{4}}\mathbf{1}_{r>a}\right)\\ &= \operatorname{tr}\left(\chi\widetilde{\chi}_a\cos t\sqrt{\Delta^0_{\widetilde{X}_i^{a,b}} - \frac{1}{4}} - \chi\widetilde{\chi}_a{}^t\mathbf{1}_{r>a}\cos t\sqrt{\Delta_{X_i^\sharp} - \frac{1}{4}}\mathbf{1}_{r>a}\right),\end{aligned}$$

where $\widetilde{\chi}_a \in \mathcal{C}_c^\infty(\widetilde{X}_i^{a,b})$ and $\widetilde{\chi}_a(r) = 1$ for $r < 2a$. Hence

$$\begin{aligned}\widetilde{\operatorname{tr}}\,\chi\cos t\sqrt{\Delta^0_{X_i^a} - \frac{1}{4}} &= \operatorname{tr}\chi\widetilde{\chi}_a\cos t\sqrt{\Delta^0_{\widetilde{X}_i^{a,b}} - \frac{1}{4}}\\ &\quad + \widetilde{\operatorname{tr}}\,\chi(1-\widetilde{\chi}_a)\cos t\sqrt{\Delta_{X_i^\#} - \frac{1}{4}}.\end{aligned} \tag{6.2}$$

When $\widetilde{X}_j^{a,b}$ is a compact manifold with boundary, the conormal behaviour of the trace at 0 was given by Ivrii[5] (in [22]; see [21, Prop. 29.1.2 and the proof of Prop. 29.3.3] or [34]). To analyse the second term on the right-hand side of (6.2) we can use the explicit formula for the kernel of $\cos t\sqrt{\Delta_{\mathbf{H}^2} - \frac{1}{4}}$ given in the proof of Proposition 5.5:

$$\cos t\sqrt{\Delta_{X_i^\sharp} - \frac{1}{4}}(z,z') = \frac{1}{2\pi\sqrt{2}}\sum_{k\in\mathbf{Z}}\frac{\partial}{\partial t}(\cosh t - \cosh d(w, w' + kh_i))_+^{-\frac{1}{2}}, \quad t > 0,$$

where z, z' are projections of w, w' to $X_i^\sharp = \langle w \mapsto w + h_i\rangle\backslash\mathbf{H}^2$. From this we will see that only the term with $k = 0$ contributes to the leading singularity of the regularized trace at $t = 0$. To study the contribution from the terms with

[5]This argument could easily be adapted to avoid the use of Ivrii's result; however, we will need the full power of it in the next lemma when $\partial X \neq \emptyset$. If $\partial X = \emptyset$, then by choosing reference operators on boundaryless surfaces we could apply the now standard small-time analysis of Hörmander instead.

$k \neq 0$ we first make a change of variables $t \mapsto s(t)$ such that $\cosh t = 1+s(t)^2/2$. We then recall (see the proof of Proposition 5.2) that

$$\cosh d(w, w+kh_i) = 1 + \frac{1}{2}\left(\frac{kh_i}{y}\right)^2 , \quad w = x+iy ,$$

and hence, to describe the singularities of the b-trace, we need to find

$$\mathrm{FP}_{\varepsilon\to 0}\frac{h_i}{2\pi}\int_a^{\varepsilon^{-1}} \sum_{k\neq 0} \frac{\partial}{\partial s}\left(s^2 - \left(\frac{kh_i}{y}\right)^2\right)_+^{-\frac{1}{2}} \frac{dy}{y^2} ,$$

in the sense of distributions. Thus it suffices to study

$$\mathrm{FP}_{\varepsilon\to 0}\frac{1}{2\pi}\int_{ah_i^{-1}}^{(h_i\varepsilon)^{-1}} \sum_{k\neq 0} \left(s^2 - \left(\frac{k}{y}\right)^2\right)_+^{-\frac{1}{2}} \frac{dy}{y^2} , \quad s>0 .$$

