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ELLIPTIC COHOMOLOGY

[after Landweber-Stong, Ochanine, Witten, and others]

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In recent years there have come hints from several directions that it is useful to think of the space $\mathcal{L}M$ of smooth loops in a manifold M as a kind of natural "thickening" of M: one identifies M with the subspace of $\mathcal{L}M$ consisting of loops collapsed to a point. Ideally one would study the space of unparametrized loops, but that is rather intractable, so one considers the space $\mathcal{L}M$ of parametrized loops instead, and works equivariantly with respect to the group of diffeomorphisms of the circle, which acts on $\mathcal{L}M$ by reparametrization. The most important stimulus to think in this way has come from the theory of strings in particle physics, and has entered mathematics mainly through the influence of Witten. In mathematics proper I would mention (a) Bismut's work (e.g. [8], cf. also [5]) on the probabilistic treatment of Atiyah-Singer index theory; (b) the relation between loop spaces and cyclic cohomology described, for example, by Goodwillie [13] and Getzler-Jones-Petrack [12], and the possible implications of Goodwillie's theorem for Waldhausen's work on the classification of manifolds; (c) Floer's definition of "middle-dimensional" cohomology groups for the loop space of a symplectic manifold.

One sign that there may be rich undiscovered geometrical properties of loop spaces is the existence of elliptic cohomology. What is known about this at present does not, strictly speaking, involve loop spaces at all, and can be obtained by standard methods of algebraic topology. In that context, however, the results are mystifying, and Witten has explained heuristically how they fall naturally into place in terms of analysis on loop spaces. I have not attempted here to describe the history of the subject, which can be found in the excellent collection [1]. The main concrete results so far seem to be the equivalent characterizations of elliptic genera ((2.1) and (3.7) below), the existence of the elliptic cohomology theory $E\ell\ell^*$, and some information about $E\ell\ell^*(BG)$ when G is a finite group.

2. GENERA

A *genus* is a rule which associates to each closed oriented manifold M a complex number $\Phi(M)$ satisfying

(i) $\Phi(M_1 \amalg M_2) = \Phi(M_1) + \Phi(M_2)$,

(ii) $\Phi(M_1 \times M_2) = \Phi(M_1).\Phi(M_2)$, and

(iii) $\Phi(M_1) = \Phi(M_2)$ if M_1 and M_2 are cobordant.

In other words, Φ is a ring-homomorphism $\Omega \to \mathbb{C}$, where Ω is Thom's oriented cobordism ring. The best-known genera are the *signature* (of the intersection form on the middle homology) and the *$\hat{A}$-genus*. They take values in $\mathbb{Z}$ and $\mathbb{Z}[\frac{1}{2}]$ respectively. The *Euler number* $\chi(M)$ is not a genus, as it is not a cobordism invariant.

Thom proved that $\Omega \otimes \mathbb{Q}$ is a polynomial ring $\mathbb{Q}[P^2, P^4, P^6, \ldots]$ on the classes of the even-dimensional complex projective spaces. To give a genus Φ is therefore the same thing as to give the formal power series

$$\log_\Phi(x) = \Sigma \frac{1}{2n+1} \Phi(P^{2n}) x^{2n+1}$$

in $\mathbb{C}[[x]]$.

The signature and the $\hat{A}$-genus possess a stronger multiplicativity property than (ii). The signature satisfies $\Phi(M) = \Phi(F).\Phi(B)$ whenever M is a fibre bundle on B with fibre F and compact *connected* structural group. The $\hat{A}$-genus has the same property when F is a spin manifold. The natural question of finding all formal power series $\log_\Phi$ for which Φ has this stronger multiplicativity is answered by

THEOREM (2.1). *The genus Φ is multiplicative for all bundles of spin manifolds with compact connected structural group if and only if $\log_\Phi$ is an elliptic integral, i.e. of the form*

$$\log_\Phi(x) = \int_0^x (1 - 2\delta t^2 + \varepsilon t^4)^{-\frac{1}{2}} dt \tag{2.2}$$

for some $\delta, \varepsilon \in \mathbb{C}$.

Such Φ are called *elliptic genera*. One can take them all together, and speak of a *universal* elliptic genus $\Phi(M)$ whose value is a function of δ and ε. The 'only if' part of (2.1) was proved by Ochanine [19], and also the 'if' part when the fibre is P^{2n+1}. The remainder was proved by Taubes [25].

The integral (2.2), if the discriminant $\varepsilon^2(\delta^2-\varepsilon)$ is non-zero, is the inverse function of an odd elliptic function s which is uniquely characterized by its period lattice L together with the fact that it vanishes at one point ω of order two in $\mathbb{C}/L$. (It has a pole at the other two.) The integral has the homogeneity property that replacing (δ, ε) by $(\lambda^2\delta, \lambda^4\varepsilon)$ changes L to $\lambda^{-1}L$ and $\Phi(M)$ to

$\lambda^{\frac{1}{2}\dim M}\Phi(M)$. Thus (cf. [24]) as a function of (L,ω) the elliptic genus $\Phi(M)$ defined by (2.2) is a *modular form* of weight $\frac{1}{2}\dim M$. One usually normalizes it so that $L = 2\pi i\mathbb{Z} + 4\pi i\tau\mathbb{Z}$ and $\omega = 2\pi i\tau$, with τ in the upper half-plane H. Then $\Phi(M)$ is a holomorphic function of $\tau \in H$ which is modular of weight $\frac{1}{2}\dim M$ under the subgroup $\Gamma_0(2)$ of $PSL_2(\mathbb{Z})$ which preserves the half-period point ω. Alternatively, as ω defines a spin-structure on the elliptic curve $\mathbb{C}/L$, one can say that there is an elliptic genus for each elliptic curve with spin-structure. The space $H/\Gamma_0(2)$ is compactified by the addition of two cusps $\tau = 0$, $\tau = i\infty$ where the curve degenerates. These points correspond to the signature and $\hat{A}$-genus. In terms of $q = e^{2\pi i\tau}$ we have

$$(2.3) \qquad s(u) = 2\tanh\tfrac{1}{2}u.\prod_{n=1}^{\infty}\left\{\frac{(1-q^n e^u)(1-q^n e^{-u})(1+q^n)^2}{(1+q^n e^u)(1+q^n e^{-u})(1-q^n)^2}\right\}.$$

It is striking that the universal elliptic genus of a spin manifold is the character of a virtual projective unitary representation of the group $\mathrm{Diff}(S^1)$ of diffeomorphisms of the circle. More precisely, if $R_\tau \in \mathrm{Diff}(S^1)$ is rotation by τ, then there are projective unitary representations $E_M^{\pm}$ such that $\Phi(M)(\tau) = \mathrm{tr}(R_\tau|E_M^+) - \mathrm{tr}(R_\tau|E_M^-)$ as hyperfunctions of τ. In particular, $\Phi(M)$ has integral coefficients when expanded in powers of q.

