WHAT IS AN ELLIPTIC OBJECT?

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1. Elliptic cohomology

A generalized cohomology theory is a sequence of contravariant functors $\{h^i\}_{i\in\mathbb{Z}}$ from spaces to abelian groups which are linked together in a well-known way. The theories that arise in nature are of two types: K-theories, and cobordism theories. (Classical cohomology can be approached in so many different ways that I shall leave it aside for the moment.)

On a compact space X the isomorphism classes of complex vector bundles form an abelian semigroup $\mathrm{Vect}(X)$ under the operation of direct sum, and $K^0(X)$ is the abelian group got by formally adjoining inverses to the semigroup $\mathrm{Vect}(X)$. Then K^0 is a homotopy functor, and the functors K^{-i}, for $i > 0$, defined — roughly — by composing K^0 with the i-fold suspension functor, have the properties of "half" a cohomology theory. That much is true for any representable homotopy functor, but the functors K^i are special because of the Bott periodicity theorem, which gives a canonical equivalence between K^i and K^{i-2} for $i \leq 0$, and enables us to define K^i for all $i \in \mathbb{Z}$ by periodicity.

There is a completely different reason, however, unrelated to Bott periodicity, why the functor K^0 forms part of a cohomology theory, and it applies in a much more general context. For any (discrete) ring A we have a contravariant functor $X \mapsto \mathrm{Mod}_A(X)$, where $\mathrm{Mod}_A(X)$ is the semigroup of isomorphism classes of bundles of finitely generated projective A-modules on X. It is a representable homotopy functor, though not a very interesting one, as it sees only the fundamental group of X. But if, instead of making the semigroup $\mathrm{Mod}_A(X)$ into a group separately for each space X, we perform the group-completion on the representing space, i.e. we look for the universal abelian-group-valued representable homotopy functor F with a transformation $\mathrm{Mod}_A \to F$, then we obtain a much more interesting functor K^0_A. In fact

$$K^0_A(X) = [X; \Omega B|\mathcal{P}_A|].$$

Here $|\mathcal{P}_A|$, the "space" of the category $\mathcal{P}_A$ of finitely generated projective A-modules, which is the representing space for $\mathrm{Mod}_A(X)$, is a topological semigroup under $\oplus$, and $\Omega B|\mathcal{P}_A|$ is the loop-space of the classifying space of $|\mathcal{P}_A|$. The remarkable thing is that for any category $\mathcal{C}$ with a composition law which, like $\oplus$, is commutative up to coherent canonical isomorphisms we can iterate the classifying space functor $|\mathcal{C}| \mapsto B|\mathcal{C}|$, and can define a cohomology theory by

$$K^i_{\mathcal{C}}(A) = [X; \Omega B^{i+1}|\mathcal{C}|].$$

When $\mathcal{C} = \mathcal{P}_A$ this is Quillen's algebraic K-theory for the ring A, and when $\mathcal{C}$ is the category of finite sets under disjoint union then we get stable cohomotopy theory [S1]. The theories $K^*_{\mathcal{C}}$ are the first basic class of cohomology theories. Classical cohomology with coefficients in an abelian group A is the case coming from the category whose objects are the elements of A, and which has no morphisms except identity morphisms.

The second basic class of cohomology theories are cobordism theories. They arise as homology rather than cohomology theories.[1] The basic example is oriented bordism, where $h_n(X)$ is defined as the cobordism classes of maps $\phi : M \to X$, where M is a compact oriented smooth n-manifold. (A cobordism between (M_0, ϕ_0) and (M_1, ϕ_1) is a cobordism W between M_0 and M_1 with a map $\phi : W \to X$ restricting to ϕ_0 and ϕ_1 at the ends.) By considering manifolds M with various kinds of additional structure, or allowing manifolds with singularities of various specified kinds, we get a variety of different homology theories. For example, framed manifolds lead to stable homotopy theory, while classical integral homology corresponds to allowing singular manifolds with arbitrary singularities of codimension two. The example we shall be interested in is *complex cobordism* MU_*, corresponding to manifolds with weakly almost-complex structure.

A noticeable difference between cobordism theories and K-theories is that, with the former, classes of any degree n are equally well represented geometrically, not just those of degree 0. For any cobordism theory h^* there is an appropriate space $\mathcal{M}_n$ of n-manifolds such that

$$h^{-n}(X) = [X; \mathcal{M}_n],$$

and $\mathcal{M}_n$ has an interpretation even for $n < 0$.

