preliminary version

ON THE MOTION OF A VISCOUS LIQUID FILLING SPACE[1]

by

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in Rennes

Introduction.[2]

I. *The theory of viscosity* leads one to allow that motions of a viscous liquid are governed by Navier's equations. It is necessary to justify this hypothesis a posteriori by establishing *the following existence theorem:* there is a solution of Navier's equations which corresponds to a state of velocity given arbitrarily at an initial instant. That is what Oseen tried to prove[3]. He only succeeded in establishing the existence of such a solution for a possibly very short time after the initial instant. One can also verify that the total kinetic energy of the liquid remains bounded[4] but it does not seem possible to deduce from this fact that the motion itself remains regular*. I have indicated a reason which makes me believe there are motions which become irregular in a finite time[5]. Unfortunately I have not succeeded in creating an example of such a singularity.

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[1] This paper has been summarized in a note which appeared in Comptes rendus de l'Academie des Sciences, February 20 1933, vol. 196, p. 527.

[2] Pages 59–63 of my Thesis (Journ. de Math. 12, 1933) announce this paper and complement this introduction.

[3] See Hydrodynamik (Leipzig, 1927), §7, p. 66. Acta mathematica vol. 34. Arkiv för matematik, astronomi och fysik. Bd. 6, 1910. Nova aeta reg. soc. scient. Upsaliensis Ser. IV, Vol. 4, 1917.

[4] l. c. 2, p. 59–60.

[5] l. c. 2, p. 60–61. I return to this subject in §20 of the present work (p. 224).

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reset from: Acta mathematica. 63. Printed July 5, 1934.

* Translator's note: "regular" is defined on p. 217.

In fact it is not paradoxical to suppose that the thing which regularizes the motion–dissipation of energy–does not suffice to keep the second derivatives of the velocity components bounded and continuous. Navier's theory assumes the second derivatives bounded and continuous. Oseen himself had already emphasised that this was not a natural hypothesis. He showed at the same time how the fact that the motion obeys the laws of mechanics could be expressed by means of integro-differential equations[1] which contain only the velocity components and their first spatial derivatives. In the course of the present work I consider a system of relations[2] equivalent to Oseen's integro-differential equations complemented by an inequality expressing dissipation of energy. Moreover, these relations may be deduced from Navier's equations, using an integration by parts which causes the higher order derivatives to disappear. And, if I have not succeeded in establishing the existence theorem stated above, I have nevertheless proved the following[3]: the relations in question always have *at least one* solution corresponding to a given initial velocity and *which is defined for an unlimited time* of which the origin is the initial instant. Perhaps that solution is not sufficiently regular to have bounded second partial derivatives at each instant, so it is not, in a proper sense of the term, a solution to Navier's equations. I propose to say that it constitutes "*a turbulent solution*".

It is moreover quite remarkable that each turbulent solution actually satisfies Navier's equations, properly said, except at certain times of irregularity. These times constitute a closed* set of measure zero. At these times alone must the continuity of the solution be interpreted in a very generous sense.

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[1] Oseen, Hydrodynamik, §6, equation (1).

[2] See relations (5.15), p. 240.

[3] See p. 241.

[4] I allow myself to cite a passage from Oseen (Hydrodynamik): "From still another point of view it seems worth the trouble to subject the singularities of the motion of a viscous liquid to careful study. If singularities appear, then we must distinguish two types of motion of a viscous liquid, regular motion, which is to say motion without singularity, and irregular motion, which is to say motion with singularity. Now in other parts of Hydraulics one distinguishes two sorts of motion, laminar and turbulent. One is tempted from now on to presume that laminar motions furnished by experiment are identical to theoretical regular motions, and that experimental turbulent motions are identified with irregular theoretical motion. Does this presumption correspond with reality? Only further research will be able to decide."

* [translator's note: The set is compact, as is proved on p. 246.]

A *turbulent solution* therefore has the following *structure*: it is composed of a *succession of regular solutions*.

If I succeed in constructing solutions to Navier's equations which become irregular, then I can say that there exist turbulent solutions which do not simply reduce to regular solutions. Likewise if this proposition is false, the notion of turbulent solution, which from then on plays no role in the study of viscous liquids, still does not lose interest: it serves to present *problems* of mathematical Physics for which *the physical cause of regularity* does not suffice to justify *the hypothesis of regularity made at the time of writing the equation*; to these problems then one can apply considerations similar to those which I introduce here.

Finally let us point out the two following facts:

Nothing allows one to assert the uniqueness of a turbulent solution which corresponds to a given initial state. (See however Supplementary information 1^o, p. 245; §33).

A solution which corresponds to an initial state sufficiently near rest never becomes irregular. (See the case of regularity pointed out in §21 and §22, p. 226 and 227.)

II. The present work concerns unlimited viscous liquids. The conclusions are quite analogous to those of another paper[2] that I devoted to plane motion of a viscous liquid enclosed within fixed convex walls; this leads to the belief that its conclusions extend to the general case of a viscous liquid in two or three dimensions bounded by arbitrary walls. (same variables(?))

The absence of walls indeed introduces some complications concerning the unknown behavior of functions at infinity[3] but greatly simplifies the exposition and brings the essential difficulties more to light. The important role played by homogeneity of the formulas is more evident (the equations in dimensions allow us to predict a priori nearly all the inequalities that we write).

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[1] In virtue of the existence theorem of §31 (p. 241) and of the uniqueness theorem of §18 (p. 222).

[2] Journal de Mathématiques, T. 13, 1934.

[3] The conditions at infinity by which we characterize those solutions of Navier's equations which we call regular, differ essentially from those of Oseen.

Recall that we have already treated the case of unlimited motions in the plane[1]. These are special[2] because the motion is regular.

Summary of the paper.

Chapter I recalls a series of propositions of analysis which are important, but which are not entirely classical.

Chapter II establishes several preliminary inequalities easily deduced from properties of Oseen's fundamental solution.

Chapter III applies the inequalities to the study of regular solutions of Navier's equation.

Chapter IV states several properties of regular solutions to be used in Chapter VI.

Chapter V establishes that for each initial state, there is at least one turbulent solution defined for unlimited time. The proof of this *existence theorem* is based on the following *principle*: one doesn't directly approach the problem of solving Navier's equations; instead one treats a neighboring problem which can be proved to have a regular solution of unlimited duration; we let the neighboring problem tend toward the original problem to construct the limit (or limits) of these solutions. There is an elementary method to apply this principle: it is the same one which I used in my study of planar motion of a viscous liquid within walls, but it is intimately bound with the structure of turbulent solutions which we have previously pointed out. Without this structure the method may not apply. Here we proceed in another fashion whose range is very likely larger, and which justifies the notion of turbulent solution, but which requires calling on some of the less ordinary theorems of chapter I.

Chapter VI studies the structure of turbulent solutions.

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[1] Thesis, Journal de Mathèmatiques 12, 1933; chapter IV p. 64-82. (One can give an interesting variation on the process used there, by using the notion of semi-regular initial state introduced in the present paper.)

[2] In this case one can base the study on the property that the maximum swirl is a decreasing function of time. (See: Comptes rendus de l'Académie des Sciences, T. 194; p. 1893; 30 mai 1932). – Wolibner has also made this remark.

I. Preliminaries

1. *Notation*

We use the letter Π' for an arbitrary domain of points in space. Π' may be the entire space, denoted by Π. ϖ will be a bounded domain in Π which has as boundary a regular surface σ. We represent an arbitrary point of Π by x, which has cartesian coordinates x_i $(i = 1, 2, 3)$ and distance r_0 to the origin, and generates volume element δx and surface element δx_1, δx_2, δx_3. Similarly we use y for a second arbitrary point of Π. r will always represent the distance between points named x and y.

We use the "silent index" convention: a term containing the same index twice represents the sum of terms obtained by successively giving that index the values 1, 2, 3.

Beginning with chapter II the symbol A denotes constants for which we do not specify the numerical value.

We systematically use capital letters for measurable functions and lower case letters for functions which are continuous and have continuous first partial derivatives.

2. *Recall the Schwarz inequality:*

$$\left[\iiint_{\Pi'} U(x)V(x)\,\delta x\right]^2 \leq \iiint_{\Pi'} U^2(x)\,\delta x \times \iiint_{\Pi'} V^2(x)\,\delta x \tag{1.1}$$

– The left side is defined whenever the right is finite. –

This inequality is the foundation of all properties stated in this chapter.

First application:

If

$$U(x) = V_1(x) + V_2(x)$$

then

$$\sqrt{\iiint_{\Pi'} U^2(x)\,\delta x} \leq \sqrt{\iiint_{\Pi'} V_1^2(x)\,\delta x} + \sqrt{\iiint_{\Pi'} V_2^2(x)\,\delta x};$$

more generally for a constant t if one has

$$U(x) = \int_0^t V(x, t')\, dt$$

then

$$\sqrt{\iiint_{\Pi'} U^2(x)\, \delta x} \leq \int_0^t dt' \sqrt{\iiint_{\Pi'} V^2(x, t')\, \delta x} \tag{1.2}$$

the left sides of these inequalities being finite when the right sides are.

Second application:

Consider n constants λ_p and n constant vectors $\vec{\alpha}_p$. Write $x + \vec{\alpha}_p$ for the translation of x by the vector $\vec{\alpha}_p$. We have

$$\iiint_{\Pi} \left[\sum_{p=1}^{p=n} \lambda_p U(x + \vec{\alpha}_p) \right]^2 \delta x \leq \left[\sum_{p=1}^{p=n} |\lambda_p| \right]^2 \times \iiint_{\Pi} U^2(x)\, \delta x;$$

(this inequality is easily proved by expanding the two squares and using the Schwarz inequality). From it, one deduces the following very useful one. Let $H(z)$ be a function. We denote by $H(y - x)$ the function obtained by substituting for coordinates z_i of z the components $y_i - x_i$ of the vector $\vec{xy}$. We have

$$\iiint_{\Pi} \left[\iiint_{\Pi} H(y - x) U(y)\, \delta y \right]^2 \delta x \leq \left[\iiint_{\Pi} |H(z)|\, \delta z \right]^2 \times \iiint_{\Pi} U^2(y)\, \delta y; \tag{1.3}$$

and the left side is finite when the two integrals on the right are finite.

3. *Strong convergence in mean.*[1]

Definition: One says that an infinity of functions $U^*(x)$ have function $U(x)$ as strong limit in mean on a domain Π' when:

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[1] See: F. Riesz, Untersuchungen über Systeme integrierbarer Funktionen, Math. Ann. vol 69 (1910). Delsarte, Mémorial des Sciences mathématiques, fasicule 57, Les groupes de transformation linéaires dans l'espace de Hilbert.

$$\lim \iiint_{\Pi'} [U^*(x) - U(x)]^2 \, \delta x = 0. \tag{1.4}$$

One then has for any square summable function $A(x)$ on Π'

$$\lim \iiint_{\Pi'} U^*(x) A(x) \, \delta x = \iiint_{\Pi'} U(x) A(x) \, \delta x. \tag{1.5}$$

From (1.4) and (1.5)

$$\lim \iiint_{\Pi'} U^{*2}(x) \, \delta x = \iiint_{\Pi'} U^2(x) \, \delta x. \tag{1.6}$$

Weak convergence in mean:

Definition: An infinity of functions $U^*(x)$ has function $U(x)$ as weak limit in mean on domain Π' when the two following conditions hold.

a) the set of numbers $\iiint_{\Pi'} U^{*2}(x) \, \delta x$ is bounded;
b) for all square summable functions $A(x)$ on Π'

$$\lim \iiint_{\Pi'} U^*(x) A(x) \, \delta x = \iiint_{\Pi'} U(x) A(x) \, \delta x.$$

Example I. The sequence $\sin x_1$, $\sin 2x_1$, $\sin 3x_1$, $\ldots$ converges weakly to zero on all domains ϖ.

Example II. If an infinity of functions $U^*(x)$ have strong limit $U(x)$ in mean on all domains ϖ, then they admit a weak limit in mean on Π when the set of quantities $\iiint_{\Pi'} U^{*2}(x) \, \delta x$ is bounded.

Example III. Let an infinity of functions $U^*(x)$ on a domain Π' converge almost everywhere to a function $U(x)$. That function is their weak limit in mean when the set of quantities $\iiint_{\Pi'} U^{*2}(x) \, \delta x$ is bounded.

One has

(1.7)
$$\lim \iiint_{\Pi_1{}'} \iiint_{\Pi_2{}'} A(x,y) U^*(x) V^*(x) \, \delta x \, \delta y = \iiint_{\Pi_1{}'} \iiint_{\Pi_2{}'} A(x,y) U(x) V(x) \, \delta x \, \delta y$$

when the $U^*(x)$ converge weakly in mean to $U(x)$ on $\Pi_1{}'$ and $V^*(x)$ to $V(x)$ on $\Pi_2{}'$ and the integral

$$\iiint_{\Pi_1{}'}\iiint_{\Pi_2{}'} A^2(x)\,\delta x\,\delta y$$

is finite. One has

$$\lim \iiint_{\Pi'} A(x)U^*(x)V^*(x)\,\delta x = \iiint_{\Pi'} A(x)U(x)V(x)\,\delta x \tag{1.8}$$

when on Π', $A(x)$ is bounded, $U(x)$ is the strong limit of the $U^*(x)$ and $V(x)$ is the weak limit of the $V^*(x)$.

It is also evident that, if the functions $U^*(x)$ converge weakly in mean to $U(x)$ on a domain $\Pi_1{}'$

$$\lim\{\iiint_{\Pi'} [U^*(x) - U(x)]^2\,\delta x - \iiint_{\Pi'} U^{*2}(x)\,\delta x + \iiint_{\Pi'} U^2(x)\,\delta x\} = 0$$

from which one gets the inequality

$$\liminf \iiint_{\Pi'} U^{*2}(x)\,\delta x \geq \iiint_{\Pi'} U^2(x)\,\delta x \tag{1.9}$$

and the *criteria for strong convergence:*

The functions $U^*(x)$ converge strongly in mean on domain Π' to $U(x)$ when they converge weakly in mean to $U(x)$ on the domain and in addition

$$\limsup \iiint_{\Pi'} U^{*2}(x)\,\delta x \leq \iiint_{\Pi'} U^2(x)\,\delta x. \tag{1.10}$$

Equivalently, the components $U_i^*(x)$ of the vector converge strongly in mean on domain Π' to those of a vector $U_i(x)$ when they converge weakly in mean to the components on the domain and in addition[1]

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[1] Recall that the symbol $U_i(x)U_i(x)$ represents the expression $\sum_{i=1}^{i=3} U_i(x)U_i(x)$.

$$(1.10') \qquad \limsup \iiint_{\Pi'} U_i^*(x) U_i^*(x)\, \delta x \leq \iiint_{\Pi'} U_i(x) U_i(x)\, \delta x.$$

The weak convergence criteria applied in Example III gives the following.

