Introduction to simplicial homotopy theory,
the art of breaking down topological spaces to points and rebuilding them up

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Contents

CHAPTER 1

Simplicial sets

1.1. Introduction

The primary goal of algebraic topology is to build algebraic probes for topological spaces in order to distinguish them from one another. However one can be more ambitious and ask for rebuilding the space out of these algebraic entities. The most natural topological spaces that our intuition can afford, are the topological spaces built out of geometric pieces such as lines, triangles, tetrahedra etc. These are the spaces that we call triangulated manifolds. Obviously this is too much to ask since such spaces are rare. So the first step would be to try approximate and compare a general topological space X with such geometric pieces. Thus we have to consider all the continuous maps $\Delta^n \to X$ because there is no preferred one. Next we have investigate how these rough images of simplexes fit together to fill the target space X. This step gets us to singular singular complex $(\mathrm{Sing}_*(X), \partial)$ where

$$\mathrm{Sing}(X)_n := \{f : \Delta^n \to X | f \text{ is continuous } \}$$

and $\Delta^n := \{(t_0, t_1, \cdots t_n) | \sum t_i = 1, 0 \leq t_i \leq 1\}$ is the geometric simplex. The singular chains complex $S_*(X)$ is the graded free abelian group whose generators are the the element of $\mathrm{Sing}_*(X)$.

The only structural data which has so far manifested is the collection of face maps which are of geometric nature. Now we can start asking many natural questions such as: can we build back the space X out of the singular complex $(\mathrm{Sing}_*(X), \partial)$. How about other (infinite dimensional)topological space such loop spaces and paths spaces related to X? The next layer of questions would be the functorial properties of the construction $(\mathrm{Sing}_*(X), \partial)$. It turns out that a continuous map gives rise to a map of simplcial sets and chain complexes and more; two continuous homotopic maps give rise to two chain homotopic maps. The proof of the last statement is quite interesting because it requires triangulating $\Delta^n \times I$. This process known as the prism operation relies on sending the simplices of each factor to some degenenrate simplices (i.e.lower geometric dimension) in the product. For a similar reason, computing the homology of the cartesian product of spaces uses the Eilenberg-Zilber map which also requires the degeneracy maps. While trying to understand further more the internal algebraic structure of the singular chain complex $(S_*(X), \partial)$, one has to look at the most important map in topology i.e. the diagonal map

$X \to X \times X$. The map induced by the diagonal on the singular chains together with Alexander-Whiteny (the left inverse of Eilenberg-Zilber map) equips $S_*(X)$ with a remarkable coassociative product. This coproduct, whose dual, the cup product, is better known) plays an essential role in Adams' cobar construction which computes the homology of the based loop space of X. We have started to convince ourselves that if we are interested in studying X, we should also consider the degeneracy maps as part of the structure. The degeneracy maps correspond to the situations where two vertices of a simplex are in fact geometrically identical therefore the geometric dimension is lower. So our holy grail will be $\mathrm{Sing}_*(X)$ with its face d_i and s_i degeneracy maps which is prototype of simplicial set. We have task ourselves with distinguishing topological spaces among the simplicial sets and rebuilding homotopy theory out of simplicial sets.

1.2. Basic notions

As explained in the introduction, we want the singular complex of a space to be prototype of simplicial sets, so naturally the geometric simplices

$$\Delta^n := \{(t_0, t_1, \cdots t_n) | \sum t_i = 1, 0 \leq t_i \leq 1\} \subset \mathbb{R}^{n+1}$$

should form an example of cosimilicial set.

Inspired by this example, we define the simplicial category $\mathbf{\Delta}$ whose objects are $[n] := \{0, \cdots n\}$ for $n = 0, 1, \cdots$. The set of morphism $\mathrm{Hom}_{\mathbf{\Delta}}([n],[m])) := \{f : [n] \to [m] | f$ **order preserving**$\}$

For instance, for each i we have the coface morphisms $d^i : [n] \to [n+1]$ and codegeneracy morphisms $s^i : [n+1] \to [n]$ defined as follows:

$$d^i(j) = \begin{cases} j \text{ if } 0 \leq j < i \\ j+1 \text{ if } i \leq j \end{cases}$$

and

$$s^i(j) = \begin{cases} j \text{ if } j < i+1 \\ j-1 \text{ if } i+1 \leq j \end{cases}$$

for $O \geq i \leq n$.

Definition 1.1. A cosimplicial set is covariant functor $X : \mathbf{\Delta} \to \mathbb{S}\mathrm{et}$ where is $\mathbb{S}\mathrm{et}$ is the category of sets. Similarly a simplicial set is covariant functor $X : \mathbf{\Delta}^{op} \to \mathbb{S}\mathrm{et}$ where is $\mathbf{\Delta}^{op}$ is the opposite category of $\mathbf{\Delta}$.

Notation 1.2. Let $f \in Hom_{\Delta}(-,-)$ be a morphism in the simplicial category. For a cosimplicial (resp. simplicial) set X, $f_* := X(f)$ (reps. $f^* := X(f)$) denotes the corresponding morphism in the category $\mathbb{S}\mathrm{et}$.

Example 1.3. The geometric simplices , $\Delta^* : n \mapsto \Delta^n$ form a cosimplicial object. For a morphism $f \in \mathrm{Hom}_{\mathbf{\Delta}}([n],[m])$, $\Delta^*(f) : \Delta_n \to \Delta_m$ is defined by $\Delta^*(f)(t_0, t_1 \cdots t_n) := (s_0 \cdots s_m)$ where $s_i = \sum_{t_j \in f^{-1}(s_i)} t_j$.

In this example $\Delta^*(d^i)$ are precisely the inclusion $\Delta^n \hookrightarrow \Delta^n$ as the i-th face.

EXAMPLES 1.4. For a topological space X, the singular complex $\mathrm{Sing}(X)$ functor $\mathrm{Sing}(X) : \mathbf{\Delta}^{op} \to \mathbb{S}\mathrm{et}$, given by the sets of singular simplices

$$\mathrm{Sing}_n(X) := C^0(\Delta^n, X)$$

form a simplicial set. Therefore we have functor $Sing : Top{-}Spaces \to \mathrm{s}\mathbb{S}\mathrm{et}$

LEMMA 1.5. *Any morphism in* $\mathrm{Hom}([n],[m])$ *has a unique factorisation of the form*

$$f = d^{i_l} \cdots d^{i_1} s^{j_1} \cdots s^{j_k}$$

where $n - k = m - l$

PROOF. An order preserving map from $[n]$ to $[m]$ is determined by its image (or its complement) and the equivalent classes of $\overset{f}{\sim}$, where $x \overset{f}{\sim} y$ iff $f(x) = f(y)$. Moreover f induces a bijection between these equivalence classes and its image. Suppose that $i_1 < i_2 \cdots < i_l$ are the distinct elements of $[m] \setminus \mathrm{Im}(f)$, and $j_1, \cdots j_k$ is a the maximal sequence for which $f(j_p) = f(j_p + 1)$. The equivalence described above has $n - k$ element and the image of f has $m - l$, therefore $n - k = m - l$. We also noticed that the map $d^{i_l} \cdots d^{i_1}$ does not have $i_1, i_2 \cdots, i_l$ in its image and the map $s^{j_1} \cdots s^{j_k}$ define an equivalence relation $\overset{\phi}{\sim}$ whose number of classes is smaller than number of classes of f. In other words, if we set $\phi = d^{i_l} \cdots d^{i_1} s^{j_1} \cdots s^{j_k}$ then $\phi(j_p) = \phi(j_p + 1)$ for all p and $i_1, i_2 \cdots, i_l$ are not in the image of ϕ. In fact $\mathrm{Im}(\phi) = \mathrm{Im}(f)$. Since ϕ is also order preserving, it establishes a bijection between its equivalence classes (which there are at most $n - k$) and its image (which has exactly $m - l$ elements). Since $m - l = n - k$, therefore ϕ has exactly the same number of equivalence classes (and the imgae) therefore $f = \phi$. $\square$

Since d^i and s^i verify the relations

$$\begin{aligned} d^i d^j &= d^{j+1} d^i \qquad i \le j \\ s^j s^i &= s^i s^{j+1} \qquad i \le j \\ s^j d^i &= \begin{cases} d^i s^{j-1} & i < j \\ 1 & i = j, j+1 \\ d^{i-1} s^j & \text{otherwise} \end{cases} \end{aligned} \tag{1.1}$$

By Lemma1.5 and relations (1.1) we have

PROPOSITION 1.6. *The morphisms of the simplicial category* $\mathbf{\Delta}$ *are generated by* d^i *and* s^i *subject to the relations (1.1).*

We call d^i's and s^i's respectively the coface and codegeneracy maps. Similarly the the morphisms of the opposition category Δ^{op} are generated by the (dual) generators $d_i : [n+1] \to [n]$ and $s_i : [n] \to [n+1]$, for $0 \leq i \leq n$ subject to relations,

$$
\begin{aligned}
d_j d_i &= d_i d_{j+1} \qquad i \leq j \\
s_i s_j &= s_{j+1} s_i \qquad i \leq j \\
d_i s_j &= \begin{cases} s_{j-1} d_i & i < j \\ 1 & i = j, j+1 \\ s_j d_{i-1} & \text{otherwise} \end{cases}
\end{aligned}
\tag{1.2}
$$

Corollary 1.7. *A simplicial set consists of a collections of sets $\{X_n\}$ together with maps $d_i : X_{n+1} \to X_n$ and $s_i : X_n \to X_{n+1}$ subject to relations (1.2).*

Proof. If $X : \Delta^{op} \to \mathbb{S}\mathrm{et}$ is a given simplicial set then above mentioned morphism $X(d_i) : X_{n+1} \to X_n$ and $X(s_i) : X_n \to X_{n+1}$ are the above-mentioned morphism. For simplicity we denote $X(d_i)$ and $X(s_i)$ by d_i and s_i. □

We call X_0 the set of vertices and X_n the set of n-simplices. We call d^i's and s^i's respectively the face and degeneracy maps.

Definition 1.8. Simplcial sets form a category $\mathrm{s}\mathbb{S}\mathrm{et}$. The morphism sets $\mathrm{Hom}_{\mathrm{s}\mathbb{S}\mathrm{et}}(X, Y)$ from a simplicial set X to Y is defined to be the set of natural transformation between X and Y as functors. This is equivalent to have a collection of maps $f_n : X_n \to Y_n$ which commute with structural maps d_i and s_i.

Definition 1.9. In a simplicial set simplex $\{X_n\}_n$, an n-simplex x is called degenerate if it belongs to the image of a degeneracy maps s_i.

Proposition 1.10. *A n-simplex x in a simplicial set $\{X_n\}$, is either nondegenerate or there is a presentation $x = s_{j_1} s_{j_2} \cdots s_{j_k} y$ with $j_1 \leq j_2 \cdots \leq j_k$ where y is unique*

Proof. **Existence of the presentation:** If x is degenerate then we are done, if not then there is i_1 and y_1 such that $x = s_{i_1} y_1$. Continuing this process and by finiteness of the dimension, we can write $x = s_{i_1} s_{i_2} \cdot s_{i_k} y$ where y is nondegenrate. Now using the relations (1.2) we can rewrite the expression in the form $x = s_{j_1} s_{j_2} \cdots s_{j_k} y$ such that $j_1 < j_2 \cdots < j_k$.

Uniquness Suppose that $s_{j_1} s_{j_2} \cdots s_{j_k} y = s_{i_1} s_{j_2} \cdots s_{i_l} z$. Let $D = d_{j_k} d_{j_{k-1}} \cdots d_{j_1}$. Then we have $D s_{j_1} s_{j_2} \cdots s_{j_k} = id$, therefore

$$y = D s_{i_1} s_{i_2} \cdots s_{i_l} z.$$

Let $S = s_{i_1} s_{i_2} \cdots s_{i_l}$, so we have $y = DSz$. Using the simplicial relations (1.2) we can write $DS = S'D'$ where D' is a product of some face maps and S' is a product of some degeneracy maps. This give $y = S'D'z$ which would contradict y being nondegenerate unless S' is void/nonexisting. Therefore we have proved that y is a face of z and symmetrically y is face of z hence $y = z$. □

1.3. Yoneda Lemma

As we know there are some topological space which are triangulate therefore they provide us a host of simplicial complexes. It turns ou that we can turn them into simplicial sets as follows.

EXAMPLE 1.11. Let $K = \{s_i\}$ be an oriented simplicial complex i.e. with a partial ordering on $Vert(K)$ such that the induced orientation on each simplex is a linear ordering. We construct a simplicial set $\{K_s(n)\}_n$ by

$$K_s(n) := \{[v_{i_O} \leq v_{i_1} \cdots \leq v_{i_n}] | v_{i_O}, v_{i_1} \cdots v_{i_n} \text{ spans a simplex in } K\}.$$

Note that in a n-simplex, we allow repeated vertices. The structural maps are given by

$$d_k([v_{i_O} \leq v_{i_1} \cdots \leq v_{i_n}]) = [v_{i_O} \leq v_{i_1} \cdots v_{i_{k-1}} \leq v_{i_{k+1}} \cdots \leq v_{i_n}]$$

and

$$s_k(([v_{i_O} \leq v_{i_1} \cdots\cdots \leq v_{i_n}]) == [v_{i_O} \leq v_{i_1} \cdots v_{i_k} \leq v_{i_k} \cdots \leq v_{i_n}]$$

We give an explicit example of the construction above. Let K be the simplicial complex consisting of the standard geometric simplex $\Delta^n = [e_0, e_1 \cdots e_n] \subset \mathbb{R}^n$ and all of its faces. We have

$$K_s(m) = \{[e_{i_0}, e_{i_1} \cdots e_{i_m}] | \quad i_0 \leq i_1 \leq \cdots \leq i_m \qquad \& \qquad 0 \leq i_k \leq n\}$$

which as a set, it is in bijection with the set of order preserving maps from $\{0, \cdots m\}$ to $\{0, \cdots n\}$.

EXAMPLES 1.12. A point $\Delta^0 = \{0\}$ as a simplicial set, has one n-simplex for each, they are $X_0 = \{[0]\}$, $X_1 = \{[0, 0]\}$, $X_2 = \{[0, 0, 0]\} \cdots$

Similarly the interval Δ^1 has , as a simplcial set, $n + 2$ n-simplices for each n, namely

$$X_0 = \{[0], [1]\}$$
$$X_1 = \{[0, 0], [0, 1], [1, 1]\}$$
$$X_2 = \{[0, 0, 0], [0, 0, 1], [0, 1, 1], [1, 1, 1]\}$$

etc.

So we define the simplicial set Δ_n whose set of m-simplices is

$$\Delta_n[m] := \mathrm{Hom}_{\boldsymbol{\Delta}}([m], [n])$$

LEMMA 1.13. *(Yoneda lemma) There is a natural bijection*

$$\mathrm{Hom}_{s\mathbb{S}\mathrm{et}}(\Delta_n, X) \simeq X_n$$

PROOF. The bijection $\psi : \mathrm{Hom}_{\mathbb{sSet}}(\Delta_n, X) \to X_n$ is given by sending a natural transformation $T : \Delta_n \to X_n$ to $\psi(T) := T(id_n) \in X_n$. Here we think of id_n as an element of $\Delta_n[n] = \mathrm{Hom}_{\mathbf{\Delta}}([n],[n])$. The inverse Υ of ψ is given by $\Upsilon_n(x) : \Delta_n \to X$ defined by $\Upsilon_n(x)(f) = X(f)(x) \in X_m$ for an element $f : [m] \to [n]$ of $\Delta_n[m]$. This defines a natural transformation if the diagram

$$\begin{array}{ccc} \Delta_n[m] & \xrightarrow{f \mapsto X(f)(x)} & X_m \\ {\scriptstyle f\mapsto g\circ f}\downarrow & & \downarrow{\scriptstyle X(g)} \\ \Delta_n[p] & \xrightarrow[h \mapsto X(h)(x)]{} & X_p \end{array} \tag{1.3}$$

is commutative for all $g : [p] \to [m] \in \mathrm{Hom}_{\mathbf{\Delta}}([p],[m])$. The commutativity is indeed a consequence of X begin a functor from the the opposite categor the identity

$$X(g)(X(f)(x) = (X(g) \circ X(f))(x) = X(g \circ f)(x)$$

□

In order to be able to define the notion of homotopy between simplicials maps we need to define the basis operation on the simplicial sets.

For a simplicial set $X = \{X_n\}_{n\geq 0}$, the n-th skeleton $\mathrm{Sk}_n(X)$ is by definition the smallest simplicial subset of X containing all the nondegenerate simplices of dimension at most n. We have a natural filtration

$$\mathrm{Sk}_0(X) \subset \mathrm{Sk}_1(X) \cdots$$

Now we can define a notion of dimension for a simplicial set. A simplicial set is said to be finite dimensional if for some n, $X = \mathrm{Sk}_n(X)$. If X is finite dimensional, the dimension of X is the smallest n for which $X = \mathrm{Sk}_n(X)$. For instance, the simplicial Δ_n is of dimension n. The boundary $\partial\Delta_n$ of Δ_n is the $(n-1)$-skeleton of X. One should think of $\partial\Delta_n$ as a simplicial model for the the sphere S^{n-1}. Intuitively we can notice that $\partial\Delta_n$ should a be union of a other simplicial subsets $\partial_i\Delta_n$. The simplicial subset $\partial_i\Delta_n$ is the simplmicial subset generated by $d^i \in \mathrm{Hom}_{\mathbf{\Delta}}([n-1],[n]) = \Delta_n[n-1]$.

PROPOSITION 1.14. *If x is a nondegenerate n-simplex then any face $d_i x$ belongs to $\mathrm{Sk}_{n-1}(X)$. As a consequence we have a push out,*

$$\begin{array}{ccc} \cup_{x\in e_n(X)}\partial\Delta_n & \xrightarrow{incl.} & \cup_{x\in e_n(X)}\Delta_n \\ {\scriptstyle \Upsilon_{n-1}}\downarrow & & \downarrow{\scriptstyle \Upsilon_n} \\ \mathrm{Sk}_{n-1}(X) & \xrightarrow[incl.]{} & \mathrm{Sk}_n(X) \end{array} \tag{1.4}$$

where $e_n(X)$ denotes the set of nondegenerate n-simplices and Υ_n is the map provided by Yoneda lemma.

PROOF. If $d_i(x)$ is nondenerate then by defining it belongs to $\mathrm{Sk}_{n-1}(X)$. Otherwise it is of the form $s_{i_1}s_{i_2}\cdots s_{i_k}(y)$ where y is nondegenerate simplicies of dimension less than $n-1$ therefore $y \in \mathrm{Sk}_{n-1}(X)$. □

1.4. Basic operations on simplcial sets

The cartesian product. For simplicial sets $X = \{X_n\}_n$ and $Y = \{Y_n\}_n$, the simplicial cartesian product is defined by

$$(X \times Y) = X_n \times Y_n.$$

The face and degeneracy maps are defined in a diagonal manner i.e.

$$d_i(x,y) = (d_i(x), d_i(y))\& \quad s_i(x,y) = (s_i(x), s_i(y)) \tag{1.5}$$

REMARK 1.15. Note that that product of two degenerate simplices in X and Y is not necessarily a degenerate simplex in $X \times Y$. You can find the importance of this observation in the following example.

EXAMPLE 1.16. As explained earlier in Example 1.11, unit interval $I = [0,1]$ can be enriched into a simplidcial set therefore we can consider the simplicial cartesian product $I \times I$. Note that the simplices of $I = \Delta_1$ are the sequences of the form $[0,\cdots 0,1,\cdots 1]$. So in the cartesian product, there are 4 0-simplices are

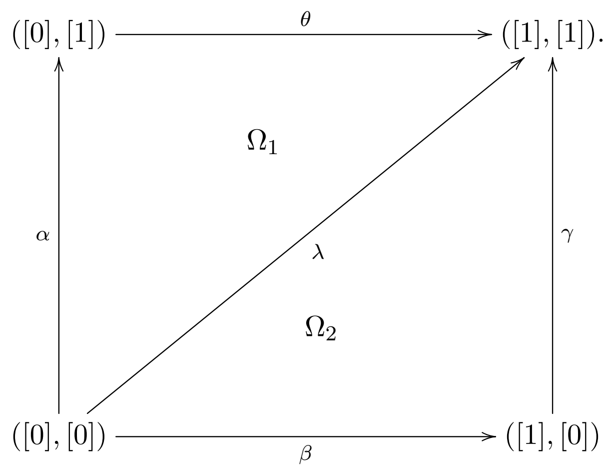
$$([0],[0]),([0],[1]),([1],[0]),([1],[1]).$$

We have 5 nondegenerate 1-simplices

$$\begin{aligned} \alpha &= ([0,0],[0,1]) \\ \beta &= ([0,1],[0,0]) \\ \gamma &= ([1,1],[0,1]) \\ \theta &= ([0,1],[1,1]) \\ \lambda &= ([0,1],[0,1]) \end{aligned} \tag{1.6}$$

and two nondegenerate 2-simplices $\Omega_1 = ([0,0,1] \times [0,1,1])$ and $\Omega_2 = ([0,1,1] \times [0,0,1])$.

(1.7)



Union For two simplicial sets $X = \{X_i\}$ and $Y = \{Y_i\}$, union $X \cup Y$ is a simplicial set whose n-simplices are

$$(X \cup Y)_n := X_n \cup Y_n.$$

The faces maps and degeneracy maps are those of X or Y.

Wedge For two simplicials sets X and Y, the wedge product $X \wedge Y$ is simplicial subset of $X \times Y$

$$X \wedge Y := (X \times [*_y]) \cup ([*_x] \cup Y)$$

Here $[*_x]$ and $[*_y]$ are the (simplicial) base points of X and Y.

1.5. Simplicial object in a category

Definition 1.17. A simplicial object in a category $\mathcal{C}$ is covariant functor $F : \mathbf{\Delta}^{op} \to \mathcal{C}$. Similarly a cosimplicial object in $\mathcal{C}$ is a covariant functor $F : \Delta \to \mathcal{C}$

For instance, a simplicial abelian group is a functor $F : \mathbf{\Delta}^{op} \to \mathbb{Z} - Module$ i.e. it consists of a collection $\{G_n\}_n$ of abelian groups G_n with group homomorphism $d_i : G_n \to G_{n-1}$ and $s_i : G_{n-1} \to G_n$ which satisfy the usual simplicial relations

Definition 1.18. By a simplicial chain complex over a unital ring R, we mean a simplicial object in the category of R-Modules. It consists of a sequence R-module C_n together with maps $d_i : C_n \to C_{n-1}$ and $S_i : C_{n-1} \to C_n$ subject to the usual simplicial identities. It has an underlying chain complex (C_*, ∂) where $\partial : C_n \to C_{n-1}$ given by $\partial = \sum (-1)^i d_i$

1.5.1. Chain complex of a simplicial set. Let R be a unital ring. We start with a simplicial set $X = \{X_n\}$. We set $C_n(X) = \oplus_{x \in X_n} R\langle x \rangle$ to be the free R-module generated by $x \in X_n$.

The face and degeneracy maps induces the face and degeneracy maps of $C_*(X)$. s In other words the functor $C_*(X) : \mathbf{\Delta}^{op} \to R - Module$ defined by

$$C_*(X)[n] := C_n(X)$$

and

$$C_*(X)(f) : C_n(X) \to C_m(X) \quad C_*(X)(f)(x) = f(x), \quad \forall x \in X_n$$

for all $f : [m] \to [n] \in \mathrm{Hom}_{\mathbf{\Delta}}$, is a simplicial chain complex. Of course $C_*(X)$ has a underlying a chain complex whose differential is given by $\partial = \sum(-1)^i d_i : C_n(X) \to C_{n-1}(X)$. We recall the standard notation for the group of cycles

$$Z_i(C) = \ker(\partial : C_i(K) \to C_{i-1}(K))$$

and boundary element

$$B_i(C) = \mathrm{Im}(\partial : C_{i+1}(C) \to C_i(K))$$

Example 1.19. The simplicial chain complex of the simplicial set $\mathrm{Sing}_n(X)$ is called the singular chain complex of X and is denoted $\{S_n(X)\}_n$.

Proposition 1.20. *The collection of simplicials sets $\{\Delta_n\}_n$ form a cosimplicial set in the category of simplicial sets.*

Proof. For $f : [m] \to [n]$, the $f* : \Delta_m \to \Delta_n$ on a k-simplex is defined by

$$h \in \Delta_m = \mathrm{Hom}_{\Delta}([k], [m]) \mapsto f \circ h \in \Delta_n = \mathrm{Hom}_{\Delta}([k], [n]).$$

The funtoriality $(f \circ g)_* = f_* \circ g_*$ is obvious. □

Definition 1.21. Similarly we can define the maps between simplicial objets $X, Y : \mathbf{\Delta}^{op} \to \mathcal{C}$ in a category $\mathcal{C}$. These are the natural transformation $X \stackrel{F}{\to} Y$. More explicitly, a map f between two simplicial R-modules $\{X_n\}$ and $\{Y_n\}_n$ consists of a sequence of R-linear maps $f_n : X_n \to Y_n$ which commute with the (simplicial) structural maps d_i and s_i.