As far as I know, the two classes of theories just mentioned exhaust the known "natural" cohomology theories. But there are other classes of theories which can be constructed from them algebraically. The most important of these are complex-orientable theories.

Definition 1.1. *A multiplicative*[2] *theory h^* is complex-oriented if there is given an element $c_1 \in h^2(\mathbb{P}^\infty_{\mathbb{C}})$ which restricts to the canonical generator of $\tilde{h}^2(S^2) \cong h^0(\text{point})$. (Here $S^2 = \mathbb{P}^1_{\mathbb{C}} \subset \mathbb{P}^\infty_{\mathbb{C}}$.)*

For such a theory we have (see [Ad])

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[1] We can pass at will between homology and cohomology theories, but it is significant that it is the homology classes which have a straightforward geometric interpretation.

[2] A theory h^* is *multiplicative* if $h^*(X)$ is an anticommutative graded ring.

Proposition 1.2.

$$h^*(\mathbb{P}^\infty_\mathbb{C}) \cong A[[c_1]],$$

where $A = h^*(\text{point})$.

Because $\mathbb{P}^\infty_\mathbb{C}$ is an H-space (its composition-law representing the tensor product of line bundles) the ring $A[[c_1]]$ is a Hopf algebra, and the diagonal map $c_1 \mapsto m(c_1 \otimes 1, 1 \otimes c_1)$ is a formal group-law associated to h^*.

In a complex-oriented theory h^* there is a canonical Thom class for any complex vector bundle, and, in particular, a sequence of elements in $h^{2n}(MU_n)$ corresponding to a transformation $MU^* \to h^*$. This means that complex cobordism is universal among complex-oriented theories. Quillen proved that its formal group law is also universal, in the sense that a law over any graded ring R comes from that of MU^* by a ring-homomorphism

$$A_{MU} = MU^*(\text{point}) \to R.$$

Alternatively expressed, a formal group-law over R is the same thing as a *genus* for weakly almost-complex manifolds, for a genus is exactly such a homomorphism.

Elliptic cohomology was conceived because of the discovery of the *elliptic genus* — actually, of the remarkable rigidity properties (see [L],[S2]) of a particular family of genera $\Phi_\tau : A_{MU} \to \mathbb{C}$ parametrized by elliptic curves $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. The Φ_τ can be assembled into $\Phi : A_{MU} \to R$, where R is a ring of modular forms. Landweber observed that if we take $R = \mathbb{Z}[\frac{1}{2}, \delta, \varepsilon, \Delta^{-1}]$, where $\Delta = \varepsilon(\delta^2 - \varepsilon)^2$, and δ and ε are the functions of the curve Σ_τ which arise when its equation is written in the form

$$y^2 = 1 - 2\delta x^2 + \varepsilon x^4,$$

then

$$\text{Ell}^*(X) = MU^*(X) \otimes_{A_{MU}} R$$

satisfies conditions that he had previously found which ensure that the functor $MU^*(\ \ ) \otimes_{A_{MU}} R$ is a cohomology theory. This was the original definition of elliptic cohomology. Since its proposal a great deal of work — especially by Hopkins [H] and his collaborators — has been devoted to finding an improved version, which ought not to require inverting the prime 2. It is now believed that the "correct" theory, which Hopkins calls tmf*, is not, in fact, quite complex-orientable, but that a tmf*-orientation of a manifold M should be a *string structure* on M in the sense described below. The coefficient ring tmf*(point) maps to the ring $\mathcal{M}_\mathbb{Z}$ of integral modular forms (i.e. modular forms whose expansion in terms of $q = e^{2\pi i\tau}$ lies in $\mathbb{Z}[[q]]$). If we tensor with the rational numbers $\mathbb{Q}$ then $\text{tmf}^*(\text{point}) \to \mathcal{M}_\mathbb{Z}$ becomes an isomorphism, but tmf*(point) is a much more subtle and complicated ring than $\mathcal{M}_\mathbb{Z}$, with a great deal of torsion. An n-manifold with a string structure has a genus in $\text{tmf}^{-n}(\text{point})$ — the image of the fundamental class $[M]$ under the map induced by $M \to$ (point) — and the image of this genus in $\mathcal{M}_{\mathbb{Z}}$ is the *Witten genus*.

