On string topology of three manifolds

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Abstract

Let $M$ be a closed, oriented and smooth manifold of dimension $d$. Let $LM$ be the space of smooth loops in $M$. In [String topology, preprint math.GT/9911159] Chas and Sullivan introduced the loop product, a product of degree $-d$ on the homology of $LM$. We aim at identifying the three manifolds with “nontrivial” loop product. This is an application of some existing powerful tools in three-dimensional topology such as the prime decomposition, torus decomposition, Seifert fiber space theorem, torus theorem.

Key words: Free loop space; Prime decomposition; JSJ-decomposition

1. Introduction and statement of main theorem

Throughout this paper $M$ denotes a connected, oriented smooth manifold unless otherwise stated. We think of $S^1$ the unit circle with a marked point, as the quotient $\mathbb{R}/\mathbb{Z}$. The marked point of $S^1$ is the image of 0 in this quotient. A loop in $M$ is a continuous map from $S^1$ to $M$. The free loop space $LM$ of $M$ is the space of all loops in $M$. There is a one-to-one correspondence between the connected components of $LM$ and the conjugacy classes of $\pi_1(M)$. If $f : S^1 \rightarrow M$ is a loop in $M$ then the image of the marked point of $S^1$ is called the marked point of the loop $f$. The integral homology of $LM$ is equipped with the loop product, $\bullet$, an associative product of degree $-d$, where $d$ is the dimension of $M$,

$$\bullet : H_i(LM) \otimes H_j(LM) \rightarrow H_{i+j-d}(LM).$$

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We give an informal description of the loop product: given two homology classes in $H_s(LM)$, choose a chain representative for each one of them so that the corresponding sets of marked points intersect transversally. By concatenating the corresponding loops of each chain at the intersection points one obtains a new chain in $LM$. We refer the reader to [6] for a homotopy theoretic definition of the loop product.

**Theorem (Chas–Sullivan [5]).** For a connected oriented smooth manifold $M$, $(H^{−d}(LM), \bullet)$ is a graded commutative algebra.

Let $p : H_*(LM) \to H_*(M)$ be the map induced by $f \mapsto f(0)$ and $i : H_*(M) \to H_*(LM)$ be the map induced by the inclusion of the constant loops. Observe that

$$p \circ i = id_{H_*(M)},$$

hence there is a canonical decomposition $H_*(LM) = i(H_*(M)) \oplus A_M$ where $A_M = \text{Ker } p$. The projection on $A_M$ is denoted by $p_{AM}$ and $p_{AM} = id_{H_*(LM)} - p$. From now on we identify $H_*(M)$ with its image under $i$.

**Proposition 1.1 (see Chas and Sullivan [5]).** $i : H_*(M) \to H_*(LM)$ is a map of graded algebras where the multiplication on $H_*(M)$ is the intersection product denoted by $\land$. Therefore $(H_*(LM), \bullet)$ is an extension of $(H_*(M), \land)$.

If $M$ is a closed manifold then the algebra $(H_*(LM), \bullet)$ has a unit, $\mu_M$, which is the image of the fundamental class of $M$ under $i$.

**Definition 1.2.** We shall say $M$ has nontrivial extended loop products if the restriction of the loop product to $A_M$ is nontrivial. Otherwise we say that $M$ has trivial extended loop products.

**Definition 1.3.** A closed oriented 3-manifold $M$ is said to be algebraically hyperbolic if it is a $K(\pi, 1)$ (aspherical) and its fundamental group has no rank 2 abelian subgroup.

Note that every finite cover of an algebraically hyperbolic manifold is also algebraically hyperbolic.

The following is the main theorem of this paper [1]:

**Theorem 1.4.** Let $M$ be an oriented closed 3-manifold. If $M$ is not algebraically hyperbolic then $M$ or a double cover of $M$ has nontrivial extended loop products.

If $M$ is algebraically hyperbolic then $M$ and all its finite covers have trivial extended loop products.

**Remark 1.5.** Although the main result as stated above concerns the closed 3-manifolds, throughout this paper we identify several classes of 3-manifolds with boundary or which have nontrivial extended loop products (see Sections 3, 5 and 6.1).

**Remark 1.6.** As we shall see many of our arguments can be generalized to higher-dimensional manifolds. We have pointed them out with some remarks in Sections 3, 5 and 6. Also we invite the reader to see Example 1.8.
Notation: The based loop space of $M$ is denoted $\Omega M$ and $\hat{\pi}_1(M)$ is the set of conjugacy classes of $\pi_1(M)$. For $x \in \pi_1(M)$, $C_x$ is the centralizer of $x$ in $\pi_1(M)$ and $[x]$ is its conjugacy class. For a conjugacy class $[\alpha]$, $(LM)_{[\alpha]}$ denotes the corresponding connected component of $LM$.

Lemma 1.7. For $x \in \pi_1(M)$ and $f \in LM$ a loop representing $x$, then there exists a short exact sequence

$$\pi_2(M) \cong \pi_1((\Omega M)) \longrightarrow \pi_1((LM)_{[\alpha]} f) \longrightarrow C_x \longrightarrow 0.$$  

(1.1)

Example 1.8 (see Chas and Sullivan [5]). Let $M$ be a closed oriented hyperbolic manifold of dimension $d \geq 3$. Since $M$ is aspherical, it follows from the long exact sequence associated with the fibration $\Omega M \to LM \xrightarrow{p} M$ that each connected component of $LM$ is also aspherical. In order to find the homotopy type of each component $(LM)_{[\alpha]}$, one has to compute the fundamental group of each component. As $\pi_2(M) = 0$, by Lemma 1.7, each component $(LM)_{[\alpha]}$ is a $K(C_x, 1)$. If $x \neq 1 \in \pi_1(M)$ then $C_x \simeq \mathbb{Z}$ as $M$ is a closed hyperbolic manifold. Hence $(LM)_{[\alpha]}$ is a $K(\mathbb{Z}, 1)$ and

$$A_M \simeq H_*(\bigcap_{[\alpha] \neq [1]} K(\mathbb{Z}, 1)).$$

Therefore the restriction of the loop product to $A_M$ is identically zero as $A_M$ is concentrated at degree at most 1.

2. 3-Manifolds with finite fundamental group

In this section, we show that closed 3-manifolds with finite fundamental group have nontrivial extended loop products.

The following construction is valid for any oriented manifold $M$. Let $p$ be the base point of $M$ of dimension $d$. Consider the 0-chain in $LM$ that consists of the constant loop at $p$. It represents a homology class $e \in H_0(LM)$. Given a chain $c \in LM$ whose set of marked points is transversal to $p$, then the loops in $c$ whose marked points are $p$, form a chain in the based loop space $\Omega M$. This induces a map of degree $-d$

$$\cap : H_*(LM) \to H_{*-d}(\Omega M),$$

where $d$ is the dimension of $M$. On the other hand $H_q(\Omega M)$ is equipped with the Pontrjagin product.

Lemma 2.1 (see Chas and Sullivan [5]). $\cap : (H_*(LM), \bullet) \to (H_{*-d}(\Omega M), \times)$ is an algebra map where $\times$ is the Pontrjagin product. In particular if $M$ is a Lie group then $\cap$ is surjective.

Proposition 2.2. $S^3$ has nontrivial extended loop products.

Proof. As $S^3$ is a Lie group, by Lemma 2.1 $\cap$ is surjective. $(H_*(\Omega S^3), \times)$ is a polynomial algebra with one generator of degree 2. Let $x_1 \in H_p(\Omega S^3)$ and $x_2 \in H_q(\Omega S^3)$ be two nonzero elements such that $p, q > 3$. Let $a_i \in \cap^{-1}(x_i)$ for $i = 1, 2$. Since $H_1(S^3) = 0$ for $i > 3$ we have $a_i \in A_{S^3}$. By Lemma 2.1, $\cap(a_1 \bullet a_2) = \cap(a_1) \times \cap(a_2) = x_1 \times x_2 \neq 0$, therefore $a_1 \bullet a_2 \neq 0$. $\square$
For an arbitrary homotopy 3-sphere we can obtain some examples of nontrivial loop product by transferring the examples found for 3-sphere in Proposition 2.2, using a suitable homotopy equivalence. The following two lemmata describe and prove the existence of such a homotopy equivalence.

**Lemma 2.3.** Let $M$ and $N$ be two closed oriented $d$-manifolds and $p \in N$. For $f : M \to N$ which is transversal to $p$ and $f^{-1}(p)$ consists of only one point, the following diagram commutes:

\[
\begin{array}{ccc}
H_*(LM) & \xrightarrow{\cap} & H_*(\Omega M) \\
|f_L| & & |f_\Omega| \\
H_*(LN) & \xrightarrow{\cap} & H_*(\Omega N)
\end{array}
\]

(2.1)

$f_\Omega$ and $f_L$ are the induced maps by $f$ on the homologies of the based loop space and the free loop space.

**Proof.** We choose $p$ and $f^{-1}(p)$, respectively, as the base point of $N$ and $M$. We show that (2.1) is commutative at the chain level. Let $k : \Delta^m \to LM$ be a $m$-simplex in $LM$. So for each $x \in \Delta^m$

\[
k(x) : \mathbb{S}^1 \to M.
\]

If $k_0 : \Delta^m \to M$,

\[
k_0(x) = k(x)(0),
\]

is transversal to $f^{-1}(p)$ then consider the set $\Lambda = \{x \in \Delta^m | k(x)(0) = f^{-1}(p)\}$.

$k|\Lambda$ is a chain in $\Omega_{f^{-1}(p)}M$ and $\cap(k) = k|\Lambda$. After composing with $f$, $f \circ k|\Lambda$ is a chain in $\Omega_p N$ and

\[
(f_\Omega \circ \cap)(k) = f \circ k|\Lambda.
\]

On the other hand $f_L(k) = f \circ k$ is a chain in $LN$. Since $k_0$ is transversal to $f^{-1}(p)$ and $f$ is transversal to $p$ thus $f \circ k_0$ is transversal to $p$. Since $f^{-1}(p)$ consists of only one point in $M$ then

\[
\{x \in \Delta^m | ((f \circ k)(x))(0) = p\} = \{x \in \Delta^m | k(x)(0) = f^{-1}(p)\} = \Lambda,
\]

therefore,

\[
(\cap \circ f_L)(k) = f \circ k|\Lambda = (f_\Omega \circ \cap)(k). \quad \square
\]

**Lemma 2.4.** Let $p_1, p_2, \ldots, p_r$ be $r$ distinct points in a homotopy 3-sphere $M$. Then there is a homotopy equivalence $f : S^3 \to M$ which is transversal to all $p_i$’s and $f^{-1}(p_i)$ consists of only one point for each $i$.

**Proof.** Consider a closed ball $D \subset S^3$. Let $f$ be a homeomorphism from $D$ to a closed ball in $D' \subset M$ where $p_i \in D'$ for all $i$. 
\( f|_{\partial D} : \partial D \to \partial D' \) can be extended to \( f : S^3 \setminus \text{Int} D' \to M \setminus \text{Int} D \) as all the obstructions which are cohomology classes in
\[
H^{q+1}(S^3 \setminus \text{Int} D', \partial D', \pi_q(M \setminus \text{Int} D))
\]
vanish since \( M \setminus \text{Int} D' \) is contractible.\(^1\)

Therefore, we have a map \( f : S^3 \to M \) which sends \( D \to D' \) homeomorphically and sends the complement of \( D \) to the complement of \( D' \) and in particular it is transversal to \( p_i \) and \( f^{-1}(p_i) \) consists of only one point in \( S^3 \). To see that \( f \) is a homotopy equivalence it is enough to observe that it has degree one. \( \square \)

**Proposition 2.5.** A closed simply connected 3-manifold \( M \) has nontrivial extended loop products.

**Proof.** Let \( p \in M \) and \( f : S^3 \to M \) be the homotopy equivalence provided by Lemma 2.4 for \( r = 1 \).

\( f \) induces the homotopy equivalences \( f_\Omega : \Omega S^3 \to \Omega M \) and \( f_L : LS^3 \to LM \). By Lemma 2.3 the following diagram is commutative:

\[
\begin{array}{ccc}
H_\ast(LS^3) & \overset{\cap S^3}{\longrightarrow} & H_\ast(\Omega f^{-1}(p)S^3) \\
\downarrow f_L & & \downarrow f_L \\
H_\ast(LM) & \overset{\cap M}{\longrightarrow} & H_\ast(\Omega pM)
\end{array}
\]  

\( (2.2) \)

Since \( f_\Omega \) and \( f_L \) are isomorphisms and \( \cap_{S^3} \) is surjective, therefore \( \cap_M \) is also surjective. Let \( x_1 \neq 0 \in H_\ast(\Omega M) \) and \( x_2 \neq 0 \in H_\ast(\Omega M) \) where \( n, m > 3 \) and \( a_i \in \cap_M^{-1}(x_i), i = 1, 2 \). Then, \( a_i \in A_M \) since \( H_k(M) = 0 \) for all \( k > 3 \). By Lemma 2.1,

\[ \cap(a_1 \cdot a_2) = \cap(a_1) \times \cap(a_2) = x_1 \times x_2 \neq 0, \]

hence \( a_1 \cdot a_2 \neq 0 \). \( \square \)

Now we construct some examples of nontrivial loop product for a 3-manifold with finite fundamental group. The homology classes are obtained by pushing forward the classes introduced in Proposition 2.5 for the universal cover.