If we put $T = s/(\varepsilon h_i)$, the expression becomes

$$\begin{aligned} &\frac{1}{2\pi}\int_1^\infty (\tau^2-1)^{-\frac{1}{2}}\tau^{-1}\left[\sum_{k\neq 0}\left(\frac{T}{\tau}-|k|\right)_+^0 \frac{1}{|k|}\right] d\tau \\ &\quad = \frac{1}{2\pi}\int_1^\infty (\tau^2-1)^{-\frac{1}{2}}\tau^{-1}\left[2\log\left(\frac{T}{\tau}\right) + 2\gamma + E\left(\frac{T}{\tau}\right)\right]\left(\frac{T}{\tau}-1\right)_+^0 d\tau , \end{aligned}$$

where the remainder E satisfies $E(t) = 0$ for $t<1$ and $E(t) = \mathcal{O}(1/t)$. Hence,

$$\int_1^\infty (\tau^2-1)^{-\frac{1}{2}}\tau^{-1}E\left(\frac{T}{\tau}\right)d\tau = \mathcal{O}\left(\frac{1}{T}\right)\int_1^T (\tau^2-1)^{-\frac{1}{2}}d\tau = \mathcal{O}\left(\frac{\log T}{T}\right) , \quad T>1 .$$

Since $T = s/(\varepsilon h_i)$, this term disappears in the distributional limit as $\varepsilon \to 0$.

Noting that the distributional finite part has to be odd in s we obtain

$$\frac{1}{2}\mathrm{sgn}(s)\log|s| + \frac{1}{2}(-\log h_i + \gamma) - \frac{1}{\pi}\int_1^\infty (\tau^2-1)^{-\frac{1}{2}}\tau^{-1}\log\tau\, d\tau .$$

Distributional differentiation with respect to s and the substitution $s = s(t)$ ($s'(0) = 1$) give the leading singular contribution as $|t|^{-1}/2$ with a classically conormal remainder.

The integral of $\chi(1-\chi_a)$ with respect to the Riemannian volume form is finite and the lemma follows as the three contributions (the compact part, $k=0$ and $k\neq 0$) add up to give the integral of χ. □

In general we have:

LEMMA 6.2. *If $\psi \in \mathcal{C}_c^\infty(\mathbf{R})$ has support sufficiently close to* 0 *and* $\psi(0) = 1$, *then*

$$\psi(t)\, 0\text{-tr} \cos t\sqrt{\Delta_X - \frac{1}{4}} - \frac{0\text{-vol}(X)}{2}|D_t|\delta_0(t) - \frac{M}{2}|t|^{\dot{-1}} \in I^{\frac{1}{4}}_{\text{phg}}(\mathbf{R},\{0\}), \tag{6.3}$$

where $|t|^{\dot{-1}}$ *is as in Lemma* 6.1.

Proof. Let us consider a reference manifold obtained using the decomposition (1.1) and $a > \max a_i + 1$ and $b > \max b_j + 1$:

$$\widetilde{X}^{a,b} = Z \cup \widetilde{X}_1^a \cup \cdots \cup \widetilde{X}_M^a \cup \widetilde{Y}_1^b \cup \cdots \cup \widetilde{Y}_N^b,$$

where using the identifications implicit in (1.2) and (1.3) we put

$$\widetilde{X}_i^a \simeq [a_i, a] \times \mathbf{R}/h_i\mathbf{Z}, \quad \widetilde{Y}_j^b \simeq [b_j, b] \times \mathbf{R}/\ell_j\mathbf{Z}.$$