2. Elliptic objects

It is a very natural question whether the elliptic cohomology classes of a space have geometric representatives in the way that K-theory classes are represented by vector bundles. The main evidence that this may be true is Witten's heuristic argument for the rigidity of the elliptic genus. The essential idea is that the elliptic genus of a compact manifold M is the Hilbert series $\Sigma q^k \dim(V_k)$ of a (virtual) graded vector space $V = \oplus V_k$ which is the index

$$\ker(\mathcal{D}_{\mathcal{L}M}) - \operatorname{coker}(\mathcal{D}_{\mathcal{L}M})$$

of a Dirac-like differential operator $\mathcal{D}_{\mathcal{L}M}$ defined not on M but on its smooth loop-space $\mathcal{L}M$. The grading on the index comes from the action of the circle $\mathbb{T}$ on $\mathcal{L}M$ by rotation of loops. This loop space perspective, however, does not by itself shed light on the modularity which is the basic property of the elliptic genus. The place where one "naturally" encounters graded vector spaces whose Hilbert series are modular is two-dimensional conformal field theory.

Definition 2.1. [3] *A conformal field theory is a Hilbert space $\mathcal{H}$ together with a trace-class operator*

$$U_{\Sigma,\xi} : \mathcal{H}^{\otimes p} \to \mathcal{H}^{\otimes q}$$

associated to each pair (Σ, ξ), where Σ is a Riemann surface which is a cobordism from an "incoming" manifold S_p consisting of p parametrized circles to a similar "outgoing" manifold S_q, and ξ is a point of the Quillen determinant line Det_Σ of the $\bar{\partial}$-operator of Σ.

The essential properties the operators $U_{\Sigma,\xi}$ are required to satisfy are

(i) $U_{\Sigma',\xi'} \circ U_{\Sigma,\xi} = U_{\Sigma' \cup \Sigma, \xi' \otimes \xi}$

when the cobordisms Σ and Σ' are concatenated, and

(ii) $U_{\Sigma,\xi} \otimes U_{\Sigma',\xi'} = U_{\Sigma \sqcup \Sigma', \xi \otimes \xi'}$

when the cobordisms are simply put side-by-side.

If $U_{\Sigma,\xi}$ depends holomorphically on (Σ, ξ) then the theory is called *chiral*. If, furthermore, we have

$$U_{\Sigma,\lambda\xi} = \lambda^m U_{\Sigma,\xi}$$

for $\lambda \in \mathbb{C}^\times$ then we say the theory is of *level* m. If, finally, we have $U_{\Sigma,\xi} = U^*_{\Sigma^*,\xi}$ for all (Σ, ξ), where Σ^* is Σ with the conjugate complex structure, then the theory is called *unitary*.

If the surface Σ is closed then $U_{\Sigma,\xi}$ is simply a complex number. For a chiral theory of level m, the restriction of $U_{\Sigma,\xi}$ to closed surfaces of genus 1 is a modular form of level m.

[3] For more details, see [S3].

Among the cobordisms from S_1 to S_1 there is a sub-semigroup formed by the annuli

$$A_q = \{z \in \mathbb{C} : |q| \leq |z| \leq 1\}$$

for $0 \leq |q| \leq 1$, with the boundary circles parametrized by

$$\theta \mapsto e^{i\theta}, \theta \mapsto qe^{i\theta}.$$

Its action on the Hilbert space $\mathcal{H}$ of a chiral theory gives $\mathcal{H}$ a grading $\mathcal{H} = \oplus_{k\geq 0}\mathcal{H}_k$ by finite dimensional subspaces.

Proposition 2.2. *If Σ_q is the torus $\mathbb{C}^\times/q^{\mathbb{Z}}$, then*

$$U_{\Sigma_q,\xi_q} = \Sigma q^k \dim(\mathcal{H}_k)$$

for any chiral theory of level m, where ξ_q is the canonical element[4] *of Det_{Σ_q} coming from the annulus A_q.*

This is easily proved by regarding the cobordism Σ_q as the composite of two annuli A_{q_1}, A_{q_2} with $q_1 q_2 = q$, but I shall not give the details here .

As chiral conformal field theories give us modular forms so naturally, we might first guess that elliptic cohomology is a K-theory made from bundles of field theories. The crudest approximation to an elliptic class is simply a graded complex vector bundle, and there is indeed a forgetful transformation

$$\mathrm{tmf}^*(X) \to K^*(X)[[q]]$$

corresponding to the q-expansion of a modular form. Nevertheless, to get further we must remember that the elliptic genus is the index of an operator not on X but on $\mathcal{L}X$. We need the notion of a conformal field theory *over* X. There is no loss in assuming that X is a smooth manifold.