**Proposition 2.6.** If \( M \) is a closed oriented 3-manifold with finite fundamental group then \( M \) has nontrivial extended loop products.

**Proof.** Let \( \tilde{M} \) be the universal cover of \( M \) and \( q \) be the covering map and \( r = \text{degree}(q) = |\pi_1(M)| < \infty \). Choose \( p \in M \) and let \( \{p_1, p_2, \ldots, p_r\} \in q^{-1}(p) \) where \( p_i \)’s are distinct.

Since \( \tilde{M} \) is homotopy equivalent to \( S^3 \), by Lemma 2.4 there is a homotopy equivalence \( f : S^3 \to \tilde{M} \) which is transversal at each \( p_i \) and \( f^{-1}(p_i) \) consists of only one point in \( S^3 \) in \( M \). Let \( m_i = f^{-1}(p_i), i = 1, 2, \ldots, r \). Hence \((q \circ f)^{-1}(p) = \{m_1, m_2, \ldots, m_r\} \) and \( q \circ f \) is transversal to \( p \). We choose \( m_1 \) as the base point of \( S^3 \).

\(^1\) This can be proved using Mayer–Vietrois exact sequence.

\(^2\) \( H_\ast(\Omega M) \cong H_\ast(\Omega S^3) \cong \mathbb{Z}[x] \) and degree \( x = 2 \).
The covering map \( q : \hat{M} \to M \) composed by \( f \) induces the maps

\[
(q \circ f)_L : H_\ast(\mathrm{LS}^3) \to H_\ast(LM)
\]

and

\[
(q \circ f)_\up{W} : H_\ast(\Omega_m, S^3) \to H_\ast(\Omega_q M),
\]

where the latter is an isomorphism.

We claim that

\[
\begin{array}{ccc}
H_\ast(\mathrm{LS}^3) & \xrightarrow{r \cap S^3} & H_\ast(\Omega_m, S^3) \\
(f \circ q)_L & \downarrow & (f \circ q)_\up{W} \\
H_\ast(LM) & \xrightarrow{\cap M} & H_\ast(\Omega_q M)
\end{array}
\]

(2.3)

is commutative. To prove this, note that \( S^3 \) has a group structure hence we have the homeomorphism \( \mathrm{LS}^3 \simeq S^3 \times \Omega S^3 \), and

\[
H_\ast(\mathrm{LS}^3) \simeq H_\ast(S^3) \otimes H_\ast(\Omega S^3).
\]

(2.4)

There are two types of homology classes in \( H_\ast(\mathrm{LS}^3) \) and we verify the commutativity for the both cases:

1. The classes that correspond to homology classes in \( H_0(S^3) \otimes H_\ast(\Omega S^3) \) under isomorphism (2.4). These classes can be represented by cycles in \( \mathrm{LS}^3 \) whose sets of marked points consist of a single point in \( S^3 \). Therefore they are mapped to 0 by \( \cap S^3 \), as we can choose a chain representative whose set of marked point, consisting of only one points, is different from all \( m_i \)’s. Therefore they are mapped to 0 by \( (f \circ q)_\up{W} \circ (r \cap S^3) \).

On the other hand their images under \( (f \circ q)_L \) have the set of marked point consisting of only one point which is different from \( p \). Thus, they are mapped to zero under \( \cap M \circ (f \circ q)_L \) and the diagram commutes.

2. The classes that by isomorphism (2.4) correspond to the homology classes in \( H_3(S^3) \otimes H_\ast(\Omega S^3) \). Such an element \( \theta \in H_\ast(\mathrm{LS}^3) \) corresponds to \( \mu \otimes b \in H_3(S^3) \otimes H_\ast(\Omega S^3) \) for some \( b \in H_\ast(\Omega S^3) \) and \( \mu \) is the fundamental class of \( S^3 \). Then

\[
\cap S^3(\theta) = b
\]

so

\[
(f \circ q)_\up{W} \circ (r \cap S^3 \theta) = r(f \circ q)_\up{W}(b)
\]

and this is exactly \( \cap M \circ (f \circ q)_L(b) \). Note that under \( f \circ q \), \( r \) copies of \( b \) which are based at each one of \( m_i \)’s, come together. Therefore \( \cap M \circ (f \circ q)_L(\theta) = r(f \circ q)_\up{W}(b) \).

To finish the proof, consider \( a_1, a_2 \in H_\ast(\mathrm{LS}^3) \) as constructed in the proof of Proposition 2.2. We have \( (q \circ f)_L(a_i) \in A_M, i = 1, 2 \) and from the commutativity of (2.3) and the fact that \( (q \circ f)_\up{W} \) is an isomorphism it follows that

\[
(q \circ f)_L(a_1) \bullet (q \circ f)_L(a_2) \neq 0.
\]

\( \Box \)
3. 3-Manifolds with non-separating 2-sphere or 2-torus

In this section, we prove the 3-manifolds with a non-separating sphere or 2-torus have nontrivial extended loop product. We recall that the action of the unit circle on $LM$ induces a map of degree 1, $\Delta : H_*(LM) \to H_{*+1}(LM)$.

**Proposition 3.1.** $S^1 \times S^2$ has nontrivial extended loop products.

**Proof.** Let $b$ and $p$ be two distinct points in $S^2$. We choose $(1, p)$ as the base point of $S^1 \times S^2$. The map $x \mapsto (x, p)$, $x \in \mathbb{S}^1$, gives rise to an element $\eta$ of $\pi_1(S^1 \times S^2)$.

Consider the map $\psi : \mathbb{S}^1 \to S^1 \times S^2$ defined by $\psi(x) = (x, b)$. Note that $\psi$ as a loop with the marked point $(1, b)$, represents a homology class $\Psi \in H_0((L(S^1 \times S^2))_{[\eta]})$.

Let $\phi : S^2 \to S^1 \times S^2$ be the map defined by $\phi(y) = (1, y)$. The images of $\psi$ and $\phi$ intersect exactly at $(1, b)$. We write $\phi(S^2)$ as a union of circles, any two of them having only the point $(1, p)$ in common. This gives rise to a one dimensional family of loops in $S^1 \times S^2$ (see Fig. 1). Note that the free homotopy type of the loops of this one-dimensional family is the one of the trivial loop. One can compose the loops of this family with a fixed loop whose marked point is $(1, p)$ and modify their free homotopy type. Suppose that we have done this modification with a fixed loop which does not meet $\psi$ and represents a nontrivial element $\mu \in \pi_1(S^1 \times S^2)$ where $\mu \neq \eta$. This new one-dimensional family of loops represents a homology class $\phi \in H_1((L(S^1 \times S^2))_{[\mu]})$.

We prove that $pA_{S^1 \times S^2}(\Delta \Psi) \bullet pA_{S^1 \times S^2}(\Delta \Phi) \neq 0$ which implies that $S^1 \times S^2$ has nontrivial loop products. Since $pA_{S^1 \times S^2}(\Delta \Psi) \bullet pA_{S^1 \times S^2}(\Delta \Phi)$ belongs to $H_0(L(S^1 \times S^2))$, it can be expressed as a sum of conjugacy classes with $+1$ or $-1$ as the coefficients. Indeed it equals $\pm [\eta \mu] \pm [\eta] \pm [\mu] \pm [1]$. Since 1, $\eta$ and $\mu$ are distinct therefore three terms out of four are distinct and hence there cannot be a complete cancellation. \[
\]

**Corollary 3.2.** An oriented 3-manifold with a non-separating 2-sphere has nontrivial extended loop products.

**Proof.** The proof is similar to the one of Proposition 3.1. The only property we used in the proof of Proposition 3.1 was that there was a non-separating two sided 2-sphere.\[\]

\[\]

\[\]

$^3$Two sided means that the normal bundle is trivial (see [9]).
Remark 3.3. Corollary 3.2 can be generalized to higher-dimensional manifolds with a nonseparating codimension one sphere. The proof is a direct generalization of the one of Corollary 3.2.

Proposition 3.4. An oriented 3-manifold \( M \) with a non-separating two sided incompressible\(^4\) torus has nontrivial extended loop products.

Proof. Let \( \phi : S^1 \times S^1 \to T \subset M \) be a homeomorphism. We set \( \phi(1, 1) \) as the base point of \( M \). Consider the one-dimensional family of loops \( \phi_t \) defined by \( \phi_t(s) = \phi(t, s) \) (longitudes of \( T \) in Fig. 2). This 1-family of loops represents a homology class \( \Phi \) in \( H_1((LM)_{[h]}) \), where \( h \) is the element of \( \pi_1(M) \) represented by \( \phi_1 \). Now consider a closed simple curve \( \psi : \mathbb{S}^1 \to M \) which meets \( T \) transversally at exactly one point \( \phi(1, 1) \). Note that \( \psi \) represents an element \( g \in \pi_1(M) \) and also gives rise to a homology class \( \Psi \in H_0((LM)_{[g]}) \).

We show that \( p_{A_M}(\Delta \Psi) \cdot p_{A_M}(\Delta \Phi) \neq 0 \in H_0(LM) \). Similar to \( S^1 \times S^2 \) we have \( p_{A_M}(\Delta \Psi) \cdot p_{A_M}(\Delta \Phi) = \pm [gh] \pm [h] \pm [g] \pm [1] \).

To prove the claim, it is sufficient to show that \([1], [h] \) and \([g] \) are distinct. Since \( T \) is \( \pi_1 \)-injective then \([h] \neq [1] \). Note that the loop \( \psi \) intersects \( T \) exactly at one point hence the intersection product of the two homology classes (in \( M \)) represented by \( \psi \) and \( T \) are nontrivial and in particular the homology classes are nontrivial, therefore \([g] \neq [1] \). Similarly \([g] \neq [h] \) because the intersection of the homology classes represented by \( T \) and \( \phi_1 \) is zero as \( T \) has a trivial normal bundle. \( \square \)

Remark 3.5. A similar argument shows that a \( n \)-dimensional manifold with a non-separating two sided \( \pi_1 \)-injective \((n - 1)\)-dimensional torus has nontrivial extended loop product.

4 Closed Seifert manifolds

In this section we consider the closed Seifert manifolds. We refer the reader to [8,9] for an introduction to Seifert manifolds.

Remark 4.1. An orientable Seifert manifold \( M \) may not be oriented as a fibration but it always has a double cover which is Seifert and is orientable as a fibration or in other words its base surface is orientable. This double cover \( \tilde{M} \) can be described as following:

\[ \tilde{M} = \{ (m, o) | m \in M \ \& \ o \ \text{an orientation for the fiber passing through } m \} \].

\(^4\) \( \pi_1 \)-injective (see [9]).
The covering map \( \tilde{M} \to \pi \to M \) is \( q(m, o) = m \). The Seifert fibration of \( \tilde{M} \) is obtained by pulling back the one of \( M \). If \( M \) has a boundary then each boundary component, a torus, gives rise to 2 boundary components, both a torus. If \( M \) has \( g \) cross caps then \( \tilde{M} \) has genus \( g - 1 \).

**Proposition 4.2.** If \( M \) is a closed oriented Seifert 3-manifold then \( M \) or a double cover of \( M \) has nontrivial extended loop products.

**Proof.** By Remark 4.1 we may assume that the base surface of \( M \) is orientable. Also we assume \( \pi_1(M) \not\cong \mathbb{Z}_2 \), since Proposition 2.6 deals with this case.

Let \( h \) be the generator of \( \pi_1(M) \) corresponding to the normal fiber according (5.1). There is a natural 3-homology class \( \Theta_M \) in \( H_3((LM)_{[h]}) \) that has a representative whose set of marked points is exactly \( M \). The loop associated with a point in \( M \) which is on a normal fiber, is the fiber passing through the point. The loop passing through a point on a singular fiber as a map is a multiple of the singular fiber.

We have \( p_{AM}(\Theta_M) = \Theta_M - p_M(\Theta_M) = \Theta_M - \mu_M \) where \( \mu_M \) is the unit of \( H_*(LM) \).

Consider a simple curve \( l \) representing an element \( x \in \pi_1(M) \), \( x \neq h, 1 \). This is possible since we are assuming \( \pi_1(M) \not\cong \mathbb{Z}_2 \). Note that \([h] \neq [x] \) as \( h \) is a central element. Consider the 0-homology class \( \tilde{x} \in H_0((LM)_{[x]}) \) represented by \( l \). We claim that \( p_{AM}(\Theta_M) \bullet p_{AM}(\tilde{x}) \neq 0 \).