Let $\widetilde{\chi}_{a,b} \in \mathcal{C}^\infty(\widetilde{X}^{a,b}, [0,1])$ which is 1 on $\widetilde{X}^{a-1,b-1}$ and 0 in a neighbourhood of $\partial\widetilde{X}^{a,b} \setminus \partial X$. Putting $X_i^{a-1} = X_i \setminus \widetilde{X}_i^{a-1}$ and $Y_j^{b-1} = Y_j \setminus \widetilde{Y}_j^{b-1}$, we also introduce $\psi_a^{X_i}$, $\psi_b^{Y_j}$ in $\mathcal{C}^\infty$, supported in X_i^{a-1}, Y_j^{b-1} respectively, and such that

$$\widetilde{\chi}_{a,b} + \sum_{i=1}^M \psi_a^{X_i} + \sum_{j=1}^N \psi_b^{Y_j} = 1.$$

The finite speed of propagation shows that for small t

$$\begin{aligned}\cos t\sqrt{\Delta_X - \frac{1}{4}} &= \cos t\sqrt{\Delta^0_{\widetilde{X}^{a,b}} - \frac{1}{4}}\widetilde{\chi}_{a,b} + \sum_{i=1}^M \cos t\sqrt{\Delta^0_{X_i} - \frac{1}{4}}\psi_a^{X_i} \\ &\quad + \sum_{j=1}^N \cos t\sqrt{\Delta^0_{Y_j} - \frac{1}{4}}\psi_b^{Y_j},\end{aligned}$$

where, as always here, Δ^0_M denotes the Dirichlet Laplacian on M.

Hence,

$$\begin{aligned}0\text{-tr} \cos t\sqrt{\Delta_X - \frac{1}{4}} &= \operatorname{tr} \cos t\sqrt{\Delta^0_{\widetilde{X}^{a,b}} - \frac{1}{4}}\widetilde{\chi}_{a,b} \\ &\quad + \sum_{i=1}^M \widetilde{\operatorname{tr}} \cos t\sqrt{\Delta^0_{X_i} - \frac{1}{4}}\psi_a^{X_i} \\ &\quad + \sum_{j=1}^N 0\text{-tr} \cos t\sqrt{\Delta^0_{Y_j} - \frac{1}{4}}\psi_b^{Y_j}.\end{aligned}$$

For the first term we use Ivrii's result on the small time behaviour of the wave trace of a compact manifold with boundary (see the proof of Lemma 6.1 above for references and some comments), Lemma 6.1 for the second term and

(6.1) for the third one. Thus, modulo $I^{\frac{1}{4}}_{\mathrm{phg}}(\mathbf{R}, \{0\})$ and the lower order cusp contribution,

$$\begin{aligned} 0\text{-tr} \cos t\sqrt{\Delta_X - \frac{1}{4}} &\equiv \frac{\operatorname{vol}(\widetilde{X}_{a-a,b-1}) + \sum_{i=1}^{M} \operatorname{vol}(X_i^{a-1}) + \sum_{j=1}^{N} 0\text{-vol}(Y_j^{b-1})}{2} \\ &\quad \times |D_t|\delta_0(t) \\ &\equiv \frac{0\text{-vol}(X)}{2} |D_t|\delta_0(t) , \end{aligned}$$

which concludes the proof of the lemma. □

Remark 6.3. The Fourier transform of $|t|^{-1}$ is given by $-2\log|\lambda| + c$ (see [20, §7.1]) and this logarithmic term destroys the classical behaviour of the trace in the presence of cusps.

If we use the notation

$$v_X(t) = 0\text{-tr} \cos t\sqrt{\Delta_X - \frac{1}{4}} ,$$

then Lemma 6.2 shows that for

$$\phi \in \mathcal{C}^\infty_c(\mathbf{R}_+) , \quad \phi(t) \geq 0 , \quad \phi(1) > 0 , \quad \phi_\gamma(t) = \frac{1}{\gamma}\phi\left(\frac{t}{\gamma}\right) ,$$

we have for some $c > 0$ and $\gamma \in (0, \gamma_0]$,

$$\left| \int v_X(t)\phi_\gamma(t)dt \right| > c\, |\, 0\text{-vol}(X)|\gamma^{-2} . \tag{6.4}$$

