Available online at www.sciencedirect.com

SCIENCE DIRECT

Topology 44 (2005) 1057–1089

TOPOLOGY

www.elsevier.com/locate/top

On string topology of three manifolds

Hossein Abbaspour*

Centre de Mathématiques, École Polytechnique, 91128 Palaiseau, France

Received 8 October 2004; accepted 4 April 2005

Abstract

Let M be a closed, oriented and smooth manifold of dimension d. Let LM be the space of smooth loops in M. In [String topology, preprint math.GT/9911159] Chas and Sullivan introduced the loop product, a product of degree $-d$ on the homology of LM. We aim at identifying the three manifolds with "nontrivial" loop product. This is an application of some existing powerful tools in three-dimensional topology such as the prime decomposition, torus decomposition, Seifert fiber space theorem, torus theorem.

Keywords: Free loop space; Prime decomposition; JSJ-decomposition

1. Introduction and statement of main theorem

Throughout this paper M denotes a connected, oriented smooth manifold unless otherwise stated. We think of $\mathbb{S}^1$ the unit circle with a *marked point*, as the quotient $\mathbb{R}/\mathbb{Z}$. The *marked point* of $\mathbb{S}^1$ is the image of 0 in this quotient. A *loop* in M is a continuous map from $\mathbb{S}^1$ to M. The *free loop space* LM of M is the space of all loops in M. There is a one-to-one correspondence between the connected components of LM and the conjugacy classes of $\pi_1(M)$. If $f : \mathbb{S}^1 \to M$ is a loop in M then the image of the marked point of $\mathbb{S}^1$ is called the *marked point* of the loop f. The integral homology of LM is equipped with the *loop product*, $\bullet$, an associative product of degree $-d$, where d is the dimension of M,

$$\bullet : H_i(LM) \otimes H_j(LM) \to H_{i+j-d}(LM).$$

* Fax: +33 1 69 33 49 63.

E-mail address: habbaspour@math.polytechnique.fr.

0040-9383/$ - see front matter
doi:10.1016/j.top.2005.04.003

We give an informal description of the loop product: given two homology classes in $H_*(LM)$, choose a chain representative for each one of them so that the corresponding sets of marked points intersect transversally. By concatenating the corresponding loops of each chain at the intersection points one obtains a new chain in *LM*. We refer the reader to [6] for a homotopy theoretic definition of the loop product.

Theorem (*Chas–Sullivan [5]*). *For a connected oriented smooth manifold M, $(H_{*-d}(LM), \bullet)$ is a graded commutative algebra.*

Let $p : H_*(LM) \to H_*(M)$ be the map induced by $f \mapsto f(0)$ and $i : H_*(M) \to H_*(LM)$ be the map induced by the inclusion of the constant loops. Observe that

$$p \circ i = id_{H_*(M)},$$

hence there is a canonical decomposition $H_*(LM) = i(H_*(M)) \oplus A_M$ where $A_M = \mathrm{Ker}\, p$. The projection on A_M is denoted by p_{A_M} and $p_{A_M} = id_{H_*(LM)} - p$. From now on we identify $H_*(M)$ with its image under i.

Proposition 1.1 (*see Chas and Sullivan [5]*). *$i : H_*(M) \to H_*(LM)$ is a map of graded algebras where the multiplication on $H_*(M)$ is the intersection product denoted by $\wedge$. Therefore $(H_*(LM), \bullet)$ is an extension of $(H_*(M), \wedge)$.*

If *M* is a closed manifold then the algebra $(H_*(LM), \bullet)$ has a *unit*, μ_M, which is the image of the fundamental class of *M* under *i*.

Definition 1.2. We shall say *M* has *nontrivial extended loop products* if the restriction of the loop product to A_M is nontrivial. Otherwise we say that *M* has *trivial extended loop products*.

Definition 1.3. A closed oriented 3-manifold *M* is said to be *algebraically hyperbolic* if it is a $K(\pi, 1)$ (aspherical) and its fundamental group has no rank 2 abelian subgroup.

Note that every finite cover of an algebraically hyperbolic manifold is also algebraically hyperbolic.
The following is the main theorem of this paper [1]:

Theorem 1.4. *Let M be an oriented closed 3-manifold. If M is not algebraically hyperbolic then M or a double cover of M has nontrivial extended loop products.*

If M is algebraically hyperbolic then M and all its finite covers have trivial extended loop products.

Remark 1.5. Although the main result as stated above concerns the closed 3-manifolds, throughout this paper we identify several classes of 3-manifolds with boundary or which have nontrivial extended loop products (see Sections 3, 5 and 6.1).

Remark 1.6. As we shall see many of our arguments can be generalized to higher-dimensional manifolds. We have pointed them out with some remarks in Sections 3, 5 and 6. Also we invite the reader to see Example 1.8.

Notation: The based loop space of M is denoted ΩM and $\hat{\pi}_1(M)$ is the set of conjugacy classes of $\pi_1(M)$. For $\alpha \in \pi_1(M)$, C_α is the centralizer of α in $\pi_1(M)$ and $[\alpha]$ is its conjugacy class. For a conjugacy class $[\alpha]$, $(LM)_{[\alpha]}$ denotes the corresponding connected component of LM.

Lemma 1.7. *For $\alpha \in \pi_1(M)$ and $f \in LM$ a loop representing α, then there exists a short exact sequence*

$$\pi_2(M) \cong \pi_1((\Omega M)) \longrightarrow \pi_1((LM)_{[\alpha]}, f) \longrightarrow C_\alpha \longrightarrow 0. \tag{1.1}$$

Example 1.8 (*see Chas and Sullivan [5]*). Let M be a closed oriented hyperbolic manifold of dimension $d \geqslant 3$. Since M is aspherical, it follows from the long exact sequence associated with the fibration $\Omega M \hookrightarrow LM \xrightarrow{p} M$ that each connected component of LM is also aspherical. In order to find the homotopy type of each component $(LM)_{[\alpha]}$, one has to compute the fundamental group of each component. As $\pi_2(M) = 0$, by Lemma 1.7, each component $(LM)_{[\alpha]}$ is a $K(C_\alpha, 1)$. If $\alpha \neq 1 \in \pi_1(M)$ then $C_\alpha \simeq \mathbb{Z}$ as M is a closed hyperbolic manifold. Hence $(LM)_{[\alpha]}$ is a $K(\mathbb{Z}, 1)$ and

$$A_M \simeq H_*\left(\coprod_{[\alpha]\neq[1]\in\hat{\pi}_1(M)} K(\mathbb{Z}, 1)\right).$$

Therefore the restriction of the loop product to A_M is identically zero as A_M is concentrated at degree at most 1.

2. 3-Manifolds with finite fundamental group

In this section, we show that closed 3-manifolds with finite fundamental group have nontrivial extended loop products.

The following construction is valid for any oriented manifold M. Let p be the base point of M of dimension d. Consider the 0-chain in LM that consists of the constant loop at p. It represents a homology class $\varepsilon \in H_0(LM)$. Given a chain $c \in LM$ whose set of marked points is transversal to p, then the loops in c whose marked points are p, form a chain in the based loop space ΩM. This induces a map of degree $-d$

$$\cap : H_*(LM) \to H_{*-d}(\Omega M),$$

where d is the dimension of M. On the other hand $H_*(\Omega M)$ is equipped with the Pontrjagin product.

Lemma 2.1 (*see Chas and Sullivan [5]*). $\cap : (H_*(LM), \bullet) \to (H_{*-d}(\Omega M), \times)$ *is an algebra map where $\times$ is the Pontrjagin product. In particular if M is a Lie group then $\cap$ is surjective.*

Proposition 2.2. *S^3 has nontrivial extended loop products.*

Proof. As S^3 is a Lie group, by Lemma 2.1 $\cap$ is surjective. $(H_*(\Omega S^3), \times)$ is a polynomial algebra with one generator of degree 2. Let $x_1 \in H_p(\Omega S^3)$ and $x_2 \in H_q(\Omega S^3)$ be two nonzero elements such that $p, q > 3$. Let $a_i \in \cap^{-1}(x_i)$ for $i = 1, 2$. Since $H_i(S^3) = 0$ for $i > 3$ we have $a_i \in A_{S^3}$. By Lemma 2.1, $\cap(a_1 \bullet a_2) = \cap(a_1) \times \cap(a_2) = x_1 \times x_2 \neq 0$, therefore $a_1 \bullet a_2 \neq 0$. $\square$

For an arbitrary homotopy 3-sphere we can obtain some examples of nontrivial loop product by transferring the examples found for 3-sphere in Proposition 2.2, using a suitable homotopy equivalence. The following two lemmata describe and prove the existence of such a homotopy equivalence.

Lemma 2.3. *Let M and N be two closed oriented d-manifolds and $p \in N$. For $f : M \to N$ which is transversal to p and $f^{-1}(p)$ consists of only one point, the following diagram commutes*:

$$\begin{array}{ccc} H_*(LM) & \xrightarrow{\cap} & H_*(\Omega M) \\ {\scriptstyle f_L}\big\downarrow & & \big\downarrow{\scriptstyle f_\Omega} \\ H_*(LN) & \xrightarrow{\cap} & H_*(\Omega N) \end{array} \tag{2.1}$$

f_Ω and f_L are the induced maps by f on the homologies of the based loop space and the free loop space.

Proof. We choose p and $f^{-1}(p)$, respectively, as the base point of N and M. We show that (2.1) is commutative at the chain level. Let $k : \Delta^m \to LM$ be a m-simplex in LM. So for each $x \in \Delta^m$

$$k(x) : \mathbb{S}^1 \to M.$$

If $k_0 : \Delta^m \to M$,

$$k_0(x) = k(x)(0),$$

is transversal to $f^{-1}(p)$ then consider the set $\Lambda = \{x \in \Delta^m | k(x)(0) = f^{-1}(p)\}$.

$k|_\Lambda$ is a chain in $\Omega_{f^{-1}(p)}M$ and $\cap(k) = k|_\Lambda$. After composing with f, $f \circ k|_\Lambda$ is a chain in $\Omega_p N$ and

$$(f_\Omega \circ \cap)(k) = f \circ k|_\Lambda.$$

On the other hand $f_L(k) = f \circ k$ is a chain in LN. Since k_0 is transversal to $f^{-1}(p)$ and f is transversal to p thus $f \circ k_0$ is transversal to p. Since $f^{-1}(p)$ consists of only one point in M then

$$\{x \in \Delta^m | ((f \circ k)(x))(0) = p\} = \{x \in \Delta^m | k(x)(0) = f^{-1}(p)\} = \Lambda,$$

therefore,

$$(\cap \circ f_L)(k) = f \circ k|_\Lambda = (f_\Omega \circ \cap)(k). \qquad \square$$

Lemma 2.4. *Let $p_1, p_2, \ldots, p_r$, be r distinct points in a homotopy 3-sphere M. Then there is a homotopy equivalence $f : S^3 \to M$ which is transversal to all p_i's and $f^{-1}(p_i)$ consists of only one point for each i.*

Proof. Consider a closed ball $D \subset S^3$. Let f be a homeomorphism from D to a closed ball in $D' \subset M$ where $p_i \in D'$ for all i.

$f|_{\partial D} : \partial D \to \partial D'$ can be extended to $f : S^3 \backslash Int\, D' \to M \backslash Int\, D$ as all the obstructions which are cohomology classes in

$$H^{q+1}(S^3 \backslash Int\, D', \partial D', \pi_q(M \backslash Int\, D))$$

vanish since $M \backslash Int\, D'$ is contractible.[1]

Therefore, we have a map $f : S^3 \to M$ which sends D to D' homeomorphically and sends the complement of D to the complement of D' and in particular it is transversal to p_i and $f^{-1}(p_i)$ consists of only one point in S^3. To see that f is a homotopy equivalence it is enough to observe that it has degree one. □

Proposition 2.5. *A closed simply connected* 3*-manifold M has nontrivial extended loop products*.

Proof. Let $p \in M$ and $f : S^3 \to M$ be the homotopy equivalence provided by Lemma 2.4 for $r = 1$.

f induces the homotopy equivalences $f_\Omega : \Omega S^3 \to \Omega M$ and $f_L : LS^3 \to LM$. By Lemma 2.3 the following diagram is commutative:

$$\begin{array}{ccc} H_*(LS^3) & \xrightarrow{\cap_{S^3}} & H_*(\Omega_{f^{-1}(p)} S^3) \\ \Big\downarrow f_L & & \Big\downarrow f_\Omega \\ H_*(LM) & \xrightarrow{\cap_M} & H_*(\Omega_p M) \end{array} \tag{2.2}$$

Since f_Ω and f_L are isomorphisms and $\cap_{S^3}$ is surjective, therefore $\cap_M$ is also surjective. Let $x_1 \neq 0 \in H_m(\Omega M)$[2] and $x_2 \neq 0 \in H_n(\Omega M)$ where $n, m > 3$ and $a_i \in \cap_M^{-1}(x_i)$, $i = 1, 2$. Then, $a_i \in A_M$ since $H_k(M) = 0$ for all $k > 3$. By Lemma 2.1,

$$\cap(a_1 \bullet a_2) = \cap(a_1) \times \cap(a_2) = x_1 \times x_2 \neq 0,$$

hence $a_1 \bullet a_2 \neq 0$. □

Now we construct some examples of nontrivial loop product for a 3-manifold with finite fundamental group. The homology classes are obtained by pushing forward the classes introduced in Proposition 2.5 for the universal cover.

