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The Selberg Zeta Function for Convex Co-Compact Schottky Groups

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Abstract: We give a new upper bound on the Selberg zeta function for a convex co-compact Schottky group acting on the hyperbolic space $\mathbb{H}^{n+1}$: in strips parallel to the imaginary axis the zeta function is bounded by $\exp(C|s|^{\delta})$ where δ is the dimension of the limit set of the group. This bound is more precise than the optimal global bound $\exp(C|s|^{n+1})$, and it gives new bounds on the number of resonances (scattering poles) of $\Gamma\backslash\mathbb{H}^{n+1}$. The proof of this result is based on the application of holomorphic L^2-techniques to the study of the determinants of the Ruelle transfer operators and on the quasi-self-similarity of limit sets. We also study this problem numerically and provide evidence that the bound may be optimal. Our motivation comes from molecular dynamics and we consider $\Gamma\backslash\mathbb{H}^{n+1}$ as the simplest model of quantum chaotic scattering.

1. Introduction

In this paper we give an upper bound for the Selberg zeta function of a convex co-compact Schottky group in terms of the dimension of its limit set. This leads to a Weyl-type upper bound for the number of zeros of the zeta function in a strip with the number of degrees of freedom given by the dimension of the limit set plus one. We also report on preliminary numerical computations which indicate that our upper bound may be sharp, and close to a possible lower bound.

Our motivation comes from the study of the distribution of quantum resonances – see [39] for a general introduction. Since the work of Sjöstrand [33] on geometric upper bounds for the number of resonances, it has been expected that for chaotic scattering systems the density of resonances near the real axis can be approximately given by a power law with the power equal to half of the dimension of the trapped set (see (1.1) below). Upper bounds in geometric situations have been obtained in [36] and [38].

Recent numerical studies in the semi-classical and several convex obstacles settings, [12, 13 and 14] respectively, have provided evidence that the density of resonances satisfies a lower bound related to the dimension of the trapped set. In complicated situations which were studied numerically, the dimension is a delicate concept and it may be that different notions of dimension have to be used for upper and lower bounds – this point has been emphasized in [14].

Generally, the zeros of dynamical zeta functions are interpreted as the classical correlation spectrum [32]. In the case of *convex co-compact hyperbolic quotients*, $X = \Gamma\backslash\mathbb{H}^{n+1}$ quantum resonances also coincide with the singularities of the zeta function – see [22]. The notion of the dimension of the trapped set is also clear as it is given by $2(1+\delta)$. Here $\delta = \dim \Lambda(\Gamma)$ is the dimension of the limit set of Γ, that is the set of accumulation points of any Γ-orbit in $\mathbb{H}^{n+1}$, $\Lambda(\Gamma) \subset \partial\mathbb{H}^{n+1}$.

Hence we may expect that

$$\sum_{|\operatorname{Im} s|\leq r,\ \operatorname{Re} s>-C} m_\Gamma(s) \sim r^{1+\delta}\,, \tag{1.1}$$

where $m_\Gamma(s)$ is the multiplicity of the zero of the zeta function of Γ at s.

Referring for definitions of Schottky groups and zeta functions to Sects. 2 and 3 respectively we have

Theorem. *Suppose that Γ is a convex co-compact Schottky group and that $Z_\Gamma(s)$ is its Selberg zeta function. Then for any $C_0 > 0$ there exists C_1 such that for $|\operatorname{Re} s| < C_0$,*

$$|Z_\Gamma(s)| \leq C_1 \exp(C_1|s|^\delta), \quad \delta = \dim \Lambda(\Gamma). \tag{1.2}$$

The proof of this result is based on the *quasi-self-similarity* of limit sets of convex co-compact Schottky groups and on the application of holomorphic L^2-techniques to the study of the determinants of the Ruelle transfer operators.

If we use the convergence of the product representation (3.1) of the zeta function for $\operatorname{Re} s$ large and apply Jensen's theorem we obtain the following

Corollary 1. *Let $m_\Gamma(s)$ be the multiplicity of a zero of Z_Γ at s. Then, for any C_0, there exists some constant C_1 such that for $r > 1$,*

$$\sum\{m_\Gamma(s) \ :\ r \leq |\operatorname{Im} s| \leq r+1,\ \ \operatorname{Re} s > -C_0\} \leq C_1 r^\delta\,, \tag{1.3}$$

where $\delta = \dim \Lambda(\Gamma)$.

We can apply the preceding results to Schottky manifolds: a hyperbolic manifold is called Schottky if its fundamental group is Schottky. The case of surfaces is of special interest: any convex co-compact hyperbolic surface is Schottky. With the description of the divisor of the zeta function through spectral data established by Patterson and Perry [22] (using the results by Bunke and Olbrich [2] in odd dimension), we can reformulate the preceding corollary nicely in the resonance setting. We do it only for surfaces (see below for short comments on higher dimensions).

Corollary 2. *Let X be a convex co-compact hyperbolic surface, $\mathcal{S}_X$ be the set of the scattering resonances of the Laplace-Beltrami operator on X and $m_X(s)$ be the multiplicity of resonance s. Then, for any C_0, there exists some constant C_1 such that for $r > 1$,*

$$\sum\{m_X(s) \ :\ s \in \mathcal{R}, r \leq |\operatorname{Im} s| \leq r+1,\ \ \operatorname{Re} s > -C_0\} \leq C_1 r^\delta\,, \tag{1.4}$$

where $2(1+\delta)$ is the Hausdorff dimension of the recurrent set for the geodesic flow on T^*X.

This corollary is stronger than the result obtained in [38] where the upper bound of the type (1.1) was given. In fact, the upper bound (1.4) is what we would obtain had we had a Weyl law of the form $r^{1+\delta}$ with a remainder $\mathcal{O}(r^{\delta})$. That local upper bounds of this type are expected despite the absence of a Weyl law has been known since [25]. F. Naud [21] has proved the existence of $\varepsilon > 0$ such that the domain $\{\mathrm{Re}\, s > \delta - \varepsilon\} \setminus \{\delta\}$ is resonance free.

Section 7 deals with numerical computations of the density of zeros. They show that (1.1) may be true. In fact, in the range of $\mathrm{Im}\, s$ used in the computation we see that the number of zeros grows fast. If the range of $\mathrm{Re}\, s$ is large (and fixed) we need very large $\mathrm{Im}\, s$ to see the upper bound of Corollary 2. The computations also show that our bound on the zeta function is optimal. For values of $Z_\Gamma(s)$ with $\mathrm{Re}\, s$ negative we see that we need very large $\mathrm{Im}\, s$ to see the onset of the upper bound. That is not surprising since we recall in Proposition 3.2 that $\log |Z_\Gamma(s)| = \mathcal{O}(|s|^{n+1})$, and that this bound is optimal (and of course $\delta < n$).

We refer to Sect. 7 for the details and present here two pictures only. We take for Γ a group generated by compositions of reflections in three symmetrically spaced circles perpendicular to the unit circle, and cutting it at the angles 30° (see Fig.3 for the 110° angle). Figure 1 shows the density of zeros of Z_Γ in that case and Fig. 2 plots the values of $\log|\log|Z_\Gamma||$.

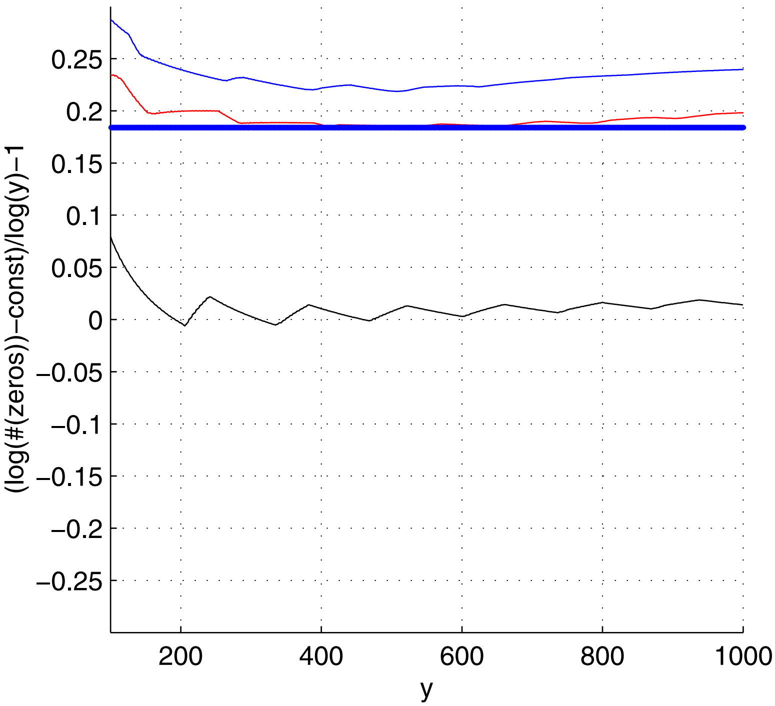

Fig. 1. The plot of $(\log N(y) - C)/\log y - 1$, where $N(y)$ is the number of zeros with $|\mathrm{Im}\, s| \le y$, for a Schottky reflection group with $\delta \simeq 0.184$. Different lines represent different strips $|\mathrm{Re}\, s| \le C_1$, and the thick blue line gives δ. The constant C is determined by least squares regression; see Sect. 7 for details

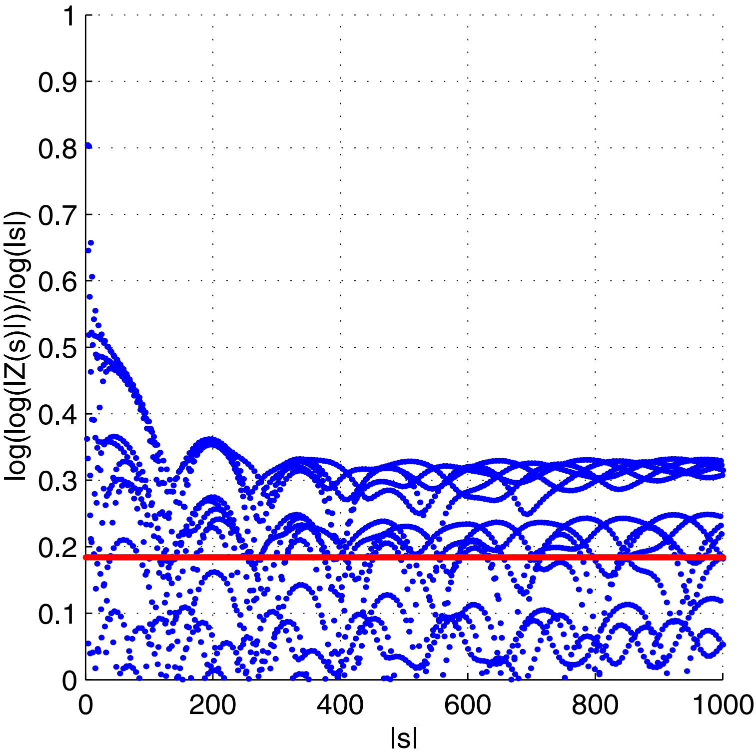

Fig. 2. Density of values of $\log|\log|Z_\Gamma||/\log|s|$ for a Schottky reflection group with $\delta \simeq 0.184$

Finally, we stress that our main theorem is most likely to be a special case of a more general statement relating the growth (and hence the density of zeros) of zeta functions to the dimensions of natural measures appearing in the underlying dynamics. Finding this general statement is an interesting problem. In that direction the methods of this paper have been applied in [34] to give bounds on zeta functions associated to the dynamics of $z \mapsto z^2 + c$, $c < -2$. Unlike in this paper the numerical study in [34] was based on the proof of the upper bound (1.2).

2. Schottky Groups

The hyperbolic geometry on the simply connected curvature -1 space $\mathbb{H}^{n+1}$ and the conformal geometry on its boundary at infinity $\partial\mathbb{H}^{n+1} = \mathbb{S}^n$ share the same automorphism group: the isometry group $\mathrm{Isom}(\mathbb{H}^{n+1})$ and the conformal group $\mathrm{Conf}(\mathbb{S}^n)$ (with the conformal structure given by the standard metric on $\mathbb{S}^n$ of curvature $+1$) are isomorphic. In particular any isometry g of $\mathbb{H}^{n+1}$ induces on $\mathbb{S}^n$ a conformal map γ, whose conformal distortion at the point $w \in \mathbb{S}^n$ will be denoted by $\|D\gamma(w)\|$. There is also a correspondence between balls $\mathcal{D}$ and spheres $\mathcal{C}$ on $\mathbb{S}^n$ (for $n = 2$, the original setting for Kleinian groups, these are discs and circles) and half-spaces $\mathcal{P}$ and geodesic hyperplanes $\mathcal{H}$ in $\mathbb{H}^{n+1}$: $\mathcal{D} = \overline{\mathcal{P}} \cap \partial\mathbb{H}^{n+1}$ and $\mathcal{C} = \overline{\mathcal{H}} \cap \partial\mathbb{H}^{n+1}$. Given a hyperplane $\mathcal{H}$ (a sphere $\mathcal{C}$ resp.), its interior hyperplane (ball) will be given by the choice of a component of $\mathbb{H}^{n+1} \setminus \mathcal{H}$ ($\mathbb{S}^n \setminus \mathcal{C}$ resp.)