In expanding \( p_{AM}(\Theta_M) \bullet p_{AM}(\tilde{x}) \in H_0(LM) \) we obtain four terms, \( \pm[x,h], \pm[h], \pm[x] \) and \( \pm[1] \). Note that \([h] \neq [x] \) since \( h \neq x \) and \( h \) is in the center. As there are at least three different conjugacy classes, any linear combination of them with coefficient \pm 1 is nonzero. Therefore \( M \) has nontrivial extended loop products. \( \square \)

5. Seifert manifolds with boundary

By torus decomposition [10,11], Seifert manifolds with incompressible boundary are among the building blocks of 3-manifolds. In this section, we provide various examples of Seifert manifold with boundary which have nontrivial extended loop product. We introduce the following notation as they will crucial from now on.

**Notation:** Let \( M \) be an oriented Seifert manifold with \( p \) exceptional fibers and \( b \) boundary components. If \( S \), the base surface of \( M \), is orientable, then it has genus \( g \geq 0 \) otherwise \( S \) is nonorientable and \( g \) is the number of cross caps in \( S \). The fundamental group of \( M \) has the following presentation (see [8,9]):

1. If \( S \) is orientable and of genus \( g \),

\[
\pi_1(M) = \left\{ a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_p, d_1, d_2 \ldots d_b, h | a_i ha_i^{-1} = h, \quad b_i h b_i^{-1} = h, \quad c_i h c_i^{-1} = h, \quad d_i h d_i^{-1} = h, \quad c_i^2 h c_i^{-2} = 1, \prod_{i=1}^{g} [a_i, b_i] \prod_{i=1}^{g} c_i \prod_{i=1}^{b} d_i = 1 \right\},
\]  

(5.1)
(2) If $S$ is nonorientable and has $g > 0$ cross caps,
\[
\pi_1(M) = \left\{ a_1, \ldots, a_g, c_1, \ldots, c_p, d_1, d_2 \ldots d_b, h | a_i h a_i^{-1} = h^{-1}, c_i h c_i^{-1} = h^{\delta_i}, d_i h d_i^{-1} = h, c_i^{|x_i|} h^{\beta_i} = 1 \prod_{i=1}^{g} a_i^2 \prod_{i=1}^{p} c_i \prod_{i=1}^{b} d_i = 1 \right\},
\]
(5.2)

where $0 < \beta_i < x_i$ are integers and $\delta_i = \pm 1$ and $h$ corresponds to a normal fiber. The $d_i$’s correspond to the boundary components. Each $c_i$ can be represented by a loop on the base surface going around the singular fiber once, $x_i$ is the multiplicity of the singular fiber corresponding to $c_i$ and we say that it has type $(x_i, \beta_i)$. Note that $(h)$ is a normal subgroup of $\pi_1(M)$ and it is central if $S$ is orientable.

Throughout this section $M$ is a compact oriented Seifert manifold with orientable base surface unless otherwise stated (see Corollary 5.6), $b \geq 1$ and $p'$ denotes the number of the singular fibers of multiplicity greater than 2. We assume\(^5\) that $b + p \geq 3$.

We present two main ideas for proving that a Seifert manifold has nontrivial extended loop product, **Two curves argument and Chas’ figure eight argument**.

The following table indicates the cases where we can apply these arguments.

<table>
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<th>Argument</th>
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<td>$p + b \geq 4$</td>
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<tr>
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5.1. $p + b \geq 3$ (Two curves argument)

**Proposition 5.1.** Let $M$ be a compact oriented Seifert manifold with $p > 2$ and $b \geq 1$. Then $M$ has nontrivial extended loop products.

**Proof.** Let $c_1, c_2$ and $c_3$ be the generators of $\pi_1(M)$ corresponding to three singular fibers. Consider two simple curves $\gamma_1$ and $\gamma_2$, on the base surface, away from singular fibers and representing respectively the free homotopy class $[c_1 c_2]$ and $[c_2 c_3]$. Moreover, $\gamma_2$ can be chosen such that it has exactly 2 intersection points with $\gamma_1$.

Since $\gamma_1$ is away from the singular point, the fibration can be trivialized over $\gamma_1$ (see Fig. 3). Therefore, we obtain a map $f : S^1 \times S^1 \rightarrow M$, where $f(0, \cdot) = \gamma_1$.

\(^5\) This assumption will be justified in Section 10.
Note that \( f \) gives rise to a homology class \( \Gamma_1 \in H_1((LM)_{c_1c_2}), \) by declaring \( f(t, 0), t \in S^1, \) as the set of marked points and \( f(t, s), s \in S^1, \) as the loop passing through \( f(t, 0). \) On the other hand, the loop \( \gamma_2 \) gives rise to a homology class \( \Gamma_2 \in H_0((LM)_{c_2c_1}). \)

We show that \( p_{A M}(\Delta \Gamma_1) \cdot p_{A M}(\Delta \Gamma_2) \neq 0 \in H_0(LM) \) which proves the proposition.

The sets of marked points of \( \Delta \Gamma_1 \in H_2((LM)_{c_1c_2}) \) and \( \Delta \Gamma_2 \in H_1((LM)_{c_2c_3}) \) intersect at two points. Each point contributes four terms in the expansion of \( p_{A M}(\Delta \Gamma_1) \cdot p_{A M}(\Delta \Gamma_2). \) All in all eight terms emerge while calculating \( p_{A M}(\Delta \Gamma_1) \cdot p_{A M}(\Delta \Gamma_2) \). Each is represented by a conjugacy class in \( \hat{\pi}_1(M) \) and a \( \pm \) sign. The conjugacy classes are

\[ \{c_1c_2c_3c_2, [c_1c_2], [c_2c_3], [1] \} \quad \text{and} \quad \{c_1c_2^2c_3, [c_1c_2], [c_2c_3], [1] \}. \]

Thus in calculating \( p_{A M}(\Delta \Gamma_1) \cdot p_{A M}(\Delta \Gamma_2) \) mod 2, only two terms remain, namely \([c_1c_2^2c_3]\) and \([c_1c_2c_3c_2]\) (see Fig. 4). To prove the claim it is sufficient to show that these two conjugacy classes are different. For this, consider the group

\[ H = \langle c_1, c_2, c_3, d|c_1c_2c_3d_1 = 1, c_i^{x_i} = 1, 1 \leq i \leq 3 \rangle, \]

where \( d_1 \) is a generator corresponding to a boundary component. Indeed \( H \) is the free product of the groups \( \mathbb{Z}_{x_1} \ast \mathbb{Z}_{x_2} \ast \mathbb{Z}_{x_3}. \)

Consider the homomorphism \( \phi : \pi_1(M) \to H \) which is the identity on \( c_1, c_2, c_3, d \) and sends the other generators to the trivial element of \( H. \) By (5.1) \( \phi \) is well defined.

Note that the images of \( c_1c_2^2c_3 \) and \( c_1c_2c_3c_2 \) under the \( \phi \) are not conjugate in \( H \) as \( c_1c_2^2c_3 \) and \( c_1c_2c_3c_2 \) are cyclically different reduced words. Therefore \([c_1c_2^2c_3]\) and \([c_1c_2c_3c_2]\) are two different conjugacy classes of \( \pi_1(M). \) \( \square \)

**Remark 5.2.** In the proof of the Proposition 5.1 one can replace a singular fiber by a boundary component, in other words a boundary component works as well as a singular fiber of multiplicity zero. To be more explicit, the curves \( \gamma_1 \) and \( \gamma_2 \) can go around a boundary component of the base surface instead of the singular fiber. In the free product \( H, \) one of the finite cyclic groups is replaced by a infinite cyclic group. The rest of the proof remains the same.
Corollary 5.3. Let $M$ be a compact oriented Seifert manifold with $p$ singular fibers and $b$ boundary components. If $p + b > 3$ then $M$ has nontrivial extended loop products.

Proof. If $p \geq 3$ then this is just Proposition 5.1. If $p < 3$, then by Remark 5.2, in the proof of Proposition 5.1 one replaces the generator of $\pi_1(M)$ coming from a singular fiber with the one corresponding to a boundary component. □

5.2. $p' + b \geq 3$ (Chas’ figure eight argument)

The main idea of the proof of this case is due to Moira Chas. She found the idea while reformulating a conjecture of Turaev (see [4]) on Lie bi-algebras of surfaces, characterizing non self-intersecting closed curves.

Proposition 5.4. Let $M$ be a compact oriented Seifert manifold with $p = 2$ singular fibers and $b = 1$ boundary component. Suppose that none of its singular fibers has multiplicity 2. Then $M$ has nontrivial extended loop products.

Proof. Let $c_1$ and $c_2$ be generators of $\pi_1(M)$ corresponding to the singular fibers. Let $\gamma_1$ be a smooth curve on the base surface, with one self-intersection point, away from singular fibers and representing the free homotopy class $[c_1 c_2^{-1}]$. Similarly, consider a curve $\gamma_2$ on the base surface, away from the singular fibers, with one intersection point and representing the free homotopy class $[c_1^{-1} c_2]$. Moreover, $\gamma_2$ can be chosen such that it has exactly 2 intersection points with $\gamma_1$.

The fibration can be trivialized over the $\gamma_1$ since it is away from the singular point. Therefore, we obtain a map $f : S^1 \times S^1 \to M$, where $f(0, \cdot) = \gamma_1$. This map gives rise to a homology class $\Gamma_1 \in H_1(LM)$, by declaring $f(t, 0), t \in S^1$, as the set of marked points. The loop passing through $f(t, 0)$ is $f(t, s), s \in \mathbb{S}^1$. All the loops of this family have the free homotopy type $[c_1 c_2^{-1}]$. The loop $\gamma_2$ gives rise to a homology class $\Gamma_2 \in H_0(LM)$. We claim that

$$p_{A_M}(\Delta \Gamma_1) \bullet p_{A_M}(\Delta \Gamma_2) \neq 0 \in H_0(LM)$$

which proves the proposition.

The sets of marked points of $\Delta \Gamma_1 \in H_2(LM)$ and $\Delta \Gamma_2 \in H_1(LM)$ intersect exactly at two points. Each point contributes four terms in the expansion of $p_{A_M}(\Delta \Gamma_1) \bullet p_{A_M}(\Delta \Gamma_2)$. All in all eight terms emerge in the expansion of $p_{A_M}(\Delta \Gamma_1) \bullet p_{A_M}(\Delta \Gamma_2)$. Each one of them is represented by a conjugacy class in $\hat{\pi}_1(M)$ and a $\pm$ sign. These conjugacy classes are

$$[c_1 c_2 c_1^{-1} c_2^{-1}], [c_1^{-1} c_2], [c_1 c_2^{-1}], [1] \quad \text{and} \quad [c_1 c_2^{-1} c_1^{-1} c_2], [c_1^{-1} c_2], [c_1 c_2^{-1}], [1].$$

Calculating $p_{A_M}(\Delta \Gamma_1) \bullet p_{A_M}(\Delta \Gamma_2) \mod 2$, only two terms remain, namely $[c_1 c_2 c_1^{-1} c_2^{-1}]$ and $[c_1 c_2^{-1} c_1^{-1} c_2]$ (see Fig. 5).

To prove the claim it is sufficient to show that these two conjugacy classes are different. For this, consider the group

$$H = \langle c_1, c_2, d_1 | c_1 c_2 d_1 = 1, c_i^{x_i} = 1, i = 1, 2 \rangle \cong \mathbb{Z}_{x_1} \ast \mathbb{Z}_{x_2},$$

where $d_1$ corresponds to one of the boundary components.
Consider the homomorphism $\phi : \pi_1(M) \to H$ which is the identity on $c_1, c_2, d_1$ and sends the other generators of $\pi_1(M)$ to the trivial element of $H$. It follows from (5.1) that $\phi$ is well defined.

Since $a_i \neq 2, i = 1, 2$, then $c_i \neq c_i^{-1}$ in $H$. The images of $c_1c_2c_1^{-1}c_2^{-1}$ and $c_1c_2^{-1}c_1^{-1}c_2$ under $\phi$ are not conjugate as they are cyclically different. Therefore $c_1c_2c_1^{-1}c_2^{-1}$ and $c_1c_2^{-1}c_1^{-1}c_2$ are not conjugate in $\pi_1(M)$ either. □

**Corollary 5.5.** Let $M$ be a compact oriented Seifert manifold with $p' + b \geq 3$. Then $M$ has nontrivial extended loop products.

**Proof.** If $p' \geq 2$ then this is just Proposition 5.4. Otherwise, by Remark 5.2, a boundary component works like a singular fiber with multiplicity zero. So in the proof of Proposition 5.4 we should replace the generators of $\pi_1(M)$ due to the singular fibers with the generators corresponding to two boundary components. □

**Corollary 5.6.** Let $M$ be a compact oriented Seifert manifold with a non-orientable base surface and $b \geq 2$ boundary components, then $M$ has nontrivial extended loop products.

