HYPERFUNCTIONS IN HYPERBOLIC GEOMETRY

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Abstract. In the framework of real hyperbolic geometry, this review note begins with the Helgason correspondence induced by the Poisson transform between eigenfunctions of the Laplace-Beltrami operator on the hyperbolic space $\mathbb{H}^{n+1}$ and hyperfunctions on its boundary at infinity $\mathbb{S}^n$. Focused on the scattering operator for real hyperbolic manifolds of finite geometry, discussion is given on the two different constructions (pseudodifferential calculus for degenerate operators, harmonic analysis for the conformal group) and some applications (Selberg zeta functions, resonances and scattering poles).

Respectfully dedicated to Professor A. Kaneko.

After the pioneering works by Selberg [39], Lax and Phillips [23, 24], Patterson [30], the spectral analysis of the Laplace-Beltrami operator on non-compact hyperbolic manifolds has been studied in numerous contributions in the last thirty years.

This short survey note aims to show how hyperfunctions appear in this theory, even if their (enlightening) role is not so central : the representation of Laplace-Beltrami eigenfunctions on the hyperbolic space $\mathbb{H}^{n+1}$ through hyperfunctions on its boundary at infinity $\mathbb{S}^n$ was introduced by Helgason [21] and may be considered as a typical example for the deep links between geometry in the interior of the hyperbolic space and geometry on its boundary at infinity (see [10]). Another unexpected venue of hyperfunctions is the resolution of Patterson conjecture by Bunke and Olbrich [7], who show the importance of considering generalized sections with hyperfunction regularity rather than distributional. However, for spaces which are hyperbolic only outside of a compact set and whose spectral analysis has been done with success by purely $\mathcal{C}^\infty$ considerations, this hyperfunction framework is of no help.

For usual non-compact manifolds (e.g. Euclidean spaces, its geometric compact perturbations or Schrödinger operators), the scattering operator (or matrix) is known to be the spectral replacement for the L^2 eigenvalues spectrum (there are most of the time finitely many, where compact configurations have a purely discrete spectrum with finite multiplicities eigenvalues accumulating at $+\infty$). For hyperbolic spaces, the construction of such an operator is of primary importance: it's the second goal of this note to present (in a very simplified way) an overview of the two different scattering operator constructions for real hyperbolic manifolds of finite geometry, each one with its own generalizations : the first one [26, 33, 11, 12, 19, 20, 24, 34] with a clear PDE and geometric analysis approach is

2010 *Mathematics Subject Classification.* 58J50, 35P25, 47A40.

Key words and phrases. hyperbolic geometry, scattering operator, hyperfunctions, resonances, Selberg's zeta function.

able to consider manifold with hyperbolic metric up to a compact set, the second one [6, 7, 8, 9] in the realm of harmonic analysis and algebraic analysis applies to locally symmetric spaces of real rank one (spaces with negative curvature) and their geometric bundles (e.g. functions and forms).

In the first section, we set up the geometry of the hyperbolic manifolds as locally symmetric Riemannian manifolds and G-homogeneous manifold, before recalling the Helgason transform in the second section. In the third section, we present the conformally compact spaces and convex co-compact groups: like the hyperbolic space, they have a smooth compactification by adding a boundary at infinity supporting a natural conformal Riemannian class. The fourth section summarizes some results concerning cusps of non-maximal rank, model spaces for the finite geometry hyperbolic manifolds : even if with a natural compactification, the conformal metric on its boundary degenerates at the cusp. The last section presents Selberg type determinant formulas, establishing strong relations between geometrical data and spectral data (resolvent or scattering resonances), before stating some location results for these discrete data.

1. Hyperbolic manifolds

For the integer n and up to isometry, the hyperbolic space $\mathbb{H}^{n+1}$ is uniquely defined as the $n+1$ dimensional simply connected Riemannian manifold with constant sectional curvature -1: it is a symmetric space G/K with $G \simeq \mathrm{Isom}^+(\mathbb{H}^{n+1})$ and K the stabilizer of any point. Among its many isometric presentations, the ball model $\mathbb{B}^{n+1} = \{z \in \mathbb{R}^{n+1}, \|z\| < 1\}$, where $\| \ \|$ is the Euclidean norm on $\mathbb{R}^{n+1}$, is endowed with the Riemannian metric $g_z = 4(1 - \|z\|^2)^{-2} dz^2$. Included in $\mathbb{R}^{n+1}$, the ball model has a natural compactification $\overline{\mathbb{B}^{n+1}}$, with boundary at infinity the sphere $\mathbb{S}^n = \partial_\infty \mathbb{B}^{n+1}$. This boundary is defined topologically as the set of equivalence classes of asymptotic geodesic rays, definition valid more generally for any Riemannian simply connected manifold with non-positive sectional curvature (a so called Hadamard manifold); only in the case of the hyperbolic space there exists natural smooth and analytic structures. Furthermore, with the representation through the ball $\mathbb{B}^{n+1}$, the boundary at infinity $\partial_\infty \mathbb{H}^{n+1} \simeq \mathbb{S}^n$ has a natural conformal structure, given by the metric with constant sectional curvature $+1$ obtained by restriction of the Euclidean metric on $\mathbb{R}^{n+1}$.