Proposition 2.6. *If M is a closed oriented* 3*-manifold with finite fundamental group then M has nontrivial extended loop products*.

Proof. Let $\tilde{M}$ be the universal cover of M and q be the covering map and $r = degree(q) = |\pi_1(M)| < \infty$. Choose $p \in M$ and let $\{p_1, p_2, \ldots, p_r\} \in q^{-1}(p)$ where p_i's are distinct.

Since $\tilde{M}$ is homotopy equivalent to S^3, by Lemma 2.4 there is a homotopy equivalence $f : S^3 \to \tilde{M}$ which is transversal at each p_i and $f^{-1}(p_i)$ consists of only one point in S^3 in M. Let $m_i = f^{-1}(p_i)$, $i = 1, 2, \ldots, r$. Hence $(q \circ f)^{-1}(p) = \{m_1, m_2, \ldots, m_r\}$ and $q \circ f$ is transversal to p. We choose m_1 as the base point of S^3.

[1] This can be proved using Mayer–Vietrois exact sequence.

[2] $H_*(\Omega M) \cong H_*(\Omega S^3) \cong \mathbb{Z}[x]$ and degree $x = 2$.

The covering map $q : \tilde{M} \to M$ composed by f induces the maps

$$(q \circ f)_L : H_*(LS^3) \to H_*(LM)$$

and

$$(q \circ f)_\Omega : H_*(\Omega_{m_1} S^3) \to H_*(\Omega_p M),$$

where the latter is an isomorphism.

We claim that

$$\begin{array}{ccc} H_*(LS^3) & \xrightarrow{r.\cap_{S^3}} & H_*(\Omega_{m_1} S^3) \\ \Big\downarrow{\scriptstyle (f\circ q)_L} & & \Big\downarrow{\scriptstyle (f\circ q)_\Omega} \\ H_*(LM) & \xrightarrow{\cap_M} & H_*(\Omega_p M) \end{array} \tag{2.3}$$

is commutative. To prove this, note that S^3 has a group structure hence we have the homeomorphism $LS^3 \simeq S^3 \times \Omega S^3$, and

$$H_*(LS^3) \simeq H_*(S^3) \otimes H_*(\Omega S^3). \tag{2.4}$$

There are two types of homology classes in $H_*(LS^3)$ and we verify the commutativity for the both cases:

(1) The classes that correspond to homology classes in $H_0(S^3) \otimes H_*(\Omega S^3)$ under isomorphism (2.4). These classes can be represented by cycles in LS^3 whose sets of marked points consist of a single point in S^3. Therefore they are mapped to 0 by $\cap_{S^3}$, as we can choose a chain representative whose set of marked point, consisting of only one points, is different from all m_i's. Therefore they are mapped to 0 by $(f \circ q)_\Omega \circ (r.\cap_{S^3})$.

On the other hand their images under $(f \circ q)_L$ have the set of marked point consisting of only one point which is different from p. Thus, they are mapped to zero under $\cap_M \circ (f \circ q)_L$ and the diagram commutes.

(2) The classes that by isomorphism (2.4) correspond to the homology classes in $H_3(S^3) \otimes H_*(\Omega S^3)$. Such an element $\theta \in H_*(LS^3)$ corresponds to $\mu \otimes b \in H_3(S^3) \otimes H_*(\Omega S^3)$ for some $b \in H_*(\Omega S^3)$ and μ is the fundamental class of S^3. Then

$$\cap_{S^3}(\theta) = b$$

so

$$(f \circ q)_\Omega \circ (r\cap_{S^3}\theta) = r(f \circ q)_\Omega(b)$$

and this is exactly $\cap_M \circ (f \circ q)_\Omega(b)$. Note that under $f \circ q$, r copies of b which are based at each one of m_i's, come together. Therefore $\cap_M \circ (f \circ q)_L(\theta) = r(f \circ q)_\Omega(b)$.

To finish the proof, consider $a_1, a_2 \in H_*(LS^3)$ as constructed in the proof of Proposition 2.2. We have $(q \circ f)_L(a_i) \in A_M$, $i = 1, 2$ and from the commutativity of (2.3) and the fact that $(q \circ f)_\Omega$ is an isomorphism it follows that

$$(q \circ f)_L(a_1) \bullet (q \circ f)_L(a_2) \neq 0. \qquad \square$$

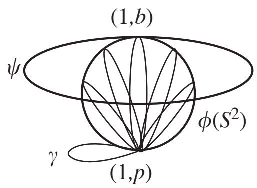

Fig. 1. $S^1 \times S^2$.

3. 3-Manifolds with non-separating 2-sphere or 2-torus

In this section, we prove the 3-manifolds with a non-separating sphere or 2-torus have nontrivial extended loop product. We recall that the action of the unit circle on LM induces a map of degree 1, $\Delta : H_*(LM) \to H_{*+1}(LM)$.

Proposition 3.1. *$S^1 \times S^2$ has nontrivial extended loop products.*

Proof. Let b and p be two distinct points in S^2. We choose $(1, p)$ as the base point of $S^1 \times S^2$. The map $x \mapsto (x, p)$, $x \in \mathbb{S}^1$, gives rise to an element η of $\pi_1(S^1 \times S^2)$.

Consider the map $\psi : \mathbb{S}^1 \to S^1 \times S^2$ defined by $\psi(x) = (x, b)$. Note that ψ as a loop with the marked point $(1, b)$, represents a homology class $\Psi \in H_0((L(S^1 \times S^2))_{[\eta]})$.

Let $\phi : S^2 \to S^1 \times S^2$ be the map defined by $\phi(y) = (1, y)$. The images of ψ and ϕ intersect exactly at $(1, b)$. We write $\phi(S^2)$ as a union of circles, any two of them having only the point $(1, p)$ in common. This gives rise to a one dimensional family of loops in $S^1 \times S^2$ (see Fig. 1). Note that the free homotopy type of the loops of this one-dimensional family is the one of the trivial loop. One can compose the loops of this family with a fixed loop whose marked point is $(1, p)$ and modify their free homotopy type. Suppose that we have done this modification with a fixed loop which does not meet ψ and represents a nontrivial element $\mu \in \pi_1(S^1 \times S^2)$ where $\mu \neq \eta$. This new one-dimensional family of loops represents a homology class $\Phi \in H_1((L(S^1 \times S^2))_{[\mu]})$.

We prove that $p_{A_{S^1\times S^2}}(\Delta\Psi) \bullet p_{A_{S^1\times S^2}}(\Delta\Phi) \neq 0$ which implies that $S^1 \times S^2$ has nontrivial loop products. Since $p_{A_{S^1\times S^2}}(\Delta\Psi) \bullet p_{A_{S^1\times S^2}}(\Delta\Phi)$ belongs to $H_0(L(S^1 \times S^2))$, it can be expressed as a sum of conjugacy classes with $+1$ or -1 as the coefficients. Indeed it equals $\pm[\eta\mu] \pm [\eta] \pm [\mu] \pm [1]$. Since 1, η and μ are distinct therefore three terms out of four are distinct and hence there cannot be a complete cancellation. □

Corollary 3.2. *An oriented 3-manifold with a non-separating 2-sphere has nontrivial extended loop products.*

Proof. The proof is similar to the one of Proposition 3.1. The only property we used in the proof of Proposition 3.1 was that there was a non-separating two sided 2-sphere.[3] □

[3] Two sided means that the normal bundle is trivial (see [9]).

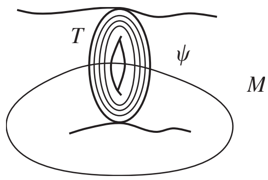

Fig. 2. Nonseparating T.

Remark 3.3. Corollary 3.2 can be generalized to higher-dimensional manifolds with a nonseparating codimension one sphere. The proof is a direct generalization of the one of Corollary 3.2.

Proposition 3.4. *An oriented* 3*-manifold M with a non-separating two sided incompressible*[4] *torus has nontrivial extended loop products.*

Proof. Let $\phi : S^1 \times S^1 \to T \subset M$ be a homeomorphism. We set $\phi(1,1)$ as the base point of M. Consider the one-dimensional family of loops ϕ_t defined by $\phi_t(s) = \phi(t,s)$ (longitudes of T in Fig. 2). This 1-family of loops represents a homology class Φ in $H_1((LM)_{[h]})$, where h is the element of $\pi_1(M)$ represented by ϕ_1. Now consider a closed simple curve $\psi : \mathbb{S}^1 \to M$ which meets T transversally at exactly one point $\phi(1,1)$. Note that ψ represents an element $g \in \pi_1(M)$ and also gives rise to a homology class $\Psi \in H_0((LM)_{[g]})$.

We show that $p_{A_M}(\Delta\Psi) \bullet p_{A_M}(\Delta\Phi) \neq 0 \in H_0(LM)$. Similar to $S^1 \times S^2$ we have $p_{A_M}(\Delta\Psi) \bullet p_{A_M}(\Delta\Phi) = \pm[gh] \pm [h] \pm [g] \pm [1]$.

To prove the claim, it is sufficient to show that $[1]$, $[h]$ and $[g]$ are distinct. Since T is π_1-injective then $[h] \neq [1]$. Note that the loop ψ intersects T exactly at one point hence the intersection product of the two homology classes (in M) represented by ψ and T are nontrivial and in particular the homology classes are nontrivial, therefore $[g] \neq [1]$. Similarly $[g] \neq [h]$ because the intersection of the homology classes represented by T and ϕ_1 is zero as T has a trivial normal bundle. □

Remark 3.5. A similar argument shows that a n-dimensional manifold with a non-separating two sided π_1-injective $(n-1)$-dimensional torus has nontrivial extended loop product.

4. Closed Seifert manifolds

In this section we consider the closed Seifert manifolds. We refer the reader to [8,9] for an introduction to Seifert manifolds.

Remark 4.1. An orientable Seifert manifold M may not be oriented as a fibration but it always has a double cover which is Seifert and is orientable as a fibration or in other words its base surface is orientable. This double cover $\tilde{M}$ can be described as following:

$$\tilde{M} = \{(m, o) | m \in M \ \& \ o \text{ an orientation for the fiber passing through } m\}.$$

[4] π_1-injective (see [9]).

The covering map $\tilde{M} \xrightarrow{q} M$ is $q(m,o)=m$. The Seifert fibration of $\tilde{M}$ is obtained by pulling back the one of M. If M has a boundary then each boundary component, a torus, gives rise to 2 boundary components, both a torus. If M has g cross caps then $\tilde{M}$ has genus $g-1$.

Proposition 4.2. *If M is a closed oriented Seifert 3-manifold then M or a double cover of M has nontrivial extended loop products.*

Proof. By Remark 4.1 we may assume that the base surface of M is orientable. Also we assume $\pi_1(M)\ncong\mathbb{Z}_2$, since Proposition 2.6 deals with this case.

Let h be the generator of $\pi_1(M)$ corresponding to the normal fiber according (5.1). There is a natural 3-homology class Θ_M in $H_3((LM)_{[h]})$ that has a representative whose set of marked points is exactly M. The loop associated with a point in M which is on a normal fiber, is the fiber passing through the point. The loop passing through a point on a singular fiber as a map is a multiple of the singular fiber.

We have $p_{A_M}(\Theta_M)=\Theta_M-p_M(\Theta_M)=\Theta_M-\mu_M$ where μ_M is the unit of $H_*(LM)$.

Consider a simple curve l representing an element $\alpha\in\pi_1(M)$, $\alpha\neq h, 1$. This is possible since we are assuming $\pi_1(M)\neq\mathbb{Z}_2$. Note that $[h]\neq[\alpha]$ as h is a central element. Consider the 0-homology class $\hat{\alpha}$ in $H_0((LM)_{[\alpha]})$ represented by l. We claim that $p_{A_M}(\Theta_M)\bullet p_{A_M}(\hat{\alpha})\neq 0$.