**Proof.** The proof is similar to the one for Proposition 5.4 but with a slight modification. Note that a boundary component serves our purposes as well as a singular fiber. Moreover the restriction of the fibration to the boundary components is oriented. Thus it can be trivialized over a simple curve representing the free homotopy type $[d_1d_2^{-1}]$ where $d_1$ and $d_2$ are generators of $\pi_1(M)$ contributed by two boundary components. The rest of the proof is similar to the one of Proposition 5.4. Calculating mod 2 we get two terms $[d_1d_2d_1^{-1}d_2^{-1}]$ and $[d_1d_2^{-1}d_1^{-1}d_2]$. We must prove that these two conjugacy classes are different. If $a_1$ is a generator of $\pi_1(M)$ due to a cross cap then consider the group

$$H = \langle d_1, d_2, a_1 | d_1d_2a_1^2 = 1 \rangle \simeq \langle d_1 \rangle \ast \langle a_1 \rangle \simeq \mathbb{Z} \ast \mathbb{Z}$$

and the homomorphism $\phi : \pi_1(M) \to H$ which is the identity on $d_1, d_2, a_1$ and sends the other generators of $\pi_1(M)$ to the trivial element. It follows from (5.2) that $\phi$ is well defined.

The images of $d_1a_1^{-2}d_1^{-1}a_1^{-2}$ and $d_1a_1^{-2}d_1^{-1}a_1^2$ in the free product $H$ are not conjugate. Therefore $d_1d_2d_1^{-1}d_2^{-1}$ and $d_1d_2^{-1}d_1^{-1}d_2$ are not conjugate in $\pi_1(M)$. □
6. Connected sum of 3-manifolds

In this section we consider the connected sum of 3-manifolds. We shall show how one can obtain some examples of nontrivial loop products in this kind of manifold. For this part, the author has benefited from conversations with numerous colleagues [2].

**Proposition 6.1.** Suppose that \( M = M_1 \# M_2 \) where \( M_i \)'s are 3-manifolds of nontrivial fundamental group. Then \( M \) has nontrivial extended loop products.

**Proof.** Let \( \Sigma \subset M \) be the 2-sphere that separates the two components \( M_1^0 \) and \( M_2^0 \), where \( M_k^0 \), for \( k \in \{1, 2\} \), is \( M_k \) with a ball removed. Just like Section 3.2, the 2-sphere \( \Sigma \) gives rise to a one-dimensional family of loops which have the same marked point \( p \in M \). We set \( p \) to be the base point of \( M \) (Fig. 6). The loops in this one-dimensional family have the free homotopy type of the trivial loop. In order to modify their free homotopy type, one can compose the loops of this one-dimensional family with a fixed loop whose marked point is \( p \). Suppose that we have done this modification using a fixed loop \( \gamma \) (Fig. 6) which represents a nontrivial element \( h \in \pi_1(M) \). The new one-dimensional family of loops represents a homology class \( \Phi \in H_1((LM)_h) \). Now consider a simple smooth curve \( \psi : \mathbb{S}^1 \to M \) which intersects \( \Sigma \) exactly at 2 points and has the free homotopy type \([x_1x_2]\) where \( x_i \neq 1 \in \pi_1(M_i) \), \( i = 1, 2 \) (Fig. 6). Note that \( \pi_1(M) = \pi_1(M_1) * \pi_1(M_2) \) and \( x_1x_2 \) is regarded as an element of this free product. We choose \( \psi \) such that it does not intersect \( \gamma \). As a loop, \( \psi \) represents a homology class \( \Psi \in H_0((LM)_{x_1x_2}) \). We show that there exist some choices of \( x_1, x_2 \) and \( h \) such that \( p_{\Lambda \psi}(\Delta \Psi) \cdot p_{\Lambda \Phi}(\Delta \Phi) \neq 0 \in H_0(LM) \).

In expanding \( p_{\mathbb{S}^1 \times \mathbb{S}^2}(\Delta \Psi) \cdot p_{\mathbb{S}^1 \times \mathbb{S}^2}(\Delta \Phi) \) we get eight terms. By passing to mod 2 only two terms remain, namely \([x_1x_2h]\) and \([x_2x_1h]\). Now we must show that there exist some choices of \( h \) such that these two conjugacy classes are different. Indeed \( h = x_1x_2 \) is a convenient choice since

\[
[x_1x_2h] = [x_1x_2x_1x_2] \quad \text{and} \quad [x_2x_1h] = [x_2x_1x_1x_2] = [x_1^2x_2^2]
\]

and the reduced words \( x_1x_2x_1x_2 \) and \( x_1^2x_2^2 \) are cyclically different. \( \square \)

**Remark 6.2.** We shall point out that in Proposition 6.1 \( M \) is not required to be closed. Moreover, Proposition 6.1 and its proof can be generalized to manifolds of any dimension \( d \geq 3 \). In the proof, \( \Sigma \) would be replaced with a \((n-1)\)-sphere and the homology classes \( \Psi \) and \( \Phi \) would be of degree 0 and \( n-2 \).
7. An injectivity lemma

Suppose that $M$ is a closed oriented 3-manifold and $\mathcal{F}$ is a collection of incompressible separating tori in $M$. Let $M \setminus \mathcal{F} = M_1 \cup M_2 \cup \cdots \cup M_n$. We associate with $(M, \mathcal{F})$ a tree of groups $(G, T)$ (see [17] and Appendix A.2 for the definitions).

The vertices of $T$ are in a one-to-one correspondence with $M_i$’s. Two vertices $v_i$ and $v_j$ are connected by an edge if the corresponding $M_i$ and $M_j$ are glued along a torus. With each vertex $v_i$ we associate the fundamental group of $\pi_1(M_i)$ and with each edge we associate the fundamental group of the corresponding torus. As the tori are incompressible there are two monomorphisms from each edge group to the vertex groups of the corresponding vertices, namely if $e$ is an edge (a torus in $M$) and $v$ a vertex of $e$ (some $M_i$ s.t. $T \subset \partial M_i$) then there is an homomorphism

$$f_e^v : G_e \to G_v$$

which is indeed the map induced by inclusion

$$i : \pi_1(T) \to \pi_1(M_i).$$

By Van-Kampen theorem, $\pi_1(M)$ is the amalgamation of $G_v$’s along $G_e$’s (see Appendix A.2). Combining all these with Lemma A.5 in Appendix A, we have,

**Lemma 7.1 (Injectivity lemma).** Let $M$, $\mathcal{F}$ and $M_i$’s be as above. Suppose that $[a]$ and $[b]$ are two distinct conjugacy classes in $\pi_1(M_i)$ such that $a$ is not conjugate to the image of any element of the edge groups in $\pi_1(M_i)$. Then $a$ and $b$ represent distinct conjugacy classes of $\pi_1(M)$.

The injectivity lemma will be very useful to show that certain examples of nontrivial loop products in a submanifold of $M$ give rise to nontrivial loop products in $M$.

8. Gluing Seifert manifolds along tori

In this section we consider the gluing of the Seifert manifolds along tori. We prove that the result of the gluing Seifert manifolds along incompressible separating tori has nontrivial extended loop products except in certain cases for which there are double covers that have nontrivial extended loop products.

The idea of the proof is to apply either the two curves argument or the figure eight argument to a Seifert piece that fulfills the conditions as indicated by the table in Section 5. Then we show that the examples of nontrivial loop product in the piece give rise to examples of nontrivial loop product in the manifold, and for that we benefit from the injectivity lemma, Lemma 7.1.

When we cannot apply either the two curves argument or the figure eight argument, we consider an appropriate double cover of the manifold which either has a nonseparating torus or one of the two arguments can be applied to a Seifert piece of its torus decomposition [10,11].

**Notation:** Let $M$ be a closed 3-manifold and $\mathcal{F} \neq \emptyset$ be a collection of tori in $M$ and

$$M \setminus \mathcal{F} = M_1 \cup M_2 \cup \cdots \cup M_n,$$

where all the $M_i$’s are oriented Seifert manifolds with incompressible boundary.
Let $b_i \geq 1$ and $p_i$ denote, respectively, the number of boundary components and singular fibers of $M_i$ and $A_i \subset \mathbb{Z}^+$ is the set of all the multiplicities of the singular fibers of $M_i$. The base surface of $M_i$ is denoted $S_i$ and the genus of $S_i$ is $g_i$ if $S_i$ is oriented otherwise $g_i$ is the number of cross caps in $S_i$.

We assume that all tori in $\mathcal{T}$ are separating since in Section 3 we proved that 3-manifolds with nonseparating torus have nontrivial extended loop products. We assume $b_i + p_i \geq 3$ if $S_i$ is orientable.

**Proposition 8.1.** Let $M$ be a closed 3-manifold described as above. Then $M$ or a double cover of $M$ has nontrivial extended loop products.

**Proof.** We assume that $g_i = 0$ when $S_i$ is orientable, as if $g_i > 0$ then there is a non-separating torus in $M$ and this case has already been studied in Section 3. Then $M$ belongs to one of the following categories:

1. $b_i \geq 3$ and $S_i$ is orientable for some $i$. In this case we use the figure eight argument. Note that if $b_i > 3$ we can also use the two curves argument.
2. $b_i \geq 2$ and $S_i$ is nonorientable for some $i$, for this case we also use the figure eight argument.
3. $p_i + b_i > 3$ and $S_i$ orientable for some $i$, we can apply the two curves argument.
4. $b_i + p_i = 3$, $1 \leq b_i \leq 2$, $S_i$ orientable and $2 \notin A_i$ for some $i$. In this case we use the figure eight argument.
5. For all $i$, $b_i + p_i = 3$, $1 \leq b_i \leq 2$, $S_i$ orientable and $2 \in A_i$ for all $i$. For this case we consider a double cover of $M$.
6. $n = |\mathcal{T}|$, $n \geq 2$:
   - $b_i = 2$, $p_i = 1$, $A_i = \{2\}$ and $S_i$ is orientable for $i \neq 1, n$;
   - $b_1 = b_n = 1$;
   - $S_i$ is nonorientable;
   - If $S_n$ is orientable then $p_n = 1$ and $2 \in A_n$.

In this case also we will find a double cover with nontrivial extended loop products.

We verify the statement for each case.

1. $b_i \geq 3$ and $S_i$ orientable for some $i$: Suppose that $b_1 \geq 3$ and $S_1$ orientable. By Corollary 5.5 $\overline{M_1}$ has nontrivial extended loop products. Let $d_1$, $d_2$ and $d_3$ be the generators of $\pi_1(\overline{M_1})$ contributed by 3 boundary components. Consider the homology classes $\Gamma_1$ and $\Gamma_2 \in H_1(\overline{LM})$ constructed in the proof of Corollary 5.5, they can be regarded as the elements of $H_0(\overline{LM})$. We claim that

$$p_{\Lambda_M}(\Delta \Gamma_1) \cdot p_{\Lambda_M}(\Delta \Gamma_2) \neq 0 \in H_0(\overline{LM}).$$

Mod 2 only two terms survive, $[d_1d_2d_1^{-1}d_2^{-1}]$ and $[d_1d_2^{-1}d_1^{-1}d_2]$, which by the proof of Corollary 5.5 are two different conjugacy classes of $\pi_1(\overline{M_1})$. So if we show that $d_1d_2d_1^{-1}d_2^{-1}$ is not conjugate to any element of the fundamental group of one of the components of $\partial \overline{M_1}$, then by the injectivity lemma $[d_1d_2d_1^{-1}d_2^{-1}]$ and $[d_1d_2^{-1}d_1^{-1}d_2]$ are different as the conjugacy classes of $\pi_1(M)$.

Suppose that

$$[d_1d_2d_1^{-1}d_2^{-1}] = [h^r d^s],$$

where $h$ is the generator of $\pi_1(\overline{M_1})$ that corresponds to a normal fiber of $M_1$ and $d$ is a generator of $\pi_1(\overline{M_1})$ due to a boundary component.
Consider the group $H$
\[
H = \langle d_1, d_2, d_3 | d_1d_2d_3 = 1 \rangle \simeq \langle d_1 \rangle * \langle d_2 \rangle
\]
and the homomorphism
\[
\phi : \pi_1(M_1) \to H
\]
which is the identity on $d_1, d_2, d_3$ and trivial on all other generators of $\pi_1(M_1)$. It follows from (5.1) that $\phi$ is well defined.

After applying $\phi$ to (8.1) we get
\[
[d_1d_2d_1^{-1}d_2^{-1}] = [1]
\]
if $d \neq d_1, d_2, d_3$ otherwise
\[
[d_1d_2d_1^{-1}d_2^{-1}] = [d^s].
\]

The first case is impossible as $d_1d_2d_1^{-1}d_2^{-1}$ is a cyclically reduced word of length 4 in the free product $\langle d_1 \rangle * \langle d_2 \rangle$, thus it represents a nontrivial conjugacy class.