Definition 2.3. *A conformal field theory over X is a rule which assigns a vector space $\mathcal{H}_\gamma$ to each smooth loop $\gamma : S^1 \to X$, and an operator*

$$U_{\Gamma,\xi} : \mathcal{H}_{\gamma_1} \otimes \ldots \mathcal{H}_{\gamma_p} \to \mathcal{H}_{\gamma_{p+1}} \otimes \ldots \mathcal{H}_{\gamma_{p+q}}$$

to each Riemann surface Σ which is a cobordism from the "incoming" loops $\gamma_1, \ldots, \gamma_p$ to the "outgoing" loops $\gamma_{p+1}, \ldots, \gamma_{p+q}$, and is equipped with a map $\Gamma : \Sigma \to X$. As before, $\xi \in \mathrm{Det}_\Sigma$, and $U_{\Gamma,\xi}$ must have the properties (i) and (ii) of Definition 2.1.

The basic example which motivates Definition 2.2 is the bundle of spinors on the loop space of an oriented Riemannian n-manifold M. The tangent bundle of $\mathcal{L}M$ has structure group $\mathcal{L}SO_n$, and this group has a projective unitary representation (see [PS] Chap.12) which is naturally regarded as its

[4] See [S3] §6. The element ξ_q differs from the "more canonical" element of Det_{Σ_q} which is unique only up to a 12^{th} root of unity by multiplication by the square of the Dedekind η-function.

"spin" representation.[5] The condition that we can make a Hilbert space bundle on $\mathcal{L}M$ associated to its tangent bundle by this projective representation is the vanishing of the characteristic classes w_2 and $\frac{1}{2}p_1$ of M. The choice of such a spinor bundle on $\mathcal{L}M$ is called a *string structure* on M. A string structure automatically extends — using the Riemannian metric of M — to a conformal field theory of level n over M. The propagation operators $U_{\Gamma,\xi}$ form a kind of connection in the spinor bundle, though it should be remembered that even when Σ is a cylinder, i.e. $\Gamma : \Sigma \to M$ is a path in $\mathcal{L}M$, the operator $U_{\Gamma,\xi}$ is a contraction operator, not a unitary isomorphism.

In my talk [S2] I speculated whether a conformal field theory over X of level m defines a class in $\mathrm{Ell}^{-m}(X)$. As far as I know, the question is still open. If something of the kind really is true then it seems to me quite remarkable, for, apart from cobordism theories, the only situations I know where we have geometric representatives for cohomology classes of all dimensions are real and complex K-theory (in virtue of Bott periodicity), and classical de Rham theory for smooth manifolds. The main evidence in support of the idea is that a level m theory over a compact $2n$-manifold X with a string structure can be "integrated" to give a virtual conformal field theory of level $m+2n$, and hence a modular form in $\mathrm{Ell}^{-m-2n}(\text{point})$: the integration process is tensoring the theory with the Dirac operator on $\mathcal{L}X$ and forming the index of the resulting coupled operator.

In the language of quantum field theory the Dirac operator in the spinor bundle on $\mathcal{L}M$ is a supersymmetry operator. To explain what this means we must first recall two more aspects of the formalism of conformal field theory. First, the group $\mathrm{Diff}(S^1)$ of diffeomorphisms of S^1 acts on the Hilbert space $\mathcal{H}$ of a conformal theory, and the action of annuli extends the action of its Lie algebra $\mathrm{Vect}(S^1)$ to the complexification, giving us a map

$$L : \mathrm{Vect}_{\mathbb{C}}(S^1) \to \mathrm{End}(\mathcal{H}).$$

In the case of a chiral theory the map L is complex-linear, but in general we write $L = L^+ + L^-$, where L^+ is $\mathbb{C}$-linear and L^--antilinear. The maps L^+ and L^- define commuting (projective) actions of $\mathrm{Vect}_{\mathbb{C}}(S^1)$ on $\mathcal{H}$.