In expanding $p_{A_M}(\Theta_M)\bullet p_{A_M}(\hat{\alpha})\in H_0(LM)$ we obtain four terms, $\pm[\alpha.h]$, $\pm[h]$, $\pm[\alpha]$ and $\pm[1]$. Note that $[h]\neq[\alpha]$ since $h\neq\alpha$ and h is in the center. As there are at least three different conjugacy classes, any linear combination of them with coefficient ± 1 is nonzero. Therefore M has nontrivial extended loop products. □

5. Seifert manifolds with boundary

By torus decomposition [10,11], Seifert manifolds with incompressible boundary are among the building blocks of 3-manifolds. In this section, we provide various examples of Seifert manifold with boundary which have nontrivial extended loop product. We introduce the following notation as they will crucial from now on.

Notation: Let M be an oriented Seifert manifold with p exceptional fibers and b boundary components. If S, the base surface of M, is orientable, then it has genus $g\geqslant 0$ otherwise S is nonorientable and g is the number of cross caps in S. The fundamental group of M has the following presentation (see [8,9]):

(1) If S is orientable and of genus g,

$$\begin{aligned}\pi_1(M)=\Big\langle & a_1,b_1,\ldots,a_g,b_g,c_1,\ldots,c_p,d_1,d_2\ldots d_b,h \,|\, a_iha_i^{-1}=h,\quad b_ihb_i^{-1}=h,\\ & c_ihc_i^{-1}=h,\quad d_ihd_i^{-1}=h,\\ & c_i^{\alpha_i}h^{\beta_i}=1,\prod_{i=1}^{g}[a_i,b_i]\prod_{i=1}^{p}c_i\prod_{i=1}^{b}d_i=1\Big\rangle,\end{aligned}\tag{5.1}$$

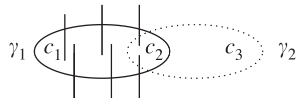

Fig. 3. γ_1, γ_2 and a trivialized fibration over γ_1.

(2) If S is nonorientable and has $g>0$ cross caps,

$$\pi_1(M)=\Big\langle a_1,\ldots,a_g,c_1,\ldots,c_p,d_1,d_2\ldots d_b,h|a_iha_i^{-1}=h^{-1},\\ c_ihc_i^{-1}=h^{\delta_i},\quad d_ihd_i^{-1}=h,\\ c_i^{\alpha_i}h^{\beta_i}=1\prod_{i=1}^{g}a_i^2\prod_{i=1}^{p}c_i\prod_{i=1}^{b}d_i=1\Big\rangle, \tag{5.2}$$

where $0<\beta_i<\alpha_i$ are integers and $\delta_i=\pm 1$ and h corresponds to a normal fiber. The d_i's correspond to the boundary components. Each c_i can be represented by a loop on the base surface going around the singular fiber once, α_i is the multiplicity of the singular fiber corresponding to c_i and we say that it has type (α_i, β_i). Note that $\langle h\rangle$ is a normal subgroup of $\pi_1(M)$ and it is central if S is orientable.

Throughout this section M is a compact oriented Seifert manifold with orientable base surface unless otherwise stated (see Corollary 5.6), $b\geqslant 1$ and p' denotes the number of the singular fibers of multiplicity greater than 2. We assume[5] that $b+p\geqslant 3$.

We present two main ideas for proving that a Seifert manifold has nontrivial extended loop product, *Two curves argument* and *Chas' figure eight argument*.

The following table indicates the cases where we can apply these arguments.

	Argument
$p+b\geqslant 4$	Two curves argument
$p'+b\geqslant 3$	Chas' figure eight argument

5.1. $p+b\geqslant 3$ (Two curves argument)

Proposition 5.1. *Let M be a compact oriented Seifert manifold with $p>2$ and $b\geqslant 1$. Then M has nontrivial extended loop products.*

Proof. Let c_1, c_2 and c_3 be the generators of $\pi_1(M)$ corresponding to three singular fibers. Consider two simple curves γ_1 and γ_2, on the base surface, away from singular fibers and representing respectively the free homotopy class $[c_1c_2]$ and $[c_2c_3]$. Moreover, γ_2 can be chosen such that it has exactly 2 intersection points with γ_1.

Since γ_1 is away from the singular point, the fibration can be trivialized over γ_1 (see Fig. 3). Therefore, we obtain a map $f: S^1\times S^1\to M$, where $f(0,\cdot)=\gamma_1$.

[5] This assumption will be justified in Section 10.

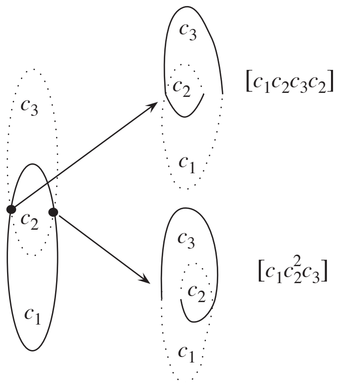

Fig. 4. Two curves argument.

Note that f gives rise to a homology class $\Gamma_1 \in H_1((LM)_{[c_1c_2]})$, by declaring $f(t,0)$, $t \in S^1$, as the set of marked points and $f(t,s)$, $s \in \mathbb{S}^1$, as the loop passing through $f(t,0)$. On the other hand the loop γ_2 gives rise to a homology class $\Gamma_2 \in H_0((LM)_{[c_2c_3]})$.

We show that $p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2) \neq 0 \in H_0(LM)$ which proves the proposition.

The sets of marked points of $\Delta\Gamma_1 \in H_2((LM)_{[c_1c_2]})$ and $\Delta\Gamma_2 \in H_1((LM)_{[c_2c_3]})$ intersect at two points. Each point contributes four terms in the expansion of $p_{A_M}(\Delta\Gamma_1)\bullet p_{A_M}(\Delta\Gamma_2)$. All in all eight terms emerge while calculating $p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2)$. Each is represented by a conjugacy class in $\hat{\pi}_1(M)$ and a $\pm$ sign. The conjugacy classes are

$$[c_1c_2c_3c_2], [c_1c_2], [c_2c_3], [1] \quad \text{and} \quad [c_1c_2^2c_3], [c_1c_2], [c_2c_3], [1].$$

Thus in calculating $p_{A_M}(\Delta\Gamma_1)\bullet p_{A_M}(\Delta\Gamma_2) \bmod 2$, only two terms remain, namely $[c_1c_2^2c_3]$ and $[c_1c_2c_3c_2]$ (see Fig. 4). To prove the claim it is sufficient to show that these two conjugacy classes are different. For this, consider the group

$$H = \langle c_1, c_2, c_3, d | c_1c_2c_3d_1 = 1, c_i^{\alpha_i} = 1, 1 \leqslant i \leqslant 3\rangle,$$

where d_1 is a generator corresponding to a boundary component. Indeed H is the free product of the groups $\mathbb{Z}_{\alpha_1} * \mathbb{Z}_{\alpha_2} * \mathbb{Z}_{\alpha_3}$.

Consider the homomorphism $\phi : \pi_1(M) \to H$ which is the identity on c_1, c_2, c_3, d_1 and sends the other generators to the trivial element of H. By (5.1) ϕ is well defined.

Note that the images of $c_1c_2^2c_3$ and $c_1c_2c_3c_2$ under the ϕ are not conjugate in H as $c_1c_2^2c_3$ and $c_1c_2c_3c_2$ are cyclically different reduced words. Therefore $[c_1c_2^2c_3]$ and $[c_1c_2c_3c_2]$ are two different conjugacy classes of $\pi_1(M)$. $\square$

Remark 5.2. In the proof of the Proposition 5.1 one can replace a singular fiber by a boundary component, in other words a boundary component works as well as a singular fiber of multiplicity zero. To be more explicit, the curves γ_1 and γ_2 can go around a boundary component of the base surface instead of the singular fiber. In the free product H, one of the finite cyclic groups is replaced by a infinite cyclic group. The rest of the proof remains the same.

Corollary 5.3. *Let M be a compact oriented Seifert manifold with p singular fibers and b boundary components. If $p+b>3$ then M has nontrivial extended loop products.*

Proof. If $p \geqslant 3$ then this is just Proposition 5.1. If $p<3$, then by Remark 5.2, in the proof of Proposition 5.1 one replaces the generator of $\pi_1(M)$ coming from a singular fiber with the one corresponding to a boundary component. $\square$

5.2. $p'+b \geqslant 3$ (Chas' figure eight argument)

The main idea of the proof of this case is due to Moira Chas. She found the idea while reformulating a conjecture of Turaev (see [4]) on Lie bi-algebras of surfaces, characterizing non self-intersecting closed curves.

Proposition 5.4. *Let M be a compact oriented Seifert manifold with $p=2$ singular fibers and $b=1$ boundary component. Suppose that none of its singular fibers has multiplicity 2. Then M has nontrivial extended loop products.*

Proof. Let c_1 and c_2 be generators of $\pi_1(M)$ corresponding to the singular fibers. Let γ_1 be a smooth curve on the base surface, with one self-intersection point, away from singular fibers and representing the free homotopy class $[c_1c_2^{-1}]$. Similarly, consider a curve γ_2 on the base surface, away from the singular fibers, with one intersection point and representing the free homotopy class $[c_1^{-1}c_2]$. Moreover, γ_2 can be chosen such that it has exactly 2 intersection points with γ_1.

The fibration can be trivialized over the γ_1 since it is away from the singular point. Therefore, we obtain a map $f: S^1 \times S^1 \to M$, where $f(0,\cdot)=\gamma_1$. This map gives rise to a homology class $\Gamma_1 \in H_1(LM)$, by declaring $f(t,0), t \in S^1$, as the set of marked points. The loop passing through $f(t,0)$ is $f(t,s), s \in \mathbb{S}^1$. All the loops of this family have the free homotopy type $[c_1c_2^{-1}]$. The loop γ_2 gives rise to a homology class $\Gamma_2 \in H_0(LM)$. We claim that

$$p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2) \neq 0 \in H_0(LM)$$

which proves the proposition.

The sets of marked points of $\Delta\Gamma_1 \in H_2(LM)$ and $\Delta\Gamma_2 \in H_1(LM)$ intersect exactly at two points. Each point contributes four terms in the expansion of $p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2)$. All in all eight terms emerge in the expansion of $p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2)$. Each one of them is represented by a conjugacy class in $\hat{\pi}_1(M)$ and a $\pm$ sign. These conjugacy classes are

$$[c_1c_2c_1^{-1}c_2^{-1}], [c_1^{-1}c_2], [c_1c_2^{-1}], [1] \quad \text{and} \quad [c_1c_2^{-1}c_1^{-1}c_2], [c_1^{-1}c_2], [c_1c_2^{-1}], [1].$$

Calculating $p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2) \bmod 2$, only two terms remain, namely $[c_1c_2c_1^{-1}c_2^{-1}]$ and $[c_1c_2^{-1}c_1^{-1}c_2]$ (see Fig. 5).

To prove the claim it is sufficient to show that these two conjugacy classes are different. For this, consider the group

$$H = \langle c_1, c_2, d_1 | c_1c_2d_1 = 1, c_i^{\alpha_i} = 1, i = 1, 2\rangle \simeq \mathbb{Z}_{\alpha_1} * \mathbb{Z}_{\alpha_2},$$

where d_1 corresponds to one of the boundary components.

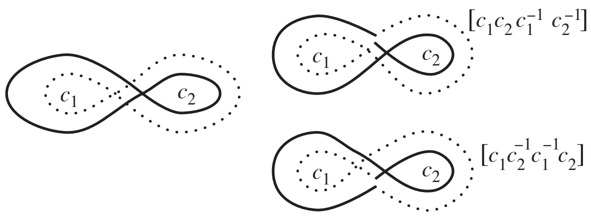

Fig. 5. Figure eight argument.

Consider the homomorphism $\phi : \pi_1(M) \to H$ which is the identity on c_1, c_2, d_1 and sends the other generators of $\pi_1(M)$ to the trivial element of *H*. It follows from (5.1) that ϕ is well defined.