In the second case, If $d = d_1$ (or $d_2$) then
\[
[d_1d_2d_1^{-1}d_2^{-1}] = [d_1^s],
\]
which is not possible either as $d_1d_2d_1^{-1}d_2^{-1}$ is a cyclically reduced word of length 4 and $d_1^s$ (or $d_2^s$) has length 1. If $d = d_3 = d_2^{-1}d_1^{-1}$ then we have
\[
[d_1d_2d_1^{-1}d_2^{-1}] = [(d_1d_2)^{-s}].
\]
As the length of $d_1d_2d_1^{-1}d_2^{-1}$ is 4, we must have $s = \pm 2$. In either case the equality does not hold since $d_1 \neq d_1^{-1}$ and $d_2 \neq d_2^{-1}$ ($H$ is a free group). Therefore $d_1d_2d_1^{-1}d_2^{-1}$ and $d_1d_2^{-1}d_1^{-1}d_2$ are not conjugate in $\pi_1(M)$.

(2) $b_i \geq 2$ and $S_i$ not orientable: Similar to the previous case we use the figure eight argument in $M_i$. Suppose that $b_1 \geq 2$ and $S_1$ is not orientable. By Corollary 5.6 $M_1$ has nontrivial extended loop products. The homology classes $\Gamma_1$ and $\Gamma_2$ in $H_*(L\overline{M}_1)$ constructed in the proof of Corollary 5.6 can be regarded as homology classes in $H_*(LM)$. We claim that
\[
p_{\lambda M}(\Delta \Gamma_1) \bullet p_{\lambda M}(\Delta \Gamma_2) \neq 0 \in H_0(LM).
\]
Mod 2 we get two terms, $[d_1a_1^2d_1^{-1}a_1^{-2}]$ and $[d_1a_1^{-2}d_1^{-1}a_1^2]$ (see the proof of Corollary 5.6). We must prove that they are distinct as the conjugacy classes of $\pi_1(M)$. For that it is sufficient to prove that one of them has no boundary representative. We show that $[d_1a_1^2d_1^{-1}a_1^{-2}]$ has no boundary representative. Suppose we have
\[
[d_1a_1^2d_1^{-1}a_1^{-2}] = [h^fd^s],
\]
where $d$ is a generator of $\pi_1(M)$ coming from a boundary component and $h$ corresponds to a normal fiber. Consider the group
\[
H = \langle d_1, d_2, a_1 | d_1d_2a_1^2 = 1 \rangle \simeq \langle d_1 \rangle * \langle a_1 \rangle
\]
and the homomorphism $\varphi : \pi_1(\overline{M}_1) \to H$ which is the identity on $d_1, d_2, a_1$ and trivial on the other generators of $\pi_1(\overline{M}_1)$. It follows from (5.2) that $\varphi$ is well defined. Applying $\varphi$ to (8.2) we have

$$[da_1^2d_1^{-1}a_1^{-2}] = [1]$$

if $d \neq d_1, d_2$, otherwise

$$[da_1^2d_1^{-1}a_1^{-2}] = [d^s].$$

The first case is impossible as $da_1^2d_1^{-1}a_1^{-2}$ is a cyclically reduced word of length 4. If $d = d_1$ then

$$[da_1^2d_1^{-1}a_1^{-2}] = [d_1^s]$$

which is again impossible in the free product $\langle d_1 \rangle * \langle a_1 \rangle$. If $d = d_2$ then

$$[da_1^2d_1^{-1}a_1^{-2}] = [(a_1^2d_1)^{-s}].$$

Once again we must have $s = \pm 2$ but this is not sufficient as $d_1 \neq d_1^{-1}$ and $a_1^2 \neq a_1^{-2}$.

(3) $p_i + b_i \geq 4$ and $S_i$ orientable for some $i$: Since (1) includes the case $b_i \geq 3$ for some $i$, we may assume that $b_i \leq 2$. So we are dealing with one of the following cases:

(i) $p_i > 2$ for some $i$,

(ii) $b_i = p_i = 2$ for some $i$.

The proof of both cases are similar and use the two curves argument, keeping in mind that a boundary component works as a singular fiber of multiplicity 0. Here we only present the proof of case (i).

Suppose that $p_i > 2$. Consider the homology classes $\Gamma_1$ and $\Gamma_2$ in $H_*(LM)$ introduced in the proof of Proposition 5.1. They can be considered as the elements of $H_*(LM)$. We claim that $p_{AM}(\Delta \Gamma_1) \cdot p_{AM}((\Delta \Gamma_2) \neq 0 \in H_0(LM)$.

The calculation is the same and the results is a sum of conjugacy classes of $\pi_1(M)$. Computing mod 2 only two terms survive, $[c_1c_2c_3c_2]$ and $[c_1c_2^2c_3]$, and we must prove that as the conjugacy classes of $\pi_1(M)$ they are different. By the proof of Proposition 5.1 we know that they are distinct as the conjugacy classes of $\pi_1(\overline{M}_1)$. So it remains to show, by injectivity lemma, that one of them has no representative in the boundary components. Suppose that

$$[c_1c_2c_3c_2] = [h^rd^s],$$

where $h$ is represented by a normal fiber and $d$ is the generator coming from a boundary component.

Consider the group,

$$H = \langle c_1, c_2, c_3 | c_1c_2c_3 = 1, c_2^a_i = 1, 1 \leq i \leq 3 \rangle,$$

where $a_i$ is the multiplicity of the singular fiber corresponding to $c_i$, and the homomorphism $\phi : \pi_1(\overline{M}_1) \to H$ which is the identity on $c_1, c_2, c_3$ and sends the other generators of $\pi_1(\overline{M}_1)$ to the trivial element of $H$. Applying $\phi$ to (8.3) we have

$$[c_1c_2c_3c_2] = [1]$$
or
\[ c_1c_2c_3c_2 = 1 \]
in \( H \). Since \( c_1c_2c_3 = 1 \) in \( H \) we conclude that \( c_2 = 1 \) which is a contradiction.

(4) \( b_i + p_i = 3, 1 \leq b_i \leq 2, S_i \) orientable and \( 2 \not\in A_i \) for some \( i \): Here we shall use the figure eight argument. Suppose \( i = 1 \). By assumption, one of the following holds:

\[ (b_1 = 1 \text{ and } p_1 = 2) \quad \text{or} \quad (b_1 = 2 \text{ and } p_1 = 1). \]

The proof is similar for both cases, considering that a boundary component behaves just like a singular fiber of multiplicity 0. So we only present the proof of the first case.

Suppose \( p_1 = 2 \) and \( b_1 = 1 \). Consider the homology classes \( \Gamma_1 \) and \( \Gamma_2 \in H_0(\hat{L}M) \) as constructed in the proof of the Proposition 5.4. They can be regarded as homology classes in \( H_0(\hat{L}M) \). We prove that \( p_{\Lambda_M}(\Delta \Gamma_1) \cdot p_{\Lambda_M}(\Delta \Gamma_2) \neq 0 \in H_0(\hat{L}M) \) which proves that \( M \) has nontrivial extended products.

Having done the same calculation mod 2 as the proof of Proposition 5.4 we get two conjugacy classes \([c_1c_2c_1^{-1}c_2^{-1}]\) and \([c_1c_2^{-1}c_1^{-1}c_2]\).

Just like the previous cases, in order to show that these two conjugacy classes are distinct, we must prove that at least one of them has no representative in the fundamental group of the boundary component.

Suppose that
\[ [c_1c_2c_1^{-1}c_2^{-1}] = [h^rd_1^s], \] (8.4)
where \( h \) is the generator corresponding to a normal fiber and \( d_1 \) is the generator corresponding to the boundary component.

Consider the group
\[ H = \langle c_1, c_2, d_1 | c_1c_2d_1 = 1, c_1^{x_1} = c_2^{x_2} = 1 \rangle \]
and the homomorphism \( \phi : \pi_1(M) \rightarrow H \) which is the identity on \( c_1, c_2, d_1 \) and sends other generators of \( \pi_1(M) \) to the trivial element of \( H \). It follows from (5.1) that \( \phi \) is well defined.

By applying \( \phi \) to (8.4) we have
\[ [c_1c_2c_1^{-1}c_2^{-1}] = [d_1^s] \]
in \( H \), or
\[ [c_1c_2c_1^{-1}c_2^{-1}] = [(c_1c_2)^s]. \]

Note that \( H \) is the free product \( \mathbb{Z}_{x_1} \ast \mathbb{Z}_{x_2} \) with \( c_1 \) and \( c_2 \) as the generators of the corresponding factors.

So if \( c_1c_2c_1^{-1}c_2^{-1} \) and \( (c_1c_2)^s \) are conjugate then \( s = 2 \) or \( -2 \). In both cases \( c_1c_2c_1^{-1}c_2^{-1} \) and \( (c_1c_2)^s \) are cyclically different since \( c_1 \) and \( c_2 \) have multiplicities different from 2.

(5) \( b_i + p_i = 3, 1 \leq b_i \leq 2, S_i \) orientable and \( 2 \not\in A_i \) for all \( i \): Here the figure eight argument does not work. The following lemma provides us a double cover of the manifold which has nontrivial extended products by previous cases or the nonseparating torus argument. Appendix B is devoted to the proof of this lemma.
Lemma 8.2. Let $M$ be as above, $|\mathcal{F}| = n$, $b_i + p_i = 3$, $1 \leq b_i \leq 2$ and $2 \in A_i$ and $S_i$ orientable for all $i$.

(i) If $n \geq 3$ then $M$ has a double cover with no nonseparating torus whose torus decomposition has a Seifert piece with 3 boundary components (Lemma B.4).

(ii) If $n = 2$, $A_1 = [2, r]$ and $A_2 = [2, s]$ where $r, s \neq 2$, then $M$ has a double cover with no non-separating torus whose torus decomposition has two singular fibers of multiplicity $r \neq 2$ (Lemma B.5).

(iii) If $n = 2$ and $A_1 = A_2 = [2]$ then $M$ has a double cover with a non-separating torus (Lemma B.6).

(6) $n = |\mathcal{F}|$, $n \geq 2$:

- $b_i = 2$, $p_i = 1$, $A_i = \{2\}$ and $S_i$ is orientable for $i \neq 1, n$;
- $b_1 = b_n = 1$;
- $S_1$ is nonorientable;
- If $S_n$ is orientable then $p_n = 2$ and $2 \in A_n$.

We will be in one of the following situations:

(i) If $\bar{M}_1$ has a singular fiber or $S_1$ has more than one cross cap: Consider $\bar{M}_1'$ the double cover of $\bar{M}_1$ which is a Seifert manifold and oriented as a fibration (see Remark 4.1). Then $\bar{M}_1'$ has either more than 2 singular fibers and exactly 2 boundary components or its base surface has a genus greater than zero. We can construct $\bar{M}$, a double cover of $M$, by gluing two copies of $\bar{M} \setminus M_1$ to $\bar{M}_1'$ along the boundary components of $\bar{M}_1'$ such that the result double covers $M$ and the restriction of the covering to the complement of $M_1'$ in $M$ is the trivial double cover. Therefore $\bar{M}$ either belongs to case (3) or its base surface has genus greater than 0. In both cases it has nontrivial extended loop products.

(ii) If $\bar{M}_1$ has no singular fiber and $S_1$ has one cross cap: then $S_1$ is indeed the Möbius strip and $\bar{M}_1$ has another Seifert fibration model whose base surface is a disk and has 2 singular fibers of multiplicity 2. If $S_n$ is nonorientable then $M$ either belongs to case (i) or $M_n$ has also the Möbius strip as its base surface. We can consider the Seifert fibration model for $M_n$ with the disk as the base surface and two singular fibers of multiplicity 2, therefore $M$ belongs into case (5). \(\square\)

9. Hyperbolic factor

In this section we analyze gluing of hyperbolic and Seifert/hyperbolic 3-manifolds along torus. We need the following lemma and the proof can be found in Appendix A (see Lemma A.4).

Lemma 9.1. Suppose that $G_1$, $G_2$ and $H$ are three groups and $H = G_1 \cap G_2$. Let $x_1 \in G_1 \setminus H$ and $x_2 \in G_2 \setminus H$ and $h \in H$ such that:

(a) $x_1^{-1} H x_1 \cap H = 1$

(b) $x_2 h \neq h x_2$

Then $x_1 x_2 h$ and $x_2 x_1 h$ are not conjugate in $G_1 \ast_H G_2$. 
Proposition 9.2. Let \( M \) be a 3-manifold which contains a separating two sided \( \pi_1 \)-injective torus \( T \). Suppose that \( M \setminus T \) has two connected components \( M_1 \) and \( M_2 \) such that:

(i) \( \overline{M}_1 \) has a hyperbolic interior with finite volume.
(ii) Either \( M_2 \) has a complete hyperbolic structure of finite volume, or else \( \overline{M}_2 \) is a Seifert manifold and \( \overline{M}_2 \neq S^1 \times S^1 \times [0, 1] \).