3. CHARACTERISTIC CLASSES AND EQUIVARIANT GENERA

PROPOSITION (3.1). *Any genus Φ can be expressed uniquely in the form*

$$(3.2) \qquad \Phi(M) = \langle\Phi(T_M),[M]\rangle ,$$

where $[M] \in H_*(M)$ is the fundamental class, T_M is the tangent bundle of M, and $\Phi : KO(\;) \to H^{\mathrm{even}}(\;;\mathbb{C})$ is a stable exponential characteristic class for real vector bundles.

Here *exponential* means that $\Phi(E \oplus F) = \Phi(E).\Phi(F)$, and *stable* means that $\Phi(\mathbb{R}) = e \neq 0$ in $H^*(\text{point};\mathbb{C}) = \mathbb{C}$.

This result of Hirzebruch is well-known [18], but recall that (3.2) defines a cobordism invariant because if $M = \partial W$ then

$$\begin{aligned}\langle\Phi(T_M),[M]\rangle &= e^{-1}\langle\Phi(T_W|M),[M]\rangle, \text{ where } e = \Phi(\mathbb{R}) ,\\ &= e^{-1}\langle d\Phi(T_W),[W]\rangle \text{ by Stokes's theorem}\\ &= 0 .\end{aligned}$$

An exponential characteristic class is determined by its value on $\eta_{\mathbb{R}}$, the real bundle underlying the universal complex line bundle η on P^∞, and $\Phi(\eta_{\mathbb{R}})$ can be an arbitrary even element of $H^*(P^\infty;\mathbb{C}) \cong \mathbb{C}[[u]]$, where $u \in H^2$. Thus $\Phi(\eta_{\mathbb{R}}) = p_\Phi(u)$, say, is a second formal power series associated with Φ. The stable tangent bundle of P^n is $(n+1)\eta_{\mathbb{R}}-2$, so (3.2) shows that $\Phi(P^n)$ is the coefficient of u^n in $e^{-2}p_\Phi(u)^{n+1}$, and Lagrange's formula for reversion of series ([10] p.125) gives

PROPOSITION (3.3). $e^2\log_\Phi(x)$ *is the inverse power series to* $u/p_\Phi(u)$, *i.e.* $e^2\log_\Phi(x) = u \Leftrightarrow u/p_\Phi(u) = x$.

Proposition (3.1) can be restated in terms of K-theory, for the Chern character gives an isomorphism $ch : K(X) \otimes \mathbb{C} \to H^{even}(X;\mathbb{C})$.

PROPOSITION (3.4).
$$\Phi(M) = \pi_!^M(\Lambda_\Phi(T_M)) ,$$
where $\Lambda_\Phi : KO \to K \otimes \mathbb{C}$ *is a stable exponential characteristic class, and* $\pi_!^M : K(M) \otimes \mathbb{C} \to \mathbb{C}$ *is the Gysin map in K-theory corresponding to* $\pi^M : M \to$ point.

By the topological Riemann-Roch theorem [11] we have $\pi_!^M(E) = \langle ch(E).\hat{A}(T_M),[M]\rangle$, and therefore $ch\Lambda_\Phi(E) = \hat{A}(E)^{-1}.\Phi(E)$.

EXAMPLES

(i) $\hat{A}(M) = \pi_!^M(1)$.

(ii) signature $(M) = \pi_!^M(\Delta(T_M))$, where $\Delta(T_M)$ is the spinor bundle on M. (As a characteristic class for real bundles one defines $\Delta(E) = \Lambda(E^{\mathbb{C}})^{\frac{1}{2}}$, where $E^{\mathbb{C}} = E \otimes \mathbb{C}$ and Λ is the exterior algebra.)

(iii) $\chi(M) = \pi_!^M(\Lambda_{-1}(T_M^{\mathbb{C}}))$, where $\Lambda_{-1} = \Lambda^{even} - \Lambda^{odd}$. This is unstable.

(iv) From (2.3) and (3.3) we find that when Φ is the universal elliptic genus

(3.5)
$$\Lambda_\Phi(E) = a(\tau)^{-\dim E}.\Delta(E) \prod_{k>0} \{\Lambda_{q^k}(E^{\mathbb{C}})S_{q^k}(E^{\mathbb{C}})\} ,$$
where
$$\Lambda_t = \Sigma t^m\Lambda^m,\ S_t = \Sigma t^m S^m, \text{ and } a(\tau) = \Pi(1+q^k)/(1-q^k) .$$

There are two advantages of (3.4) over (3.1). The first is that the Atiyah-Singer index theorem provides an analytic construction of $\pi_!^M(E)$ when M is a spin manifold. In fact $\pi_!^M(E)$ is the index of the Dirac operator D_E which maps sections of $\Delta^+(T_M) \otimes E$ to sections of $\Delta^-(T_M) \otimes E$. In particular $\hat{A}(M)$ is an integer when M is a spin manifold, and more generally

PROPOSITION (3.6). *For a spin manifold* M *the universal elliptic genus belongs to* $\mathbb{Z}[[q]]$.