Any isometry on $\mathbb{B}^{n+1}$ extends continuously on the compactification $\overline{\mathbb{B}^{n+1}}$, conformally acting on the boundary $\mathbb{S}^n = \partial_\infty \mathbb{B}^{n+1}$: the resulting map $\mathrm{Isom}(\mathbb{H}^{n+1}) \to \mathrm{Conf}(\mathbb{S}^n)$ is a group isomorphism, preserving the identity components consisting of orientation preserving transformations. We will denote by G any of these connected isomorphic groups: $G \simeq \mathrm{Isom}^+(\mathbb{H}^{n+1}) \simeq \mathrm{Conf}^+(\mathbb{S}^n)$. The hyperbolic space $\mathbb{H}^{n+1}$ and its boundary $\mathbb{S}^n$ are G-homogeneous manifolds. Let us choose base points O, (O, v) and ∞ in the space $\mathbb{H}^{n+1}$, its unitary tangent bundle $S^1\mathbb{H}^{n+1}$ and its boundary $\mathbb{S}^n$ *resp.*: in the ball model $\mathbb{B}^{n+1}$, it could be $O = 0 \in \mathbb{B}^{n+1}$, $(O, v) = (0, (1, 0, \dots, 0))$ and $\infty = (1, 0, \dots, 0)$. Let us denote by K, M, P the isotropy subgroups of the base points *resp.*: $\mathbb{B}^{n+1} = G/K$, $S^1\mathbb{H}^{n+1} = G/M$ and $\mathbb{S}^n = G/P$. The Lie group decompositions $G = KAN$, $P = MAN$ and $G/K = P/M$ will be used. The group G acts on any usual functional space induced by a G-bundle E with base $B = \mathbb{H}^{n+1}$ or $B = \mathbb{S}^n$: $\mathcal{C}^k(E), k \in \mathbb{N} \cup \{\infty\}$ for smooth sections of order k, $\mathcal{C}^{-\infty}(E)$ for distribution sections, $\mathcal{C}^\omega(E)$ for analytic sections, $\mathcal{C}^{-\omega}(E)$ for hyperfunction sections.

A hyperbolic manifold M is a locally symmetric manifold with constant sectional curvature. It can be represented as the quotient $\Gamma\backslash\mathbb{H}^{n+1}$ for a discrete group Γ without torsion elements in $\mathrm{Isom}(\mathbb{H}^{n+1})$. The limit set Λ_Γ is the derived set in $\partial_\infty\mathbb{H}^{n+1}$ of any Γ-orbit in $\mathbb{H}^{n+1}$: it is the complementary of the maximal open set Ω_Γ in $\partial\mathbb{H}^{n+1}$ where the Γ-action is totally discontinuous. The quotient $B = \Gamma\backslash\Omega_\Gamma$ is a smooth manifold while $M \cup B = \Gamma\backslash(\mathbb{H}^{n+1} \cup \Omega_\Gamma)$ is a smooth manifold with boundary.

We will consider those M (or Γ) of finite geometry [4]: this class includes any group Γ admitting for its action on $\mathbb{H}^{n+1}$ a finitely-sided fundamental polyhedron, *i. e.* a domain whose boundary is a union of totally geodesic faces. The manifold M is compact or with finite volume (with cusps of maximal rank) if and only if the limit set Λ_Γ is $\partial_\infty\mathbb{H}^{n+1}$: if not, either B is compact and the manifold M is said *convex co-compact*, or the manifold has infinite volume cusps. The last two cases will be further described in the subsequent sections 3 and 4.

2. Eigenfunctions and the Poisson-Helgason transform

The Laplace-Beltrami operator $\Delta_{\mathbb{H}^{n+1}}$ on $\mathbb{H}^{n+1}$ defined for $\varphi, \psi \in \mathcal{C}_0^2(\mathbb{H}^{n+1})$ by

$$\int_{\mathbb{H}^{n+1}} \varphi(z)\Delta_{\mathbb{H}^{n+1}}\psi(z)d\,\mathrm{vol}_{\mathbb{H}^{n+1}}(z) = \int_{\mathbb{H}^{n+1}} g_z(\nabla_{\mathbb{H}^{n+1}}\varphi(z), \nabla_{\mathbb{H}^{n+1}}\psi(z))d\,\mathrm{vol}_{\mathbb{H}^{n+1}}(z)$$

has a local expression

$$\Delta_{\mathbb{B}^{n+1}} := \frac{(1-\|z\|^2)^2}{4}\Delta_{\mathbb{R}^{n+1}} + (n-1)\frac{1-\|z\|^2}{4}\langle z, \nabla_{\mathbb{R}^{n+1}}\rangle$$

where $\Delta_{\mathbb{R}^{n+1}}$ and $\nabla_{\mathbb{R}^{n+1}}$ are the Laplace-Beltrami and gradient operators on the Euclidean space $\mathbb{R}^{n+1}$. The Poisson kernel

$$P(z,\xi) := \frac{1-\|z\|^2}{\|z-\xi\|^2}, \quad z \in \mathbb{B}^{n+1}, \xi \in \mathbb{S}^n$$

gives eigenfunctions of the Laplace operator $\Delta_{\mathbb{B}^{n+1}}$ parametrized by $\xi \in \mathbb{S}^n$:

$$(\Delta_{\mathbb{B}^{n+1}} - s(n-s))[P(z,\xi)^s] = 0, \quad z \in \mathbb{B}^{n+1}.$$

We will denote $\Delta_M - s(n-s)$ shortly by $\Delta_M(s)$.

As $\mathbb{S}^n$ is a compact analytic manifold, its hyperfunction space $\mathcal{C}^{-\omega}(\mathbb{S}^n)$ is defined as the space of analytic functionals in the Martineau sense: we will denote the duality bracket $\langle\ ,\ \rangle$ on $\mathcal{C}^{-\omega}(\mathbb{S}^n) \times \mathcal{C}^{\omega}(\mathbb{S}^n)$ so that

$$\langle T, \varphi\rangle = \int_{\mathbb{S}^n} \varphi(\xi)T(\xi)d\,\mathrm{vol}_{\mathbb{S}^n}(\xi), \quad T \in \mathcal{C}^{-\omega}(\mathbb{S}^n), \varphi \in \mathcal{C}^{\omega}(\mathbb{S}^n).$$

The space $\mathcal{C}^{-\omega}(\mathbb{S}^n)$ may be considered also as the global section space of the sheaf of hyperfunctions on $\mathbb{S}^n$. Helgason [21] for the hyperbolic plane $\mathbb{H}^2$ and Minemura [28] for any $\mathbb{H}^{n+1}$ proved the following representation of the eigenfunctions of the Laplace-Beltrami operator on $\mathbb{B}^{n+1}$ by hyperfunctions on its boundary at infinity:

Theorem 1. *Let s be a complex number. The Poisson-Helgason transform P_s defined on $\mathcal{C}^{-\omega}(\mathbb{S}^n)$ by*

$$P_sT(z) := \langle T, P(z,\cdot)^s\rangle = \int_{\mathbb{S}^n} \left[\frac{1-\|z\|^2}{\|z-\xi\|^2}\right]^s T(\xi)d\,\mathrm{vol}_{\mathbb{S}^n}(\xi), \quad T \in \mathcal{C}^{-\omega}(\mathbb{S}^n), z \in \mathbb{B}^{n+1}$$

is a G-equivariant map from $\mathcal{C}^{-\omega}(\mathbb{S}^n)$ to $\mathcal{C}^{\omega}(\mathbb{B}^{n+1})$, with image included in the kernel $\mathrm{Ker}\,\Delta_{\mathbb{B}^{n+1}}(s)$*. For $\Re\mathrm{e}\, s > n/2$, the map P_s is an isomorphism between*

$\mathcal{C}^{-\omega}(\mathbb{S}^n)$ and $\operatorname{Ker}\Delta_{\mathbb{B}^{n+1}}(s)$, with inverse B_s given by a generalized boundary value. For $u \in \operatorname{Ker}\Delta_{\mathbb{B}^{n+1}}(s)$, the hyperfunction $B_s u$ is defined by

$$\langle B_s u, \varphi\rangle = \frac{\sqrt{\pi}2^{s-n}\Gamma(s)}{\Gamma(s-n/2)} \lim_{x\to 0^+} x^{s-n} \int_{\mathbb{S}^n} u((1-x)\xi)\varphi(\xi) d\operatorname{vol}_{\mathbb{S}^n}, \quad \varphi \in \mathcal{C}^{\omega}(\mathbb{S}^n).$$

The operators P_s, B_s defined on $\{\Re e\, s > n/2\}$ have a meromorphic continuation to $\mathbb{C}$ with poles in $-\mathbb{N}$

This representation of any eigenfunction on a symmetric space X of non-compact type through a hyperfunction on its minimal boundary, known as the Helgason conjecture, has been proved in full generality by Kashiwara, Kowata, Minemura, Okamoto, Oshima and Tanaka [22].

To have a formulation of the Poisson-Helgason transform defined more independently from the choice of the hyperbolic model, it is useful to introduce the one-dimensional representation $\pi_s : man \in P \mapsto a^s \in \mathbb{C}^*$, the line bundle $L_s = [G \times_{\pi_s} \mathbb{C}]/P \to \mathbb{S}^n = G/P$ and the induced representation on G by π_s, the so called principal series G-representations $\mathcal{C}^\infty(\mathbb{S}^n, L_s) \simeq \{f : G \to \mathbb{C}, f(gman) = a^s f(g)\}$. Furthermore, the line bundle L_s is isomorphic to the (conformal) s-densities bundle $|\Lambda|^s(\mathbb{S}^n)$: $\mathcal{C}^\infty(\mathbb{S}^n, L_s) = \{f(\xi)|d\xi|^s\}$. Distributional section space $\mathcal{C}^{-\infty}(\mathbb{S}^n, L_s)$ is defined as the dual of the smooth section space $\mathcal{C}^\infty(\mathbb{S}^n, L_s)$.

The Poisson-Helgason transform P_s can be viewed as a G-module map from $\mathcal{C}^{-\omega}(\mathbb{S}^n, L_s)$ to $\ker\Delta_{\mathbb{H}^{n+1}}(s)$. Because of the identity of the G-modules $\ker\Delta_{\mathbb{H}^{n+1}}(s)$ and $\ker\Delta_{\mathbb{H}^{n+1}}(n-s)$, the G-modules $\mathcal{C}^\infty(\mathbb{S}^n, L_s)$ and $\mathcal{C}^\infty(\mathbb{S}^n, L_{n-s})$ are isomorphic: the Knapp-Stein operator $J_n(s)$ realizes an intertwining isomorphism between them, preserving the analytic sections and with extension to generalized section spaces by duality. It is a family of pseudodifferential operators of order $2s-n$, meromorphic on $\mathbb{C}$. The scattering operator on $\mathbb{H}^{n+1}$ is defined as the Knapp-Stein operator up to a normalizing factor

$$\mathcal{S}_{\mathbb{H}^{n+1}}(s) := \frac{2^{2(s-n)}}{\sqrt{\pi}} \frac{\Gamma(s-n/2)}{\Gamma(s)} J_n(s) : \mathcal{C}^*(\mathbb{S}^n, L_s) \to \mathcal{C}^*(\mathbb{S}^n, L_{n-s})$$

It is meromorphic on $\mathbb{C}$ and because of the prefactor choice it, obeys the functional equation

$$S_{\mathbb{H}^{n+1}}(s) S_{\mathbb{H}^{n+1}}(n-s) = \operatorname{Id}, \quad s \in \mathbb{C} \tag{1}$$

and describes the boundary values of eigenfunctions on $\mathbb{B}^{n+1}$ with smooth boundary value $T \in \mathcal{C}^\infty(\mathbb{S}^n)$: for $r \in \mathbb{R}$ and $\xi \in \mathbb{S}^n$,

$$[P_{\frac{n}{2}+ir} T]((1-x)\xi) = x^{\frac{n}{2}-ir} T(\xi) + x^{\frac{n}{2}+ir} [\mathcal{S}_{\mathbb{H}^{n+1}}(\tfrac{n}{2}+ir)T](\xi) + o(x^{\frac{n}{2}}), \quad x \to 0^+. \tag{2}$$

3. Convex co-compact hyperbolic and conformally compact manifolds

Conformally compact manifolds are a generalization of the convex co-compact hyperbolic manifolds: geometrically, they have asymptotically constant negative sectional curvature at infinity.

Definition 1. *The complete Riemannian manifold (M, g) is said* conformally compact *if there exists a compact manifold $\overline{M}$ with boundary $B = \partial\overline{M}$, a smooth function x on $\overline{M}$ defining the boundary B ($B = x^{-1}(0)$ and dx non zero on B) such that the metric $x^2 g$ extends smoothly to a metric $\overline{g}$ on $\overline{M}$. The boundary B is called* boundary at infinity $\partial_\infty M$ *of the manifold M.*

The conformal class $\gamma_B = [\overline{g}_{|B}]$ is uniquely determined, independently from the boundary defining function x. If g is asymptotically hyperbolic, for any Riemannian tensor h_0 representing the class γ_B, there is a unique defining function x so that the metric $\overline{g} = x^2 g$ has in a neighborhood of B the expansion $\overline{g} = dx^2 + h_0 + o(1)$.