Let us review the definition of a Schottky group (see [16, 18, 30] and references given there). Let k, ℓ be integers with $0 \leq k \leq \ell, k + \ell \geq 3$ and $\mathcal{D}_i, i = 1, \dots, k + \ell$ be a collection of mutually disjoint topological balls on the sphere $\mathbb{S}^n$. We suppose that, for each $i = 1, \dots, k$, there exists a conformal map γ_i such that

$$\gamma_i(\mathbb{S}^n \setminus \mathcal{D}_i) = \overline{\mathcal{D}_{i+\ell}},$$

and, for $i = k+1, \dots, \ell$, there exists a conformal symmetry γ_i such that $\gamma_i(\mathbb{S}^n \setminus \mathcal{D}_i) = \overline{\mathcal{D}_i}$. The Schottky marked group

$$\Gamma = \Gamma(\mathcal{D}_1, \dots, \mathcal{D}_{\ell+k}, \gamma_1, \dots, \gamma_\ell),$$

is the group of conformal transformations generated by the $\gamma_1, \dots, \gamma_\ell$. We take $k+\ell \geq 3$ to exclude elementary groups. If in addition the closures $\overline{\mathcal{D}_i}$ are mutually disjoint, which will be assumed here, the Schottky group is convex co-compact.

If, for $i = 1, \dots, \ell$, Γ_i denotes the cyclic group generated by γ_i, the group Γ is the free product $\Gamma_1 * \dots * \Gamma_\ell$, with fundamental domain $\mathbb{S}^n \setminus \cup_{i=1}^{\ell+k} \mathcal{D}_i$ for its action on the sphere $\mathbb{S}^n$. If we introduce, for $j = \ell + 1, \dots, \ell + k$, the transformation $\gamma_j = \gamma_{j-\ell}^{-1}$, every non-trivial element $\gamma \in \Gamma$ is uniquely written as $\gamma = \gamma(1)\gamma(2)\dots\gamma(N)$ with each $\gamma(I)$ in $\{\gamma_1, \dots, \gamma_{\ell+k}\}$ and $\gamma(I)\gamma(I+1) \neq 1$, $I = 1, \dots, N-1$. The uniquely defined integer N is the word length $|\gamma|$ of γ (with respect to the generators set $\{\gamma_1, \dots, \gamma_\ell\}$).

Let us discuss some particular cases. We suppose that each $\mathcal{D}_i$ is a geometric ball, boundary at infinity of an hyperbolic half-space $\mathcal{P}_i$: the marked Schottky group is said to be *classical* and can be described as an isometry group of the interior hyperbolic space $\mathbb{H}^{n+1}$.

If $k = 0$, the group Γ is called a Schottky *reflection* group. For $i = 1, \dots, \ell$, the symmetry γ_i is the conformal symmetry with respect to the sphere $\partial\mathcal{D}_i$ and is omitted in the marking : $\Gamma = \Gamma(\mathcal{D}_1, \dots, \mathcal{D}_\ell)$. The corresponding hyperbolic isometry group is the Schottky marked reflection group $\Gamma(\mathcal{P}_1, \dots, \mathcal{P}_\ell)$, generated by the hyperbolic symmetries $s_i, i = 1, \dots, \ell$ with respect to the hyperplane $\mathcal{H}_i = \partial\mathcal{P}_i$ (with infinite boundary $\partial\mathcal{D}_i$). Figure 3 shows the fundamental domain of a reflection group in $\mathbb{H}^2$ with $\ell = 3$.

If $k = \ell$, the Schottky group contains only orientation preserving transformations. If g_i is the hyperbolic isometry of $\mathbb{H}^{n+1}$ with action at infinity given by γ_i, then the classical Schottky group has a hyperbolic marking

$$\Gamma = \Gamma(\mathcal{P}_1, \dots, \mathcal{P}_{2\ell}, g_1, \dots, g_\ell).$$

The Schottky domain $\mathbb{H}^{n+1} \setminus \cup_{i=1}^{2\ell} \mathcal{P}_i$ is a fundamental domain for the action of Γ on $\mathbb{H}^{n+1}$.

A group is said to be a Schottky group if it admits a presentation induced by a configuration of balls as described above. The subgroup Γ^+ of orientation preserving transformations of a Schottky group Γ is Schottky: for the reflection Schottky group $\Gamma = \Gamma(\mathcal{D}_1, \dots, \mathcal{D}_\ell)$, we have

$$\Gamma^+ = \Gamma(\mathcal{D}_1, \dots \mathcal{D}_{\ell-1}, \gamma_\ell\mathcal{D}_1, \dots, \gamma_\ell\mathcal{D}_{\ell-1}, \gamma_\ell\gamma_1, \dots, \gamma_\ell\gamma_{\ell-1}),$$

where γ_i denotes the conformal symmetry with respect to the sphere $\partial\mathcal{D}_i$. Such a group was called *symmetrical* by Poincaré [26].

An oriented hyperbolic manifold M is said to be (classical) Schottky if its fundamental group $\pi_1(M)$ (realized as a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^{n+1})$) admits a (classical) Schottky marking.

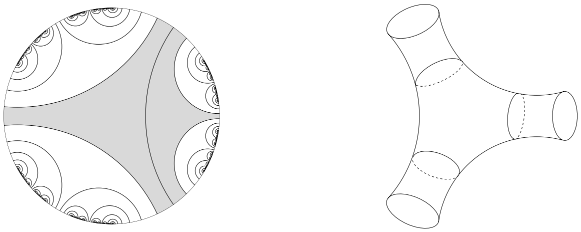

Fig. 3. Tessellation in $\mathbb{H}^2$ by the group, Γ_θ, $\theta = 110°$, generated by symmetries in three symmetrically placed lines each cutting the unit circle in an $110°$ angle, with the fundamental domain of its Schottky subgroup of direct isometries, Γ_θ^+, and the associated Riemann surface $\Gamma_\theta^+\backslash\mathbb{H}^2$. The dimension of the limit set is $\delta = 0.70055063\ldots$

Non-trivial elements of Γ are either symmetries or hyperbolic. For a hyperbolic element $\gamma \in \Gamma$, there exists $\alpha \in \mathrm{Isom}(\mathbb{H}^{n+1})$ such that, in the Poincaré model $\mathbb{H}^{n+1} \simeq \mathbb{R}^{n+1}_+ = \mathbb{R}_+ \times \mathbb{R}^n$,

$$\alpha^{-1}\gamma\alpha(x, y) = e^{\ell(\gamma)}(x, O(\gamma)y), \quad (x, y) \in \mathbb{R}^{n+1}_+, \quad O(\gamma) \in O(n), \quad \ell(\gamma) > 0. \quad (2.1)$$

If $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^{n+1})$, the conjugacy classes of hyperbolic elements,

$$[\gamma_1] = [\gamma_2] \iff \exists\, \beta \in \Gamma \;\; \beta\gamma_1\beta^{-1} = \gamma_2,$$

are in one-to-one correspondence with closed geodesics of $X = \Gamma\backslash\mathbb{H}^{n+1}$. The primitive geodesics correspond to conjugacy classes of primitive elements of Γ (that is, elements which are not non-trivial powers). The magnification factor $\exp \ell(\gamma)$ in (2.1) gives the length $\ell(\gamma)$ of the closed geodesic.

The limit set, $\Lambda(\Gamma)$ of a discrete subgroup, Γ, of $\mathrm{Isom}(\mathbb{H}^{n+1})$, is defined as the set in $\overline{\mathbb{H}^{n+1}} = \mathbb{H}^{n+1} \cup \partial\mathbb{H}^{n+1}$ of accumulation points of any Γ-orbit in $\mathbb{H}^{n+1}$: the limit set $\Lambda(\Gamma)$ is included in the boundary $\partial\mathbb{H}^{n+1}$. In the convex co-compact case it has a particularly nice structure; furthermore, for Schottky groups, it is totally disconnected and included in $\mathcal{D} = \cup_{i=1}^{\ell+k}\mathcal{D}_i$. The aspects relevant to us come from the work of Patterson and Sullivan – see [35] and references given there. As will be discussed in more detail in Sect. 4, the limit set has a quasi-self-similar structure and a finite Hausdorff measure at dimension $\delta = \delta(\Gamma)$.

The limit set is related to the *trapped set*, K, of the usual scattering [33, 36], that is the set of points in phase space such that the trajectory through that point does not escape to infinity in either direction: if π^* is the projection from $T^*\mathbb{H}^{n+1}$ on $T^*(\Gamma\backslash\mathbb{H}^{n+1})$, the trapped set K is the union of the projections $\pi^*(C_{\xi\eta})$, where $C_{\xi\eta}$ is the geodesic line (ξ, η) with extremities ξ and η, distinct points of the limit set $\Lambda(\Gamma)$. In particular, we have

$$\dim K = 2(\delta + 1),$$

see [38].

To stress the connection to closed geodesics on $\Gamma\backslash\mathbb{H}^{n+1}$ let us also mention that, generalizing earlier results of Guillopé [7] and Lalley [10], Perry [23] showed that

$$\sharp\{[\gamma] \;:\; \gamma \text{ primitive}, \ell(\gamma) < r\} \sim \frac{e^{\delta r}}{\delta r}.$$

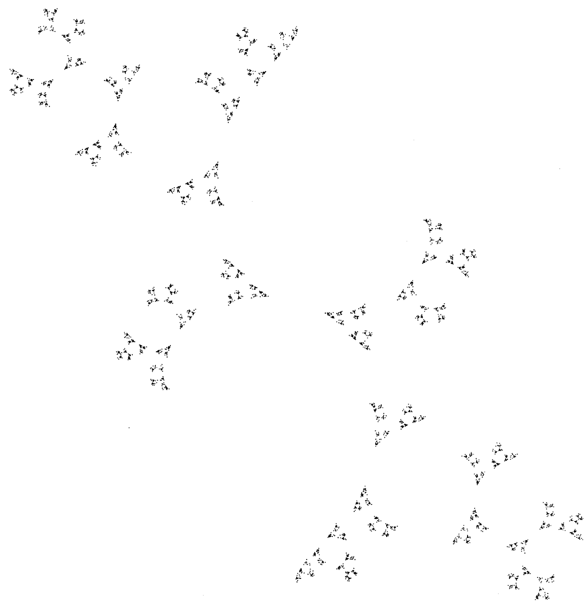

Fig. 4. A typical limit set for a convex co-compact Schottky group, $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$, taken from [18]

3. Properties of the Selberg Zeta Function

For Γ, a discrete subgroup of $\mathrm{Conf}(\mathbb{H}^{n+1})$, the Selberg zeta function is defined as follows:

$$Z_\Gamma(s) = \prod_{\{[\gamma]\}} \prod_{\alpha \in \mathbb{N}_0^n} \left(1 - e^{-i\langle \theta(\gamma), \alpha\rangle} e^{-(s+|\alpha|)\ell(\gamma)}\right). \tag{3.1}$$

Here, $\gamma \in \Gamma$ are hyperbolic, $\exp(\ell(\gamma) + i\theta_j(\gamma))$ are the eigenvalues of the derivative of the action of γ on $\mathbb{S}^n$ at the repelling fixed point $f_+(\gamma)$ ($\exp(i\theta_j(\gamma))$ are the eigenvalues of the isometry $O(\gamma)$ in the normal form (2.1)) and $[\gamma]$ its conjugacy class in Γ. The first product in (3.1) goes over the primitive conjugacy classes. The real exp $\ell(\gamma)$ is called the dilation factor of γ (we have always $\ell(\gamma) > 0$ because we consider the fixed repelling point). An element is called primitive if it is *not* a non-trivial power of another element.

In terms of hyperbolic geometry, the isometry γ keeps invariant the geodesic line $(f_+(\gamma), f_+(\gamma^{-1}))$ in its action on the hyperbolic space $\mathbb{H}^{n+1}$, whose projection on $\Gamma\backslash\mathbb{H}^{n+1}$ is a closed geodesic of length $\ell(\gamma)$ and holonomy spectrum $\theta_j(\gamma)$, $j = 1, \dots, n$. The induced correspondence between conjugacy classes of Γ and closed geodesics of $\Gamma\backslash\mathbb{H}^{n+1}$ is one to one. The word length $|[\gamma]|$ of the conjugacy class $[\gamma]$ is the minimum of the word length of the elements in this conjugacy class.

For the Schottky group $\Gamma = \Gamma(\mathcal{D}_1, \dots, \mathcal{D}_{\ell+k}, \gamma_1, \dots, \gamma_\ell)$, we define the following map $T = T_\Gamma$ on $\mathcal{D} = \bigcup_{i=1}^{\ell+k} \mathcal{D}_i$:

$$T : \mathcal{D} \longrightarrow \mathbb{S}^n, \quad T(x) = \gamma_i(x), \quad x \in \mathcal{D}_i. \tag{3.2}$$

We need to find an open neighbourhood of the limit set where T is strictly expanding in the following sense : F defined on V is said to be (strictly) expanding on $V \subset \mathbb{S}^n$ with respect to the metric $\|\ \ \|$ if there exists $\theta \geq 1$ ($\theta > 1$) such that

$$\|DF(v)\xi\| \geq \theta\|\xi\|, \quad v \in V, \xi \in T_v V.$$

In the case when Γ is a symmetrical Schottky group, we can suppose that up to a conformal identification $\partial\mathcal{D}_\ell$ is a great circle of the sphere $\mathbb{S}^n$. For the metric on $\mathbb{S}^n$ we can take the metric induced by its embedding in $\mathbb{R}^{n+1}$. The inversion, σ_ℓ, is the restriction to $\mathbb{S}^n$ of the symmetry on $\mathbb{R}^{n+1}$ with respect to the euclidean hyperplane containing $\partial\mathcal{D}_\ell$, hence an isometry. Each inversion σ_i, $i = 1, \ldots, \ell - 1$ is expanding on the ball $\mathcal{D}_i$. Hence, the map T is expanding on $\mathcal{D} = \bigcup_{i=1}^{\ell-1} \mathcal{D}_i \cup \sigma_\ell \mathcal{D}_i$, strictly expanding on any open set precompact in $\mathcal{D}$.