Since $\alpha_i \neq 2$, $i = 1, 2$, then $c_i \neq c_i^{-1}$ in *H*. The images of $c_1c_2c_1^{-1}c_2^{-1}$ and $c_1c_2^{-1}c_1^{-1}c_2$ under ϕ are not conjugate as they are cyclically different. Therefore $c_1c_2c_1^{-1}c_2^{-1}$ and $c_1c_2^{-1}c_1^{-1}c_2$ are not conjugate in $\pi_1(M)$ either. □

Corollary 5.5. *Let M be a compact oriented Seifert manifold with $p' + b \geqslant 3$. Then M has nontrivial extended loop products.*

Proof. If $p' \geqslant 2$ then this is just Proposition 5.4. Otherwise, by Remark 5.2, a boundary component works like a singular fiber with multiplicity zero. So in the proof of Proposition 5.4 we should replace the generators of $\pi_1(M)$ due to the singular fibers with the generators corresponding to two boundary components. □

Corollary 5.6. *Let M be a compact oriented Seifert manifold with a non-orientable base surface and $b \geqslant 2$ boundary components, then M has nontrivial extended loop products.*

Proof. The proof is similar to the one for Proposition 5.4 but with a slight modification. Note that a boundary component serves our purposes as well as a singular fiber. Moreover the restriction of the fibration to the boundary components is oriented. Thus it can be trivialized over a simple curve representing the free homotopy type $[d_1d_2^{-1}]$ where d_1 and d_2 are generators of $\pi_1(M)$ contributed by two boundary components. The rest of the proof is similar to the one of Proposition 5.4. Calculating mod 2 we get two terms $[d_1d_2d_1^{-1}d_2^{-1}]$ and $[d_1d_2^{-1}d_1^{-1}d_2]$. We must prove that these two conjugacy classes are different. If a_1 is a generator of $\pi_1(M)$ due to a cross cap then consider the group

$$H = \langle d_1, d_2, a_1 | d_1d_2a_1^2 = 1\rangle \simeq \langle d_1\rangle * \langle a_1\rangle \simeq \mathbb{Z} * \mathbb{Z}$$

and the homomorphism $\varphi : \pi_1(M) \to H$ which is the identity on d_1, d_2, a_1 and sends the other generators of $\pi_1(M)$ to the trivial element. It follows from (5.2) that φ is well defined.

The images of $d_1a_1^2d_1^{-1}a_1^{-2}$ and $d_1a_1^{-2}d_1^{-1}a_1^2$ in the free product *H* are not conjugate. Therefore $d_1d_2d_1^{-1}d_2^{-1}$ and $d_1d_2^{-1}d_1^{-1}d_2$ are not conjugate in $\pi_1(M)$. □

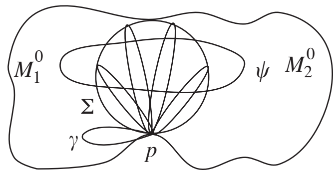

Fig. 6. $M = M_1 \# M_2$.

6. Connected sum of 3-manifolds

In this section we consider the connected sum of 3-manifolds. We shall show how one can obtain some examples of nontrivial loop products in this kind of manifold. For this part, the author has benefited from conversations with numerous colleagues [2].

Proposition 6.1. *Suppose that $M = M_1 \# M_2$ where M_i's are 3-manifolds of nontrivial fundamental group. Then M has nontrivial extended loop products.*

Proof. Let $\Sigma \subset M$ be the 2-sphere that separates the two components M_1^0 and M_2^0, where M_k^0, for $k \in \{1, 2\}$, is M_k with a ball removed. Just like Section 3.2, the 2-sphere Σ gives rise to a one-dimensional family of loops which have the same marked point $p \in M$. We set p to be the base point of M (Fig. 6). The loops in this one-dimensional family have the free homotopy type of the trivial loop. In order to modify their free homotopy type, one can compose the loops of this one-dimensional family with a fixed loop whose marked point is p. Suppose that we have done this modification using a fixed loop γ (Fig. 6) which represents a nontrivial element $h \in \pi_1(M)$. The new one-dimensional family of loops represents a homology class $\Phi \in H_1((LM)_{[h]})$. Now consider a simple smooth curve $\psi : \mathbb{S}^1 \to M$ which intersects Σ exactly at 2 points and has the free homotopy type $[x_1 x_2]$ where $x_i \neq 1 \in \pi_1(M_i)$, $i = 1, 2$ (Fig. 6). Note that $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$ and $x_1 x_2$ is regarded as an element of this free product. We choose ψ such that it does not intersect γ. As a loop, ψ represents a homology class $\Psi \in H_0((LM)_{[x_1 x_2]})$. We show that there exist some choices of x_1, x_2 and h such that $p_{A_M}(\Delta\Psi) \bullet p_{A_M}(\Delta\Phi) \neq 0 \in H_0(LM)$.

In expanding $p_{A_{S^1 \times S^2}}(\Delta\Psi) \bullet p_{A_{S^1 \times S^2}}(\Delta\Phi)$ we get eight terms. By passing to mod 2 only two terms remain, namely $[x_1 x_2 h]$ and $[x_2 x_1 h]$. Now we must show that there exist some choices of h such that these two conjugacy classes are different. Indeed $h = x_1 x_2$ is a convenient choice since

$$[x_1 x_2 h] = [x_1 x_2 x_1 x_2] \quad \text{and} \quad [x_2 x_1 h] = [x_2 x_1 x_1 x_2] = [x_1^2 x_2^2]$$

and the reduced words $x_1 x_2 x_1 x_2$ and $x_1^2 x_2^2$ are cyclically different. □

Remark 6.2. We shall point out that in Proposition 6.1 M is not required to be closed. Moreover, Proposition 6.1 and its proof can be generalized to manifolds of any dimension $d \geqslant 3$. In the proof, Σ would be replaced with a $(n-1)$-sphere and the homology classes Ψ and Φ, would be of degree 0 and $n-2$.

7. An injectivity lemma

Suppose that M is a closed oriented 3-manifold and $\mathscr{T}$ is a collection of incompressible separating tori in M. Let $M\backslash\mathscr{T} = M_1 \cup M_2 \cup \cdots \cup M_n$. We associate with $(M, \mathscr{T})$ a tree of groups (G, T) (see [17] and Appendix A.2 for the definitions).

The vertices of T are in a one-to-one correspondence with M_i's. Two vertices v_i and v_j are connected by an edge if the corresponding $\overline{M_i}$ and $\overline{M_j}$ are glued along a torus. With each vertex v_i we associate the fundamental group of $\pi_1(\overline{M_i})$ and with each edge we associate the fundamental group of the corresponding torus. As the tori are incompressible there are two monomorphisms from each edge group to the vertex groups of the corresponding vertices, namely if e is an edge (a torus in M) and v a vertex of e (some M_i s.t. $T \subset \partial\overline{M_i}$) then there is an homomorphism

$$f_e^v : G_e \to G_v$$

which is indeed the map induced by inclusion

$$i : \pi_1(T) \to \pi_1(\overline{M_i}).$$

By Van-Kampen theorem, $\pi_1(M)$ is the amalgamation of G_v's along G_e's (see Appendix A.2).

Combining all these with Lemma A.5 in Appendix A, we have,

Lemma 7.1 (*Injectivity lemma*). *Let M, $\mathscr{T}$ and M_i's be as above. Suppose that $[a]$ and $[b]$ are two distinct conjugacy classes in $\pi_1(\overline{M_i})$ such that a is not conjugate to the image of any element of the edge groups in $\pi_1(\overline{M_i})$. Then a and b represent distinct conjugacy classes of $\pi_1(M)$.*

The injectivity lemma will be very useful to show that certain examples of nontrivial loop products in a submanifold of M give rise to nontrivial loop products in M.

8. Gluing Seifert manifolds along tori

In this section we consider the gluing of the Seifert manifolds along tori. We prove that the result of the gluing Seifert manifolds along incompressible *separating* tori has nontrivial extended loop products except in certain cases for which there are double covers that have nontrivial extended loop products.

The idea of the proof is to apply either the *two curves argument* or the *figure eight argument* to a Seifert piece that fulfills the conditions as indicated by the table in Section 5. Then we show that the examples of nontrivial loop product in the piece give rise to examples of nontrivial loop product in the manifold, and for that we benefit from the injectivity lemma, Lemma 7.1.

When we cannot apply either the two curves argument or the figure eight argument, we consider an appropriate double cover of the manifold which either has a nonseparating torus or one of the two arguments can be applied to a Seifert piece of its torus decomposition [10,11].

Notation: Let M be a closed 3-manifold and $\mathscr{T} \neq \emptyset$ be a collection of tori in M and

$$M\backslash\mathscr{T} = M_1 \cup M_2 \cup \cdots \cup M_n,$$

where all the $\overline{M_i}$'s are oriented Seifert manifolds with incompressible boundary.

Let $b_i \geqslant 1$ and p_i denote, respectively, the number of boundary components and singular fibers of $\overline{M_i}$ and $A_i \subset \mathbb{Z}^+$ is the set of all the multiplicities of the singular fibers of $\overline{M_i}$. The base surface of M_i is denoted S_i and the genus of S_i is g_i if S_i is oriented otherwise g_i is the number of cross caps in S_i.

We assume that all tori in $\mathscr{T}$ are separating since in Section 3 we proved that 3-manifolds with nonseparating torus have nontrivial extended loop products. We assume $b_i + p_i \geqslant 3$ if S_i is orientable.

Proposition 8.1. *Let M be a closed 3-manifold described as above. Then M or a double cover of M has nontrivial extended loop products.*

Proof. We assume that $g_i = 0$ when S_i is orientable, as if $g_i > 0$ then there is a non-separating torus in M and this case has already been studied in Section 3. Then M belongs to one of the following categories:

(1) $b_i \geqslant 3$ and S_i is orientable for some i. In this case we use the figure eight argument. Note that if $b_i > 3$ we can also use the two curves argument.
(2) $b_i \geqslant 2$ and S_i is nonorientable for some i, for this case we also use the figure eight argument.
(3) $p_i + b_i > 3$ and S_i orientable for some i, we can apply the two curves argument.
(4) $b_i + p_i = 3$, $1 \leqslant b_i \leqslant 2$, S_i orientable and $2 \notin A_i$ for some i. In this case we use the figure eight argument.
(5) For all i, $b_i + p_i = 3$, $1 \leqslant b_i \leqslant 2$, S_i orientable and $2 \in A_i$ for all i. For this case we consider a double cover of M.
(6) $n = |\mathscr{T}|$, $n \geqslant 2$:
- $b_i = 2$, $p_i = 1$, $A_i = \{2\}$ and S_i is orientable for $i \neq 1, n$;
- $b_1 = b_n = 1$;
- S_1 is nonorientable;
- If S_n is orientable then $p_n = 1$ and $2 \in A_n$.

In this case also we will find a double cover with nontrivial extended loop products.

We verify the statement for each case.

(1) $b_i \geqslant 3$ and S_i orientable for some i: Suppose that $b_1 \geqslant 3$ and S_1 orientable. By Corollary 5.5 $\overline{M_1}$ has nontrivial extended loop products. Let d_1, d_2 and d_3 be the generators of $\pi_1(\overline{M_1})$ contributed by 3 boundary components. Consider the homology classes Γ_1 and $\Gamma_2 \in H_1(L\overline{M_1})$ constructed in the proof of Corollary 5.5, they can be regarded as the elements of $H_*(LM)$. We claim that

$$p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2) \neq 0 \in H_0(LM).$$

Mod 2 only two terms survive, $[d_1 d_2 d_1^{-1} d_2^{-1}]$ and $[d_1 d_2^{-1} d_1^{-1} d_2]$, which by the proof of Corollary 5.5 are two different conjugacy classes of $\pi_1(\overline{M_1})$. So if we show that $d_1 d_2 d_1^{-1} d_2^{-1}$ is not conjugate to any element of the fundamental group of one of the components of $\partial\overline{M_1}$, then by the injectivity lemma $[d_1 d_2 d_1^{-1} d_2^{-1}]$ and $[d_1 d_2^{-1} d_1^{-1} d_2]$ are different as the conjugacy classes of $\pi_1(M)$.

Suppose that

$$[d_1 d_2 d_1^{-1} d_2^{-1}] = [h^r d^s], \tag{8.1}$$

where h is the generator of $\pi_1(\overline{M_1})$ that corresponds to a normal fiber of M_1 and d is a generator of $\pi_1(\overline{M_1})$ due to a boundary component.

Consider the group H

$$H = \langle d_1, d_2, d_3 | d_1 d_2 d_3 = 1 \rangle \simeq \langle d_1 \rangle * \langle d_2 \rangle$$

and the homomorphism

$$\phi : \pi_1(\overline{M_1}) \to H$$

which is the identity on d_1, d_2, d_3 and trivial on all other generators of $\pi_1(\overline{M_1})$. It follows from (5.1) that ϕ is well defined.

After applying ϕ to (8.1) we get

$$[d_1 d_2 d_1^{-1} d_2^{-1}] = [1]$$

if $d \neq d_1, d_2, d_3$ otherwise

$$[d_1 d_2 d_1^{-1} d_2^{-1}] = [d^s].$$

The first case is impossible as $d_1 d_2 d_1^{-1} d_2^{-1}$ is a cyclically reduced word of length 4 in the free product $\langle d_1 \rangle * \langle d_2 \rangle$, thus it represents a nontrivial conjugacy class.