Lemma 1. If an infinity of functions $U^*(x)$ [or vectors $U_i^*(x)$] converge almost everywhere on domain Π' to a function $U(x)$ [or a vector $U_i^*(x)$] and satisfy inequality (1.10) [or (1.10')], then they converge strongly in mean.

Theorem of F. Riesz: An infinity of functions $U^*(x)$ have a weak limit in mean on domain Π' if the two following conditions are satisfied:

a) the set of numbers $\iiint_{\Pi'} U^{*2}(x)\, \delta x$ is bounded;

b) for each square summable function $A(x)$ on Π' the quantities $\iiint_{\Pi'} U^*(x) A(x)\, \delta x$ have a single limiting value.

Condition b) may be replaced by the following:

b') for each cube c with sides parallel to the coordinate axes and rational vertices, the quantities $\iiint_{\Pi'} U^*(x)\, \delta x$ have a single limiting value.

The proof of this theorem makes use of the work of Lebesgue on summable functions.

4. *Cantor's diagonal method.*

Consider a countable infinity of quantities each dependent on integer indices n: a_n, b_n, $\ldots (n = 1, 2, 3 \ldots)$. Suppose the a_n are bounded, the b_n are bounded, etc. Cantor's diagonal method allows us to find a sequence of integers m_1, m_2, …, such that each of the sequences a_{m_1}, a_{m_2}, …; b_{m_1}, b_{m_2}, …; … converge to a limit.

Recall this process briefly: one constructs a first sequence of integers n_1^1, n_2^1, n_3^1 … such that the quantities $a_{n_1^1}$, $a_{n_2^1}$, $a_{n_3^1}$, … converge to a limit; one then constructs using elements of the first sequence a second sequence n_1^2, n_2^2, n_3^2 …, such that the quantities $b_{n_1^2}$, $b_{n_2^2}$, $b_{n_3^2}$, …

converge to a limit; etc. One then chooses m_p equal to n_p^p, which is the p-th term of the diagonal of the infinite table of n_i^j.

Application: The following results from the theorem of the preceeding paragraph.

Fundamental theorem of F. Riesz Let an infinity of functions $U^*(x)$ on a domain Π' be such that the quantities $\iiint_{\Pi'} U^{*2}(x)\,\delta x$ are bounded. Then one can always extract from them a sequence of functions which have a weak limit in mean.

In fact condition a) is satisfied and Cantor's diagonal process allows construction of a sequence of functions $U^*(x)$ which satisfy condition b).

5. *Various modes of continuity of a function with respect to a parameter.*

Let a function $U(x,t)$ depend on a parameter t. We say it is *uniformly continuous* in t when the following three conditions hold:

a) it is continuous with respect to x_1, x_2, x_3, t;

b) for each particular value t_0 of t the maximum of $U(x,t_0)$ is finite;

c) given a positive number ϵ, one can find a positive η such that the inequality $|t-t_0| < \eta$ implies

$$|U(x,t) - U(x,t_0)| < \epsilon.$$

The maximum of $|U(x,t)|$ on Π is then a continuous function of t.

We say that $U(x,t)$ is *strongly continuous* in t when, for each particular value t_0 of t, $\iiint_{\Pi} U^2(x,t_0)\,\delta x$ is finite and for each ϵ there is η such that the inequality $|t-t_0| < \eta$ implies

$$\iiint_{\Pi} [U(x,t) - U(x,t_0)]^2\,\delta x < \epsilon.$$

The integral $\iiint_{\Pi} U^2(x,t_0)\,\delta x$ is therefore a continuous function of t. Conversely we learn from lemma 1 that a function $U(x,t)$ continuous with respect to variables x_1, x_2, x_3, t is strongly continuous in t when the preceeding integral is a continous function of t.

6. *Relations between a function and its derivatives*

Consider two functions $u(x)$ and $a(x)$ with continuous first derivatives, with the functions and the first derivatives square summable on Π. s is the surface of a sphere S with center at the origin and for which the radius r_0 may become arbitrarily large. Let

$$\varphi(r_0) = \iint_s u(x)a(x)\,\delta x_i;$$

We have*

$$\varphi(r_0) = \iiint_S \left[u(y)\frac{\partial a(y)}{\partial y_i} + \frac{\partial u(y)}{\partial y_i}a(y)\right]\,\delta y.$$

The second expression shows that $\varphi(r_0)$ tends to a limit $\varphi(\infty)$ when r_0 increases indefinitely. The first expression for $\varphi(r_0)$ gives us

$$|\varphi(r_0)| \leq \iint_s |u(x)a(x)|\frac{x_i\,\delta x_i}{r_0}$$

from which

$$\int_0^\infty |\varphi(r_0)|\,dr_0 \leq \iiint_\Pi |u(x)a(x)|\,\delta x.$$

As a result $\varphi(\infty) = 0$. In other words

$$\iiint_\Pi \left[u(y)\frac{\partial a(y)}{\partial y_i} + \frac{\partial u(y)}{\partial y_i}a(y)\right]\,\delta y = 0 \tag{1.11}$$

and from this we have more generally

$$\iiint_{\Pi-\varpi} \left[u(y)\frac{\partial a(y)}{\partial y_i} + \frac{\partial u(y)}{\partial y_i}a(y)\right]\,\delta y = -\iint_\sigma u(x)a(x)\,\delta y_i. \tag{1.12}$$

Choose as domain ϖ a sphere of infinitely small radius and center x and take[1] in (1.12) $a(y) = \frac{1}{4\pi}\frac{\partial(\frac{1}{r})}{\partial y_i}$; add the

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[1] r is the distance between the points x and y.

*`translator's note:` It seems that δx_1 means dx_2dx_3 etc.

relations for values 1, 2, 3 of i to obtain the important indentity

$$u(x) = \frac{1}{4\pi}\iiint \frac{\partial(\frac{1}{r})}{\partial y_i}\frac{\partial u}{\partial y_i}\,\delta y. \tag{1.13}$$

We now take $a(y) = \frac{y_i - x_i}{r^2}u(y)$ in (1.11) and add these relations for values 1, 2, 3 of i, giving

$$2\iiint_\Pi \frac{y_i - x_i}{r^2}\frac{\partial u}{\partial y_i}u(y)\,\delta y = -\iiint_\Pi \frac{1}{r^2}u^2(y)\,\delta y.$$

By applying the Schwarz inequality to the left side we get the useful inequality

$$\iiint_\Pi \frac{1}{r^2}u^2(y)\,\delta y \leq 4\iiint_\Pi \frac{\partial u}{\partial y_i}\frac{\partial u}{\partial y_i}\,\delta y. \tag{1.14}$$

7. *Quasi-derivatives.*

Let $u^*(x)$ be an infinity of square summable functions with continuous square summable first derivatives on Π. Suppose that the derivatives $\frac{\partial u^*}{\partial x_1}$, $\frac{\partial u^*}{\partial x_2}$, $\frac{\partial u^*}{\partial x_3}$ converge weakly in mean on Π to functions $U_{,1}$, $U_{,2}$, $U_{,3}$. Let $U(x)$ be the measurable function defined almost everywhere by

$$U(x) = \frac{1}{4\pi}\iiint_\Pi \frac{\partial(\frac{1}{r})}{\partial y_i}U_{,i}\,\delta y.$$

We have

(1.15)
$$\iiint_\varpi [u^*(x) - U(x)]^2\,\delta x = -\iiint_\Pi\iiint_\Pi K_{ij}(y,y')\left[\frac{\partial u^*}{\partial y_i} - U_{,i}(y)\right]\left[\frac{\partial u^*}{\partial y_j{}'} - U_{,j}(y')\right]\delta y\,\delta y'$$

where[1]

[1] r' is the distance between the points x and y'.

$$K_{ij}(y,y') = \frac{1}{16\pi^2}\iiint_{\varpi} \frac{\partial(\frac{1}{r})}{\partial y_i}\frac{\partial(\frac{1}{r'})}{\partial y_i{}'}\,\delta x_i.$$

This expression for K allows an easy proof that the integral

$$\iiint_{\Pi}\iiint_{\Pi} K_{ij}(y,y')K_{ij}(y,y')\,\delta y\,\delta y'$$

is finite, so the right side of (1.15) is defined. It tends to zero by (1.7). Therefore the $u^*(x)$ have $U(x)$ as strong limit in mean on all domains ϖ. And, if the integrals $\iiint_{\Pi} U^{*2}(x)\,\delta x$ are bounded, $U(x)$ is the weak limit in mean of the $u^*(x)$ on Π (Cf. §3. Example II, p. 199). One then gets from (1.11) the equality

$$\iiint_{\Pi}\left[U(y)\frac{\partial a}{\partial y_i} + U_{,i}(y)a(y)\right]\,\delta y = 0. \tag{1.16}$$

We make the following definition:

Definition of quasi-derivatives: Consider two square summable functions $U(y)$ and $U_{,i}(y)$ on Π. We say that $U_{,i}(y)$ is the quasi-derivative of $U(x)$ with respect to y_i when (1.16) holds. Recall that in (1.16) $a(y)$ is any square summable function with continuous square summable first derivatives on Π.

Let us summarize the results of preceeding paragraph.

Lemme 2. Suppose we have an infinity of continuous functions $u^*(x)$ with continuous first derivatives. Suppose the integrals $\iiint_{\Pi} u^{*2}(x)\,\delta x$ are bounded and that each of the derivatives $\frac{\partial u^*(x)}{\partial y_i}$ has a weak limit in mean $U_{,i}(x)$ on Π. Then the $u^*(x)$ converge in mean to a function $U(x)$ for which the $U_{,i}(x)$ are the quasi-derivatives. This convergence is strong on all domains ϖ. It is weak[1] on Π.

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[1] Or strong.

Following our definition of quasi-derivatives we are going to define the *quasi-divergence* $\Theta(x)$ of a vector $U_i(x)$ with square summable components on Π. When it exists, it is a square summable function with

$$\iiint_\Pi \left[U_i(y)\frac{\partial a}{\partial y_i} + \Theta(y)a(y)\right] \delta y = 0. \tag{1.17}$$

8. *Approximation of a measurable function by a sequence of regular functions.* Let $\epsilon > 0$. We choose a positive continuous function $\lambda(s)$ defined for $0 \le s$, identically zero for $1 \le s$ and having derivatives of all orders such that

$$4\pi\int_0^1 \lambda(\sigma^2)\sigma^2\, d\sigma = 1.$$

If $U(x)$ is summable on all domains ϖ, let

$$\overline{U(x)} = \frac{1}{\epsilon^3}\iiint_\Pi \lambda\left(\frac{r^2}{\epsilon^2}\right) U(y)\,\delta y \tag{1.18}$$

(r = distance between x and y)

$\overline{U(x)}$ has derivatives of all orders

$$\frac{\partial^{l+m+n}\overline{U(x)}}{\partial x_1^l \partial x_2^m \partial x_3^n} = \frac{1}{\epsilon^3}\iiint_\Pi \frac{\partial^{l+m+n}\lambda\left(\frac{r^2}{\epsilon^2}\right)}{\partial x_1^l \partial x_2^m \partial x_3^n} U(y)\,\delta y. \tag{1.19}$$

If $U(x)$ is bounded on Π then we clearly have

$$\min U(x) \le \overline{U(x)} \le \max U(x). \tag{1.20}$$

If $U(x)$ is square summable on Π the inequality (1.3) applied to (1.18) gives*

$$\iiint_\Pi \overline{U(x)}^2\,\delta x \le \iiint_\Pi U^2(x)\,\delta x. \tag{1.21}$$

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[1] To fix ideas we take $\lambda(s) = Ae^{\frac{1}{s-1}}$, A any suitable constant, $0 < s < 1$.

* [`translator's note:` The bar was extended too far in the original.]

The same applied to (1.19) proves that the partial derivatives of $\overline{U(x)}$ are square summable on Π.

Finally note that we have, if $U(x)$ and $V(x)$ are square summable on Π

$$\iiint_\Pi \overline{U(x)} V(x)\, \delta x = \iiint_\Pi U(x) \overline{V(x)}\, \delta x. \tag{1.22}$$

If $V(x)$ is continuous $\overline{V(x)}$ tends uniformly to $V(x)$ on all domains ϖ when ϵ tends to zero. One therefore has from (1.22)

$$\lim \iiint_\Pi \overline{U(x)} V(x)\, \delta x = \iiint_\Pi U(x) V(x)\, \delta x.$$

From this one deduces that $\overline{U(x)}$ converges weakly in mean to $U(x)$ on Π when ϵ approaches zero. Inequality (1.21) and the criteria for strong convergence on p. 200 similarly give a more precise conclusion:

Lemma 3. Let $U(x)$ be square summable on Π. $\overline{U(x)}$ converges strongly in mean to $U(x)$ on Π when ϵ tends to zero.

Similarly one establishes the following proposition.

Generalization of lemma 3. Suppose a sequence of functions $U_\epsilon(x)$ converge strongly (or weakly) in mean on Π to a limit $U(x)$ as ϵ tends to zero. Then the functions $\overline{U_\epsilon(x)}$ converge strongly (or weakly) to the same limit.

9. *Some lemmas on quasi-derivatives.*

Let $U(x)$ be square summable on Π. Suppose that for all square summable functions $a(x)$ having square summable derivatives of all orders

$$\iiint_\Pi U(x) a(x)\, \delta x = 0$$

then

$$\iiint_\Pi U(x) \overline{U(x)}\, \delta x = 0$$

from which one gets as ϵ tends to zero

$$\iiint_\Pi U^2(x)\,\delta x = 0.$$

The function $U(x)$ is therefore zero almost everywhere.

That fact allows us to establish the following propositions. 1) When the quasi-derivative of a function with respect to x_i exists, it is unique. (We consider two functions identical if they are equal almost everywhere.) 2) The quasi-divergence of a vector is unique if it exists.

Lemma 4. Let $U(x)$ have a quasiderivative $U_i(x)$. Then I claim that $\frac{\partial \overline{U(x)}}{\partial x_i} = \overline{U_i(x)}$.