The hyperbolic space $\mathbb{H}^{n+1}$ and the convex co-compact $\Gamma\backslash\mathbb{H}^{n+1}$ are the basic geometric examples of such manifolds: if $x(m) = \mathrm{e}^{-d(0,m)}, m \in \mathbb{H}^{n+1}$, the metric $g_{\mathbb{H}^{n+1}}$ takes the form $x^{-2}(dx^2 + (1-x^2)^2 d\omega^2/4)$ where $d\omega$ is the standard metric on the sphere $\mathbb{S}^n$. The locally symmetric space $\Gamma\backslash\mathbb{H}^{n+1}$ admits a finite cover by two types of sets: open hyperbolic balls in $\mathbb{H}^{n+1}$ and half-balls in $\overline{\mathbb{H}^{n+1}}$ with equatorial boundary on $\partial_\infty\mathbb{H}^{n+1}$. This decomposition in model balls will be crucial for the realization of the resolvent $\Delta_M(s)^{-1}$ by cutting and gluing localized resolvents $\chi_2\Delta_{\mathbb{H}^{n+1}}(s)^{-1}\chi_2$ of the *free model* $\mathbb{H}^{n+1}$. Other examples of asymptotically constant curvature -1 manifolds are given by Schwarzschild [37] metrics and the problem of defining a conformally compact Einstein metric on M inducing a given conformal structure γ_B on its boundary at infinity B has attracted much attention [1].

According to Lax and Phillips [23, 24], the spectrum of the Laplace-Beltrami operator Δ_M consists of a discrete set of eigenvalues, all of finite multiplicity in $(0, n^2/4)$ and an absolutely continuous spectrum $[n^2/4, +\infty)$. The generalized eigenfunctions corresponding to the spectral value $\lambda = n^2/4 + r^2$ or $s = n/2 + \mathrm{i}r$ ($r \in \mathbb{R}$) have an asymptotics describe as in (2) by a scattering operator, which can be built along two different ways: the first one through the resolvent analysis [26, 33, 19], the second one by Γ-equivariant analysis on the boundary $\mathbb{S}^n$ [31, 6].

3.1. **Pseudodifferential calculus.** The resolvent operator $R_M(s) := \Delta_M(s)^{-1} : L^2(M) \to L^2(M)$ is defined as a meromorphic, bounded operator valued function on $\Re e\, s > n/2$ with singularities at the s values corresponding to L^2 eigenvalues λ ($s = n/2 + \sqrt{n^2/4 - \lambda}$). In a collar neighborhood of the boundary with the coordinate system $(x, y_1, \ldots, y_n)$, the Laplace-Beltrami operator is of the form

$$\Delta_M = -(x\partial_x)^2 + nx\partial_x + x^2\Delta_{h_0} + xP_\infty(x\partial_x, x\partial_{y_1}, \ldots, x\partial_{y_n}). \tag{3}$$

There exists a pseudodifferential calculus, the so-called 0-calculus introduced by Mazzeo and Melrose, which contains all differential operators generated by vector fields V on $\overline{M}$ of 0-type: V is of 0-type if its restriction $V_{|B}$ is null. The local expression (3) indicates that the Laplacian is an elliptic operator for this calculus, which is perfectly suited to build a good parametrix for $\Delta_M(s)$. In the case of a space with constant curvature, such a construction [19] can be done also directly by using reference models, the ball $\{z \in \mathbb{B}^{n+1}; \|z\| < r\}$ ($r < 1$) with the (explicit) resolvent on $\mathbb{H}^{n+1}$ and the half-ball $\{z \in \mathbb{B}^{n+1}; \|z - (1, 0, \ldots, 0)\| < r\}$ with a resolvent built with the help of the indicial equation appearing in (3).

Theorem 2 ([26, 33, 12])**.** *Let M be a conformally compact manifold asymptotically hyperbolic with even metric (i. e. the metric $\overline{g}$ has only even powers in its Taylor expansion in the variable x defining the boundary B with $|dx|_{x^2g} = 1$).*

The resolvent $R_M(s) : \mathcal{C}_0^\infty(M) \to L^2_{\mathrm{loc}}(M)$ has a meromorphic continuation to $s \in \mathbb{C}$ so that

$$R_M(s)\left(\bigcap_{N\geq 0} x^N \mathcal{C}^\infty(\overline{M})\right) \subset x^s \mathcal{C}^\infty(\overline{M})$$

whose singular set $\mathcal{R}_M$ is called the resonance set*. The polar part is of finite rank at any singular point $s \in \mathcal{R}_M$.*

The resolvent $R_M(s)$ permits to solve the Dirichlet problem at infinity [11]: for $s \in \{\Re e\, s \geq n/2\} \setminus (n/2 + \mathbb{N})$ and $T \in \mathcal{C}^\infty(|\Lambda|^s(B))$, there exists a smooth $u \in \mathcal{C}^\infty(M)$ such that

$$\Delta_M(s)u = 0 \text{ in } M, \quad [x^{-s}u]_{|x=0} = T \text{ on the boundary.} \tag{4}$$

The resulting operator P_s such that $u = P_s T$ is the unique solution of (4) is called the *Poisson operator* and inherits from the resolvent a meromorphic continuation to all of $\mathbb{C}$, with poles in $\mathcal{R}_M \cup (n/2 + \mathbb{N})$: the last poles due to integer difference of the roots of the indicial equation induced by (3). The scattering operator $S_M(s)$ describes the asymptotics of the Poisson operator, as in the particular case of the hyperbolic space $\mathbb{H}^{n+1}$: for $r \in \mathbb{R}^*$ and $T \in \mathcal{C}^\infty(|\Lambda|^{n/2+ir}(B))$

$$[P_{\frac{n}{2}+ir}T](x,b) = x^{\frac{n}{2}-ir}T(b) + x^{\frac{n}{2}+ir}[\mathcal{S}_M(\tfrac{n}{2}+ir)T](b) + o(x^{\frac{n}{2}}), \quad b \in B, x \simeq 0.$$

Furthermore, the scattering operator $\mathcal{S}_M(s)$ is defined as a meromorphic function of pseudodifferential operators of order $2s - n$.

$$S_M(s) : \mathcal{C}^\infty(B, |\Lambda|^s(B)) \longrightarrow \mathcal{C}^\infty(B, |\Lambda|^{n-s}(B))$$

with a functional equation similar to the one (1) for the full hyperbolic space $\mathbb{H}^{n+1}$.