In the second case, If $d = d_1$ (or d_2) then

$$[d_1 d_2 d_1^{-1} d_2^{-1}] = [d_1^s],$$

which is not possible either as $d_1 d_2 d_1^{-1} d_2^{-1}$ is a cyclically reduced word of length 4 and d_1^s (or d_2^s) has length 1. If $d = d_3 = d_2^{-1} d_1^{-1}$ then we have

$$[d_1 d_2 d_1^{-1} d_2^{-1}] = [(d_1 d_2)^{-s}].$$

As the length of $d_1 d_2 d_1^{-1} d_2^{-1}$ is 4, we must have $s = \pm 2$. In either case the equality does not hold since $d_1 \neq d_1^{-1}$ and $d_2 \neq d_2^{-1}$ (H is a free group). Therefore $d_1 d_2 d_1^{-1} d_2^{-1}$ and $d_1 d_2^{-1} d_1^{-1} d_2$ are not conjugate in $\pi_1(M)$.

(2) $b_i \geqslant 2$ and S_i not orientable: Similar to the previous case we use the figure eight argument in M_i. Suppose that $b_1 \geqslant 2$ and S_1 is not orientable. By Corollary 5.6 $\overline{M_1}$ has nontrivial extended loop products. The homology classes Γ_1 and Γ_2 in $H_*(L\overline{M_1})$ constructed in the proof of Corollary 5.6 can be regarded as homology classes in $H_*(LM)$. We claim that

$$p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2) \neq 0 \in H_0(LM).$$

Mod 2 we get two terms, $[d_1 a_1^2 d_1^{-1} a_1^{-2}]$ and $[d_1 a_1^{-2} d_1^{-1} a_1^2]$ (see the proof of Corollary 5.6). We must prove that they are distinct as the conjugacy classes of $\pi_1(M)$. For that it is sufficient to prove that one of them has no boundary representative. We show that $[d_1 a_1^2 d_1^{-1} a_1^{-2}]$ has no boundary representative. Suppose we have

$$[d_1 a_1^2 d_1^{-1} a_1^{-2}] = [h^r d^s], \tag{8.2}$$

where d is a generator of $\pi_1(\overline{M_1})$ coming from a boundary component and h corresponds to a normal fiber. Consider the group

$$H = \langle d_1, d_2, a_1 | d_1 d_2 a_1^2 = 1 \rangle \simeq \langle d_1 \rangle * \langle a_1 \rangle$$

and the homomorphism $\varphi : \pi_1(\overline{M_1}) \to H$ which is the identity on d_1, d_2, a_1 and trivial on the other generators of $\pi_1(M_1)$. It follows from (5.2) that φ is well defined. Applying φ to (8.2) we have

$$[d_1 a_1^2 d_1^{-1} a_1^{-2}] = [1]$$

if $d \neq d_1, d_2$, otherwise

$$[d_1 a_1^2 d_1^{-1} a_1^{-2}] = [d^s].$$

The first case is impossible as $d_1 a_1^2 d_1^{-1} a_1^{-2}$ is a cyclically reduced word of length 4. If $d = d_1$ then

$$[d_1 a_1^2 d_1^{-1} a_1^{-2}] = [d_1^s]$$

which is again impossible in the free product $\langle d_1 \rangle * \langle a_1 \rangle$. If $d = d_2$ then

$$[d_1 a_1^2 d_1^{-1} a_1^{-2}] = [(a_1^2 d_1)^{-s}].$$

Once again we must have $s = \pm 2$ but this is not sufficient as $d_1 \neq d_1^{-1}$ and $a_1^2 \neq a_1^{-2}$.

(3) $p_i + b_i \geqslant 4$ and S_i orientable for some i: Since (1) includes the case $b_i \geqslant 3$ for some i, we may assume that $b_i \leqslant 2$. So we are dealing with one of the following cases:

(i) $p_i > 2$ for some i,
(ii) $b_i = p_i = 2$ for some i.

The proof of both cases are similar and use the two curves argument, keeping in mind that a boundary component works as a singular fiber of multiplicity 0. Here we only present the proof of case (i).

Suppose that $p_1 > 2$. Consider the homology classes Γ_1 and Γ_2 in $H_*(L\overline{M_1})$ introduced in the proof of Proposition 5.1. They can be considered as the elements of $H_*(LM)$. We claim that $p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2) \neq 0 \in H_0(LM)$.

The calculation is the same and the results is a sum of conjugacy classes of $\pi_1(M)$. Computing mod 2 only two terms survive, $[c_1c_2c_3c_2]$ and $[c_1c_2^2c_3]$, and we must prove that as the conjugacy classes of $\pi_1(M)$ they are different. By the proof of Proposition 5.1 we know that they are distinct as the conjugacy classes of $\pi_1(\overline{M_1})$. So it remains to show, by injectivity lemma, that one of them has no representative in the boundary components. Suppose that

$$[c_1c_2c_3c_2] = [h^r d^s], \tag{8.3}$$

where h is represented by a normal fiber and d is the generator coming from a boundary component.

Consider the group,

$$H = \langle c_1, c_2, c_3 | c_1c_2c_3 = 1, c^{\alpha_i} = 1, 1 \leqslant i \leqslant 3 \rangle,$$

where α_i is the multiplicity of the singular fiber corresponding to c_i, and the homomorphism $\phi : \pi_1(\overline{M_1}) \to H$ which is the identity on c_1, c_2, c_3 and sends the other generators of $\pi_1(\overline{M_1})$ to the trivial element of H. Applying ϕ to (8.3) we have

$$[c_1c_2c_3c_2] = [1]$$

or

$$c_1c_2c_3c_2 = 1$$

in H. Since $c_1c_2c_3 = 1$ in H we conclude that $c_2 = 1$ which is a contradiction.

(4) $b_i + p_i = 3$, $1 \leqslant b_i \leqslant 2$, S_i orientable and $2 \notin A_i$ for some i: Here we shall use the figure eight argument. Suppose $i = 1$. By assumption, one of the following holds:

$$(b_1 = 1 \text{ and } p_1 = 2) \quad \text{or} \quad (b_1 = 2 \text{ and } p_1 = 1).$$

The proof is similar for both cases, considering that a boundary component behaves just like a singular fiber of multiplicity 0. So we only present the proof of the first case.

Suppose $p_1 = 2$ and $b_1 = 1$. Consider the homology classes Γ_1 and $\Gamma_2 \in H_*(L\overline{M_1})$ as constructed in the proof of the Proposition 5.4. They can be regarded as homology classes in $H_*(LM)$. We prove that $p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2) \neq 0 \in H_0(LM)$ which proves that M has nontrivial extended products.

Having done the same calculation mod 2 as the proof of Proposition 5.4 we get two conjugacy classes $[c_1c_2c_1^{-1}c_2^{-1}]$ and $[c_1c_2^{-1}c_1^{-1}c_2]$.

Just like the previous cases, in order to show that these two conjugacy classes are distinct, we must prove that at least one of them has no representative in the fundamental group of the boundary component.

Suppose that

$$[c_1c_2c_1^{-1}c_2^{-1}] = [h^r d_1^s], \tag{8.4}$$

where h is the generator corresponding to a normal fiber and d_1 is the generator corresponding to the boundary component.

Consider the group

$$H = \langle c_1, c_2, d_1 | c_1c_2d_1 = 1, c_1^{\alpha_1} = c_2^{\alpha_2} = 1\rangle$$

and the homomorphism $\phi : \pi_1(\overline{M_1}) \to H$ which is the identity on c_1, c_2, d_1 and sends other generators of $\pi_1(M)$ to the trivial element of H. It follows from (5.1) that ϕ is well defined.

By applying ϕ to (8.4) we have

$$[c_1c_2c_1^{-1}c_2^{-1}] = [d_1^s]$$

in H, or

$$[c_1c_2c_1^{-1}c_2^{-1}] = [(c_1c_2)^s].$$

Note that H is the free product $\mathbb{Z}_{\alpha_1} * \mathbb{Z}_{\alpha_2}$ with c_1 and c_2 as the generators of the corresponding factors.

So if $c_1c_2c_1^{-1}c_2^{-1}$ and $(c_1c_2)^s$ are conjugate then $s = 2$ or -2. In both cases $c_1c_2c_1^{-1}c_2^{-1}$ and $(c_1c_2)^s$ are cyclically different since c_1 and c_2 have multiplicities different from 2.

(5) $b_i + p_i = 3$, $1 \leqslant b_i \leqslant 2$, S_i orientable and $2 \in A_i$ for all i: Here the figure eight argument does not work. The following lemma provides us a double cover of the manifold which has nontrivial extended products by previous cases or the nonseparating torus argument. Appendix B is devoted to the proof of this lemma.

Lemma 8.2. *Let M be as above, $|\mathscr{T}|=n$, $b_i+p_i=3$, $1\leqslant b_i\leqslant 2$ and $2\in A_i$ and S_i orientable for all i.*

(i) *If $n\geqslant 3$ then M has a double cover with no nonseparating torus whose torus decomposition has a Seifert piece with 3 boundary components* (*Lemma* B.4).
(ii) *If $n=2$, $A_1=\{2,r\}$ and $A_2=\{2,s\}$ where $r,s\neq 2$, then M has a double cover with no non-separating torus whose torus decomposition has a Seifert piece with two singular fibers of multiplicity $r\neq 2$* (*Lemma* B.5).
(iii) *If $n=2$ and $A_1=A_2=\{2\}$ then M has a double cover with a non-separating torus* (*Lemma* B.6).

(6) $n=|\mathscr{T}|$, $n\geqslant 2$:

- $b_i=2$, $p_i=1$, $A_i=\{2\}$ and S_i is orientable for $i\neq 1,n$;
- $b_1=b_n=1$;
- S_1 is nonorientable;
- If S_n is orientable then $p_n=2$ and $2\in A_n$.

We will be in one of the following situations:

(i) *If $\overline{M_1}$ has a singular fiber or S_1 has more than one cross cap*: Consider M_1' the double cover of $\overline{M_1}$ which is a Seifert manifold and oriented as a fibration (see Remark 4.1). Then M_1' has either more than 2 singular fibers and exactly 2 boundary components or its base surface has a genus greater than zero. We can construct $\tilde{M}$, a double cover of M, by gluing two copies of $M\backslash M_1$ to M_1' along the boundary components of M_1' such that the result double covers M and the restriction of the covering to the complement of M_1' in M is the trivial double cover. Therefore $\tilde{M}$ either belongs to case (3) or its base surface has genus greater than 0. In both cases it has nontrivial extended loop products.

(ii) *If $\overline{M_1}$ has no singular fiber and S_1 has one cross cap*: then S_1 is indeed the Möbius strip and $\overline{M_1}$ has another Seifert fibration model whose base surface is a disk and has 2 singular fibers of multiplicity 2. If S_n is nonorientable then M either belongs to case (i) or M_n has also the Möbius strip as its base surface. We can consider the Seifert fibration model for M_n with the disk as the base surface and two singular fibers of multiplicity 2, therefore M belongs into case (5). □

9. Hyperbolic factor

In this section we analyze gluing of hyperbolic and Seifert/hyperbolic 3-manifolds along torus. We need the following lemma and the proof can be found in Appendix A (see Lemma A.4).

Lemma 9.1. *Suppose that G_1, G_2 and H are three groups and $H=G_1\cap G_2$. Let $x_1\in G_1\backslash H$ and $x_2\in G_2\backslash H$ and $h\in H$ such that*:

(a) $x_1^{-1}Hx_1\cap H=1$,
(b) $x_2h\neq hx_2$.

Then x_1x_2h and x_2x_1h are not conjugate in $G_1 *_H G_2$.

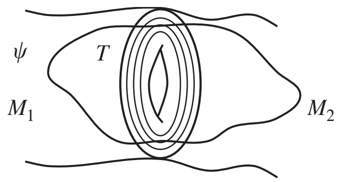

Fig. 7. Separating torus T.

Proposition 9.2. *Let M be a 3-manifold which contains a separating two sided π_1-injective torus T. Suppose that $M \backslash T$ has two connected components M_1 and M_2 such that*:

(i) *$\overline{M}_1$ has a hyperbolic interior with finite volume.*
(ii) *Either M_2 has a complete hyperbolic structure of finite volume, or else $\overline{M}_2$ is a Seifert manifold and $\overline{M}_2 \neq S^1 \times S^1 \times [0, 1]$.*

Then M has nontrivial extended loop products.

Let $\phi : S^1 \times S^1 \to T \subset M$ be a homeomorphism. We choose $\phi(1, 1)$ as the base point. Just like the previous case, ϕ gives rise to a one dimensional family of loops ϕ_t, $t \in S^1$ (longitudes of T in Fig. 7). This 1-family of loops represents a homology class $\Phi \in H_1((LM)_{[h]})$, where $h \in \pi_1(M)$ is the element represented by ϕ_1.

Now consider a simple smooth curve $\psi : \mathbb{S}^1 \to M$ which intersects T exactly at 2 points and it has the free homotopy type $[x_1 x_2]$ where $x_i \in \pi_1(M_i)$, $i = 1, 2$. Note that $\pi_1(M) = \pi_1(\overline{M}_1) *_{\pi_1(T)} \pi_1(\overline{M}_2)$ and $x_1 x_2$ is regarded as an element of this amalgamated free product.