The second point is that $\mathrm{Vect}_{\mathbb{C}}(S^1)$ is the even part of a Lie superalgebra $\mathbb{V}(S^1)$ whose odd part is the space $\Omega^{-\frac{1}{2}}(S^1)$ of $(-\frac{1}{2})$-forms on S^1 (two of which can be multiplied pointwise to give a vector field). The class of conformal field theories which are "half-supersymmetric" in the sense that there is given a $\mathbb{C}$-antilinear action on $\mathbb{V}(S^1)$ extending the L^--action of $\mathrm{Vect}_{\mathbb{C}}(S^1)$ is important for elliptic cohomology, for the action of the odd element $(d\theta)^{-\frac{1}{2}}$ of $\mathbb{V}(S^1)$ on

[5] More precisely, there is a positive energy and a negative energy spin representation, differing by changing the orientation of the circle. We actually want the negative energy choice, which makes the theory *antichiral*, in the sense that the operators depend antiholomorphically on the complex structure of the surface.

$\mathcal{H}$ has an index which is a virtual chiral conformal field theory. Indeed the space of these half-supersymmetric theories seems to be the correct model of the space of virtual chiral conformal theories, just as the space of Fredholm operators is the best model of the space of virtual finite-dimensional vector spaces.

When we have a string structure $\{\mathcal{H}_\gamma\}$ on a manifold M the Dirac operator acts — in principle — on the Hilbert space $\mathcal{H}$ of sections of the spinor bundle $\{\mathcal{H}_\gamma\}$ over $\mathcal{L}M$ which are square-summable for a measure on $\mathcal{L}M$ which forces them to be concentrated in an extremely small neighbourhood of the point loops. The group $\mathrm{Diff}(S^1)$ acts on $\mathcal{L}M$, and the bundle $\{\mathcal{H}_\gamma\}$ is equivariant with respect to it, so we expect $\mathrm{Diff}(S^1)$ to act on $\mathcal{H}$. One would like to say that this action is part of a conformal field theory structure on $\mathcal{H}$ which is half-supersymmetric in the above sense: the Dirac operator should be the action on $\mathcal{H}$ on the element $(d\theta)^{-\frac{1}{2}}$ of the superalgebra. In fact that is too much to hope for; but as far as homotopy theory is concerned one can proceed much more formally, replacing the space of sections of the bundle $\{\mathcal{H}_\gamma\}$ on $\mathcal{L}M$ by the space $\mathcal{H}$ of *jets* of sections along the subspace M of point loops.[6] On this Hilbert space $\mathcal{H}$ one can much more plausibly define the half-supersymmetric conformal field theory structure, as I have attempted to sketch in [S2]. We can do this even after tensoring the spinor bundle with an arbitrary chiral conformal field theory over M as defined in 2.3. This is the "integration" operation referred to above. The Dirac operator itself is mapped to the Witten genus, while the original elliptic genus is the image of the Dirac operator tensored with the chiral — rather than antichiral — spinor bundle. If we could do this for a *family* of manifolds M rather than just a single one then we should have related the space of level $m+n$ conformal field theories to the n-fold loop space of the space of level m theories.

Unfortunately, one could not expect to use conformal field theories over X by themselves to define $\mathrm{Ell}^*(X)$. The essential reason is that, like the loop space $\mathcal{L}X$, they are not defined *locally* on X, and so do not have the basic Mayer-Vietoris property of a cohomology theory. Another disconcerting fact is that a chiral conformal field theory is a rigid object which is not determined up to isomorphism by its modular form (e.g. an even unimodular lattice of rank k gives rise to a conformal field theory of level k, and non-isomorphic lattices with the same modular form give non-isomorphic theories). It is plausible, however, that chiral theories with the same modular form are connected in the space of half-supersymmetric theories.

Before describing the recent progress in understanding elliptic objects it seems worth mentioning one other hint — first pointed out by Grojnowski

[6] I should mention at this point a large body of work by Gorbounov, Malikov, Schechtman, and Vaintrob, e.g. [MSV],[GMS], who have developed a notion of "chiral de Rham complex" defined on the formal neighbourhood of M in $\mathcal{L}M$.

[G] — that the field theory approach may be on the right track. This is the question of G-equivariant elliptic cohomology for a compact group G: we know that the definition of equivariant K-theory in terms of G-vector-bundles is one of the things that makes K-theory such a useful tool. There is a general notion of a G-equivariant quantum field theory — in physicists' language, a "theory with gauged G-symmetry" — which in the present situation reduces to the following.