In particular f induces a map of chain complexes $C_*(f) : C_*(X) \to C_*(Y)$ which on generator is given by $x \in X_n \mapsto f(x_n) \in Y_n$

Definition 1.22. A simplicial chain homotopy between two simplcial chain maps $g, f : (C_*, \partial_C = \sum(-1)^i d_i) \to (D_*, \partial_D = \sum(-1)^i d_i)$ is a sequence of $h_i : C_n \to D_{n+1}$, for $0 \le i \le n$ such

$$
\begin{aligned}
d_0 h_0 &= f \\
d_{n+1} h_n &= h \\
d_i h_j &= h_{j-1} d_i \quad i < j \\
d_{j+1} h_j &= d_{j+1} h_{j+1} \\
d_i h_j &= h_j d_{i-1} \quad i > j+1 \\
s_i h_j &= h_{j+1} s_i \quad i \leq j \\
s_i h_j &= h_j s_{i-1} \quad i > j
\end{aligned} \tag{1.8}
$$

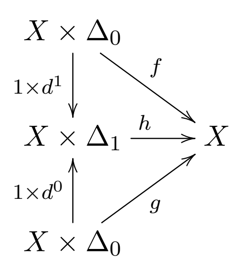
If these relations above holds then $h = \sum_{i=0}(-1)^i h_i$ is a chain homotopy in the usual sense i.e

$$\partial_D h + h \partial_C = f - g$$

1.6. Simplicial homotopy

DEFINITION 1.23. Two simplicial maps $f, g : X \to Y$ are said to be homotopic, we write $f \sim g$, if there is a simplicial map $h : X \times \Delta^1 \to Y$ such that $h|_{X \times [0]} = f$ and $h|_{X \times [1]} = f$. Here [0] and [1] are singleton as simplicial sets (see Example 1.12) In other words we have a commutative diagram

(1.9)



Here $d^0, d^1 : \Delta_0 = \mathrm{Hom}_\Delta(-, [0]) \to \Delta_1 = \mathrm{Hom}_\Delta(-, [1])$ are induced are induced by $d^0, d^1 : [0] \to [1]$.

PROPOSITION 1.24. *For composable the simplicial maps f_1 and f_2, If $f_1 \sim g_1$ and $f_2 \sim g_2$ the $f_1 \circ f_2 \sim g_1 \circ g_2$*

PROOF. it is worth to detail □

PROPOSITION 1.25. *Let X and Y be two simplicial abelian groups. A simplicial homotopy $h : X \times \Delta_1 \to Y$ between maps of simplicial abelian groups $f, g : X \to Y$, induces a simplicial chain homotopy between induced maps $C_*(f)$ and $C_*(g)$ on the the Moore complexes.*

PROOF. For each n and $0 \leq i \leq n$, Let $\eta_i^n : [n] \to [1]$ be simplicial morphism defined $\eta_i^n(j) = 0$ if and only if $j \leq i$. One should. think of η_i^n as a n-simplex in Δ_1. We observe that

$$
d_i(\eta_j^n) = \begin{cases} \eta_{j-1}^{n-1} & i \leq j \\ \eta_j^{n-1} & i > j \end{cases} \tag{1.10}
$$

and

$$s_i(\eta_j^n) = \begin{cases} \eta_{j+1}^{n+1} & i \le j \\ \eta_j^{n+1} & i > j \end{cases} \tag{1.11}$$

Let $h_i : X_n \to Y_{n+1}$, $0 \le i \le n$, be the map defined by

$$h_i(a) := h((s_i(a), \eta_i^{n+1})) \tag{1.12}$$

Now one can check easy that relations (1.13) hold. For instance,

$$d_0 h_0(a) = d_0 h((s_0(a), \eta_0^{n+1})) = h_0((d_0 s_0(a), d_0 \eta_0^{n+1}) = h_0(a, ([1, 1 \cdots 1]) = f(a)$$

For $i < j$

$$d_i h_j(a) := d_i h((s_j(a), \eta_j^{n+1})) = h(d_i s_j(a), d_i \eta_j^{n+1}) = h(s_{j-1} d_i(a), \eta_{j-1}^{n-1}) = h_{j-1}(d_i(a))$$

□

Corollary 1.26. *Let X and Y be two simplicial sets. A simplicial homotopy. $H : X \times \Delta_1 \to Y$ between simplicial maps $f, g : X \to Y$, induces a chain homotopy between $C_*(f)$ and $C_*(g) : C_*(X) \to C_*(Y)$.*

Proof. Apply Proposition1.25 to the simplicial groups $C_*(X)$ and $C_*(Y)$ □

The proof of the following result is identical to that of Proposition 1.25.

Corollary 1.27. *A simplicial homotopy $h : X \times \Delta_1 \to Y$ between. simplicial maps $f, g : X \to Y$ is equivalent to a collection of maps $h_i : X_n \to Y_{n+i}$ which satisfies the identites*

$$\begin{aligned} d_0 h_0 &= f \\ d_{n+1} h_n &= h \\ d_i h_j &= h_{j-1} d_i \quad i < j \\ d_{j+1} h_j &= d_{j+1} h_{j+1} \\ d_i h_j &= h_j d_{i-1} \quad i > j+1 \\ s_i h_j &= h_{j+1} s_i \quad i \le j \\ s_i h_j &= h_j s_{i-1} \quad i > j \end{aligned} \tag{1.13}$$

1.7. Adjunction

Let $T : \mathcal{C} \to \mathcal{D}$ and $S : \mathcal{D} \to \mathcal{C}$ be two covariant functors and

$$\phi_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, S(B)) \to \mathrm{Hom}_{\mathcal{D}}(T(A), B)$$

and

$$\psi_{A,B} : \mathrm{Hom}_{\mathcal{D}}(T(A), B) \to \mathrm{Hom}_{\mathcal{C}}(A, S(B))$$

be two natural transformations between bi-functors $\mathrm{Hom}(A, S(B))$ and $\mathrm{Hom}(T(A), B)$ on the category $\mathcal{C}^{op} \times \mathcal{D}$. Being natural transformation amount to the identity

$$
\begin{aligned}
f \circ \phi(g) &= \phi(s(f) \circ g) \\
\psi(f \circ h) &= s(f) \circ \psi(h)
\end{aligned}
\tag{1.14}
$$

for $g \in \mathrm{Hom}_{\mathcal{C}}(A, S(B)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(A, S(B))$ and $f : B \to B'$ and

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(A, S(B)) & \underset{\psi_{A,B}}{\overset{\phi_{A,B}}{\rightleftarrows}} & \mathrm{Hom}_{\mathcal{D}}(T(A), B) \\
\downarrow{\scriptstyle S(f)\circ -} & & \downarrow{\scriptstyle f\circ -} \\
\mathrm{Hom}_{\mathcal{D}}(A, S(B')) & \underset{\psi_{A,B'}}{\overset{\phi_{A,B'}}{\rightleftarrows}} & \mathrm{Hom}_{\mathcal{C}}(T(A), B')
\end{array}
\tag{1.15}
$$

Similarly, for $k : A' \to A$, $g : A \to S(B)$, $h : T(A) \to B$

$$
\begin{aligned}
\phi(g \circ k) &= \phi(g)T(k) \\
\psi(h \circ T(k)) &= \psi(h) \circ k
\end{aligned}
\tag{1.16}
$$

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(A, S(B)) & \underset{\psi_{A,B}}{\overset{\phi_{A,B}}{\rightleftarrows}} & \mathrm{Hom}_{\mathcal{D}}(T(A), B) \\
\downarrow{\scriptstyle -\circ k} & & \downarrow{\scriptstyle -\circ T(k)} \\
\mathrm{Hom}_{\mathcal{D}}(A', S(B)) & \underset{\psi_{A',B}}{\overset{\phi_{A',B}}{\rightleftarrows}} & \mathrm{Hom}_{\mathcal{C}}(T(A'), B)
\end{array}
\tag{1.17}
$$

Remark 1.28. Using theidentities (1.14) and (1.16) we can prove that the natural transfor mation ϕ and ψ are natural with respect S and T. In other words two natural $\tau : S \to S'$ and $\sigma : T \to T'$ gives rises to a ma natural transformation ϕ' and ψ' .

Definition 1.29. We say that the functors T and S for an adjunction if $\phi_{A,B} \circ \psi_{A,B} = id$ and $\psi_{A,B} \circ \phi_{A,B} = id$.
We call T a left adjoint of S and S is a right adjoint of T.

The natural transformations ϕ and ψ provide us two natural transformations $\Psi : 1 \to ST$ and $\Phi : TS \to 1$ called respectively unit and counit called .

They are defined by

$$\Phi_B := \phi_{S(B),B}(id_{S(B)}) \in \mathrm{Hom}(TS(B), B)$$

and

$$\Psi_A = \psi(1_{T(A)}) \in \mathrm{Hom}(A, ST(A)).$$

The natural transformations ϕ and ψ provide us two natural transformations $\Psi : 1 \to ST$ and $\Phi : TS \to 1$ called respectively unit and counit called .

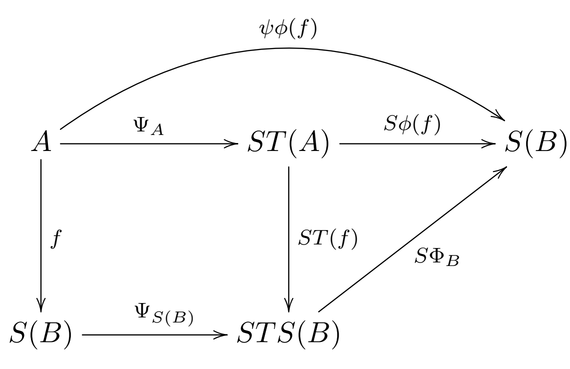
Notice that the defining the natural transformations Ψ and Φ does not require the idnentity $\phi \circ \psi = id$ and $\phi \circ \psi = id$.

Proposition 1.30. *$\psi \circ \phi = 1$ if and only if the composition $S \xrightarrow{\Psi S} STS \xrightarrow{S\Phi} S$ is the identity natural transformation.*

Similarly, $\phi \circ \psi = 1$ if and only the composition $T \xrightarrow{\Psi S} TST \xrightarrow{S\Phi} T$ is the identity natural transformation.

Proof. It follows from the commutative diagram below.

(1.18)



□

Definition 1.31. The adjunction given by ϕ and ψ is called an equivalence if $\phi \circ \psi = id$ and $\psi \circ \phi = id$

Proposition 1.32. *The equivalence adjunctions are natural with respect to the natural transformation $\tau : T' \to T$ and $\sigma : S \to S'$. More precisely if $\tau : T' \to T$ exists then $\sigma : S \to S'$ making the diagram below commutative, and vice versa.*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(A,S(B)) & \underset{\psi_{A,B}}{\overset{\phi_{A,B}}{\rightleftarrows}} & \mathrm{Hom}_{\mathcal{D}}(T(A),B) \\ {\scriptstyle \sigma(B)\circ -}\downarrow & & \downarrow{\scriptstyle -\circ\tau(A)} \\ \mathrm{Hom}_{\mathcal{D}}(A,S'(B)) & \underset{\psi'_{A,B}x}{\overset{\phi'_{A,B}}{\rightleftarrows}} & \mathrm{Hom}_{\mathcal{C}}(T'(A),B)\end{array} \tag{1.19}$$

PROOF. If τ is given, in order to find $\sigma : S(B) \to S'(B)$, one should chase digram from upper-left corner for $A := S(B)S$. we will get

$$\sigma(B) = \psi'_{S(B),B}(\phi_{S(B),B}(id_{S(B)})\tau(S(B))),$$

One can then check that σ makes the diagram commutative for all A and B. Similarly τ can be defined in terms of σ by

$$\tau(A) = \phi'(\sigma(T(A)) \circ \psi_{A,T(A)}).$$

□

1.8. Geometric realization and adjunction

We already know that there is a functor $S : TopSpace \to \mathrm{s}\mathbb{S}\mathrm{et}$ which is given by the singular simplices $\mathrm{Sing}_n(X) = C^0(\Delta^n, X)$ of X. Now we intend to introduce a left adjoint for S.

DEFINITION 1.33. Geometric realization of a simplicial set $K = \{K_n\}_n$ is the set of equivalence relation

$$T(K) = |K| := \sqcup_{n\geq 0} K_n \times \Delta^n / \sim$$

where the equivalence relation is generated by the relations $(d_i x, p) \sim (x, d^i p)$ and $(s_i x, p) \sim (x, s^i p)$.

The topology of $T(k)$ is quotient topology of $\sqcup K_n \times \Delta^n$ which itself is equipped with the thweak topology (on the union). This means that $U \subset \sqcup_{n\geq 0} K_n \times \Delta^n i$ is open if and only if for each n, $U \cap K_n \times \Delta^n i$ is open in $K_n \times \Delta^n i$. Here $K_n \times \Delta^n$ has product topology.

DEFINITION 1.34. A pair $(k, w) \in |K|$ is called an ideal point if k is nondegenerate n-simplex and w is in the interior of the geometric n-simplex.

PROPOSITION 1.35. *Each class in $|K|$ has a unique representative (x, p) which is an ideal point. where x is nondegenerate and p in the interior $\dot{\Delta}^n$ of the geometric simplex Δ^n.*

Proof. Let a be a class in $|K|$, starting with a representative (z, q') we can assume that q' is interior otherwise it can be wirtten $q' = d^{i_1} \cdots d^{i_l} q$ where q is in the interior of a geometric simplex. Then

$$a = [(z, q')] = [(z, d^{i_1} \cdots d^{i_l} q)] = [(d_{i_l} \cdots d_{i_1} z, q)]$$

So we can suppose that a has a representative (y, q) where q is in the interior. By Proposition 1.10, either y is non degenerate or $y = s_{i_1} s_{i_2} \cdots s_{i_k}(x)$ where z is nondegenerate. In either case, a has (y, q) or $(x, s^{i_k} \cdots s^{i_2} s^{i_1}(q))$ as representative. In either case the first coordinate is nondegenerate. Also note that if q is interior then $s^{i_k} \cdots s^{i_2} s^{i_1}(q)$ is also in interior. This is because the codegeneracy maps of the geometric simplices are of the form $(t_0, \cdots, t_n) \mapsto (t_0, \cdots, t_i + t_{i+1}, \cdot, t_n)$, so if all $t_i > 0$ the same is true for its image. □

Corollary 1.36. *$|K|$ is a CW-complex.*

Proof. By Proposition 1.35 K is a union of open cell whose boundaries are included in lower dimension cells. The topology of $|K|$ is the weak topology which is the topology of CW-complexes. □

Theorem 1.37. *For all simplicial sets K and L, there is a natural bijection. $|K \times L|$ and $|K| \times |L|$. Moreover this bijection is a homeomorphism if $|K|$ or $|L|$ is locally finite or if they are both countable.*

Proof. We introduce $\pi_1 \times \pi_2 : |K \times L| \to |K| \times |L|$ as follows: For a class $x \in |K \times L|$ we choose a representative (k, l, w) where (k, l) is nondegenerate n-simplex and w is in interior of a simplex Δ^n. Note that this does not mean that k or l are nondegenerate. Nonetheless the pairs (k, w) (l, w), which are not necessarily ideal points, represent respectively two classes in $\pi_1(x) \in |K|$ and $\pi_2(x) \in |L|$.

Now we construct the inverse of $\pi_1 \times \pi_2$. Let (k, u) and (l, v) be two ideal points representing two classes in $x \in |k|$ and $y \in |l|$ and $u = (t_0, \cdots t_m)$ and $v = (t'_0 \cdots t'_n)$. We set

$$u^p := \sum_1^p t_i \quad \& \quad v^q := \sum_1^q t'_i.$$

u^p's and v^q's are strictly increasing sequence out of which we can reconstruct the sequences t_i's and $t'i$'s by subtracting consecutive terms. Being strictly increasing is a consequence of having ideal points as representative.

We can consider the set $\{u^p\}_p \cup \{v^q\}_q$ and write its elements in a increasing sequence $r^0 < r_1 \cdots < r_a$. Note that the sequence $t''_i := r_i - r_{i-1}$ is very unlikely to be on the nose $t_p = u^p - u^{p-1}$ and $t'q = v^q - v^{q-1}$ but this can be corrected by taking carefully their consecutive sums. The latter corresponds to the codegeneracy maps for geometric simplicials. More accurately, let $i_1 < \cdots < i_{a-m}$ where $r_{i_k} \notin \{u^p\}_p$ and $j_1 < \cdots < i_{a-n}$ where $r_{j_k} \notin \{v^q\}_q$. Notice that we have $\sum t''_i = 1$ so $w := (t''_1, \cdots t''_a) \in \Delta^a$ and we have

$$u = s^{i_1} \cdots s^{i_{a-m}} w$$

and

$$v = s^{j_1} \cdots s^{j_{a-n}} w.$$

The inverse η of $(\pi_1 \times \pi_2)$ is given by

$$\eta(x, y) := [s_{i_{a-m}} \cdots s_{i_1} k, s_{i_{j_1}} \cdots s_{i_{a-n}} l, w)].$$

The identity $(\pi_1 \times \pi_2)\eta = id$ is pretty clear. The identity $\eta(\pi_1 \times \pi_2) = id$ is also easily verifiable. For the ideal point $(k, l, w) \in |K \times L|$, k and l are not necessarily nondegenerate. In that case $k = s_{i_m} \cdots s_{i_1} k'$ and $l = s_{j_n} \cdots s_{j_1} l'$. We have

$$\pi_1([(k, l, w)]) = [(k, w)] = [(k', s^{i_1} \cdots s^{i_m} w)]$$

and

$$\pi_2([(k, l, w)]) = [((l', s^{j_1} \cdots s^{j_n} w)]$$

Note that both $k', s^{i_1} \cdots s^{i_m} w)$ and $((l', s^{j_1} \cdots s^{j_n} w)$ are ideal points because the codegeneracy maps of the geometric simplex send the interiors to the interiors. Applying the algorithm defining η to $(k', s^{i_1} \cdots s^{i_m} w)$ and $(l', s^{j_1} \cdots s^{j_n} w)$ gives us back $(s_{i_m} \cdots s_{i_1} k', s_{j_n} \cdots s_{j_1} l', w)$ because the w is common preimage (under codegeneracy maps) of $s^{i_1} \cdots s^{i_m} w$ and $s^{j_1} \cdots s^{j_n} w$. This proves the identity.

Note that we can equip $|K| \times |L|$ with the topology of of CW-complex(weak topology) because the product of two open cell is cell and with respect CW-topology η (and its inverse $\pi_1 \times \pi_2$) is a homeomorphism because it is true cell by cell. Therefore if the product (weak) topology of (CW-complexes) on $|K| \times |L|$ coincides weak topology (of CW-complexes) on $|K| \times |L|$ then we have a homeomorphism in that sense too. This happens under the assumptions of the theorem. □

COROLLARY 1.38. *If $h : K \times \Delta_1 \to L$ is the homotopy between simplicial map $f, g : K \to L$ then the continuous maps $|f|, |g| : |K| \to |L|$ are homotopic.*

PROOF. By apply the geometric realization to H we obtain a continuous map $|H| : |K| \times \mathbf{\Delta}^1 \to |L|$ which is homotopy between $|f| = |H|_{|K| \times \{0\}}$ and $|g| = |H|_{|K| \times \{1\}}$. □

COROLLARY 1.39. *If $\mathcal{C}$ is category has a initial (or terminal) element then $|N\mathcal{C}|$ is contractible topological space.*

PROOF. This is a consequence of the previous result and Proposition 1.55. □

As mentioned earlier there the functors $S : Top - Space \to$ s$\mathbb{S}$et whose n-simplices are the s continuous maps $\sigma : \mathbf{\Delta}^n \to X$ with integer coefficients. We have also the geometric realization functor $T :$ s$\mathbb{S}$et $\to Top - Space$.

We define adjunction natural transformations

$$\phi : \mathrm{Hom}_{\mathrm{s}\mathbb{S}\mathrm{et}}(K, S(X)) \to \mathrm{Hom}_{Top}(|K|, X)$$

and

$$\psi : \mathrm{Hom}_{Top}(|K|, X) \to \mathrm{Hom}_{\mathrm{s}\mathbb{S}\mathrm{et}}(K, S(X))$$

by:

(a) For $f : K_* \to S_*(X)$, the continuous $\phi(f) : |K| \to X$ is given by

$$\phi(f)([k, u]) = f(k)(u) \quad \forall [k, u] \in |K|$$

which is well-defined because f is a map of simplicial sets.

(b) For $g \in \mathrm{Hom}_{Top}(|K|, X)$, the simplicial map is given by $\psi(g) : K \to S(X)$

$$\forall k \in K_n \quad \psi(g)(k)(u) := g([k, u]) \quad \forall u \in \mathbf{\Delta}^n.$$

One can see that for all f, $\psi(\phi(f))(k)(u) = \phi(f)([k, u]) = f(k)(u)$ implying that $\psi(\phi(f)) = f$, and similarly $\phi \circ \psi = id$.

The associated the counit natural transformation $\Phi_X : |S(X)| = TS(X) \to X$ is given by

$$\Phi_X([k, u]) = \phi(1_{S(X)}([k, u]) = k(u)$$

where k is a singular chain and $u \in \mathbf{\Delta}^n$. The unit natural transformation $\Psi_K : K \to ST(K)$ is

$$\Psi_K(k)(u) = \psi(1_{TK}) = id([k, u]).$$

Proposition 1.40. *For all simplicial set K, the unit $\Psi_K : K \to ST(K)$ is injective. Similarly, for all topological space X, $\Phi_X : TS(X) \to X$ is surjective.*

Proof. TO come soon □

The following results would be useful for comparing the model categories of simplicial sets and topological spaces.

Proposition 1.41. *The maps ϕ and ψ preserves homotopies.*

Proof. If $F : K \times \Delta^1 \to S(X)$ a simplicial homotopy then $T(F) : TK \times \mathbf{\Delta}^1 \to TS(X)$ is homotopy of continuous maps, so is the composition

$$\Phi_X \circ TF : TK \times \mathbf{\Delta}^1 \to TS(X) \to X.$$

Note that $\Phi_X \circ TF = \phi(1_{S(X)}) \circ T(F) \overset{\text{by (1.14)}}{=} \phi(1.F) = \phi(F)$ and $\phi(F)$ is homotopy between $\phi(F)|_{TK\times[0]}$ and $\phi(F)|_{TK\times[1]}$.

Similarly if $H : T(K) \times \mathbf{\Delta}^1 \to X$ is homotopy of continuous maps then $SH : ST(K \times \Delta_1) = ST(K) \times ST(\Delta_1) \to S(X)$ is a simplcial map (a homotopy) so is its precomposition with a simplicial map $\Psi_{K\times\Delta^1}$. Therefore we have a simplicial map

$$SH \circ \Psi_{K\times I} : K \times \Delta_1 \to S(X)$$

which is a simplicial homotopy. Moreover $SH \circ \Psi_{K\times I} = SH \circ \psi(1_{K\times I}) \overset{\text{by (1.14)}}{=} \psi(H)$, therefore $\psi(H)$ is a homotopy between $\psi|_{K\times d^1(\Delta_0)}$ and $\psi|_{K\times d^0(\Delta_0)}$ □

THEOREM 1.42. *The unit and counit natural transformations induce an isomorphism in homologies.*

PROOF. We have $H_*(K) := H_*(C_*(K))$ where is $C_*(K)$ is the simplicial complex of K. We will in Section 1.9) that $H_*(C_*(K)) \simeq H_*(C_*(K)/D_*(K))$ where $D_*(K)$ is teh subcomplex generated by the degenerate simplices. On the other hand the generators of $C_*(K)/D_*(K)$ are precisellt the cells of CW-complex $T(K)$ therefore $H_*(C_*(K)/D_*(K))$ is precesly the cellular homology of TK which isomorphic to the singular homology $H_*(ST(K))$, we conclude that Ψ_K (an inclusion) induces an isomorphism $H_*(K) \simeq H_*(ST(K))$.

As for $(S\Phi_X)* : H_*(TS(X)) = H_*(STS(X)) \to H_*(X) = H_*(S(X)$, from the identity $S\Phi \circ \Psi S = id$ and the previous result that $\Psi \circ S$ induces an isomorphism on homology for the case $K = S(X)$, it follows that $S\Phi$ also induces an isomorphism in homology.

□

1.9. Dold-Kan correspondence

The aim of this section is to prove the Dold-Kan correspondence which states that the category of simplicial objects in an abelian category $\mathcal{A}$ is equivalent to the category of positively graded chain complexes of $\mathcal{A}$. As a byproduct we prove that the complex of nondegenerate subcomplex of a simplicial complex is chain homotopic to the simplicial complex itself.

Let $A = \{A_n\}_n$ be a simplicial abelian group we continue to use A_* to denote its Moore complex equipped with differential $\partial = \sum_{i=0}^{n}(-1)^i d_i$. The normalized complex NA_* of A_* is defined by

$$NA_n := \cap_{i=0}^{n-1} \ker(d_i)$$

which is submcomplex complex of the Moore complex of A_* meaning that $\partial(NA_*) \subset NA_*$ (a consequence of the simplicial identites). We have in fac

$$\partial|_{NA_n} = (-1)^n d_n.$$

Let DA_* be subcomplex of the Moore complex generated by the elements of the image of the degeneracy maps. We call DA_* the degenerate subcomplex and its quotient the degenerate complex .

We consider the composition of the inclusion followed by the natural projection,

$$\phi : NA_* \hookrightarrow A_* \twoheadrightarrow A_*/DA_*.$$

Obviously ϕ is a chain map.