3.2. **Distributions on the conformal boundary $\partial_\infty \mathbb{H}^{n+1}$.** Bunke and Olbrich [7, 8] study restrictions and extensions of generalized sections (distribution or hyperfunction type) on the boundary $\partial_\infty \mathbb{H}^{n+1}$ with respect to its partition $\partial_\infty \mathbb{H}^{n+1} = \Omega_\Gamma \sqcup \Lambda_\Gamma$. The scattering operator is a by-product of this general and complete analysis, which applies to any geometric bundle with base a locally symmetric space of rank one (real, complex or quaternionic hyperbolic). The generality of the method has its drawback: it doesn't apply to perturbation (even on compact sets) of such locally symmetric spaces.

Let us define the Poincaré exponent δ_Γ: it is the convergence abscissa for any of the series

$$\sum_{\gamma \in \Gamma} \mathrm{e}^{-sd(m,\gamma m)}, \quad m \in \mathbb{H}^{n+1}, \quad \sum_{\gamma \in \Gamma} \|\gamma'(z)\|^{-s}, \quad z \in \mathbb{B}^{n+1} \cup \Omega_\Gamma.$$

In the first series, the distance d is the hyperbolic distance in $\mathbb{H}^{n+1}$ and the γ appear as isometries on $\mathbb{H}^{n+1}$, while in the second γ operates as conformal transformations with conformal dilation $\|\gamma'(z)\|$. The Poincaré exponent δ_Γ lies in the interval $[0, n]$ and for finite geometry groups, the exponent δ_Γ coincides with the Haussdorf dimension for the limit set Λ_Γ or the topological entropy of the geodesic flow on the unit tangent bundle S^1M.

The extension operator between Γ-invariant sections, either of distribution ($\# = \infty$) or hyperfunction ($\# = \omega$) type,

$$\mathrm{ext}_\Gamma(s) : \mathcal{C}^{-\#}(\Omega_\Gamma, L_s)^\Gamma \to \mathcal{C}^{-\#}(\mathbb{S}^n, L_s)^\Gamma, \quad \Re e\, s > \delta_\Gamma$$

plays a central role. It is realized as the dual of the push-down operator

$$\pi_\Gamma(-s) : \varphi \in \mathcal{C}^\#(\mathbb{S}^n, L_{-s}) \to \sum_{\gamma \in \Gamma} \gamma^* \varphi \in \mathcal{C}^\#(\Omega_\Gamma, L_{-s})^\Gamma \tag{5}$$

where the series is convergent, either for $\mathcal{C}^\infty$ smooth or analytic sections.

Based on the identification $B \simeq \Gamma\backslash\Omega$, the isomorphism $\mathcal{C}^*(B, L_s) \simeq \mathcal{C}^*(\Omega_\Gamma, L_s)^\Gamma$ is basic for the definition of the scattering operator: in the distribution case, it is defined through a partition of unity adapted to a B cover by open sets U trivializing the covering map $\Gamma\backslash\Omega \to B$, while in the hyperfunction case we can use a cover by closed sets (in this case the topology of the hyperfunction spaces on open sets or compact sets must be treated with care).

In a first step and for $\Re e\, s > \delta_\Gamma$, the scattering operator for Γ is the map $S_\Gamma(s) : \mathcal{C}^*(B, L_s) \to \mathcal{C}^*(B, L_{n-s})$ obtained by the preceding identifications applied to the domain and codomain of the composition

$$\mathcal{C}^{-\#}(\Omega_\Gamma, L_s)^\Gamma \xrightarrow{\mathrm{ext}_\Gamma(s)} \mathcal{C}^{-\#}(\mathbb{S}^n, L_s)^\Gamma \xrightarrow{S_{\mathbb{H}^{n+1}}(s)} \mathcal{C}^{-\#}(\mathbb{S}^n, L_{n-s})^\Gamma \xrightarrow{\mathrm{res}} \mathcal{C}^{-\#}(\Omega_\Gamma, L_{n-s})^\Gamma$$

so that

$$\mathcal{S}_\Gamma(s) = \mathrm{res} \circ \mathcal{S}_{\mathbb{H}^{n+1}}(s) \circ \mathrm{ext}_\Gamma(s), \quad \Re e\, s > \delta_\Gamma.$$

If $\delta_\Gamma < n/2$, the extension map and the scattering operator satisfy on the strip $|\Re e\, s - n/2| < n/2 - \delta_\Gamma$ the functional equations

$$\mathrm{ext}_\Gamma(s) = \mathcal{S}_{\mathbb{H}^{n+1}}(n-s) \circ \mathrm{ext}_\Gamma(n-s) \circ \mathcal{S}_\Gamma(s), \quad \mathcal{S}_\Gamma(s)\mathcal{S}_\Gamma(n-s) = 1.$$

which, through analytic Fredholm theory, gives the meromorphic continuation for the scattering operator and the extension operator to all $\mathbb{C}$. The preceding equations show how crucial is the extension operator in the definition of the scattering operator.