It suffices to prove that

$$\iiint_\Pi \frac{\partial \overline{U(x)}}{\partial x_i} a(x)\,\delta x = \iiint_\Pi U_i(x) a(x)\,\delta x.$$

Because one easily deduces from (1.18) that

$$\frac{\partial \overline{a(x)}}{\partial x_i} = \overline{\frac{\partial a(x)}{\partial x_i}}$$

and this formula with (1.11), (1.16), and (1.22) justify the transformations

$$\iiint_\Pi \frac{\partial \overline{U(x)}}{\partial x_i} a(x)\,\delta x = -\iiint_\Pi \overline{U(x)} \frac{\partial a(x)}{\partial x_i}\,\delta x = -\iiint_\Pi U(x) \overline{\left(\frac{\partial a(x)}{\partial x_i}\right)}\,\delta x =$$

$$-\iiint_\Pi U(x) \frac{\partial \overline{a(x)}}{\partial x_i}\,\delta x = \iiint_\Pi U_i(x)\overline{a(x)}\,\delta x = \iiint_\Pi \overline{U_i(x)} a(x)\,\delta x. \qquad Q.E.D.$$

Lemma 5. Suppose that two square summable functions $U(x)$ and $V(x)$ have quasi-derivatives $U_i(x)$ and $V_i(x)$ on Π. I claim that

$$\iiint_\Pi [U(x)V_i(x) + U_i(x)V(x)]\,\delta x = 0. \tag{1.23}$$

This is obtained by applying lemma 3 to the formula

$$\iiint_\Pi [U(x)\overline{V_i(x)} + U_i(x)\overline{V(x)}]\,\delta x = 0.$$

which follows from 1.16 and lemma 4.

Lemma 6. If a vector $U_i(x)$ has quasi-divergence $\Theta(x)$ then the divergence of $\overline{U_i(x)} = \overline{\Theta(x)}$.

(The proof is very much analogous to that of lemma 4.)

Lemme 7. Suppose a vector $U_i(x)$ has quasi-divergence 0, and that

$$\iiint_\Pi U_i(x) a_i(x)\,\delta x = 0$$

for all square summable vectors $a_i(x)$ which have 0 divergence and square summable derivatives of all orders on Π. Then I claim that $U_i(x) = 0$.

In fact lemma 4 allows us to choose $a_i(x) = \overline{U_i(x)}$ because when ϵ tends to 0 the relation $\iiint_\Pi U_i(x)\overline{U_i(x)}\,\delta x = 0$ reduces to

$$\iiint_\Pi U_i(x) U_i(x)\,\delta x = 0.$$

Corollary. An infinity of vectors $U_i^*(x)$, of quasi-divergence 0 has, on Π, a unique weak limit in mean if the two following conditions hold:

a) the numbers $\iiint_\Pi U_i^*(x) U_i^*(x)\,\delta x = 0$ are bounded

b) for all square summable vectors $a_i(x)$ which have 0 divergence and square summable derivatives of all orders on Π, the quantities $\iiint_\Pi U_i^*(x) a_i(x)\,\delta x$ have a single limiting value.

If not, then the fundamental theorem of F. Riesz (p. 202) allows extraction from the sequence $U_i^*(x)$ two subsequences having distinct limits. This contradicts lemma 7.

II. Infinitely slow motion.

10. The "linearised Navier equations" are the following

$$\nu \Delta u_i(x,t) - \frac{\partial u_i(x,t)}{\partial t} - \frac{1}{\rho}\frac{\partial p(x,t)}{\partial x_i} = -X_i(x,t) \left[\Delta = \frac{\partial^2}{\partial x_k \partial x_k}\right] \tag{2.1}$$

$$\frac{\partial u_j(x,t)}{\partial x_j} = 0.$$

ν and ρ are given constants, $X_i(x,t)$ is a vector which represents external forces, $p(x,t)$ is the pressure, and $u_i(x,t)$ the speed of the molecules of the liquid.

The problem posed by the theory of viscous liquids is the following: Construct for $t > 0$ the solution of (2.1) which has given initial values $u_i(x,0)$.

We recall the solution of this problem and some of its properties. Write

$$W(t) = \iiint_\Pi u_i(x,t)u_i(x,t)\,\delta x$$

$$J_m^2(t) = \iiint_\Pi \frac{\partial^m u_i(x,t)}{\partial x_k \partial x_l \ldots} \frac{\partial^m u_i(x,t)}{\partial x_k \partial x_l \ldots}\,\delta x.$$

$V(t) =$ Maximum of $\sqrt{u_i(x,t)u_i(x,t)}$ at time t.

$D_m(t) =$ Maximum of the function $\left| \frac{\partial^m u_i(x,t)}{\partial x_1^h \partial x_2^k \partial x_3^l} \right|$ at time t $(h+k+l=m)$.

We make the following assumptions: The functions $u_i(x,t)$ and their first derivatives are continuous, $\frac{\partial u_j(x,0)}{\partial x_j} = 0$, the quantities $W(0)$ and $V(0)$ are finite, $|X_i(x,t) - X_i(y,t)| < r^{\frac{1}{2}} C(x,y,t)$, where $C(x,y,t)$ is a continuous function, and $\iiint_\Pi X_i(x,t)X_i(x,t)\,\delta x$ is a continuous function of t, or is less than a continuous function of t.

From now on the letters A and A_m denote constants and functions with index m for which we do not specify the numerical value.

11. *First case:* $X_i(x,t) = 0$.

The theory of heat gives the following solution to system (2.1):

$$u_i'(x,t) = \frac{1}{(2\sqrt{\pi})^3} \iiint_\Pi \frac{e^{-\frac{r^2}{4\nu t}}}{(\nu t)^{\frac{3}{2}}} u_i(y,0)\,\delta y; \quad p'(x,t) = 0. \tag{2.2}$$

The integrals $u_i'(x,t)$ are uniformly continuous in t (cf. §5, p. 202) for $0 < t$, and from this one has

(2.3) $$V(t) < V(0).$$

If $J_1(0)$ is finite, inequality (1.14) and the Schwarz inequality (1.1) applied to (2.2) give a second bound on $V(t)$:

$$V^2(t) < 4J_1^2(0)\frac{1}{(4\pi)^3}\iiint_\Pi \frac{e^{-\frac{\mathbf{r}^2}{2\nu t}}}{(\nu t)^3} r^2\,\delta y,$$

which is to say

(2.4) $$V(t) < \frac{AJ_1(0)}{(\nu t)^{\frac{1}{4}}}.$$

Inequality (1.3) applied to (2.2) proves:

(2.5) $$W(t) < W(0);$$

the integrals $u_i'(x,t)$ are strongly continuous in t (cf. §5, p. 202) including $t=0$. Inequality (1.3) applied to

$$\frac{\partial u_i'(x,t)}{\partial x_k} = \frac{1}{(2\sqrt{\pi})^3}\iiint_\Pi \frac{\partial}{\partial x_k}\left[\frac{e^{-\frac{r^2}{4\nu t}}}{(\nu t)^{\frac{3}{2}}}\right] u_i(y,0)\,\delta y$$

proves that

(2.6) $$J_1(t) < J_1(0);$$

the first derivatives $\frac{\partial u_i'}{\partial x_k}$ are strongly continuous in t, including $t=0$ if $J_1(0)$ is finite.

For analogous reasons the derivatives of all orders of $u_i'(x,t)$ are uniformly and strongly continuous in t for $t>0$ and more precisely

(2.7) $$D_m(t) < \frac{A_m\sqrt{W(0)}}{(\nu t)^{\frac{2m+3}{4}}},$$

(2.8) $$J_m(t) < \frac{A_m\sqrt{W(0)}}{(\nu t)^{\frac{m}{2}}}.$$

12. *Second particular case* $u_i'(x,0)=0$.

Oseen's fundamental solution[1], $T_{ij}(x,t)$, furnishes the following solution to system (2.1):

[1] See: Oseen: Hydrodynamik §5; Acta mathematica vol. 34.

[`translator's note:` which gives on p. 41

$t_{jk} = \delta_{jk}\frac{1}{2\nu}\frac{E(r,t^{(0)}-t)}{t^{(0)}-t} + \frac{\partial^2\Phi}{\partial x_j\partial x_k}$, $\Phi = \frac{1}{r}\int_0^r E(\alpha, t^{(0)}-t)\,d\alpha$, $E(r,t^{(0)}-t) = \frac{e^{-\frac{\mathbf{r}^2}{4\nu(t^{(0)}-t)}}}{\sqrt{t^{(0)}-t)}}$.]

$$u_i''(x,t) = \int_0^t dt' \iiint_\Pi T_{ij}(x-y, t-t') X_j(y,t')\, \delta y \tag{2.9}$$

$$p''(x,t) = -\frac{\rho}{4\pi}\frac{\partial}{\partial x_j}\iiint_\Pi \frac{1}{r} X_j(y,t)\, \delta y$$

We have

$$|T_{ij}(x-y,t-t')| < \frac{A}{[r^2+\nu(t-t')]^{\frac{3}{2}}} \tag{2.10}$$

$$\left|\frac{\partial^m T_{ij}(x-y,t-t')}{\partial x_1^h \partial x_2^k \partial x_3^l}\right| < \frac{A_m}{[r^2+\nu(t-t')]^{\frac{m+3}{2}}}; \quad (t'<t).$$

We remark in the first place that integrals (1.2) and (1.3) applied with (2.10) to the formula

$$\frac{\partial u_i''(x,t)}{\partial x_k} = \int_0^t dt' \iiint_\Pi \frac{\partial T_{ij}(x-y,t-t')}{\partial x_k} X_j(y,t')\, \delta y \tag{2.11}$$

prove that the first derivatives $\frac{\partial u_i''}{\partial x_k}$ are strongly continuous in t for $t \geq 0$, and that

$$J_1(t) < A\int_0^t \frac{dt'}{\sqrt{\nu(t-t')}}\sqrt{\iiint_\Pi X_i(x,t')X_i(x,t')\,\delta x}. \tag{2.12}$$

This done, we add to previously stated hypotheses the assumption that the maximum of $\sqrt{X_i(x,t')X_i(x,t')}$ at time t is a continuous function of t, or is less than a continuous function of t. Then there is no difficulty in deducing from (2.9) that $u_i''(x,t)$ and $\frac{\partial u_i''}{\partial x_k}$ are uniformly continuous in t for $t \geq 0$, and more precisely for example

$$D_1(t) < A\int_0^t \frac{dt'}{\sqrt{\nu(t-t')}} \max\sqrt{X_i(x,t')X_i(x,t')}. \tag{2.13}$$

Inequality (2.13) may be complemented as follows. We have

$$\frac{\partial u_i''(x,t)}{\partial x_k} - \frac{\partial u_i''(y,t)}{\partial y_k} = \int_0^t dt' \iiint_{\varpi} \frac{\partial T_{ij}(x-z,t-t')}{\partial x_k} X_j(z,t')\,\delta z$$
$$- \int_0^t dt' \iiint_{\varpi} \frac{\partial T_{ij}(y-z,t-t')}{\partial y_k} X_j(z,t')\,\delta z$$

$$+ \int_0^t dt' \iiint_{\Pi-\varpi} \left[\frac{\partial T_{ij}(x-z,t-t')}{\partial x_k} - \frac{\partial T_{ij}(y-z,t-t')}{\partial y_k}\right] X_j(z,t')\,\delta z,$$

ϖ being the domain of points at distance less than $2r$ to x or y. We apply the formula of finite differences to the bracket

$$\left[\frac{\partial T_{ij}(x-z,t-t')}{\partial x_k} - \frac{\partial T_{ij}(y-z,t-t')}{\partial y_k}\right]$$

and majorize the preceeding three integrals by replacing the various functions there by the majorants of their absolute values. We easily verify

$$\left|\frac{\partial u_i''(x,t)}{\partial x_k} - \frac{\partial u_i''(y,t)}{\partial y_k}\right| < \tag{2.14}$$
$$< A r^{\frac{1}{2}} \int_0^t \frac{dt'}{[\nu(t-t')]^{\frac{3}{4}}} \max \sqrt{X_i(x,t')X_i(x,t')}.$$

— We say that *a function* $U(x,t)$ *satisfies condition* H if an inequality analogous to the preceeding holds:

$$|U(x,t) - U(y,t)| < r^{\frac{1}{2}} C(t), \tag{2.15}$$

where $C(t)$ is smaller than a continuous function of t. We call the weakest possible $C(t)$, the condition H coefficient. —

Now suppose that the functions $X_i(x,t)$ satisfy condition H with coefficient $C(t)$. Then the second derivatives $\frac{\partial^2 u_i''(x,t)}{\partial x_k \partial x_l}$, given by the formulas

$$\frac{\partial^2 u_i''(x,t)}{\partial x_k \partial x_l} = \int_0^t dt' \iiint_\Pi \frac{\partial^2 T_{ij}(x-y,t-t')}{\partial x_k \partial x_l} [X_j(y,t') - X_j(x,t')]\,\delta y,$$

are then uniformly continuous in t and

$$D_2(t) < A\int_0^t \frac{C(t')\,dt'}{[\nu(t-t')]^{\frac{3}{4}}}. \tag{2.16}$$

More generally:

Suppose the m-th order derivatives of the $X_i(x,t)$ with respect to x_1, x_2, x_3 exist, are continuous, and are smaller than some continuous functions $\varphi_m(t)$. Then the derivatives of order $m+1$ of the $u_i''(x,t)$ exist and are uniformly continuous in t. We have

$$D_{m+1}(t) < A\int_0^t \frac{\varphi_m(t')\,dt'}{\sqrt{\nu(t-t')}} \tag{2.17}$$

and finally the derivatives of order $m+1$ satisfy condition H with coefficient

$$C_{m+1}(t) < A\int_0^t \frac{\varphi_m(t')\,dt'}{[\nu(t-t')]^{\frac{3}{4}}}. \tag{2.18}$$

If further

$$\iiint_\Pi \left[\frac{\partial^m X_i(x,t)}{\partial x_1^h \partial x_2^k \partial x_3^l}\right]^2 \delta x < \psi_m^2(t),$$

where $\psi_m(t)$ is a (positive) continuous function, then the derivatives of order $m+1$ with respect to x_1, x_2, x_3 of the $u_i(x,t)$ are strongly continuous in t and satisfy the inequality

$$J_{m+1}(t) < A\int_0^t \frac{\psi_m(t')\,dt'}{\sqrt{\nu(t-t')}} \tag{2.19}$$

Now suppose that the derivatives of order m of the functions $X_i(x,t)$ with respect to x_1, x_2, x_3 exist, are smaller in absolute value than a continuous function of t, and satisfy condition H with coefficient $\theta_m(t)$. Then the derivatives of order $m+2$ of the $u_i(x,t)$ exist, are uniformly continuous, and satisfy the inequality

$$D_{m+2}(t) < A \int_0^t \frac{\theta_m(t')\,dt'}{[\nu(t-t')]^{\frac{3}{4}}}. \tag{2.20}$$

13. *General case.*

To obtain solutions $u_i(x,t)$ of (2.1) corresponding to given initial values $u_i(x,0)$, it suffices to superpose the two preceeding particular solutions, taking

$$u_i(x,t) = u_i'(x,t) + u_i''(x,t);\; p(x,t) = p''(x,t).$$

We propose to complete the information of the two preceeding paragraphs by establishing that $u_i(x,t)$ is strongly continuous in t and is majorised by $W(t)$.