Then \( M \) has nontrivial extended loop products.

Let \( \phi : S^1 \times S^1 \rightarrow T \subset M \) be a homeomorphism. We choose \( \phi(1, 1) \) as the base point. Just like the previous case, \( \phi \) gives rise to a one dimensional family of loops \( \phi_t, t \in S^1 \) (longitudes of \( T \) in Fig. 7).

This 1-family of loops represents a homology class \( \Phi \in H_1((LM)_[h]) \), where \( h \in \pi_1(M) \) is the element represented by \( \phi_1 \).

Now consider a simple smooth curve \( \psi : S^1 \rightarrow M \) which intersects \( T \) exactly at 2 points and it has the free homotopy type \( [x_1x_2] \) where \( x_i \in \pi_1(M_i), i = 1, 2 \). Note that \( \pi_1(M) = \pi_1(\overline{M}_1) * \pi_1(T) * \pi_1(\overline{M}_2) \) and \( x_1x_2 \) is regarded as an element of this amalgamated free product.

As a loop \( \psi \) represents a homology class \( \Psi \in H_0((LM)_{[x_1x_2]}) \). We prove that there exist choices of \( x_1, x_2 \) and \( h \) such that \( p_{AM}(\Delta \Psi) \bullet p_{AM}(\Delta \Phi) \neq 0 \in H_0(LM) \).

In computing \( p_{AM}(\Delta \Psi) \bullet p_{AM}(\Delta \Phi) \) we get eight terms. By passing to mod 2 only two terms survive, namely \( [x_1x_2, h] \) and \( [x_2x_1, h] \). Now we must show that there are some choices of \( x_1, x_2 \) and \( h \) such that these two conjugacy classes are different. For that we use Lemma 9.1.

In our case \( G_i = \pi_1(\overline{M}_i), i = 1, 2 \) and \( H = \pi_1(T) \). Since \( \overline{M}_1 \) has a hyperbolic interior of finite volume \( \pi_1(T) \) consists of parabolic elements of \( PSL(2, \mathbb{C}) \) with a common fixed point. Then \( x_1^{-1}(\pi_1(T))x_1 \cap \pi_1(T) = 1 \) for \( x_1 \in \pi_1(\overline{M}_1) \setminus \pi_1(T) \) since conjugation with an element outside of \( H \) changes the fixed point. Therefore there exists a choice of \( x_1 \).

If \( \overline{M}_2 \) has a hyperbolic interior with finite volume then it follows from the same reasoning as before that there is a choice of \( x_2 \) so that (b) is satisfied. If \( \overline{M}_2 \) is a Seifert manifold, all we have to do is it to modify the embedding \( \phi \) so that \( h \) is not in the center of \( \pi_1(\overline{M}_2) \) which is generated by a power of \( h \). Therefore under the hypothesis above there are choices of \( x_1, x_2 \) and \( y \) such that the conditions of Lemma 9.1 are satisfied.

10. Proof of main theorem: part I

In this section we prove the first part of the main theorem, namely that if a closed oriented manifold \( M \) is not algebraically hyperbolic, then \( M \) or a double cover of \( M \) has nontrivial extended loop products.
For this we use the prime decomposition [14] and the torus decomposition for 3-manifolds [10,11]. We refer the reader to [8,9] for the basic materials in the three-dimensional topology.

If $M$ is not algebraically hyperbolic then at least one of the following holds:

10.1. $M$ is not aspherical

By the prime decomposition theorem, every 3-manifold $M$ can be written as

$$M = (K_1\#K_2\#\ldots\#K_p)\#(L_1\#L_2\#\ldots\#L_q)\#(\#_1(S^1 \times S^2)),$$

where $K_i$’s are closed aspherical irreducible 3-manifolds with infinite $\pi_1$ and $L_i$ are closed 3-manifolds that are finitely covered by a homotopy 3-sphere. If $M$ is not aspherical then one of the following holds:

(i) If $M$ is a connected sum of manifolds with nontrivial fundamental group then Proposition 6.1 says that $M$ has nontrivial extended loop products.

(ii) $M$ has a non-separating 2-sphere or in other words there is a $S^2 \times S^1$ factor in its prime decomposition, then by Corollary 3.2 $M$ has nontrivial extended loop products.

(iii) $M$ has finite fundamental group; Then by Proposition 2.6 $M$ has nontrivial extended loop products.

Thus in either cases $M$ has nontrivial extended loop product.

10.2. $M$ is aspherical and $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$

We turn to the case when $M$ is aspherical 3-manifold with rank 2 abelian groups. If follows from the argument in 10.1 that $M$ is either irreducible or is the connected sum of an irreducible 3-manifold and homotopy 3-sphere.\(^6\)

10.2.1. $M$ is irreducible

By the torus theorem [16] and Seifert fiber space theorem [3,7] one of the following holds:

(a) $M$ is a closed Seifert manifold. If $M$ is Seifert manifold then by Proposition 4.2 $M$ or a double cover of $M$ has nontrivial extended loop products.

(b) The collection of tori in the torus decomposition of $M$ is nonempty. One of the following occurs:

(i) $M$ contains a nonseparating torus, then by Proposition 3.4 $M$ has nontrivial extended loop products.

(ii) $M$ has no nonseparating torus and the torus decomposition of $M$ has only Seifert pieces: This case has been treated in Section 8. We should verify that $M$ satisfies the conditions of Proposition 8.1. We use the notation introduced in Section 8 and $\mathcal{T}$ is provided by the torus composition of $M$.

First of all $g_i$’s are all zero as we have assumed that $M$ has no nonseparating torus. If $p_i + b_i \leq 2$ for $S_i$ orientable, then one of the following holds:

(1) $b_i = 1$ and $p_i = 1$,
(2) $b_i = 1$ and $p_i = 0$,
(3) $b_i = 2$ and $p_i = 0$.

\(^6\)Note that such a connected sum is again aspherical.
(1) and (2) cannot occur since the boundary component of $\overline{M_i}$ is incompressible. In the latter case, it follows that $\overline{M_j} \simeq S^1 \times S^1 \times [0, 1]$ which contradicts the minimality of the collection $\mathcal{F}$ as one can extend the Seifert fibration of one of the neighboring $\overline{M_j}$ to $\overline{M_i}$.

(iii) $M$ has no nonseparating torus and the torus decomposition of $M$ includes an atoroidal component:

- Suppose that $\mathcal{F}$ is the collection of tori provided by the torus decomposition of $M$ and each torus in $\mathcal{F}$ is separating.

By Thurston’s theorem on geometrization of Haken manifolds (see [18]) atoroidal components of $M \setminus \mathcal{F}$ are hyperbolic. Let $M_1$ be a hyperbolic component of $M \setminus \mathcal{F}$ and $M_2$ is another component that is attached to $M_1$ along a torus $T$. By Proposition 9.2, $M' = \overline{M_1} \cup T_1 \cup \overline{M_2}$ has nontrivial extended loop products.

Consider the same homology classes $\Gamma$ and $H$ in $H_*(LM')$ constructed in the proof of Proposition 9.2. They can be regarded as the elements of $H_*(LM)$. We claim that $p_{\Lambda M}(\Delta(\Gamma)) \bullet p_{\Lambda M}(\Delta H) \neq 0$, thus $M$ has nontrivial extended loop products.

The calculation is the same as Proposition 9.2, mod 2 only two terms survive, $[hg_1g_2]$ and $[hg_2g_1]$ and we must prove that are different as the conjugacy classes of $\pi_1(M)$. By the proof of Proposition 9.2 we know that $hg_1g_2$ and $hg_2g_1$ are not conjugated in $\pi_1(M')$. So by the injectivity lemma (Lemma 7.1), we only need to show that $hg_1g_2$ is not conjugate to an element of the fundamental group of the boundary of $M'$.

Let $T_1 \cup T_2 \cup \cdots \cup T_k = \partial M'$ and suppose that $hg_1g_2$ is conjugate to an element of $\pi_1(T_i)$, $i > 1$ in $\pi_1(M')$. One of $T_i \subset \overline{M_1}$ or $T_i \subset \overline{M_2}$ holds. If $hg_1g_2$ is conjugate to an element to $\pi_1(T_i)$ then it is conjugate to an element in $\pi_1(M_1)$. But this is a contradiction since $hg_1g_2 = (hg_1)g_2$ as an element of $\pi_1(M') = \pi_1(\overline{M_1}) \ast \pi_1(T) \pi_1(\overline{M_2})$ is a cyclically reduced word of length 2 and it cannot be conjugate to an element one of the factors.

10.2.2. $M$ is a connected sum of an irreducible 3-manifold and a homotopy 3-sphere

Suppose that $M = N \# P$ where $N$ is irreducible aspherical and $P \neq S^3$ is a homotopy 3-sphere. In fact $M$ is obtained from $N$ by replacing a standard ball $D \subset N$ with a fake ball $D'$ namely $M = (N \setminus D) \cup_{S^2} D'$. Let $f : N \to M$ be a homotopy equivalence which sends $D$ to $D'$ and is the identity on the complement of $D$ in $N$. Since $\overline{\mathbb{Z}} \oplus \mathbb{Z} \subset \pi_1(M)$ hence $\overline{\mathbb{Z}} \oplus \mathbb{Z} \subset \pi_1(N)$. By Seifert fiber space and torus theorem either $N$ is Seifert or the collection of tori in its torus decomposition is nonempty.

If $N$ is a Seifert manifold then $\partial N$, the 3-homology class constructed in the proof of Proposition 4.2, cannot be extended directly to a homology class in $M$ but we claim that $p_{\Lambda M}(f_L(\partial N)) \bullet p_{\Lambda M}(f_L(z)) \neq 0$ which proves that $M$ has nontrivial extended loop products. The proof is exactly the one in Section 4, note that we can choose a representative for $z$ (see the proof of Proposition 4.2) such that its marked point is in the complement of $D$.

If $N$ has a nontrivial torus decomposition it follows from the case 10.2.1, that $N$ or a double cover of $N$ has nontrivial extended loop products. In the first case, the homology classes in $LN$ constructed in 10.2.1 can be regarded as a homology classes in $LM$ because they can have chain representatives which are in the complement of $D$ and the loop products calculated there, are also nontrivial in $M$ as $\pi_1(M) \simeq \pi_1(N)$. In the latter case, a double cover $\tilde{N}$ of $N$ gives rise to $\tilde{M}$ a double cover of $M$ where the covering on $D'$ is the trivial double covering and again the examples of nontrivial loop product found for $\tilde{N}$ in Section 8 gives rise to examples of nontrivial loop product for $\tilde{M}$ as we can always do the construction outside of a ball which is to be replaced with a fake ball.
It is not known to us whether the double cover case is necessary. However, one can give a complete description of the all the homology classes of $LM$ in those cases. For instance, let $M$ be one of the 3-manifolds listed in part (5) or (6) of the proof of Proposition 8.1. Since $M$ is irreducible with infinite fundamental group then each connected component $(LM)_{[x]} = K(C_2, 1)$. It is possible to compute the centralizers $C_x$ since $\pi_1(M)$ is given by a free product with amalgamation of a trees of groups (see Section 7 and Appendix A). The centralizer $C_x, x \neq 1$, is:

(i) $\mathbb{Z}$ if $x$ does not belong to any factor $\pi_1(M_i)$.
(ii) $\pi_1(M_i)$ if $x$ corresponds to a power $h$ represented by a normal fiber of $M_i$.
(iii) $\mathbb{Z} \oplus \mathbb{Z}$ if $x$ belongs to one of the factors $\pi_1(M_i)$ and it does not correspond to any generator due to a singular fiber or a power of the normal fiber of any components.
(iv) $\mathbb{Z}$ if $x$ corresponds to a generator due to a singular fiber.

So $A_M$ has the homological dimension at most two. Moreover, one can find chain representatives for the homology classes of $H_*(((LM)_{[x]}].$ If $\gamma$ is a smooth curve representing $x \in \pi_1(M)$, then $\gamma$ represents a generator $g \in H_0(((LM)_{[x]})$ and $\Delta g$ is a generator of $H_1(((LM)_{[x]}).$ When $x$ belongs to a factor $\pi_1(M_i)$, other generators of $H_1(((LM)_{[x]}$ can be obtained by trivializing the fibration over $\gamma$, and then one gets the generators of $H_2(((LM)_{[x]}$ by applying $\Delta$ to them.

One cannot get a nontrivial example of loop in a Seifert piece of the 3-manifolds in case (5) and (6), but one can hope to get a nontrivial example of loop product by considering the product of a 2-homology class and a 1-homology class in a component $(LM)_{[x]}$, where $x$ is not in a factor. For that, we have to carry a computation similar to the one in Section 9.