In fact, when $w \in I^{\frac{1}{4}}_{\mathrm{phg}}(\mathbf{R}, \{0\})$ then $\hat{w}(\lambda) \leq C$ and by the Plancherel theorem we have $\int w(t)\phi_\gamma(t)dt = \mathcal{O}(\gamma^{-1})$. If we use (6.3) with ψ identically 1 near 0 then for small γ we can replace $v_X(t)$ by $b\ 0\text{-vol}(X)t^{-2}$ on the support of ϕ_γ (noting that $|D_t|\delta_0(t) \sim 2bt^{-2}$ as $t \searrow 0$). From this, (6.4) follows with $c = b\int t^{-2}\phi(t)dt - \varepsilon$ for any $\varepsilon > 0$ and for $\gamma_0 = \gamma_0(\varepsilon)$ small enough.

Theorem 1.2 is now an immediate consequence of Theorem 5.1, (6.4), and Proposition 4.2 of [59]. For the convenience of the reader we will briefly recall the argument, simplifying it slightly for our special case.

The left-hand side of (6.4) can be rewritten and estimated as follows:

$$\left| \sum_{\zeta \in \widetilde{\mathcal{R}}_X} \hat{\phi}(-\gamma\zeta) \right| \leq C \int_0^\infty \frac{1}{(1+\gamma\tau)^M} d\widetilde{N}(\tau) = MC \int_0^\infty \frac{1}{(1+\tau)^{M+1}} \widetilde{N}\left(\frac{\tau}{\gamma}\right) d\tau ,$$

where we use the same notation as in the proof of Proposition 5.1 and also put $\widetilde{N}(r) = \sharp\{\zeta \in \widetilde{\mathcal{R}}_X : |\zeta| \leq r\}$ where the counting is done according to multiplicity (that is, $\widetilde{N}(r) = \sum_{|\frac{1}{2}+is|\leq r} m_s(R_X)$). Since by Theorem 2 of [18], $\widetilde{N}(r) \leq Cr^2 + C$, it follows that

$$\begin{aligned} c\,|\,0\text{-vol}(X)|\gamma^{-2} &\leq C\int_0^\infty \frac{1}{(1+\tau)^{M+1}}\widetilde{N}\left(\frac{\tau}{\gamma}\right)d\tau \\ &\leq C\left(\widetilde{N}\left(\frac{\lambda}{\gamma}\right)\right) + \int_\lambda^\infty \frac{1}{(1+\tau)^{M+1}}\left(\frac{\tau}{\gamma}\right)^2 d\tau) \\ &\leq C\widetilde{N}\left(\frac{\lambda}{\gamma}\right) + C\gamma^{-2}\int_\lambda^\infty \frac{\tau^2}{(1+\tau)^{M+1}}d\tau\,. \end{aligned}$$

We then put $M = 3$ and fix λ to be large enough so that $C\int_\lambda^\infty \tau^2/(1+\tau)^4 d\tau < \frac{1}{2}c|\,0\text{-vol}(X)|$. This shows that $\widetilde{N}(r) \geq r^2/C$ for $r > C$ and hence the same estimate holds for $N(r)$ which then gives Theorem 1.2.

We will now follow [35] and prove Theorem 1.3, that is, the Weyl law for the relative scattering phase. Melrose's idea (which he applied in obstacle scattering) can be roughly described as follows. If $d\widetilde{\sigma}_X/d\lambda \geq 0$, then Proposition 3.1, Lemma 6.2, and Hörmander's Tauberian argument would immediately give the Weyl asymptotics for σ_X. That is however unlikely to hold. Nevertheless, if we apply Propositions 3.1 and 5.1 *formally*, then (see the proof of Proposition 5.1):

$$2\pi\frac{d\widetilde{\sigma}_X}{d\lambda}(\lambda) \text{ “=” } \sum_{\zeta\in\widetilde{\mathcal{R}}_X}\frac{2\operatorname{Im}\zeta}{(\lambda-\operatorname{Re}\zeta)^2+(\operatorname{Im}\zeta)^2} - \sum_{\zeta\in\widetilde{\mathcal{R}}_{Y^0}}\frac{2\operatorname{Im}\zeta}{(\lambda-\operatorname{Re}\zeta)^2+(\operatorname{Im}\zeta)^2}\,.$$