This strong continuity is evident in the case where $X_i(x,t)$ is zero outside of a domain ϖ. When x moves indefinitely far away, $u_i''(x,t)$, $\frac{\partial u_i''(x,t)}{x_k}$ and $p(x,t)$ approach zero as $(x_i x_i)^{-\frac{3}{2}}$, $(x_i x_i)^{-2}$ and $(x_i x_i)^{-1}$ respectively, and it suffices to integrate

$$\nu u_i \Delta u_i - \frac{1}{2}\frac{\partial}{\partial t}(u_i u_i) - \frac{1}{\rho} u_i \frac{\partial p}{\partial x_i} = -u_i X_i$$

to obtain the *the relation of dissipation of energy*

$$\nu \int_0^t J_1^2(t')\,dt' + \frac{1}{2}W(t) - \frac{1}{2}W(0) = \int_0^t dt' \iiint_\Pi u_i(x,t')X_i(x,t')\,\delta x \tag{2.21}$$

from which we get the inequality

$$\frac{1}{2}W(t) \leq \frac{1}{2}W(0) + \int_0^t dt' \sqrt{W(t')} \sqrt{\iiint_\Pi X_i(x,t')X_i(x,t')\,\delta x}.$$

$W(t)$ is therefore less than or equal to the solution $\lambda(t)$ of the equation

$$\frac{1}{2}\lambda(t) = \frac{1}{2}W(0) + \int_0^t dt' \sqrt{\lambda(t')} \sqrt{\iiint_\Pi X_i(x,t')X_i(x,t')\,\delta x}$$

which is to say

$$\sqrt{W(t)} \leq \int_0^t \sqrt{\iiint_\Pi X_i(x,t')X_i(x,t')\,\delta x}\,dt' + \sqrt{W(0)}. \tag{2.22}$$

When $X_i(x,t)$ is not zero outside a domain ϖ, one can approach the functions $X_i(x,t)$ by a sequence of functions $X_i^*(x,t)$ zero outside domains ϖ^*, and establish by the preceeding that relations (2.21) and (2.22) still hold. Then (2.21) shows that $W(t)$ is continuous. The $u_i(x,t)$ are therefore strongly continuous in t for $t \geq 0$.

14. $u_i(x,t) = u_i'(x,t)+u_i''(x,t)$ is the only solution to the problem posed in paragraph 10, for which $W(t)$ is less than a continuous function of t. This proposition results from the following

Uniqueness theorem The system

$$\nu\Delta u_i(x,t) - \frac{\partial u_i(x,t)}{\partial t} - \frac{1}{\rho}\frac{\partial p(x,t)}{\partial x_i} = 0; \quad \frac{\partial u_j(x,t)}{\partial x_j} = 0 \tag{2.23}$$

has just one solution defined and continuous for $t \geq 0$, zero for $t = 0$, such that $W(t)$ is less than a continuous function of t. This solution is $u_i(x,t) = 0$.

In fact the functions

$$v_i(x,t) = \int_0^t \overline{u_i(x,t)}\, dt', \quad q(x,t) = \int_0^t \overline{p(x,t)}\, dt'$$

are solutions to the same system (2.23). The derivatives

$$\frac{\partial^m v_i(x,t)}{\partial x_1^h \partial x_2^k \partial x_3^l} \quad \text{and} \quad \frac{\partial^{m+1} v_i(x,t)}{\partial t \partial x_1^h \partial x_2^k \partial x_3^l}$$

exist and are continuous. One evidently has $\Delta q = 0$ and it follows that

$$\nu\Delta\Delta v_i - \frac{\partial}{\partial t}(\Delta v_i) = 0.$$

The theory of heat allows us to deduce that $\Delta v_i = 0$. Further, inequalities (1.2) and (1.21) show that the integral $\iiint_\Pi v_i(x,t)v_i(x,t)\,\delta x$ is finite. Therefore $v_i(x,t) = 0$. And then $u_i(x,t) = 0$.

We state a corollary to be used in the following paragraph.

Lemma 8. Suppose we have for $\Theta \leq t < T$ the system of relations

$$\nu\Delta u_i(x,t) - \frac{\partial u_i(x,t)}{\partial t} - \frac{1}{\rho}\frac{\partial p(x,t)}{\partial x_i} = -\frac{\partial X_{ik}(x,t)}{\partial x_k}; \quad \frac{\partial u_j(x,t)}{\partial x_j} = 0.$$

Suppose the derivatives $\frac{\partial^2 X_{ik}(x,t)}{\partial x_j \partial x_l}$ are continuous and the integrals

$$\iiint_\Pi X_{ik}(x,t)X_{ik}(x,t)\,\delta x,\ \iiint_\Pi u_i(x,t)u_i(x,t)\,\delta x$$

less than some continuous functions of t for $\Theta \leq t < T$. We have then

$$u_i(x,t) = \frac{1}{(2\sqrt{\pi})^3}\iiint_\Pi \frac{e^{-\frac{\mathbf{r}^2}{4\nu t}}}{(\nu t)^{\frac{3}{2}}} u_i(y,t_0)\,\delta y +$$

$$\frac{\partial}{\partial x_k}\int_{t_0}^t dt' \iiint_\Pi T_{ik}(x-y,t-t')X_{jk}(y,t)\,\delta y;$$

$$p(x,t) = -\frac{\rho}{4\pi}\frac{\partial}{\partial x_k}\iiint_\Pi \frac{1}{r}X_{ik}(y,t)\,\delta y;\ (\Theta \leq t_0 < t < T).$$

III. Regular motions.

15. *Definitions:* Motions of viscous liquids are governed by Navier's equations

$$\nu\Delta u_i(x,t) - \frac{\partial u_i(x,t)}{\partial t} - \frac{1}{\rho}\frac{\partial p(x,t)}{\partial x_i} = u_k(x,t)\frac{\partial u_i(x,t)}{\partial x_k};\ \frac{\partial u_k(x,t)}{\partial x_k} = 0, \tag{3.1}$$

where ν and ρ are constants, p is the pressure, u_i the components of the velocity. We set

$$W(t) = \iiint_\Pi u_i(x,t)u_i(x,t)\,\delta x,$$

$$V(t) = \max\sqrt{u_i(x,t)u_i(x,t)}.$$

We say that *a solution $u_i(x,t)$ of this system is regular* in an interval of time[1] $\Theta < t < T$ if in this interval the functions u_i, the corresponding p and the derivatives $\frac{\partial u_i}{\partial x_k}$, $\frac{\partial^2 u_i}{\partial x_k \partial x_l}$, $\frac{\partial u_i}{\partial t}$, $\frac{\partial p}{\partial x_i}$ are continuous with respect to x_1, x_2, x_3, t and if in addition the functions $W(t)$ and $V(t)$ are less than some continuous functions of t for $\Theta < t < T$.

We use the following conventions.

The function $D_m(t)$ will be defined for each value of t in a neighborhood in

[1] The case where $T = +\infty$ is not excluded.

which the derivatives exist and are uniformly continuous in t; it will be the upper bound of their absolute values.

The function $C_0(t)$ [or $C_m(t)$] will be defined for all values of t in a neighborhood in which the functions $u_i(x,t)$ [or the derivatives $\frac{\partial^m u_i(x,t)}{\partial x_1^h \partial x_2^k \partial x_3^l}$] satisfy the same condition H; it will be the coefficient.

Finally the function $J_m(t)$ will be defined for each value of t in a neighborhood in which the derivatives $\frac{\partial^m u_i(x,t)}{\partial x_1^h \partial x_2^k \partial x_3^l}$ exist and are strongly continuous in t. We set

$$J_m^2(t) = \iiint_\Pi \frac{\partial^m u_i(x,t)}{\partial x_k \partial x_l \dots} \frac{\partial^m u_i(x,t)}{\partial x_k \partial x_l \dots} \, \delta x.$$

Lemma 8 (p. 216) applies to regular solutions to system (3.1) and gives us the relations

$$u_i(x,t) = \frac{1}{(2\sqrt{\pi})^3} \iiint_\Pi \frac{e^{-\frac{\mathbf{r}^2}{4\nu t}}}{(\nu t)^{\frac{3}{2}}} u_i(y,t_0)\, \delta y + \tag{3.2}$$

$$\frac{\partial}{\partial x_k} \int_{t_0}^t dt' \iiint_\Pi T_{ik}(x-y, t-t') u_j(y,t) u_k(y,t)\, \delta y;$$

$$p(x,t) = -\frac{\rho}{4\pi} \frac{\partial^2}{\partial x_k \partial x_j} \iiint_\Pi \frac{1}{r} u_k(y,t) u_j(y,t)\, \delta y; \ (\Theta < t_0 < t < T). \tag{3.3}$$

Paragraphs 11 and 12 allow us to conclude from (3.2) that the functions $u_i(x,t)$ are uniformly and strongly continuous in t for $\Theta < t < T$, $C_0(t)$ is defined for $\Theta < t < T$ and we have [cf. (2.7) and (2.18)]

$$C_0(t) < \frac{A\sqrt{W(t_0)}}{\nu(t-t_0)} + A \int_{t_0}^t \frac{V^2(t')\, dt'}{[\nu(t-t_0)]^{\frac{3}{4}}}.$$

This result with (3.2) shows that $D_1(t)$ exists for $\Theta < t < T$ and gives the inequality [cf. (2.7) and (2.16)]

$$D_1(t) < \frac{A\sqrt{W(t_0)}}{[\nu(t-t_0)]^{\frac{3}{4}}} + A \int_{t_0}^t \frac{V(t') C_0(t')\, dt'}{[\nu(t-t_0)]^{\frac{3}{4}}}.$$

We proceed by recurrence:

The existence of $D_1(t), \ldots, D_{m+1}(t)$ guarantees that of $C_{m+1}(t)$ and one has [cf. (2.7) and (2.18)]

$$C_{m+1}(t) < \frac{A_m \sqrt{W(t_0)}}{[\nu(t-t_0)]^{\frac{m+3}{2}}} + A_m \int_{t_0}^{t} \frac{V(t')D_{m+1}(t') + \sum_{\alpha+\beta=m+1} D_\alpha(t')D_\beta(t')}{[\nu(t-t')]^{\frac{3}{4}}}\, dt'.$$

The existence of $D_1(t), \ldots, D_{m+1}(t), C_0(t), \ldots, C_{m+1}(t)$ guarantees that of $D_{m+2}(t)$ and one can majorize this last function using the preceeding [cf. (2.7) and (2.20)].

The functions $D_m(t)$ and $C_m(t)$ are therefore defined for $\Theta < t < T$, however large m may be.

Further, paragraphs 11 and 12 allow us to deduce from (3.2) the existence of $J_1(t)$ for all values of t and we have [cf. (2.8) and (2.19)]

$$J_1(t) < \frac{A\sqrt{W(0)}}{[\nu(t-t_0)]^{\frac{1}{2}}} + A\int_{t_0}^{t} \frac{W(t')D_1(t')}{\sqrt{\nu(t-t')}}\, dt'.$$

More generally the existence of $D_1(t), \ldots, D_m(t),\ J_1(t), \ldots, J_{m-1}(t)$ guarantees that of $J_m(t)$ [cf. (2.8) and (2.19)].

It is now easy for us to establish by the intermediary (3.3) that $p(x,t)$ and its derivatives $\frac{\partial^m p(x,t)}{\partial x_k \partial x_j \ldots}$ are uniformly and strongly continuous in t for $\Theta < t < T$. By Navier's equations it is the same for the functions $\frac{\partial u_i}{\partial t}$, $\frac{\partial^{m+1} u}{\partial t \partial x_k \partial x_j \ldots}$.

More generally, equations (3.1) and (3.3) allow us to reduce the study of the order $n+1$ derivatives with respect to t to the study of the order n derivatives with respect to t. So finally we achieve the following *theorem*.

If the functions $u_i(x,t)$ are a regular solution of Navier's equations for $\Theta < t < T$, then all their partial derivatives exist, and the derivatives as well as the $u_i(x,t)$ are uniformly and strongly continuous in t for $\Theta < t < T$.

16. The preceeding paragraph teaches us more: we learn that it is possible to bound the functions $u_i(x,t)$ and their partial derivatives of all orders by means of just $W(t)$ and $V(t)$. The result is:

Lemma 9. Let $u_i^*(x,t)$ be an infinity of solutions to Navier's equations, all regular in the same interval (Θ, T). Suppose the various $V^*(t)$ and $W^*(t)$ all less than one function of t, continuous in (Θ, T). Then one can extract a subsequence such that the $u_i^*(x,t)$ and each of their derivatives converge respectively to certain functions $u_i(x,t)$ and their derivatives. Each of the convergences is uniform on all domains ϖ for $\Theta + \eta < t < T - \eta$ $(\eta > 0)$. The functions $u_i(x,t)$ are a regular solution of Navier's equations in (Θ, T).

In fact, Cantor's diagonal method (§4, p. 201) allows the extraction of a sequence of functions $u_i^*(x,t)$ which, with their derivatives, converge for any given rational values of x_1, x_2, x_3, t. This subsequence has the properties stated in the lemma.

17. The quantities $W(t)$ and $J_1(t)$ — which from now on we write simply as $J(t)$ — are linked by an important relation. It is obtained by replacing X_i in (2.21) by $u_k \frac{\partial u_i}{\partial x_k}$ and remarking that

$$\iiint_\Pi u_i(x,t')u_k(x,t')\frac{\partial u_i(x,t')}{\partial x_k}\,\delta x = \frac{1}{2}\iiint_\Pi u_k(x,t')\frac{\partial u_i(x,t')u_i(x,t')}{\partial x_k}\,\delta x = 0;$$

It is the *"energy dissipation relation"*

$$\nu\int_{t_0}^{t} J^2(t')\,dt' + \frac{1}{2}W(t) = \frac{1}{2}W(t_0). \tag{3.4}$$

This relation and the two paragraphs above show that the functions $W(t)$, $V(t)$, and $J(t)$ play an essential role. We will especially point out, of all the inequalities one can deduce from chapter II, some which involve these three functions without any longer occupying ourselves with the quantities $C_m(t)$, $D_m(t)$, ...