The case $\delta_\Gamma > n/2$ can be solved by the embedding trick, already used by Mandouvalos [25]. Let N be an integer bigger than n: the spaces $\mathbb{H}^{n+1}, \mathbb{S}^n$ and groups $\mathrm{Conf}(H_n), \mathrm{Isom}(\mathbb{H}^{n+1})$ are included in similar objects for the value N, either by geometric inclusion (e.g. $\mathbb{H}^{n+1}$ is a totally geodesic subspace of $\mathbb{H}^{N+1}$, with inclusion $\iota : \mathbb{S}^n \hookrightarrow \mathbb{S}^N$ of their boundary at infinity) or extension of elements (e.g. any conformal map on $\mathbb{S}^n$ has a natural extension to $\mathbb{S}^N$ such that $\mathrm{Conf}(\mathbb{H}^{n+1}) \hookrightarrow \mathrm{Conf}(\mathbb{H}^{N+1})$ is a group homomorphism). The limit sets $\Lambda_\Gamma^n, \Lambda_\Gamma^N$ of Γ acting on $\mathbb{H}^{n+1}$ and its extension on $\mathbb{H}^{N+1}$ coincide up to identification : $\Lambda_\Gamma^N = \iota(\Lambda_\Gamma^n)$, so that $\iota(\Omega_\Gamma^n) = \iota(\mathbb{S}^n) \cap \Omega_\Gamma^N$. The natural restriction from the bundle $L_s \to \mathbb{S}^N$ to the bundle $L_s \to \mathbb{S}^n$ induces a natural map on the spaces of regular sections $\mathcal{C}^\infty(\mathbb{S}^N, L_s) \to \mathcal{C}^\infty(\mathbb{S}^n, L_s)$ and by duality the push forward maps

$$\iota_* : \mathcal{C}^{-\infty}(\mathbb{S}^n, L_s) \to \mathcal{C}^{-\infty}(\mathbb{S}^N, L_s), \quad \mathcal{C}^{-\infty}(\Omega_\Gamma^n, L_s) \to \mathcal{C}^{-\infty}(\Omega_\Gamma^N, L_s).$$

The support $\mathrm{supp}\, \iota_*(u)$ in $\mathbb{S}^N$ is a subset of $\iota(\mathbb{S}^n)$ for any $u \in \mathcal{C}^{-\infty}(\mathbb{S}^n, L_s)$. For $v \in \mathcal{C}^{-\infty}(\mathbb{S}^N, L_s)$ such that $\langle v, \varphi \rangle = 0$ for any test section φ zero on $\mathbb{S}^N$, we have a pull back $\iota_{N,n}^*(v)$ defined according to $\langle \iota_{N,n}^*(v), \varphi \rangle = \langle u, \widetilde{\varphi} \rangle$ where $\widetilde{\varphi}$ is any smooth extension to $\mathbb{S}^N$ of $\varphi \in \mathcal{C}^\infty(\mathbb{S}^n)$. The extension operators on $\mathbb{S}^n$ and $\mathbb{S}^N$ satisfy for $\Re e\, s >> 0$

$$\mathrm{ext}_\Gamma^n(s)(u) = \iota_{N,n}^*(\mathrm{ext}_\Gamma^N(s)[\iota_*(u)]), \quad u \in \mathcal{C}^{-\infty}(\Omega_\Gamma^n, L_s),$$

and the meromorphic continuation to $\mathbb{C}$ of the operator $\mathrm{ext}_\Gamma^N(s)$ implies the simultaneous continuation for $\mathrm{ext}_\Gamma^n(s)$ and the scattering operator $\mathcal{S}_\Gamma(s)$ acting on $\mathcal{C}^\infty(B, L_s)$.

4. Spaces with non-maximal rank cusps

As the models of the ball and half-ball have been essential in the construction of the convex co-compact parametrix, the (involved) spectral geometric analysis of the pure cusp C_p is crucial for proving the (usual) properties for the scattering operator $S_{C_p}(s)$: pseudodifferential operator meromorphic family of the localized operator $\chi\mathcal{S}_{C_p}(s)\chi$ (χ any smooth compactly supported function), the functional equation and the description of the asymptotics for eigenfunctions.

Let us begin by describing the geometry of a hyperbolic k-rank cusp $C_p = \Gamma_p\backslash\mathbb{H}^{n+1}$: every element of the discrete group Γ_p is a parabolic isometry fixing the point $p \in \partial_\infty\mathbb{H}^{n+1}$ and Γ_p contains a commensurable lattice L_p of rank k, the so called rank of the cusp. Let us introduce the half-space model $\mathbb{R}^{n+1}_+ \simeq \{(t,u) \in (0,\infty)\times\mathbb{R}^n\}$ with metric $(dt^2+du^2)/t^2$ for the hyperbolic space $\mathbb{H}^{n+1}$: by adding a point ∞ to the boundary $H_0 = \{0\}\times\mathbb{R}^n$ of the half-space $(0,\infty)\times\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ we recover the compactification of the ball model with its boundary $\partial_\infty\mathbb{H}^{n+1} \simeq \mathbb{S}^n \simeq H_0\cup\{\infty\}$. To describe precisely the geometry of the cusp, we conjugate Γ_p by an isometry sending p on ∞. The conjugate discrete group, denoted by Γ_∞, fixes the point ∞ and every (flat) horosphere $H_t = \{t\}\times\mathbb{R}^n, t>0$, centered at ∞; it acts also discontinuously on H_0, the complementary of $\Lambda_{\Gamma_\infty} = \{\infty\}$ in $\partial_\infty\mathbb{H}^{n+1}$. According to Bieberbach, there is a Riemannian product $H_1 \simeq Y\times Z$ where the Euclidean factors $Y\simeq\mathbb{R}^{n-k}$ and $Z\simeq\mathbb{R}^k$ are invariant under the Γ_∞ action such that every $\gamma\in\Gamma_\infty$ has a compatible decomposition:

$$\gamma = (\alpha_\gamma, t_\gamma\circ\rho_\gamma), \quad \alpha_\gamma\in SO(Y), \rho_\gamma\in SO(Z), t_\gamma \text{ translation on } Z,$$

with $\rho_\gamma^m = 1$ for some m. If there exists m such that $(\alpha_\gamma,\rho_\gamma)^m = 1$ for all γ, then the cusp is said *rational*, otherwise *irrational*. The cusp $C_p = \Gamma_p\backslash\mathbb{H}^{n+1}$ and the quotient $L_p\backslash\mathbb{H}^{n+1}$ are products $(0,\infty)\times F$ and $(0,\infty)\times F_k$, where F and F_k are flat bundles with respective bases the flat manifold $T = \Gamma\backslash Z$ and the flat torus $T_k = L_p\backslash Z$.