Definition 2.4. *A two-dimensional G-equivariant chiral conformal field theory assigns a Hilbert space $\mathcal{H}_{S,P}$ to each oriented 1-manifold S equipped with a principal G-bundle P, and an operator*

$$U_{\Sigma,Q,\xi} : \mathcal{H}_{S_0,P_0} \to \mathcal{H}_{S_1,P_1}$$

to each conformal cobordism Σ from S_0 to S_1 equipped with a holomorphic $G_{\mathbb{C}}$-bundle reducing to P_0 and P_1 at the ends. The operator should have properties analogous to those of definition 2.1, and should depend holomorphically on the pair (Σ, Q). As usual, $\xi \in \mathrm{Det}_{\Sigma,Q}$.

When G is a connected group this means that $\mathcal{H}_{S,P}$ is a positive-energy projective representation of the loop group $\mathcal{L}G$. The attractive idea that $\mathrm{Ell}^*_G(\text{point})$ should be some kind of representation ring of $\mathcal{L}G$ has been pursued further by Devoto [Dv1] and Ando [An], but for lack of a satisfactory equivariant version of Landweber's theorem there is still no real candidate for the equivariant elliptic theory Ell^*_G. For finite groups G the field theory point of view seems to fit with what is known ([HKR],[Dv2]) about $\mathrm{Ell}^*(BG)$, but again I know no definite theorem.

The work of Stolz and Teichner [ST] has moved the idea of an elliptic object forward in several important ways. One is their focus on the space of the half-supersymmetric theories already mentioned. But their main contribution concerns the Mayer-Vietoris property. The problem with the definition 2.1 of a conformal field theory is that it does not incorporate any sense in which the Hilbert space $\mathcal{H}$ associated to the circle S^1 is local with respect to S^1. If $\mathcal{H}$ could be reconstructed from objects $\mathcal{H}_I$ associated to small subintervals I of S^1 then we might be able to think of a conformal field theory over X as a local object on X, and could hope to construct a cohomology theory.

The simplest sense in which $\mathcal{H}$ could be local would be if one could associate a Hilbert space $\mathcal{H}_I$ to each closed subinterval I of the circle so that $\mathcal{H} \cong \otimes\mathcal{H}_{I_i}$ when the circle is the union of intervals I_i meeting only at their ends. Locality of this simple kind — which would hold, for example, if $\mathcal{H}$ were the symmetric or exterior algebra on $L^2(S^1)$ — is easily seen to be impossible in conformal field theory. In the simplest conformal field theories, the space $\mathcal{H}$ is a "renormalized" symmetric or exterior algebra on a space of functions such as $L^2(S^1)$, where the renormalization depends on a *polarization*

$$L^2(S^1) = L^2(S^1)^+ \oplus L^2(S^1)^-$$

of $L^2(S^1)$ into positive and negative frequency parts. The projection operators defining the polarization need to be given only up to Hilbert-Schmidt perturbations: they are singular integral operators on S^1 with kernels whose supports can be chosen in an arbitrarily small neighbourhood of the diagonal in $S^1 \times S^1$, but which cannot be supported exactly *on* the diagonal. The effect for the locality of $\mathcal{H}$ is that if $S^1 = I \cup J$, where I and J are *open* intervals, then $\mathcal{H}$ can be reconstructed from Hilbert spaces $\mathcal{H}_I, \mathcal{H}_J$ and a von Neumann algebra $A_{I\cap J}$ associated to $I \cap J$ which acts on both $\mathcal{H}_I$ and $\mathcal{H}_J$. The reconstruction is by means of Connes's notion of the tensor product of bimodules over von Neumann algebras. If A, B, C are von Neumann algebras, M is an (A, B)-bimodule, and N is a (B, C)-bimodule, then Connes defines a (A, C)-bimodule $M *_B N$. Two important features of his theory are the existence of a neutral B-bimodule B_0 with the property that $M *_B B_0 \cong M$ and $B_0 *_B N \cong N$, and the fact that a (B, B)-bimodule gives us a Hilbert space $M{*_B} = M *_{(B^{\mathrm{op}} \otimes B)} B_0$.[7]

The relevance of the Connes tensor product to the locality of loop group representations, and hence to two-dimensional conformal field theory, was first realized by Wassermann [W]. In the light of his work the following definition — essentially that of Stolz and Teichner — seems appropriate.