Before writing the fundamental inequalities, we make the *definition:*

A solution $u_i(x,t)$ of Navier's equations will be called regular for $\Theta \le t < T$ when it is regular for $\Theta < t < T$ and if in addition the following conditions are satisfied: The functions $u_i(x,t)$ and $\frac{\partial u_i(x,t)}{\partial x_j}$ are continuous with respect to x_1, x_2, x_3, t also for $t = \Theta$, they are strongly continuous in t also for $t = \Theta$, and the $u_i(x,t)$ remain bounded when t approaches Θ.

In these conditions the relation (3.2) holds for $\Theta \leq t_0 < t < T$ (the value Θ was not allowed to be t_0 until now). Chapter II allows us to deduce *two fundamental inequalities.* In these, the symbol $\{B; C\}$ is the smaller of B and C, and A', A'', A''' are numerical constants. The inequalities are

$$V(t) < A' \int_{t_0}^{t} \{\frac{V^2(t')}{\sqrt{\nu(t-t')}}; \frac{W(t')}{[\nu(t-t')]^2}\} \, dt' + \{V(t_0); \frac{A''' J(t_0)}{[\nu(t-t_0)]^{\frac{1}{4}}}\} \tag{3.5}$$

$$J(t) < A'' \int_{t_0}^{t} \frac{J(t')V(t')}{\sqrt{\nu(t-t')}} \, dt' + J(t_0) \qquad (\Theta \leq t_0 < t < T). \tag{3.6}$$

18. *Comparison of two regular solutions.*

We consider two solutions of Navier's equations, u_i and $u_i + v_i$, regular for $\Theta < t < T$. We have

$$\nu \Delta v_i - \frac{\partial v_i}{\partial t} - \frac{1}{\rho} \frac{\partial q}{\partial x_i} = v_k \frac{\partial u_i}{\partial x_k} + (u_k + v_k) \frac{\partial v_i}{\partial x_k}; \qquad \frac{\partial v_k}{\partial x_k} = 0.$$

Let

$$w(t) = \iiint_\Pi v_i(x,t) v_i(x,t) \, \delta x; \qquad j^2(t) = \iiint_\Pi \frac{\partial v_i(x,t)}{\partial x_k} \frac{\partial v_i(x,t)}{\partial x_k} \, \delta x.$$

We apply (2.21) which has already given us the fundamental relation (3.4). Here it gives

$$\nu j^2(t) + \frac{1}{2} \frac{dw}{dt} = \iiint_\Pi v_i v_k \frac{\partial u_i}{\partial x_k} \, \delta x + \iiint_\Pi v_i (u_k + v_k) \frac{\partial v_i}{\partial x_k} \, \delta x.$$

Now we have

$$\iiint_\Pi v_i (u_k + v_k) \frac{\partial v_i}{\partial x_k} \, \delta x = \frac{1}{2} \iiint_\Pi (u_k + v_k) \frac{\partial (v_i v_i)}{\partial x_k} \, \delta x = 0$$

and

$$\iiint_\Pi v_i v_k \frac{\partial u_i}{\partial x_k} \, \delta x = - \iiint_\Pi \frac{\partial v_i}{\partial x_k} v_k u_i \, \delta x < j(t) \sqrt{w(t)} V(t).$$

Therefore

$$\nu j^2(t) + \frac{1}{2}\frac{dw}{dt} < j(t)\sqrt{w(t)}V(t)$$

from which

$$2\nu\frac{dw}{dt} < v(t)V^2(t)$$

and finally

$$w(t) < w(t_0)e^{\frac{1}{2\nu}\int_{t_0}^{t} V^2(t')\,dt'} \qquad (\Theta < t_0 < t < T). \tag{3.7}$$

From this important relation we get in particular

A uniqueness theorem: Two regular solutions of Navier's equations for $\Theta \leq t < T$ are necessarily identical for these t if their initial velocities are the same for $t = \Theta$.

19. Suppose we are given a *regular initial state,* which is to say a continuous vector $u_i(x,t)$ with continuous first derivatives, having zero divergence and such that the quantities $W(0)$, $V(0)$, $J(0)$ are finite. The goal of this paragraph is to establish the following proposition.

Existence theorem: To each regular initial state $u_i(x,0)$ there corresponds a solution $u_i(x,t)$ to Navier's equations, defined for $0 \leq t < \tau$ and which reduces to $u_i(x,0)$ for $t = 0$.

We form *successive approximations*

$$u_i^{(0)}(x,t) = \frac{1}{(2\sqrt{\pi})^3}\iiint_\Pi \frac{e^{-\frac{\mathbf{r}^2}{4\nu t}}}{(\nu t)^{\frac{3}{2}}} u_i(x,0)\,\delta y,$$

....

$$u_i^{(n+1)}(x,t) = \frac{\partial}{\partial x_k}\int_0^t dt' \iiint_\Pi T_{ij}(x-y,t-t')u_k^{(n)}(y,t')u_j^{(n)}(y,t')\,\delta y + u_i^{(0)}(x,t),$$

....

First we write inequalities which follow from (2.3) and (2.13):

$$V^0(t) \leq V(0)$$

$$V^{(n+1)}(t) \leq A'\int_0^t \frac{[V^{(n)}(t')]^2}{\sqrt{\nu(t-t')}}\,dt' + V(0).$$

These show that we have for all n

$$V^{(n)}(t) \leq \varphi(t) \qquad \text{for } 0 \leq t \leq \tau,$$

if $\varphi(t)$ is a continuous function satisfying

$$\varphi(t) \geq A' \int_0^t \frac{\varphi^2(t)}{\sqrt{\nu(t-t')}}\, dt' + V(0).$$

We choose $\varphi(t) = (1+A)V(0)$ which gives τ the value

$$\tau = A\nu V^{-2}(0). \tag{3.8}$$

Then let

$$v^{(n)}(t) = \max \sqrt{[u_i^{(n)}(x,t) - u_i^{(n+1)}(x,t)][u_i^{(n)}(x,t) - u_i^{(n+1)}(x,t)]}$$

at time t.

We have

$$v^{(1)}(t) < A' \int_0^\tau \frac{V^2(0)}{\sqrt{\nu(t-t')}}\, dt' = AV(0)$$

$$v^{(n+1)}(t) < A \int_0^\tau \frac{\varphi(t')v^{(n)}(t')}{\sqrt{\nu(t-t')}}\, dt' = AV(0) \int_0^\tau \frac{v^{(n)}(t')}{\sqrt{\nu(t-t')}}\, dt'$$

From this we get that the functions $u_i^{(n)}(x,t)$ converge uniformly to continuous limits $u_i(x,t)$ for $0 \leq t \leq \tau$.

One shows without difficulty that in the interior of the interval, each of the derivatives of the $u_i^{(n)}(x,t)$ converges uniformly to the corresponding derivative of the $u_i(x,t)$; the reasoning is too close to that of paragraph 15 to repeat it. The functions $u_i(x,t)$ therefore satisfy Navier's equations for $0 < t < \tau$.

We verify that the integral $W(t) = \iiint_\Pi u_i(x,t)u_i(x,t)\, \delta x$ is less than a continuous function of t. Inequalities (2.5) and (2.12) give the following, where A_0 is a constant

$$\sqrt{W^{(0)}(t)} \leq \sqrt{W(0)}$$

$$\sqrt{W^{(n+1)}(t)} \leq A_0 \int_0^t \frac{\varphi(t')\sqrt{W^{(n)}(t)}}{\sqrt{\nu(t-t')}}\, dt' + \sqrt{W(0)}.$$

By the theory of linear equations there is a positive function $\theta(t)$ satisfying

$$\theta(t) = A_0 \int_0^t \frac{\varphi(t')\theta(t)}{\sqrt{\nu(t-t')}}\, dt' + \sqrt{W(0)}.$$

We have $W^{(n)}(t) \leq \theta^2(t)$, so $W(t) \leq \theta^2(t)$.

It rests upon us to make precise how the $u_i(x,t)$ behave when t tends to zero. We already know that they reduce to the given $u_i(x,0)$, remaining continuous for $t=0$. To show that they remain strongly continuous in t when t approaches zero, it suffices by lemma 1 to prove

$$\limsup_{t\to 0} W(t) \leq W(0).$$

This inequality is clear since $\theta^2(0) = W(0)$. One shows in the same way that the $\frac{\partial u_i(x,t)}{\partial x_k}$ are strongly continuous in t, even for $t=0$.

At this point the proof of the existence theorem announced above is complete.

But formula (3.8) furnishes a second result: Let us say that a *a solution of Navier's equations,* regular in a interval (Θ, T), *becomes irregular at time* T when T is finite and it is impossible to extend the regular solution to any larger interval (Θ, T'). Formula (3.8) reveals

A first characterization of irregularities If a solution of Navier's equations becomes irregular at time T, then $V(t)$ becomes arbitrarily large as t tends to T, and more precisely

$$V(t) > A\sqrt{\frac{\nu}{T-t}}. \tag{3.9}$$

20. *It will be important to know whether there are solutions which become irregular.* If these cannot be found to exist, then the regular solution corresponding to a regular initial state $u_i(x,0)$ will exist for all positive values of t.

No solution can become irregular if inequality (3.9) is incompatible with the fundamental relations (3.4), (3.5) and (3.6), but this is not an issue as one sees by choosing

$$(3.10) \qquad V(t) = A_0'[\nu(T-t)]^{-\frac{1}{2}}; W(t) = A_0''[\nu(T-t)]^{\frac{1}{2}}; J(t) = \frac{\sqrt{A_0''}}{2}[\nu(T-t)]^{-\frac{1}{4}}$$

and from this check that for all sufficiently large values of the constants A_0' and A_0'' inequality (3.9) and relation (3.4) are satisfied, as well as the following two inequalities which are stronger than (3.5) and (3.6)

$$V(t) < A' \int_{t_0}^{t} \{\frac{V^2(t')}{\sqrt{\nu(T-t')}}; \frac{W(t')}{[\nu(T-t')]^2}\} \, dt' + \{V(t_0); \frac{A''' J(t_0)}{[\nu(t-t_0)]^{\frac{1}{4}}}\}$$

$$J(t) < A'' \int_{t_0}^{t} \frac{J(t')V(t')}{\sqrt{\nu(T-t')}} \, dt' + J(t_0) \qquad (t_0 < t < T).$$

Navier's equations certainly have a solution which becomes irregular and for which $W(t)$, $V(t)$ and $J(t)$ are of the type (3.10) if the system

$$(3.11) \qquad \nu \Delta U_i(x) - \alpha \left[U_i(x) + x_k \frac{\partial U_i(x)}{\partial x_k} \right] - \frac{1}{\rho} \frac{\partial P(x)}{\partial x_i} = U_k(x) \frac{\partial U_i(x)}{\partial x_k};$$

$$\frac{\partial U_k(x)}{\partial x_k} = 0,$$

where α is a positive constant, has a nonzero solution with the $U_i(x,t)$ bounded and the integrals $\iiint_\Pi U_i(x,t)U_i(x,t)\,\delta x$ finite. It is

$$(3.12) \qquad u_i(x,t) = [2\alpha(T-t)]^{-\frac{1}{2}} U_i[(2\alpha(T-t))^{-\frac{1}{2}} x] \qquad (t < T)$$

(λx is the point with coordinates λx_1, λx_2, λx_3.)

Unfortunately I have not made a successful study of system (3.11). We therefore leave in suspense the matter of knowing whether irregularities occur or not.

21. *Various consequences of the fundamental relations* (3.4), (3.5) *and* (3.6). Suppose we have a solution to Navier's equations, regular for $\Theta \leq t < T$ and which becomes irregular as t tends to T, where T is not $+\infty$. From the fundamental relations (3.4) and (3.5) we get the inequality

$$V(t) < A' \int_{t_0}^{t} \{ \frac{V^2(t')}{\sqrt{\nu(t-t')}}; \frac{W(t')}{[\nu(t-t')]^2} \} \, dt' + \{ V(t_0); \frac{A''' J(t_0)}{[\nu(t-t_0)]^{\frac{1}{4}}} \} \tag{3.13}$$

$$(\Theta \le t_0 < t < T).$$

We suppose there is a continuous function $\varphi(t)$ in $0 < t \le \tau$, satisfying the inequality

$$\varphi(t) \ge A' \int_{0}^{t} \{ \frac{\varphi^2(t')}{\sqrt{\nu(t-t')}}; \frac{W(t')}{[\nu(t-t')]^2} \} \, dt' + \{ V(t_0); \frac{A''' J(t_0)}{[\nu(t-t_0)]^{\frac{1}{4}}} \}. \tag{3.14}$$

We then have

$$V(t) < \varphi(t - t_0) \tag{3.15}$$

for values of t common to the two intervals (t_0, T) and $(t_0, t_0 + \tau)$. Then the first characterisation of irregularity implies

$$t_0 + \tau < T. \tag{3.16}$$

Further suppose we know a function $\psi(t)$ such that

$$\psi(t) \ge A'' \int_{0}^{t} \frac{\varphi(t')\psi(t')}{\sqrt{\nu(T-t')}} \, dt' + J(t_0) \qquad (0 < t \le \tau). \tag{3.17}$$

Then inequality (3.6) gives

$$J(t) < \psi(t - t_0) \quad \text{pour} \quad t_0 < t \le t_0 + \tau. \tag{3.18}$$

The first characterisation of irregularities follows from (3.16) if we choose

$$\varphi(t) = (1 + A)V(t_0) \quad \text{and} \quad \tau = A\nu V^{-2}(t_0).$$

The choice $\varphi(t) = (1 + A)V(t_0)$ and $\tau = +\infty$ satisfies (3.14) if

$$V(t_0)) > \int_{0}^{\infty} \{ \frac{AV^2(t_0)}{\sqrt{\nu t'}}; \frac{AW(t_0)}{(\nu t')^2} \} \, dt'$$

i.e. when $\nu^{-3} W(t_0) V(t_0) < A$. So:

First case of regularity: A regular solution never becomes irregular if the quantity $\nu^{-3} W(t) V(t)$ is less than a certain constant A either initially or at any other instant at which the solution has not become irregular.