The boundary at infinity $\partial_\infty C_p$ of the cusp C_p is non-compact, but has a natural compactification. Let us consider the simple case $p=\infty$ and $\Gamma_\infty = L_p$ a lattice, so that $\Gamma_\infty\backslash\mathbb{H}^{n+1} \simeq (0,\infty)\times\mathbb{R}^{n-k}\times T_k$. Let us denote $\mathcal{C}_\pm(R) = \{(t,y,z)\in(0,\infty)\times Y\times T_k, \pm(t^2+\|y\|^2-R^2)>0\}$, so that the cusp $\Gamma_\infty\backslash\mathbb{H}^{n+1}$ is covered by the two charts $\mathcal{C}_-(2R), \mathcal{C}_+(R)$ with the metric $(dt^2+dy^2+dz^2)/t^2$. The first chart $\mathcal{C}_-(2R)$ can be completed with the boundary at infinity $\partial_\infty\mathcal{C}_-(2R) = \{0\}\times\{\|y\|<2R\}\times T_k$ and related defining function t : its infinity is of the type of conformally compact hyperbolic space. For the other chart $\mathcal{C}_+(R)$ covering the cusp region, the mapping [35] $(t,y,z)\mapsto(s = t/(t^2+\|y\|^2), v = -y/(t^2+\|y\|^2), z)$ maps it isometrically on the set $\mathcal{C}_-(R^{-1})$ endowed with the metric $(ds^2+dv^2+(s^2+\|v\|^2)dz^2)/s^2$, completed with the boundary $\{s=0\}\times\{\|y\|<R\}\times T_k$ and related defining function s: the metric $s^2 g_{\mathcal{C}_+(R)}$ extends smoothly to a metric on $\partial_\infty\mathcal{C}_+(R)$, except on the cuspidal torus $c_p = \{s=0, v=0\}\simeq T_k$ where it degenerates to the non-definite two-tensor ds^2+dv^2.

For a rational cusp, Guillarmou [15] analyses carefully the singularities of the scattering operator kernel around the cuspidal torus c_p, using the Φ-pseudodifferential calculus for fibered boundary manifold set up by Mazzeo and Melrose [27]. In particular, he introduces Schwartz type spaces of smooth functions which are asymptotically constant on the cusp c_p : for example, a function of the Schwartz space $\mathcal{S}_{\rm scc}(\partial_\infty\mathcal{C}_p)$ has only $s^\alpha v^\beta$ terms in its Taylor expansion relative to the variables (s,v,z) of the local model around the cusp c_p given above. The scattering operator operates well on these spaces. For an irrational cusp, Guillarmou and Mazzeo [17] use the flat fibration structure (replacing the product structure valid for a rational cusp) to do a Fourier mode decomposition for the construction of the resolvent and *in fine* the scattering operator : due to the cusp irrationality, there is accumulation of Fourier modes in the neighborhood of 0 which need special care. The scattering operator is proved to act on $H^\infty(\partial\mathcal{C}_p)$ functions. The resolvent and scattering operators for any finite geometry hyperbolic manifold M (and black box compact perturbation) can be deduced then by a parametrix built by gluing resolvents on model spaces.

Bunke and Olbrich [9] apply the same strategy as in the convex co-compact case, by defining jointly the meromorphic continuation for the extension $\mathrm{ext}_\Gamma(s)$ and scattering operators $\mathcal{S}_\Gamma(s)$. However, the push-down operator (5) is not surjective for a cusp group Γ_p : its image is characterized as a space of sections with asymptotic expansions, which are very similar to Guillarmou $\mathcal{S}_{\mathrm{scc}}(\partial_\infty\mathcal{C}_p)$ space. The analysis done for the pure cusps is extended to any finite geometry hyperbolic space M by gluing with partitions of unity adapted to the M covering by model open sets. Use of this smooth partition of unity prohibits consideration of hyperfunction sections.

5. Scattering operator and Selberg zeta function

For a space M with constant sectional curvature -1, the Selberg zeta function is defined by a convergent product for $\Re e\, s > n$.

$$Z_M(s) = \prod_{\text{closed geodesic } C} \exp\left[-\sum_{m\geq 1}\frac{1}{m}\frac{\mathrm{e}^{-ms\ell(C)}}{\det[1-\mathcal{P}(C)^m]}\right]$$

where $\ell(C)$ is the length of the closed geodesic C and $\mathcal{P}(C)$ the Poincaré map along C. This Selberg function can be expressed in term of the fundamental group $\Gamma = \pi_1(M)$ viewed as a group of conformal maps on $\mathbb{S}^n$

$$Z_M(s) = \prod_{\text{hyp. prim. } [\gamma]} \exp\left[-\sum_{m\geq 1}\frac{1}{m}\frac{{\gamma'_-}^{ms}}{\det[1-(\gamma'_- A_\gamma)^m]}\right]$$

where γ'_- is the conformal dilatation of γ at the attractive fixed point f_γ^- and A_γ is the rotational part : if $f_\gamma^- = 0$ in the half-space model, $\gamma(t,u) = \gamma'_-(t, A_\gamma y)$.

If M is compact, the Selberg trace formula can be expressed [38, 40, 41, 5] as an identity between the Selberg zeta function and regularized determinants for Laplace-Beltrami operators

$$Z_M\left(\frac{n}{2}+\lambda\right) = \det{}_\zeta\Big[\Delta_M - \frac{n^2}{4}+\lambda^2\Big]\left[\det{}_\zeta\left[\sqrt{\Delta_{\mathbb{S}^{n+1}}+\frac{n^2}{4}}+\lambda\right]\mathrm{e}^{P_n(\lambda)}\right]^{-\chi(M)}.$$

Here $\chi(M)$ is the Euler characteristic of M and $\det_\zeta$is the regularized determinant obtained by zeta-regularization: the trace $\mathrm{tr}\, A^z$ (well defined for $\Re e\, z << 0$) has a meromorphic continuation to $\mathbb{C}$, regular in $z=0$ so that $\det_\zeta A = \exp(\partial_z[\mathrm{tr}\, A^z]_{|z=0})$.