As a loop ψ represents a homology class $\Psi \in H_0((LM)_{[x_1 x_2]})$. We prove that there exist choices of x_1, x_2 and h such that $p_{A_M}(\Delta\Psi) \bullet p_{A_M}(\Delta\Phi) \neq 0 \in H_0(LM)$.

In computing $p_{A_{S^1\times S^2}}(\Delta\Psi) \bullet p_{A_{S^1\times S^2}}(\Delta\Phi)$ we get eight terms. By passing to mod 2 only two terms survive, namely $[x_1 x_2 h]$ and $[x_2 x_1 h]$. Now we must show that there are some choices of x_1, x_2 and h such that these two conjugacy classes are different. For that we use Lemma 9.1.

In our case $G_i = \pi_1(\overline{M}_i)$, $i = 1, 2$ and $H = \pi_1(T)$. Since $\overline{M}_1$ has a hyperbolic interior of finite volume $\pi_1(T)$ consists of parabolic elements of $PSL(2, \mathbb{C})$ with a common fixed point. Then $x_1^{-1}(\pi_1(T))x_1 \cap \pi_1(T) = 1$ for $x_1 \in \pi_1(\overline{M}_1) \backslash \pi_1(T)$ since conjugation with an element outside of H changes the fixed point. Therefore there exists a choice of x_1.

If $\overline{M}_2$ has a hyperbolic interior with finite volume then it follows from the same reasoning as before that there is a choice of x_2 so that (b) is satisfied. If $\overline{M}_2$ is a Seifert manifold, all we have to do is it to modify the embedding ϕ so that h is not in the center of $\pi_1(\overline{M}_2)$ which is generated by a power of h. Therefore under the hypothesis above there are choices of x_1, x_2 and y such that the conditions of Lemma 9.1 are satisfied.

10. Proof of main theorem: part I

In this section we prove the first part of the main theorem, namely that if a closed oriented manifold M is not algebraically hyperbolic, then M or a double cover of M has nontrivial extended loop products.

For this we use the prime decomposition [14] and the torus decomposition for 3-manifolds [10,11]. We refer the reader to [8,9] for the basic materials in the three-dimensional topology.

If M is not algebraically hyperbolic then at least one of the following holds:

10.1. *M is not aspherical*

By the prime decomposition theorem, every 3-manifold M can be written as

$$M = (K_1 \# K_2 \# \ldots \# K_p) \# (L_1 \# L_2 \ldots \# L_q) \# (\#_1^r (S^1 \times S^2)), \tag{10.1}$$

where K_i's are closed aspherical irreducible 3-manifolds with infinite π_1 and L_i are closed 3-manifolds that are finitely covered by a homotopy 3-sphere. If M is not aspherical then one of the following holds:

(i) If M is a connected sum of manifolds with nontrivial fundamental group then Proposition 6.1 says that M has nontrivial extended loop products.
(ii) M has a non-separating 2-sphere or in other words there is a $S^2 \times S^1$ factor in its prime decomposition, then by Corollary 3.2 M has nontrivial extended loop products.
(iii) M has finite fundamental group; Then by Proposition 2.6 M has nontrivial extended loop products.

Thus in either cases M has nontrivial extended loop product.

10.2. *M is aspherical and* $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$

We turn to the case when M is aspherical 3-manifold with rank 2 abelian groups. If follows from the argument in 10.1 that M is either irreducible or is the connected sum of an irreducible 3-manifold and homotopy 3-sphere.[6]

10.2.1. *M is irreducible*

By the torus theorem [16] and Seifert fiber space theorem [3,7] one of the following holds:

(a) M is a closed Seifert manifold. If M is Seifert manifold then by Proposition 4.2 M or a double cover of M has nontrivial extended loop products.

(b) The collection of tori in the torus decomposition of M is nonempty. One of the following occurs:

(i) M contains a nonseparating torus, then by Proposition 3.4 M has nontrivial extended loop products.
(ii) M has no nonseparating torus and the torus decomposition of M has only Seifert pieces: This case has been treated in Section 8. We should verify that M satisfies the conditions of Proposition 8.1. We use the notation introduced in Section 8 and $\mathscr{T}$ is provided by the torus composition of M.

First of all g_i's are all zero as we have assumed that M has no nonseparating torus. If $p_i + b_i \leqslant 2$ for S_i orientable, then one of the following holds:

(1) $b_i = 1$ and $p_i = 1$,
(2) $b_i = 1$ and $p_i = 0$,
(3) $b_i = 2$ and $p_i = 0$.

[6] Note that such a connected sum is again aspherical.

(1) and (2) cannot occur since the boundary component of $\overline{M_i}$ is incompressible. In the latter case, it follows that $\overline{M_i} \simeq S^1 \times S^1 \times [0, 1]$ which contradicts the minimality of the collection $\mathscr{T}$ as one can extend the Seifert fibration of one of the neighboring $\overline{M_j}$ to $\overline{M_i}$.

(iii) M has no nonseparating torus and the torus decomposition of M includes an atoroidal component: suppose that $\mathscr{T}$ is the collection of tori provided by the torus decomposition of M and each torus in $\mathscr{T}$ is separating.

By Thurston's theorem on geometrization of Haken manifolds (see [18]) atoroidal components of $M\backslash\mathscr{T}$ are hyperbolic. Let M_1 be a hyperbolic component of $M\backslash\mathscr{T}$ and M_2 is another component that is attached to M_1 along a torus T. By Proposition 9.2, $M' = \overline{M_1} \cup T_1 \cup \overline{M_2}$ has nontrivial extended loop products.

Consider the same homology classes Γ and H in $H_*(LM')$ constructed in the proof of Proposition 9.2. They can be regarded as the elements of $H_*(LM)$. We claim that $p_{A_M}(\Delta(\Gamma)) \bullet p_{A_M}(\Delta H) \neq 0$, thus M has nontrivial extended loop products.

The calculation is the same as Proposition 9.2, mod 2 only two terms survive, $[hg_1g_2]$ and $[hg_2g_1]$ and we must prove that are different as the conjugacy classes of $\pi_1(M)$. By the proof of Proposition 9.2 we know that hg_1g_2 and hg_2g_1 are not conjugate in $\pi_1(M')$. So by the injectivity lemma (Lemma 7.1), we only need to show that hg_1g_2 is not conjugate to an element of the fundamental group of the boundary of M'.

Let $T_1 \cup T_2 \cup \cdots \cup T_k = \partial M'$ and suppose that hg_1g_2 is conjugate to an element of $\pi_1(T_i)$, $i > 1$ in $\pi_1(M')$. One of $T_i \subset \overline{M_1}$ or $T_i \subset \overline{M_2}$ holds. If hg_1g_2 is conjugate to an element to $\pi_1(T_i)$ then it is conjugate to an element in $\pi_1(M_1)$. But this is a contradiction since $hg_1g_2 = (hg_1)g_2$ as an element of $\pi_1(M') = \pi_1(\overline{M_1}) *_{\pi_1(T)} \pi_1(\overline{M_2})$ is a cyclically reduced word of length 2 and it cannot be conjugate to an element of one of the factors.

10.2.2. M is a connected sum of an irreducible 3-manifold and a homotopy 3-sphere

Suppose that $M = N\#P$ where N is irreducible aspherical and $P \neq S^3$ is a homotopy 3-sphere. In fact M is obtained from N by replacing a standard ball $D \subset N$ with a fake ball D' namely $M = (N\backslash D) \cup_{S^2} D'$. Let $f : N \to M$ be a homotopy equivalence which sends D to D' and is the identity on the complement of D in N. Since $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$ hence $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(N)$. By Seifert fiber space and torus theorem either N is Seifert or the collection of tori in its torus decomposition is nonempty.

If N is a Seifert manifold then Θ_N, the 3-homology class constructed in the proof of Proposition 4.2, cannot be extended directly to a homology class in M but we claim that $p_{A_M}(f_L(\Theta_N)) \bullet p_{A_M}(f_L(\alpha)) \neq 0$ which proves that M has nontrivial extended loop products. The proof is exactly the one in Section 4, note that we can choose a representative for α (see the proof of Proposition 4.2) such that its marked point is in the complement of D.

If N has a nontrivial torus decomposition it follows from the case 10.2.1, that N or a double cover of N has nontrivial extended loop products. In the first case, the homology classes in LN constructed in 10.2.1 can be regarded as a homology classes in LM because they can have chain representatives which are in the complement of D and the loop products calculated there, are also nontrivial in M as $\pi_1(M) \simeq \pi_1(N)$. In the latter case, a double cover $\tilde{N}$ of N gives rise to $\tilde{M}$ a double cover of M where the covering on D' is the trivial double covering and again the examples of nontrivial loop product found for $\tilde{N}$ in Section 8 gives rise to examples of nontrivial loop product for $\tilde{M}$ as we can always do the construction outside of a ball which is to be replaced with a fake ball.

It is not known to us whether the double cover case is necessary. However, one can a give a complete description of the all the homology classes of *LM* in those cases. For instance, let *M* be one of the 3-manifolds listed in part (5) or (6) of the proof of Proposition 8.1. Since *M* is irreducible with infinite fundamental group then each connected component $(LM)_{[\alpha]} = K(C_\alpha, 1)$. It is possible to compute the centralizers C_α since $\pi_1(M)$ is given by a free product with amalgamation of a trees of groups (see Section 7 and Appendix A). The centralizer C_α, $\alpha \neq 1$, is:

(i) $\mathbb{Z}$ if α does not belong to any factor $\pi_1(M_i)$.
(ii) $\pi_1(M_i)$ if α corresponds to a power *h* represented by a normal fiber of M_i.
(iii) $\mathbb{Z} \oplus \mathbb{Z}$ if α belongs to one of the factors $\pi_1(M_i)$ and it does not correspond to any generator due to a singular fiber or a power of the normal fiber of any components.
(iv) $\mathbb{Z}$ if α corresponds to a generator due to a singular fiber.

So A_M has the homological dimension at most two.[7] Moreover, one can find chain representatives for the homology classes of $H_*((LM)_{[\alpha]})$. If γ is a smooth curve representing $\alpha \in \pi_1(M)$, then γ represents a generator $\Gamma \in H_0((LM)_{[\alpha]})$ and $\Delta\Gamma$ is a generator of $H_1((LM)_{[\alpha]})$. When α belongs to a factor $\pi_1(\overline{M_1})$, other generators of $H_1((LM)_{[\alpha]})$ can be obtained by trivializing the fibration over γ, and then one gets the generators of $H_2((LM)_{[\alpha]})$ by applying Δ to them.

One cannot get a nontrivial example of loop in a Seifert piece of the 3-manifolds in case (5) and (6), but one can hope to a get a nontrivial example of loop product by considering the product of a 2-homology class and a 1-homology class in a component $(LM)_{[\alpha]}$, where α is not in a factor. For that, we have to carry a computation similar to the one in Section 9.

11. Proof of main theorem: Part II, algebraically hyperbolic manifolds

In this section we prove the second part of the main theorem, namely that algebraically hyperbolic 3-manifolds have trivial extended loop products.

It follows from Lemma 1.7 that

Lemma 11.1. *If M is a* $K(\pi, 1)$ (*aspherical*) *then LM is homotopy equivalent to*

$$\coprod_{[\alpha]\neq[1]\in\hat{\pi}_1(M)} K(C_\alpha, 1),$$

where we choose only one representative for each conjugacy class and C_α *is the centralizer of* α *in* $\pi_1(M)$.

Therefore to prove the second part of the theorem we need a good understanding of the centralizers.

Lemma 11.2. *If a closed oriented* 3*-manifold M is algebraically hyperbolic then the fundamental group of M does not have any nontrivial finite subgroups and the centralizer* C_α *of an element* $\alpha \in \pi_1(M)$ *is isomorphic to an additive subgroup of* $\mathbb{Q}$.

[7] Note that in case (ii), $H_3(K(C_\alpha, 1)) = K(\pi_1(M), 1) = H_3(M_i) = 0$ as M_i is aspherical and not closed.

Proof. Let $\mathbb{Z}_p$ be a nontrivial finite subgroup of $\pi_1(M)$. Then $K(\mathbb{Z}_p, 1)$ is a covering of M, so it has to be a 3-manifold. One knows that $K(\mathbb{Z}_p, 1)$ has homology in infinitely many dimensions. Therefore $\pi_1(M)$ has no finite subgroup.