However, the map T is not expanding on $\mathcal{D}$ in general. To circumvent that we need to consider refinements, $\mathcal{D}^N$, defined by recurrence:

$$\mathcal{D}^1 = \mathcal{D}, \quad \mathcal{D}^N = T^{-1}(\mathcal{D}^{N-1}) \cap \mathcal{D}^{N-1}, \quad N > 1.$$

Each set $\mathcal{D}^N$ is a disjoint union $\cup_{i=1}^{d_N} \mathcal{D}_i^N$. The collection of sets $\{\mathcal{D}_i^N\}_i$ coincides with the collection

$$\{\mathcal{D}_\gamma\}_{\gamma\in\Gamma_N}, \quad \Gamma_N = \{\gamma \in \Gamma \ : \ |\gamma| = N\},$$

where

$$\mathcal{D}_\gamma = \gamma(N)^{-1} \ldots \gamma(2)^{-1} \mathcal{D}_{\gamma(1)} \quad \text{if} \quad \gamma = \gamma(1) \ldots \gamma(N),$$

with $\mathcal{D}_{\gamma_i} = \mathcal{D}_i$, $i = 1, \ldots, 2\ell$. The iterated map, T^N, is defined on $\mathcal{D}^N$, and

$$T^{|\gamma|}{}_{|\mathcal{D}_\gamma} = \gamma.$$

The map T is strictly expanding on $\mathcal{D}^N$ for N big enough as explained in the following lemma (see Lemma 9.2 in Lalley [10] for a similar result).

Lemma 3.1. *Let Γ be a Schottky group and $\mathcal{D}$, T defined as in* (3.2). *There exist an integer $N \geq 1$, a metric $\| \ \|_\Gamma$, defined on $\mathcal{D}^N$, and a real $\beta > 1$ such that $\|DT(w)\|_\Gamma \geq \beta$, $w \in \mathcal{D}^N$. The metric can be taken analytic on $\mathcal{D}^N$.*

Proof. Let us recall that any Möbius transformation γ of $\mathbb{R}^k$ which does not fix the point at infinity, ∞, has an isometric sphere S_γ, and that γ is strictly contracting on any compact subset of its exterior (the unbounded component of $\mathbb{R}^k \setminus S_\gamma$). The sphere S_γ is centered at $\gamma^{-1}\infty$ and if r_γ is its radius, we have (see [30])

$$||\gamma x - \gamma y|| = \frac{r_\gamma^2 \, ||x - y||}{||x - \gamma^{-1}\infty|| \, ||y - \gamma^{-1}\infty||}, \quad x, y \in \mathbb{R}^k \setminus \{\gamma^{-1}\infty\}. \tag{3.3}$$

Up to a conformal transformation, we can suppose that Γ is a subset of $\mathrm{Conf}(\mathbb{R}^n)$, with the point at infinity in its ordinary set. No non-trivial element in Γ fixes ∞ and, taking in (3.3) as x, y points in the upper half-plane $\mathbb{H}^{n+1}$ (and the Poincaré extension of γ to $\mathbb{R}^{n+1}$), we deduce that the set of radii $\{r_\gamma, \gamma \in \Gamma\}$ accumulates only at 0.

For $\gamma = \gamma(1) \ldots \gamma(|\gamma|)$, we have

$$\gamma^{-1}(\infty) \in \mathcal{D}_\gamma \subset \mathcal{D}_{\gamma(2)\ldots\gamma(|\gamma|)} \subset \ldots \subset \overline{\mathcal{D}_{\gamma(|\gamma|-1)\gamma(|\gamma|)}} \subset \mathcal{D}_{\gamma(|\gamma|)},$$

hence there exists N_0 such that the isometric sphere S_γ is included in $\mathcal{D}_{\gamma(|\gamma|)}$ if $|\gamma| \geq N_0$. For such a γ, the interior of the isometric sphere $S_{\gamma^{-1}}$ is included in $\mathcal{D}_{\gamma(1)^{-1}}$: its exterior contains all the $\overline{\mathcal{D}_{\gamma_i}}$, $\gamma_i \neq \gamma(1)^{-1}$, hence γ^{-1} is strictly contracting on $\cup_{\gamma_i \neq \gamma(1)^{-1}} \mathcal{D}_i$. As $\gamma(\mathcal{D}_\gamma) \subset \cup_{\gamma_i \neq \gamma(1)^{-1}} \mathcal{D}_i$, the map $\gamma = T^{|\gamma|}{}_{|\overline{\mathcal{D}_\gamma}}$ is expanding on $\mathcal{D}_\gamma \subset \mathcal{D}^{|\gamma|}$. We have just proved the existence of $\eta_0 > 1$ such that $\|DT^{N_0}(w)\| \geq \eta_0$, $w \in \mathcal{D}^{N_0}$, hence there exist constants $C > 0, \theta > 1$ such that $\|DT^p(w)\| \geq C\theta^p$, $w \in \mathcal{D}^p$, $p \geq 1$.

Taking an integer N such that $C\theta^N > 1$, we define on $\mathcal{D}^N$ the metric (introduced by Mather [17])

$$\|V\|_\Gamma = \sum_{p=0}^{N-1} \|DT^p(w)V\|, \quad V \in T_w\mathcal{D}^N,$$

which concludes the proof. □

Let us fix now an integer N as in Lemma 3.1. Let $\widetilde{\mathbb{S}}^n$ be a Grauert tube of $\mathbb{S}^n$, that is a complex n-manifold containing $\mathbb{S}^n$ as a totally real submanifold (that is all we need). Let us then choose open neighbourhoods[1] D_i, $i = 1, \ldots, d_N$ of $\mathcal{D}_i^N$ in $\widetilde{\mathbb{S}}^n$. By further shrinking, we can suppose that the open sets D_i are mutually disjoint, and that the real analytic maps T and $\|DT\|_\Gamma$ extend holomorphically to $D = \cup_{i=1}^{d_N} D_i$, with $\|DT\|_\Gamma \geq \widetilde{\beta}$ for some $\widetilde{\beta} > 1$. The open sets D_i can be chosen to be a union $D_i = \cup_{k=1}^{\delta_i} D_{ik}$ of open sets, each one biholomorphic to the ball $B_{\mathbb{C}^n}(0, 1)$ in $\mathbb{C}^n$.

With this formalism in place we define the Ruelle transfer operator

$$\mathcal{L}(s)u(z) = \sum_{Tw=z} [DT(w)]^{-s}u(w), \quad z \in D,$$

$$u \in H^2(D), \quad H^2(D) = \{u \text{ holomorphic in } D \ : \ \iint_D |u(z)|^2 dm(z) < \infty\}, \tag{3.4}$$

$$[DT(w)] \text{ is holomorphic in } D, \quad [DT(w)]\restriction_{\mathbb{S}^n} = |\det DT|^{\frac{1}{n}}.$$

The only difference from the standard definition lies in choosing L^2 spaces of holomorphic functions instead of Banach spaces. However we still obtain the analogue of a (special case of a) result of Ruelle [31] and Fried [5]:

Proposition 3.2. *Suppose that $\mathcal{L}(s) : H^2(D) \to H^2(D)$ is defined by* (3.4). *Then for all $s \in \mathbb{C}$ $\mathcal{L}(s)$ is a trace class operator and*

$$|\det(I - \mathcal{L}(s))| \leq \exp(C|s|^{n+1}). \tag{3.5}$$

Proof. The proof is based on estimates of the characteristic values, $\mu_\ell(\mathcal{L}(s))$. We will show that there exists $C > 0$ such that

$$\mu_\ell(\mathcal{L}(s)) \leq Ce^{C|s| - \ell^{\frac{1}{n}}/C}. \tag{3.6}$$

To see how that is obtained and how it implies (3.5) let us first recall some basic properties of characteristic values of a compact operator $A : H_1 \to H_2$, where H_j's are Hilbert spaces. We define $\|A\| = \mu_0(A) \geq \mu_1(A) \geq \cdots \geq \mu_\ell(A) \to 0$, to be the eigenvalues of $(A^*A)^{\frac{1}{2}} : H_1 \to H_1$, or equivalently of $(AA^*)^{\frac{1}{2}} : H_2 \to H_2$. The min-max principle shows that

$$\mu_\ell(A) = \min_{\substack{V \subset H_1 \\ \operatorname{codim} V = \ell}} \max_{\substack{v \in V \\ \|v\|_{H_1} = 1}} \|Av\|_{H_2}. \tag{3.7}$$

[1] We drop the index N in the open sets D_i^N for the purpose of notational simplicity.

The following rough estimate will be enough for us here: suppose that $\{x_j\}_{j=0}^{\infty}$ is an orthonormal basis of H_1, then

$$\mu_\ell(A) \leq \sum_{j=\ell}^{\infty} \|Ax_j\|_{H_2}. \tag{3.8}$$

To see this we will use $V_\ell = \text{span}\, \{x_j\}_{j=\ell}^{\infty}$ in (3.7): for $v \in V_\ell$ we have,

$$\|Av\|_{H_2} = \left\| \sum_{j=\ell}^{\infty} \langle v, x_j \rangle_{H_1} A x_j \right\|_{H_2} \leq \sum_{j=\ell}^{\infty} |\langle v, x_j \rangle_{H_1}| \; \left\| Ax_j \right\|_{H_2} \leq \|v\| \sum_{j=\ell}^{\infty} \|Ax_j\|_{H_2},$$

from which (3.7) gives (3.8).

We will also need some real results about characteristic values. The first is the *Weyl inequality* (see [6], and also [33, Appendix A]). It says that if $H_1 = H_2$ and $\lambda_j(A)$ are the eigenvalues of A, $|\lambda_0(A)| \geq |\lambda_1(A)| \geq \cdots \geq |\lambda_\ell(A)| \to 0$, then for any N,

$$\prod_{\ell=0}^{N} (1 + |\lambda_\ell(A)|) \leq \prod_{\ell=0}^{N} (1 + \mu_\ell(A)).$$

In particular if the operator A is of trace class, that is if, $\sum_\ell \mu_\ell(A) < \infty$, then the determinant

$$\det(I + A) \stackrel{\text{def}}{=} \prod_{\ell=0}^{\infty} (1 + \lambda_\ell(A)),$$

is well defined and

$$|\det(I + A)| \leq \prod_{\ell=0}^{\infty} (1 + \mu_\ell(A)). \tag{3.9}$$

We also need to recall the following standard inequality about characteristic values (see [6]):

$$\mu_{\ell_1+\ell_2}(A + B) \leq \mu_{\ell_1}(A) + \mu_{\ell_2}(B). \tag{3.10}$$

We finish the review, as we started, with an obvious equality: suppose that $A_j : H_{1j} \to H_{2j}$ and we form $\bigoplus_{j=1}^{J} A_j : \bigoplus_{j=1}^{J} H_{1j} \to \bigoplus_{j=1}^{J} H_{2j}$, as usual, $\bigoplus_{j=1}^{J} A_j(v_1 \oplus \cdots \oplus v_J) = A_1 v_1 \oplus \cdots \oplus A_J v_J$. Then

$$\sum_{\ell=0}^{\infty} \mu_\ell \left(\bigoplus_{j=1}^{J} A_j \right) = \sum_{j=1}^{J} \sum_{\ell=0}^{\infty} \mu_\ell(A_j). \tag{3.11}$$

With these preliminary facts taken care of, we see that (3.6) implies (3.5). In fact, (3.9) shows that

$$|\det(I - \mathcal{L}(s))| \leq \prod_{\ell=0}^{\infty} (1 + e^{C|s| - \ell^{\frac{1}{n}}/C}) \leq e^{C_1 |s|^{n+1}}.$$

Hence it remains to establish (3.6). For that we will write

$$H^2(D) = \bigoplus_{i=1}^{d_N} H^2(D_i),$$

and introduce, for $i, j = 1, \ldots, d^N$, the operator $\mathcal{L}_{ij}(s) : H^2(D_i) \to H^2(D_j)$, nonzero only when $T(D_i)$ and D_j are not disjoint, where

$$\mathcal{L}_{ij}(s)u(z) \stackrel{\text{def}}{=} [Df_{ij}(z)]^s u(f_{ij}(z)), \quad z \in D_j, \quad f_{ij} = (T\restriction_{D_i})^{-1}\restriction_{D_j}, \tag{3.12}$$

where $[Df]$ is defined as in (3.4). From (3.10) and a version of (3.11) we then have

$$\mu_\ell(\mathcal{L}(s)) \le \max_{1\le i,j\le d_N} 2\mu_{[\ell/C]}(\mathcal{L}_{ij}(s)).$$