In obstacle scattering only the first term appears and $\operatorname{Im}\zeta > 0$. Hence, formally, $\widetilde{\sigma}'_X(\lambda)$ is "positive" and the Tauberian argument can be applied. By an ingenious regularization argument which we will review below, this can in fact be made rigorous. In our case we have an additional "negative" contribution coming from the ends, Y_j^0, but it can be controlled explicitly.

We start with:

LEMMA 6.4. *Let* $\psi \in \mathcal{C}_c^\infty(\mathbf{R};[0,1])$ *satisfy*

$$\psi(\tau) = 1\,, \ \textit{for } |\tau| < 2 \quad \textit{and} \ \ \psi(\tau) = 0\,, \quad \textit{for } |\tau| > 3\,, \tag{6.5}$$

and define for $\lambda \in \mathbf{R}$

$$\frac{d\widetilde{\sigma}^1_M}{d\lambda}(\lambda) = \frac{1}{2\pi}\sum_{\zeta\in\widetilde{\mathcal{R}}_M}\psi\left(\frac{|\zeta|}{\lambda}\right)\frac{2\operatorname{Im}\zeta}{(\lambda-\operatorname{Re}\zeta)^2+(\operatorname{Im}\zeta)^2}\,, \tag{6.6}$$

$\widetilde{\sigma}^1_M(0) = 0\,,\ M = X, Y_j^0$. *Then*

$$\frac{d\widetilde{\sigma}^1_{Y_j^0}}{d\lambda}(\lambda) = \mathcal{O}(\lambda)\,, \quad \widetilde{\sigma}^1_X(\lambda) = \mathcal{O}(\lambda^2)\,. \tag{6.7}$$

Proof. We use again the explicit expression for the resonant set of a half cylinder with the Dirichlet boundary condition given in Lemma 3.4 of [18]:

$$\widetilde{\mathcal{R}}_{Y_j^0} = \frac{2\pi}{\ell_j}\mathbf{Z} + \left(2\mathbf{N}_0 + \frac{3}{2}\right) i\,.$$

Hence we only need to estimate (after some rescaling and with $a, b > 0$),

$$\begin{aligned}\sum_{n\in\mathbf{N},m\in\mathbf{Z}} \psi\left(\frac{|m+ia(n+b)|}{\lambda}\right)\frac{a(n+b)}{(\lambda-m)^2+a^2(n+b)^2} \\ \leq \sum_{n=1}^{4|\lambda|/a}\sum_{m\in\mathbf{Z}}\frac{a(n+b)}{(\lambda-m)^2+a^2(n+b)^2} \\ \leq \sum_{n=1}^{4|\lambda|/a}\left(\frac{1}{a}+\int\frac{a(n+b)}{x^2+a^2(n+b)^2}dx\right) = \mathcal{O}(\lambda)\,.\end{aligned}$$

The second part of (6.7) follows from the standard bound

$$\int_\alpha^\beta \frac{v\,dt}{(t-u)^2+v^2} \leq \pi\,,$$

and the optimal polynomial bound on the number of resonances of X given by Theorem 2 of [18]. □

Following [35] we now put

$$\widetilde{\sigma}_X^2(\lambda) = \widetilde{\sigma}_X(\lambda) - \widetilde{\sigma}_X^1(\lambda) + \widetilde{\sigma}_{Y^0}^1(\lambda)\,. \tag{6.8}$$

The following lemma is proved exactly as (48) in [35] is proved, with Lemma 6.2 used in place of (35) there:

LEMMA 6.5. *If $\widetilde{\sigma}_X^2$ is defined by* (6.8), *then* $\widetilde{\sigma}_X^2 \in S^2(\mathbf{R})$.