The second advantage of (3.4) is to make clear that when a compact Lie group G acts smoothly on M the value of *any* genus is naturally an element $\Phi_G(M)$ of the complex character ring $R(G) \otimes \mathbb{C} = K_G(\text{point}) \otimes \mathbb{C}$. For any exponential transformation in K-theory is a unique polynomial in the exterior powers, and makes equally good sense for G-vector-bundles [4]; and the Gysin map is also defined for K_G. (Thus when M is spin the index of D_E is a virtual representation of G.)

We can now state an equivalent version of Theorem (2.1).

THEOREM (3.7). *Elliptic genera Φ are characterized by the property that if a compact connected group* G *acts on a spin manifold* M *then* $\Phi_G(M)$ *is constant as a function on* G.

Remark. G may or may not act on the spin bundle of M, but if not - the so-called *odd* case - then a double covering $\tilde{G}$ of G acts, and $\Phi_G(M)$ is a constant function on $\tilde{G}$. But it is zero as it takes opposite values on the two elements of $\ker : \tilde{G} \to G$.

Proof that (3.7) $\Longrightarrow$ (2.1). Consider a spin fibration $f : M \to B$ with fibre F and structural group G. We have $\pi_!^M = \pi_!^B \cdot f_!$ by the functoriality of Gysin maps, and $T_M = f^*T_B \oplus T_{M/B}$, where $T_{M/B}$ is the tangent bundle along the fibres, so

$$\Phi(M) = \pi_!^M(\Lambda_\Phi(T_M)) = \pi_!^B f_!(f^*\Lambda_\Phi(T_B).\Lambda_\Phi(T_{M/B}))$$
$$= \pi_!^B(\Lambda_\Phi(T_B).f_!\Lambda_\Phi(T_{M/B})) \ .$$

Now $f_!\Lambda_\Phi(T_{M/B})$ is an element of $K(B) \otimes \mathbb{C}$ whose augmentation is $\pi_!^F\Lambda_\Phi(T_F) = \Phi(F)$. It is clearly the image of the equivariant index of F under the map $R(G) \to K(B)$ defined by the principal bundle of $M \to B$. But by (3.7) it is constant and hence equal to its augmentation. So $\Phi(M) = \Phi(F).\Phi(B)$.

Proof that (2.1) $\Longrightarrow$ (3.7). It is enough to prove (3.7) when G is the circle $\mathbb{T}$. Let F be a spin manifold with circle action. We must show that $\Phi_{\mathbb{T}}(F) = \Phi(F)$ when Φ satisfies (2.1). It suffices to show that $\phi = \mathrm{ch}(\Phi_{\mathbb{T}}(F) - \Phi(F))$ vanishes in $H^*(P^\infty) = \mathbb{C}[[u]]$. Let F_n be the bundle on P^n with fibre F associated to the $\mathbb{T}$-action. Our formula shows that the discrepancy $\delta_n = \Phi(F_n) - \Phi(F)\Phi(P^n)$ is the coefficient of u^n in $\phi.e^{-2}\cdot p_\Phi(u)^{n+1}$. By Lagrange's formula, already used to obtain (3.3), we find that if $x = u/p_\Phi(u)$ then

$$e^2\phi = \Sigma\delta_n x^n \ / \ \Sigma\Phi(P^n)x^n \ .$$

So $\phi = 0$ if and only if all δ_n are zero.

We shall not prove (2.1) or (3.7) here, but make only the following remarks. Ochanine proved that the power series $\log_\Phi$ is an elliptic integral if and only if the genus Φ vanishes for all fibre bundles with fibre P^{2n+1}. (Notice that $\Phi(P^{2n+1}) = 0$ because P^{2n+1} is null-cobordant.) His method was a direct computation, but very ingeniously arranged. On the other hand Taubes considered the equivariant universal elliptic genus Φ_q for a spin manifold M as a function of $\lambda \in \mathbb{T}$ and $q = e^{2\pi i\tau}$. Following an idea of Witten (to which I shall return in §4) he observed that $\lambda \mapsto \Phi_q(M)_\lambda$ extends to a meromorphic function on $\mathbb{C}^\times$ which satisfies $\Phi_q(M)_\lambda = \pm\Phi_q(M)_{q\lambda}$, i.e. that it is an elliptic function defined on the torus $\mathbb{C}^\times / (q^2)$. Then he proved it has no poles, and is therefore constant. The meromorphicity and ellipticity follow at once from the localization theorem in equivariant K-theory [7], which can be stated as follows.

PROPOSITION (3.8). *Let* M *be a manifold with* $\mathbb{T}$*-action, and* F *the submanifold of fixed points of* $\mathbb{T}$. *Then the value of the character* $\pi_!^M(E)$ *at* $\lambda \in \mathbb{T}$ *is given by*

$$\pi_!^M(E)_\lambda = \pi_!^F((E|F)_\lambda . \Delta_{-1}(N_F)_\lambda^{-1}) ,$$

providing λ *is not of finite order. Here* N_F *is the normal bundle to* F *in* M, $\Delta_{-1} = \Delta^+ - \Delta^-$, *and on the right-hand-side elements of* $K_{\mathbb{T}}(F) \otimes \mathbb{C}$ *are identified with functions from* $\mathbb{T}$ *to* $K(F) \otimes \mathbb{C}$.

To obtain the elliptic genus Φ_q we take $E = \Delta(T_M)\Pi\{\Lambda_{q^k}(T_M^{\mathbb{C}})S_{q^k}(T_M^{\mathbb{C}})\}$, ignoring the constant factor $a(\tau)$ in (3.5). By (3.8) we find

$$\Phi_q(M)_\lambda = \pi_!^F\{\xi.\Psi(N_F)_\lambda\} ,$$

where $\xi \in K(F) \otimes \mathbb{C}$ is independent of λ, and Ψ is the exponential operation given on line bundles η by

$$\Psi(\eta) = \frac{1+\eta}{1-\eta} \Pi \frac{(1+q^k\eta)(1+q^k\eta^{-1})}{(1-q^k\eta)(1-q^k\eta^{-1})} .$$

Now $(N_F)_\lambda = \Sigma\lambda^i N_i$, say. So $\Psi(N_F)_\lambda = \Pi\Psi(\lambda^i N_i)$. Because $\Psi(\eta)$ as a meromorphic function of $\eta \in \mathbb{C}^\times$ changes sign when η is replaced by $q\eta$ we can conclude that $\Psi(\lambda^i N_i)$, and hence $\Phi_q(M)_\lambda$, has the same property as a function of λ.