To estimate $\mu_k(\mathcal{L}_{ij}(s))$, let us recall that D_i was taken as a union of open sets $D_{ik}, k = 1, \ldots, \delta_i$ biholomorphic to $B_{\mathbb{C}^n}(0, 1)$: as $f_{ij}(D_j)$ is relatively compact in D_i, we can find $\rho \in (0, 1)$ (independent of $i, j = 1, \ldots, d_N$) such that $f_{ij}(D_j) \subset D_i^\rho$, where $D_i^\rho = \cup_{k=1}^{\delta_i} D_{ik}^\rho$ with $D_{ik}^\rho \subset D_{ik}$ the pullback of the ball $B_{\mathbb{C}^n}(0, \rho)$ through the biholomorphism of D_{ik} onto $B_{\mathbb{C}^n}(0, 1)$. The map $\mathcal{L}_{ij}(s)$ is the composition

$$H^2(D_i) \xrightarrow{R} \oplus_{k=1}^{\delta_i} H^2(D_{ik}) \xrightarrow{\oplus R_{ik}^\rho} \oplus_{k=1}^{\delta_i} H^2(D_{ik}^\rho) \xrightarrow{\pi^\rho} H^2(D_i^\rho) \xrightarrow{\mathcal{L}_{ij}^\rho(s)} H^2(D_j)\ ,$$

where R and R_{ik}^ρ are the natural restrictions, π^ρ is the orthogonal projection on the space $H^2(D_i^\rho)$ immersed in $\oplus_{k=1}^{\delta_i} H^2(D_{ik}^\rho)$ by the natural restrictions and $\mathcal{L}_{ij}^\rho(s)$ is defined by the same formula (3.12) as $\mathcal{L}_{ij}(s)$. The maps R and π^ρ are bounded, while the norm of $\mathcal{L}_{ij}^\rho(s)$ is bounded by $Ce^{C|s|}$. The bounds on the singular values of R_{ik}^ρ, given up to a bounded factor by the following lemma, give the bound

$$\mu_\ell(\mathcal{L}_{ij}(s)) \le Ce^{C|s|-\ell^{1/n}/C},$$

for some C, which completes the proof of (3.6). □

Lemma 3.3. *Let $\rho \in (0, 1)$ and $R^\rho : H^2(B_{\mathbb{C}^n}(0, 1)) \to H^2(B_{\mathbb{C}^n}(0, \rho))$ induced by the restriction map of $B_{\mathbb{C}^n}(0, 1)$ to $B_{\mathbb{C}^n}(0, \rho)$. Then, for any $\widetilde{\rho} \in (\rho, 1)$ there exits a constant C such that*

$$\mu_\ell(R^\rho) \le C\widetilde{\rho}^{\ell^{1/n}}.$$

Proof. We use (3.8) with the standard basis $(x_\alpha)_{\alpha\in\mathbb{N}^n}$ of $H^2(B_{\mathbb{C}^n}(0, 1))$:

$$x_\alpha(z) = c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad \int_{B_{\mathbb{C}^n}(0,1)} |x_\alpha(z)|^2 dm(z) = 1, \quad \alpha \in \mathbb{N}_0^n, \tag{3.13}$$

for which we have

$$\|R^\rho(x_\alpha)\|^2 = \int_{B_{\mathbb{C}^n}(0,\rho)} |x_\alpha(w)|^2 dm(w) = \rho^{2(|\alpha|+n)}.$$

The number of α's with $|\alpha| \le m$ is approximately m^n and hence by (3.8) we have

$$\mu_\ell(R^\rho) \le C \sum_{Ck\ge\ell^{1/n}} k^{n-1}\rho^k \le C\widetilde{\rho}^{\ell^{1/n}}. \quad \square$$

The next proposition is a modification of standard zeta function arguments – see [27 and 28] for the discussion of the hyperbolic case.

Proposition 3.4. *Let* $\mathcal{L}(s)$ *be defined by* (3.4)*. Then, if* Z_Γ *is the zeta function* (3.1) *corresponding to the group* Γ*,*

$$Z_\Gamma(s) = \det(I - \mathcal{L}(s)).$$

Proof. For s fixed and $z \in \mathbb{C}$,

$$h(z) \stackrel{\text{def}}{=} \det(I - z\mathcal{L}(s))$$

is, in view of (3.6) and (3.9), an entire function of order 0. For $|z|$ sufficiently small $\log(I - z\mathcal{L}(s))$ is well defined and we have

$$\det(I - z\mathcal{L}(s)) = \exp\left(-\sum_{m=1}^{\infty} \frac{z^m}{m} \operatorname{tr}(\mathcal{L}(s)^m)\right). \tag{3.14}$$

The correspondence between the conjugacy classes of hyperbolic elements and the periodic orbits of T is particularly simple for Schottky groups and we recall it in the form given in [28] (where it is given in the more complicated setting of co-compact groups):

> Conjugacy classes of Γ with contraction factor $\exp \ell(\gamma)$ and word length $\|[\gamma]\|$ are in one to one correspondence with periodic orbits $\{x, Tx, \cdots, T^{m-1}x\}$ such that $[DT^m(x)] = \exp \ell(\gamma)$, and $m = \|[\gamma]\|$. For prime closed geodesics we have the same correspondence with primitive periodic orbits of T.

It is not needed for us to recall the precise definition of the *word length*. Roughly speaking it is the number of generators of Γ needed to write down γ.

To evaluate $\operatorname{tr}(\mathcal{L}(s)^m)$ we write

$$\operatorname{tr}\left(\mathcal{L}(s)^m\right) = \sum_{(i_1,\cdots,i_m)} \operatorname{tr}\left(\mathcal{L}_{i_1 i_2}(s) \circ \cdots \circ \mathcal{L}_{i_m i_1}(s)\right),$$

where in the notation of (3.12) we have

$$\mathcal{L}_{i_1 i_2}(s) \circ \cdots \circ \mathcal{L}_{i_m i_1}(s) u(z) = [D(f_{i_1 i_2} \circ \cdots \circ f_{i_m i_1})(z)]^s u(f_{i_1 i_2} \circ \cdots \circ f_{i_m i_1}(z)),$$
$$f_{i_1 i_2} \circ \cdots \circ f_{i_m i_1} \ : \ D_{i_1} \ \longrightarrow \ D_{i_1}.$$

The trace of this operator is non-zero only if $f_{i_1 i_2} \circ \cdots \circ f_{i_m i_1}$ has a fixed point in D_{i_1}. Since this transformation corresponds to an element of Γ that fixed point is unique. Let us call this element γ^{-1}. Since it corresponds to a given periodic point, x, of T^n (corresponding to a fixed point of $f_{i_1 i_2} \circ \cdots \circ f_{i_m i_1}$), γ is determined uniquely by x and n:

$$\gamma = \gamma(x, n), \quad T^n x = x.$$

By conjugation and a choice of coordinates $z = (z_1, \ldots, z_n)$ it can be put into the form

$$\gamma(z) = e^{\ell(\gamma)}(e^{i\theta_1(\gamma)} z_1, \cdots, e^{i\theta_n(\gamma)} z_n),$$

and the trace can be evaluated on the Hilbert space $H^2(B_{\mathbb{C}^n}(0,1))$. Using the basis (3.13) we can write the kernel of $\mathcal{L}_{i_1i_2}\circ\cdots\circ\mathcal{L}_{i_{m-1}i_m}$ as

$$\mathcal{L}_{i_1i_2}\circ\cdots\circ\mathcal{L}_{i_{m-1}i_m}(z,w) = |\gamma'(0)|^{-s}\sum_{\alpha\in\mathbb{N}_0^n} c_\alpha(\gamma^{-1}(z))^\alpha\overline{w}^\alpha$$
$$= \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha e^{-(s+|\alpha|)\ell(\gamma)-i\langle\theta(\gamma),\alpha\rangle}z^\alpha\overline{w}^\alpha.$$

The evaluation of the trace is now clear.

Returning to (3.14), we obtain for Re s sufficiently large (using $\{[\gamma]\}$'s to denote the conjugacy classes of primitive elements of Γ),

$$\det(I-z\mathcal{L}(s)) = \exp\left(-\sum_{n=1}^\infty\frac{z^n}{n}\sum_{T^nx=x}\sum_{\alpha\in\mathbb{N}_0^n}e^{-(s+|\alpha|)\ell(\gamma(x,n))-i\langle\theta(\gamma(x,n)),\alpha\rangle}\right)$$
$$= \exp\left(-\sum_{n=1}^\infty\sum_{\substack{\{[\gamma]\}\\ |[\gamma]|=n}}\sum_{k=1}^\infty\frac{z^{nk}}{k}\sum_{\alpha\in\mathbb{N}_0^n}e^{-k((s+|\alpha|)\ell(\gamma)-i\langle\theta(\gamma),\alpha\rangle)}\right)$$
$$= \prod_{\{[\gamma]\}}\prod_{\alpha\in\mathbb{N}_0^n}\left(1-z^{|\gamma|}e^{-i\langle\theta(\gamma),\alpha\rangle}e^{-(s+|\alpha|)\ell(\gamma)}\right)$$

which in view of (3.1) proves the proposition once we put $z=1$. □

Remark. The proof above is inspired by the work on the distribution of resonances in Euclidean scattering - see [37, Prop. 2]. The Fredholm determinant method and the use of Weyl inequalities in the study of resonances were introduced by Melrose [20] and developed further by many authors – see [33, 39], and references given there. That was done at about the same time as David Fried (across the Charles River from Melrose) was applying the Grothendieck-Fredholm theory to multidimensional zeta-functions [5]. In both situation the enemy is the exponential growth for complex energies s, which is eliminated thanks to analyticity properties of the kernel of the operator.

Finally, we remark that in view of the lower bounds on the number of zeros of Z_Γ obtained in [8] in dimension two, and in [24] in general, we see from Proposition 3.4 that the upper bound (3.5) is optimal for any Γ.

4. Applications of Quasi-Self-Similarity of $\Lambda(\Gamma)$

In this section we will review the results on convex co-compact Schottky groups (coming essentially from [35]) and apply them to refine the domain D used in the definition of the transfer operator (3.4).

We start with a more general definition of convex co-compact subgroups of $\mathrm{Isom}(\mathbb{H}^{n+1})$. A discrete subgroup is called *convex co-compact* if

$$\Gamma\backslash C(\Lambda(\Gamma)) \text{ is compact},\quad C(\Lambda(\Gamma)) \stackrel{\text{def}}{=} \text{convex hull}(\Lambda(\Gamma)). \tag{4.1}$$

Here, the convex hull is meant in the sense of the hyperbolic metric on $\mathbb{H}^{n+1}$: $\Lambda(\Gamma)\subset\partial\mathbb{H}^{n+1}$, and Γ acts on it in the usual way. In particular this implies that $\Gamma\backslash C(\Lambda(\Gamma))$ has a compact fundamental domain in $\mathbb{H}^{n+1}$.

The first result gives a quasi-self-similarity for arbitrary convex co-compact groups:

Proposition 4.1. *Suppose that* $\Gamma \subset \text{Isom}(\mathbb{H}^{n+1})$ *is convex co-compact in the sense of* (4.1). *Then there exist* $c > 0$ *and* $r_0 > 0$ *such that for any* $x_0 \in \Lambda(\Gamma)$ *and* $r < r_0$ *there exists a map* $g : B_{\mathbb{S}^n}(x_0, r) \to \mathbb{S}^n$ *with the properties*

$$\begin{gathered} g(\Lambda(\Gamma) \cap B_{\mathbb{S}^n}(x_0, r)) \subset \Lambda(\Gamma), \\ cr^{-1} d_{\mathbb{S}^n}(x, y) \leq d_{\mathbb{S}^n}(g(x), g(y)) \leq c^{-1} r^{-1} d_{\mathbb{S}^n}(x, y), \quad x, y \in B_{\mathbb{S}^n}(x_0, r). \end{gathered} \tag{4.2}$$

Proof. We proceed following the argument in [35, Sect.3]. Let us fix $z_0 \in C(\Lambda(\Gamma))$. If L is the geodesic ray through z_0 and x_0, then

$$\exists\, C > 0\ \forall z \in L\ \exists\, \gamma \in \Gamma \ \ d(\gamma^{-1} z_0, z) < C.$$

This follows from the compactness of $\Gamma \backslash C(\Lambda(\Gamma))$: for any point on the ray, z, there exists an element of the orbit of z_0 within a finite distance from z. We can now choose $z = z(r)$ on the ray L so that $d(z, z_0) = \log(1/r)$, and then γ such that $d(\gamma^{-1} z_0, z_0) = \log(1/r) + \mathcal{O}(1)$.