THEOREM 1.43. *The chain map ϕ is indeed an isomorphism of complexes.*

Proof. We first filter NA_* by subcomplexes N_jA_*

$$N_jA_n := \cap_{i=0}^{j} \ker(d_i) \subset NA_n.$$

Similarly we filter DA_* by D_jA_* by setting $D_jA_n \subset DA$ to be the subcomplex generated by $\mathrm{Im}(s_i)$ for $i \leq j$ and consider the restriction ϕ to these subcomplexes,

$$\phi_j := \phi|_{N_jA_*} : N_jA_* \to D_jA_*.$$

We prove by induction on j that ϕ_j, $j < n$, is an isomorphism. $j = 0$: We have $N_0A_n = \ker(d_0)$ and for all classes $[x] \in A_n/D_0A_n$, $d_0(x - s_0d_0)x$ because $d_0s_0 = id$. In the quotient A_n/D_0A_n

$$[x] = [x - s_0d_0x]$$

proving that ϕ_0 is surjective. For $x_0 \in N_0A_n$ if $\phi_0(x) = 0 \in A_n/D_0A_n$ then $x = s_0y$ for some $y \in A$, and as a consequence, $0 = d_0x = d_0s_0y = y$ hence $y = 0$ and ϕ_0 is injective.

Now we suppose that for all $k < j$, $\phi_k : N_kA_n \to A_n/D_kA_n$ is an isomorphism for $n > k$. We have the commutative diagram

$$\begin{array}{ccc} N_{j-1}A_n & \xrightarrow{\phi_{j-1}} & A_n/D_{j-1}(A_n) \\ \uparrow & & \downarrow \\ N_jA_n & \xrightarrow[\phi_j]{} & A_n/D_jA_n \end{array} \tag{1.20}$$

where the vertical arrow on the left is surjective. A class $[x] \in A_n/D_jA_n$ has representative and ultimately a presentative in $N_{j-1}A_n$ because by hypothesis ϕ_{j-1} is surjective. Let $y \in N_{j-1}A_n$ be representative for $[x]$, replace y by $y - s_jd_jy$. We have $d_j(y - s_jd_jy) = 0$ and $[y] = [y - s_jd_jy] \in A_n/D_jA_n$ therefore $[x] = \phi_j(y - s_jd_jy)$ proving thatϕ_j is surjective.

As for the injectivity, consider the commutative diagram whose top arrow is exact.

(1.21)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{n-1}/D_{j-1}A_{n-1} & \xrightarrow{s_j} & A_n/D_{j-1}A_n & \longrightarrow & A_n/D_jA_n & \longrightarrow & 0 \\ & & \uparrow \phi j-1 & & \uparrow \phi j-1 & & \uparrow \phi j & & \\ & & N_{j-1}A_{n-1} & \hookrightarrow & N_{j-1}A_n & \hookleftarrow & N_jA_n & & \end{array}$$

If for $x \in N_jA_n$, $\phi_j(x) = 0$, then using the commutative diagram above we can conclude that $x = s_j(y)$ for some $y \in N_{j-1}A_{n-1}$. Since $d_jx = 0$, we have $y = d_js_j(y) = d_jx = 0$. Thus ϕ_j is injective. □

We jut proved that normalised complex (which is subcomplex) is isomorphic to the quotient complex $A * /DA*$ where there is no degeneracy maps, i.e. essentially a complex. In the following we explain how can we recover the simplicial complex from of it normalized complex.

Let $B_n := \oplus_{[n]\twoheadrightarrow[k]} NA_k$ where direct sum is taken over surjective morphism in simplicial category $\mathbf{\Delta}$. We recall that by Proposition 1.5, a surjection is a composition of the codege neracy maps s^i's. We use the notation $(x, [n] \overset{\sigma}{\twoheadrightarrow} [k])$ to denote the elements of B_n

It turns out that $n \mapsto B_n$ is a simplcial abelian group: For $f : [m] \to [n] \in \mathrm{Hom}_{\mathbf{\Delta}}$ and for each index map $\sigma : [n] \to [k]$, using Lemma 1.5 we decompose the decompostion map $\sigma \circ f : [m] \to [k]$ as the composition of surjection g and injection τ i.e. $\sigma \circ f = \tau \circ g$ where τ is injective and g is surjective

$$
\begin{array}{c}
[m] \xrightarrow{f} [n] \overset{\sigma}{\twoheadrightarrow} [k] \\
|| \\
[m] \overset{g}{\twoheadrightarrow} [l] \overset{\tau}{\rightarrowtail} [k]
\end{array}
\tag{1.22}
$$

We define the action of f on $(x, [n] \overset{\sigma}{\twoheadrightarrow} [k]) \in B_n$ to be

$$f^*(x, [n] \overset{\sigma}{\twoheadrightarrow} [k]) := (\tau^*(x), [m] \overset{g}{\twoheadrightarrow} [l])$$

In other words to every morphism f in the category $\mathbf{\Delta}$ we have associated a morphism

$$f^* : B_n \to B_m$$

which will prove that it is functorial

Proposition 1.44. *The collection of set $\{B_n\}_n$ together with induced maps f^* as above, is a simplicial set.*

Proof. let $f : [p] \to [m]$ and $g : [m] \to [n]$ be two composable morphisms in $\mathbf{\Delta}$ and $(x, [n] \overset{\sigma}{\twoheadrightarrow} [k]))$ an element of B_n. Then $f^*(g^*((x, [n] \overset{\sigma}{\twoheadrightarrow} [k])$ is defined by the series of (unique) epic-monic decomposition displayed in the diagram below,

$$
\begin{array}{c}
[p] \xrightarrow{f} [m] \xrightarrow{g} [n] \overset{\sigma}{\twoheadrightarrow} [k] \\
|| \\
[p] \xrightarrow{f} [m] \overset{\sigma'}{\twoheadrightarrow} [l] \overset{\tau}{\rightarrowtail} [k] \\
|| \\
[p] \overset{f'}{\twoheadrightarrow} [t] \overset{\sigma''}{\rightarrowtail} [l] \overset{\tau}{\rightarrowtail} [k] \\
|| \\
[p] \overset{f'}{\twoheadrightarrow} [t] \overset{\tau\sigma''}{\rightarrowtail} [k] \\
|| \\
[p] \overset{gf}{\twoheadrightarrow} [n] \overset{\sigma}{\rightarrowtail} [k]
\end{array}
\tag{1.23}
$$

we have $f^*(g^*((x,[n]\overset{\sigma}{\twoheadrightarrow}[k])=((\sigma'')^*\tau^*(x),f')$ and from the diagram we see that $(f',\tau\sigma'')$ is the epi-monic decomposition of gf hence $(gf)*(x,[n]\overset{\sigma}{\twoheadrightarrow}[k])=((\sigma'')^*\tau^*(x),f')$.

This proves that $f^*(g^*(x,[n]\overset{\sigma}{\twoheadrightarrow}[k]))=(gf)^*(x,[n]\overset{\sigma}{\twoheadrightarrow}[k])$ □

PROPOSITION 1.45. *For a simplicial abelian group $A=\{A_n\}$, the natural map $\psi_n : B_n := \oplus_{[n]\to[k]}NA_k \to A_n$, given on the generators of B_n by*

$$\psi_n : (x,[n]\overset{\sigma}{\twoheadrightarrow}[k])\mapsto \sigma^*(x)\in A_n, s$$

is an isomorphism for simplicial sets.

PROOF. First we verify that ψ is a map of simplicial sets. Let $f:[m]\to[n]\in \mathrm{Hom}_{\mathbf{\Delta}}$,

$$f^*(\psi_n(x,[n]\overset{\sigma}{\twoheadrightarrow}[k]))=f^*(\sigma^*(x)).$$

On the other hand

$$\psi_m(f^*((x,[n]\overset{\sigma}{\twoheadrightarrow}[k]))=\psi(\tau^*(x),[m]\overset{g}{\twoheadrightarrow}[l]) \tag{1.24}$$

where $\tau\circ g=f\sigma$ is the pic-monic decomposition of $f\circ\sigma$, and then

$$\begin{aligned}\psi_m(f^*((x,[n]\overset{\sigma}{\twoheadrightarrow}[k]))&=\psi_m(\tau^*(x),[m]\overset{g}{\twoheadrightarrow}[l])\\ &=g^*\tau^*(x)\overset{A\text{ being a simplicial set}}{=}(\tau\circ g)^*(x)\\ &=(f\sigma)^*(x)\overset{A\text{ being a simplicial set}}{=}f^*(\sigma^*(x))\\ &=f^*\psi_n(((x,[n]\overset{\sigma}{\twoheadrightarrow}[k]))\end{aligned} \tag{1.25}$$

Now we prove by induction on n that ψ is an isomorphism. It is clear that $B_0=NA_0=A_0$ and the only surjection out of $[0]$ is the identity map.

Suppose that ψ_j is an isomorphism for $j<n$: The image of ψ_n include all the degenerat simplices $x=s_i(y)$ because $y\in A_{n-1}$ therefore $y=\psi_{n-1}(z)$ and $x=s_i\psi_{n-1}(z)=\psi_n(s_i(z))$.

We also claim that the ψ induces an isormorphism between the normalised complex of B_* and NA_*. To this end we compute the NB_n,

$$NB_n=\cap\ker_{i=0}^{n-1}(d_i)$$

For $(x,[n]\overset{\sigma}{\twoheadrightarrow}[k])$, if $\sigma\neq id$ we can write $\sigma=s^{j_1}\cdots s^{j_{n-k}}$ where $j_1<j_2\cdots<j_{n-k}$. Then $d^{j_{n-k}}(x,[n]\overset{\sigma}{\twoheadrightarrow}[k])=(x,s^{j_1}s^{j_2}\cdots s^{j_{n-k-1}})$. This is because $s^{j_1}\cdots s^{j_{n-k}}\circ d^{j_{n_k}}=s^{j_1}\cdots s^{j_{n-k-1}}$ therefore its epi-monic decomposition is $id\circ s^{j_1}\cdots s^{j_{n-k-1}}$. This means if $(x,\sigma)\neq 0$ is in NB_n then $\sigma=id$ and in that case $0=d_i(x,id)=d^{i^*}(x,id)=(d^{i^*}(x),id)=(d_ix,id)$ because $id_{n-1}\circ d^i$ is epi-monic decomposition of $d^i\circ id_n$. Thus $x\in NA_n$ and NB_n is isomorphic to NA_n.

Using the isomorphism in Theorem 1.43, that we have a natural exact sequence of $0\to DA_*\to A_*\to NA_*\to 0$ which is split because NA_* is a subcomplex of A_*. Since ψ_n is surjective on degenerate simplices DA_n

of A_n and an isomorphism (hence surjective) on normalised complexes, we conclude that ϕ_n is surjective.

As for the injectivity of ϕ_n: Suppose that $\phi_n(((x_{\sigma'},[n] \overset{\sigma'}{\twoheadrightarrow} [k_{\sigma'}])_{\sigma'})) = 0$. For a surjection $\sigma \neq id$, $\sigma = s^{j_1} \cdots s^{j_{n-k}}$ and σ has a right inverse $d_\sigma := d^{j_{n-k}} \cdots d^{j_1}$. Using this fact $d_\sigma^*(x_\sigma,[n] \overset{\sigma}{\twoheadrightarrow} [k_\sigma]) = (x_\sigma, id)$. We fixe an index σ in $(x_{\sigma'},[n] \overset{\sigma'}{\twoheadrightarrow} [k_{\sigma'}])$,

$$\phi_{n-1}(d_\sigma((x_{\sigma'},[n] \overset{\sigma'}{\twoheadrightarrow} [k_{\sigma'}])) = d_\sigma \phi_n((x_{\sigma'},[n] \overset{\sigma'}{\twoheadrightarrow} [k_{\sigma'}])) = 0$$

therefore by injectivity of ϕ_{n-1} we conclude that

$$d_\sigma((x_{\sigma'},[n] \overset{\sigma'}{\twoheadrightarrow} [k_{\sigma'}])_{\sigma'}) = 0.$$

First notice that since left inverse d_σ is unique (i.e. σ) there is only one component of $d_\sigma((x_{\sigma'},[n] \overset{\sigma'}{\twoheadrightarrow} [k_{\sigma'}])'_\sigma)$ which corresponds to the identity map and that is precisely x_σ, hence $x_\sigma = 0$. Since σ was arbitary, we have $((x_{\sigma'},[n] \overset{\sigma'}{\twoheadrightarrow} [k_{\sigma'}]) = 0$ and ϕ_n is injective. □

THEOREM 1.46. ***Dold-Kan correspondence*** *The nomalisation functor $N : \mathrm{sAb} \to \mathrm{Ch}_+$, from the simplicial abelian groups to the category of positively graded complexes, is an isomorphism of complexes. The inverse $\Gamma : \mathrm{Ch}_+ \to \mathrm{sAb}$ to N is given by*

$$\Gamma(C)_n =:= \bigoplus_{[n] \twoheadrightarrow [k]} C_k$$

PROOF. The content of Proposition 1.45 is essentially that there is a natural isomorphism $\Gamma \circ N \simeq id$.

It remains to prove that there is natural of isomorphism of complexes $N \circ \Gamma \simeq id$. To that end, we prove that $\Gamma(C)/D(\Gamma(C)) \simeq C$ as complexes and since there is natural isomorphism $N(\Gamma(C))$ we get the isomorphism that we want.

Suppose that $(x,[n] \overset{\sigma}{\twoheadrightarrow} [k]) \in \Gamma(C)_n = \oplus_{[n]\twoheadrightarrow[k]} C_k$. Since σ is surjective, we can write $\sigma = s^{j_1}s^{j_2}\cdots s^{j_k}$, $k \geq 1$. Then we get $(s^{j_k})^*(x, s^{j_1}s^{j_2}\cdots s^{j_{k-1}}) = (x,\sigma)$. This means that in $\Gamma(C)$ all the components are degenerate except the one which corresponds to the identity morphism $\sigma = id$ therefore $\Gamma(C)/D(\Gamma(C)) \simeq C$. □

THEOREM 1.47. *For a simplicial abelian group $A = \{A_n\}_n$, the normalised complex NA_* is homotopy equivalent to the Moore complex A_*.*

PROOF. We introduce a nested sequence N_jA_* of subcomplexes of A_* which stabilises degree-wise to NA_* in a finite length. Moreover we prove that each inclusion is a homotopy equivalence. We set $N_{-1}A_* = A_*$ and for $0 \leq j \leq n-1$

$$N_jA_n := \begin{cases} \cap_{k=0}^{j} \ker(d_k) \text{ if } n \geq j+2 \\ NA_n \text{ otherwise} \end{cases}$$

Notice that. $N_{n-1}A_n = NA_n$ and each N_jA_* is subcomplex of A_* because

- if $n < j+2$, then for $x \in NA_n = N_jA_n$ then $dx \in NA_{n-1} = N_jA_{n-1}$, since NA_* is a subcomplex.
- if $n \geq j+2$ then for $x \in N_jA_n$ and $k \leq j$

$$d_k dx = d_k(\sum_{i=j+1}^{n} (-1)^i d_i x) = \sum_{i=j+1}^{n} (-1)^i d_{i-1} d_k x = 0$$

It turns out that the inclusion $i_j : N_{j+1}A_* \subset N_jA_*$ has a homotopy inverse,

$$r_j(x) = \begin{cases} x - s_{j+1}d_{j+1}(x) \text{ if } n \geq j+2 \\ x \text{ otherwise} \end{cases}$$

First of all note that for $x \in N_jA_n$

- $d_{j+1}(x - s_{j+1}d_{j+1}(x)) = d_{j+1}(x) - d_{j+1}s_{j+1}d_{j+1}(x) = d_{j+1}x - d_{j+1}x = 0$,
- for $k < j+1$, $d_k(x - s_{j+1}d_{j+1}(x)) = d_k(x) - d_k s_{j+1} d_{j+1}(x) = 0 - s_j d_k d_{j+1}(x) = -s_j d_j d_k(x) = 0$,

implying that $\mathrm{Im}(r_j) \subset N_{j+1}A_*$. The second step is to verify that r_j is a chain map: For $x \in N_jA_n$, $n \geq j+2$,

(1.26)

$$dr_j(x) = \sum_{i=j+2}^{n} (-1)^k d_k(x - s_{j+1}d_{j+1}x) = \sum_{i=j+2}^{n} (-1)^k d_k(x) - (-1)^{j+2} d_{j+2}s_{j+1}d_{j+1}x$$
$$- \sum_{i=j+3}^{n} (-1)^k d_k s_{j+1} d_{j+1} x = \sum_{i=j+2}^{n} (-1)^k d_k(x) - (-1)^{j+2} d_{j+1}x - \sum_{i=j+3}^{n} (-1)^k d_k s_{j+1} d_{j+1} x$$
$$\sum_{i=j+1}^{n} (-1)^k d_k(x) - \sum_{i=j+3}^{n} (-1)^k d_k s_{j+1} d_{j+1} x$$

and

(1.27)

$$r_j(dx) = dx - s_{j+1}d_{j+1}dx = \sum_{i=j+1}^{n} (-1)^k d_k x - \sum_{i=j+1}^{n} (-1)^k s_{j+1}d_{j+1}d_k x$$
$$= \sum_{i=j+1}^{n} (-1)^k d_k x - (-1)^{j+1} s_{j+1}d_{j+1}d_{j+1}x - (-1)^{j+2} s_{j+1}d_{j+1}d_{j+2}x - \sum_{i=j+3}^{n} (-1)^k s_{j+1}d_{j+1}d_k x$$
$$= \sum_{i=j+1}^{n} (-1)^k d_k x - (-1)^{j+1} s_{j+1}d_{j+1}d_{j+1}x - (-1)^{j+2} s_{j+1}d_{j+1}d_{j+1}x - \sum_{i=j+3}^{n} (-1)^k s_{j+1}d_{j+1}d_k x$$
$$= \sum_{i=j+1}^{n} (-1)^k d_k x - (-1)^{j+1} s_{j+1}d_{j+1}d_{j+1}x - \sum_{i=j+3}^{n} (-1)^k s_{j+1}d_{j+1}d_k x = dr_j(x)$$

It is clear that $r_j \circ i_j = id$. We claim that that $i_j \circ r_j$ is homotop to the identity map via the homotopy $h_j : N_j A_n \to N_j A_{n+1}$

$$h_j(x) = \begin{cases} (-1)^j s_{j+1}(x) \text{ if } n \geq j+1 \\ 0 \text{ otherwise} \end{cases} \tag{1.28}$$

To see that,

(1.29)

$$\begin{aligned}
\partial h_j(x) + h_j \partial(x) &= \sum_{k=j+1}^{n+1} (-1)^{j+k} d_k s_{j+1}(x) + \sum_{k=j+1}^{n} (-1)^{j+k} s_{j+1} d_k(x) \\
&= (-1)^{j+j+1} d_{j+1} s_{j+1}(x) + (-1)^{j+j+2} d_{j+2} s_{j+1}(x) + \sum_{k=j+3}^{n+1} (-1)^{j+k} d_k s_{j+1}(x) \\
&+ (-1)^{j+j+1} s_{j+1} d_{j+1}(x) + \sum_{k=j+2}^{n} (-1)^{j+k} s_{j+1} d_k(x) \\
&= -x + x - s_{j+1} d_{j+1}(x) + \sum_{k=j+3}^{n+1} (-1)^{j+k} d_k s_{j+1}(x) + \sum_{k=j+2}^{n} (-1)^{j+k} s_{j+1} d_k(x) \\
&= -s_{j+1} d_{j+1}(x) + \sum_{k=j+3}^{n+1} (-1)^{j+k} d_k s_{j+1}(x) + \sum_{k=j+2}^{n} (-1)^{j+k} d_{k+1} s_{j+1}(x) \\
&= -s_{j+1} d_{j+1}(x) \\
&= i_j \circ r_j(x) - x
\end{aligned}$$

The homotopy inverse $f : A_* \to NA_*$ to the inclusion $i : NA_* \to A_*$ is given degree-wise by $A_n \to NA_n$

$$f_n := r_{n-2} \circ \cdots \circ r_0 \circ r_{-1}$$

and the chain homotopy is given by

(1.30)

$$h_{-1} + i_{-1} h_0 r_{-1} + \cdots + i_{-1} \cdots r_{k-1} h_k i_{k-1} \cdots r_{-1} + \cdots + i_{-1} \cdots r_{n-3} h_{n-2} i_{n-3} \cdots r_{-1}$$

‘ □

1.10. Internal hom and simplicial fonction space

The morphism set of two objects in a category does not necessary the category. However this turns out to be turn be for category of presheaves on a category. Here we explain this phenomena for the simplicial set (i.e. presheaves on Δ^{op}) but everything extends to any category of presheaves.

Definition 1.48. For two simplcial sets X and Y, the internal hom-set $\hom(X, Y) \in s\mathbb{S}\text{et}$ is a simplicial set whose set n-simplices is

$$\hom(X, Y)_n := \text{Hom}_{s\mathbb{S}\text{et}}(X \times \Delta_n, Y)$$

The simplicial structure of $\hom(X,Y)$ is consequence of the cosimplicial structure of the collection $\{\Delta_n\}_n$ (see Proposition†1.20): For $f:[m]\to[n]$, and $\theta \in \hom(X,Y)_n$

$$f^*(\theta)\in \hom(X,Y)_m$$

is given by

$$(x,h)\in X_k\times \Delta_m[k]\mapsto \theta(x,f\circ h)\in Y_k$$

i.e.

$$f^*(\theta):=\theta(id\times f_*).$$

The functoriality is clear because

$$g^*f^*(\theta)=g^*(f^*(\theta))=g^*(\theta(id,f_*))=\theta(id,f_*)(id,g_*)=\theta(id,(fg)_*)=(fg)^*(\theta)$$

With every morphism set, comes an evaluation.

Proposition 1.49. *The evaluation map* $ev_*: X\times \hom(X,Y)\to Y$ *given by*

$$ev_n(x,g):=g(x,id_n)\quad x\in X_n,\quad g\in \hom(X,Y)_n$$

is a simplicial map.

Proof. For a map $f:[m]\to[n]$ and $(x,\theta)\in X_n\times \mathrm{Hom}_{s\mathbb{S}et}(X\times\Delta_n,Y)$

$$\begin{aligned} ev_m(f^*(x,\theta)) &= ev_m(f^*(x),f^*(\theta))=f^*(\theta)(f^*(x),id_m)\\ &=\theta(id,f_*)(f^*(x),id_m)=\theta(f^*(x),f_*(id))=\\ &=\theta(f^*(x),f)\end{aligned}\tag{1.31}$$

$$\begin{aligned} f^*(ev_n(x,\theta)) &= f^*(\theta(x,id_n)) \overset{\theta \text{ being a simplicial map}}{=} \theta(f^*(x,id_n)\\ &=\theta(f^*(x),f^*(id_n))=\theta(f^*(x),f)\end{aligned}$$

□

Proposition 1.50. *In the category of simplicial sets, the cartesian product is the left adjoint of the internal hom functor i.e. we have a natural bijection*

$$\mathrm{Hom}_{s\mathbb{S}et}(K,\hom(X,Y))\simeq \mathrm{Hom}_{s\mathbb{S}et}(K\times X,Y)$$

Proof. We define

$$\phi:\mathrm{Hom}_{s\mathbb{S}et}(K,\hom(X,Y))\to \mathrm{Hom}_{s\mathbb{S}et}(K\times X,Y)$$

$$\phi(\theta)(k,x):=ev(x,\theta(k))=ev(1\times\theta)(x,k).$$

which is a simplicial map because eve and θ are. The inverse

$$\psi:\mathrm{Hom}_{s\mathbb{S}et}(K\times X,Y)\to \mathrm{Hom}_{s\mathbb{S}et}(K,\hom(X,Y))$$

is given by

$$\psi(h)(k)(x,g)=h(g^*(k),x)$$

where $h \in \mathrm{Hom}_{\mathrm{s}\mathbb{S}\mathrm{et}}(K \times X, Y)$, $k \in K_k$ and $(x, g) \in X_n \times \Delta_k[n] \in Y$. We have

$$\begin{aligned}\psi(\phi(\theta))(k)(x,g) &= \phi(\theta)(g^*(k),x) = \theta(g^*(k))(x,id) = g^*(\theta)(x,id) \\ &= \theta(id \times g_*)(x,id) = \theta(x,g),\end{aligned}$$

hence $\psi \circ \phi = id$. Similarly,

$$\phi(\psi(h))(k,x) = \psi(h)(k)(x,id) = h(id^*(k),x) = h(k,x),$$

and $\phi \circ \psi = id$. □

1.11. All about the nerve of a category: Part I

There is a general construction which allows to give a combinatorial model for the classifying space of the groups.

Suppose that $\mathcal{C}$ is a small category. We define a simplicial set $N\mathcal{C}$ by setting $N\mathcal{C}_0 = \mathrm{obj}(\mathcal{C})$ and $N\mathcal{C}_n$ to nbe the set of n composable morphism i.e

$$N\mathcal{C}_n := \{(X_0 \stackrel{f_0}{\to} X_1 \cdots X_{n-1} \stackrel{f_{n-1}}{\to} X_n)| \quad f_i \in \mathrm{Hom}_{N\mathcal{C}}(X_i, X_{i+1}), \quad X_i \in \mathrm{obj}(\mathcal{C})\}.$$

The face et degeneracy maps are given as follows: For $n > 1$,

$$\begin{aligned} d_0(f_0, \cdots f_{n-1}) &= (f_1, \cdots f_{n-1}) \\ d_n(f_0, \cdots f_{n-1}) &= (f_0, \cdots f_{n-2}) \\ d_i(f_0, \cdots f_{n-1}) &= ((f_0, \cdots f_{i+1} \circ f_i, \cdots, f_{n-1}), \quad 0 < i < n \\ s_i(f_0, \cdots f_{n-1}) &= ((f_0, \cdots id_{X_i}, \cdots f_{n-1}) \end{aligned} \tag{1.32}$$

For $n = 1$, $d_0(f_0) = t(f_0)$ and $d_1(f_0) = s(f_0)$ where t and s stand for the target (image) and source (domain) of f.