4. WITTEN'S EXPLANATION: ANALYSIS ON LOOP SPACES

One would like to see how the universal elliptic genus arises naturally. A satisfactory account must explain (i) its *rigidity* under group actions, (ii) its *modularity*, and (iii) why it is a virtual representation of $\mathrm{Diff}(S^1)$. Witten [26],[27] has given what is obviously the right explanation, but in terms of the mathematics of two-dimensional conformal quantum field theory, which has not yet been fully developed.

Without mentioning field theory one can at least say that formally the elliptic genus is the equivariant index of a natural differential operator on $\mathcal{L}M$. The operator commutes with the circle action on $\mathcal{L}M$ which rotates the loops, so its index is a virtual representation of $\mathbb{T}$. In fact each character $q \mapsto q^k$ of $\mathbb{T}$ occurs with finite multiplicity, and only *positive energy* characters (those with $q \geq 0$) occur at all. So the index is a formal power series $\Sigma a_k q^k$ with $a_k \in \mathbb{Z}$. This is the elliptic genus. But to define the operator even formally we must digress to consider the theory of spinors on $\mathcal{L}M$.

Suppose that M is an oriented Riemannian manifold. At a loop $\gamma \in \mathcal{L}M$ the tangent space T_γ to $\mathcal{L}M$ is the space of tangent vector fields to M along γ, i.e. the space of sections of γ^*T_M on S^1. Covariant differentiation along γ is a skew operator $D/D\theta$ in T_γ, and gives a decomposition $T_\gamma^{\mathbb{C}} = W \oplus N \oplus \overline{W}$, where $N = \ker(D/D\theta)$ is finite dimensional, and $-iD/D\theta$ is positive-definite on W. If we define an operator J_γ in T_γ as $i \oplus 0 \oplus (-i)$ with respect to $W \oplus N \oplus \overline{W}$, then T_γ is *polarized* in the sense of

DEFINITION (4.1). *(i) A* polarization *of a real pre-Hilbert space* E *is a class of skew operators* $J : E \to E$, *any two differing by a trace-class operator, such that* $J^2 + 1$ *is of trace-class.*

(ii) The subgroup of the orthogonal group of E *which preserves the polarization is called the* restricted orthogonal group $O_{res}(E)$.

The polarization reduces the structural group of the tangent bundle $T_{\mathcal{L}M}$ to $O_{res}(E)$, where $E = \mathcal{L}\mathbb{R}^n$, and $n = \dim M$.

The group $O_{res}(E)$ has two connected components, and has the homotopy type of $\varinjlim O_{2n}/U_n$, i.e. of the loop space of $\varinjlim SO_{2n}$ [20] (12.4). In particular it makes sense to ask whether the bundle $T_{\mathcal{L}M}$ is orientable. Its classifying map $\mathcal{L}M \to BO_{res}(E) \simeq \varinjlim SO_{2n}$ is clearly got by looping the classifying map $M \to BSO_{2n}$ of T_M, and as $w_2 \in H^2(BSO_{2n};\mathbf{F}_2)$ transgresses to the generator of $H^1(SO_{2n};\mathbf{F}_2)$ we have

PROPOSITION (4.2). $\mathcal{L}M$ *is orientable if and only if* M *is spin.*

To define spinors on $\mathcal{L}M$ we recall that the group $O_{res}(E)$ possesses a projective spin representation ([20] Chap.12) on a Hilbert space $\Delta(E) = \Delta^+(E) \oplus \Delta^-(E)$. It is defined on a central extension of $O_{res}(E)$ by $\mathbb{T}$ described topologically by the generator of $H^2(O_{res};\mathbb{Z})$, which is the double transgression of the Pontrjagin class $p_1 \in H^4(BSO_{2n};\mathbb{Z})$. Hence we have

PROPOSITION (4.3). *There is a bundle of spinors on* $\mathcal{L}M$ *if and only if* $p_1(M) = 0$.

A spinor bundle $\Delta(T_{\mathcal{L}M})$ on $\mathcal{L}M$ is automatically a module for the Clifford bundle of $T_{\mathcal{L}M}$, so *formally* one can define a Dirac operator $\mathcal{D}$ acting on sections of Δ, and also an operator $\mathcal{D}_E$ acting on sections of $E \otimes \Delta$ when E is any other bundle on $\mathcal{L}M$. Assuming this makes sense let us blindly apply the localization theorem (3.8) to calculate the index of $\mathcal{D}_E$ when E is $\mathbb{T}$-equivariant. The fixed points are the constant loops $M \subset \mathcal{L}M$, so (3.8) gives

$$\operatorname{index} \mathcal{D}_E = \pi_!^M((E|M).\Delta_{-1}(N_M)^{-1}) .$$

The fibre of the normal bundle N_M at $m \in M$ is $\mathcal{L}T_{M,m}/T_{M,m}$, where $T_{M,m}$ is the tangent space to M at m. By Fourier series $N_M^{\mathbb{C}}$ is identified with $\bigoplus_{k\neq 0} q^k T_M^{\mathbb{C}}$, and $\Delta_{-1}(N_M)$ with $\Lambda_{-1}(\bigoplus_{k>0} q^k T_M^{\mathbb{C}}) = S(\bigoplus_{k>0} q^k T_M^{\mathbb{C}})^{-1}$. Thus

(4.4)
$$\operatorname{index} \mathcal{D} = \pi_!^M\{\prod_{k>0} S_{q^k}(T_M^{\mathbb{C}})\} .$$

(It is usual to include the not obviously meaningful factor $(\det(\oplus\, q^k T_M^{\mathbb{C}}))^{-\frac{1}{2}}$ in $\Delta(N_M)$. This amounts to a way of normalizing the projective representation. As $\det(T_M^{\mathbb{C}})$ is trivial the factor is interpreted simply as $q^{-\frac{1}{2}\dim M.\Sigma k} = q^{-\frac{1}{2}\zeta(1)\dim M} = q^{1/24 \dim M}$. When it is inserted the expression (4.4) can be shown [28] to be a modular function under $PSL_2(\mathbb{Z})$ when $p_1(M) = 0$.)