If M is convex co-compact with even dimension, Guillarmou [16] proves the formula

$$\text{(6)}\quad \det{}_{\mathrm{KV}} S_M\left(\frac{n}{2}+\lambda\right) = \frac{Z_M\left(\frac{n}{2}+\lambda\right)}{Z_M\left(\frac{n}{2}-\lambda\right)}\left[\frac{\det_\zeta\left(\sqrt{\Delta_{\mathbb{S}^n}+\frac{n^2}{4}}+\lambda\right)\mathrm{e}^{Q_n(\lambda)}}{\det_\zeta\left(\sqrt{\Delta_{\mathbb{S}^n}+\frac{n^2}{4}}-\lambda\right)\mathrm{e}^{Q_n(-\lambda)}}\right]^{-\chi(M)}$$

Here the Kontsevich-Vershik determinant $\det_{\mathrm{KV}}$ is obtained through the generalized trace $\operatorname{tr}_{\mathrm{KV}} A = \operatorname{tr}(AP^z)_{|z=0}$ where P is a positive self-adjoint elliptic differential operator of positive order and A is in some restricted class of pseudodifferential operators: Guillarmou proved that this regularization process applies well to $\partial_s S(s)S^{-1}(s)$.

The product formula (6) indicates the strong link between the divisor of the Selberg zeta function and the scattering singularities or poles of the resolvent as stressed by Patterson and Perry [32].

The multiplicity $m_M(\rho)$ of a resolvent resonance $\rho \in \mathcal{R}_M$ is the rank of the polar part of order 1 of the resolvent $\Delta_M(s)^{-1}$. For the scattering singularities, it is convenient to consider the normalized scattering operator

$$\widetilde{\mathcal{S}}_M(s) = \frac{\Gamma(s-n/2)}{\Gamma(n/2-s)} 2^{2s-n}(1+\Delta_{h_0})^{n/2-s}\mathcal{S}_M(s)(1+\Delta_{h_0})^{n/2-s}$$

whose every pole has a polar of finite rank. The multiplicity $\nu_M(\rho)$ of a scattering resonance ρ is defined by $\nu_M(\rho) = -\operatorname{tr}\operatorname{Res}_\rho[\partial_s \widetilde{\mathcal{S}}_M(s)\widetilde{\mathcal{S}}_M(n-s)]$; for $k \in \mathbb{N}$, the scattering operator $\widetilde{\mathcal{S}}_M(n/2+k)$ is a differential operator of order $2k$ with kernel dimension d_k, as introduced by Graham and Zworski [11].

The following theorem describes the relations between the multiplicities of resonances (scattering or resolvent) and kernel dimension d_k :

Theorem 3 ([13, 18])**.** *Let M be a convex co-compact hyperbolic manifold. Then, for any complex ρ,*

$$\nu_M(\rho) = m_M(\rho) - m_M(n-\rho) + \sum_{k\in\mathbb{N}} \left(\mathbf{1}_{n/2-k}(\rho) - \mathbf{1}_{n/2+k}(\rho)\right) d_k.$$

Location of resolvent resonances in the complex plane is largely an open problem: no Weyl law (except for cylindrical hyperbolic manifolds) is known, but some upper bounds and lower bounds have been established by Zworski and the author [20], Zworski [42], Perry [36], Borthwick [2], Naud [29]. In the following theorem, we present some of these results, whose proof strategies may be quite different: parametrix construction (1,2,3), trace formula of Poisson type (3,4), dynamical systems theory (5).

Theorem 4. *Let M be a m-dimensional manifold with constant curvature -1 outside a compact set. Let $N_M(R)$ and $N_M(\sigma,\delta,T)$ be the number of resonances, multiplicities included, in the disk $D_R = \{|\rho| \le R\}$ and the rectangle $R_{\sigma,\delta,T} = \{\sigma \le \Re e\,\rho \le \delta, |\Im m\,\rho| \le T\}$* resp.

1 [2] *Let M be a convex co-compact manifold with constant curvature -1 outside a compact set. Then $N_M(R) = \mathcal{O}(R^m)$.*

2 [14] *Let M be a compact perturbation of a finite geometry manifold. Then $N_M(r) = \mathcal{O}(R^{m+1})$.*

3 [20] *Let M be a hyperbolic surface with finite geometry. Then $N_M(R) = \Omega(R^2)$.*

4 [36] *Let M be a convex co-compact hyperbolic manifold. Then $N_M(R) = \Omega(R^m)$.*

5 [3] *Generically in the set of conformally compact Riemannian metrics with a fixed hyperbolic neighborhood at infinity, $\overline{\lim}_{R\to\infty}(\log N_M(R)/\log R) = m$.*

6 [29] *Let M be a non-elementary hyperbolic surface with infinite area and without cusp. Then, there exists an $\varepsilon > 0$ such that $N_M(\delta-\varepsilon,\delta,T) = 1$.*

and a function τ *defined on* $(\delta/2, 1)$ *with* $\sigma(\delta/2) = \delta, \tau(\sigma) < \delta$ *such that* $N_M(\sigma, \delta, T) = \mathcal{O}(T^{1+\tau(\sigma)})$.

There is another spectral interpretation of the Z_M divisor for M convex co-compact: conjectured by Patterson, it is based on cohomological dimensions for Γ-module of generalized functions on $\mathbb{S}^n$ with support in the limit set Λ_Γ. It has been proved by Bunke and Olbrich [7]

Theorem 5. *Let* $\Gamma \subset \operatorname{Conf} \mathbb{S}^n$ *a discrete torsion free convex co-compact group. For* $s \in \mathbb{C}$*, the order of the Selberg zeta function* Z_Γ *is given by*

$$\operatorname{ord}_s Z_\Gamma = \sum_{p=0}^{\infty} (-1)^p \dim H^p(\Gamma, \mathcal{C}^{-\omega}_{\Lambda_\Gamma}(\mathcal{O}_s L)),$$

where $\mathcal{O}_s L$ *is the space of germs of holomorphic sections* $\sigma \to F_\sigma \in \mathcal{C}^{-\omega}(\mathbb{S}^n, L_\sigma)$.

The initial conjecture concerned distribution sections: its resolution show the importance to settle it in the framework of hyperfunctions.

Acknowledgements

This note is a written exposition of a talk given at the conference organized on December 14-16, 2011 in honor of Professor Akira Kaneko, who has been my first teacher in graduate studies. I am grateful to the organizers for their invitation.

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