Long memory random fields

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1 Introduction

A random field $X = (X_n)_{n\in\mathbb{Z}^d}$ is usually said to exhibit long memory, or strong dependence, or long-range dependance, when its covariance function $r(n),\ n \in \mathbb{Z}^d$, is not absolutely summable : $\sum_{n\in\mathbb{Z}^d} |r(n)| = \infty$. An alternative definition involves spectral properties : a random field is said to be strongly dependent if its spectral density is unbounded. These two points of view are closely related but not equivalent.

Generalizing a hypothesis widely used in dimension 1, most studies on long-range dependent random fields assume that the covariance function behaves at infinity as

$$r(h) \underset{h\to\infty}{\sim} |h|^{-\alpha} L(|h|)\, b\left(\frac{h}{|h|}\right) , \qquad 0 < \alpha < d , \tag{1}$$

where L is slowly varying at infinity and where b is continuous on the unit sphere of $\mathbb{R}^d$, $|.|$ denoting the l^1-norm on $\mathbb{R}^d$.

Even if the form (1) is not exactly isotropic because of the presence of the function b defined on the unit sphere, the long memory is due to the term $|h|^{-\alpha}$ which depends only on the norm. So we will call isotropic this kind of long-range dependence. Let us focus on the spectral domain to precise this notion of isotropy.

Definition 1. *A stationary random field exhibits isotropic long memory if it admits a spectral density which is continuous everywhere except at 0 where*

$$f(x) \sim |x|^{\alpha-d} L\left(\frac{1}{|x|}\right) b\left(\frac{x}{|x|}\right) , \qquad 0 < \alpha < d , \tag{2}$$

where L is slowly varying at infinity and where b is continuous on the unit sphere in $\mathbb{R}^d$.

Conditions (1) and (2) are linked by a result of [Wai65] who proved that if the covariance of a random field satisfies (1) and if its spectral density is continuous outside 0, then this random field exhibits isotropic long memory according to definition 1.

Conditions (1) and (2) are regular ways for a random field to be strongly dependent. Now, it is easy to build long memory random fields which fail to satisfy these conditions, either by filtering white noises through unbounded filters like some special AR filters or by aggregating random parameters short memory random fields. Besides, non-isotropic long memory fields naturally arise in statistical mechanics in relatively simple situations of phase transition.

So, the aim of the paper is to give a presentation as complete as possible of isotropic or non-isotropic long memory random fields.

In the first section, we present families of models presenting different kinds of long memory with special glance to Ising model and Gaussian systems in the more specific domain of statistical mechanics.

In the second section, we present a review of the available limit theorems. The first part is devoted to the convergence of partial sums and the second part to the empirical process. We present some well-known results concerning the isotropic long-memory setting : the asymptotic behaviour of the partial sums investigated by [DM79] for Gaussian subordinated fields and by [Sur82] for functionals of linear fields ; the convergence of the empirical process for linear fields proved in [DLS02]. We also give the asymptotic behaviour of the partial sums and of the empirical process in some non-isotropic long memory cases. For these new results, we explain the scheme of proof, based on a spectral convergence theorem. In both situations of isotropic and non-isotropic strong dependence, we observe, like in dimension $d = 1$, a non standard rate of convergence and a non standard limiting process.

2 Modeling long memory stationary random fields

We present two classes of long-memory stationary random fields. The first class is a straightforward generalization of models now widely used for random processes ($d = 1$). The second one comes from mechanical statistics and is for that reason specifically adapted to dimensions $d > 1$.

2.1 Filtering and aggregation

Filtering white noises through unbounded filters or aggregating random coefficients ARMA processes are the two main ways leading to long-memory processes. Since the pioneer works of [GJ80], [Gra80] and [Hos81], these methods have been generalized and improved, providing large families of long-memory one-dimensional processes. See for instance [BD91] for filtered processes and [OV04] for aggregation schemes. These methods are easily extended to dimensions $d > 1$. In fact they lead to rather close covariance structures, but the aggregation method produces only Gaussian random fields. Both provide useful simulation methods.