The nerve construction is functorial because a functor preserves the composition and the identity maps.

Proposition 1.51. *A functor $F : \mathcal{C} \to \mathcal{D}$ between small categories, induces a map of simplicial set $Nf : N\mathcal{C} \to N\mathcal{D}$ in a natural manner.*

Proof. The induced maps on the vertices is given by $x \mapsto F(x)$ and on n-simplices is given by $(f_0, \cdots f_{n-1}) \mapsto (F(f_0), \cdots F(f_{n-1}))$. □

Example 1.52. **Classifying space of a group**
A groups G can be thought of as a category $\mathcal{G}$ with one object $*$ and the morphism $\mathrm{Hom}_{cG}(*, *) = G^{op}$. Here op means that the compostion rule for the morphism is $(g, h) \mapsto hg$ Then nerf $N\mathcal{G}$ is a simplicial in which

$$\begin{aligned} d_0(g_0, \cdots g_{n-1}) &= (g_1, \cdots g_{n-1}) \\ d_n(g_0, \cdots g_{n-1}) &= (g_0, \cdots g_{n-2}) \\ d_i(g_0, \cdots g_{n-1}) &= ((g_0, \cdots g_i g_{i+1}, \cdots, g_{n-1}), \quad 0 < i < n \\ s_i(f_0, \cdots f_{n-1}) &= (f_0, \cdots id_G, \cdots f_{n-1}) \end{aligned} \tag{1.33}$$

Note that this construction is natural with respect group homomorphism.

EXAMPLE 1.53. Let $\mathcal{I}$ be the category with two ordered objects 0 and 1 where the nonempty morphism sets are singleton: $\mathrm{Hom}(0,0) = \{id_0\}$, $\mathrm{Hom}(1,1) = \{id_1\}$ and $\mathrm{Hom}(0,1) = \{w\}$. We can easily check that the nerve of this category is the simplicial Δ_1.

EXAMPLE 1.54. The simplicila set Δ_n can be identifies as the nerve of the category $[n]$. This is the category associated to the ordered set $\{0, 1 \cdots n\}$ where t

$$\mathrm{Hom}_{[n]}(p,q) = \begin{cases} i_{pq} & \text{if } p \leq q \\ \emptyset \text{ otherwise} \end{cases} \tag{1.34}$$

On should think of i_{pq} as the inclusion of $\{0, 1 \cdots p\}$ in $\{0, 1 \cdots q\}$. It is esily seen that $\Delta_n = N[n]$.

Since the simplicial map $d^i : [n] \to [n+1]$ and $s^i : [n+1] \to [n]$ are order preseverving they can be seen as functors between $[n+1]$ and $[n]$ therefore $N(d^i)$ and $N(s^i)$ are maps simplicial sets which ultimatemately turns the collection $\{\Delta_n\}$ into a cosimplicial objet in the category of simplicial sets s$\mathbb{S}$et.

Note that $N(d^i) : \Delta_n \to \Delta_{n+1}$ and $N(s^i) : \Delta_{n+1} \to \Delta_n$ are respectively the post-composition with d^i and s^i i.e for $f \in \mathrm{Hom}_{\boldsymbol{\Delta}}([k],[n])$ and $g \in \mathrm{Hom}_{\boldsymbol{\Delta}}([k],[n+1])$

$$N(d^i)(f) = d^i \circ f \text{ and } N(s^i)(g) = s^i \circ g.$$

PROPOSITION 1.55. *A natural transformation $F \xrightarrow{T} G$ between two functors induces a simplicial homotopy $N\mathcal{C} \times \Delta_1 \to N\mathcal{D}$ between F and G.*

PROOF. Let $\mathcal{I}$ be the category with two objects in Example 1.53. We claim that that the natural T induces a functor $G : \mathcal{C} \times \mathcal{I} \to \mathcal{D}$:

$$\begin{aligned} H(x,0) &= G(x) \quad x \in \mathrm{obj}(\mathcal{C}) \\ H(x,1) &= F(x) \quad x \in \mathrm{obj}(\mathcal{C}) \\ H(f, Id_0) &= G(f) \quad f \in \mathrm{Hom}_{\mathcal{C}} \\ H(f, Id_0) &= G(f) \quad f \in \forall\, \mathrm{Hom}_{\mathcal{C}} \\ H(f,w) &= T_y \circ F(f) = G(f) \circ T_x \quad \forall f \in \mathrm{Hom}_{\mathcal{C}}(x,y) \end{aligned} \tag{1.35}$$

H being a functor is essentially consequence of the commutativity of the diagrams of the form

$$\begin{array}{ccc} F(x) & \xrightarrow{T_x} & G(x) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F((y) & \xrightarrow{T_y} & G(y) \end{array} \tag{1.36}$$

Now by applying the nerv efunctor N to H, we obtain a simplicial map $NH : N(\mathcal{C}) \times \times \Delta_1 = N(\mathcal{C}) \times N(\mathcal{I}) = N(\mathcal{C} \times \mathcal{I}) \to N\mathcal{D}$ □

COROLLARY 1.56. *The nerf of a category with final objet (initial) object is a contractible simplicial set.*

PROOF. Having an initial objects gives rise to a natural transformation between the identity functor and constant functor. □

DEFINITION 1.57. For two categories $\mathcal{C}$ and $\mathcal{D}$, let $\mathrm{Func}(C, D)$ be the category whose objets are functors $F : C \to D$ and its morphisms are the natural transformations between functors.

PROPOSITION 1.58. *The nerve functor* $N : Small - categories \to \mathrm{s}\mathbb{S}\mathrm{et}$

PROOF. We shall prove that $N : \mathrm{Func}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathrm{s}\mathbb{S}\mathrm{et}}(N\mathcal{C}, N\mathcal{D})$ is a bijection. We construct an invese N^{-1} for N. Let $\phi : N\mathcal{C} \to N\mathcal{D}$ be a map of simplicials sets. Since the 0-simplicies of $N\mathcal{C}$ and $N\mathcal{D}$ are the objects of cC and $\mathcal{D}$, we obtain a map $N^{-1}(\phi) = \phi_0 : obj(\mathcal{C}) = N\mathcal{C}_0 \to obj(\mathcal{D}) = N\mathcal{D}_0$. Note the $f : X \to Y$ be a morphism in $\mathcal{C}$. Since $N\mathcal{C}_1$ is the set of morphism, then we ϕ_1 is indeed a maps from morphisms of $\mathcal{C}$ to the morphisms of $\mathcal{D}$ and we define $N^{-1}(\phi)(f) := \phi_1(f)$. Now we have to show that ϕ_0 and ϕ_1 constitute a functor. First notice that for a morphism $f : X \to Y$, the morphism $\phi(f)$ is a morphism from $\phi_0(X) \to \phi_0(Y)$. This is consequence of of ϕ being a simplicial map

$$source(\phi_1(f)) = d_0\phi_1(f) = \phi_0(d_0 f) = \phi_0(X)$$

and similarly for the target. We have also $\phi_1(id_A) = id_{\phi_0(A)}$, for $A \in obj(\mathcal{C})$ because ϕ commutes with the degeneracy maps. Finally we should prove that for two composable morphisms $X \overset{f_0}{\to} Y \overset{f_1}{\to} Z$, $\phi_1(g \circ f) = \phi_1(g) \circ \phi_1(f)$.This follows from that the fact $\sigma := (f_0, f_1)$ is defines a 2-simplex in $N\mathcal{C}$ and since ϕ is a map of simplicial sets. We have

$$d_0\phi_2(\sigma) = \phi_1(d_1\sigma) = \phi_1(f)$$

and

$$d_2\phi_2(\sigma) = \phi_1(d_2\sigma) = \phi_1(g),$$

therefore $\phi_2(\sigma) = (\phi_1 f), \phi_1(g)$. Again, ϕ being a simplicial map, we

$$\phi_1(g \circ f) = \phi_1(d_1\sigma) = d_1\phi_2(\sigma) = \phi_1(g) \circ \phi_1(f). \tag{1.37}$$

hence (ϕ_0, ϕ_1) form a functor. The identity $N^{-1}N = id$ is obvious. TO prove that $N \circ N^{-1} = id$, take a simplicial map $\phi : N\mathcal{C} \to N\mathcal{D}$. By definition of N^{-1}, $N \circ N^{-1}(\phi)$ on 0 and 1-simplicies is identical to ϕ and this suffices because a simpliciam map betwen nerves is determined by it effect on 0 and 1-simplices, therefore

$$N \circ N^{-1}(\phi) = \phi.$$

□

PROPOSITION 1.59. *Let $\mathcal{C}$ and $\mathcal{D}$ be two small categories. Then we have an isomorphism of simplicial set*

$$N\mathrm{Func}(\mathcal{C}, \mathcal{D}) \simeq \hom(NC, ND) \tag{1.38}$$

Here $\hom(NC, ND) \in \mathrm{s}\mathbb{S}\mathrm{et}$ *is the simpclicial function space introduced in Section 1.10*

PROOF.

(1.39)

$$\begin{aligned}\hom(N\mathcal{C},\mathcal{D})_n &= \mathrm{Hom}_{\mathrm{s}\mathbb{S}\mathrm{et}}(N\mathcal{C}, \times\Delta_n, N\mathcal{D}) \overset{\text{by Example 1.54}}{\simeq} \mathrm{Hom}_{\mathrm{s}\mathbb{S}\mathrm{et}}(N\mathcal{C}\times N[n], N\mathcal{D})\\ &\simeq \mathrm{Hom}_{\mathrm{s}\mathbb{S}\mathrm{et}}(N(\mathcal{C}\times[n]), N\mathcal{D})\\ &\overset{\text{by Prop 1.58}}{\simeq} \mathrm{Func}(\mathcal{C}\times[n],\mathcal{D}) \simeq \mathrm{Func}([n],\mathrm{Func}(\mathcal{C},\mathcal{D})) = N(\mathrm{Func}(\mathcal{C},\mathcal{D}))_n\end{aligned}$$

The fact that these isomorphisms are compatible avec les simplicial map is easily verifiable and is actually discussed in Example 1.54. □

One can naturally ask if the nerve function is part of an adjunction. It turns out that the functor N has a left adjoint $T : \mathrm{s}\mathbb{S}\mathrm{et} \to small-Cat$ called the fundamental category functor and for simplicial set X, subsequently $T(X)$ is called the fundamental category of the simplicial set X..

DEFINITION 1.60. Let $X = \{X_n\}_n$ be a simplicial set. Let fundamental category $T(X)$ of X be the category whose set of objects is X_0 with morphism sets

$$Hom_{T(X)}(x, y) = \{f \in X_1 | \quad d_1 f = x \quad \& \quad d_0 f = y\}/\sim$$

where $\sim$ is defined by the identities $d_1\sigma = d_0\sigma \circ d_2\sigma$ for all $\sigma \in X_2$.

It is not hard to see that for $x \in X_0$, we have $s_0(x) = id_x \in T(X)$. Indeed if $f \in \mathrm{Hom}_{T(X)}(x, x)$ for $\sigma = s_0(f)$ we have

$$d_0\sigma = d_1\sigma = f \text{ and } d_2\sigma = d_2 s_0 f = s_0 d_1 f = s_0(x)$$

and the 2-simplex σ imposes the relation $f \circ s_0(x) = f$. Similarly for 2-simplex $\tau = s_1(f)$ we obtain that $s_0(x) \circ f = f$, therefore $s_0(x)$ is the identity morphism.

PROPOSITION 1.61. *The functor* $T : \mathrm{s}\mathbb{S}\mathrm{et} \to Small - Catergoies$ *is left-adjoint for the nerve functor* N.

PROOF. To prove that statment we should a natural bijection $\mathrm{Func}(TX, \mathcal{C}) \simeq \mathrm{Hom}_{\mathrm{s}\mathbb{S}\mathrm{et}}(X, N\mathcal{C})$.

Let $\phi : X \to N\mathcal{C}$ be simplicial maps. This includes the maps $\phi_0 : X_0 \to Obj(\mathcal{C})$ and $\phi_1 : X_1 \to \mathrm{Hom}_{\mathcal{C}}$ which can be though of as map $\phi_0 : Obj(T(X)) \to obj(\mathcal{C})$ and $\phi_1 : \mathrm{Hom}_{T(X)} \to \mathrm{Hom}_{\mathcal{C}}$ and $T(X)$ to $\mathcal{C}$. Note that. ϕ_1 is a first defined on the generators of $\mathrm{Hom}_{T(X)}$ and then it is extend by compositing law, now we need to prove this is well-defined with respect to the equaivalance relation$\sim$ and it sends the identity morphism to the identity.

Note that the maps ϕ_0 and ϕ_1 come with the following commutative diagrams:

$$(1.40)\quad \begin{array}{ccc} X_1 & \xrightarrow{d_0} & X_0 \\ \downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi_0} \\ \mathrm{Hom}_{\mathcal{C}} & \xrightarrow[\text{target}]{} & obj(\mathcal{C}) \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{d_1} & X_0 \\ \downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi_0} \\ \mathrm{Hom}_{\mathcal{C}} & \xrightarrow[\text{source}]{} & obj(\mathcal{C}) \end{array} \qquad \begin{array}{ccc} X_0 & \xrightarrow{s_0} & X_1 \\ \downarrow{\scriptstyle\phi_0} & & \downarrow{\scriptstyle\phi_1} \\ obj(\mathcal{C}) & \xrightarrow[s_0: A \mapsto (A \xrightarrow{id_A} A)]{} & \mathrm{Hom}_{\mathcal{C}} \end{array}$$

The upper right diagram implies that for $x \in X_0 = Obj(T(X))$, we have $\phi_1(id_x) = \phi_1(s_0(x)) = s_0(\phi_0(x)) = id_{\phi_0(x)}$ therefore ϕ_1 sends the identity morphism to the identity, as it should (as a functor). The two other diagrams above show that for a 1-simplex σ, the morphism $\phi_1(\sigma)$ is a morphism from $\phi_0(d_1\sigma)$ to $\phi_0(d_0\sigma)$ as it should be. The only remaining part is to prove that under that the composition law under $\sim$ is sent to the composition of the morphisms. Let σ be a 2-simplex. Then $\phi_2(\sigma)$ is 2-simplex of the form (f, g) where f and g are composable morphisms in $\mathrm{Hom}_{\mathcal{C}}$. Becaucse $\phi_2 \circ d_i = d_i \circ \phi_2$ we conclude that

$$(1.41)\qquad g = d_0\phi_2(\sigma) = \phi_1(d_0\sigma) \quad f = d_2\phi_2(\sigma) = \phi_1(d_2\sigma),$$

therefore $d_0\phi_2(\sigma) \circ d_2\phi_2(\sigma) = g \circ f = d_1(\phi_2(\sigma)) = \phi_1(d_1\sigma)$. So we have constructed a map $\Psi : \mathrm{Hom}_{s\mathbb{S}\mathrm{et}}(X, N\mathcal{C}) \to \mathrm{Func}(T(X), \mathcal{C})$ which essentially looks like $\{\phi_i\}_{i\geq 0} \to (\phi_0, \phi_1)$. It is not hard that to see that Ψ is injective. This follows from that fact that an simplex in $N\mathcal{C}$ is completely determined by it 0 and 1 dimensional faces. In other words if for two simplicial maps $\phi = \{\phi_i\}_{i\geq 0}$ and $\psi = \{\psi_i\}_{i\geq 0}$, we have $\phi_0 = \psi_0$ and $\phi_1 = \psi_1$ then $\phi = \psi$.

As for surjectivity of Ψ, let $F : T(X) \to \mathcal{C}$ be a functor. The first two component $\phi_0 : X_0 \to N\mathcal{C}_0$ and $\phi_1 : X_1 \to N\mathcal{C}_1$ of the simplicial map $\{\phi_i\}_{i\geq 0}$ are the maps given by F on sets of object and morphisms. The higher components $\phi_n : X_n \to N\mathcal{C}_n$ is given by

(1.42)

$$\phi_n(\sigma) := (F(d_2 d_3 \cdots d_n\sigma), \cdots, F(d_2 d_3 d_0^{n-3}\sigma), F(d_2 d_0^{n-2}\sigma), F(d_0^{n-1}\sigma))$$

This formule can be easily proved by induction. The naturality of the bijection Ψ is easily seen. $\square$

1.12. Kan complexes

As mentioned earlier, we are guided by the singular chains as the main examples of simplicial set offered by topology. So we intend to characterize simplicial set coming from topological spaces. To that end we introduce the notion of Kan complexes.

Definition 1.62. Let K be a simplicial set. A sequence of n $(n-1)$-simplicies $x_0, x_1, \cdots x_{k-1}, -, x_{k+1}, \cdots x_n$ is said to be compatible if for all $i < j$, $i \neq k$, $i \neq k$

$$d_i x_j = d_{j-1} x_i.$$

We say that K is a Kan complex if every compatible sequence has an extension i.e. there is an n-simplex x such that for all $i \neq k$

$$d_i x = x_i$$

PROPOSITION 1.63. *For all topological space X then singular chain* $\text{Sing}(X) = \{\text{Sing}_n(X)\}_n$ *is a Kan complex.*

PROOF. We have to prove the extension for a compatible sequence made of generators i.e. continuous maps from the geometric simplex Δ^{n-1} to X. Given a compatible sequence of continuous maps $x_i : \Delta_{n-1} \to X$ with the property

$$d_i x_j = d_{j-1} x_i \tag{1.43}$$

which means

$$x_j(s_0, s_1 \cdots s_{i-1}, 0, s_i, \cdots s_{n-2}) = x_i(s_0, s_1 \cdots s_{j-2}, 0, s_{j-1}, \cdots s_{n-2}) \tag{1.44}$$

We define $f : \cup_{i\neq k} \partial_i \Delta^n \to X$

$$\partial_i \Delta^n = t_i^{-1}(\{0\}) \cap \Delta^n \subset \Delta^n$$

as follows: $f(t_0, \cdots t_{i-1}, 0, t_{i+1}, \cdots t_n) = x_i(t_0, \cdots t_{i-1}, t_{i+1}, \cdots t_n)$. We have to check that f is well-defined on $\partial_i \Delta^n \cap \partial_j \Delta^n$. For $i < j-1$ this is equivalent to the identity

$$x_i(t_0, \cdots t_{i-1}, t_{i+1}, \cdots t_{j-1}, 0, t_{j+1} \cdots t_n) = x_j(t_0, \cdots t_{i-1}, 0, t_{i+1}, \cdots t_{j-1}, t_{j+1}, \cdots t_n)$$

which follows from the (1.44) for

$$\begin{aligned} s_0 = t_0, \cdots, s_{i-1} = t_{i-1} \\ s_i = t_{i+1}, \cdots, s_{j-2} = t_{j-1} \\ s_{j-1} = t_{j+1}, \cdots, s_{n-2} = t_n \end{aligned} \tag{1.45}$$

For $i = j-1$ the f being well-defines amounts to the identity

$$x_i(t_0, \cdots t_{i-1}, 0, t_{i+2} \cdots t_n) = x_{i+1}(t_0, \cdots t_{i-1}, 0, t_{i+2}, \cdots t_n)$$

which follows from (1.44) for $s_k = t_k$ □

There is another description of the extension property for for compatible sequences in terms of simplicial maps. For a fix n and $0 \leq i \leq n$, as we mentioned earlier the i-th face $\partial_i \Delta_n$ of Δ_n is the subcomplex generated by $d_i \in \text{Hom}_{\mathbf{\Delta}}([n-1], [n])$. If we have a sequence of $(n-1)$-simplices $x_0, \cdots x_{k-1}, x_{k+1}, \cdots x_n$, then just like Yoneda Lemma we a have a sequence of simplicial maps $\bar{x}_i : \partial_i \Delta_n \to X$ given by sending a k-simplex $d^i \circ f \in \partial_i \Delta^n[k]$ to

$$\bar{x}_i(d^i f) = f^*(x_i) \in X_k,$$

here $f : [k] \to [n-1]$. One cas natural ask if the collection $\bar{x}_i$ gives rise to a well-defined a simplicial map from the i-th horn

$$\Lambda_n^k := \cup_{i\neq k} \partial_i \Delta_n$$

Proposition 1.64. *$x_0, \cdots x_{k-1}, x_{k+1}, \cdots x_n$, of $(n-1)$-simplices in X are compatible if and only the collection of maps $\bar{x}_i : \partial_i \Delta_n \to X$ extends to a simplicial maps $\bar{x} : \cup_{i \neq k} \partial_i \Delta_n$. When the sequence is compatible, the existence of a n-simlex $x \in X_n$ with $\partial_i x = x_i$ is equivalente to the existence of an extension of $\bar{x}$ from Λ_n^k to the entier Δ_n.*

Proof. Suppose that x_i's are compatible. We just have to prove that on the intersection $\partial_i \Delta_n \cap \partial_j \Delta_n$ the simplicial maps $\bar{x}_i$ and $\bar{x}_j$ coincide: Suppose that $d^i f = d^j g$ for certain $f, g : [k] \to [n-1]$. We can assume that $i < j$. Then using Proposition 1.5, we can write $f = d^{j-1} h$ $g = d^i h$ for a unique morphism $h : [k-2] \to [k-1]$. We have

$$\begin{aligned} \bar{x}_i(d^i f) = f^*(x_i) = (d^{j-1}h)^*(x_i) = h^*(d^{j-1})^*(x_i) = h^*(d_{j-1}(x_i))x(\\ = h^*(d_i x_j) = h^*(d^i)^*(x_j) = (d^i h)^*(x_j) = g^*(x_j) = \bar{x}_j(d^j g) \end{aligned} \tag{1.46}$$

Conversely, if $\bar{x}_i$ extends to a map on Λ_n^k, then $\bar{x}_i$ and $\bar{x}_j$ must coincide on the intersection $\partial_i \Delta_n \cap \partial_j \Delta_n$ which contains $d^i d^{j-1} = djd^i$. The identity $\bar{x}_i(d^i d^{j-1}) = \bar{x}_j(d^j d^i)$ is equaivalente to the identity $d_{j-1} x_i = d_i x_j$.

□

Theorem 1.65. *(J. C. Moore) Any simplicial group G is s Kan complex.*

Proof. Suppose that $x_0, \cdots x_{k-1}, x_{k+1}, \cdots x_n$, is a compatible sequence. Let

$$\begin{aligned} y_0 &= s_0(x_0) \\ y_i &= y_{i-1}(s_i d_i w_{i-1})^{-1} s_i(x_i) \quad 0 < i < k \\ y_n &= y_{k-1}(s_{n-1} d_n(y_{k-1}))^{-1} s_{n-1}(x_n) \\ y_i &= y_{i+1}(s_{i-1} d_i y_{i+1})^{-1} s_i(x_i) \quad k < i < n \end{aligned} \tag{1.47}$$

y_{k+1} is the desired extension of x_i's. □

1.13. Homotopies in Kan complexes and Homotopy groups of simplicial sets

For topological spaces we can construct homotopy groups. So we expect to be able to introduce a notion of homotopy groups which will be isomorphic to the the homotopy groups of the geometric realisation.

Definition 1.66. Let K be a simplcial set. Two n-simplices x and x' are said to be homotop and we write $x \sim x'$, if for all $0 \leq i \leq n$,

$$d_i x = d_i x'$$

and there is an $n+1$-simplex y such that

$$\begin{aligned} d_n y &= x \\ d_{n+1} y &= x' \\ d_i y = s_{n-1} d_i x &= s_{n-1} d_i x' \end{aligned} \tag{1.48}$$

We say that y is a homotopy between x and x'.

PROPOSITION 1.67. *If K is a Kan complex the $\sim$ is an equivalence relation.*

PROOF. First we have to prove $x \sim x$ for all x. If $y = S_n(x)$, then we have $d_n y = d_{n+1} y = x$ and $d_i y = d_i s_n(x) = s_{n-1} d_i x$.

To prove the symmetry and transitivity we prove that if $x \sim x'$ and $x \sim x''$ then $x' \sim x''$. Let y' be a homotopy between x and x' and y'' a homotopy between x and x'':

$$\begin{aligned} d_n y' = x, \quad d_{n+1} y' = x', \quad d_i y' = s_{n-1} d_i x = s_{n-1} d_i x' \\ d_n y'' = x, \quad d_{n+1} y'' = x'', \quad d_i y'' = s_{n-1} d_i x = s_{n-1} d_i x'' \end{aligned} \tag{1.49}$$

Consider the compatible sequence

$$\alpha_0 = d_0 s_n s_n x', \cdots, \alpha_k = d_k s_n s_n x' \cdots ..., \alpha_n = d_{n-1} s_n s_n x, \alpha_n = y', \alpha_{n+1} = y'', -.$$

There is a z such that $d_i z = \alpha_i$ for $0 \le i \le n+1$. Let $w = d_{n+2} z$, then we have

$$\begin{aligned} d_n w = d_n d_{n+2} z = d_{n+1} d_n z = d_{n+1} y' = x' \\ d_{n+1} w = d_{n+1} d_{n+2} z = d_{n+1} d_{n+1} z = d_{n+1} y'' = x'' \end{aligned} \tag{1.50}$$

and

$$d_i w = d_i d_{n+2} z = d_{n+1} d_i z = d_{n+1} d_i s_n s_n x' = d_i d_n s_n s_n x' = d_i s_n x' = s_{n-1} d_i x'$$

so w is a homotopy between x' and x''. □

A good theory of homotopy groups requires long exact sequence therefore a relative theory.