One can satisfy (3.14) and (3.17) by a choice of the type

$$\varphi(t) = AJ(t_0)[\nu(t-t_0)]^{-\frac{1}{4}}; \quad \psi(t) = (1+A)J(t_0); \quad \tau = A\nu^3 J^{-4}(t_0). \tag{3.19}$$

This gives

A second characterisation of irregularities: If a solution of Navier's equations becomes irregular at time T, then $J(t)$ grows indefinitely as t tends to T; and more precisely

$$J(t) > \frac{A\nu^{\frac{3}{4}}}{(T-t)^{\frac{1}{4}}}.$$

Inequalities (3.15) and (3.19) show that a solution regular at t remains regular until $t_0+\tau$ and that one has

$$V(t_0+\tau) < A\nu^{-1}J^2(t_0).$$

The fundamental relation (3.4) further gives

$$W(t_0+\tau) < W(t_0).$$

Therefore

$$\nu^{-3}W(t_0+\tau)V(t_0+\tau) < A\nu^{-4}W(t_0)J^2(t_0).$$

An application of the first case of regularity to the time $t_0+\tau$ now gives

A second case of regularity: A regular solution never becomes irregular if

$$\nu^{-4}W(t)J^2(t)$$

is less than a certain constant A either initially or at all other previous instants at which the solution has not become irregular.

22. One similarly establishes the following results, of which the preceeding are particular cases.

Characterisation of irregularities: If a solution becomes irregular at time T, one has

$$\{\iiint_\Pi [u_i(x,t)u_i(x,t)]^{\frac{p}{2}}\,\delta x\}^{\frac{1}{p}} > \frac{A(1-\frac{3}{p})\nu^{\frac{1}{2}(1+\frac{3}{p})}}{(T-t)^{\frac{1}{2}(1-\frac{3}{p})}} \qquad (p>3).$$

Case of regularity: A regular solution never becomes irregular if at some time

$$[AW(t)]^{p-3} \iiint_\Pi [u_i(x,t)u_i(x,t)]^{\frac{p}{2}}\, \delta x\} < A(1-\frac{3}{p})^3 \nu^{3(p-2)} \qquad (p>3).$$

The case of regularity which we are pointing out shows how a solution always remains regular if its initial velocity state is sufficiently near rest. More generally, consider a velocity state to which corresponds a solution which never becomes irregular. For all initial states sufficiently near there corresponds a solution which also never becomes irregular. The proof makes use of those results of paragraph 34 which concern behavior of solutions to Navier's equations for large values of t.

IV. Semi-regular initial states.

23. We will be led by the current of Chapter VI to consider initial states which are not regular in the sense of paragraph 17. We begin their study with the remark, that inequality (3.7) allows a uniqueness theorem which is more general than that of paragraph 18. To this end we make a definition.

We say that a solution of Navier's equations is semi-regular for $\Theta \leq t < T$ if it is regular for $\Theta < t < T$ and the two following conditions hold.

The integral $\int_\Theta^t V^2(t')\, dt'$ is finite when $\Theta < t < T$.

The $u_i(x,t)$ have $u_i(x,\Theta)$ as strong limit in mean as t tends to Θ.

— We call "initial velocity state" any vector $u_i(x,\Theta)$, with quasi-divergence zero. — The theorem given by inequality (3.7) is the following.

Uniqueness theorem: Two solutions of Navier's equations which are semi-regular for $\Theta \leq t < T$, are necessarily identical for all values of t if their initial velocity states at time Θ are equal almost everywhere.

We say that *an initial velocity state* $u_i(x,0)$ is semi-regular if there corresponds a semi-regular solution $u_i(x,t)$ on an interval $0 \leq t < \tau$.

24. Suppose a vector $U_i(x)$ has quasi-divergence zero, components square summable on Π, and quasi-derivatives square summable on Π. We are going to establish that the velocity field $U_i(x)$ is a semi-regular initial state.

Let

$$W(0) = \iiint_{\Pi} U_i(x) U_i(x)\, \delta x \quad \text{et} \quad J^2(0) = \iiint_{\Pi} U_{i,j}(x) U_{i,j}(x)\, \delta x.$$

The functions $\overline{U_i(x)}$ constitute a regular initial state, as shown by lemma 6 and paragraph 8 (p. 209 et 206). Let $u_i^*(x,t)$ be the regular solution which corresponds to the initial state $\overline{U_i(x)}$. We have, in virtue of inequality (1.21) and the energy dissipation relation (3.4) that

$$W^*(t) < W(0). \tag{4.1}$$

Lemma 4 shows us that $\frac{\partial \overline{U_i(x)}}{\partial x_j} = U_{i,j}(x)$. Thus we have from (1.21)

$$J^*(0) < J(0).$$

Relations (3.15), (3.18), and (3.19) allow us to deduce from this that in some interval $(0,\tau)$ the various solutions $u_i^*(x,t)$ are regular and satisfy inequalities

$$V^*(t) < AJ(0)(\nu t)^{-\frac{1}{4}}; \qquad J^*(t) < (1+A)J(0). \tag{4.2}$$

We have further

$$\tau = A\nu^3 J^{-4}(0). \tag{4.3}$$

Inequalities (4.1) and (4.2) let us apply lemma 9 (p. 220). There is a length ϵ in the definition (1.18) of $\overline{U(x)}$. It is possible to make this tend to zero in such a way that for $0 < t < \tau$ the functions $u_i^*(x,t)$ and all their derivatives converge respectively to certain functions $u_i(x,t)$ and to their derivatives. These $u_i(x,t)$ are a regular solution to Navier's equations for $0 < t < \tau$. By (4.1) and (4.2) this solution satisfies the three inequalities

$$W(t) \leq W(0); \qquad V(t) \leq AJ(0)(\nu t)^{-\frac{1}{4}}; \qquad J(t) \leq (1+A)J(0). \tag{4.4}$$

The integral $\int_0^t V^2(t')\, dt'$ is therefore finite for $0 < t < \tau$. Now we must specify how the $u_i(x,t)$ behave as t tends to zero.

Let $a_i(x)$ be any vector of divergence zero, for which the components as well as all their derivatives are square summable on Π. From Navier's equations we get

$$\iiint_\Pi u_i^*(x,t)a_i(x)\,\delta x = \iiint_\Pi \overline{U_i^*(x)}a_i(x)\,\delta x +$$

$$\nu\int_0^t dt' \iiint_\Pi u_i^*(x,t')\Delta a_i(x)\,\delta x + \int_0^t dt' \iiint_\Pi u_k^*(x,t')u_k^*(x,t')\frac{\partial a_i(x)}{\partial x_k}\,\delta x.$$

Then passing to the limit

$$\iiint_\Pi u_i(x,t)a_i(x)\,\delta x = \iiint_\Pi U_i^*(x)a_i(x)\,\delta x +$$

$$\nu\int_0^t dt' \iiint_\Pi u_i(x,t')\Delta a_i(x)\,\delta x + \int_0^t dt' \iiint_\Pi u_k^*(x,t')u_k^*(x,t')\frac{\partial a_i(x)}{\partial x_k}\,\delta x.$$

This last relation shows that

$$\iiint_\Pi u_i(x,t)a_i(x)\,\delta x \qquad \text{tends to} \qquad \iiint_\Pi U_i^*(x)a_i(x)\,\delta x$$

when t tends to zero. In these conditions $u_i(x,t)$ has a unique weak limit in mean, which is $U_i(x)$ (cf. Corollary to lemma 7, p. 209). But the inequality $W(t) \leq W(0)$ allows us to use the criteria for strong convergence announced on p. 200, and we also note that the $u_i(x,t)$ converge strongly in mean to the $U_i(x)$ as t tends to zero.

$u_i(x,t)$ is therefore a semi-regular solution[1] for $0 \leq t < \tau$ and it corresponds to the initial state $U_i(x)$.

25. By analogous reasoning one can treat the two other cases pointed out in the theorem below.

Existence theorem: Let the vector $U_i(x)$ have quasi-divergence zero, with

[1] One can similarly check that the functions $\frac{\partial u_j(x,t)}{\partial x_j}$ converge strongly in mean to the $U_{i,m}(x)$ as t tends to zero.

the components are square summable on Π. One can verify that the initial velocity state which it defines is semi-regular

a) if the functions $U_i(x)$ have square summable quasi-derivative on Π;

b) if the functions $U_i(x)$ are bounded;

c) or finally if the integral $\iiint_\Pi [U_i(x)U_i(x)]^{\frac{p}{2}}\,\delta x$ is finite for some value of p larger than 3.

N. B. This theorem and the existence theorem of paragraph 19 evidently do not let us study the behavior at infinity for a solution with initial state in the neighborhood of a given initial state.

V. Turbulent solutions.

26. Let $u_i(x,0)$ be a regular initial state. We have not succeeded in proving that the corresponding regular solution to Navier's equations is defined for all values of t after the initial instant $t=0$. But consider the system

$$\text{(5.1)} \qquad \nu\Delta u_i(x,t) - \frac{\partial u_i(x,t)}{\partial t} - \frac{1}{\rho}\frac{\partial p(x,t)}{\partial x_i} = \overline{u_k(x,t)}\frac{\partial u_i(x,t)}{\partial x_k}; \qquad \frac{\partial u_j(x,t)}{\partial x_j} = 0.$$

This system is very near Navier's equations when the length[1] ϵ is very short. All we have said in Chapter III on Navier's equations is applicable without modification, other than the inconclusive considerations of paragraph 20. Thus we know many properties of system (5.1) which are independent of ϵ. Further, the Schwarz inequality (1.1) gives us

$$\overline{U_k(x,t)} < A_0\epsilon^{-\frac{3}{2}}\sqrt{W(t)},$$

A_0 being a numerical constant. This new inequality and the energy dissipation relation (3.4) allows us to write the following beside inequality (3.5) if a solution to system (5.1) is regular for $0 \le t < T$, then

[1] Recall this length was introduced in §8 (p, 206), when we defined the symbol $\overline{U(x)}$.

$$V(t) < A' A_0 \epsilon^{-\frac{3}{2}} \sqrt{W(0)} \int_0^t \frac{V(t')\,dt'}{\sqrt{\nu(t-t')}} + V(0) \qquad (0 < t < T).$$

From this we get that on all intervals of regularity $(0, T)$, $V(t)$ remains less than the continuous function $\varphi(t)$ on $0 \leq t$, which satisfies the Volterra linear integral equation

$$\varphi(t) = A'' A_0 \epsilon^{-\frac{3}{2}} \sqrt{W(0)} \int_0^t \frac{\varphi(t')\,dt'}{\sqrt{\nu(t-t')}} + V(0).$$

$V(t)$ therefore remains bounded when, T being finite, t tends to T. That contradicts the first caracterization of irregularity (p. 224). In other words, *the unique solution to equations (5.1) corresponding to a given regular initial state is defined for all time after the initial instant.*

27. Given a motion which satisfies equations (5.1), we will need results on its *repartition of kinetic energy:* $\frac{1}{2}u_i(x,t)u_i(x,t)$. These must be independent[1] of ϵ.

Consider two constant lengths R_1 and R_2 ($R_1 < R_2$) and introduce the following function $f(x)$

$$f(x) = 0 \quad \text{for} \quad r_0 \leq R_1;$$

$$f(x) = \frac{r_0 - R_1}{R_2 - R_1} \quad \text{for} \quad R_1 \leq r_0 \leq R_2; \qquad (r_0^2 = x_i x_i)$$

$$f(x) = 1 \quad \text{for} \quad R_2 \leq r_0.$$

A calculation analogous to that giving the energy dissipationn relation (2.21) here gives

$$\nu \int_0^t dt' \iiint_\Pi f(x) \frac{\partial u_i(x,t')}{\partial x_k} \frac{\partial u_i(x,t')}{\partial x_k}\,\delta x + \frac{1}{2} \iiint_\Pi f(x) u_i(x,t) u_i(x,t)\,\delta x =$$

$$= \frac{1}{2} \iiint_\Pi f(x) u_i(x,0) u_i(x,0)\,\delta x - \nu \int_0^t dt' \iiint_\Pi \frac{\partial f(x)}{\partial x_k} u_i(x,t) \frac{\partial u_i(x,t)}{\partial x_k}\,\delta x +$$

[1] They will apply equally to solutions of Navier's equations.

$$+\frac{1}{\rho}\int_0^t dt' \iiint_\Pi \frac{\partial f(x)}{\partial x_k} p(x,t')u_i(x,t')\,\delta x +$$

$$+\frac{1}{2}\int_0^t dt' \iiint_\Pi \frac{\partial f(x)}{\partial x_k}\overline{u_k(x,t')}u_i(x,t')u_i(x,t')\,\delta x.$$

From this we get the inequality

$$\frac{1}{2}\iiint_{r_0>R_2} u_i(x,t)u_i(x,t)\,\delta x < \frac{1}{2}\iiint_{r_0>R_1} u_i(x,0)u_i(x,0)\,\delta x +$$

$$+\frac{\nu\sqrt{W(0)}}{R_2-R_1}\int_0^t J(t')\,dt' + \frac{1}{\rho}\frac{\sqrt{W(0)}}{R_2-R_1}\int_0^t dt'\sqrt{\iiint_\Pi p^2(x,t')\,\delta x} + \tag{5.2}$$

$$+\frac{\sqrt{W(0)}}{R_2-R_1}\int_0^t dt'\sqrt{\iiint_\Pi\left[\frac{1}{2}u_i(x,t')u_i(x,t')\right]^2\delta x}.$$

We majorize the last three terms. By the Schwarz inequality

$$\int_0^t J(t')\,dt' < \sqrt{\int_0^t J^2(t')\,dt'}\sqrt{t} < \sqrt{\frac{W(0)}{2\nu}}\sqrt{t}. \tag{5.3}$$

Further (cf. (3.3)):

$$\frac{1}{\rho}p(x,t') = \frac{1}{4\pi}\iiint_\Pi \frac{\partial(\frac{1}{r})}{\partial x_j}\frac{\partial u_i(y,t')}{\partial y_k}\overline{u_k(y,t')}\,\delta y, \tag{5.4}$$

from which

$$\frac{1}{\rho^2}\iiint_\Pi p^2(x,t')\,\delta x = \frac{1}{4\pi}\iiint_\Pi\iiint_\Pi \overline{u_k(x,t')}\frac{\partial u_i(x,t')}{\partial x_k}\frac{1}{r}\overline{u_j(y,t')}\frac{\partial u_i(y,t')}{\partial y_j}\,\delta x\,\delta y.$$