If x_γ is the end point of the geodesic ray through z_0 and $\gamma^{-1} z_0$, then for a fixed C_1, the ball $B_{\mathbb{S}^n}(x_\gamma, C_1 r)$ covers $B_{\mathbb{S}^n}(x_0, r)$. The action of γ on $B_{\mathbb{S}^n}(x_\gamma, C_1 r)$ satisfies (4.2): $\Lambda(\Gamma)$ is Γ-invariant, and the other property follows by putting γ into the normal form (2.1). Since z_0 was fixed and we have no dependence on γ, the proof is completed. □

Using the self-similarity we will obtain neighbourhoods of $\Lambda(\Gamma)$, $D = D(h)$, which can be used in place of the fixed domain D of Sect. 3. We will use the upper half space model for $\mathbb{H}^{n+1}$, and assume (as we may) that $\Lambda(\Gamma) \Subset \mathbb{R}^n \subset \mathbb{C}^n$. When talking about the expanding property of T near $\Lambda(\Gamma)$ we will use the metric obtained in Lemma 3.1. The distance we use below is given in terms of that (analytic) metric.

Proposition 4.2. *For any* $h > 0$*, sufficiently small we can find* $D(h) = \cup_j D_j(h)$*, an open neighbourhood of* $\Lambda(\Gamma)$ *in* $\mathbb{C}^n$*, with* $D_j(h)$ *its connected components, and such that*

$$T(D_i(h)) \cap D_j(h) \neq \emptyset \implies d_{\mathbb{C}^n}((T\restriction_{D_i(h)})^{-1}(D_j(h)), \partial D_i(h)) > h/C. \tag{4.3}$$

In addition there exists K *independent of* h *such that*

$$D_j(h) \text{ is a union of at most } K \text{ balls of radius } h. \tag{4.4}$$

Proof. We start by considering

$$\widetilde{D}(h) = \{z \ : \ d_{\mathbb{C}^n}(\Lambda(\Gamma), z) < (1 - \eta)h\},$$

where $0 < \eta < 1$ will be chosen later.

Let h be small enough, so that $\widetilde{D}(h) \subset D$, where D is as in (3.4). Let $\widetilde{D}_j(h)$ be the connected components of $\widetilde{D}(h)$. Then (4.3) holds due to the expanding property of T on D and the fact that T preserves $\Lambda(\Gamma)$, $\Lambda(\Gamma) \cap \widetilde{D}_j(h) \neq \emptyset$.

The self-similarity provided in Proposition 4.1 shows that each connected component is contained in a ball of radius bounded by $K_1 h$, and that the distances between $\widetilde{D}_j(h)$'s are bounded from below by h/K_2, with K_1, K_2 fixed.

In fact, $\Lambda(\Gamma)$ is totally disconnected and for a sufficiently small ϵ, the ϵ-neighbourhood of $\Lambda(\Gamma)$ has more than one connected components, each contained in a ball, of radius at most K_0, and separated from the others by the distance at least $1/K_0$. By applying the self-similarity transformation with $r = h(c\epsilon)^{-1}$, we see that the diameter of each connected component of $\widetilde{D}(h)$ is bounded by $2K_1h$, $K_1 = K_0(c^2\epsilon)^{-1}$. Similarly we obtain a separation condition.

We now modify $\widetilde{D}(h)$ so that (4.4) holds. That is done by modifying each $\widetilde{D}_j(h)$. Observe that

$$\widetilde{D}_j(h) = \bigcup_{z\in\Lambda_j(h)} B_{\mathbb{C}^n}(z, (1-\eta)h), \quad \Lambda_j(h) \stackrel{\text{def}}{=} \Lambda(\Gamma)\cap\widetilde{D}_j(h).$$

Choose $z_0 \in \Lambda_j(h)$. Since for z in $B_{\mathbb{C}^n}(z_0, \eta h)$, $B_{\mathbb{C}^n}(z, (1-\eta)h) \subset B_{\mathbb{C}^n}(z_0, h)$, we have

$$B_{\mathbb{C}^n}(z_0, h) \cup \bigcup_{z\in\Lambda_j(h)\setminus B_{\mathbb{C}^n}(z_0,\eta h)} B_{\mathbb{C}^n}(z, (1-\eta)h) \supset \widetilde{D}_j(h).$$

If $z_0, \cdots, z_k$, are chosen, we then find

$$z_{k+1} \in \Lambda_j(h) \setminus \bigcup_{m=0}^{k} B_{\mathbb{C}^n}(z_m, h\eta),$$

so that now

$$\bigcup_{m=0}^{k+1} B_{\mathbb{C}^n}(z_m, h) \cup \bigcup_{z\in\Lambda_j(h)\setminus\bigcup_{m=0}^{k+1} B_{\mathbb{C}^n}(z_m,\eta h)} B_{\mathbb{C}^n}(z, (1-\eta)h) \supset \widetilde{D}_j(h).$$

This process has to terminate in a uniformly bounded number of steps, as the number of points separated by $h\eta$ in a set of diameter K_1h is uniformly bounded (independently of h, by $C^n(K/\eta)^{2n}$, where C depends on the metric; this can be seen, for instance, by volume comparisons). Hence

$$D_j(h) \stackrel{\text{def}}{=} \bigcup_{m=0}^{K} B_{\mathbb{C}^n}(z_m, h) \supset \widetilde{D}_j(h).$$

We now choose η small enough depending on the expansion constant of T and the separation constant, so that (4.3) holds and that $D_j(h)$'s are mutually disjoint. □

5. Estimates in Terms of the Dimension of $\Lambda(\Gamma)$

In the definition (3.4) and Proposition 3.4 we used the neighbourhood $\mathcal{D}$ of $\Lambda(\Gamma)$ given by Lemma 3.1.

It is clear from the proof that we can, in place of $\mathcal{D}$ use any neighbourhood for which (4.3) holds. For the proof of the Theorem stated in Sect. 1 we will modify D_j's in the definition of $\mathcal{L}(s)$ in a way dependent on the size of s: we will use Proposition 4.2 with $h = 1/|s|$. The self-similarity structure of $\Lambda(\Gamma)$ will show that we can choose $D_j = D_j(h)$ to be a union of $\mathcal{O}(h^{-\delta})$ disjoint balls of radii $\sim h$. A modification of the argument used in the proof of Proposition 3.2 will then give (1.2).

We start with the following lemma which is a more precise version of the argument already used in the proof of Proposition 3.2:

Lemma 5.1. *Suppose that $\Omega_j \subset \mathbb{C}^n$, $j = 1, 2$, are open sets, and $\Omega_1 = \bigcup_{k=1}^{K} B_{\mathbb{C}^n}(z_k, r_k)$. Let g be a holomorphic mapping defined on a neighbourhood, $\widetilde{\Omega}_1$ of Ω_1 with values in Ω_2, satisfying*

$$d_{\mathbb{C}^n}(g(\Omega_1), \partial\Omega_2) > 1/C_0 > 0, \quad 0 < \|Dg(z)\| < 1, z \in \Omega_1.$$

If

$$A \ : \ H^2(\Omega_2) \longrightarrow H^2(\Omega_1), \quad Au(z) \stackrel{\text{def}}{=} u(g(z)), z \in \Omega_1,$$

then for some C_1 depending only on r_k's, K, $d_{\mathbb{C}^n}(g(\Omega_1), \partial\Omega_2)$, and $\min_{\widetilde{\Omega}_1} \|Dg\|_{\mathbb{C}^n \to \mathbb{C}^n}$, we have

$$\mu_\ell(A) \leq C_1 e^{-\ell^{1/n}/C_1},$$

where $\mu_\ell(A)$'s are the characteristic values of A.

Proof. We define a new Hilbert space

$$\mathcal{H} \stackrel{\text{def}}{=} \bigoplus_{k=1}^{K} H^2(B_k), \quad B_k = B_{\mathbb{C}^n}(z_k, r_k),$$

and a natural operator

$$J \ : \ H^2(\Omega_1) \longrightarrow \mathcal{H}, \quad (Ju)_k = u\restriction_{B_k}.$$

We easily check that $J^*J : H^2(\Omega_1) \to H^2(\Omega_1)$ is invertible with constants depending only on K. Hence

$$\mu_\ell(A) = \mu_\ell((J^*J)^{-1}J^*JA) \leq \|(J^*J)^{-1}\| \, \|J^*\| \mu_\ell(JA).$$

We then notice that

$$\mu_{k\ell}(JA) \leq k \max_{1 \leq k \leq K} \mu_\ell(A_k),$$

where

$$A_k \ : \ H^2(\Omega_2) \longrightarrow H^2(B_k), \quad A_k u(z) = u(g_k(z)), \quad g_k = g\restriction_{B_k}.$$

To estimate the characteristic values of A_k we observe that we can extend g_k to a larger ball, $\widetilde{B}_k$ (contained in $\widetilde{\Omega}_1$) and such that the image of its closure still lies in Ω_2 (since we know that $\min_{\widetilde{\Omega}_1} \|Dg\|_{\mathbb{C}^n \to \mathbb{C}^n}$ is strictly less than 1). That gives us the operators $R_k : H^2(\widetilde{B}_k) \to H^2(B_k)$, $R_k u = u\restriction_{B_k}$, and $\widetilde{A}_k$ defined as A_k but with B_k replaced by $\widetilde{B}_k$. We now have $A_k = R_k \widetilde{A}_k$ and consequently,

$$\mu_\ell(A_k) \leq \|\widetilde{A}_k\| \mu_\ell(R_k).$$

Lemma 3.3 gives $\mu_\ell(R_k) \leq C_2 \exp(-\ell^{1/n}/C_2)$ completing the proof. □

Proof of Theorem. As outlined in the beginning of the section we put $h = 1/|s|$, where $|s|$ is large but $|\mathrm{Re}\, s|$ is uniformly bounded. In Proposition 4.2 each $D_j(h)$ is given as a union of (a fixed number of) balls with respect to some fixed metric for which T is uniformly expanding. For h small each ball in the family can be replaced by a linearly distorted ball with all the constants uniform. Hence we can apply Lemma 5.1 with g given by rescaled f_{ij} (defined in (3.12)).

The now classical results of Patterson and Sullivan [35] on the dimension of the limit set show that the total number of the balls is $\mathcal{O}(h^{-\delta})$: what we are using here is the fact that the Hausdorff measure of $\Lambda(\Gamma)$ is finite.

We can now apply the same procedure as in the proof of Proposition 3.2 using Lemma 5.1. What we have gained is a bound on the weight: since $|\mathrm{Re}\, s| \leq C$ and $[Df_{ij}]$ is real on the real $\mathbb{S}^n$,

$$|[Df_{ij}(z)]^s| \leq C \exp(|s| |\arg[Df_{ij}(z)]|) \leq C \exp(C_1 |s| |\mathrm{Im}\, z|) \leq C_2, \quad z \in D_j(h).$$

We write $\mathcal{L}(s)$ as a sum of fixed number of operators $\mathcal{L}_{ij}(s)$ each of which is a direct sum of $\mathcal{O}(h^{-\delta})$ operators. The balls and contractions are uniform after rescaling by h and hence the characteristic values of each of these operators satisfy the bound $\mu_\ell \leq C\gamma^l$, $0 < \gamma < 1$. Using (3.9) and (3.11) we obtain the bound

$$\log |\det(I - \mathcal{L}(s))| \leq C P(h) = \mathcal{O}(h^{-\delta}),$$

and this is (1.2). □

Proof of Corollary 1. The definition of $Z_\Gamma(s)$ (3.1) shows that for $\mathrm{Re}\, s > C_1$ we have $|Z_\Gamma(s)| > 1/2$. The Jensen formula then shows that the left-hand side of (1.3) is bounded by

$$\sum \{ m_\Gamma(s) \ : \ |s - ir - C_1| \leq C_2 \} \leq 2 \max_{\substack{|s| \leq r + C_3 \\ |\mathrm{Re}\, s| \leq C_0}} \log |Z_\Gamma(s)| + C_4,$$

and (1.3) follows from (1.2). □

6. Schottky Manifolds and Resonances

We recall that a complete Riemannian manifold of constant curvature -1 is said to be Schottky if its fundamental group is Schottky. In low dimensions Schottky manifolds can be described geometrically.

Proposition 6.1. *Any convex co-compact hyperbolic surface is Schottky.*

This result is proved by Button [3] and for the reader's convenience we sketch the proof.

Proof. Any convex co-compact, non-elementary surface X is topologically described by two integers (g, f) : its numbers g of holes and f of funnels, with the conditions $g \geq 0$, $f \geq 1$ and $f \geq 3$ if $g = 0$. For any such pair (g, f), there does exist a Schottky surface of this type and we choose for each type (g, f) such a surface $X_{g,f}$. The projection onto $X_{g,f}$ of the boundary of the Schottky domain is a collection $\mathcal{L}_1, \ldots, \mathcal{L}_\ell$, of mutually disjoint geodesic lines.