DEFINITION 1.68. Let L be a simplicial subset of K. Two n-simplices are said to homotop relative L, $x \overset{L}{\sim} x'$ if

(1) $d_i x = d_i x'$, $1 \le i \le n$.
(2) If y is a homotopy between $d_0 x$ and $d_0 x'$, there is a $(n+1)$-simplex $w \in K_{n+1}$ such that

$$d_0 w = y, \quad d_n w = x, \quad d_{n+1} w = x' \quad d_{0<i<n-1} w = s_{n-1} d_i x = s_{n-1} d_i x'$$

We say that w is a relative homotopy between x and x'.

PROPOSITION 1.69. *Suppose that $L \subset K$ are both Kan complexes. Then $\overset{L}{\sim}$ is a homotopy equivalence.*

PROOF. First of all $x \overset{L}{\sim} x$ because if $d_0 x \in L$ then as we saw $y = s_{n-1} d_0 x$ is self-homotopy of $d_0 x$. For $w = s_n s x$, we have $1 \le i \le n$, $d_i w = s_{n-1} d_i x$ and $d_n w = d_{n+1} = x$, $d_0 w = y$ therefore w is a relative homotopy between x and x.

As for symmetry and transitivity, suppose $x \overset{L}{\sim} x'$ and $x \overset{L}{\sim} x''$. Let w' and w'' be the corresponding relative homotopies ie.e

$$\begin{aligned} d_n w'' = x, \quad d_{n+1} w' = x', \quad d_i w' = s_{n-1} d_i x = s_{n-1} d_i x', \quad 1 \le i \le n \\ d_n y'' = x, \quad d_{n+1} y'' = x'', \quad d_i y'' = s_{n-1} d_i x = s_{n-1} d_i x'' \quad 1 \le i \le n \end{aligned} \tag{1.51}$$

and $y' = d_0 w'$ and $y'' = d_0 w''$ provide respectively the homotopies $d_0 x \overset{L}{\sim} d_0 x'$ and $d_0 x \overset{L}{\sim} d_0 x''$ in L. Similarly to the proof of Proposition 1.67 there is a $z \in L_{n+1}$ such that $d_i = d_i s_{n-1} s_{n-1} d_0 x'$, $0 \le i < n-1$, $d_{n-1} z = y'$ and $d_{n-1} z = y''$.

Now the sequence $z, d_1 s_n s_n x', \cdots, d_{n-1} s_n s_n x', w', w'', -$ is a compatible one, so there is a V such that $d_i v$, $0 \le i \le n+1$ are, in order, the elements of the above sequence. Then $w = d_{n+2} v$ does the job, i.e. it is a relative homotopy between x' and x''.

□

We are ready to the define the homotopy groups of a Kan complex K. To that end we need to introduce the right notion of basepoint. Choose $x_0 \in K$, consider the simplicial subset generated by x_0. This simplicial subset has exactly one n-simplex $s_{n-1} s_{n-2} \cdots s_0(x)$ for all $n \ge 1$. We continue to denote this simplicial subset with x_0. We set

$$\tilde{K}_n := \{x \in K | d_i x = x_0, \quad 0 \le i \le n\}.$$

Then the n-homotopy group of K is

$$\pi_n(K, x_0) := \tilde{K}_n / \sim$$

The relative homotopy groups is defines in a similar manner,

$$\tilde{K}_n(L) := \{x \in K_n | \quad d_i x = x_0 \quad 1 \le i \le n, \quad d_0 x \in L_{n-1}\},$$

and

$$\pi_n(K, L, x_0) := \tilde{K}_n(L) / \sim .$$

Definition 1.70. For homotopy classes $a = [x]$ and $b = [y] \in \pi_n(K, x_0)$ We define product

$$[x].[y] = [d_n z]$$

where z is the extension of the compatible sequence $x_0, \cdots x_0, x, -, y$.

Proposition 1.71. *The product defined above is well-defined.*

Proof.
- First we prove that for given representatives x and y, the class of $[d_n z]$ is independent of the choice of z. Suppose that z and z' are two extension for the sequence $x_0, \cdots x_0, x, -, y$. Let $w \in K_{n+2}$ be an extension for the suite $x_0 \cdots, s_n x, -, z, z')$ then

$d_n w$ is homotopy between $d_n z$ and $d_n z'$ because we have

$$\begin{aligned} d_n d_n w &= d_n d_{n+1} w = d_n z \\ d_{n+1} d_n w &= d_n d_{n+2} w = d_n z. \\ d_{n-1} d_n w &= d_{n-1} d_{n-1} w = d_{n-1} s_n x = s_{n-1} d_{n-1} x = x_0. \\ d_{i \le n-1} d_n w &= d_{n-1} d_i w = x_0 \end{aligned} \tag{1.52}$$

- We prove that the classes a and b of the product $a.b$ is independent of the choice of the representatives x and y. Suppose that w is a homotopy between y and y' i.e.

$$d_{n+1} w = y \quad \& \quad d_n w = y' \& \quad d_i w = x_0$$

- Let z' be an extension for the sequence $x_0, \cdots x_0, x, -, y'$. The sequence $x_0, \cdots x_0, s_{n-1} x, z', -, w$ is compatible therefore it has an extension of the sequence $u \in K_{n+2}$. We claim that $d_{n+1} u$ is an extension $x_0, \cdots x_0, x, -, y$. This is because

$$\begin{aligned} d_{n-1} d_{n+1} u &= d_n d_{n-1} u = d_n s_{n-1} x = x \\ d_{n+1} d_{n+1} u &= d_{n+1} d_{n+2} u = d_{n+1} w = y \\ d_{i<n-1} d_{n+1} u &= d_n d_i u = x_0. \end{aligned} \tag{1.53}$$

Therefore $d_n d_{n+1} u$ is the product of the classes a and b using the representatives x and y.On the other hand

$$d_n d_{n+1} u = d_n d_n u = d_n z'$$

which is the product of the classes using the representatives x and y'.

□

Similarly we define the product on the relative homotopy classes $\pi_n(K, L, x_0)$. Let $[x]$ and $[y]$ two relative classes. Then $d_0 x$ and $d_0 y$ in L_{n-1} and represents two homotopy classes in $\pi_{n-1}(L, x_0)$ where we can define the their product

$$[d_0 x].[d_0 y] = [d_{n-1} z]$$

where z is the extension of the compatible sequence $x_0, \cdots, x_0, d_0 x, -, d_0 y$. The sequence $z, x_0, \cdots, x_0, x, -, y$ is also compatible therefore has an extension w, we set

$$[x].[y] = [d_n w].$$

It turns out this product is well-defined as well.

Proposition 1.72. *For a Kan complex K, $(\pi_n(K, x_0), \cdot)$ is a group.*

Proof. It is quite clear that that $[x_0]$ is a neutral element. We prove that existence of the inverse by showing that we can divide. For the two classes $[x]$ and $[y] \in \pi_n(K, x_0)$ we consider the compatible sequence $x_0, \cdots x_0, -, x, y$ of n-simplices which can be extend to a $(n+1)$-simplex z, then by the definition of the product we have

$$[d_{n-1} z][x] = [y].$$

As for the associativity, we consider the three extensions of three compatible sequences,

Extension/faces$n+1$	0 to $(n-2)$th faces	d_{n-1}	d_n	d_{n+1}	d_{n+2}
$w_{n-1} \in K_{n+1}$	$x_0, \cdots, x_0$	x	$-$	y	$\emptyset$
$w_{n+1} \in K_{n+1}$	$x_0, \cdots, x_0$	$d_n w_{n-1}$	$-$	z	$\emptyset$
$w_{n+2} \in K_{n+1}$	$x_0, \cdots, x_0$	y	$-$	z	$\emptyset$
$u \in K_{n+2}$	$x_0, \cdots, x_0$	w_{n-1}	$-$	w_{n+1}	w_{n+2}

Then we have

$$([x].[y]).[z] = [d_n w_{n-1}][z] = [d_n w_{n+1}] = [d_n d_{n+1} u] = [d_n d_n u] \tag{1.54}$$

We can observe that $d_n u$ is an extension of the sequence

$$x_0, \cdots x_0, x, -, d_n w_{n+2},$$

therefore

$$[x].[d_n w_{n+2}] = [d_n d_n u]$$

On the other hand $[d_n w_{n+2}] = [y][z]$, hence the associativity.

□

THEOREM 1.73. *For all pair of $L \subset K$ of Kan complexes, there is a natural long exact sequence of groups*

(1.55)

$$\cdots \longrightarrow \pi_{n+1}(K, L, x_0) \xrightarrow{\partial} \pi_n(L, x_0) \xrightarrow{i} \pi_n(K, x_0) \xrightarrow{j} \pi_n(K, L, x_0) \longrightarrow \cdots$$

Where $\partial[x] = [d_0 x]$ and i and j are induced by inclusion.

PROOF. Obviously the i and j are group homomorphism. It follows from our definition of the the production on the relative homotopy groups that ∂ is also a group homomorphism.

(1) Exactness at (∂, i):

- Im$(\partial) \subset \ker(i)$: We have $i\partial[x] = i[d_0 x]$. Let y be an extension of the sequence of $n+1$-simplices $-, x_0, x_0, \cdots, x_0, x$, the $d_0 z$ is a homotopy between $d_0 x$ and x_0 in L_n.
-ker$(i) \subset$ Im(∂):

-ker$(i) \subset$ Im(∂): If $i[x] = [x_0]$ i.e. $x \simeq x_0 \in K_n$ there is a homotopy y in K_n between x and $_0$. This implies that the sequence of $(n+1)$-simplices $y, x_0 \cdots, x_0, -$ are compatible, hence expandable to z. Take a look at $d_{n+2} z$, we have

$$\begin{aligned} d_0 d_{n+2} z &= d_{n+1} d_0 z = d_{n+1} y = x \\ d_{1 \le i \le n+1} d_{n+2} z &= d_{n+1} d_i z = d_{n+} x_0 = x_0 \end{aligned} \tag{1.56}$$

therefore $d_{n+2} z$ represents a class in $\pi_{n+1}(K, L, x_0)$ and $\partial[d_{n+2} z] = [d_0 d_{n+2} u] = [y]$.

(2) Exactness at (j, ∂):
- $\mathrm{Im}(j) \subset \ker \partial$ because $\partial j[x] = [d_0 x] = [x_0]$ by definition of $\tilde{K}_n$.

-As for $\ker \partial \subset \mathrm{Im}(j)$: If for $x \in \pi_n(K, x_0)$, $d_0 x \simeq x_0 \in \tilde{K}_n(L)$ then there is $z \in L_n$ such that

$$d_n z = d_0 x \quad \text{and } d_{0 \leq i < n} z = x_0.$$

Clearly $z, x_0 \cdots, x_0, -, x$ are compatible so there is an extension $y \in K_{n+1}$. It is easily verified that y is relative homotopy between $d_n y$ and x. Note that

$$d_{1 \leq i \leq n} d_n y = d_{n-1} d_i y = d_{n-1} d_i y = x_0 d_0 d_n y = d_{n-1} d_0 y = d_{n-1} z = x_0$$

therefore $d_n y$ represents a class in $\pi_n(K, x_0)$. Moreover $d_0 y = z$ is the homotopy in L_{n-1} between $d_0 x$ and x_0 as it should be, and

$$[x] = j([d_n y])$$

(3) Exactness at (i, i)):
- $\mathrm{Im}(i) \subset \ker(j)$: For $x \in \tilde{L}_n$, the sequence

$$-, x_0, \cdots, x_0, x$$

is extendable to $z \in L_{n+1}$. Indeed $d_0 z$ is a homottopy in L between x_0 and $d_0 x$ and z is a relative homotopy between x_0 and x.
- $\ker(j) \subset \mathrm{Im}(i)$: Suppose that $j([x]) = [x_0]$ in $\pi_n(K, L, x_0)$ then there is a relative homotopy w between x and x_0 and $z := d_0 w \in L_n$ is a homotopy between x_0 and $d_0 x = x_0$ in L. The $n+1$ n-simplices $z, x_0, \cdots, x_0, -$ are compatible therefore there is an $n+1$-simplex v such tht

$$d_0 v = z, d_{0 \leq i \leq n} z = x_0$$

It turns out that

$$s_{n-1} z, x_0, \cdots, x_0, v, w, -$$

are compatible and extendable to $t \in K_{n+2}$. Then we claim that $d_{n+2} t$ is a homotopy between x an $d_{n+1} v \in L$, hence x is in the image of i. To see that

$$\begin{aligned} d_n d_{n+2} t &= d_{n+1} d_n t = d_{n+1} v \\ d_{n+1} d_{n+2} t &= d_{n+1} d_{n+1} t = d_{n+1} w = x \\ d_{0<i<n} d_{n+2} t &= d_{n+1} d_i t d_{n+2} x_0 = x_0 \\ d_1 d_{n+2} t &= d_{n+1} d_0 t = d_{n+1} s_{n-1} z = s_{n-1} d_n z = s_{n-1} x_0 = x_0 \end{aligned} \tag{1.57}$$

□

It may not be immediately clear to the reader why our definition of homotopy groups is related to the standard topological definition. Here we answer to the this question.

By Yoneda lemma, a n-simplex x gives rise to a simplicial map $\bar{x} : \Delta_n \to K$. In particular, if $\Delta_i x = x_0$, for all i, then $\bar{x}(\partial \Delta_n) \subset x_0$.

PROPOSITION 1.74. *There is a bijection between elements of $\pi_n(K, x_0)$ and the homotopy class of maps $\bar{x} : (\Delta_n, \partial\Delta_n) \to (K, x_0)$. Here the homotopies are relative to $\partial\Delta_n$) and x_0.*

PROOF. Suppose that $x, y \in \tilde{K_n}$ are homotpo. Then there is $z \in K_{n+1}$ such that $x = d_n z$, $y = d_{n+1} z$ and $d_{i<n} z = x_0$. We construct a homotopy $h_i : \Delta_n[q] \to L_{q+1}$, $0 \leq i \leq q$, between $\bar{x}$ and $\bar{y}$ as follows: We use the fact Δ_n is generated by on element namely $id_n \in \Delta_n[n]$ and then we extend to other element of Δ_n using the equations (see (1.13)) uniqueness of the presentation of the morphism in the category $\mathbf{\Delta}$.

$$
\begin{aligned}
h_i(id_n) &= s_i(x) \quad 0 \leq i < n, \\
h_n(id_n) &= z
\end{aligned}
\tag{1.58}
$$

It remains to prove that $h_i(\partial\Delta_n) \subset x_0$. To that end, note that $\partial\Delta_n$ is generated by $d^i \in \mathrm{Hom}_\Delta([n-1],[n])$ and its elements are of the form $d^i \circ f$ where $f \in \mathrm{Hom}_{\mathbf{\Delta}}([k],[n])$. For instance, for $f = id_n$ and $0 \leq j \leq n-1$ we have

(1.59)

$$
h_j(d^i \circ id_n) = h_j(d_i(id_n)) = \begin{cases} d_i h_{j+1}(id_n) = d_i s_j(x) & \text{if } j \geq i = \begin{cases} d_i h_n(id_n) = d_i z = x_0, \text{ if } i < n \\ d_i s_{j+1}(x) = s_j d_i(x) = s_j(x_0) = x_0 \end{cases} \\ d_{i+1} h_j(id_n) & \text{if } j < i = \begin{cases} d_n h_{n-1}(id_n) = d_n s_{n-1}(x) = x_0, \quad i = n-1 \\ d_{i+1} s_j(x) = s_j d_i(x) = s_j(x_0) = x_0 \end{cases} \end{cases}
$$

Similar computation and result hold for all $f \in \mathrm{Hom}_{\mathbf{\Delta}}(-,[n])$. Coversely, suppose that h is a homotopy relative to $\partial\Delta^n$ between $\bar{x}$ and $\bar{y}$. Let $z_i := h_i(id_n)$, for $0 \leq i \leq n$. Then using the relations ((1.13)) and hypothesis that $h_i(d^i(id_n)) \in h_i(\partial\Delta_n) = x_0$, we have,

$$
\begin{aligned}
d_i z_j &= x_0, \quad i \neq j, j+1 \\
d_i z_i &= d_i z_{i-1} \\
d_0 z_0 &= x \\
d_{n+1} z_n &= y
\end{aligned}
\tag{1.60}
$$

To complete the proof we need the following lemma.

LEMMA 1.75. *Suppose that $z \in K_{n+1}$ and $d_i z = x_0$ for $i \neq i, i+1$, and $d_r z$ and $d_{r+1} z \in \tilde{K_n}$. Then we have a homotopy $d_r z \sim d_{r+1} z$ in K_n.*

Proof of the Lemma For $r = n$ it is obvious by definition. So we suppose that $r < n$.

Let $w \in K_{n+2}$ be the extension for the sequence

$$\alpha_0 := x_0, \cdots, \alpha_{r-1} := x_0, \alpha_r = s_{r+1} d_{r+1} z, \alpha_{r+1} := z, \alpha_{r+2} := s_r d_{r+1} z, -, \alpha_{i > r+3} := x_0.$$

Then $t = d_r w$ satisfies the

$$d_{r+1} t = d_r z, \quad d_{r+2} t = d_{r+1} z \text{ and } d_i t = x_0, \neq r+1, r+2.$$

By repeating this process we increase the indices from r to n where the results is obvious.

To finish the proof of the Proposition, it suffices to applies the lemma to to the each z_i for $r = i$ and use the identit $d_i z_i = d_i z_{i-1}$,

$$x = d_0 z \sim d_1 z_0 = d_1 z_1 \sim d_2 z_1 \cdots d_n z_{n-1} = d_n z_n \sim d_{n+1} z_n = y$$

□

An immediate consequence of the previous result, Proposition 1.74, is the invariance of homotopy groups under homotopic maps.

COROLLARY 1.76. *If $f, g : K \to L$ are homotopic then $f_* = g_* : \pi_*(K) \to \pi_*(L)$.*

The first and most important example of Kan complex is the singular complex $\mathrm{Sing}(X) = \{\mathrm{Sing}_n(X)\}_n$ of a topology space (See Proposition 1.63). So it is natural to ask if $\pi_*(\mathrm{Sing}(X)$ is related to the (topological) homotopy groups $\pi_*(X)$.

THEOREM 1.77. *For a topological spaces X we have a natural isomorphism of groups*

$$\pi_n(S(X), S(a)) \simeq \pi_n(X, a)$$

where $a \in X$ is basepoint, $S(X) := \mathrm{Sing}(X)$ is the singular (Kan) complex de X and T is the geometric realization.

PROOF. By the adjunction property of S and T, we have a bijection

$$\mathrm{Hom}_{Top}(D_n, X) \simeq \mathrm{Hom}_{Top}(T\Delta_n, X) \simeq \mathrm{Hom}_{\mathrm{s\mathbb{S}et}}(\Delta_n, S(X)).$$

Here D_n is the n-dimensional disk. Since $T(\partial\Delta_n) \overset{homeo}{\simeq} \partial\mathbb{D}_n \simeq S^{n-1}$, The functoriality of the above bijection implies that we have a bijection for pairs,

$$\mathrm{Hom}_{Top}((T\Delta_n, T\partial\Delta_n), (X, a)) \simeq \mathrm{Hom}_{\mathrm{s\mathbb{S}et}}((\Delta_n, \partial\Delta_n), (S(X), S(a)).$$

which Proposition 1.41 conserves the homotopy relation, therefore ϕ induces an isomorphism

$$\pi_n(S(X), S(a)) \simeq \pi_n(X, a)$$

Verifying that this is conserves the group structure is not hard and is left to a reader who wants to indulge oneself. □

THEOREM 1.78. *(Moore's Theorem) For a simplicial abelian group G_* we have a natural group isomorphism*

$$\pi_n(G_*, 0) \simeq H_n(N(G_*))$$

induced by the identity map, where $N(G_*)$ is the normalized complex of G and 0 is the simplicial basepoint generated by neutral element of G_0.

PROOF. Let 0 be the neutral elements of G_0, since all the degeneracy maps S_i are simplicial maps, $s_{i-1} \cdots s_0(0)$ is the neutral element of G_i. Therefore our simplicial base point consists of the neutral element of G_i's.

The identity map $\tilde{K}_n \to N(G)_n$ is well-define because if $x \in \tilde{K}_n$, then $d_i x = 0$ for $0 \leq n \leq n-1$. Morever, since $d_n x = 0$ the identity map induces a well-define map $\tilde{K}_n \to H_n(N(G))$.

Suppose that $x \sim y \in \tilde{K}_n$ via homotopy $\sigma \in K_{n+1}$, i.e.

$$d_n\sigma = x, \quad d_{n+1}\sigma = y, \quad d_i\sigma = 0.$$

Let $z := \sigma - s_n(y)$. Then $d_n z = x - y$, $d_i z = 0$ for $i = n+1$ and $0 \leq i < n$, therefore $\partial z = x - y$ and x and y are homologous. Therefore we have an induced map

$$\pi_n(G, 0) \to H_n(N(G)).$$

The surjectivity of this map is clear because the elements of $H_n(N(G))$ are represented by n-simplices σ such that $d_i\sigma = 0$ for all i, therefore $\sigma \in \tilde{K}_n$.

As for injectivity, suppose that $\sigma \in \tilde{K}_n$ repreent 0 in $H_n(N(G))$ i.e. there is $z \in N(G)_{n+1}$ such that $d_{n+1}z = \sigma$. Now it is clear that $\sigma \sim 0 = d_n z$ because $d_i z = 0$ for $0 \leq o \leq n-1$.

It remains to prove that the identity map is group homomorphism. To end we should prove that for $x, y \in \tilde{K}_n$, there is a $(n+1)$-simplex z such that

$$\begin{aligned} d_i z &= 0 \quad 0 \leq in-1 \\ d_{n-1}z &= x \\ d_n z &= x + y \end{aligned} \tag{1.61}$$

Indeed $z = s_{n-1}(x) + s_n(y)$ does the job. □

1.14. Kan fibrations

A simplicial map $p : E \to B$ is called a Kan fibration if for any compatible n $(n-1)$-simplices $x_0, \cdots, x_{k-1}, -, x_{k+1}, \cdots, x_n$ with an extension $y \in B_n$ for $p(x_i)$, then there is an extension $x \in E_{n+1}$ for x_i's such that $y = p(x)$. It is clear that:

Proposition 1.79. *E is a Kan complex if and only if $p : E \to *$ is a Kan fibration. Here $*$ is a the simplicial singleton.*

One can introduce the notion of fibre for a Kan fibration $p : E \to B$ by setting $F = \partial^{-1}(x_0)$ where $x_0 \subset B$ is the simplicial base point (i.e the simplicial subset generated by a 0-simplex $x_0 \in B_0$.).

Proposition 1.80. *The fiber F of a Kan fibration $p : E \to B$ is a Kan complex.*

Proof. Let $x_0, \cdots, x_{k-1}, -, x_{k+1}, \cdots, x_n$ be a compatible sequence in F, then for all i, $p(x_i) = x_0$ and x_0 is a an obvious extension of $p(x_i)$'s in B. Since p is a Kan fibration then there is an extension $x \in E$ for x_i's such that $p(x) = x_0$. The latter means that $x \in F$, in other wors x is an extension of x_i's in F, this proves that F is a Kan complex. □

LEMMA 1.81. *Suppose that $p : E \to B$ is a Kan fibration. Let $x_{i_1}, \cdots x_{i_r}$, $i_1 < i_2 < \cdots < i_r$, be a sequence of q-simplices in E such that $d_{i_s} x_{i_t} = d_{i_t - 1} x_{i_s}$ for $s < t$. Assume that there is an extension $y \in B_{q+1}$ for $p(x_{i_j})'s$, i.e.*

$$d_{i_j} y = p(x_{i_j}).$$

There is an extension $x \in E$ for x_{i_j}'s, i.e.

$$d_{i_j} x = p(x_{i_j}).$$

PROOF. One day □

PROPOSITION 1.82. *Let $p : E \to B$ be a Kan fibertion:*

(1) If E is Kan complex and p is onto, then B is also a Kan complex.

(2) If B is a Kan complex then E is also.

PROOF. Proof of (1): If $y_0, \cdots, y_{k-1}, -, y_{k+1}, \cdots, y_n$ is a sequence of compatible $(n-1)$-simplices in B. Then there is x_0 such that. $y_0 = p(x_0)$. Since $d_0 y_1 = d_0 y_0$, by applying Lemma 1.81 to sequence x_0 (with $y := y_1$ satisfying $d_0 y_1 = p(d_0 x_0)$) there is a $x_1 \in E$ such that $p(x_1) = y_1$ and $d_0 x_1 = d_0 x_0$. By repeating this process, at each stage we find $x_i \in E$ such that $p(x_i) = y_i$ and $d_j x_i = d_{i-1} x_j$ for all $j < i$. By doing so we are lifting y_i's to a compatible sequence of $(n-1)$-simpkices in E which a is Kan complex. Let x be an extension of x_i's in E, then $p(x)$ is desired extension of y_i's.

Proof of (2): If x_i's is a compatible sequence in E then $p(x_i)$ is a compatible sequence in B therefore has an extension y in B. Since p is Kan fibration, y can be lifted to an extension x in E of x_i's. □

One naturally expects a long exacts sequence of homotopy groups associated to a fibration. the groups homomorphism $q : \pi_n(B, b_0) \to \pi_n(E, F, a_0)$. where $b_0 = p(a_0)$ and $a_0 \in E_0$,is defined as follows: For $y \in \tilde{B}_n$, we have $d_i y = b_0$, so we can think of y as an extension of the compatible sequence $-, b_0, b_0, \cdots, b_0$ which is image of the compatible sequence $-, a_0, a_0, \cdots a_0$,, p being a Kan fibration there is an n-simplex x such that $p(x) = y$ and $d_{i>0} x = a_0$. Now $p(d_0 x) = d_0 p(x) = d_0 y = b_0$ therefore $d_0 x \in F$ and x defines a homotopy class in the the relative homotopy group $\pi_n(E, F, a_0)$. The map $q : \pi_n(B, b_0) \to \pi_n(E, F, a_0)$,

$$q([x]) = [y]$$

is well-defined meaning that it conserves the homotopy relation, because the homotopies which are also extensions, can be lifted via p. On the other hand $p : E \to B$ induces a map $p : \pi_n(E, F, a_0) \to \pi_n(B, b_0)$. It is clear that $qp = id$ and $pq = id$ proving that q and q are bijections, and since p is a group homomorphism, q is so.