At present, however, we are concerned not with $\mathcal{D}$ itself but with $\mathcal{D}_\Delta$, i.e. $\mathcal{D}$ with coefficients in the spin bundle of $\mathcal{L}M$. This is the operator whose index is the signature in finite dimensions. The localization formula gives

$$\text{index}\, \mathcal{D}_\Delta = \pi_!^M\{\Delta(T_M)\ \Pi\ \Lambda_{q^k}(T_M^{\mathbb{C}})S_{q^k}(T_M^{\mathbb{C}})\}\ .$$

This is the universal elliptic genus, except for the uninteresting factor $a(\tau)^{-\dim M}$ (cf. (3.5)).

To define $\mathcal{D}_\Delta$ one does not actually need the bundle Δ globally on $\mathcal{L}M$, but only the product $\Delta \otimes \Delta$, graded by $\Delta \otimes \Delta^{\pm}$. In finite dimensions $\Delta(E) \otimes \Delta(E)$ is just $\Lambda(E^{\mathbb{C}})$, and the grading is that given by the Hodge star operator. Its existence does not require a spin structure, but only an orientation. Correspondingly $\mathcal{D}_\Delta$ is defined - formally - on $\mathcal{L}M$ when M is spin, even if $p_1(M) \neq 0$.

I shall attempt below to make the use of the localization formula on $\mathcal{L}M$ seem a little more reasonable, but first I shall give Witten's explanation of the rigidity of the elliptic genus under group actions. As we know from §3, the essential point is that, for a spin manifold M with $\mathbb{T}$-action, $\Phi_q(M)_\lambda$ extends from a function of $\lambda \in \mathbb{T}$ to one defined on the torus $\mathbb{C}^\times/q$. To see this, consider for each $\lambda \in \mathbb{T}$ the *twisted* loop space $\mathcal{L}_\lambda M$ consisting of smooth maps $\gamma : \mathbb{R} \to M$ such that $\gamma(\theta + 2\pi) = \lambda.\gamma(\theta)$. There is, we assume, a corresponding family of Dirac operators $\mathcal{D}_{\Delta,\lambda}$. But whereas $\mathcal{L}M$ has a $\mathbb{T} \times \mathbb{T}$-action got by independently rotating the loops and translating them by the $\mathbb{T}$-action on M, the twisted space $\mathcal{L}_\lambda M$ has a natural action of the torus $T_\lambda = \mathbb{R}^2/2\pi L_\lambda$, where if $\lambda = e^{i\ell}$ the lattice L_λ is spanned by $(0,1)$ and $(1,-\ell)$. The characters of T_λ form a discrete set, and each occurs with finite multiplicity in the index of $\mathcal{D}_{\Delta,\lambda}$. The homotopy invariance of the index makes us expect that the multiplicity should be independent of λ. But as λ moves once round $\mathbb{T}$ the character $(q,\mu) \mapsto q^a\mu^b$ of $\mathbb{T} \times \mathbb{T} = T_1$ moves continuously to $(q,\mu) \mapsto q^{a+b}\mu^b = q^a(q\mu)^b$. Thus $\Phi_q(M)_\mu = \Phi_q(M)_{q\mu}$.

The idea of the Dirac operator $\mathcal{D}$ motivates the definition of the elliptic genus and accounts for its rigidity. It does not explain the modularity or the role of $\text{Diff}(S^1)$: indeed $\text{Diff}(S^1)$ does not act by isometries on $\mathcal{L}M$, and certainly does not commute with $\mathcal{D}$. To go further one cannot avoid conformal field theory.

One approach is in terms of functional integrals. On a finite dimensional manifold M recall [6] that the index of the Dirac operator D is the *supertrace* of e^{tD^2} for any $t > 0$. More precisely, D is an operator of degree 1 on the mod 2 graded space $\Gamma^+ \oplus \Gamma^-$ of spinor fields. So D^2, which is the Laplacian, preserves the grading and is negative semidefinite. The supertrace is defined by

$$\text{str}(e^{tD^2}) = \text{tr}(e^{tD^2}|\Gamma^+) - \text{tr}(e^{tD^2}|\Gamma^-)\ .$$

The trace and also the supertrace of e^{tD^2} can be calculated as integrals over $\mathcal{L}M$, although that is a long story [5],[8]. If a group G acts on M the character-valued index of D at $g \in G$ is given by a corresponding integral over the twisted loop space $\mathcal{L}_g M$ already mentioned.

Optimists will believe that an analogous discussion applies to the Dirac operator $\mathcal{D}$ on $\mathcal{L}M$. The index ought to be an integral over $\mathcal{L}\mathcal{L}M$, i.e. over the space of maps of a torus into M, and the equivariant index likewise. In general terms the fact that the elliptic genus of M is a modular form, i.e. a function of a *torus*, is explained in this way. A mathematical treatment of the theory by functional integration seems, however, out of sight at present, so I shall not pursue it further.