Let X be any hyperbolic convex co-compact surface . The surface X is homeomorphic to some $X_{g,f}$. Pushing back on X the geodesic lines $\mathcal{L}_i$, $i = 1, \ldots, \ell$ of $X_{g,f}$ and cutting

X along these curves, we obtain in the hyperbolic plane a domain whose boundary is the union of paired mutually disjoint curves $\mathcal{C}_i, \mathcal{C}_{\ell+i}, i = 1, \dots, \ell$, each one with a pair of points at infinity. These point pairs determine intervals, which are mutually disjoint (the curves $\mathcal{C}_j, j = 1, \dots, 2\ell$ don't intersect). The intervals are paired with an hyperbolic transformation, so give a Schottky group, which coincide with the fundamental group of the surface X. □

Proof of Corollary 2. For a Schottky manifold, the fundamental group is Schottky, and hence, $X = \Gamma\backslash\mathbb{H}^{n+1}$, $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^{n+1})$. We then introduce its zeta function Z_X as the zeta function Z_Γ of the group Γ. Following Patterson and Perry [22] we introduce the spectral sets $\mathcal{P}_X$ and $\mathcal{S}_X$ defined by the Laplace-Beltrami operator Δ_X on X:

$$\mathcal{P}_X = \{s \ : \ \mathrm{Re}\, s > n/2, s(n-s) \text{ is a } L^2 \text{ eigenvalue of } \Delta_X\},$$
$$\mathcal{S}_X = \{s \ : \ \mathrm{Re}\, s < n/2, s \text{ is a singularity of the scattering matrix } S_X\}.$$

Moreover, each complex s in $\mathcal{P}_X$ has a multiplicity denoted by $m_X(s)$, each s in $\mathcal{S}_X$ a pole multiplicity denoted by $m_X^-(s)$. In the case of surfaces, the divisor of the Selberg zeta function Z_X is given by the following formula:

$$-\chi_X \sum_{k=0}^{\infty}(2k+1)[-k] + m_X\left(\frac{n}{2}\right)\left[\frac{n}{2}\right] + \sum_{s\in\mathcal{P}_X} m_X(s)[s] + \sum_{s\in\mathcal{S}_X} m_X^-(s)[s],$$

where χ_X is the Euler characteristic of X, see [22, Theorem 1.2]. The zeta function, Z_X, is entire and in any half-plane $\{\mathrm{Re}\, s > -C_0\}$, the formula above shows that the bounds on the number of its zeros provide bounds on the number of resonances. The dimension of the limit set, δ depends only on Γ and, as shown in [35], it gives the Hausdorff dimension of the recurrent set for the geodesic flow on T^*X by the formula $2(1+\delta)$. □

For a convex co-compact hyperbolic manifold X, Patterson and Perry give a formula for the divisor of the zeta function Z_X in any (even) dimension, but it does not imply (in the non-Schottky case) that the zeta function is entire. In the case of Schottky groups, the zeta function Z_X is entire, as it was shown in Proposition 3.4. Hence we concluded that Corollary 2 holds also for Schottky manifolds.

We conclude with some remarks about Kleinian groups in dimension $n+1=3$. Schottky 3-manifolds are geometrically described by Maskit [15]:

Proposition 6.2. *A hyperbolic convex co-compact, non compact 3-manifold is Schottky if and only if its fundamental group is a free group of finite type.*

While non-compact surfaces of finite geometric type always have a free fundamental group, that is not true for the 3-manifold. For instance, if Γ is a co-compact surface group, the 3-manifold $\mathbb{H}^3/\,\Gamma$ is convex co-compact with a non-free fundamental group. Quasi-fuchsian groups (that is, deformation of such a Γ in $\mathrm{Isom}(\mathbb{H}^3)$) give similar examples.

Finally, we note that the bound on the number of zeros of Z_Γ established here for Schottky groups is valid for any group Γ, for which an expanding Markov partition can be built. Anderson and Rocha [1] construct such a Markov partition for any function group. This class of groups does not exhaust all convex co-compact groups (the complement in the 3-sphere of a regular neighbourhood of a graph is not in this class) and it is not known if all convex co-compact Kleinian groups admit an expanding Markov partition.

7. Numerical Results

7.1. Discussion. In this section, we present *empirical* numerical results on the distribution of zeros of the zeta function $Z(s)$ for the simple case of the group Γ_θ. As a hyperbolic geometry group, Γ_θ is generated by reflections s_0, s_1, s_2 in three symmetrically placed geodesics in the Poincaré disc, which intersect its boundary, the unit circle, at angles θ – see Fig. 3 where $\theta = 110^\circ$. The corresponding conformal symmetries are denoted by ϕ_0, ϕ_1, ϕ_2.

Numerical computations of the zeta function in that case have been already performed by Jenkinson-Pollicott [9]. Their goal was to find an efficient way of computing the dimension of limit sets (see also the earlier work of McMullen [19]). Table 1 gives

Table 1. Dimensions of the limit set for relevant values of θ

θ	$\delta = \dim\ \Lambda(\Gamma_\theta)$
10°	0.116009447786
20°	0.151183682038
30°	0.183983061248
40°	0.217765810254

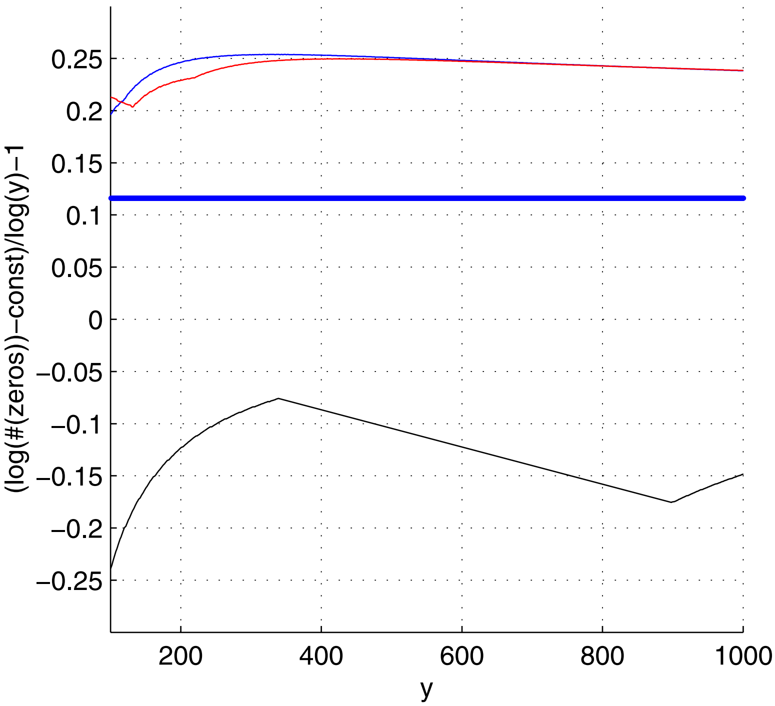

Fig. 5. This plot shows $\frac{\log(|\{s\in[x_0,x_1]\times[y_0,y]:Z(s)=0\}|)-C}{\log(y)} - 1$ as a function of y, for different values of x_0: The thin blue line is for $x_0 = -0.2$, the red line for $x_0 = -0.1$, and the black line for $x_0 = +0.1$. The thick horizontal line indicates the dimension of the corresponding limit set. In this plot, $\theta = 10^\circ$. The constant C is determined by least squares regression, as explained in this section

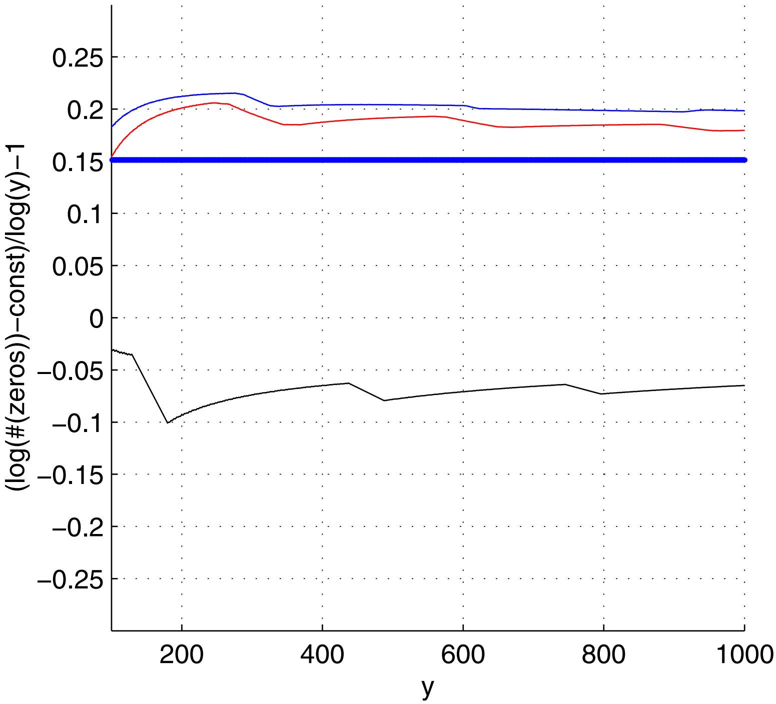

Fig. 6. This plot shows $\frac{\log(|\{s\in[x_0,x_1]\times[y_0,y]:Z(s)=0\}|)-C}{\log(y)} - 1$ as a function of y, for different values of x_0: The thin blue line is for $x_0 = -0.2$, the red line for $x_0 = -0.1$, and the black line for $x_0 = +0.1$. The thick horizontal line indicates the dimension of the corresponding limit set. In this plot, $\theta = 20°$. The constant C is determined by least squares regression, as explained in this section

the (approximate) dimensions of the limit sets for the relevant angles, calculated as the largest real zero of $Z(s)$ [9] using Newton's method.

Figures 1 and 5–7 show

$$\frac{\log(|\{s \in [x_0, x_1] \times [y_0, y] : Z(s) = 0\}|) - C}{\log(y)} - 1 \tag{7.1}$$

as a function of y, where the constant C is chosen to minimize the usual mean square error

$$\mathrm{err}(C', C) = \sum_{k=1}^{N} \left(|\{s \in [x_0, x_1] \times [y_0, y_k] : Z(s) = 0\}| - C' \log(y_k) - C\right)^2,$$

defined using the numerically computed data $\{(y_k, |\{s \in [x_0, x_1] \times [y_0, y_k] : Z(s) = 0\}|) : k = 1, ..., N\}$. In each plot, the value of x_0 is varied to test the dependence of the distribution on the region in which we count: The blue line corresponds to $x_0 = -0.2$, the red line $x_0 = -0.1$, and the black line $x_0 = +0.1$. The data show that most of the zeros

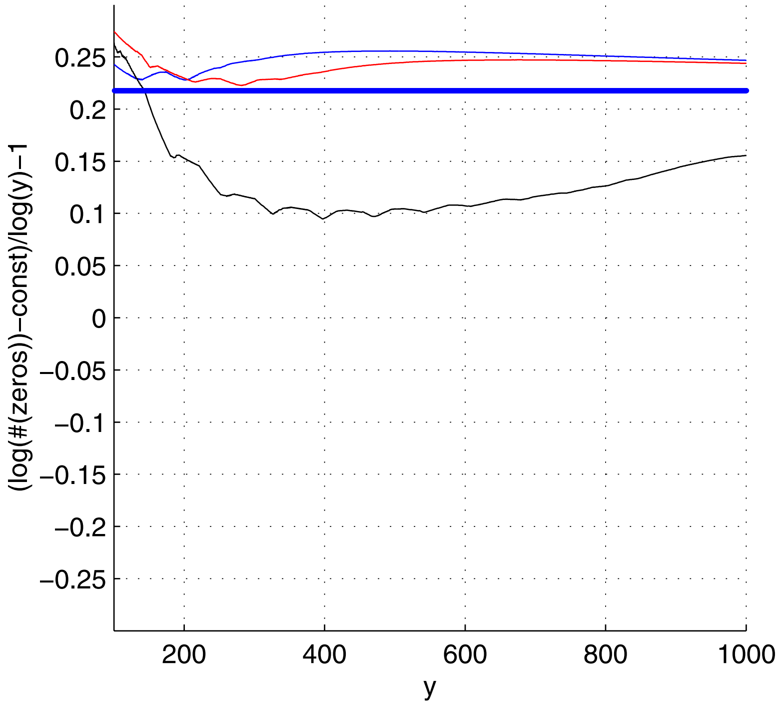

Fig. 7. This plot shows $\frac{\log(|\{s\in[x_0,x_1]\times[y_0,y]:Z(s)=0\}|)-C}{\log(y)} - 1$ as a function of y, for different values of x_0: The thin blue line is for $x_0 = -0.2$, the red line for $x_0 = -0.1$, and the black line for $x_0 = +0.1$. The thick horizontal line indicates the dimension of the corresponding limit set. In this plot, $\theta = 40°$. The constant C is determined by least squares regression, as explained in this section

lie in the left half plane. Based on the theorems proved in earlier sections, we expect the curves to be bounded above by the dimension (the thick blue line) asymptotically. This is not the case, except for the black line, which represents zeros with $\mathrm{R}e(s) > x_0 = +0.1$. Note that the value of x_1 is not very important because $Z(s) - 1$ decays very rapidly for large $\mathrm{R}e(s)$. Thus, we set $x_1 = 10$ throughout. The value of y_0 is fixed at -0.1, to avoid integrating over any zeros.

Similarly, Fig. 2 and 8–10 show $\frac{\log(\log(|Z(s)|))}{\log(|s|)}$ as a function of $|s|$, for a large number of points in the rectangle $[-0.2, 1.0] \times [0, 10^3]$. In this case, we also expect the curves to be asymptotically bounded by the dimension. This is also not the case. The only reasonable explanation, barring errors in the numerical calculations, is that the asymptotic upper bound is accurate only for very large values of $\mathrm{I}m(s)$, and we were not able to calculate $Z(s)$ reliably for such values. These results also show that $Z(s)$ has plenty of zeros in regions of interest.