Definition 2.5. *A three-tier conformal field theory over X consists of the data of Definition 2.2 together with*

(i) *a bundle of von Neumann algebras $\{A_x\}_{x\in X}$ on X, and*

(ii) *an (A_x, A_y)-bimodule $\mathcal{H}_\gamma$ for each path γ from x to y.*

The properties the bimodules must have are that

$$\mathcal{H}_\gamma \cong \mathcal{H}_{\gamma_2} *_{A_z} \mathcal{H}_{\gamma_1}$$

if the path γ from x to y is the concatenation of γ_1 from x to z and γ_2 from z to y, and that

$$\mathcal{H}_{\gamma_0} \cong \mathcal{H}_\gamma *_{A_x}$$

if the path γ from x to x is regarded as a closed path γ_0.

One can presumably construct a cohomology theory based on 3-tier conformal field theories of any chosen level, but, as far as I know, little has yet been proved, especially about why the theories at different levels should be related by suspension.

Apart from the Stolz-Teichner programme there is another quantum field theory approach to elliptic cohomology which has been proposed by Baas, Dundas, and Rognes [BDR]. In one important sense it is much less ambitious: it aims only to construct elliptic objects of degree zero, relying on the

[7] I have taken the notation $M{*_B}$ from Quillen, who uses it in an algebraic setting for the quotient of M which equalizes the left and right B-actions.

machinery of algebraic K-theory to produce the theory in other dimensions. I shall try to say briefly how it fits in to the general philosophy of this talk.

We can give a definition of a general d-dimensional quantum field theory along the lines of 2.1, but using manifolds equipped with a Riemannian rather than just a conformal structure. (Of course we shall have a Hilbert space $\mathcal{H}_S$ assigned to each compact oriented Riemannian $(d-1)$-manifold S, subject to $\mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2} \cong \mathcal{H}_{S_1 \sqcup S_2}$.) Among these theories are the conformal ones, and — much more specially still — the so-called *topological* field theories, for which the vector spaces and the operators depend only on the smooth structure of the manifolds, without any metric at all. On the space of all quantum field theories we have the *renormalization group flow*: a theory is a functor from a cobordism category to vector spaces, and for any $t > 0$ we can compose the theory with the functor from the cobordism category to itself which multiplies the metric of every manifold by t.

When $d = 1$ a quantum field theory is precisely a semigroup of trace-class operators in a Hilbert space — self-adjoint operators if the theory is unitary. The renormalization group flow retracts the space of 1-dimensional theories (with its natural topology) to the subspace of topological theories, which is simply the space of finite-dimensional complex vector spaces.

We can also define *supersymmetric* unitary 1-dimensional theories. Such a theory is a mod 2 graded Hilbert space with a trace-class semigroup whose generator is given as the square of a self-adjoint operator of degree 1. Up to homotopy, this is the space of Fredholm operators $\mathbb{Z} \times BU$, i.e. the representing space for K-theory.

When $d = 2$ we can not assume that the space of quantum field theories is homotopy equivalent to the space of topological theories. It nevertheless seems interesting to consider the space of 2-dimensional topological theories, and better, in the light of the discussion above, the space of "3-tier" unitary topological theories. The general definition of a 3-tier d-dimensional quantum field theory — of which Definition 2.5 is a specialization — is as a structure that assigns

(i) a linear category $\mathcal{C}_Z$ to each closed $(d-2)$-manifold Z,

(ii) a functor $F_Y : \mathcal{C}_{Z_0} \to \mathcal{C}_{Z_1}$ to each $(d-1)$-dimensional cobordism from Z_0 to Z_1, and

(iii) a transformation of functors $U_X : F_{Y_0} \to F_{Y_1}$ to each d-dimensional cobordism X between cobordisms Y_0 and Y_1 from Z_0 to Z_1.

These data must satisfy natural conditions which I shall not spell out. (There is a discussion of the 3-dimensional case in Lecture 3 of [S4].) In Definition 2.5 the category associated to a point x is the category of modules for the von Neumann algebra A_x, and the functors are defined in the usual way by bimodules. Now in the topological 2-dimensional case we expect the whole structure to be determined by the category assigned to a point, so that a theory reduces to a semisimple $\mathbb{C}$-linear category, i.e. a "module" over the category of finite dimensional vector spaces, or, in the language of [BDR] a "two vector space". These objects define module spectra over the complex K-theory spectrum, and the K-theory of that ring-spectrum is the elliptic theory proposed in [BDR].

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