The primary difficulty in defining a genuine Dirac operator $\mathcal{D}$ on $\mathcal{L}M$ is to find a suitable Hilbert space $\mathcal{H}$ of spinor fields on which it will act. Conformal field theory, however, predicts the presence of an elaborate structure on the space $\mathcal{H}$ which to an earthbound mathematical eye is quite unexpected. In good cases the natural (projective) action of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ should extend canonically to a projective unitary action of $\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$ (= $\mathrm{Diff}_L \times \mathrm{Diff}_R$, say) inside which the geometrical $\mathrm{Diff}(S^1)$ is the diagonal subgroup. This makes a great mathematical simplification: one reason is that the natural action of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ is neither of positive nor of negative energy, whereas Diff_L and Diff_R act with positive and negative energy respectively. Better still, $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is *supersymmetric* with respect to the action of Diff_R, in the following sense. First recall [15] that the Lie algebra $\mathrm{Vect}(S^1)$ of $\mathrm{Diff}(S^1)$ is the even part of a Lie superalgebra whose odd part is the space $\Omega^{-\frac{1}{2}}(S^1)$ of smooth $(-\frac{1}{2})$-densities on S^1: the natural pointwise product of two elements of $\Omega^{-\frac{1}{2}}$ is a vector field. To say that $\mathcal{H}$ is supersymmetric under Diff_R means that the action of its Lie algebra is the even part of an action of the superalgebra on the mod 2 graded space $\mathcal{H}$. Thus if $D_\alpha : \mathcal{H}^{\pm} \to \mathcal{H}^{\mp}$ is the action of $\alpha = \alpha(\theta)d\theta^{-\frac{1}{2}}$ then D_α^2 is the action on $\mathcal{H}$ of the vector field $\alpha(\theta)^2 d/d\theta$ belonging to Vect_R. The Dirac operator $\mathcal{D}$ is simply D_α when $\alpha = d\theta^{-\frac{1}{2}}$. It therefore commutes with Diff_L, and its index is a virtual representation of Diff_L. But it does not commute with Diff_R or with the geometric action of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$.

The existence of $\mathcal{H}$ for a general manifold M is problematical. One can, however, construct a good approximation to it with the right formal properties by restricting attention to the normal bundle N_M of M in $\mathcal{L}M$. We have already remarked that the fibre of the normal bundle is $V = \mathcal{L}\mathbb{R}^n/\mathbb{R}^n$, so let us first consider the Dirac operator on V.

We can define a Hilbert space of square-summable functions $L^2(V)$ by prescribing that it contains all functions $v \mapsto f(v)e^{-\frac{1}{2}\langle v,Av\rangle}$, where $A = (-(d/d\theta)^2)^{\frac{1}{2}}$ and f is a polynomial on V. A better description of $L^2(V)$ is got by using $d/d\theta$ to polarize V, so that $V_{\mathbb{C}} = W \oplus \overline{W}$, and we have two Bosonic Fock spaces $S(W)$ and $S(\overline{W})$, each a unitary representation of $\mathrm{Diff}(S^1)$ – cf. [22]. Then the Heisenberg group ([20] p.188) of V acts unitarily on S(W) by operators $\{U_f\}_{f\in V}$ which obey the Weyl relation

$$U_f U_g = e^{iS(f,g)} U_{f+g} ,$$

where $S(f,g) = \int \langle f, dg \rangle$. The matrix elements of this action identify $S(W) \otimes S(\bar{W})$ with a space of functions on V which is precisely $L^2(V)$. This shows that $\mathrm{Diff}(S^1)$ acts unitarily on $L^2(V)$, and also that the action extends to $\mathrm{Diff}_L \times \mathrm{Diff}_R$.

To understand the Dirac operator on V first observe that if $V_1 = \Omega^{\frac{1}{2}}(S^1;\mathbb{R}^n)$ then $V \oplus V_1$ is a graded module for the Lie superalgebra $\mathrm{Vect}(S^1) \oplus \Omega^{-\frac{1}{2}}(S^1)$; an element $\omega \in \Omega^{-\frac{1}{2}}$ maps V_1 to V by multiplication, and V to V_1 by $f \mapsto \omega df$. The skew form S on V fits together with the natural symmetric form on V_1 to define a "super-skew" form on $V \oplus V_1$ which is invariant under $\mathrm{Vect} \oplus \Omega^{-\frac{1}{2}}$. Thus $V \oplus V_1$ has a Weyl/Clifford algebra $A(V) \otimes C(V_1)$ whose natural irreducible module is $S(W) \otimes \Lambda(W_1)$, where $V_{1,\mathbb{C}} = W_1 \oplus \bar{W}_1$ is the polarization. It follows that the superalgebra $\mathrm{Vect} \oplus \Omega^{-\frac{1}{2}}$ acts on $S(W) \otimes \Lambda(W_1)$.

In obtaining (4.4) we took the fibre of the spin bundle of $\mathcal{L}M$ to be $\Delta(\Omega^0(S^1;\mathbb{R}^n)) = \Lambda(W)$, but we now see that the correct fibre is $\Delta(\Omega^{\frac{1}{2}}(S^1;\mathbb{R}^n)) = \Lambda(W_1)$. We should also stress that these spaces are not isomorphic to their complex conjugates. Let us now define the space of L^2 spinors on V to be $\mathcal{H}_0 = L^2(V) \otimes \Lambda(\bar{W}_1) = S(W) \otimes S(\bar{W}) \otimes \Lambda(\bar{W}_1)$. We now know that this is acted on by $\mathrm{Diff}_L \times \mathrm{Diff}_R$, and is supersymmetric with respect to Diff_R. We define the Dirac operator $\mathcal{D}_0$ in $\mathcal{H}_0$ to be the action of the element $d\theta^{-\frac{1}{2}}$ of $\Omega_R^{-\frac{1}{2}}$. Because its square is $(d/d\theta)_R$ its index is simply S(W).

We can now at least define rigorously the Dirac operator $\mathcal{D}$ on N_M. There is a bundle $\mathcal{H}_M$ on M with fibre $\mathcal{H}_0$, and the space of L^2 spinors on N_M is the space of sections of $\Delta(M) \otimes \mathcal{H}_M$. We define $\mathcal{D} = \mathcal{D}_M + \mathcal{D}_N$, where $\mathcal{D}_M$ is the usual Dirac operator of M tensored with $\mathcal{H}_M$, and $\mathcal{D}_N$ is induced by the operator $\mathcal{D}_0$ fibrewise in $\mathcal{H}_M$. It is easy to calculate the index of $\mathcal{D}$, for the bundle $\mathcal{H}_M$ is bigraded by the rotation action of $\mathbb{T}_L \times \mathbb{T}_R$, and is finite dimensional in each bidegree; and $\mathcal{D}^2$ preserves the bigrading. So the index of $\mathcal{D}$ is the index of the Dirac operator of M tensored with the bundle index $(\mathcal{D}_N)$ on M, i.e. tensored with $S(W) = S(\oplus\, q^k T_M^{\mathbb{C}})$. This goes some distance towards justifying the use of the localization formula in (4.4).