7.2. Implementation notes. To count the number of zeros of $Z(s)$ in a given region Ω in the complex plane, we rely on the Argument Principle:

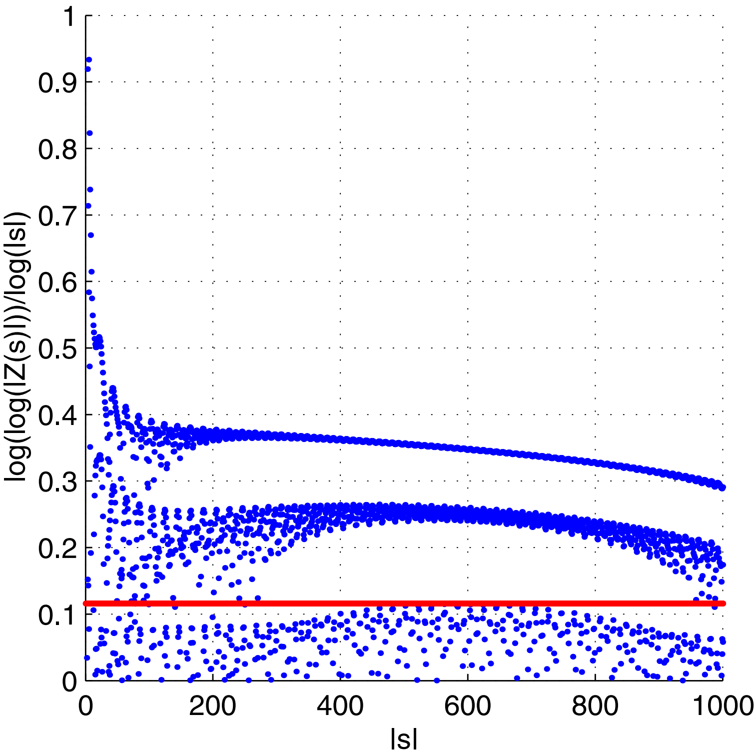

Fig. 8. This plot shows $\log(\log(|Z(s)|))/\log(|s|)$ for a large number of points in the rectangle $[-0.2, 1] \times [0, 10^3]$. The horizontal line indicates dimension. Here, $\theta = 10°$

$$|\{s \in \Omega : Z(s) = 0\}| = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{Z'(s)}{Z(s)} ds. \tag{7.2}$$

To evaluate $Z(s)$, our main technical tool comes from Jenkinson and Pollicott [9], though we note that the essential ideas were used in Eckhardt, et. al. [4] and date back to Ruelle [31].

First, some notation: Let us denote symbolic sequences on the three characters 0, 1, 2 of length $|\sigma| = n$ by σ. That is, $\sigma = (\sigma(0), \sigma(1), ..., \sigma(n))$, $\sigma(k) \in \{0, 1, 2\}$, and $\sigma(0) = \sigma(n)$. Symbolic sequences represent periodic orbits : to each sequence σ we associate a composition of reflections $\phi_\sigma = \phi_{\sigma(n)} \circ ... \circ \phi_{\sigma(1)} : D_{\sigma(0)} \to D_{\sigma(0)}$. As ϕ_σ is a contraction of $D_{\sigma(0)}$ into itself, it has a unique fixed point z_σ.

It is shown in Jenkinson and Pollicott [9] that $Z(s) = \lim_{M\to\infty} Z_M(s)$, where

$$Z_M(s) = 1 + \sum_{N=1}^{M} \sum_{r=1}^{N} \frac{(-1)^r}{r!} \sum_{n \in P(N,r)} \prod_{k=1}^{r} \frac{1}{n_k} \sum_{|\sigma|=n_k} \frac{|\phi'_\sigma(z_\sigma)|^s}{1 - \phi'_\sigma(z_\sigma)}, \tag{7.3}$$

where $P(N, r)$ is the set of all r-tuples of positive integers $(n_1, ..., n_r)$ such that $n_1 + ... + n_r = N$. The series (in N) converges absolutely in $\{s : \mathrm{R}e(s) > -a\}$ for some positive a.

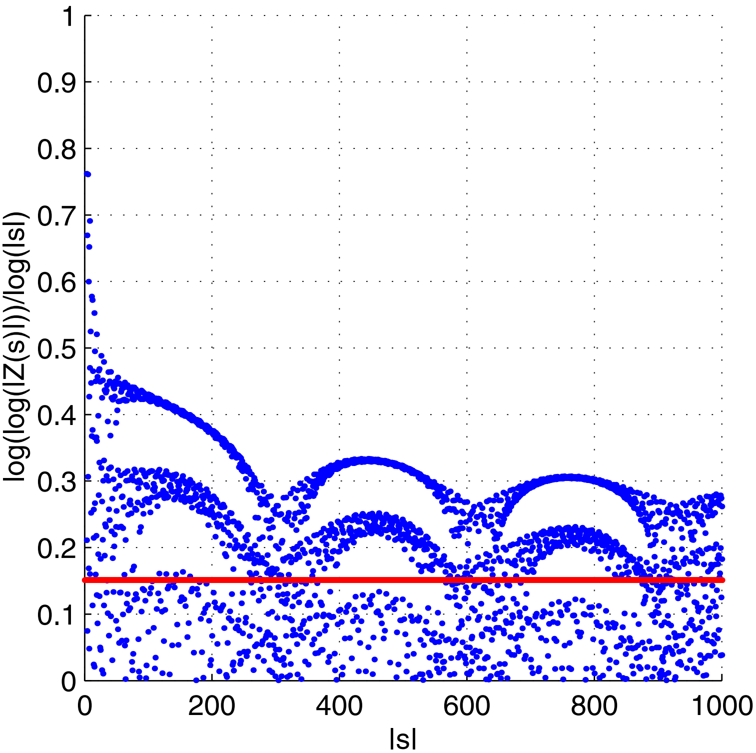

Fig. 9. This plot shows $\log(\log(|Z(s)|))/\log(|s|)$ for a large number of points in the rectangle $[-0.2, 1] \times [0, 10^3]$. The horizontal line indicates dimension. Here, $\theta = 20^\circ$

Equation (7.3) lets us evaluate $Z(s)$ for reasonable values of s in a straightforward manner. In addition, we found two simple and useful observations during the course of this calculation:

(1) Define

$$a_n(s) = -\frac{1}{n} \sum_{|\sigma|=n} \frac{|\phi'_\sigma(z_\sigma)|^s}{1 - \phi'_\sigma(z_\sigma)} \tag{7.4}$$

and

$$B_{N,r}(s) = \frac{1}{r!} \sum_{n \in P(N,r)} \prod_{k=1}^{r} a_{n_k}(s). \tag{7.5}$$

Then the recursion relation

$$B_{N,r}(s) = \frac{1}{r} \sum_{n=1}^{N-r+1} B_{N-n,r-1}(s) \cdot a_n(s), \tag{7.6}$$

with initial conditions $B_{N,1}(s) = a_N(s)$, provides an efficient way to evaluate the sum in (7.3). A similar relation can be derived for $Z'(s)$ by differentiation.

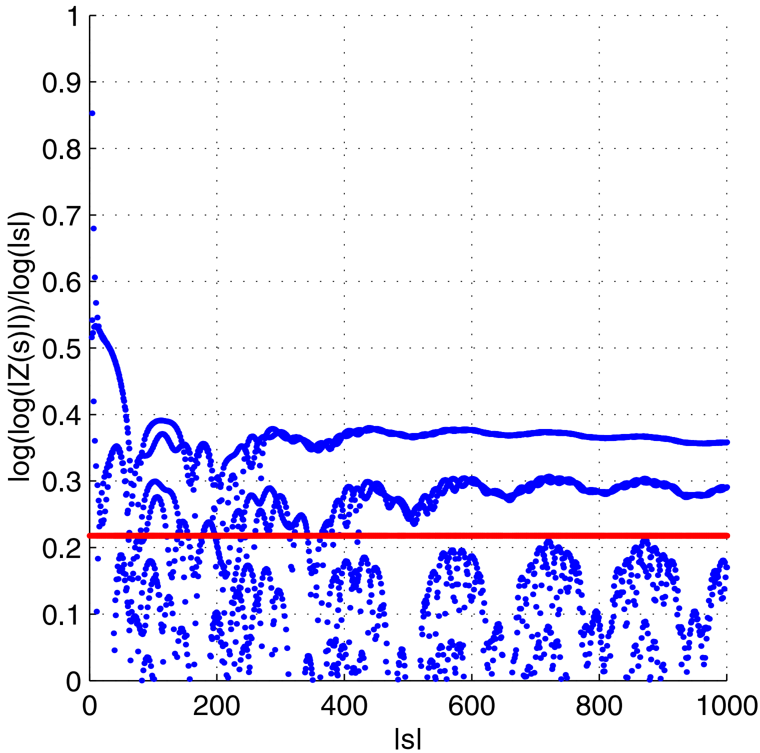

Fig. 10. This plot shows $\log(\log(|Z(s)|))/\log(|s|)$ for a large number of points in the rectangle $[-0.2, 1] \times [0, 10^3]$. The horizontal line indicates dimension. Here, $\theta = 40°$

(2) Recall that the maps ϕ_σ are compositions of linear fractional transformations. Identifying these transformations with elements of $GL(2, \mathbb{R})$ in the usual way, we can compute the numbers $\phi'_\sigma(z_\sigma)$ via matrix multiplications. However, long matrix products can become numerically unstable for larger values of $|\sigma|$. An alternative involves the observation that the matrices $A_\sigma = A_{\sigma(n)} \cdot \ldots \cdot A_{\sigma(1)}$ corresponding to the maps ϕ_σ have distinct nonzero real eigenvalues. Let us denote these eigenvalues by λ_+ and λ_- so that $|\lambda_+| > |\lambda_-|$. Then a simple calculation shows that $\phi'_\sigma(z_\sigma) = \lambda_-/\lambda_+$. This becomes simply $(-1)^{|\sigma|}/\lambda_+^2$ if we normalize the determinants of the generators A_0, A_1, and A_2. The larger eigenvalue λ_+ can be easily computed using a naive power method:

(a) Choose a random v_0.
(b) For each $k \geq 0$, set $v_{k+1} = A_\sigma v_k/||A_\sigma v_k||$ and $\lambda_+^{(k)} = \langle A_\sigma v_k, v_k\rangle$.
(c) Iterate until the sequence $(\lambda_+^{(k)})$ converges, up to some prespecified error tolerance.

The resulting algorithm is slightly less efficient than direct matrix multiplication, but it is much less susceptible to the effects of round-off error.

Note that it is certainly possible, even desirable, to apply to this problem modern linear algebraic techniques, such as those implemented in ARPACK [11]. But, we found that the power method suffices in these calculations.

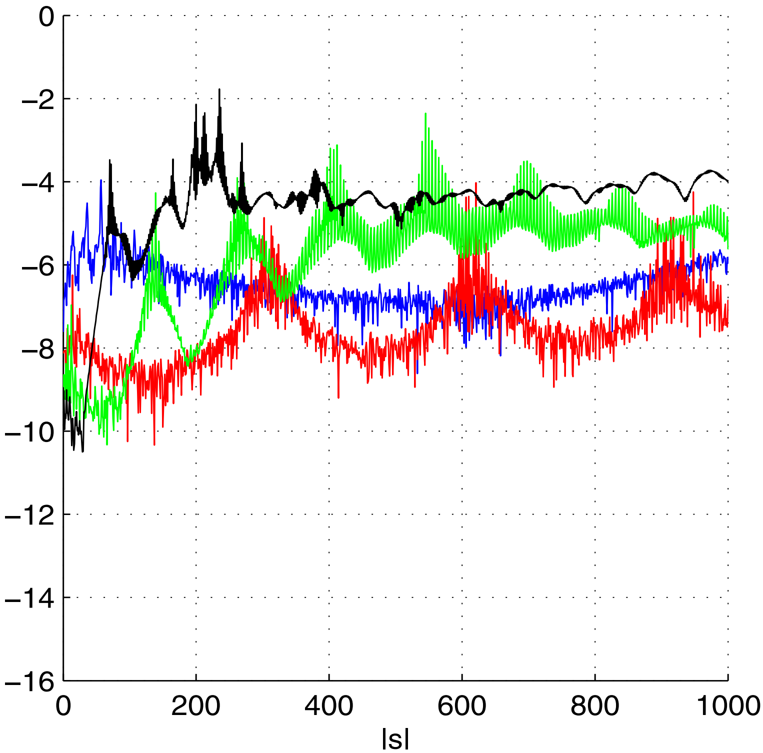

Fig. 11. Logarithmic plot (base 10) of the modified relative error $\frac{|R_{12}(s)-R_{13}(s)|}{1+|R_{12}(s)|+|R_{13}(s)|}$ along the line $\mathrm{Re}(s) = -0.2$, where $R_N(s) = Z'_N(s)/Z_N(s)$. The blue curve is $\theta = 10^\circ$, the red curve $\theta = 20^\circ$, the green curve $\theta = 30^\circ$, and the black curve $\theta = 40^\circ$

These two simple observations let us calculate the values of $Z(s)$ for a wide range of values in an efficient manner. When combined with adaptive gaussian quadrature, Eq. (7.3) allows us to evaluate the relevant contour integrals.