Relation (1.14) and the Schwarz inequality (1.1) give

$$\sum_i\left[\iiint_\Pi \frac{1}{r}\overline{u_j(y,t')}\frac{\partial u_i(y,t')}{\partial y_j}\partial y\right]^2 < 4J^4(t').$$

Further

$$\sum_i \left[\iiint_\Pi \overline{u_k(x,t')\frac{\partial u_i(x,t')}{\partial x_k}}\,\delta x\right]^2 < W(t')J(t');$$

therefore

$$\frac{1}{\rho^2}\iiint_\Pi p^2(x,t')\,\delta x < \frac{1}{2\pi}\sqrt{W(t')}J^3(t');$$

and it follows[1]

$$\frac{1}{\rho}\int_0^t dt'\sqrt{\iiint_\Pi p^2(x,t')\,\delta x} < \frac{[W(0)]^{\frac{1}{4}}}{\sqrt{2\pi}}\int_0^t J^{\frac{3}{2}}(t')\,dt' < \frac{W(0)}{\sqrt{2\pi}(2\nu)^{\frac{3}{4}}}t^{\frac{1}{4}}. \tag{5.5}$$

From (1.13) we get

$$\frac{1}{2}u_i(x,t')u_i(x,t') = -\frac{1}{4\pi}\iiint_\Pi \frac{\partial(\frac{1}{r})}{\partial x_k}u_i(y,t')\frac{\partial u_i(y,t')}{\partial y_k}\,\delta y.$$

This formula is analogous to (5.4). By calculations like the preceeding it leads to

$$\int_0^t dt'\sqrt{\iiint_\Pi\left[\frac{1}{2}u_i(x,t')u_i(x,t')\right]^2\,\delta x} < \frac{W(0)}{\sqrt{2\pi}(2\nu)^{\frac{3}{4}}}t^{\frac{1}{4}}. \tag{5.6}$$

Using the majorants (5.3), (5.5), and (5.6) in (5.2) we obtain

$$\frac{1}{2}\iiint_{r_0>R_2} u_i(x,t')u_i(x,t')\,\delta x < \frac{1}{2}\iiint_{r_0>R_1} u_i(x,0)u_i(x,0)\,\delta x +$$

$$+\frac{W(0)\sqrt{\nu t}}{\sqrt{2}(R_2-R_1)} + \frac{W^{\frac{3}{2}}(0)t^{\frac{1}{4}}}{2^{\frac{1}{4}}\pi^{\frac{1}{2}}\nu^{\frac{3}{4}}(R_2-R_1)}. \tag{5.7}$$

---

[1] We use the inequality

$$\int_0^t J^{\frac{3}{2}}(t')\,dt' < \left[\int_0^t J^2(t')\,dt'\right]^{\frac{3}{4}} t^{\frac{1}{4}}$$

which is a particular case of "Hölder's inequality"

$$\left|\int_0^t \varphi(t')\psi(t')\,dt'\right| < \left[\int_0^t \varphi^p(t')\,dt'\right]^{\frac{1}{p}}\left[\int_0^t \psi^q(t')\,dt'\right]^{\frac{1}{q}} \qquad (\frac{1}{p}+\frac{1}{q}=1; 1<p, 1<q).$$

This inequality shows how kinetic energy remains localized at finite distance.

28. Suppose we have given at $t = 0$ an arbitrary initial vector $U_i(x)$, with components square summable on Π and quasi-divergence zero. The vector $\overline{U_i(x)}$ is a regular initial state (cf. lemma 6 and paragraph 8). Write $u_i^*(x,t)$ for the corresponding regular solution to equations (5.1). It is defined for all t. *The object of this chapter is to study the limits which this regular solution may have as ϵ tends to zero.*

We will use the following three properties of the $u_i^*(x,t)$.

1^o) Let $a_i^*(x,t)$ be an arbitrary vector of divergence zero, of which all components and all their derivatives are uniformly and strongly continuous in t. By (5.1):

$$\iiint_\Pi u_i^*(x,t)a_i(x,t)\,\delta x = \iiint_\Pi \overline{U_i^*(x,t)}a_i(x,0)\,\delta x +$$

$$(5.8)\qquad + \int_0^t dt' \iiint_\Pi u_i^*(x,t)\left[\nu\Delta a_i(x,t') + \frac{\partial a_i(x,t')}{\partial t'}\right]\delta x +$$

$$+ \int_0^t dt' \iiint_\Pi \overline{u_i^*(x,t)}u_i^*(x,t)\frac{\partial a_i(x,t')}{\partial x_k}\,\delta x.$$

2^o) The energy dissipation relation and (1.21) give

$$(5.9)\qquad \nu\int_{t_0}^t J^{*2}(t')\,dt' + \frac{1}{2}W^*(t) = \frac{1}{2}W^*(t_0) < \frac{1}{2}W(0).$$

By definition

$$(5.10)\qquad W(0) = \iiint_\Pi U_i(x)U_i(x)\,\delta x.$$

3^o) Inequality (5.7) and the inequality $W^*(0) < W(0)$ justify the following proposition

Le η be an arbitrarily small constant with $(0 < \eta < W(0))$. We let $R_1(\eta)$ be the length for which

$$\iiint_{r_0 > R_1(\eta)} U_i(x) U_i(x)\, \delta x = \frac{\eta}{2}$$

and write $s(\eta, t)$ for the sphere with center at the origin with radius

$$R_2(\eta, t) = R_1(\eta, t) + \frac{4}{\eta}\left[\frac{W(0)\sqrt{\nu t}}{\sqrt{2}} + \frac{W^{\frac{3}{2}} t^{\frac{1}{4}}}{2^{\frac{1}{4}} \pi^{\frac{1}{2}} \nu^{\frac{3}{4}}}\right].$$

We have

$$\limsup_{s \to 0} \iiint_{\Pi - s(\eta, t)} u_i^*(x,t) u_i^*(x,t)\, \delta x \leq \eta. \tag{5.11}$$

29. Let ϵ tend to zero through a countable sequence of values $\epsilon_1, \epsilon_2, \ldots$ Consider the corresponding functions $W^*(t)$. This is a bounded set of functions and each is decreasing. Cantor's diagonal method (§4) allows us to extract from the sequence $\epsilon_1, \epsilon_2, \ldots$ a subsequence $\epsilon_{l_1}, \epsilon_{l_2}, \ldots$ such that the $W^*(t)$ converge for all rational values of t. The $W^*(t)$ therefore converge to a decreasing function, except maybe at points of discontinuity of the limit. The points discontinuity of a decreasing function are countable. A second application of Cantor's method allows us to extract from $\epsilon_{l_1}, \epsilon_{l_2}, \ldots$ a subsequence $\epsilon_{m_1}, \epsilon_{m_2}, \ldots$ such that the corresponding $W^*(t)$ converge[1] for all t. We write $W(t)$ for the decreasing function which is their limit. (This definition does not contradict (5.10).)

The inequality $W^*(t) < W(0)$ shows that each of the integrals

$$\int_{t_1}^{t_2} dt' \iiint_{\varpi} u_i^*(x,t')\, \delta x; \quad \int_{t_1}^{t_2} dt' \iiint_{\varpi} \overline{u_k^*(x,t')} u_i^*(x,t')\, \delta x$$

is less than a bound independent of ϵ. By a third use of Cantor's diagonal method we can therefore extract from the sequence $\epsilon_{m_1}, \epsilon_{m_2}, \ldots$ a subsequence $\epsilon_{n_1}, \epsilon_{n_2}, \ldots$ such that each of these integrals has a unique

[1] In other words we use Helly's theorem.

limit when t_1 and t_2 are rational and ϖ is a cube with sides parallel to the axes and with vertices having rational coordinates. The inequality $W^*(t) < W(0)$ and the hypotheses made on the $a_i(x,t)$ imply that the integrals

$$\int_0^t dt' \iiint_\Pi u_i^*(x,t') \left[\nu\Delta a_i(x,t') + \frac{\partial a_i(x,t')}{\partial t'}\right] \delta x;$$

$$\int_0^t dt' \iiint_\Pi \overline{u_k^*(x,t')} u_k^*(x,t') \frac{\partial a_i(x,t')}{\partial x_k} \delta x$$

have a unique limit. This result, with (5.8) shows that

$$\iiint_\Pi u_i^*(x,t) a_i(x,t)\, \delta x$$

converges to a unique limit, for all $a_i(x,t)$ and t. Therefore (cf. Corollary to lemma 7) the $u_i^*(x,t)$ converge weakly in mean to some limit $U_i(x,t)$ for each value of t.

Also, given a sequence of values of ϵ which tend to zero, one can extract from them a subsequence such that *the* $W^*(t)$ *converge to a unique limit* $W(t)$ and that *the* $u_i^*(x,t)$ *have for each value of* t *a unique weak limit in mean:* $U_i(x,t)$. We suppose from here on that ϵ tends to zero through a sequence of values ϵ^* such that these two conditions hold.

Remark I By (1.9)

$$W(t) \geq \iiint_\Pi U_i(x,t) U_i(x,t)\, \delta x.$$

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[1] In fact these hypotheses imply the following. Given t, a number η (> 0) and a function $\delta(x,t)$ equal to one of the derivatives of the $a_i(x,t)$, one can find an integer N and two discontinuous functions $\beta(x,t)$ and $\gamma(x,t)$ with the following properties. $\beta(x,t)$ and $\gamma(x,t)$ remain constant when x_1, x_2, x_3, t vary without hitting(?) any multiple of $\frac{1}{N}$, and each of them is zero outside of a domain ϖ, and(?)

$$\int_0^t dt' \iiint_\Pi [\delta(x,t') - \beta(x,t')]^2\, \delta x < \eta; \quad [\delta(x,t') - \gamma(x,t')] < \eta \quad \text{for} \quad 0 < t' < t.$$

Remark II. The vector $U_i(x,t)$ clearly has quasi-divergence zero.

30. Inequality (5.9) gives us

$$\nu \int_0^\infty [\liminf J^*(t')]^2 \, dt' < \frac{1}{2} W(0).$$

Thus the $\liminf J^*(t)$ can only be $+\infty$ for a set of values of t of measure zero. Suppose t_1 is in the complement of this set. One can extract from the sequence of values ϵ^* considered here a subsequence[1] ϵ^{**} such that on Π the corresponding functions $\frac{\partial u_i^{**}(x,t_1)}{\partial x_j}$ converge weakly in mean to a limit $U_{i,j}(x,t_1)$ (cf. Fundamental Theorem of F. Riesz, p. 202).

Lemma 2 allows us to conclude that *the $U_i(x,t_1)$ have quasi-derivatives which are the* $U_{i,j}(x,t_1)$. We set

$$J(t_1) = \iiint_\Pi U_{i,j}(x,t_1) U_{i,j}(x,t_1) \, \delta x.$$

We have (cf. (1.9))

$$J(t_1) \leq \liminf J^*(t_1).$$

Using this inequality in (5.9) we obtain

$$\nu \int_{t_0}^t J^2(t') \, dt' + \frac{1}{2} W(t) \leq \frac{1}{2} W(t_0) \leq \frac{1}{2} W(0) \qquad (0 \leq t_0 \leq t). \tag{5.12}$$

Lemma 2 teaches us finally that on all domains ϖ the $u_i^{**}(x,t_1)$ converge strongly in mean to the $U_i(x,t)$;

$$\lim_{\epsilon^{**} \to 0} \iiint_\varpi u_i^{**}(x,t_1) u_i^{**}(x,t_1) \, \delta x = \iiint_\Pi U_i(x,t_1) U_i(x,t_1) \, \delta x.$$

Choosing ϖ to be $s(\eta, t_1)$ and taking account of (5.11) we get

---

[1] The subsequence we choose is a function of t_1.

$$\limsup \iiint_{\Pi} u_i^{**}(x,t_1)u_i^{**}(x,t_1)\,\delta x \leq \iiint_{s(\eta,t_1)} U_i(x,t_1)U_i(x,t_1)\,\delta x + \eta.$$

From this, since η is arbitrarily small and since $W^*(t_1)$ has a limit

$$\lim_{\epsilon^*\to 0} \iiint_{\Pi} u_i^*(x,t_1)u_i^*(x,t_1)\,\delta x \leq \iiint_{\Pi} U_i(x,t_1)U_i(x,t_1)\,\delta x. \tag{5.13}$$

We apply the strong convergence criterion from p. 200. Note that *on* Π *the* $u_i^*(x,t)$ *converge strongly in mean to the* $U_i(x,t)$ *for all values* t_1 *of* t *not belonging to the set of measure zero on which* $\liminf J^*(t) = +\infty$.

For all these values of t the two sides of (5.13) are equal, i.e.

$$W(t_1) = \iiint_{\Pi} U_i(x,t_1)U_i(x,t_1)\,\delta x. \tag{5.14}$$

The functions $\overline{u_i^*(x,t_1)}$ also converge strongly in mean to $U_i(x,t_1)$ (cf. Generalisation of lemma 3, p. 207). The integral which figures in (5.8)

$$\iiint_{\Pi} \overline{u_k^*(x,t')}u_i^*(x,t')\frac{\partial a_i(x,t')}{\partial x_k}\,\delta x$$

therefore converges to

$$\iiint_{\Pi} U_k(x,t')U_i(x,t')\frac{\partial a_i(x,t')}{\partial x_k}\,\delta x$$

for almost all values of t' (cf. (1.8)). Further, this integral is less than

$$3W(0)\max\left|\frac{\partial a_i(x,t')}{\partial x_k}\right|$$

Lebesgue's theorem concerning passage to the limit under the $\int$ sign gives

$$\lim_{\epsilon^* \to 0} \int_0^t dt' \iiint_\Pi \overline{u_k^*(x,t')} u_i^*(x,t') \frac{\partial a_i(x,t')}{\partial x_k}\, \delta x =$$

$$\int_0^t dt' \iiint_\Pi U_k(x,t') U_i(x,t') \frac{\partial a_i(x,t')}{\partial x_k}\, \delta x,$$

By lemma 5 the right hand side of this can be put into the form

$$-\int_0^t dt' \iiint_\Pi U_k(x,t') U_{i,k}(x,t') a_i(x,t')\, \delta x.$$

From the beginning of this paragraph we can claim that the other terms in (5.8) similarly converge. We obtain the limits by substituting $U_i(x,t)$ for $u_i^*(x,t)$ and $U_i(x)$ for $\overline{U_i(x)}$. This gives

$$\iiint_\Pi U_i(x,t) a_i(x,t)\, \delta x = \iiint_\Pi U_i(x,t) a_i(x,0)\, \delta x$$

$$+\int_0^t dt' \iiint_\Pi U_i(x,t) \left[\nu \Delta a_i(x,t') + \frac{\partial a_i(x,t')}{\partial t'} \right] \delta x \tag{5.15}$$

$$-\int_0^t dt' \iiint_\Pi U_i(x,t) U_{i,k}(x,t') a_i(x,t')\, \delta x.$$

31. These results lead to the following definition. We say that a vector $U_i(x,t)$ defined for $t \geq 0$ constitutes *a turbulent solution to Navier's equations* when the following conditions are realised, where values of t that we call *singular* form a set of measure zero.