5. ELLIPTIC COHOMOLOGY

We now return to classical algebraic topology. In the majority of significant applications the *integrality* properties of a genus are crucial. These are best discussed in terms of generalized cohomology theories. Let us consider multiplicative cohomology theories h^* in which 2 is invertible in h^0(point). We shall denote the graded ring h^*(point) by R.

An *orientation* of h^* is an odd element $\xi \in h^2(P^\infty)$ which restricts to the canonical generator of $h^2(P^1)$. (*Odd* means that $\xi \mapsto -\xi$ under complex conjugation in P^∞.) Choosing an orientation, if it is possible, gives one at once a great deal of structure ([2], [21]), in particular:

(i) Chern classes $c_i^{(h)}(E) \in h^{2i}(X)$ for complex vector bundles E on X;

(ii) a Thom isomorphism $h^i(X) \to \tilde{h}^{i+n}(E^+)$ for each oriented real n-dimensional vector bundle E on X;

(iii) Gysin maps $f_!: h^i(M) \to h^{i-n}(B)$ for each fibration $f : E \to B$ whose fibre is an oriented n-manifold;

(iv) an R-valued genus Φ_h, where $\Phi_h(M) = \pi_!^M(1) \in R^{-\dim M}$;

(v) a graded odd formal group law F over R, where $F(x,y) = \Sigma a_{ij}x^iy^j \in R[[x,y]]$ is defined by $c_1^{(h)}(L \otimes M) = F(c_1^{(h)}(L), c_1^{(h)}(M))$ for any two complex line bundles L and M.

Here E^+ denotes the Thom space of E, $\tilde{h}^*(X) = h^*(X,\text{point})$, and to say that F is graded and odd means that $a_{ij} \in R^{-2(i+j-1)}$ and $F(-x,-y) = -F(x,y)$.

Because of property (ii) there is a universal oriented theory Ω^* with a canonical transformation $\Omega^* \to h^*$ to any other; Ω^* is oriented cobordism, and its coefficient ring $\Omega^*(\text{point}) = R_\Omega$ is Thom's ring with $\frac{1}{2}$ adjoined. The universal map $R_\Omega \to h^*(\text{point})$ is Φ_h.

EXAMPLE. Let $R = \mathbb{C}[\delta,\varepsilon]$ be the graded ring of modular forms for $\Gamma_0(2)$, where $\delta \in R^{-4}$, $\varepsilon \in R^{-8}$ are as in (2.2), and let $h^*(X) = H^*(X;R)$. (Thus $h^i(X) = \oplus H^{i+j}(X;R^{-j})$.) The element s(u) of (2.3), regarded as an element of $H^2(P^\infty;R)$ by taking u as the generator of $H^2(P^\infty;\mathbb{Z})$, is an orientation, and leads to the elliptic genus. To see that the coefficient of u^k in $s(u) = s(u,\tau)$ does belong to R^{-2k+2} observe that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$ then

$$s\left(\frac{u}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \frac{1}{c\tau+d}\, s(u,\tau) ,$$

because both sides have the same double periodicity in u, the same zeros, and the same derivative at u = 0. The formal group law of h^* is Euler's addition theorem for the integral (2.2):

$$F(x,y) = \frac{xr(y)+yr(x)}{1-\varepsilon x^2y^2}, \tag{5.1}$$

where $r(x) = (1-2\delta x^2+\varepsilon x^4)^{\frac{1}{2}}$. Notice that F is defined over the subring $\mathbb{Z}[\frac{1}{2}][\delta,\varepsilon]$.

Quillen made the very fruitful observation (cf. [2], [21]) that the category of oriented cohomology theories over $\mathbb{Z}[\frac{1}{2}]$ is very nearly equivalent to the algebraic category of pairs (R,F), where R is an anticommutative graded ring over

$\mathbb{Z}[\frac{1}{2}]$ and F is a graded odd formal group law over R. In particular, R_Ω is the base-ring of Lazard's universal formal group. As far as I know, the categories may even be exactly equivalent, but there is no known general method of assigning a cohomology theory to a pair (R,F). Nevertheless, the law F defines a homomorphism $R_\Omega \to R$, and if R is *flat* over R_Ω then $\Omega^*_R = \Omega^* \otimes_{R_\Omega} R$ is the desired theory. Landweber [16] observed that for Ω^*_R to be a cohomology theory it is enough for R to be flat when localized at those prime ideals of R_Ω which are invariant under the group of automorphisms of the universal formal group. He determined these ideals explicitly: at each prime p of $\mathbb{Z}$ a group law has an integer invariant called its *height* [21], and there is a unique invariant prime ideal $I_{p,k}$ for each p and each height k such that F reduced modulo $I_{p,k}$ is the universal group law of height k in characteristic p. Landweber gave a simple algebraic condition on an R_Ω-module R which ensures that Ω^*_R is a cohomology theory, and in [17] it was applied to prove

THEOREM (5.2). *Let* $R = \mathbb{Z}[\frac{1}{2}][\delta,\varepsilon,\varepsilon^{-1}]$, *regarded as an* R_Ω*-module by the elliptic genus, i.e. by the formal group law (5.1). Then* $E\ell\ell^* = \Omega^* \otimes_{R_\Omega} R$ *is a cohomology theory.*

This is elliptic cohomology. In fact Landweber's theorem produces a p-local theory $\Omega^*_{(p,k)}$ from Ω^* for each (p,k), related to the group laws of height k over p-local rings: they are called Johnson-Wilson theories [21]. K-theory is an amalgam of all $\Omega^*_{(p,k)}$ with k = 1, and elliptic theory of all those with k = 2.