Via the isomorphism q, the connecting $\partial : \pi_{n+1}(E, F, a_0) \to \pi_n(F, b_0)$ becomes $\partial_{\#} : \pi_{n+1}(B, a_0) \to \pi_n(F, b_0)$,

$$\partial_{\#}([y]) = [d_0 x]).$$

Using the isomorphism q, the long exact sequence 1.73 transforms in:

PROPOSITION 1.83. *For a Kan fibration $p : E \to B$ with fibre F, we have a long exact sequence of groups*

$$\cdots \to \pi_{n+1}(F, a_0) \xrightarrow{i} \pi_{n+1}(E, a_0) \xrightarrow{p} \pi_{n+1}(B, b_0) \xrightarrow{\partial_\#} \pi_n(F, a_0) \to \cdots$$

1.15. Universal cover

In this section we give a construction of the universal cover of a Kan complex K.

We suppose that K is a connected Kan complex with just one 0-simples x_0. Let $\pi = \pi_1(K, x_0)$ be the fundamental group of K (here x_0 also denotes the subcomplex generated by x_0). Define $\tilde{K}$ by

$$\tilde{K}_n = K_n \times \pi$$

equipped with the degeneracy and degeneracy maps

(1) $d_i(x, a) = (d_i x, a)$ for. $i < n$
(2) $d_n(x, a) = (d_n x, (d_0^{n-1} x)^{-1} a)$, here d_0^{n-1} is $(n-1)$-th iteration of d_0.
(3) $s_i(x, a) = (s_i x, a)$

Let x_0 be the (simplicial) base point of K and $\bar{x}_0 = (x_0, 1)$ be the simplicial basepoint of $\tilde{K}$ where $1 \in \pi$ is the neutral element. Then we have

(1) $\tilde{K_n}$ is a simplicial set.
(2) The natural projection $\pi : \tilde{K} \to K$ is a Kan fiberation.
(3) $\pi_n(F, \bar{x}_0) = 1$ for $n \geq 1$ and $\pi_0(F, \bar{x}_0) \simeq \pi$.
(4) The connecting map $\partial : \pi_1 = \pi_1(K, x_0) \to \pi_1 = \pi_0(F, \bar{x}_0)$ of the fiberation long exact sequence

$$\cdots \to \pi_1(F, \bar{x}_0) \to \pi_1(\tilde{K}, \bar{x}_0) \to \pi_1(K, x_0) \to \pi_0(F, \bar{x}_0) \to \cdots$$

is an isomorphism.

(5) Conclude that $\pi_n(\tilde{K}, \bar{x}_0) \simeq \pi_n(K, x_0)$ for $n \geq 2$.

Proof of the (1) and (2) are easy and left to the reader.

1.16. Minimal Complex

Existence of a minimal complex is an key ingredient for proving various theorems such as Whitehead and Hurewicz theorems.

DEFINITION 1.84. A Kan complex is called Minimal if $d_i x = d_i y$ for all $i \neq k$, implies that $d_k x = d_k y$.

PROPOSITION 1.85. *A Kan complex is minimal if and only if homotopy equivalence relation is indeed the equality.*

PROOF. Suppose that K is minimal and $x \sim y \in K_n$. Then there exist $w \in K_{n+1}$ such that $d_n w = x$, $d_{n+1} w = y$ and $d_{i<n} w = s_{n-1} d_i x$. The latter implies that

$$d_i s_n(x) = s_{n-1} d_i x = d_i w \text{ for } i < n$$

and we have $d_n s_n(x) = x = d_n w$, therefore by minimality

$$x = d_{n+1} s_n(x) = d_{n+1} w = y.$$

Conversely, suppose that for $(n+1)$-simplices x and y we have $d_i x = d_i y$ for all $i \neq k$. In order to prove that $d_k x = d_k y$ it suffices to $d_k x \sim d_k y$.

First the case $k \leq n$, note that $s_n d_0 x, \cdots, s_n d_{k-1} x -, s_n d_{k+1} x, \cdots s_n d_n x, , x, y$ is a compatible sequence therefore extendable by a $(n+2)$-simplex. It is easily checked that $d_k z$ is homotopy beween $d_k x$ and $d_k y$. As for the case $k = n+1$, let the $(n+2)$-simplex z be an extension for $s_{n-1} d_0 x, \cdots, s_{n-1} d_{n-1} x, x, y, -$ then $d_{n+2} z$ is a homotopy between $d_{n+1} x$ and $d_{n+1} y$.

□

The fundamental result of this section is that every Kan complex K has minimal subcomplex which is deformation retract of K. This requires a lemma.

LEMMA 1.86. *Let x and y be two degenerate n-simplices. If $d_i x = d_i y$ for all i, then $x = y$.*

PROOF. First notice that a degenerate simplex x, is of the form form $x = s_i d_i x$. To see, write $x = s_i z$ for z and i, then $z = d_i s_i z = d_i x$ hence $x = s_i d_i x$.

If $x = s_i d_i x$ and $y = s_i d_i y$ now it is clear that $x = y$ because by hypothesis $d_i x = d_i y$. If $x = s_i d_i x$ and $y = s_j d_j y$ for $i < j$, then

$$\begin{aligned} x &= s_i d_i x = s_i d_i y = s_i d_i s_j d_j y = s_i s_{j-1} d_i d_j y \\ &= s_j s_i d_i d_j y \end{aligned} \tag{1.62}$$

Therefore x is in the image of s_j thus by the argument in the begining, $x = s_j d_j x$ and we are back to the case $i = j$. □

THEOREM 1.87. *Every Kan complex K admits a minimal K' which is deformation retract of K.*

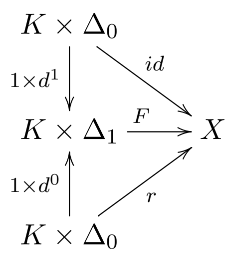
PROOF. We construct the simplices of K' by induction. The vertices of K'_0 is made by choosing a representative for each classes of $\pi_0(K)$ Suppose that K'_i, $i < n$, are constructed for . To define K'_n, first we consider the set of n-simplices x with $d_i x \in K'_{n-1}$. Then we pick one representative from each homotopy classe of this set, and when it is possible we chose a degenerate representative.

First we prove that K' is a subcomplex. The stability under the face maps is true by construction. Stability under degeneracy maps is also proved by induction. Let $x \in K'_n$, then for a fixed degeneracy map s_i, all the face $d_j(s_i(x))$ belongs, by induction, to K'_n. Therefore, by construction of K'_{n+1}, $s_i(x)$ is homotopic to a simplex y in K'_{n+1}. By the construction K'_{n+1}, y

should be degenerate since otherwise we would have chosen the degenerate representative $s_i(x)$ instead of y and we would be done. So have $s_i(x) \sim y$, which also means $d_k(s_i(x) = d_k y$ for all k and by Lemma 1.86, $s_i(x) = y$ hence $s_i(x) \in K'_{n+1}$. By construction K' is a minimal comples because the homotopy implies equality.

Now we prove that K' is a deformation retract of K. To that end we construct a simplicial homotopy $H : K \times \Delta_1 \to K$ between the identity map $id : K \simeq K \times \Delta_0 \to K$ and a retraction $r : K \simeq K \times \Delta_1 \to K$ whose image is in. K'

(1.63)



We construct H on $\mathrm{Sk}_n(K)$ by induction on n. For $n = 0$, FH on $\mathrm{Sk}_0(K) \times \Delta_1$ is defined by

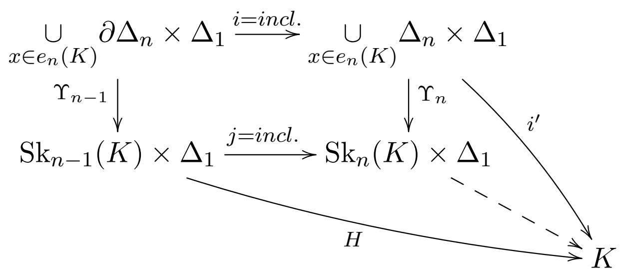
(1.64)

$H(x, 0) = x$

$H(x, 1) = m,$ where m is the unique 0-simplex in K' which is in the same connected component a

$H(s(x), id) = \sigma$ where σ is a 1-simplex with $d_0\sigma = x$ and $d_1\sigma = m$

(1.65)



We suppose that $H : \mathrm{Sk}_{n-1}(K) \times \Delta_1 \to K$ is constructed. Since we have a push-out diagram (see Proposition 1.14), in order to extend H to $Sk_n(K) \times \Delta_1$ it suffices to extend. $H \circ \Upsilon_n$ from $\bigcup_{x\in e_n(K)} \partial\Delta_n \times \Delta_1$ to $\bigcup_{x\in e_n(K)} \Delta_n \times \Delta_1$. Note that $\Delta_n \times \Delta_1$ is union of simplicial subsets which are generated by the images of the maps $\sigma_j : [n+1] \to [1] \times [n]$ given by its image (which is ordered)

$$\mathrm{Im}(\sigma_j) = ((0,0), (1,0) \cdots, (j, 0,), (j, 1) \cdots (n, 1))$$

These generators have all of their faces in the boundary complex $\partial_n \times \Delta_1$ except for $d^i\sigma_j \in \partial\Delta_n \times \Delta_1$, for $i \neq j, j+1$ and $j \neq 0, n+1$. Moreover,

$d_0\sigma_0 \in \Delta_n \times d_0\Delta_1 = \Delta_n \times 1$ and $d_{n+1}\sigma_0 \in \Delta_n \times d_1\Delta_1 = \Delta_n \times 0$

$$d_{j+1}\sigma_j = d_{j+1}\sigma_{j+1}.$$

Now we prove the existence of i' as extension of $H \circ \Upsilon_{n-1}$. Since $(n+1)$-simplex σ_0 has all of its faces, except for one, in $\partial\Delta_n \times \Delta_1$ and K is a Kan complex then $H \circ \Upsilon_{n-1}|_{\sigma_0}$ can be extended to σ_0. Now σ_1 has all of its faces in $\partial\Delta_n \times \Delta_1$ except for two, one of which is shared with σ_0. Therefore $H \circ \Upsilon_n$ is defined on all of the faces σ_1 except for one face, again since K is a Kan complex, we can extend $H \circ \Upsilon_{n-1}$ to all of σ_1 and so on. By repeating this process we can extend $H \circ \Upsilon_{n-1}$ to all of $\Delta_n \times \Delta_1$. Finally the restriction of the newly extended H to $\Delta_n \times d^0\Delta_1$ takes its in the minimal subcomplex because by the induction hypothesis all of its faces $H|_{\partial\Delta_n \times d^0\Delta_1}$ are in the minimal complex. □

PROPOSITION 1.88. *Let M be a minimal complex and $f, g : M \to L$ two homotopic simplicial map. If f is an isomorphism then g is also.*

PROOF. Let $h = \{h_q\}_q$, be a homotopy between f and g, here $h_q : M_i \to L_{i+1}$, $0 \le q \le i$.
Proof of injectivity:
Suppose that $x, y \in M_0$ and $g(x) = g(y)$. For the two 1-simplices $h_0(x)$ and $h_0(y)$, We have $d_1h_0(x) = g(x) = g(y) = d_1h_0(y)$ and by minimality of L we have $f(x) = d_1h_0(x) = d_1h_0(y) = f(y)$.

Now suppose that we have proved the injectivity of g on M_i for $i \le q$. let x and y be two q-simplices with $g(x) = g(y)$. Since g is a simplicial map, we deduce that $g(d_ix) = g(d_iy)$, therefore by hypothesis (on injectivity) $d_ix = d_iy$. Using these identities we have

$$\begin{aligned} d_ih_q(x) &= h_{q-1}(d_ix) = h_{q-1}(d_iy) = d_ih_q(y), \quad \text{for } i < q \\ d_{q+1}h_q(x) &= g(x) = g(y) = d_{q+1}h_q(y) \end{aligned}$$

therefore by the minimality of L, we get $d_qh_q(x) = d_qh_q(y)$ which implies

$$d_qh_{q-1}(x) = d_qh_{q-1}(y) \tag{1.66}$$

because h being a simplicial homotopy satisfies the identity $d_qh_{q-1} = d_qh_q$. We repeat this process for $q-1$ instead of q, more precisely

$$\begin{aligned} &d_ih_{q-1}(x) = h_{q-2}(d_ix) = h_{q-2}(d_iy) = d_ih_{q-1}(y), \quad \text{for } i < q-1 \\ &d_qh_{q-1}(x) = d_qh_{q-1}(y) \quad \text{(by (1.66))} \end{aligned}$$

and then by minimality we get $d_{q-1}h_{q-1}(x) = d_{q-1}h_{q-1}(y)$, hence $d_{q-1}h_{q-2}(x) = d_{q-1}h_{q-2}(y)$. By repeating this process we can finally get to the identity

$$d_0h_0(x) = d_0h_0(y)$$

which is to say $f(x) = f(y)$, therefore $x = y$

Proof of injectivity:
If $x \in L_0$ then choose $z \in L_1$ such that $d_1 z = x$, and then choose $x \in M_0$ such that $f(x) = d_0 z$. We have

$$d_0 h_0(x) = f(x) = d_0 z,$$

so by minimality $d_1 h_0(x) = d_1 z$ which implies $g(x) = y$ hence the surjectivity of g on L_0. Now we complete the proof of the surjectivity by an induction: suppose that g is surjective on L_i for $i < q$ and $y \in L_q$. Choose x_i's such that $g(x_i) = d_i y$, $0 \leq i \leq q$.

Let $z_q \in L_{q+1}$ be an extension to for the sequence

$$h_{q-1}(x_0), \cdots h_{q-1}(x_{q-1}), -, y.$$

We choose $z_{i<q} \in L_{q+1}$ by descending recurrence relation as follows: $z_j \in L_{q+1}$ is the extension for sequence

$$h_{j-1}(x_0), \cdots h_{j-1}(x_{j-1}), -, d_{j+1}z_{j+1}, h_j(x_{j+2}), \cdots, h_j(x_q)$$

(in particular $d_{j+1}z_j = d_{j+1}z_{j+1}$). Now choose x such that $f(x) = d_0 z_0$, we have, for $i > 0$

(1.67)
$$f(d_i x) = d_i f(x) = d_i d_0 z_0 = d_0 d_{i+1} z_0 = d_0 h_0(x_0) = d_0 h_0(x_i) = f(x_i),$$

for $i = 0$,

$$f(d_0 x) = d_0 f(x) = d_0 d_0 z_0 = d_0 d_1 z_0 = d_0 d_1 z_1 = d_0 d_0 z_1 = d_0 h_0(x_0) = f(x_0)$$

and by injectivity $d_i x = x_i$ for all i.

We have

$$\begin{aligned} d_0 h_0(x) &= f(x) = d_0 z_0 \\ d_i h_0(x) &= h_0(d_{i-1}x) = h_0(x_{i-1}) = d_i z_0 \quad \text{for } i > 1 \end{aligned} \tag{1.68}$$

therefore by minimality $d_1 z_0 = d_1 h_0(x)$. We continue this process using the identity $d_{j+1}z_j = d_{j+1}z_{j+1}$, we can prove that $d_i z_j = d_i h_j(x)$ for all i and as a consequence

$$g(x) = d_{q+1} h_q(x) = d_{q+1} z_q = y,$$

proving g is surjective.

□

COROLLARY 1.89. *Suppose that M and L are minimal complexes. If $f : M \to L$ to is a homotopy equivalence then f is an isomorphism.*

PROOF. Suppose the g is the homotopical inverse of f then $f \circ g \simeq id_L$ and $g \circ f \simeq id_M$. Then by the previous result, Proposition 1.88, $f \circ g$ and $g \circ f$ are isomorphisms hence f and g are isomorphisms. □

1.17. Simplicial Postnikov system

The Postnikov system is a way of decomposition a simplicial K set by means of an inverse system of simplicial subsets $K^{(n)}$ whose first n-th homotopy groups are identical to those of K. A major tool in proving various theorems about the homotopy type of the simplicial sets.

Definition 1.90. For a simplicial set $K^{(n)}$ is a simplicial defined as the equivalence classes

$$K_q^{(n)} := K_q / \overset{n}{\sim},$$

where $x \overset{n}{\sim} y$ if $\bar{x}|_{\Delta_q[p]} = \bar{y}|_{\Delta_q[p]}$ for all $p \leq n$. In other words,

$$x \overset{n}{\sim} y \Longleftrightarrow \bar{x}|_{\mathrm{Sk}_n(\Delta_q)} = \bar{y}|_{\mathrm{Sk}_n(\Delta_q)}$$

Here $\bar{x} = \Upsilon_q(x) : \Delta_n \to K$ is the simplicial map provided by the Yoneda Lemma. It is clear that

$$K_q^{(n)} = K_q, \quad \text{for } n \geq q.$$

For our convenience in formulating he statements, we introduce

$$K^{\infty} := K.$$

When $n \geq m$ we have the natural projection maps $p_m^n : K^{(n)} \to K^{(m)}$ which are obviously simplicial maps. The reader may have noticed that this definition does not requires K to be a Kan complex.

Proposition 1.91. *Let K be a Kan complex.*

(1) For all n (∞ included), $K^{(n)}$ is a Kan complex.

(2) For all $n \geq m$ (∞ included), the simplicial map $p_m^n : K^{(n)} \to K^{(m)}$ is Kan fibration.

Proof. By virtue of Proposition 1.82, it is clear (2)for $n = \infty$ implies (1). As for (2), suppose that $x_1, \cdots, x_{k-1}, -, x_{k+1}, \cdots, x_{q+1} \in K_q^{(n)}$ is a compatible sequence and $y \in K_q^{(m)}$ such that $d_i y = p(x_i)$.

- If $q \leq m$ then $K_q^{(n)} = K_q^{(m)} = K_q$, therefore x_i are basically the element of K_q. Let $z \in K_{q+1}$ be a representative for y, then it is clear that $d_i z = x_i = p(x_i) \in K_q^{(n)} = K_q^{(m)} = K_q$ so the class represented by z in $K_q^{(n)}$ does the job.
- If $q > m$, there are two possibilities:
  - $n = \infty$: Since $K_q^{(n)} = K_q$ and K is a Kan complex, then there exist $x \in K_{q+1}$ such that $d_i x = x_i$. Let $z \in K_{q+1}$ be a representative for $y \in K_{q+1}^{(m)}$, then we claim that $x \overset{m}{\sim} z$, and this implies that $p(x) = y$ and we are done. As for the claim, first noticed that $p(d_i x) = p(x_i) = d_i y = p(d_i z)$ therefore for all i

$$d_i x \overset{m}{\sim} d_i z. \tag{1.69}$$

Since $m < q$, all the m-iterated faces of the q-simplices x and z are respectively are $m-1$-iterated faces of, respectively, $\{d_i x\}_i$ and $\{d_i z\}_i$. Let's spell the reasoning in more details: The relation (1.69) means that for all simplicial morphism $f : [m] \to [q]$ we $f^* d^{i*}(x) = f^* d^{i*}(z)$ i.e.

$$(d^i f)^*(x) = (d^i f)^*(z), \quad \text{for all } i \tag{1.70}$$

On the other hand $x \overset{m}{\sim} z$ is equivalent to $g^*(x) = g^*(z)$ for any simplicial morphism $g : [m] \to [q+1]$. Note that any such simplicial morphism has a unique decomposition (see Lemma 1.5) which by some d^i therefore it can be written of the form $g = d^i f$, now the claim follows from (1.70).

- $n < \infty$: Since we just prove the case $n = \infty$ so we can use the result that $K^{(n)}$'s are all Kan complxes. Now the proof of this case is exactly the same as case $n = \infty$ of $q < m$ since the only hypothesis that we used was $K^{(n)} = K$ being a Kan complex.

□

Definition 1.92. The n-th Eilenberg-McLane space $E_{n+1}(K)$ of a Kan complex K is the fiber of the fibration

$$p = p_n^\infty : K \to K^{(n)}, \tag{1.71}$$

i.e. we have a diagram

$$E_{n+1}(K) \hookrightarrow K \overset{p}{\to} K^{(n)}. \tag{1.72}$$

Once a basepoint $x_0 \in K$ is fixed, $E_{n+1}(K)$ consists of the simplices in K with faces of dimension less than n falling into the simplicial basepoint. So as a result

$$E_{n+1}(K)_{q \le n} = x_0.$$

and

$$\pi_{q \le n}(E_{n+1}(K)) = 0 \tag{1.73}$$

Proposition 1.93. *Let K be Kan complex, x_o basepoint for K and $m \le n$.*

(i) $p_* := (p_m^n)* : \pi_q(K^{(n)}) \to \pi_q(K^{(m)})$ *is an isomorphism for* $q \le m$.

(ii) $\pi_q(K^{(m)}) = 0$ *for all* $q > m$.

(iii) The map $\pi_q(E_{m+1}(K^{(n)})) \overset{\sim}{\to} \pi_q(K^{(n)})$ *induced by inclusion is an isomorphism for* $q > m$.

(iv) $\pi_q(E_{m+1}(K^{(n)})) = 0$ *if* $q > n$ *or* $q \le m$. *In particular* $E_n(K^{(n)})$ *is McLane-Eilenberg space.*

Proof. (i): it follows from the homotopy groups long exact sequence associated to the fibration

$$p_m^n : K^{(n)} \overset{p}{\to} K^{(m)}. \tag{1.74}$$

One can easily see that the fibre is $E_{m+1}(K^{(n)})$ and by (1.73) i.e. $\pi_{i\leq m}(E_{m+1}(K^{(n)})) = 0$ and (i) follows.

Proof of (ii): Let $[x] \in \pi_q(K^{(m)})$, where $x \in K_q$ is a representative. Then by definition $\bar{d_i x}|_{\mathrm{Sk}_m(\Delta_{q-1})} = \bar{x_0}|_{\mathrm{Sk}_m(\Delta_{q-1})}$. Because $m \leq q-1$ this implies that $\bar{x}|_{\mathrm{Sk}_m(\Delta_q)} = \bar{x_0}|_{\mathrm{Sk}_m(\Delta_q)}$. Said in more detail, every nondegenrate $f \in \mathrm{Sk}_m(\Delta_q)$ has a decomposition which starts with one codegenracy map d^i (see proof of (2) in Proposition 1.91 for a similar situation) .

(iii) is a consequence of (ii) by taking again into the fibration

$$E_{m+1}(K^{(n)}) \hookrightarrow\to K^{(n)} \xrightarrow{p} K^{(m)}.$$

Proof of (iv):The case $q \leq m$ has already been proved (1.73). The case $q < n$ follows from (ii) and (iii). $\square$

1.18. Hurewicz and Whitehead theorem in simplicial setting

In this section we prove a series of theorems on comparing homotopy and homology groups. All over this section $\sim$ denote the homotopy equivalence relation used for defining homotopy groups.

PROPOSITION 1.94. *Let K be a Kan complex. Then we have a group isomorphism*

$$H_0(K) = \mathbb{Z}\pi_0(K) \tag{1.75}$$

Here $\mathbb{Z}\pi_0(K)$ stands for the free abelian group generated by the elements of $\pi_0(K)$.

PROOF. The canonical projection map $K_0 \to K_0/\sim$ induces a map $p : C_0(K) \to \mathbb{Z}\pi_0(K)$. It is obvious that $B_0(K)$ is in the kernel of p because every 1-simplex σ, $d_0\sigma$ and $d_1\sigma$ are homotopic (by definition), therefore we have map $p : H_0(K) \to \mathbb{Z}\pi_0(K)$ which is obvious surjective. The injectivity follows from the substitution principal.

LEMMA 1.95. *(Substitution principal) Let F be a free abelian group with basis B and $\{x_i\}_{i=0\cdot k}$ in B be a list of elements in B (repetition allowed), and assume that*

$$\sum_{i=1}^{k} m_i x_i = 0$$

for some $m_i \in \mathbb{Z}$. If G is any abelian group and $\{y_i\}_{i=0\cdot k}$ a list in G with the property that $(x_i = x_j \Rightarrow y_i = y_j)$, the

$$\sum_{i=1}^{k} m_i y_i = 0$$

$\square$

Definition 1.96. The reduced simplicial homology $\tilde{H}_*(K)$ of a simplicial set K is the homology of the quotient complex

$$\tilde{C}_n(K) := C_n(K)/C_n(x_0)$$

where x_0 is a (simplicial) basepoint)

It follows from that long exact sequence associate to the short exact sequence

$$0 \to C_*(x_0) \to C_*(K) \to \tilde{C}_*(K) \to 0$$

that

$$\tilde{H}_{n>0}(K) \simeq H_n(K).$$

The Hurewicz map $h : \pi_n(K, x_0) \to \tilde{H}_n(K)$, is the identity on the generators. Indeed if x represents a class in $\pi_n(K, x_0)$ then by definition, $d_i x = x_0$ for all i therefore we have $\partial x = 0 \in \tilde{C}_*(K)$. Similarly one can define the relative Hurewicz map $h : \pi_n(K, L, x_0) \to H_n(K, L)$.