For each positive t the functions $U_i(x,t)$ are square summable on Π and the vector $U_i(x,t)$ has quasi-divergence zero.

The function

$$\int_0^t dt' \iiint_\Pi U_i(x,t)\left[\nu\Delta a_i(x,t') + \frac{\partial a_i(x,t')}{\partial t'}\right]\delta x - \iiint_\Pi U_i(x,t)a_i(x,t)\,\delta x$$

$$-\int_0^t dt' \iiint_\Pi U_k(x,t')U_{i,k}(x,t')a_i(x,t')\,\delta x$$

is constant ($t \geq 0$). (Equivalently, (5.15) holds.) For all positive values of t except possibly for certain *singular* values, the functions $U_i(x,t)$ have quasi-derivatives $U_{i,j}(x,t)$ which are square summable on Π.

Set

$$J^2(t) = \iiint_\Pi U_{i,j}(x,t)U_{i,j}(x,t)\,\delta x,$$

$J(t)$ is thus defined for almost all positive t.

There exists a function $W(t)$ defined for $t \geq 0$ which has the two following properties.

$$\text{the function } \nu\int_0^t J^2(t')\,dt' - \frac{1}{2}W(t) \text{ is nonincreasing}$$

and $\iiint_\Pi U_i(x,t)U_i(x,t)\,\delta x \leq W(t)$, the inequality holding except for certain *singular* times, but $t = 0$ is not a singular time.

We say that *such a solution corresponds to initial state* $U_i(x)$ when we have $U_i(x,0) = U_i(x)$.

The conclusion of this chapter can then be formulated as follows.

Existence theorem: Suppose an initial state $U_i(x)$ is given such that the functions $U_i(x)$ are square summable on Π and that the vector having components $U_i(x)$ has quasi-divergence zero. There corresponds to this initial state at least one turbulent solution, which is defined for all values of $t > 0$.

VI. Structure of a turbulent solution.

32. It remains to establish what connections exist between regular solutions and turbulent solutions to Navier's equations. It is entirely clear that any regular solution is a fortiori a turbulent solution. We are going to look for those cases in which a turbulent solution is regular. To this end we generalize the reasoning of paragraph 18 (p. 221).

Comparison of a regular solution and a turbulent solution: Let $a_i(x,t)$ be a solution to Navier's equations, defined and semi-regular for $\Theta \leq t < T$. We suppose that it becomes irregular when t tends to T, at least in the case when T is not equal to $+\infty$. Consider a turbulent solution $U_i(x,t)$ defined for $\Theta \leq t$, where Θ is not a singular time. The symbols $W(t)$ and $J(t)$ correspond to the turbulent solution. Set

$$w(t) = W(t) - 2\iiint_\Pi U_i(x,t)a_i(x,t)\,\delta x + \iiint_\Pi a_i(x,t)a_i(x,t)\,\delta x$$

$$j^2(t) = J^2(t) - 2\iiint U_{i,j}(x,t)\frac{\partial a_i(x,t)}{\partial x_j}\,\delta x + \iiint_\Pi \frac{\partial a_i(x,t)}{\partial x_j}\frac{\partial a_i(x,t)}{\partial x_j}\,\delta x.$$

Recall that

$$\nu\int_0^t dt' \iiint_\Pi \frac{\partial a_i(x,t')}{\partial x_j}\frac{\partial a_i(x,t')}{\partial x_j}\,\delta x + \frac{1}{2}\iiint_\Pi a_i(x,t)a_i(x,t)\,\delta x$$

is constant in t and that

$$\nu\int_\Theta^t J^2(t')\,dt' + \frac{1}{2}W(t)$$

is nonincreasing. Consequently

$$\nu\int_\Theta^t j^2(t')\,dt' + \frac{1}{2}w(t) + 2\nu\int_\Theta^t dt' \iiint_\Pi U_{i,j}(x,t)\frac{\partial a_i(x,t)}{\partial x_j}\,\delta x + \iiint_\Pi U_i(x,t)a_i(x,t)\,\delta x \tag{6.1}$$

is nonincreasing. Taking account of relation (5.15) and of that for $a_i(x,t)$

is a semi-regular solution to Navier's equations. Note that the nonincreasing function (6.1) is up to a constant(?) nearly equal to

$$(6.2)\quad \nu \int_{\Theta}^{t} j^2(t')\,dt' + \frac{1}{2}w(t) + 2\nu \int_{\Theta}^{t} dt' \iiint_{\Pi} [a_k(x,t') - U_k(x,t')] U_{i,j}(x,t') a_i(x,t')\,\delta x.$$

Now we have for each nonsingular value of t

$$\iiint_{\Pi} [a_k(x,t') - U_k(x,t')] \frac{\partial a_i(x,t')}{\partial x_k} a_i(x,t')\,\delta x =$$

$$\frac{1}{2} \iiint_{\Pi} [a_k(x,t') - U_k(x,t')] \frac{\partial a_i(x,t') a_i(x,t')}{\partial x_k}\,\delta x = 0.$$

The integral

$$\iiint_{\Pi} [a_k(x,t') - U_k(x,t')] U_{i,k}(x,t') a_i(x,t')\,\delta x$$

may therefore be written

$$\iiint_{\Pi} [a_k(x,t') - U_k(x,t')] \left[U_{i,k}(x,t') - \frac{\partial a_i(x,t')}{\partial x_k} \right] a_i(x,t')\,\delta x$$

and so it is less in absolute value than

$$\sqrt{w(t')}\, j(t') V(t'),$$

where $V(t')$ is the greatest length of the vector $a_i(x,t)$ at time t'. Since (6.2) is not increasing, it is a fortiori the same for the function

$$\nu \int_{\Theta}^{t} j^2(t')\,dt' + \frac{1}{2}w(t) - \int_{\Theta}^{t} \sqrt{w(t')}\, j(t') V(t')\,dt'.$$

Now

$$\nu \int_{\Theta}^{t} j^2(t')\,dt' - \int_{\Theta}^{t} \sqrt{w(t')}\, j(t') V(t')\,dt' + \frac{1}{4\nu} \int_{\Theta}^{t} w(t') V^2(t')\,dt'$$

manifestly cannot decrease. It follows that the function

$$\frac{1}{2}w(t) - \frac{1}{4\nu}\int_\Theta^t w(t')V^2(t')\,dt'$$

is nonincreasing. From this we get the inequality generalizing (3.7)

$$w(t) \le w(\Theta)e^{\frac{1}{2\nu}\int_\Theta^t V^2(t')\,dt'} \qquad (\Theta < t < T). \tag{6.3}$$

Suppose in particular that the solutions $U_i(x,t)$ and $a_i(x,t)$ correspond to the same initial state. Then $w(\Theta) = 0$ and by (6.3) $w(t) = 0$. therefore $U_i(x,t) = a_i(x,t)$ for $\Theta \le t < T$. The uniqueness theorems of paragraphs 18 and 23 (p. 222 and 228) are special cases of this result.

33. *Regularity of turbulent solutions in certain time intervals.*

Consider a turbulent solution $U_i(x,t)$ defined for $t \ge 0$. For each nonsingular time the vector $U_i(x,t)$ is a semi-regular initial state (cf. p.231 existence theorem, case a)). The uniqueness theorem that we are going to establish will have the following consequence. Consider a nonsingular time, i.e. a time chosen outside of a certain set of measure zero. Then this is the origin of an interval of time in the interior of which the turbulent solution coincides with a regular solution, and this coincidence does not end as long as the regular solution remains so. This result, complemented by some other easy ones, gives us the next theorem.

Structure theorem.

For a vector $U_i(x,t)$ to be a turbulent solution to Navier's equations for $t \ge 0$, it is necessary and sufficient that it have the following three properties.

a) By an *interval of regularity* we mean any interval $\overline{\Theta_l T_l}$ of time in the interior of which the vector $U_i(x,t)$ is a regular solution to Navier's equations, and such that this is true for no interval containing $\overline{\Theta_l T_l}$. Let O be the open set which is the union of the intervals of regularity

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[1] I have not been able to establish a uniqueness theorem stating that to a given initial state, there corresponds a unique turbulent solution.

(no two of which have a point in common). O differs from the half axis $t \geq 0$ only by a set of measure zero.

b) The function $\iiint_\Pi U_i(x,t)U_i(x,t)\,\delta x$ is decreasing on the set consisting of O and $t = 0$.

c) As t' tends to t the $U_i(x,t')$ must converge weakly in mean to the $U_i(x,t)$.

Supplementary information

1) A turbulent solution corresponding to a semi-regular initial state coincides with the semi-regular solution having that initial state, for as long a time as the semi-regular solution exists.

2) Make t increase to T_l in an interval of regularity. Then the solution $U_i(x,t)$ which is regular for $\Theta_l < t < T_l$ becomes irregular.

This structure theorem allows us to *summarize our work* in these terms: We have tried to establish the existence of a solution to Navier's equations corresponding to a given initial state. We have had to give up regularity of the solution at a set of times of measure zero. At these times the solution is only subject to a very weak continuity condition (c) and to condition (b) expressing the nonincrease of kinetic energy.

Remark: If system (3.11) has a nonzero solution $U_i(x)$ then we can very simply construct a turbulent solution $U_i(x,t)$ equal to

$$[2\alpha(T-t)]^{-\frac{1}{2}}U_i\left[[2\alpha(T-t)]^{-\frac{1}{2}}x\right] \text{ for } t < T$$

and to 0 for $t > T$. This has a single irregular time T.

34. *Supplementary information on intervals of regularity and behavior of solutions to Navier's equations for large time.*

Chapter IV gives inequality (4.3) in addition to the existence theorem used in the previous paragraph. This results in the following proposition. Consider a turbulent solution $U_i(x,t)$. Let t be a nonsingular time and T_l a later time. We have

$$J(t') > A_1 \nu^{\frac{3}{4}} (T_l - t')^{-\frac{1}{4}},$$

A_1 being a certain numerical constant. Using this lower bound for $J(t')$ in the inequality

$$\nu \int_0^{T_l} J^2(t')\, dt' < \frac{1}{2} W(0)$$

we get

$$2 A_1 \nu^{\frac{5}{2}} T_l^{\frac{1}{2}} < \frac{1}{2} W(0).$$

All singular times occur prior to

$$\theta = \frac{W^2(0)}{16 A_1^4 \nu^5}. \tag{6.4}$$

In other words, there is an interval of regularity that contains θ and which extends to $+\infty$. A motion which is regular up to time θ never becomes irregular.

It is easy to make this more precise.

Let $\overline{\Theta_l T_l}$ be an interval of regularity of finite length. All times t interior to this interval are nonsingular. By (4.3)

$$J(t') > A_1 \nu^{\frac{3}{4}} (T_l - t')^{-\frac{1}{4}} \text{ for } \Theta_l < t' < T_l.$$

(cf. Second characterisation of irregularity, p. 227.)

Using this lower bound on $J(t')$ in

$$\nu \sum_l \int_{\Theta_l T_l} J^2(t')\, dt' \leq \frac{1}{2} W(0).$$

Summing over all the intervals of finite length we get

$$2 A_1^2 \nu^{\frac{5}{2}} \sum_l{}' \sqrt{(T_l - \Theta_l)} < \frac{1}{2} W(0). \tag{6.5}$$

In Chapter IV we found the pair of inequalities (4.3) (4.4). These imply the following. Consider a turbulent solution, a nonsingular time t', and a later time t. We have

$$\text{either } t - t' > A_1^4 \nu^3 J^{-4}(t'), \text{ or } J(t) < (1+A)J(t')$$

in other words[1]

$$J(t') > \{A_1 \nu^{\frac{3}{4}} (t-t')^{-\frac{1}{4}}; \frac{1}{1+A} J(t)\}.$$

Using this lower bound for $J(t')$ in

$$\nu \int_0^t J^2(t')\, dt' \leq \frac{1}{2} W(0);$$

we get

$$\nu \int_0^t \{A_1^2 \nu^{\frac{3}{2}} (t-t')^{-\frac{1}{2}}; \frac{1}{(1+A)^2} J^2(t)\}\, dt' \leq \frac{1}{2} W(0). \tag{6.6}$$

This gives an upper bound for $J(t)$ for values of t larger than θ. However this bound has a rather complicated analytic expression.

We content ourselves by remarking that (6.6) gives the less precise

$$\nu \int_0^t \{A_1^2 \nu^{\frac{3}{2}} t^{-\frac{1}{2}}; \frac{1}{(1+A)^2} J^2(t)\}\, dt' \leq \frac{1}{2} W(0).$$

This can most simply be expressed as

$$J^2(t) < \frac{(1+A)^2}{2} \frac{W(0)}{\nu t} \text{ for } t > \frac{W^2(0)}{4 A_1^4 \nu^5}. \tag{6.7}$$

Complementing this result on asymptotic behavior of $J(t)$ there is another on $V(t)$. Inequalities (4.3) and (4.4) give

$$V(t) < AJ(t')[\nu(t-t')]^{-\frac{1}{4}} \text{ for } t - t' < A_1^4 \nu^3 J^{-4}(t').$$

[1] Recall that the symbol $\{B; C\}$ denotes the smaller of B and C.

By (6.7) this last inequality is satisfied for $t' = \frac{1}{2}t$ if one takes $t > A\frac{W^2(0)}{\nu^5}$. One therefore has for these t

$$V(t) < A\sqrt{W(0)}(\nu t)^{-\frac{3}{4}}.$$

In summary there exist some constants A such that

$$J(t) < A\sqrt{W(0)}(\nu t)^{-\frac{1}{2}} \text{ and } V(t) < A\sqrt{W(0)}(\nu t)^{-\frac{3}{4}} \text{ for } t > A\frac{W^2(0)}{\nu^5}.$$

N. B. I am ignoring the case in which $W(t)$ necessarily tends to 0 as t becomes indefinitely large.