For the second part we first prove that every finitely generated subgroup of C_α is isomorphic to $\mathbb{Z}$. Suppose $\beta_1, \beta_2, \ldots, \beta_n$ are some elements of C_α. Consider the finitely generated subgroup $\langle \beta_1, \beta_2, \ldots, \beta_n, \alpha\rangle \subset C_\alpha$, it is not a free product since it has a nontrivial center. Therefore by the compact realization theorem (see [15]) it is isomorphic to the fundamental group of a compact, oriented irreducible 3-manifold N. The infinite cyclic subgroup $\langle\alpha\rangle$ is in the center of $\pi_1(N)$ and therefore it is normal. Thus by the Seifert fibered space theorem [3,7], N is a Seifert manifold. Since $\pi_1(M)$ has no finite subgroup or rank 2 abelian subgroup, N has to be a solid torus,[8] so $\langle \beta_1, \beta_2, \ldots, \beta_n, \alpha\rangle = \pi_1(N) \simeq \mathbb{Z}$ which implies $\langle \beta_1, \beta_2, \ldots, \beta_n\rangle = \mathbb{Z}$.

To prove that C_α is isomorphic to a subgroup of $\mathbb{Q}$, note that because C_α is countable and every finitely generated subgroup is cyclic, we can write $C_\alpha = G_1 \cup G_2 \cdots$ where each G_i is an infinite cyclic subgroup and $G_i \subset G_{i+1}$. Let x_i be a generator for G_i; then we have $x_i = n_i x_{i+1}$ for some n_i. We construct a map $\varphi : C_\alpha \to \mathbb{Q}$ by letting $\varphi(x_1) = 1$ and $\varphi(x_{i+1}) = \frac{1}{n_1 n_2 \ldots n_i}$. This is a well-defined map because of our choice n_i's. Moreover φ is injective and to see that note that if $x \in C_\alpha$ then $x \in G_i$ for some i. So $x = kz_i$ for some i and $k \in \mathbb{Z}$ and $\varphi(x) = \frac{k}{n_1 n_2 \ldots n_i}$. Therefore $\varphi(x) = 0$ implies $k = 0$ or $x = 0$.

Lemma 11.3. *If G is an additive subgroup of $\mathbb{Q}$ then $K(G, 1)$ has homological dimension one.*

Proof. One can write $G = \lim_{n\to\infty} G_n$, where G_i's are cyclic subgroups of G. So $K(G, 1) = \lim_{n\to\infty} K(G_n, 1)$. All $K(G_n, 1) \simeq K(\mathbb{Z}, 1)$ has the homological dimension 1 therefore the direct limit $K(G, 1)$ has the homological dimension one since taking homology commutes with taking direct limit. □

Proposition 11.4. *If M is algebraically hyperbolic then any finite cover of M has trivial extended loop products.*

Proof. Let $\widetilde{M}$ be a finite cover of M. Note that $\widetilde{M}$ is also algebraically hyperbolic. Since $\widetilde{M}$ is a $K(\pi', 1)$ then its free loop space is homotopy equivalent to

$$\coprod_{[\alpha]\in\hat{\pi}'} K(C'_\alpha, 1),$$

where we choose only one representative for each conjugacy and π'_α is the centralizer of α in π'. So

$$A_{\widetilde{M}} \simeq H_*\left(\coprod_{[\alpha]\in\hat{\pi}', \alpha\neq 1} K(C'_\alpha, 1)\right) \simeq \bigoplus_{[\alpha]\in\hat{\pi}', \alpha\neq 1} H_*(K(C'_\alpha, 1)).$$

By Lemmas 11.2 and 11.3, each $K(C'_\alpha, 1)$, $\alpha \neq 1$, has homological dimension 1. Therefore $\bullet$ on $A_{\widetilde{M}}$ is zero as $A_{\widetilde{M}}$ is concentrated in degree 1. □

[8] N cannot be $S^2 \times S^1$ since it is irreducible.

Acknowledgements

Much of the work in this paper was done as the author's thesis under the supervision of Professor Dennis Sullivan in The Graduate Center of CUNY. The author expresses his deep gratitude to Dennis Sullivan for his guidance. Also the author would like to thank Moira Chas, Siddhartha Gadil, Kasra Rafi, Hossein Namazi and Reza Chamanara for many helpful conversations.

Appendix A. Free products with amalgamation and trees of groups

A.1. Some basic facts about free products with amalgamation

We recall some definitions and theorems from group theory related to the free product with amalgamation. We follow the terminology and notations of [12,13].

Let G_1, G_2 with a common subgroup H then we can form $G_1 *_H G_2$ the amalgamated free product of G_1 and G_2 over H.

A sequence of elements $g_1, g_2, \ldots, g_n \in G_1 *_H G_2$ is called *reduced* if:

(1) each g_i is in one of the factors G_1 or G_2;
(2) g_i, g_{i+1} come from different factors; So if $n > 1$ then $g_i \notin H$ for all i;
(3) if $n = 1$ then $g_1 \neq 1$.

$g_1 g_2 \cdots g_n \in G_1 *_H G_2$ is called a *reduced word* if $g_1, g_2, \ldots, g_n$ is a reduced sequence.

Theorem A.1 (*see Lyndon and Schupp [12, p. 187]*). *If $c_1, c_2, \ldots, c_n$, $n \geqslant 1$, is a reduced sequence in $G = G_1 *_H G_2$, then the product $c_1 c_2 \cdots c_n \neq 1$ in G.*

A sequence of elements $g_1, g_2, \ldots, g_n \in G_1 *_H G_2$ is called *cyclically reduced* if all cyclic permutations of the sequence $g_1, g_2, \ldots, g_n$ are also reduced and $g_1 g_2 \cdots g_n \in G_1 *_H G_2$ is called a *cyclically reduced word* if $g_1, g_2, \ldots, g_n$ is a cyclically reduced sequence. The following theorem describes the conjugacy classes of $G_1 *_H G_2$.

Theorem A.2 (*see Magnus et al. [13, p. 212]*). *Let $G = G_1 *_H G_2$. Then every element of G is conjugate to a cyclically reduced word. Moreover, suppose that g is a cyclically reduced element of G. Then*:

(i) *If g is conjugate to a cyclically reduced word $p_1 p_2 \ldots p_r$ where $r \geqslant 2$; then g can be obtained by cyclically permuting $p_1, p_2, \ldots, p_r$ and then conjugating by an element of H.*
(ii) *If g is conjugate to an element h in H, then g is in some factor and there is a sequence of $h, h_1, \ldots, h_t, g$ where h_i is in H and consecutive terms of the sequence are conjugate in some factor.*
(iii) *If g is conjugate to an element g' in some factor but not in a conjugate of H then g and g' are in the same factor and are conjugate in that factor.*

There is no canonical way of presenting the elements of $G_1 *_H G_2$ uniquely. Once a choice of right coset representatives for G_1/H and G_2/H is made then every element can be represented in a unique way. Let $c_1, \ldots, c_n$ be a right coset representative system for G_1/H and G_2/H. Here is an informal description of the representation:

Let g be an element in $G_1 *_H G_2$ and $g = g_1 g_2 \dots g_r$ be a reduced presentation of g. Start with g_r, there is an $h_r \in H$ and $c_r \in G_1/H$ or G_2/H (depending in which factor is g_r) such that $g_r = h_r c_r$. Now by replacing g_r by $h_r c_r$ we have $g = g_1 g_2 \dots g_{r-1} h_r c_r$. Again there is a $h_{r-1} \in H$ and c_{r-1} such that

$$g_{r-1} h_r = h_{r-1} c_{r-1}.$$

By replacing $g_{r-1} h_r$ by $h_{r-1} c_{r-1}$ we have $g = g_1 g_2 \dots g_{r-2} h_{r-1} c_{r-1} c_r$. Continuing this procedure we end up with $g = h_1 c_1 c_2 \dots c_r$. This is the desired presentation.

Theorem A.3 (*see Magnus et al.* [13, p. 201]). *Let $G = G_1 *_H G_2$ where H is a mutual subgroup of G_1 and G_2. Suppose a specific right coset representative system for G_1/H and G_2/H have been selected. Then with each element of g of G we can associate a unique sequence*[9] $(h, c_1, c_2, \dots, c_r)$ *such that*:

(i) *h is an element, possibly* 1, *of H*;
(ii) *c_i is a coset representative of G_1/H or G_2/H*;
(iii) $c_i \neq 1$;
(iv) *c_i and c_{i+1} are not both in G_1 or G_2*;
(v) *$g = h' c'_1 c'_2 \dots c'_r$ in G.*

Lemma A.4. *Suppose that G_1, G_2 and H are three groups and $H = G_1 \cap G_2$. Let $g_1 \in G_1 \backslash H$ and $g_2 \in G_2 \backslash H$ and $h \in H$ such that*:

(i) $g_1^{-1} H g_1 \cap H = 1$,
(ii) $g_2 h \neq h g_2$.

Then $h g_1 g_2$ and $h g_2 g_1$ are not conjugate in $G_1 *_H G_2$.

Proof. Suppose that $[hg_1g_2] = [hg_2g_1]$. It is clear that $[hg_1g_2] = [g_2hg_1]$ and $[hg_2g_1] = [g_2g_1h]$.

So $g_2(hg_1)$ is conjugate to $g_2(g_1h)$. Since $g_2(g_1h)$ is cyclically reduced (of length 2, $p_1 = g_2$ and $p_2 = g_1h$) by Theorem A.2 part (ii), $g_2(g_1h)$ is conjugate to $g_2(g_1h)$ or $(g_1h)g_2$ by an element in H.

In the latter case we have $g_2(hg_1) = h_1(g_1h)g_2h_1^{-1}$ which is equivalent to

$$g_2^{-1}(h_1g_1h)^{-1}g_2(hg_1h_1) = 1.$$

But this contradicts Theorem A.1 because g_2^{-1}, $(h_1g_1h)^{-1}$, g_2, (hg_1h_1) is a reduced sequence.

Therefore there should be an element $h_1 \in H$ such that $g_2hg_1 = h_1g_2g_1hhh_1^{-1}$ or

$$(g_2h)(g_1h_1) = (h_1g_2)(g_1h). \tag{A.1}$$

By Theorem A.1 (also see the construction), (A.1) implies that g_1h_1 and g_1h are in the same left coset in G_1/H or in other words, there exists $h_2 \in H$ such that

$$g_1h_1 = h_2g_1h,$$

or equivalently

$$h_1h^{-1} = g_1^{-1}h_2g_1.$$

[9] This presentation is called *reduced form* (see [13]).

So $h_1h^{-1} = g_1^{-1}h_2g_1 \in g_1^{-1}Hg_1 \cap H = 1$ hence $h_2 = h_1h^{-1} = 1$ which implies $h_1 = h$. By substituting $h_1 = h$ in (A.1) and simplifying, it follows

$$g_2h = hg_2$$

which is a contradiction. Therefore $[hg_1g_2] \neq [hg_2g_1]$. $\square$

A.2. Tree of groups

A *tree of groups* (G, T) consists of a finite tree T and G a collection of groups

$$G = \{G_e\}_{e\in \text{edge } T} \coprod \{G_v\}_{v\in \text{vert } T},$$

a group G_v for every vertex $v \in \text{vert } T$, a group G_e for every edge $e \in \text{edge } T$ and a monomorphism

$$f_e^v : G_e \to G_v$$

if v is a vertex of the edge e.

We call G_v a *vertex group* if v is a vertex of T and G_e an *edge group* if e is an edge of T. If v is a vertex of the edge e then G_e is considered as a subgroup of G_v.

Suppose that (G, T) is a tree of groups, v_0 a vertex of T. Let $n = |\text{edge } T|$. Consider a sequence of trees T_i, $1 \leqslant i \leqslant n$ such that

(i) $|\text{edge } T_i| = i$, $0 \leqslant i \leqslant n$,
(ii) $\text{vert } T_0 = \{v_0\}$,
(iii) $T_i \subset T_{i+1}$ and
(iv) $T_n = T$.

Let $e_1, e_2, \ldots, e_n$ be the sequences of the edges and $v_0, v_1, \ldots, v_n$ be the sequence of vertices such that

- $\text{edge } T_1 = \{e_1\}$,
- $\text{vert } T_1 = \{v_0, v_1\}$,
- $\text{edge} T_{i+1} = \text{edge} T_i \cup \{e_{i+1}\}$, $\quad 1 \leqslant i \leqslant n-1$,
- $\text{vert } T_{i+1} = \text{vert } T_i \cup \{v_{i+1}\}$ $\quad 1 \leqslant i \leqslant n-1$.

Let $G_{T_0} = G_{v_0}$ and

$$G_{T_i} = G_{T_{i-1}} *_{G_{e_n}} G_{v_n}, \quad 1 \leqslant i \leqslant n.$$

$G_T = G_{T_n}$ is called the amalgamation of G_v's along G_e's and is independent of the choice of the sequence T_i and depends only on (G, T) and there is an inclusion

$$\varphi_k : G_k \to G_T,$$

where k is a vertex or an edge.