11. Proof of main theorem: Part II, algebraically hyperbolic manifolds

In this section we prove the second part of the main theorem, namely that algebraically hyperbolic 3-manifolds have trivial extended loop products.

It follows from Lemma 1.7 that

Lemma 11.1. If $M$ is a $K(\pi, 1)$ (aspherical) then $LM$ is homotopy equivalent to

$$\bigcoprod_{[x] \neq [1] \in \pi_1(M)} K(C_x, 1),$$

where we choose only one representative for each conjugacy class and $C_x$ is the centralizer of $x$ in $\pi_1(M)$.

Therefore to prove the second part of the theorem we need a good understanding of the centralizers.

Lemma 11.2. If a closed oriented 3-manifold $M$ is algebraically hyperbolic then the fundamental group of $M$ does not have any nontrivial finite subgroups and the centralizer $C_x$ of an element $x \in \pi_1(M)$ is isomorphic to an additive subgroup of $\mathbb{Q}$.

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Note that in case (ii), $H_3(K(C_2, 1)) = K(\pi_1(M), 1) = H_3(M) = 0$ as $M_i$ is aspherical and not closed.
Proof. Let \( \mathbb{Z}_p \) be a nontrivial finite subgroup of \( \pi_1(M) \). Then \( K(\mathbb{Z}_p, 1) \) is a covering of \( M \), so it has to be a 3-manifold. One knows that \( K(\mathbb{Z}_p, 1) \) has homology in infinitely many dimensions. Therefore \( \pi_1(M) \) has no finite subgroup.

For the second part we first prove that every finitely generated subgroup of \( C_2 \) is isomorphic to \( \mathbb{Z} \). Suppose \( \beta_1, \beta_2, \ldots, \beta_n \) are some elements of \( C_2 \). Consider the finitely generated subgroup \( \langle \beta_1, \beta_2, \ldots, \beta_n, x \rangle \subset C_2 \), it is not a free product since it has a nontrivial center. Therefore by the compact realization theorem (see [15]) it is isomorphic to the fundamental group of a compact, oriented irreducible 3-manifold \( N \). The infinite cyclic subgroup \( \langle x \rangle \) is in the center of \( \pi_1(N) \) and therefore it is normal. Thus by the Seifert fibered space theorem [3,7], \( N \) is a Seifert manifold. Since \( \pi_1(M) \) has no finite subgroup or rank 2 abelian subgroup, \( N \) has to be a solid torus,\(^8\) so \( \langle \beta_1, \beta_2, \ldots, \beta_n, x \rangle = \pi_1(N) \cong \mathbb{Z} \) which implies \( \langle \beta_1, \beta_2, \ldots, \beta_n \rangle = \mathbb{Z} \).

To prove that \( C_2 \) is isomorphic to a subgroup of \( \mathbb{Q} \), note that because \( C_2 \) is countable and every finitely generated subgroup is cyclic, we can write \( C_2 = G_1 \cup G_2 \cdots \) where each \( G_i \) is an infinite cyclic subgroup and \( G_i \subset G_{i+1} \). Let \( x_i \) be a generator for \( G_i \); then we have \( x_i = n_i x_{i+1} \) for some \( n_i \). We construct a map \( \varphi : C_2 \to \mathbb{Q} \) by letting \( \varphi(x_1) = 1 \) and \( \varphi(x_{i+1}) = \frac{1}{n_i n_2 \cdots n_j} \). This is a well-defined map because of our choice \( n_i \)'s. Moreover \( \varphi \) is injective and to see that note that if \( x \in C_2 \), then \( x \in G_i \) for some \( i \). So \( x = k z_i \) for some \( i \) and \( k \in \mathbb{Z} \) and \( \varphi(x) = \frac{k}{n_1 n_2 \cdots n_i} \). Therefore \( \varphi(x) = 0 \) implies \( k = 0 \) or \( x = 0 \).

Lemma 11.3. If \( G \) is an additive subgroup of \( \mathbb{Q} \) then \( K(G, 1) \) has homological dimension one.

Proof. One can write \( G = \lim_{n \to \infty} G_n \), where \( G_i \)'s are cyclic subgroups of \( G \). So \( K(G, 1) = \lim_{n \to \infty} K(G_n, 1) \). All \( K(G_n, 1) \cong K(\mathbb{Z}, 1) \) has the homological dimension 1 therefore the direct limit \( K(G, 1) \) has the homological dimension one since taking homology commutes with taking direct limit.

Proposition 11.4. If \( M \) is algebraically hyperbolic then any finite cover of \( M \) has trivial extended loop products.

Proof. Let \( \tilde{M} \) be a finite cover of \( M \). Note that \( \tilde{M} \) is also algebraically hyperbolic. Since \( \tilde{M} \) is a \( K(\pi', 1) \) then its free loop space is homotopy equivalent to

\[
\coprod_{[z] \in \tilde{\pi}'} K(C'_z, 1),
\]

where we choose only one representative for each conjugacy and \( \pi'_z \) is the centralizer of \( z \) in \( \pi' \). So

\[
A_{\tilde{M}} \cong H_* \left( \coprod_{[z] \in \tilde{\pi}', x \neq 1} K(C'_z, 1) \right) \cong \bigoplus_{[z] \in \tilde{\pi}', x \neq 1} H_*(K(C'_z, 1)).
\]

By Lemmas 11.2 and 11.3, each \( K(C'_z, 1), x \neq 1, \) has homological dimension 1. Therefore \( \bullet \) on \( A_{\tilde{M}} \) is zero as \( A_{\tilde{M}} \) is concentrated in degree 1.

\(^8\) \( N \) cannot be \( S^2 \times S^1 \) since it is irreducible.
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Appendix A. Free products with amalgamation and trees of groups

A.1. Some basic facts about free products with amalgamation

We recall some definitions and theorems from group theory related to the free product with amalgamation. We follow the terminology and notations of [12,13].

Let $G_1, G_2$ with a common subgroup $H$ then we can form $G_1 \ast_H G_2$ the amalgamated free product of $G_1$ and $G_2$ over $H$.

A sequence of elements $g_1, g_2, \ldots, g_n \in G_1 \ast_H G_2$ is called reduced if:

1. each $g_i$ is in one of the factors $G_1$ or $G_2$;
2. $g_i, g_{i+1}$ come from different factors; So if $n > 1$ then $g_i \notin H$ for all $i$;
3. if $n = 1$ then $g_1 \neq 1$.

$g_1 g_2 \cdots g_n \in G_1 \ast_H G_2$ is called a reduced word if $g_1, g_2, \ldots, g_n$ is a reduced sequence.

**Theorem A.1** (see Lyndon and Schupp [12, p. 187]). If $c_1, c_2, \ldots, c_n, n \geq 1$, is a reduced sequence in $G = G_1 \ast_H G_2$, then the product $c_1 c_2 \cdots c_n \neq 1$ in $G$.

A sequence of elements $g_1, g_2, \ldots, g_n \in G_1 \ast_H G_2$ is called cyclically reduced if all cyclic permutations of the sequence $g_1, g_2, \ldots, g_n$ are also reduced and $g_1 g_2 \cdots g_n \in G_1 \ast_H G_2$ is called a cyclically reduced word if $g_1, g_2, \ldots, g_n$ is a cyclically reduced sequence. The following theorem describes the conjugacy classes of $G_1 \ast_H G_2$.

**Theorem A.2** (see Magnus et al. [13, p. 212]). Let $G = G_1 \ast_H G_2$. Then every element of $G$ is conjugate to a cyclically reduced word. Moreover, suppose that $g$ is a cyclically reduced element of $G$. Then:

1. If $g$ is conjugate to a cyclically reduced word $p_1 p_2 \ldots p_r$ where $r \geq 2$; then $g$ can be obtained by cyclically permuting $p_1, p_2, \ldots, p_r$ and then conjugating by an element of $H$.
2. If $g$ is conjugate to an element $h$ in $H$, then $g$ is in some factor and there is a sequence of $h, h_1, \ldots, h_1, g$ where $h_i$ is in $H$ and consecutive terms of the sequence are conjugate in some factor.
3. If $g$ is conjugate to an element $g'$ in some factor but not in a conjugate of $H$ then $g$ and $g'$ are in the same factor and are conjugate in that factor.

There is no canonical way of presenting the elements of $G_1 \ast_H G_2$ uniquely. Once a choice of right coset representatives for $G_1/H$ and $G_2/H$ is made then every element can be represented in a unique way. Let $c_1, \ldots, c_n$ be a right coset representative system for $G_1/H$ and $G_2/H$. Here is an informal description of the representation:
Let $g$ be an element in $G_1 \ast_H G_2$ and $g = g_1 g_2 \ldots g_r$ be a reduced presentation of $g$. Start with $g_r$, there is an $h_r \in H$ and $c_r \in G_1 / H$ or $G_2 / H$ (depending in which factor is $g_r$) such that $g_r = h_r c_r$. Now by replacing $g_r$ by $h_r c_r$ we have $g = g_1 g_2 \ldots g_{r-1} h_r c_r$. Again there is a $h_{r-1} \in H$ and $c_{r-1}$ such that

$$g_{r-1} h_r = h_{r-1} c_{r-1}.$$ 

By replacing $g_{r-1} h_r$ by $h_{r-1} c_{r-1}$ we have $g = g_1 g_2 \ldots g_{r-2} h_{r-1} c_{r-1} c_r$. Continuing this procedure we end up with $g = h_1 c_1 c_2 \ldots c_r$. This is the desired presentation.

**Theorem A.3** (see Magnus et al. [13, p. 201]). Let $G = G_1 \ast_H G_2$ where $H$ is a mutual subgroup of $G_1$ and $G_2$. Suppose a specific right coset representative system for $G_1 / H$ and $G_2 / H$ have been selected. Then with each element of $g$ of $G$ we can associate a unique sequence\(^9\) $(h, c_1, c_2, \ldots, c_r)$ such that:

(i) $h$ is an element, possibly 1, of $H$;
(ii) $c_i$ is a coset representative of $G_1 / H$ or $G_2 / H$;
(iii) $c_i \neq 1$;
(iv) $c_i$ and $c_{i+1}$ are not both in $G_1$ or $G_2$;
(v) $g = h c_1 c_2 \ldots c_r$ in $G$.

**Lemma A.4.** Suppose that $G_1$, $G_2$ and $H$ are three groups and $H = G_1 \cap G_2$. Let $g_1 \in G_1 \setminus H$ and $g_2 \in G_2 \setminus H$ and $h \in H$ such that:

(i) $g_1^{-1} H g_1 \cap H = 1$,
(ii) $g_2 h \neq h g_2$.

Then $h g_1 g_2$ and $h g_2 g_1$ are not conjugate in $G_1 \ast_H G_2$.

**Proof.** Suppose that $[h g_1 g_2] = [h g_2 g_1]$. It is clear that $[h g_1 g_2] = [g_2 h g_1]$ and $[h g_2 g_1] = [g_2 g_1 h]$. So $g_2 (h g_1)$ is conjugate to $g_2 (g_1 h)$. Since $g_2 (g_1 h)$ is cyclically reduced (of length 2, $p_1 = g_2$ and $p_2 = g_1 h$) by Theorem A.2 part (ii), $g_2 (g_1 h)$ is conjugate to $g_2 (g_1 h)$ or $(g_1 h) g_2$ by an element in $H$.

In the latter case we have $g_2 (h g_1) = h_1 (g_1 h) g_2 h_1^{-1}$ which is equivalent to

$$g_2^{-1} (h_1 g_1 h)^{-1} g_2 (h_1 h_1) = 1.$$ 

But this contradicts Theorem A.1 because $g_2^{-1}$, $(h_1 g_1 h)^{-1}$, $g_2$, $(h_1 h_1)$ is a reduced sequence.

Therefore there should be an element $h_1 \in H$ such that $g_2 h g_1 = h_1 g_2 g_1 h h h_1^{-1}$ or

$$(g_2 h) (g_1 h_1) = (h_1 g_2) (g_1 h). \quad (A.1)$$

By Theorem A.1 (also see the construction), (A.1) implies that $g_1 h_1$ and $g_1 h$ are in the same left coset in $G_1 / H$ or in other words, there exists $h_2 \in H$ such that

$$g_1 h_1 = h_2 g_1 h,$$

or equivalently

$$h_1 h^{-1} g_1^{-1} h_2 g_1.$$ 

---

\(^9\)This presentation is called *reduced form* (see [13]).
So \( h_1h^{-1} = g_1^{-1}h_2g_1 \in g_1^{-1}Hg_1 \cap H = 1 \) hence \( h_2 = h_1h^{-1} = 1 \) which implies \( h_1 = h \). By substituting \( h_1 = h \) in (A.1) and simplifying, it follows
\[
g_2h = hg_2
\]
which is a contradiction. Therefore \([hg_1g_2] \neq [hg_2g_1]\). \( \square \)

A.2. Tree of groups

A tree of groups \((G, T)\) consists of a finite tree \(T\) and \(G\) a collection of groups
\[
G = \{G_e\}_{e \in \text{edge } T} \coprod \{G_v\}_{v \in \text{vert } T},
\]
a group \(G_v\) for every vertex \(v \in \text{vert } T\), a group \(G_e\) for every edge \(e \in \text{edge } T\) and a monomorphism
\[
f_e^v : G_e \rightarrow G_v
\]
if \(v\) is a vertex of the edge \(e\).