Proposition 1.97. *The Hurewicz maps are well-defined group homomorphisms.*

Proof. First we treat the non-relative case $h : \pi_n(K, x_0) \to \tilde{H}_n(K)$, the proof of the relative case is very much similar. Suppose that $w \in K_{n+1}$ is homotopy between x and $y \in \tilde{K}_n$. Then we have $\partial w = (-1)^n(y-x) \in \tilde{C}_n(K)$ and h is well-defined.

As for h being a homomorphism, suppose that $w \in Kn + 1$ is the simplex defining the product of $x, y \in \tilde{K}_n$ i.e. $d_{n-1}w = x$, $d_{n+1}w = y$ and $d_i w = x_0$. and by definition $[x] \cdot [y] = [d_n w]$.

We have that

$$0 = \partial w = (-1)^n(d_n w - (x + y) \in \tilde{C}_n(K) = (-1)^n(h([x \cdot y]) - h([x] - h([y])$$

hence,

$$h([x \cdot y]) = h([x]) + h([y])$$

□

The proof of the next result is standard enough to be left to the reader.

Proposition 1.98. *The Hurewiczs maps induce a map of long exact sequences.*

$$\begin{array}{ccccccc}
\cdots \pi_{n+1}(K,L) & \xrightarrow{d_0} & \pi_n(L) & \xrightarrow{inclu.} & \pi_n(K) & \longrightarrow & \pi_n(K,L)\cdots \\
\downarrow h & & \downarrow h & & \downarrow h & & \downarrow h \\
\cdots \tilde{H}_{n+1}(K,L) & \xrightarrow[\partial]{} & \tilde{H}_n(L) & \xrightarrow{inclu.} & \tilde{H}_n(K) & \longrightarrow & \tilde{H}_n(K,L)\cdots
\end{array} \tag{1.76}$$

Theorem 1.99. *Let K be a connected simplicial set, then h induced an isomorphism*

$$h : \pi_1(K)/[\pi_1(K), \pi_1(K)] \to \tilde{H}_1(K) \simeq H_1(K)$$

PROOF. We can assume that K is minimal because it does not change the homotopy type. So K has only one 0-simplex and $K_1 = \tilde{K}_1$. First of all since the image of h is an abelian group, h induces a well-defined on the quotient i.e

$$h : \pi_1(K)/[\pi_1(K), \pi_1(K)] \to H_1(K)$$

We construct an inverse $\tilde{j} : H_1(K) \to \pi_1(K)/[\pi_1(K), \pi_1(K)]$ for h. Indeed the inverser is given by

$$j : \tilde{Z}_1(K)) \to \pi_1(K)/[\pi_1(K), \pi_1(K)]$$

which is induced by the natural projection map $K_1 \to K_1/\sim$ on the generators and then extended linearly to all of the group. We only have to show that $j(\mathrm{Im}(\partial) \subset [\pi_1, \pi_1]$. So let $\sigma \in K_2$, hen by definition of the product on π_1 we

$$[d_0\sigma] \cdot [d_2\sigma] = [d_1\sigma]$$

and

(1.77)

$$j(\partial\sigma) = j(d_0\sigma)j(-d_1\sigma)j(d_2\sigma) = [d_0\sigma][d_1\sigma]^{-1}[d_2\sigma] = [d_0\sigma]([d_0\sigma][d_2\sigma])^{-1}[d_2\sigma]$$
$$= [[d_0\sigma], [d_2\sigma]^{-1}] \in [\pi_1(K), \pi_1(K)]$$

It is quite clear that $\tilde{j} \circ h = id$ and $h \circ \tilde{j} = id$ □

DEFINITION 1.100. A Kan complex is called n-connected if $\pi_{i\leq n}(K) = 0/$

THEOREM 1.101. *Let K be a $(n-1)$-connected Kan complex then $H_{i<n}(K) = 0$ and*

$$h : \pi_n(K) \to H_n(K)$$

is an isomorphism.

PROOF. IF necessary we can replace K the minimal subcomplex of K because they have the same the same homotopy and homology groups. So assume that K is a minimal complex, therefor it has only one i-simplex for all $i < n$, and clearly $H_i(K) = 0$ for $0 < i < n$.

Another consequence is

$$\tilde{C}_n(K) = \tilde{Z}_n(K) = \mathbb{Z}(K_n \setminus \{x_0\})$$

is the free abelian group generated by the all n-simplices except for the one falling in the basepoint. Similarly to the proof of Theorem 1.101, w construct an inverse $\tilde{j}$ for h which is induced by the natural projection

$$j : K_n \to K_n/\sim$$

for the generator of $\tilde{C}_n(K) = \tilde{Z}_n(K)$ and then extended linearly. In other to prove that j induces a well-defined map $\tilde{j}$ on the reduced homology, we have to prove that the elements in the image $j \circ \partial(\tilde{C}_{n+1})$ are homotopic to x_0.

The proof relies on the following lemmata which basically give alternatives definitions of the groups law on π_n.

LEMMA 1.102. *Let v_{n+1} be a $(n+1)$-simplex such that*

$$d_i v_{n+1} = x_0, \text{ for } i = n+1 \text{ and } i < n-2.$$

Then $[d_n v_{n+1}][d_{n-2}v_{n+1}] = [d_{n-1}v_{n+1}]$ in $\pi_n(K)$

PROOF. Let $x := d_{n-1}v_n$, $y = d_n v_n$, $w = d_{n-2}v_n$ Let v_{n-1} be the $(n+1)$-simplex extending the compatible sequence $x_0 \cdots, x_0, -, x, w$, tha denote $t := d_{n-1}v_{n-1}$, hence

$$[t].[w] = [x] \tag{1.78}$$

Let r be $(n+2)$-simplex extending

$$x_0, \cdots, x_0, s_n(w), v_{n-1}, -, v_{n+1}, s_{n-2}(w)$$

and let $v_n = d_n r$. We have $d_{i \leq n-2}v_n = x_0$, $d_{n-1}v_n = t$, $d_n v_n = y$, or in other words

$$[t] = [y],$$

and using we get (1.78) $[y].[w] = [x]$ as desired. □

LEMMA 1.103. *Let v_n be a $(n+1)$-simplices such that such that $d_i v_n = x_0$ for $i = n-1$ and $i < n-2$. Then $[d_{n-1}v_n][d_n v_n] = [d_{n+1}v_n]$ in $\pi_n(K)$.*

PROOF. We put $w = d_{n-1}v_n$, $y = d_n v_n$, $z = d_{n+1}v_n$. Choose an extension v_{n-1} for the sequence $x_0, \cdots, x_0, w, x_0, -, x_0$ and let $t = d_{n-1}v_{n-1}$. By previous lemma we have

$$[t][w] = 1 = [x_0].$$

Let $(n+2)$-simplex r be the extension of $x_0 \cdots, x_0, s_{n-2}(w), v_{n-1}, v_n, -, s_n(z)$ Then $(n+1)$-simplex $v_{n+1} := d_{n+1}r$ implies the identity

$$[t][z] = [y].$$

Putting the two obtained identity together we conclude that $[w][y] = [z]$. □

Completing the proof of Theorem 1.101: Here we give detailed proof for $n = 2$, the proof higher dimension is similar and left to the reader. Below we write everything in terms n to give a clue for higher dimension, but at the end threader should pout $n = 2$.

So let v_{n+2} be a $(n+1)$-simplex. Let $w := d_{n-2}v_{n+2}$, $x := d_{n-1}v_{n+2}$, $y := d_n v_{n+2}$ and $z := d_{n+1}v_{n+2}$. We intend to prove that $[w]^{-1}[x][y]^{-1}[z]$ is homotopic to the degenerate 2-simplex x_0.

Let $v_{n-2} \in K_3$ be an extension of $x_0 \cdots, x_0, -, x_0, x_0, w$ and $t = d_{n-2}v_{n-2}$, then by the previous lemma

$$[t] = [w].$$

Let $v_{n-1} \in K_{n+1}$ be an extension of $x_0, \cdots, x_0, t, x_0, -, x$ and $u := d_n v_{n-1}$ then

$$[t][u] = [x].$$

Finally let r be an extension of $x_0, \cdots, x_0, v_{n-2}, v_{n-1}, s_n(y), -, v_{n+2}$. Then $d_{n+1}r$ defines gives rise to the identity

$$[u][z] = [y].$$

Now putting the three obtained identities above we get

$$[w][x]^{-1}[y][z]^{-1} = [x_0] \in \pi_n(K),$$

which is equivalent to

$$j(\partial v_{n+2}) = [x_0] = 1 \in \pi_n(K).$$

Now that $\tilde{j}$ is well-defined, verifying $\tilde{j} \circ h = id$ and $h \circ \tilde{j} = id$ is quite easy. □

Corollary 1.104. *If K is a 1-connected Kan complex and $H_{i>0}(K) = 0$ then K is contractible.*

Proof. By applying Theorem 1.101 we have $\pi_i(M) = 1$ for all i. Let K' be a minimal complex for K. Since K' is a deformation retract of K then, we have $\pi_(K') = 1$ for all. This means that K' has only one simplex in each dimension, which has to be the degenerate one or in other words $K' = x_0$ and K is a deformation retract of a (simplicial) point hence contractible.

□

The relative version of the Hurewicz theorem can be proved in a similar manner.

Theorem 1.105. *Let $L \subset K$ be a pair of Kan complexes. If $\pi_{i \leq n-1}(K, L) = 1$ then $H_{i \leq n-1}(K, L) = 0$ and $h : \pi_n(K, L) \to H_n(K, L)$ is an isomorphism.*

Theorem 1.106. *Let $f : K \to L$ be an inclusion of 1-connected simplicial spaces and $n > 1$. The followings statement are equivalente.*

(1) $f_* : \pi_i(K) \to \pi_i(L)$ is an isomorphism for $i < n$ and epimoprhism for $i = n$
(2) $f_* : H_i(K) \to H_i(L)$ is an isomorphism for $i < n$ and epimoprhism for $i = n$

Proof. $(1 \Rightarrow 2)$: Using the exact sequence (1.73), we see immediately that $\pi_i(K, L) = 1$ for $0 \leq i \leq n$, therefore by the relative version of Hurewicz theorem, $H_i(K, L) = 0$ for $0 \leq i \leq n$. Thus the natural map $H_*(L) \to H_*(K)$ surjective because of the long exact sequence involving the homologies of K and L and the relative homology $H_*(K, L)$.

$(2 \Rightarrow 1)$: Similarly we have $H_i(K, L) = 0$ for $0 \leq i \leq n$. Since By the relative Hurewicz theorem the fist nontrivial relative homotopy group is isomorphic to the relative homology group. Since we assume that K and L are 1-connected, we can initiate applying the relative Hurewicz theorem inductively: $\pi_1(K, L) = 1$, therefore $\pi_2(K, L) \simeq H_2(K, L) = 0$ and so on.

□

1.19. Geometric realisation of Kan complexes

PROPOSITION 1.107. *Let K be a connected Kan complex with a basepont x_0. Then maps induced by the adjuntion isomorphisl Ψ,*

$$\Psi_*(K) : \pi_1(K, x_0) \to \pi_1(ST(K), ST(x_0))$$

is an isomorphism.

PROOF. As usual, we can assume that K is minimal. Since K has a single 0-simplex, the fundamental group $\pi_1(K, x_0)$ has a simple description. It has one generator for each nondegenerate 1-simplex and on relation for each nondegenerate 2-simplex. Thanks to the Van-Kampen theorem, we have exactly the same description for the fundamental group $\pi_1(ST(K), ST(x_0))$ of CW-complex $TS(X)$ where 1-cells (generators) and 2-cells (relations) are in one-to-one correspondance with nondegenerate 1 and 2-simplices.

□

THEOREM 1.108. *Let K be be a connected Kan complex and X a connected topological space.*

1. $\Psi_*(K) : \pi_n(K, x_0) \to \pi_n(ST(K), TS(x_0))$ *is an isomorphism for all* $n \geq 1$.
2. $\Phi_* : \pi_n(TS(X), TS(a)) \to \pi_n(X, a)$ *is an isomorphism.*

Here the maps Ψ and Φ are the bijection of the adjunction between the functor S and T in Section 1.8.

PROOF. (1): The case $n = 1$ is the previous theorem therefore we assume that $n \geq 2$. We may assume that K is minimal. Otherwise we replace K by one of its minimal subcomplex which is a deformation retract of K. Therefore it has only one 0-simplex and we can use the construction of the universal cover $\tilde{K}$ of K and we have that

for $n \geq 2$. Let F be fibre of the natural projection $p : \tilde{K} \to K$ which is Kan fibration. By Theorem 1.42, the (inclusion) unit map $\Psi : \tilde{K} \to ST(K)$ of the adjunction, induces an isomorphism in homology groups, therefore by Theorem 1.106 Ψ induces an isomorphism in homotopy groups

$$\pi_n(\tilde{K}, x_0) \simeq \pi_n(ST(\tilde{K}, ST(x_0))$$

Clearly we have an isomorphism $\pi_n(F, \bar{x}_0) \simeq \pi_n(ST(F,)ST(\bar{x})_0))$. Now using the naturality of homotopy group long exact sequence associated to the Kan fibrations, in our case $F \to \tilde{K} \to K$ and $ST(F) \to ST(\tilde{K}) \to ST(K)$, we conclude that

Proof of (2): By (1) we have an isomorphism

$$\pi_n(S(X)) \overset{\Psi(S(X))_*}{\simeq} \pi_n(ST(S(X)).$$

Since $S\Phi \circ \Psi S = id$ we conclude that $(S\Phi)_*(X) : \pi_n(STS(X) \to \pi_n(S(X))$ is an isomorphism. By Theorem 1.77 we have isomorphisms

$$\pi_n(ST(S(X)) \overset{\phi_{\Delta_n,(TS(X))}}{\simeq} \pi_n(TS(X))$$

and

$$\pi_n(S(X)) \overset{\phi_{\Delta_n,(X)}}{\simeq} \pi_n(X).$$

Now the result follows. □

CHAPTER 2

An introduction to model categories and derived functors

The classical references for this subject are Hovey's book [**Hov99**] and the Dwyer-Spaliński manuscript [**DS95**]. The reader who gets to know the notion of model category for the first time should not worry about the word "closed", which now has only a historical bearing. From now on we drop the word "closed" from "closed model category".

Definition 2.1. A *model category* is a category $\mathbf{C}$ endowed with three classes of morphisms $\mathcal{C}$ (*cofibrations*), $\mathcal{F}$ (*fibrations*) and $\mathcal{W}$ (*weak equivalences*) such that the following conditions hold:

(MC1) $\mathbf{C}$ is closed under finite limits and colimits.

(MC2) Let $f, g \in Mor(\mathbf{C})$ such that fg is defined. If any two among f, g and fg are in $\mathcal{W}$, then the third one is in $\mathcal{W}$.

(MC3) Let f be a retract of g, meaning that there is a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ B & \longrightarrow & D & \longrightarrow & B \end{array}$$

in which the two horizontal compositions are identities. If $g \in \mathcal{C}$ (resp. $\mathcal{F}$ or $\mathcal{W}$), then $f \in \mathcal{C}$ (resp. $\mathcal{F}$ or $\mathcal{W}$).

(MC4) For a commutative diagram as below with $i \in \mathcal{C}$ and $p \in \mathcal{F}$, the morphism f making the diagram commutative exists if

(1) $i \in \mathcal{W}$ (left lifting property (LLP) of fibrations $f \in \mathcal{F}$ with respect to acyclic cofibrations $i \in \mathcal{W} \cap \mathcal{C}$).

(2) $p \in \mathcal{W}$ (right lifting property (RLP) of cofibrations $i \in \mathcal{C}$ with respect to acyclic fibrations $p \in \mathcal{W} \cap \mathcal{F}$).

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array} \tag{2.1}$$

In the above we call the elements of $\mathcal{W} \cap \mathcal{C}$ (resp. $\mathcal{W} \cap \mathcal{F}$) *acyclic cofibrations* (resp. *acyclic fibrations*).

(MC5) Any morphism $f : A \to B$ can be written as one of the following:

(1) $f = pi$ where $p \in \mathcal{F}$ and $i \in \mathcal{C} \cap \mathcal{W}$;
(2) $f = pi$ where $p \in \mathcal{F} \cap \mathcal{W}$ and $i \in \mathcal{C}$.

In fact, in a model category the lifting properties characterize the fibrations and cofibrations.

PROPOSITION 2.2. *In a model category:*

(i) The cofibrations are the morphisms which have the RLP with respect to acyclic fibrations.
(ii) The acyclic cofibrations are the morphisms which have the RLP with respect to fibrations.
(iii) The fibrations are the morphisms which have the LLP with respect to acyclic cofibrations.
(iv) The acyclic fibrations in C are the maps which have the LLP with respect to cofibrations.

It follows from (MC1) that a model category $\mathbf{C}$ has an initial object $\emptyset$ and a terminal object $*$. An object $A \in Obj(\mathbf{C})$ is called *cofibrant* if the morphism $\emptyset \to A$ is a cofibration and is said to be *fibrant* if the morphism $A \to *$ is a fibration.

Example 1: For any unital associative ring R, let $\mathrm{Ch}(R)$ be the category of non-negatively graded chain complexes of left R-modules. The following three classes of morphisms endow $\mathrm{Ch}(R)$ with a model category structure:

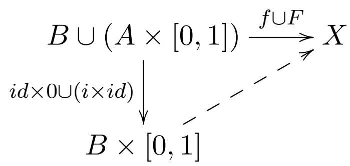
(1) Weak equivalences $\mathcal{W}$: these are the quasi-isomorphims, i.e. maps of R-complexes $f = \{f_k\}_{k\geq 0} : \{M_k\}_{k\in\mathbb{Z}} \to \{N_k\}_{k\geq 0}$ inducing an isomorphism $f_* : H_*(M) \to H_*(N)$ in homology.
(2) Fibrations $\mathcal{F}$: f is a fibration if it is (componentwise) surjective, i.e. for all $k \geq 0$, $f_k : M_k \to N_k$ is surjective.
(3) Cofibrations $\mathcal{C}$: $f = \{f_k\}$ is a cofibration if for all $k \geq 0$, $f_k : M_k \to N_k$ is injective with a projective R-module as its cokernel. Here we use the standard definition of projective R-modules, i.e. modules which are *direct summands of free R-modules.*

Example 2: The category **Top** of topological spaces can be given the structure of a model category by defining a map $f : X \to Y$ to be

(i) a weak equivalence if f is a homotopy equivalence;
(ii) a cofibration if f is a Hurewicz cofibration;
(iii) a fibration if f is a Hurewicz fibration.

Let A be a closed subspace of a topological space B. We say that the inclusion $i : A \hookrightarrow B$ is a *Hurewicz cofibration* if it has the homotopy extension property that is for all maps $f : B \to X$, any homotopy F :

$A \times [0,1] \to X$ of $f|_A$ can be extended to a homotopy of $f : B \to X$.



A *Hurewicz fibration* is a continuous map $E \to B$ which has the homotopy lifting property with respect to all continuous maps $X \to B$, where $X \in \mathbf{Top}$.

Example 3: The category **Top** of topological spaces can be given the structure of a model category by defining $f : X \to Y$ to be

(i) a weak equivalence when it is a weak homotopy equivalence.
(ii) a cofibration if it is a retract of a map $X \to Y'$ in which Y' is obtained from X by attaching cells,
(iii) a fibration if it is a Serre fibration.

We recall that a *Serre fibration* is a continuous map $E \to B$ which has the homotopy lifting property with respect to all continuous maps $X \to B$ where X is a CW-complex (or, equivalently, a cube).

Cylinder, path objects and homotopy relation. After setting up the general framework, we define the notion of homotopy. A *cylinder object* for $A \in Obj(\mathbf{C})$ is an object $A \wedge I \in Obj(\mathbf{C})$ with a *weak equivalence* $\sim: A \wedge I \to A$ which factors the natural map $id_A \sqcup id_A : A \coprod A \to A$:

$$id_A \sqcup id_A : A \coprod A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$$

Here $A \coprod A \in Obj(\mathbf{C})$ is the colimit, for which one has two structural maps $in_0, in_1 : A \to A \coprod A$. Let $i_0 = i \circ in_0$ and $i_1 = i \circ in_1$. A cylinder object $A \wedge I$ is said to be *good* if $A \coprod A \to A \wedge I$ is a cofibration. By (MC5), every $A \in Obj(\mathbf{C})$ has a good cylinder object.

Definition 2.3. Two maps $f, g : A \to B$ are said to be *left homotopic* $f \overset{l}{\sim} g$ if there is a cylinder object $A \wedge I$ and $H : A \wedge I \to B$ such that $f = H \circ i_0$ and $g = H \circ i_1$. A left homotopy is said to be *good* if the cylinder object $A \wedge I$ is good. It turns out that every left homotopy relation can be realized by a good cylinder object. In addition one can prove that if B is a fibrant object, then a left homotopy for f and g can be refined into a *very good one*, i.e. $A \wedge I \to A$ is a fibration.

It is easy to prove the following:

Lemma 2.4. *If A is cofibrant, then left homotopy $\overset{l}{\sim}$ is an equivalence relation on* $\mathrm{Hom}_{\mathbf{C}}(A, B)$.

Similary, we introduce the notion of path objects which will allow us to define right homotopy relation. A *path object* for $A \in Obj(\mathbf{C})$ is an object $A^I \in Obj(\mathbf{C})$ with a weak equivalence $A \overset{\sim}{\to} A^I$ and a morphism $p : A^I \to A \times A$ which factors the diagonal map

$$(id_A, id_A) : A \overset{\sim}{\to} A^I \overset{p}{\to} A \times A.$$

Let $pr_0, pr_1 : A \times A \to A$ be the structural projections. Define $p_i = pr_i \circ p$. A path object A^I is said to be *good* if $A^I \to A \times A$ is a fibration. By (MC5) every $A \in Obj(\mathbf{C})$ has a good path object.

DEFINITION 2.5. Two maps $f, g : A \to B$ are said to be *right homotopic* $f \overset{r}{\sim} g$ if there is a path object B^I and $H : A \to B^I$ such that $f = p_0 \circ H$ and $g = p_1 \circ H$. A right homotopy is said to be *good* if the path object P^I is good. It turns out that every right homotopy relation can be refined into a good one. In addition one can prove that if B is a cofibrant object then a right homotopy for f and g can be refined into a *very good one*, i.e. $B \to B^I$ is a cofibration.

LEMMA 2.6. *If B is fibrant, then right homotopy $\overset{r}{\sim}$ is an equivalence relation on* $\mathrm{Hom}_{\mathbf{C}}(A, B)$.

One naturally asks whether being right and left homotopic are related. The following result answers this question.

LEMMA 2.7. *Let $f, g : A \to B$ be two morphisms in a model category* $\mathbf{C}$.

(1) *If A is cofibrant then $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$.*
(2) *If B is fibrant then $f \overset{r}{\sim} g$ implies $f \overset{l}{\sim} g$.*

Cofibrant and Fibrant replacement and homotopy category. By applying (MC5) to the canonical morphism $\emptyset \to A$, there is a cofibrant object (not unique) QA and an *acyclic fibration* $p : QA \overset{\sim}{\to} A$ such that $\emptyset \to QA \overset{p}{\to} A$. If A is cofibrant we can choose $QA = A$.

LEMMA 2.8. *Given a morphism $f : A \to B$ in $\mathbf{C}$, there is a morphism $\tilde{f} : QA \to QB$ such that the following diagram commutes:*

$$\begin{array}{ccc} QA & \xrightarrow{\tilde{f}} & QA \\ \downarrow{\scriptstyle p_A} & & \downarrow{\scriptstyle p_B} \\ A & \xrightarrow{f} & B \end{array} \tag{2.2}$$

The morphism $\tilde{f}$ depends on f up to left and right homotopy, and is a weak equivalence if and only f is. Moreover, if B is fibrant then the right or left homotopy class of $\tilde{f}$ depends only on the left homotopy class of f.

Similarly one can introduce a fibrant replacement by applying (MC5) to the terminal morphism $A \to *$ and obtain a fibrant object RA with an *acyclic cofibration* $i_A : A \to RA$.

LEMMA 2.9. *Given a morphism $f : A \to B$ in $\mathbf{C}$, there is a morphism $\tilde{f} : RA \to RB$ such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle i_A} & & \downarrow{\scriptstyle i_B} \\ RA & \xrightarrow{\tilde{f}} & RB \end{array} \tag{2.3}$$

The morphism $\tilde{f}$ depends on f up to left and right homotopy, and is a weak equivalence if and only f is. Moreover, if A is cofibrant then right or left homotopy class of $\tilde{f}$ depends only on the right homotopy class of f.

REMARK 2.10. For a cofibrant object A, RA is also cofibrant because the trivial morphism $(\emptyset \to RA) = (\emptyset \to A \stackrel{i_A}{\to} RA)$ can be written as the composition of two cofibrations, therefore is a cofibration. In particular, for any object A, RQA is fibrant and cofibrant. Similarly, QRA is a fibrant and cofibrant object.

LEMMA 2.11. *Suppose that $f : A \to X$ is a map in $\mathbf{C}$ between objects A and X which are both fibrant and cofibrant. Then f is a weak equivalence if and only if f has a homotopy inverse, i.e. if and only if there exists a map $g : X \to A$ such that the composites gf and fg are homotopic to the respective identity maps.*

Putting the last three lemmas together, one can make the following definition:

DEFINITION 2.12. The *homotopy category* $\mathrm{Ho}(\mathbf{C})$ of a model category $\mathbf{C}$ has the same objects as $\mathbf{C}$ and the morphism set $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(A, B)$ consists of the (right or left) homotopy classes of the morphisms in $\mathrm{Hom}_{\mathbf{C}}(RQA, RQB)$. Note that since RQA and RQB are fibrant and cofibrant, the left and right homotopy relations are the same. There is a natural functor $H_{\mathbf{C}} : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ which is the identity on the objects and sends a morphism $f : A \to B$ to the homotopy class of the morphism obtained in $\mathrm{Hom}_{\mathbf{C}}(RQA, RQB)$ by applying consecutively Lemma 2.8 and Lemma 2.9.