Note that:

(1) To calculate the Selberg zeta function $Z_2(s)$ for closed geodesics on the quotient space $\Gamma\backslash\mathbb{H}^2$, we simply sum over periodic orbits of even length, and additionally use $a_2(n,s) = 2a(n,s)$ instead of $a(n,s)$ in the recursion relations above. This counts the number of equivalence classes of orbits correctly.

(2) The work of Pollicott and Rocha [29] revolves around a closely-related trace formula:

$$Z(s) = 1 + \sum_{N=1}^{\infty} \sum_{r=1}^{N} (-1)^r \sum_{\{[\sigma_1],\ldots,[\sigma_r]\} \in P_\Gamma(N,r)} \prod_{k=1}^{r} \frac{|\phi'_{\sigma_k}(z_{\sigma_k})|^s}{1-\phi'_{\sigma_k}(z_{\sigma_k})}, \tag{7.7}$$

where $P_\Gamma(N,r) = \{\{[\sigma_1],\ldots,[\sigma_r]\} : |\sigma_1| + \ldots + |\sigma_r| = N, \sigma_k \text{ primitive}\}$, and $[\sigma]$ is the equivalence class of σ under shifts. The primary difference between (7.3) and

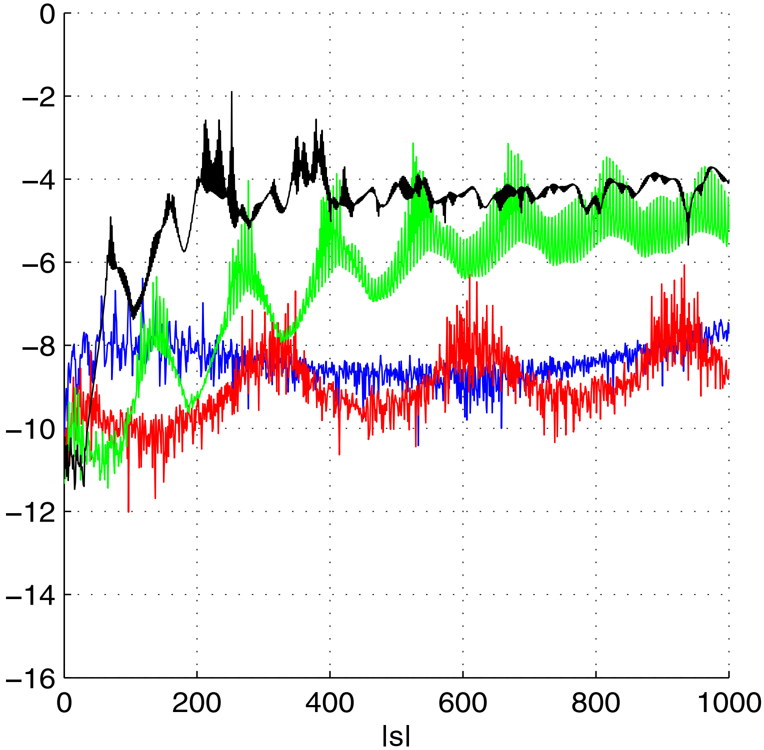

Fig. 12. Logarithmic plot (base 10) of the modified relative error $\frac{|R_{12}(s)-R_{13}(s)|}{1+|R_{12}(s)|+|R_{13}(s)|}$ along the line $\mathrm{Re}(s) = -0.1$, where $R_N(s) = Z_N'(s)/Z_N(s)$. The blue curve is $\theta = 10°$, the red curve $\theta = 20°$, the green curve $\theta = 30°$, and the black curve $\theta = 40°$

(7.7) is that the latter sums over sets of equivalence classes of primitive periodic orbits (equivalent up to shifts), whereas the former sums over all periodic orbits. While it is possible to enumerate primitive periodic orbits efficiently, for example by a simple sieve method, Eq. (7.3) still provides a better numerical algorithm, as it is easier to implement and results in faster and more stable code.

7.3. Error analysis. Figures 11–13 show the logarithms (base 10) of the modified relative errors

$$\frac{|R_{12}(s) - R_{13}(s)|}{1 + |R_{12}(s)| + |R_{13}(s)|}, \tag{7.8}$$

on the lines $x_0+i[0, 10^3]$, for $x_0 \in \{-0.2, -0.1, 0.1\}$ and where $R_N(s) = Z_N'(s)/Z_N(s)$. This formula interpolates between the absolute and the relative errors, and measures the convergence of the integrand in (7.2). These results lend some weight to the reliability (i.e. convergence) of the values of $Z_N'(s)/Z_N(s)$ used in the calculations above.

Note added in proof. Using some of the techniques of this paper, H. Christianson has recently generalized the theoretical results of [34] (where only quadradic functions with real Julia sets were treated) to the case any hyperbolic rational function.

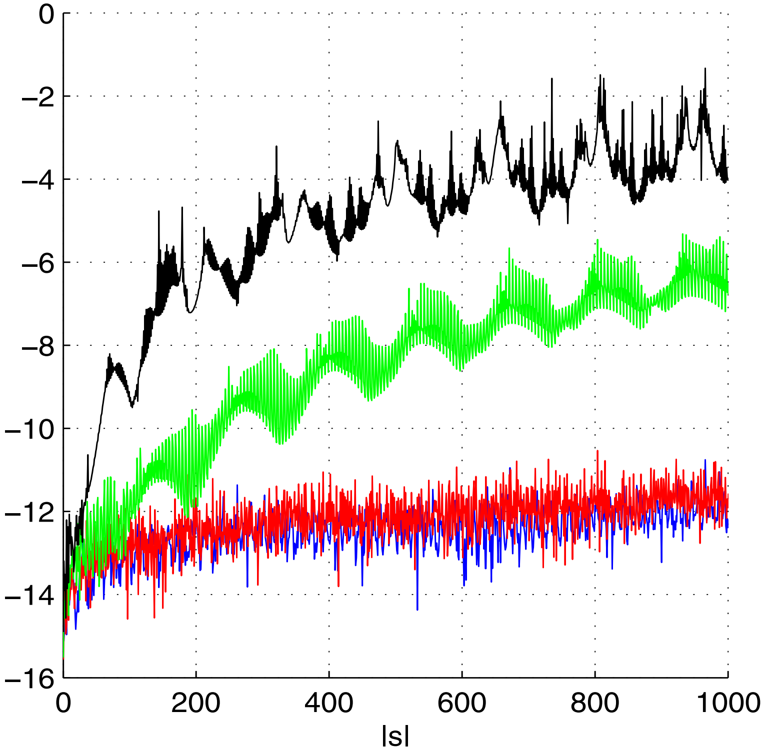

Fig. 13. Logarithmic plot (base 10) of the modified relative error $\frac{|R_{12}(s)-R_{13}(s)|}{1+|R_{12}(s)|+|R_{13}(s)|}$ along the line R$e(s) = +0.1$, where $R_N(s) = Z'_N(s)/Z_N(s)$. The blue curve is $\theta = 10°$, the red curve $\theta = 20°$, the green curve $\theta = 30°$, and the black curve $\theta = 40°$

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References

1. Anderson, J.W., Rocha, A.C.: Analyticity of Hausdorff dimension of limit sets of Kleinian groups. Ann. Acad. Sci. Fennicae **22**, 349–364 (1997)
2. Bunke, U., Olbrich, M.: Group cohomology and the singularities of the Selberg zeta function associated to a Kleinian group. Ann. Math. **149**, 627–689 (1999)
3. Button, J.: All Fuchsian Schottky groups are classical Schottky groups. Geometry and Topology Monographs, **1**, 1998, The Epstein Birthday Schrift, pp. 117–125
4. Eckhardt, B., Russberg, G., Cvitanović, P., Rosenqvist, P.E., Scherer, P.: Pinball Scattering. In: *Quantum Chaos*, Cambridge: Cambridge Univ. Press, 1995, pp. 405–433
5. Fried, D.: The Zeta functions of Ruelle and Selberg. Ann. Sc. Éc. Norm. Sup **19**, 491–517 (1986)
6. Gohberg, I.C., Krein, M.G.: *An introduction to the theory of linear non-self-adjoint operators*. Translation of Mathematical Monographs, **18**, Providence, RI: A.M.S., 1969

7. Guillopé, L.: Sur la distribution des longueurs des géodésiques fermées d'une surface compacte à bord totalement géodésique. Duke Math. J. **53**, 827–848 (1986)
8. Guillopé, L., Zworski, M.: Scattering asymptotics for Riemann surfaces. Ann. Math. **145**, 597–660 (1997)
9. Jenkinson, O., Pollicott, M.: Calculating Hausdorff dimension of Julia sets and Kleinian limit sets. Am. J. Math. **124**, 495–545 (2002)
10. Lalley, S.: Renewal theorems in symbolic dynamics, with applications to geodesic flows, noneuclidean tessellations and their fractal limits. Acta Math. **163**, 1–55 (1989)
11. Lehoucq, R.B., Sorensen, D.C., Yang, C.: *ARPACK User's Guide: Solution of Large-Scale Eigenvalue Problems with Implicitly Restarted Arnoldi Methods*. SIAM, 1998
12. Lin, K.: Numerical study of quantum resonances in chaotic scattering. J. Comp. Phys. **176**, 295–329 (2002)
13. Lin, K., Zworski, M.: Quantum resonances in chaotic scattering. Chem. Phys. Lett. **355**, 201–205 (2002)
14. Lu, W., Sridhar, S., Zworski, M.: Fractal Weyl laws for chaotic open systems. Phys. Rev. Lett. **91**, 154101 (2003)
15. Maskit, B.: A characterization of Schottky groups. J. Analyse Math. **19**, 227–230 (1967)
16. Maskit, B.: *Kleinian groups*. Grundlehren Math. Wiss. **287**, Berlin: Springer-Verlag, 1988
17. Mather, J.N.: Characterization of Anosov diffeomorphisms. Nederl. Akad. Wet. Proc., Ser. A **71**, 479–483 (1968)
18. McMullen, C.: The classification of conformal dynamical systems. In: *Current Developments in Mathematics, 1995*, Cambridge: International Press, 1995, pp. 323–360
19. McMullen, C.: Hausdorff dimension and conformal dynamics III: Computation of dimension. Am. J. Math. **120**, 691–721 (1998)
20. Melrose, R.B.: Polynomial bounds on the number of scattering poles. J. Funct. Anal. **53**, 287–303 (1983)
21. Naud, F.: Expanding maps on Cantor sets, analytic continuation of zeta functions with applications to convex co-compact surfaces. Preprint, 2003
22. Patterson, S.J., Perry, P.: Divisor of the Selberg zeta function for Kleinian groups in even dimensions, with an appendix by C. Epstein. Duke. Math. J. **326**, 321–390 (2001)
23. Perry, P.: Asymptotics of the length spectrum for hyperbolic manifolds of infinite volume. Geom. Funct. Anal. **11**, 132–141 (2001)
24. Perry, P.: A Poisson formula and lower bounds on the number of resonances for hyperbolic manifolds. Int. Math. Res. Notices **34**, 1837–1851 (2003)
25. Petkov, V., Zworski, M.: Breit-Wigner approximation and the distribution of resonances. Commun. Math. Phys. **204**, 329–351 (1999); Erratum, Commun. Math. Phys. **214**, 733–735 (2000)
26. Poincaré, H.: Mémoire sur les groupes Kleinéens. Acta Math. **3**, 49–92 (1883)
27. Pollicott, M.: Kleinian groups, Laplacian on forms and currents at infinity. Proc. Am. Math. Soc. **110**, 269–279 (1990)
28. Pollicott, M.: Some applications of thermodynamic formalism to manifolds with constant negative curvature. Adv. Math. **85**, 161–192 (1991)
29. Pollicott, M., Rocha, A.C.: A remarkable formula for the determinant of the Laplacian. Inv. Math. **130**, 399–414 (1997)
30. Ratcliffe, J.G.: *Foundations of hyperbolic manifolds*. Graduate Texts in Mathematics. **149**, Berlin: Springer-Verlag, 1994
31. Ruelle, D.: Zeta-functions for expanding Anosov maps and flows. Inv. Math. **34**, 231–242 (1976)
32. Rugh, H.H.: The correlation spectrum for hyperbolic analytic maps. Nonlinearity **5**, 1237–1263 (1992)
33. Sjöstrand, J.: Geometric bounds on the density of resonances for semiclassical problems. Duke Math. J. **60**, 1–57 (1990)
34. Strain, J., Zworski, M.: Growth of the zeta function for a quadratic map and the dimension of the Julia set. Preprint, 2003
35. Sullivan, D.: The density at infinity of a discrete group of hyperbolic motions. Publ. Math. IHÉS **50**, 259–277 (1979)
36. Wunsch, J., Zworski, M.: Distribution of resonances for asymptotically euclidean manifolds. J. Diff. Geom. **55**, 43–82 (2000)
37. Zworski, M.: Sharp polynomial bounds on the number of scattering poles. Duke Math. J. **59**, 311–323 (1989)
38. Zworski, M.: Dimension of the limit set and the distribution of resonances for convex co-compact hyperbolic surfaces. Inv. Math. **136**, 353–409 (1999)
39. Zworski, M.: Resonances in Physics and Geometry. Notices Am. Math. Soc. **46**, 319–328 (1999)

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