The following lemma is a generalization of Theorem A.2 part (iii).

Lemma A.5. *Let (G, T) be a tree of groups and v_0 a vertex of T and G_T be the amalgamation of G_v's along G_e's. Suppose that a and b are not conjugate in G_{v_0} and a is not conjugate in G_{v_0} to an element of any edge group $G_e \subseteq G_{v_0}$ where v_0 is a vertex of e. Then a and b are not conjugate in G_T.*

Proof. T_i's, v_i's and e_i's are as above.

We prove the lemma by induction. We prove that for every i, $1 \leqslant i \leqslant n$

(1) $[a]$ and $[b]$ are distinct as conjugacy classes of G_{T_i}.
(2) a is not conjugate in G_{T_i} to an element in the vertex group G_w, where $w \neq v_0$ is a vertex of T_i.

We verify the statement for $i = 1$. By the assumption and part (iii) of Theorem A.2 it follows that the first statement of induction is true for $i = 1$. For the second part, suppose that a is conjugate in G_{T_1} to an element of G_{v_1}, then again by the part (ii) and (iii) of Theorem A.3, a has to be conjugate to an element of G_{e_1} in G_{v_0} which contradicts our assumption.

Suppose that the statement is true for i. By the second statement of the induction for i, a is not conjugate in G_{T_i} to an element of $G_{e_{i+1}} \subseteq G_{T_i}$. Therefore by part (iii), a and b represent different conjugacy classes in $G_{T_{i+1}} = G_{T_i} *_{G_{e_{i+1}}} G_{v_{i+1}}$, proving the first statement of the induction for $i + 1$.

To prove the second statement for $i + 1$, suppose that by contrary a is conjugate to an element of G_w, where $w \neq v_0$ and $w \in T_{i+1}$. Since $G_{T_{i+1}} = G_{T_i} *_{G_{e_{i+1}}} G_{v_{i+1}}$ and a is not conjugate in G_{T_i} to an element of $G_{e_{i+1}}$ then again by part (iii) of Theorem A.2 we must have $w \in T_i$ and a is conjugate in G_{T_i} to an element of G_w. This contradicts the assumption of the induction for i. Hence the second statement of the induction holds for $i + 1$.

Finally for $i = n$ we get the statement of the lemma. □

Appendix B. Finite covers of certain Seifert manifolds with boundary

This appendix is devoted to proof of Lemma 8.2.

Construction B.1. *Suppose M is a compact orientable Seifert manifold with $g = 0$, $p = 1$ and $b = 2$ and the singular fiber has type multiplicity* 2.

Then M has a double cover M' with $g' = 0$, $p' = 0$ and $b' = 3$ boundary components and two of the boundary components are identified under the covering map:

Consider $D \subset \mathbb{R}^2$ the unit disk centered at the origin and F a union of two open disks of radius $1/3$ centered at $(1/2, 0)$ and $(-1/2, 0)$.

Let $M' = N \times S^1$ where $N = D \backslash F$. Then $\mathbb{Z}_2$ acts on M' freely where the action on both factors is realized by 180° rotation.

Indeed $M = M'/\mathbb{Z}^2$ and M' is the desired double covering (Fig. 8). M' has 3 boundary components where two of them are identified under the covering map.

Construction B.2. *Let M be a compact orientable Seifert manifold with $g = 0$, $b = 1$ and $p = 2$ singular fibers of multiplicity* 2. *Then M has a double cover M' with $g' = 0$, $b' = 1$ and $p' = 2$ and the singular fibers are of type* (r, s):

Let N be the unit sphere $S^2 \backslash D \subset \mathbb{R}^3$, where D is a set of n open disks located on the equator in xy-plane which is invariant under $2\pi/s$ rotation about z-axis and 180° rotation about x-axis; and If s is even then

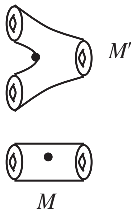

Fig. 8. Double cover M', •'s show the location of the singular fibers.

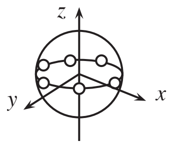

Fig. 9. N.

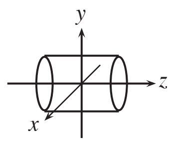

Fig. 10. Annulus $C \subset \mathbb{R}^3$.

D does not meet the x-axis otherwise D meets the x-axis exactly at one point. The $2s$-Dihedral group $D_{2s} = \langle a, b | a^2 = 1, b^s = 1, aba = b^{-1} \rangle$, acts on $P = N \times S^1$ freely (see Fig. 9):

(i) $\mathbb{Z}_2$ acts on P by 180° rotation about x-axis in the first factor and 180° rotation in the second factor.
(ii) $\mathbb{Z}_s$ acts on P freely; a fixed generator of $\mathbb{Z}_s$ acts by $2\pi/s$ rotation about z-axis in the first factor and $2r\pi/s$ rotation in the second factor.

Let $M' = P/\mathbb{Z}_s$, then M' is a Seifert manifold with two singular fibers of type (r, s) and one boundary component and base surface a disk. The action of $\mathbb{Z}_2$ on P descends to M' as the two action commutes. In fact

$$M = M'/\mathbb{Z}_2$$

and M' is the double described in the statement.

Construction B.3. *Let M be a compact Seifert manifold with $g = 0$, $b = 1$ and $p = 2$ singular fibers of type $(2, 1)$. Then M has a double cover M' with $g' = 0$, $b' = 2$ and $p' = 0$ and the boundary components of M are identified homeomorphically under the covering map.*

Consider the annulus $C = S^1 \times [-1, 1] \subset \mathbb{R}^3$, where S^1 is the unit circle in (x, y)-plane and centered at the origin (Fig. 10).

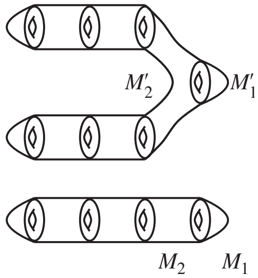

Fig. 11. Double cover $\widetilde{M}$.

$\mathbb{Z}_2$ acts on C with exactly two fixed points. The action is realized by 180° rotation about y-axis. $\mathbb{Z}_2$ acts on

$$M' = C \times \mathbb{S}^1$$

freely where the action on the first factor is described above and on the second factor is the rotation by 180°. Indeed,

$$M \simeq M'/\mathbb{Z}_2$$

and the boundary components of M' are identified homeomorphically under action.

Lemma B.4. *Let M be a closed oriented 3-manifold and T be the collection of tori provided by torus decomposition of M. Suppose that $M \backslash T = M_1 \cup M_2 \cup \cdots \cup M_n$ with all M_i's Seifert manifold such that*:

(1) $n \geqslant 3$ *and* $g_i = 0$ *for all* i's;
(2) $b_1 = b_n = 1$ *and* $b_i = 2$ *for* $i \neq 1, n$;
(3) $p_1 = 2$, $A_1 = \{2, r\}$;
(4) $p_2 = 1$ *and* $A_2 = \{2\}$;

Then M has a double cover $\widetilde{M}$ such that its torus decomposition has a Seifert component with 3 *boundary components*.

Proof. Let M_1' be the double cover of $\overline{M_1}$ provided by Construction B.2 and M_2' be the double cover of M_2 by Construction B.1. M_1' has one boundary component S with an induced involution by the covering map. M_2' has 3 boundary components T_1, T_2 and T_3 where T_2 and T_3 are identified under the covering map and T_1 has an induced involution.

Glue M_1' and M_2' along S and T_1 and then two copies of $\overline{M_3} \cup \overline{M_4} \cdots \overline{M_n}$ to $M_1' \cup_{S=T_1} M_2'$ along T_2 and T_3 in such a way that the result is a covering of M. This is the double covering which has a Seifert component with two singular fibers of multiplicity r and three boundary components (Fig. 11). □

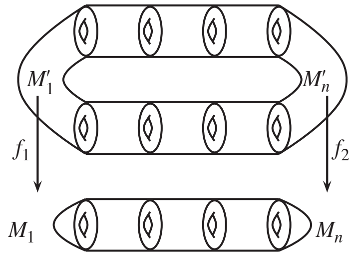

Fig. 12. Double cover $\widetilde{M}$ with a nonseparating torus.

Lemma B.5. *Let M be a closed oriented 3-manifold and T be the collection of tori provided by torus decomposition of M. Suppose that $M\backslash T = M_1 \cup M_2$ with all M_i's Seifert manifold such that*:

(1) $b_1 = 1$, $p_1 = 2$ *and* $A_1 = \{2, r\}$;
(2) $b_2 = 1$, $p_2 = 2$ *and* $A_2 = \{2, s\}$.

Then M has a double cover whose torus decomposition 2 Seifert components with 2 singular fibers of multiplicity r and s.

Proof. Let M'_1 and M'_2 be the double covers of $\overline{M_1}$ and $\overline{M_2}$ provided by Construction 2. M'_1 and M'_2 have one boundary component where the covering map induces an involution on the boundary. Glue M'_1 and M'_2 along the boundary in such a way that the result is a double cover of M. This is the desired 2-cover. □

Lemma B.6. *Let M be a closed oriented 3-manifold and T be the collection of tori provided by torus decomposition of M. Suppose that $M\backslash T = M_1 \cup M_2 \cup \cdots \cup M_n$, $n \geqslant 2$, with all M_i's Seifert manifold*; *and*

(1) $b_1 = b_n = 1$; $b_i = 2$ *for* $i \neq 1, n$;
(2) $p_1 = p_n = 2$;
(3) $A_1 = A_n = \{2\}$.

Then M has a double cover with a non-separating torus.

Proof. The 2-cover M is constructed as follows: Let $M'_1 \xrightarrow{f_1} \overline{M}_1$ and $M'_n \xrightarrow{f_2} \overline{M}_n$ be the double covers of $\overline{M_1}$ and $\overline{M_n}$ provided by Construction 3. Consider two copies of $M\backslash(M_1 \cup M_n)$ and glued them to $M'_1 \cup M'_n$ so that the result $\widetilde{M}$ double-covers M (Fig. 12) in such a ways that covering map restricted to M'_1 and M'_n is f_1 and f_2 and on the rest of $\widetilde{M}$ it is the trivial covering map. This can be done since the covering map f_1 (f_n) identifies different boundary components of M'_1 (resp. M'_n). □

References

[1] H. Abbaspour, The loop product for 3-manifolds, C. R. Math. Acad. Sci. Paris 338 (9) (2004) 713–718.
[2] M. Bendersky, M. Chas, D. Sullivan, Private communication, CUNY Graduate Center.
[3] A. Casson, D. Jungreis, Convergence groups and Seifert fibered 3-manifolds, Invent. Math. 118 (3) (1994) 441–456.

[4] M. Chas, Combinatorial Lie bialgebras of curves on surfaces, Topology 43 (3) (2004) 543–568.
[5] M. Chas, D. Sullivan, String topology, preprint math.GT/9911159.
[6] R.L. Cohen, J.D.S. Jones, A homotopy theoretic realization of string topology, Math. Ann. 324 (4) (2002) 773–798.
[7] G. Gabai, Convergence groups are Fuchsian groups, Ann. of Math. (2) 136 (3) (1992) 447–510.
[8] J. Hempel, 3-manifolds, Annals of Mathematics Studies, vol. 48, Princeton University Press, Princeton, NJ, 1976.
[9] W. Jaco, Lectures on three-manifold topology, CBMS Regional Conference Series in Mathematics, vol. 43, American Mathematical Society, Providence, RI, 1980.
[10] W.H. Jaco, P.B. Shalen, Seifert fibered spaces in 3-manifolds, Memoirs Amer. Math. Soc. 21 (220) (1979).
[11] K. Johannson, Homotopy equivalences of 3-manifolds with boundaries, Lecture Notes in Mathematics, vol. 761, Springer, Berlin, 1979.
[12] R.C. Lyndon, P.E. Schupp, Combinatorial group theory, Classics in Mathematics, Springer, Berlin, 2001 (reprint of the 1977 edition).
[13] W. Magnus, A. Karrass, D. Solitar, Combinatorial Group Theory, Wiley, New York, 1966.
[14] J. Milnor, A unique decomposition theorem for 3-manifolds, Amer. J. Math. 84 (1962) 1–7.
[15] G.P. Scott, Finitely generated 3-manifold groups are finitely presented, J. London Math. Soc. 6 (2) (1973) 437–440.
[16] P. Scott, A new proof of the annulus and torus theorems, Amer. J. Math. 102 (2) (1980) 241–277.
[17] J.-P. Serre, Trees, Springer, Berlin, New York, 1980 (translated from the French by John Stillwell).
[18] W.P. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. (N.S.) 6 (3) (1982) 357–381.