To complete the argument in our setting we choose $\phi \in \mathcal{S}(\mathbf{R})$ such that $\phi > 0$, $\hat{\phi}(0) = 1$, and $\hat{\phi} \in \mathcal{C}_c^\infty((-1,1))$. We then put $\phi_a(\tau) = \phi(\tau/a)/a$. We can then apply Lemma 6.2 with $\psi = \hat{\phi}_a$ and a small, together with Remark 6.3, to see that $\widetilde{\sigma}_X * \phi_a(\lambda) = 0\text{-vol}(X)\lambda^2/4\pi - M\lambda\log\lambda/\pi + \mathcal{O}(\lambda)$, $\lambda > 0$. On the other hand the positivity of $(\widetilde{\sigma}_X^1)'(\lambda)$ shows (see (50) in [35] or Lemma 17.5.6 of [21]) that

$$\widetilde{\sigma}_X^1(\lambda) - \widetilde{\sigma}_X^1 * \phi_a(\lambda) = \mathcal{O}(\lambda)\,,$$

and, from Lemmas 6.4 and 6.5, we also have that

$$\widetilde{\sigma}_{Y_j^0}^1(\lambda) - \widetilde{\sigma}_{Y_j^0}^1 * \phi_a(\lambda) = \mathcal{O}(\lambda)\,, \quad \widetilde{\sigma}_X^2(\lambda) - \widetilde{\sigma}_X^2 * \phi_a(\lambda) = \mathcal{O}(\lambda)\,.$$

Hence summing these expressions we obtain

$$\widetilde{\sigma}_X(\lambda) = \frac{1}{4\pi}\, 0\text{-vol}(X)\lambda^2 - \frac{M}{\pi}\lambda \log \lambda + \mathcal{O}(\lambda)\,,$$

which is Theorem 1.3.

Remark 6.6. Under the general assumptions of this paper the remainder estimate seems optimal (we could glue a large sphere to the compact part of the surface). In the case when the set of closed trajectories has measure zero (see §29.1 of [21] and references given there) a better result should hold and that could be improved yet further under the constant negative curvature assumption.

The dependence of the lower bound on the number of resonances (Theorem 1.2) on the 0-vol(X) is an unfortunate indication of the weakness of our method. Another one is discussed in the following:

Example 7.1. We would like to indicate, in an oversimplified situation, that without some control on the location of resonances the singularity of the wave trace at 0 is insufficient for obtaining asymptotics for $N(r)$. Thus let

$$\widetilde{\mathcal{R}} = \pm\sqrt{\frac{2}{a}\mathbf{N}} \cup i\sqrt{\frac{1}{b}\mathbf{N}}\,, \quad a, b > 0\,,$$

so that $N(r) = \sharp\{\lambda \in \widetilde{\mathcal{R}} : |\lambda| \leq r\}$ satisfies

$$N(r) = (a+b)r^2 + \mathcal{O}(1)\,.$$

On the other hand,

$$\sum_{\lambda \in \widetilde{\mathcal{R}}} e^{i\lambda t} \sim_{t \to 0^+} 2(b-a)t^{-2},$$

and hence there is no immediate connection between the asymptotics.

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Note added in proof: After this paper was written new optimal lower bounds for the number of resonances were obtained in other settings. J. Sjöstrand gave the optimal lower bound h^{-n}/C for semi-classical Schrödinger operators in fixed neighbourhoods of energy levels satisfying certain natural conditions (Sem. E.D.P., École Polytechnique, November, 1996). His proof used a new local semiclassical version of the Poisson formula. For Schrödinger operators with linear matrix valued potentials satisfying some ellipticity conditions, L. Nedelec also succeeded in obtaining the optimal lower bound $r^{3n/2}/C$, with the dimension n being even (to appear in Math. Res. Lett.).