Theorem (5.2) is proved quite simply by pure algebra, but the real challenge is to find a geometrical description of $E\ell\ell^*$. A clue may be provided by a striking theorem of Hopkins-Kuhn-Ravenel [14] which I shall now state.

Let G be a finite group. It is well known [3] that $K^*(BG)$ is a completion of the character ring R(G). More precisely, for each prime p one can define a "character" $\chi_p(\xi)$ for each $\xi \in K^*(BG)$. The character is a function on the set G_p of conjugacy classes of G of p-power order, with values in the algebraic closure $\bar{\mathbb{Q}}_p$ of the p-adic numbers, and it induces an *isomorphism*

$$K(BG)_p \otimes_{\mathbb{Z}_p} \bar{\mathbb{Q}}_p \to \mathrm{Map}(G_p;\bar{\mathbb{Q}}_p)$$

where $K^*(BG)_p$ is the p-adic completion of $K^*(BG)$. For elliptic cohomology the analogue of G_p is $G_p^{(2)}$, the set of conjugacy classes of *pairs of commuting elements* of p-power order in G, and the analogue of $\bar{\mathbb{Q}}_p$ is $\bar{R}$, the algebraic closure of the quotient field of the p-adic completion of $R = E\ell\ell^*(\mathrm{point})$.

THEOREM (5.3) [14]. *For any finite group* G *there is an isomorphism*

$$E\ell\ell^*(BG)_p \otimes_R \bar{R} \to \mathrm{Map}(G_p^{(2)};\bar{R}) ,$$

where $E\ell\ell^*(BG)_p$ *is the p-adic completion of* $E\ell\ell^*(BG)$.

This is very suggestive because the elliptic genus of a manifold M was supposed to be an integral over the space of maps of a torus into M, while the set of pairs of commuting elements of G is the set of homotopy classes of maps of a torus into BG.

6. SPECULATION ABOUT THE DEFINITION OF ELLIPTIC COHOMOLOGY

For any space X let $\mathcal{P}_X$ be the category whose objects are the points of X and whose morphisms from x_0 to x_1 are the paths in X from x_0 to x_1, two such paths being identified if they differ only by reparametrization. A functor from $\mathcal{P}_X$ to finite dimensional vector spaces is essentially the same thing as a vector bundle on X with a connection. (The functor must be continuous in a suitable sense.) It is well known how K-theory is constructed from such objects.

I have described elsewhere [23] a category $\mathcal{C}$ whose objects are all compact oriented one-dimensional manifolds, and whose morphisms from S_0 to S_1 are pairs (Σ,α), where Σ is a Riemann surface with boundary $\partial\Sigma$, and α is an isomorphism between ∂X and $S_1 - S_0$. Two pairs $(\Sigma,\alpha),(\Sigma',\alpha')$ are identified if they are isomorphic. For any space X one can now define a category $\mathcal{C}_X$. Its objects are pairs (S,s), where S is an object of $\mathcal{C}$ and $s : S \to X$ is a map. Its morphisms from (S_0,s_0) to (S_1,s_1) are triples (Σ,α,σ), where $(\Sigma,\alpha) : S_0 \to S_1$ is a morphism in $\mathcal{C}$, and $\sigma : \Sigma \to X$ is a map compatible with (s_0,s_1). The category $\mathcal{C}_X$ is a natural analogue of the category $\mathcal{P}_X$ which gives rise to vector bundles.

It is appropriate to consider functors from $\mathcal{C}$ to the category $\mathcal{V}$ of topological vector spaces and trace-class maps. If such a functor E is holomorphic in the natural sense then $E(S^1)$ is a positive energy representation of $\mathrm{Diff}(S^1)$ of finite type. More precisely, as is familiar in the representation theory of $\mathrm{Diff}(S^1)$, one must consider *projective* representations of $\mathcal{C}$ of some definite positive integral level k. Imposing a further condition - the *contraction* condition below - on the functor E ensures that the character of $E(S^1)$ is a modular form of weight k.

Now let us define an *elliptic object* of level k on X as a projective functor $E : \mathcal{C}_X \to \mathcal{V}$ of level k which is holomorphic and satisfies the contraction condition. Such an object consists of an infinite dimensional vector bundle on the loop space $\mathcal{L}X$, equivariant under $\mathrm{Diff}(S^1)$, together with some additional data amounting to a kind of connection. The primary example is the spin bundle of $\mathcal{L}X$, which is defined when X is a spin manifold with $p_1 = 0$.

I have nothing precise to say about elliptic objects, but it seems to me quite likely that the objects of each level lead to an interesting cohomology theory, and that the theories for different levels are related by "Bott maps". That would fit in well with Theorem (5.3), for just as elements of K(BG) are related to flat bundles, so elements of $E\ell\ell^*(BG)$ seem to be related to flat elliptic objects, i.e. ones such that the operator associated to (Σ,α,σ) depends on σ only up to homotopy, and is therefore a homomorphism $\pi_1(\Sigma) \to G$.

I should mention that the category $\mathcal{C}$ can be modified by equipping the Riemann surfaces Σ with chosen spin structures. That is certainly needed to obtain genuine elliptic cohomology.

Finally I return to the "contraction property". This is motivated by the path-integral point of view. If a surface Σ is a morphism from S to itself then the trace of the operator $E(\Sigma):E(S)\to E(S)$ associated to Σ must depend only on the *closed* surface $\check{\Sigma}$ obtained by attaching the two boundary pieces of Σ to each other. Thus if Σ_τ is the annulus $\{z \in \mathbb{C} : |e^{i\tau}| \leq |z| \leq 1\}$ then $\check{\Sigma}_\tau \cong \check{\Sigma}_{\tau'}$, when $\tau' = -1/\tau$, and therefore the trace of $E(\Sigma_\tau)$ is invariant under $\tau \mapsto -1/\tau$.

Brylinski [9] has proposed a similar approach to elliptic cohomology.

Postscript. After giving this talk, I learnt of the work [29] which gives a good account of the Dirac operator on loop space from the path integral point of view.

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