We call \(G_v\) a vertex group if \(v\) is a vertex of \(T\) and \(G_e\) an edge group if \(e\) is an edge of \(T\). If \(v\) is a vertex of the edge \(e\) then \(G_e\) is considered as a subgroup of \(G_v\).

Suppose that \((G, T)\) is a tree of groups, \(v_0\) a vertex of \(T\). Let \(n = |\text{edge } T|\). Consider a sequence of trees \(T_i, 1 \leq i \leq n\) such that
\begin{itemize}
  \item (i) \(|\text{edge } T_i| = i, \ 0 \leq i \leq n\),
  \item (ii) \(\text{vert } T_0 = \{v_0\}\),
  \item (iii) \(T_i \subset T_{i+1}\) and
  \item (iv) \(T_n = T\).
\end{itemize}

Let \(e_1, e_2, \ldots, e_n\) be the sequences of the edges and \(v_0, v_1, \ldots, v_n\) be the sequence of vertices such that
\begin{itemize}
  \item edge \(T_1 = \{e_1\}\),
  \item \(\text{vert } T_1 = \{v_0, v_1\}\),
  \item edge \(T_{i+1} = \text{edge } T_i \cup \{e_{i+1}\}\), \(1 \leq i \leq n - 1\),
  \item \(\text{vert } T_{i+1} = \text{vert } T_i \cup \{v_{i+1}\}\) \(1 \leq i \leq n - 1\).
\end{itemize}

Let \(G_{T_0} = G_{v_0}\) and
\[
G_{T_i} = G_{T_{i-1}} \ast G_{e_n} G_{v_n}, \ \ 1 \leq i \leq n.
\]

\(G_T = G_{T_n}\) is called the amalgamation of \(G_v\)'s along \(G_e\)'s and is independent of the choice of the sequence \(T_i\) and depends only on \((G, T)\) and there is an inclusion
\[
\varphi_k : G_k \rightarrow G_T,
\]
where \(k\) is a vertex or an edge.

The following lemma is a generalization of Theorem A.2 part (iii).
Lemma A.5. Let $(G, T)$ be a tree of groups and $v_0$ a vertex of $T$ and $G_T$ be the amalgamation of $G_v$'s along $G_e$'s. Suppose that $a$ and $b$ are not conjugate in $G_{v_0}$ and $a$ is not conjugate in $G_{v_0}$ to an element of any edge group $G_e \subseteq G_{v_0}$ where $v_0$ is a vertex of $e$. Then $a$ and $b$ are not conjugate in $G_T$.

Proof. $T_i$'s, $v_i$'s and $e_i$'s are as above.

We prove the lemma by induction. We prove that for every $i$, $1 \leq i \leq n$

1. $[a]$ and $[b]$ are distinct as conjugacy classes of $G_{T_i}$.
2. $a$ is not conjugate in $G_{T_i}$ to an element in the vertex group $G_w$, where $w \neq v_0$ is a vertex of $T_i$.

We verify the statement for $i = 1$. By the assumption and part (iii) of Theorem A.2 it follows that the first statement of induction is true for $i = 1$. For the second part, suppose that $a$ is conjugate in $G_{T_1}$ to an element of $G_{v_1}$, then again by the part (ii) and (iii) of Theorem A.3, $a$ has to be conjugate to an element of $G_{e_1}$ in $G_{v_0}$ which contradicts our assumption.

Suppose that the statement is true for $i$. By the second statement of the induction for $i$, $a$ is not conjugate in $G_{T_i}$ to an element of $G_{e_{i+1}} \subseteq G_{T_i}$. Therefore by part (iii), $a$ and $b$ represent different conjugacy classes in $G_{T_{i+1}} = G_{T_i} * G_{e_{i+1}} G_{v_{i+1}}$, proving the first statement of the induction for $i + 1$.

To prove the second statement for $i + 1$, suppose that by contrary $a$ is conjugate to an element of $G_w$, where $w \neq v_0$ and $w \in T_{i+1}$. Since $G_{T_{i+1}} = G_{T_i} * G_{e_{i+1}} G_{v_{i+1}}$ and $a$ is not conjugate in $G_{T_i}$ to an element of $G_{e_{i+1}}$ then again by part (iii) of Theorem A.2 we must have $w \in T_i$ and $a$ is conjugate in $G_{T_i}$ to an element of $G_w$. This contradicts the assumption of the induction for $i$. Hence the second statement of the induction holds for $i + 1$.

Finally for $i = n$ we get the statement of the lemma. \(\square\)

Appendix B. Finite covers of certain Seifert manifolds with boundary

This appendix is devoted to proof of Lemma 8.2.

Construction B.1. Suppose $M$ is a compact orientable Seifert manifold with $g = 0$, $p = 1$ and $b = 2$ and the singular fiber has type multiplicity 2.

Then $M$ has a double cover $M'$ with $g' = 0$, $p' = 0$ and $b' = 3$ boundary components and two of the boundary components are identified under the covering map:

Consider $D \subset \mathbb{R}^2$ the unit disk centered at the origin and $F$ a union of two open disks of radius $1/3$ centered at $(1/2, 0)$ and $(-1/2, 0)$.

Let $M' = N \times S^1$ where $N = D \setminus F$. Then $\mathbb{Z}_2$ acts on $M'$ freely where the action on both factors is realized by $180^\circ$ rotation.

Indeed $M = M'/\mathbb{Z}_2$ and $M'$ is the desired double covering (Fig. 8). $M'$ has 3 boundary components where two of them are identified under the covering map.

Construction B.2. Let $M$ be a compact orientable Seifert manifold with $g = 0$, $b = 1$ and $p = 2$ singular fibers of multiplicity 2. Then $M$ has a double cover $M'$ with $g' = 0$, $b' = 1$ and $p' = 2$ and the singular fibers are of type $(r, s)$:

Let $N$ be the unit sphere $S^2 \setminus D \subset \mathbb{R}^3$, where $D$ is a set of $n$ open disks located on the equator in $xy$-plane which is invariant under $2\pi/s$ rotation about $z$-axis and $180^\circ$ rotation about $x$-axis; and If $s$ is even then
Fig. 8. Double cover $M'$, •’s show the location of the singular fibers.

Fig. 9. $N$.

Fig. 10. Annulus $C \subset \mathbb{R}^3$.

$D$ does not meet the $x$-axis otherwise $D$ meets the $x$-axis exactly at one point. The 2s-Dihedral group $D_{2s} = \langle a, b| a^2 = 1, b^s = 1, aba = b^{-1} \rangle$, acts on $P = N \times S^1$ freely (see Fig. 9):

(i) $\mathbb{Z}_2$ acts on $P$ by $180^\circ$ rotation about $x$-axis in the first factor and $180^\circ$ rotation in the second factor.

(ii) $\mathbb{Z}_s$ acts on $P$ freely; a fixed generator of $\mathbb{Z}_s$ acts by $2\pi/s$ rotation about $z$-axis in the first factor and $2r\pi/s$ rotation in the second factor.

Let $M' = P/\mathbb{Z}_s$, then $M'$ is a Seifert manifold with two singular fibers of type $(r, s)$ and one boundary component and base surface a disk. The action of $\mathbb{Z}_2$ on $P$ descends to $M'$ as the two action commutes. In fact

$$M = M'/\mathbb{Z}_2$$

and $M'$ is the double described in the statement.

**Construction B.3.** Let $M$ be a compact Seifert manifold with $g = 0$, $b = 1$ and $p = 2$ singular fibers of type $(2, 1)$. Then $M$ has a double cover $M'$ with $g' = 0$, $b' = 2$ and $p' = 0$ and the boundary components of $M$ are identified homeomorphically under the covering map.

Consider the annulus $C = S^1 \times [-1, 1] \subset \mathbb{R}^3$, where $S^1$ is the unit circle in $(x, y)$-plane and centered at the origin (Fig. 10).
\( \mathbb{Z}_2 \) acts on \( C \) with exactly two fixed points. The action is realized by \( 180^\circ \) rotation about \( y \)-axis. \( \mathbb{Z}_2 \) acts on

\[
M' = C \times S^1
\]

freely where the action on the first factor is described above and on the second factor is the rotation by \( 180^\circ \). Indeed,

\[
M \simeq M'/\mathbb{Z}_2
\]

and the boundary components of \( M' \) are identified homeomorphically under action.

**Lemma B.4.** Let \( M \) be a closed oriented 3-manifold and \( T \) be the collection of tori provided by torus decomposition of \( M \). Suppose that \( M \setminus T = M_1 \cup M_2 \cup \cdots \cup M_n \) with all \( M_i \)'s Seifert manifold such that:

1. \( n \geq 3 \) and \( g_i = 0 \) for all \( i \)'s;
2. \( b_1 = b_n = 1 \) and \( b_i = 2, \text{for } i \neq 1, n \);
3. \( p_1 = 2, A_1 = \{2, r\} \);
4. \( p_2 = 1 \) and \( A_2 = \{2\} \);

Then \( M \) has a double cover \( \tilde{M} \) such that its torus decomposition has a Seifert component with 3 boundary components.

**Proof.** Let \( M'_1 \) be the double cover of \( \overline{M}_1 \) provided by Construction B.2 and \( M'_2 \) be the double cover of \( M_2 \) by Construction B.1. \( M'_1 \) has one boundary component \( S \) with an induced involution by the covering map. \( M'_2 \) has 3 boundary components \( T_1, T_2 \) and \( T_3 \) where \( T_2 \) and \( T_3 \) are identified under the covering map and \( T_1 \) has an induced involution.

Glue \( M'_1 \) and \( M'_2 \) along \( S \) and \( T_1 \) and then two copies of \( \overline{M}_3 \cup \overline{M}_4 \cdots \overline{M}_n \) to \( M'_1 \cup_{T_1} M'_2 \) along \( T_2 \) and \( T_3 \) in such a way that the result is a covering of \( M \). This is the double covering which has a Seifert component with two singular fibers of multiplicity \( r \) and three boundary components (Fig. 11). \( \Box \)
Fig. 12. Double cover $\tilde{M}$ with a nonseparating torus.

**Lemma B.5.** Let $M$ be a closed oriented 3-manifold and $T$ be the collection of tori provided by torus decomposition of $M$. Suppose that $M \setminus T = M_1 \cup M_2$ with all $M_i$'s Seifert manifold such that:

1. $b_1 = 1$, $p_1 = 2$ and $A_1 = \{2, r\}$;
2. $b_2 = 1$, $p_2 = 2$ and $A_2 = \{2, s\}$.

Then $M$ has a double cover whose torus decomposition 2 Seifert components with 2 singular fibers of multiplicity $r$ and $s$.

**Proof.** Let $M'_1$ and $M'_2$ be the double covers of $\overline{M}_1$ and $\overline{M}_2$ provided by Construction 2. $M'_1$ and $M'_2$ have one boundary component where the covering map induces an involution on the boundary. Glue $M'_1$ and $M'_2$ along the boundary in such a way that the result is a double cover of $M$. This is the desired 2-cover. □

**Lemma B.6.** Let $M$ be a closed oriented 3-manifold and $T$ be the collection of tori provided by torus decomposition of $M$. Suppose that $M \setminus T = M_1 \cup M_2 \cup \cdots \cup M_n$, $n \geq 2$, with all $M_i$'s Seifert manifold, and

1. $b_i = b_n = 1$; $b_i = 2$ for $i \neq 1, n$;
2. $p_1 = p_n = 2$;
3. $A_1 = A_n = \{2\}$.

Then $M$ has a double cover with a non-separating torus.

**Proof.** The 2-cover $M$ is constructed as follows: Let $M'_1 \xrightarrow{f_1} \overline{M}_1$ and $M'_n \xrightarrow{f_2} \overline{M}_n$ be the double covers of $\overline{M}_1$ and $\overline{M}_n$ provided by Construction 3. Consider two copies of $M \setminus (M_1 \cup M_n)$ and glued them to $M'_1 \cup M'_n$ so that the result $\tilde{M}$ double-covers $M$ (Fig. 12) in such a ways that covering map restricted to $M'_1$ and $M'_n$ is $f_1$ and $f_2$ and on the rest of $\tilde{M}$ it is the trivial covering map. This can be done since the covering map $f_1$ ($f_n$) identifies different boundary components of $M'_1$ (resp. $M'_n$). □

**References**