Filtering

Let us consider a zero-mean white noise $(\epsilon_n)_{n\in\mathbb{Z}^d}$ with spectral representation

$$\epsilon_n = \int_{[-\pi,\pi]^d} \mathrm{e}^{\mathrm{i}<n,\lambda>} \mathrm{d}Z(\lambda) \ ,$$

where the control measure of Z has constant density $\sigma^2/(2\pi)^d$ on $[-\pi,\pi]^d$, and the random field X obtained from ϵ by the filtering operation

$$X_n = \int_{[-\pi,\pi]^d} \mathrm{e}^{\mathrm{i}<n,\lambda>} a(\lambda) \mathrm{d}Z(\lambda) \ , \tag{3}$$

where $a \in L^2([-\pi,\pi]^d)$.

The spectral density of the induced field is

$$f_X(\lambda) = \frac{\sigma^2}{(2\pi)^d} |a(\lambda)|^2 \ , \quad \forall \lambda \in [-\pi,\pi]^d \ , \tag{4}$$

and long-memory is achieved when a is unbounded at certain frequencies.

Example 1 (Long memory ARMA fields). ARMA fields are obtained when $a(\lambda) = \frac{Q}{P}(\mathrm{e}^{\mathrm{i}\lambda})$ where P and Q are polynomial functions. Denoting by L_j the lag operator for index j , i.e.

$$L_j X_{n_1,n_2\dots,n_d} = X_{n_1,\dots,n_{j-1},n_j-1,n_{j+1},\dots,n_d} \ ,$$

we can write an ARMA field in the most popular way

$$P(L_1,\dots,L_d) X_{n_1,\dots,n_d} = Q(L_1,\dots,L_d) \epsilon_{n_1,\dots,n_d} \ . \tag{5}$$

If $P(\mathrm{e}^{\mathrm{i}\lambda}) \neq 0$ for all $\lambda \in [-\pi,\pi]^d$, (5) admits a unique stationary solution (cf. for instance [Ros85] and [Guy93]).

But contrary to the one dimensional case, this condition is not necessary when $d > 1$, and there exist stationary fields having an ARMA representation (5) with $P(\mathrm{e}^{\mathrm{i}\lambda}) = 0$ at some frequencies λ. In this case, the induced field X exhibits long memory since its spectral density, given by (4), is unbounded.

The following ARMA representation in dimension $d = 5$ is a trivial example of this phenomena :

$$X_{n_1,\dots,n_5} - \frac{1}{5}(X_{n_1-1,n_2,\dots,n_5} + X_{n_1,n_2-1,n_3,n_4,n_5} + \dots + X_{n_1,\dots,n_5-1}) = \epsilon_{n_1,\dots,n_5} \ .$$

This representation admits a stationary solution since the filter $a(\lambda_1,\dots,\lambda_5) = \left(1 - \frac{1}{5}(\mathrm{e}^{\mathrm{i}\lambda_1} + \dots + \mathrm{e}^{\mathrm{i}\lambda_5})\right)^{-1}$ is in $L^2([-\pi,\pi]^5)$, and the induced field X is strongly dependent because its spectral density is unbounded at $\lambda = 0$.

Example 2 (Fractional integration). Generalizing the FARIMA processes defined by

$$(I-L)^{\alpha} X_n = \epsilon_n \ ,$$

we consider random fields of the form

$$\left(P(L_1,\ldots,L_d)\right)^{\alpha} X_{n_1,\ldots,n_d} = \epsilon_{n_1,\ldots,n_d} \ ,$$

where P is a polynomial having roots on the unit circle and where $\alpha > 0$ is chosen such that $a(\lambda) = \left(P(\mathrm{e}^{\mathrm{i}\lambda})\right)^{-\alpha} \in L^2([-\pi,\pi]^d)$.

