Elliptic Cohomology

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The term *elliptic cohomology* was first introduced in 1986 by P. S. Landweber, D. C. Ravenel, and R. E. Stong (cf. [2] and [3]) to designate a cohomology theory obtained by tensoring the oriented cobordism theory with a ring of modular forms in characteristic $\neq 2$. In the past decade numerous publications devoted to elliptic cohomology have appeared. On the one hand, they connect elliptic cohomology to other generalized cohomology theories, most notably to Morava's K-theories, on the other hand they attempt to give a geometric interpretation of elliptic cohomology. Despite a multitude of very interesting results, there seems to be no agreement on what exactly is *the* elliptic cohomology. The situation is however quite well-understood *outside the prime* 2. In what follows we describe the Landweber-Ravenel-Stong theory.

Let $\Omega_*^{\mathrm{SO}}(\ )$ be the oriented bordism theory with coefficient ring Ω_*^{SO}, and let

$$\varphi : \Omega_*^{\mathrm{SO}} \longrightarrow \mathbf{Z}[\tfrac{1}{2}, \delta, \varepsilon]$$

be the level 2 elliptic genus (ref. to the "Elliptic Genera" art. in this Encyclopædia) with logarithm

$$g(z) = \int_0^z \frac{dt}{\sqrt{1 - 2\delta t^2 + \varepsilon t^4}}.$$

The ring $\mathcal{M}_* \equiv \mathbf{Z}[\frac{1}{2}, \delta, \varepsilon]$ is a ring of level 2 modular forms in characteristic $\neq 2$ and can be viewed, via φ, as an Ω_*^{SO}-module. It also has a natural grading, for which $\deg \delta = 2$, $\deg \varepsilon = 4$.

Theorem. *Let $\pi \in \mathcal{M}_*$ be any homogeneous element of positive degree. Then the functor*

$$X \longmapsto \Omega_*^{\mathrm{SO}}(X) \otimes_{\Omega_*^{\mathrm{SO}}} \mathcal{M}_*[\pi^{-1}] \tag{1}$$

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is a periodic homology theory (elliptic homology) with coefficient ring $\mathcal{M}_*[\pi^{-1}]$.

A similar construction using oriented cohomology leads to a multiplicative periodic cohomology theory (elliptic cohomology). The proof of this theorem was first given by Landweber, Ravenel, and Stong under the assumption that π is one of the factors of $\Delta = \varepsilon(\delta^2 - \varepsilon)^2$, and was based on the one hand on Landweber's Exact Functor Theorem, and on the other hand on interesting congruences for Legendre polynomials. The general form stated above is due to Jens Franke [1] who showed that the exactness of the functor (1) is a consequence of a theorem of Deuring and Eichler saying that the height of the formal group of an elliptic curve in positive characteristic is always 1 or 2.

References

[1] Jens Franke. On the construction of elliptic cohomology. *Math. Nachr.*, 158:43–65, 1992.

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[3] Peter S. Landweber, Douglas C. Ravenel, and Robert E. Stong. Periodic cohomology theories defined by elliptic curves. In *The Čech Centennial (Boston, MA, 1993)*, Contemp. Math., 181. Amer. Math. Soc., Providence, RI, 1995.