Localization functor. Here we give a brief conceptual description of the homotopy category of a model category. This description relies only on the class of weak equivalences and suggests that weak equivalences encode most of the homotopic properties of the category. Let W be a subset of the morphisms in a category $\mathbf{C}$. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be a *localization of* $\mathbf{C}$ *with respect to* W if the elements of W are sent to isomorphisms and if F is universal for this property, i.e. if $G : \mathbf{C} \to \mathbf{D}'$ is any another localizing functor then G factors through F via a functor $G' : \mathbf{D} \to \mathbf{D}'$ for which $G'F = G$. It follows from Lemma 2.11 and a little work that:

THEOREM 2.13. *For a model category* $\mathbf{C}$, *the natural functor* $H_{\mathbf{C}} : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ *is a localization of* $\mathbf{C}$ *with respect to the weak equivalences.*

Derived and total derived functors. In this section we introduce the notions of *left derived functor* LF and *right derived functor* RF of a functor $F : \mathbf{C} \to \Delta$ on a model category $\mathbf{C}$. In particular, we spell out sufficient conditions for the existence of LF and RF which provide us a factorization of F via the homotopy categories. If Δ happens to be a model category, then we also introduce the notion of *total derived functor* and provide some sufficient conditions for its existence.

All functors considered here are covariant, however see Remark 2.17.

DEFINITION 2.14. For a functor $F : \mathbf{C} \to \Delta$ on a model category $\mathbf{C}$, we consider all pairs (G, s) where $G : \mathrm{Ho}(\mathbf{C}) \to \Delta$ is a functor and $s : GH_{\mathbf{C}} \to F$ is a natural transformation. The *left derived functor* of F is such a pair (LF, t) which is universal from the left, i.e. for any other such pair (G, s) there is a unique natural transformation $t' : G \to LF$ such that $t(t'H_{\mathbf{C}}) : GH_{\mathbf{C}} \to F$ is s.

Similarly one can define the right derived functor $RF : \mathrm{Ho}(\mathbf{C}) \to \Delta$ which provides a factorization of F and satisfies the usual universal property from the right. A *right derived functor* of F is a pair (RF, t) where $RF : \mathrm{Ho}(\mathbf{C}) \to \Delta$ and t is a natural transformation $t : F \to RFH_{\mathbf{C}}$ such that for any pair (G, s) there is a unique natural transformation $t' : RF \to G$ such that $(t'H_{\mathbf{C}})t : F \to GH_{\mathbf{C}}$ is s.

The reader can easily check that the derived functors of F are unique up to canonical equivalence. The following result tells us when do derived functors exist.

PROPOSITION 2.15.
(1) *Suppose that $F : \mathbf{C} \to \Delta$ is a functor from a model category $\mathbf{C}$ to a category Δ, which transforms acyclic cofibrations between cofibrant objects into isomorphims. Then (LF, t), the left derived functor of F, exists. Moreover, for any cofibrant object X the map $t_X : LF(X) \to F(X)$ is an isomorphism.*
(2) *Suppose that $F : \mathbf{C} \to \Delta$ is a functor from a model category $\mathbf{C}$ to a category Δ, which transforms acyclic fibrations between fibrant objects into isomorphisms. Then (RF, t), the right derived functor of F, exists. Moreover, for all fibrant object X the map $t_X : RF(X) \to F(X)$ is an isomorphism.*

DEFINITION 2.16. Let $F : \mathbf{C} \to \Delta$ be a functor between two model categories. The *total left derived functor* $\mathbb{L}F : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\Delta)$ is the left derived functor of $H_\Delta F : \mathbf{C} \to \mathrm{Ho}(\Delta)$. Similarly one defines the *total right derived functor* $\mathbb{R}F : \mathbf{C} \to \Delta$ to be the right derived functor of $H_\Delta F : \mathbf{C} \to \mathrm{Ho}(\Delta)$.

REMARK 2.17. Till now we have defined and discussed the derived functor for covariant functors. We can define the derived functors for contravariant functors as well, for that we only have to work with the opposite category of the source of the functor. A morphism $A \to B$ in the opposite category is a cofibration (resp. fibration, weak equivalence) if and only if the corresponding morphism $B \to A$ is a fibration (resp. cofibration, weak equivalence).

We finish this section with an example.

Example 4: Consider the model category $\mathrm{Ch}(R)$ of Example 1 in Section **??** and let M be a fixed R-module. One defines the functor $F_M : \mathrm{Ch}(R) \to \mathrm{Ch}(\mathbb{Z})$ given by $F_M(N_*) = M \otimes_R N_*$ where $N_* \in \mathrm{Ch}(R)$ is a complex of R-modules. Let us check that $F = H_{\mathrm{Ch}(R)} F_M : \mathrm{Ch}(R) \to \mathrm{Ch}(\mathbb{Z})$ satisfies the conditions of Proposition 2.15.

Note that in $\mathrm{Ch}(R)$ every object is fibrant and a complex A_* is cofibrant if for all k, A_k is a projective R-module. We have to show that an acyclic cofibration $f : A_* \to B_*$ between cofibrant objects A and B is sent by F to an isomorphism. So for all k, we have a short exact sequence $0 \to A_* \to B_* \to B_*/A_* \to 0$ where for all k, B_k/A_k is also projective. Since f is a quasi-isomorphism the homology long exact sequence of this short exact sequence tells us that the complex B_*/A_* is acyclic. The lemma below shows that B_*/A_* is in fact a projective complex. Therefore we have $B_* \simeq A_* \oplus B_*/A_*$. So $F_M(B_*) \simeq F_M(A_*) \oplus F_M(B_*/A_*) \simeq F_M(A_*) \oplus \bigoplus_n F_M(D(Z_{n-1}(B_*/A_*), n))$. Here $Z_*(X_*) := \ker(d : X_* \to X_{*+1})$ stands for the graded module of the cycles in a given complex X_*, and the complex $D(X, n)_*$ is defined as follows: To any R-module X and a positive integer n, one can associate a complex $\{D(X, n)_k\}_{k \geq 0}$,

$$D(X, n)_k = \begin{cases} 0, \text{ if } k \neq n, n-1, \\ X, \text{ if } k = n, n-1, \end{cases}$$

where the only nontrivial differential is the identity map.

It is a direct check that each $F_M(D(Z_{n-1}(B_*/A_*), n))$ is acyclic, and therefore $H_{\mathrm{Ch}(\mathbb{Z})}(F_M(B))$ is isomorphic to $H_{\mathrm{Ch}(\mathbb{Z})}(F_M(A))$ in the homotopy category $\mathrm{Ho}(\mathrm{Ch}(\mathbb{Z}))$.

LEMMA 2.18. *Let $\{C_k\}_{k \geq 0}$ be an acyclic complex where each C_k is a projective R-module. Then $\{C_k\}_{k \geq 0}$ is a projective complex, i.e. any level-wise surjective chain complex map $D_* \to E_*$ can be lifted via any chain complex map $C_* \to E_*$.*

PROOF. It is easy to check that if X is a projective R-module then $D_n(X)$ is a projective complex. Let $C_*^{(m)}$ be the complex

$$C_k^{(m)} = \begin{cases} C_k, \text{ if } k \geq m, \\ Z_k(C), \text{ if } k = m-1, \\ 0 \text{ otherwise.} \end{cases}$$

Here $Z_k(C)$ denotes the space of cycles in C_k, and $B_k(C)$ is the space of boundary elements in C_k. The acyclicity condition implies that we have an isomorphism $C_*^{(m)}/C_*^{(m+1)} \simeq D(Z_{m-1}(C), m)$. Note that $Z_0(C) = C_0$ is a projective R-module and $C_* = C^{(1)} = C^{(2)} \oplus D_1(Z_0(C))$. Now $D_1(Z_0(C))$

is a projective complex and $C^{(2)}$ also satisfies the assumption of the lemma and vanishes in degree zero. Therefore by applying the same argument one sees that $C^{(2)} = C^{(3)} \oplus D(Z_1(C), 2)$. Continuing this process one obtains $C_* = D(Z_0(C), 1) \oplus D(Z_1(C), 2) \cdots \oplus D(Z_{k-1}, k) \oplus \cdots$ where each factor is a projective complex, thus proving the statement. □

We finish this example by computing the left derived functor. For any R-module N let $K(N, 0)$ be the chain complex concentrated in degree zero where there is a copy of N. Since every object is fibrant, a fibrant-cofibrant replacement of $K(N, 0)$ is simply a cofibrant replacement. A cofibrant replacement P_* of $K(N, 0)$ is exactly a projective resolution (in the usual sense) of N in the category of R-modules. In the homotopy category of $\mathrm{Ch}(R)$, $K(N, 0)$ and P are isomorphic because by definition $\mathrm{Hom}_{Ho(\mathrm{Ch}(R))}(K(N, 0), P)$ consists of the homotopy classes of

$$\mathrm{Hom}_{\mathrm{Ch}(R)}(RQK(N, 0), RQP_*) = \mathrm{Hom}_{\mathrm{Ch}(R)}(P_*, P_*)$$

which contains the identity map. Therefore by Proposition 2.15

$$\mathbb{L}F(K(N, 0)) \simeq \mathbb{L}F(P_*)$$

and $\mathbb{L}F(P_*)$ and the definition of total derived functor is isomorphic to $H_{\mathrm{Ch}(R)}F(P_*) = M \otimes_R P_*$. In particular,

$$H_*(\mathbb{L}F(K(N, 0)) = \mathrm{Tor}_*^R(N, M),$$

where Tor_*^R is the usual Tor_R in homological algebra. We usually denote the derived functor $\mathbb{L}F(N) = N \otimes_R^L M$. Similarly one can prove that the contravariant functor $N_* \mapsto \mathrm{Hom}_R(N_*, M)$ has a total right derived functor, denoted by $\mathrm{RHom}_R(N_*, M)$, and

$$H^*(\mathrm{RHom}_R(K(N, 0), M)) \simeq \mathrm{Ext}_R^*(N, M)$$

is just the usual Ext functor (see Remark 2.17).

2.0.1. Hinich's theorem and Derived category of DG modules. The purpose of this section is to introduce a model category and derived functors of DG-modules over a fixed differential graded $\mathbf{k}$-algebra. From now on we assume that $\mathbf{k}$ is a field. The main result is essentially due to Hinich [**Hin97**], who introduced a model category structure for algebras over a vast class of operads.

Let $C(\mathbf{k})$ be the category of (unbounded) complexes over $\mathbf{k}$. For $d \in \mathbb{Z}$ let $M_d \in C(\mathbf{k})$ be the complex

$$\cdots \to 0 \to \mathbf{k} = \mathbf{k} \to 0 \to 0 \cdots$$

concentrated in degrees d and $d + 1$.

THEOREM 2.19. *(Hinich) Let* $\mathbf{C}$ *be a category which admits finite limits and arbitrary colimits and is endowed with two right and left adjoint functors* $(\#, F)$

$$\#: C \rightleftarrows C(\mathbf{k}) : F \tag{2.4}$$

such that for all $A \in Obj(A)$ *the canonical map* $A \to A \coprod F(M_d)$ *induces a quasi-isomorphism* $A^{\#} \to (A \coprod F(M_d))^{\#}$*. Then there is a model category structure on* $\mathbf{C}$ *where the three distinct classes of morphisms are:*

1. *Weak equivalences* $\mathcal{W}$*:* $f \in Mor(\mathbf{C})$ *is in* $\mathcal{W}$ *if* $f^{\#}$ *is a quasi-isomorphism.*
2. *Fibrations* $\mathcal{F}$*:* $f \in Mor(\mathbf{C})$ *is in* $\mathcal{F}$ *if* $f^{\#}$ *is (componentwise) surjective.*
3. *Cofibrations* $\mathcal{C}$*:* $f \in Mor(\mathbf{C})$ *is a cofibration if it satisfies the LLP property with respect to all acyclic fibrations* $\mathcal{W} \cap \mathcal{F}$*.*

As an application of Hinich's theorem, one obtains a model category structure on the category $Mod(A)$ of (left) differential graded modules over a differential graded algebra A. Here $\#$ is the forgetful functor and F is given by tensoring $F(M) = A \otimes_{\mathbf{k}} M$.

COROLLARY 2.20. *The category* $Mod(A)$ *of DG* A*-modules is endowed with a model category structure where*

- (i) *weak equivalences are the quasi-isomorphisms.*
- (ii) *fibrations are level-wise surjections. Therefore all objects are fibrant.*
- (iii) *cofibrations are the maps that have the left lifting property with respect to all acyclic fibrations.*

In what follows we give a description of cofibrations and cofibrant objects. An excellent reference for this part is [**FHT95**].

DEFINITION 2.21. An A-module P is called a *semi-free extension* of M if P is a union of an increasing family of A-modules $M = P(-1) \subset P(0) \subset \cdots$ where each $P(k)/P(k-1)$ is a free A-modules generated by cycles. In particular P is said to be a *semi-free A-module* if it is a semi-free extension of the trivial module 0. A *semi-free resolution of an A-module morphism* $f : M \to N$ is a semi-free extension P of M with a quasi-isomorphism $P \to N$ which extends f.

In particular a *semi-free resolution of an A-module M* is a semi-free resolution of the trivial map $0 \to M$.

The notion of a semi-free module can be traced back to [**GM74**], and [**Dri04**] is another nice reference for the subject. A $\mathbf{k}$-complex (M, d) is called a *semi-free complex* if it is semi-free as a differential $\mathbf{k}$-module. Here $\mathbf{k}$ is equipped with the trivial differential. In the case of a field $\mathbf{k}$, every positively graded $\mathbf{k}$-complex is semi-free. It is clear from the definition that a finitely generated semi-free A-module is obtained through a finite sequence of extensions of some free A-modules of the form $A[n]$, $n \in \mathbb{Z}$. Here $A[n]$ is A after a shift in degree by $-n$.

LEMMA 2.22. *Let M be an A-module with a filtration $F_0 \subset F_1 \subset F_2 \cdots$ such that F_0 and all F_{i+1}/F_i are semifree A-modules. Then M is semifree.*

PROOF. Since F_k/F_{k-1} is semifree, it has a filtration $\cdots P_l^k \subset P_{l+1}^k \cdots$ such that P_l^k/P_{l+1}^k is generated as an (A,d)-module by cycles. So one can write $F_k/F_{k-1} = \oplus_l(A \otimes Z'_k(l))$ where $Z'_k(l)$ are free (graded) $\mathbf{k}$-modules such that $d(Z_k(l)) \subset \oplus_{j\leq l} Z_k(j)$. Therefore there are free $\mathbf{k}$-modules $Z_k(l)$ such that

$$F_k = F_{k-1} \bigoplus_{l\geq 0} Z'_k(l)$$

and

$$d(Z_k(l)) \subset F_{k-1} \bigoplus_{j<l} A \otimes Z_k(j).$$

In particular M is the free $\mathbf{k}$-module generated by the union of all basis elements $\{z_\alpha\}$ of $Z_k(l)$'s. Now consider the filtration $P_0 \subset P_1 \cdots$ of free $\mathbf{k}$-modules constructed inductively as follows: P_0 is generates as $\mathbf{k}$-module by the z_α's which are cycles, i.e. $dz_\alpha = 0$. Then P_k is generated by those z_α's such that $dz_\alpha \in A \cdot P_{k-1}$. This is clearly a semifree resolution if we prove that $M = \cup_k P_k$. For that, we show by induction on degree that for all α, z_α belongs to some P_k. Suppose that $z_\alpha \in Z_k(l)$. Then $dz_\alpha \in \oplus A.Z_i(j)$ where $i < k$ or $i = k$ and $j < l$. By the induction hypothesis all z_β's in the sum dz_α are in some P_{m_β}. Therefore $z_\alpha \in P_m$ where $m = \max_\beta m_\beta$ and this finishes the proof. □

REMARK 2.23. If we had not assumed that $\mathbf{k}$ is a field but only a commutative ring then we could still have put a model category structure on $Mod(A)$. This is a special case of the Schwede-Shipley theorem [**SS00**, Theorem 4.1]. More details are provided on pages 503-504 of [**SS00**].

PROPOSITION 2.24. *In the model category of A-modules, a mapf $: M \to N$ is a cofibration if and only if it is a retract of a semi-free extension $M \hookrightarrow P$. In particular, an A-module M is cofibrant if and only if it is a retract of a semi-free A-module, i.e. if and only if it is a direct summand of a semi-free A-module.*

Here is a list of properties of semi-free modules which allow us to define the derived functor by means of semi-free resolutions.

PROPOSITION 2.25.
- *(i) Any morphism $f : M \to N$ of A-modules has a semi-free resolution. In particular every A-module has a semi-free resolution.*
- *(ii) If P is a semi-free A-module, $\mathrm{Hom}_A(P, -)$ preserves quasi-isomorphisms.*
- *(iii) Let P and Q be semi-free A-modules and $f : P \to Q$ be a quasi-isomorphism. Then*

$$g \otimes f : M \otimes_A P \to N \otimes_A Q$$

is a quasi-isomorphism if $g : M \to N$ *is a quasi-isomorphism.*

(iv) *Let* P *and* Q *be semi-free* A*-modules and* $f : P \to Q$ *be a quasi-isomorphism. Then*

$$\mathrm{Hom}_R(g, f) : \mathrm{Hom}_A(Q, M) \to \mathrm{Hom}_A(P, N)$$

is a quasi-isomorphism if $g : M \to N$ *is a quasi-isomorphism.*

The second statement in proposition 2.25 implies that a quasi-isomorphism $f : M \to N$ between semi-free A-modules is a homotopy equivalence, i.e. there is a map $f' : N \to M$ such that $ff' - id_N = [d_N, h']$ and $f'f - id_M = [d_M, h]$ for some $h : M \to N$ and $h' : N \to M$. In fact part (iii) and (iv) follow easily from this observation.

The properties listed above imply that the functors $-\otimes_A M$ and $\mathrm{Hom}_A(-, M)$ preserve enough weak equivalences, ensuring that the derived functors $\otimes^L_A$ and $\mathrm{RHom}_A(-, M)$ exist for all A-modules M.

Since we are interested in Hochschild and cyclic (co) homology, we switch to the category of DG A-bimodules. This category is the same as the category of DG A^e-modules. Therefore one can endow A-bimodules with a model category structure and define the derived functors $- \otimes^L_{A^e} M$ and $\mathrm{RHom}_{A^e}(-, M)$ by means of fibrant-cofibrant replacements.

More precisely, for two A-bimodules M and N we have

$$\mathrm{Tor}^{A^e}_*(M, N) = H_*(P \otimes_{A^e} N)$$

and

$$\mathrm{Ext}^*_{A^e}(M, N) = H^*(\mathrm{Hom}_{A^e}(P, N))$$

where P is cofibrant replacement for M.

By Proposition 2.25 every A^e-module has a semi-free resolution. There is an explicit construction of the latter using the two-sided bar construction. For right and left A-modules P and M, let

$$B(P, A, M) = \bigoplus_{k\geq 0} P \otimes (s\bar{A})^{\otimes k} \otimes M \tag{2.5}$$

equipped with the following differential:

- if $k = 0$,

$$D(p[\quad]m) = dp[\quad]n + (-1)^{|p|}p[\quad]dm$$

- if $k > 0$

$$
\begin{aligned}
&D(p[a_1,\cdots,a_k]m)\\
&= d_0(p[a_1,\cdots,a_k]m) + d_1(p[a_1,\cdots,a_k]m)\\
&= dp[a_1,\cdots,a_k]m - \sum_{i=1}^{k}(-1)^{\epsilon_i}p[a_1,\cdots,da_i,\ldots a_k]m\\
&\qquad\qquad +(-1)^{\epsilon_{k+1}}p[a_1,\cdots,a_k]dm\\
&\quad +(-1)^{|p|}pa_1[a_2,\cdots,a_k]m + \sum_{i=2}^{k}(-1)^{\epsilon_i}p[a_1,\cdots,a_{i-1}a_i,\ldots a_k]m\\
&\qquad\qquad -(-1)^{\epsilon_k}p[a_1,\cdots,a_{k-1}]a_k m,
\end{aligned}
$$

where

$$\epsilon_i = |p| + |a_1| + \cdots |a_{i-1}| - i + 1.$$

Let $P = A$ and $\epsilon_M : B(A,A,M) \to M$ be defined by

$$\epsilon_M(a[a_1,\cdots,a_k])m) = \begin{cases} 0, \text{ if } k \geq 1,\\ am, \text{ if } k = 0.\end{cases} \tag{2.6}$$

It is clear that ϵ_M is a map of left A-modules if M.

Lemma 2.26. *In the category of left A-modules, $\epsilon_M : B(A,A,M) \to M$ is a semi-free resolution.*

Proof. We first prove that this is a resolution. Let $h : B(A,A,M) \to B(A,A,M)$ be defined by

$$h(a[a_1,a_2,\cdots a_k]m) = \begin{cases} [a,a_1,\cdots a_k]m, \text{ if } k \geq 1,\\ [a]m, \text{ if } k = 0.\end{cases} \tag{2.7}$$

On can easily check that $[D,h] = id$ on $\ker \epsilon_M$, which implies $H_*(\ker(\epsilon_M)) = 0$. Since ϵ_M is surjective, ϵ_M is a quasi-isomorphism. Now we prove that $B(A,A,M)$ is a semifree A-module. Let $F_k = \bigoplus_{i\leq k} A \otimes T(s\bar{A})^{\otimes i} \otimes M$. Since $d_1(F_{k+1}) \subset F_k$, then F_{k+1}/F_k is isomorphic as a differential graded A-module to $(A \otimes (sA)^{\otimes k} \otimes M, d_0) = (A,d) \otimes_{\mathbf{k}} ((sA)^{\otimes k}, d) \otimes (M,d)$. The latter is a semifree (A,d)-module since $((sA)^{\otimes k}, d) \otimes_{\mathbf{k}} (M,d)$ is a semifree $\mathbf{k}$-module via the filtration

$$0 \hookrightarrow \ker(d\otimes 1 + 1 \otimes d) \hookrightarrow ((sA)^{\otimes k}, d) \otimes_{\mathbf{k}} (M,d).$$

Therefore $B(A,A,M)$ is semi-free by Lemma 2.22.

□

Corollary 2.27. *The map $\epsilon_A : B(A,A) := B(A,A,\mathbf{k}) \to \mathbf{k}$ given by*

$$\epsilon_k(a[a_1,a_2\cdots a_n]) = \begin{cases} \epsilon(a), \ \textit{if } n = 0,\\ 0 \ \textit{otherwise}\end{cases}$$

is a resolution. Here $\epsilon : A \to \mathbf{k}$ *is the augmentation of* A*. In other words* $B(A,A)$ *is acyclic.*

PROOF. In the previous lemma, let $M = \mathbf{k}$ be the differential A-module with trivial differential and the module structure $a.k := \epsilon(a)k$. □

LEMMA 2.28. *In the category* $Mod(A^e)$*,* $\epsilon_A : B(A,A,A) \to A$ *is a semifree resolution.*

PROOF. The proof is similar to the proof of the previous lemma. First of all, it is obvious that this is a map of A^e-modules. Let $F_k = \bigoplus_{i\leq k} A \otimes T(s\bar{A})^{\otimes i} \otimes A$. Then F_{k+1}/F_k is isomorphic as a differential graded A-module to $(A \otimes (sA)^{\otimes k} \otimes A, d_0) = (A,d) \otimes_{\mathbf{k}} ((sA)^{\otimes k}, d) \otimes (A,d)$. The latter is semi-free as A^e-module since $((sA)^{\otimes k}, d)$ is a semi-free $\mathbf{k}$-module via the filtration $\ker d \hookrightarrow (sA)^{\otimes k}$. □

Since the two-sided bar construction $B(A,A,A)$ provides us with a semi-free resolution of A we have that

$$HH_*(A,M) = H_*(B(A,A,A) \otimes_{A^e} M) = \mathrm{Tor}_*^{A^e}(A,M)$$

and

$$HH^*(A,M) = H^*(Hom_{A^e}(B(A,A,A),M)) = \mathrm{Ext}^*_{A^e}(A,M).$$

In some special situations, for instance that of Calabi-Yau algebras, one can choose smaller resolutions to compute Hochschild homology or cohomology.

The following result will be useful.

LEMMA 2.29. *If* $H^*(A)$ *is finite dimensional then for all finitely generated semi-free* A*-bimodules* P *and* Q*,* $H^*(P)$*,* $H^*(Q)$ *and* $H^*(\mathrm{Hom}_{A^e}(P,Q))$ *are also finite dimensional.*

PROOF. Since A has finite dimensional cohomology, we see that $H^*(A \otimes A^{op})$ is finite dimensional. Similarly P (or Q) has finite cohomological dimension since it is obtained via a finite sequence of extensions of free bimodules of the form $(A \otimes A^{op})[n]$. We also have $\mathrm{Hom}_{A^e}(A \otimes A^{op}, A \otimes A^{op}) \simeq A \otimes A^{op}$, and $A \otimes A^{op}$ is a free A-bimodule of finite cohomological dimension. Since $\mathrm{Hom}_{A^e}(P,Q)$ is obtained through a finite sequence of extensions of shifted free A-bimodules, we obtain that it has finite cohomological dimension. □

CHAPTER 3

∞-categories are dark and full of terrors

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