As an example, consider, for a fixed positive integer k, the model

$$(I - L_1 L_2^k)^{\alpha} X_{n_1,n_2} = \epsilon_{n_1,n_2} \ ,$$

where $0 < \alpha < 1/2$. The spectral density of X is

$$f_X(\lambda_1,\lambda_2) = \frac{\sigma^2}{4\pi^2} \left| 1 - \mathrm{e}^{\mathrm{i}(\lambda_1 + k\lambda_2)} \right|^{-2\alpha} \ ,$$

where σ^2 is the variance of the white noise ϵ. The field X exhibits non-isotropic long memory since f_X is unbounded all over the line $\lambda_1 + k\lambda_2 = 0$ and fails to satisfy (2). Using well known results on FARIMA processes (cf [BD91]) easily leads to:

$$\begin{cases} \rho(h,kh) = \prod_{0<j\leq h} \frac{j-1+\alpha}{j-\alpha} & h = \pm 1, \pm 2, \ldots \\ \rho(h,l) = 0 & \text{if } l \neq kh \ , \end{cases}$$

where ρ denotes the correlation function of X. The field X has a non summable correlation function in the direction $l = kh$ since $\rho(h,kh)$ is asymptotically proportional to $h^{2\alpha-1}$. Compared to (1), this confirms that X is a non-isotropic long memory random field.

Aggregation

Let us consider a sequence $(X^{(q)})_{q\geq 1}$ of independent copies of the field

$$P(L_1,\ldots,L_d) X_{n_1,\ldots,n_d} = \epsilon_{n_1,\ldots,n_d} \ , \tag{6}$$

where P is a polynomial function with random coefficients such that P has almost surely no roots on the unit sphere and $(\epsilon_n)_{n\in\mathbb{Z}^d}$ is a zero-mean white noise with variance σ^2.

The representation (6) admits almost surely the solution :

$$X_n = \sum_{j\in\mathbb{Z}^d} c_j \epsilon_{n-j} \ , \tag{7}$$

where $(c_j)_{j\in\mathbb{Z}^d}$ are the coefficients of the Laurent expansion of P^{-1}. The field X given by (7) belongs to L^2 if and only if

$$\sum_{j\in\mathbb{Z}^d} \mathbb{E}(|c_j|^2) < \infty \ , \tag{8}$$

and its spectral density is

$$f(\lambda) = \frac{\sigma^2}{(2\pi)^d}\mathbb{E}\left|P^{-1}\left(\mathrm{e}^{\mathrm{i}\lambda}\right)\right|^2 \ . \tag{9}$$

Now, from the central limit theorem, the finite dimensional distributions of $N^{-1/2}\sum_{q=0}^{N} X_n^{(q)}$ converge as $N\to\infty$ to the so-called aggregated field Z

$$Z_n = \lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{q=0}^{N} X_n^{(q)} \ , \quad n\in\mathbb{Z}^d \ .$$

This process is Gaussian and has the same second order characteristics as the $X^{(q)}$'s. In particular, its spectral density is (9) and long memory is obtained when $\mathbb{E}\left|P^{-1}\left(\mathrm{e}^{\mathrm{i}\lambda}\right)\right|^2$ is unbounded.

Example 3. Let us consider, in dimension $d=2$, the AR representation

$$X_{n,m} - aX_{n-1,m} - bX_{n,m-1} + abX_{n-1,m-1} = \epsilon_{n,m} \ , \tag{10}$$

where a and b are independant and where a (resp. b) has on $[0,1]$ the density

$$(1-x)^\alpha \Phi_1(x) \ , \quad (\text{resp. } (1-x)^\beta \Phi_2(x)) \ , \tag{11}$$

where $0<\alpha,\beta<1$ and where $\Phi_j, \ j=1,2$ are bounded, continuous at $x=1$, with $\Phi_1(1)\Phi_2(1)\neq 0$.

It is easily checked that the above random parameters AR fields satisfy all the required conditions to lead to an aggregated random field with long memory (see for instance [OV04]). The spectral density of the aggregated field Z is a tensorial product and

$$f(\lambda_1,\lambda_2) \sim c|\lambda_1|^{\alpha-1}|\lambda_2|^{\beta-1} \quad \text{when } \lambda\to 0 \ ,$$

where c is a positive constant. Therefore Z exhibits long memory.

Example 4. Consider the AR representation

$$X_{n,m} - aX_{n+k,m-1} = \epsilon_{n,m} \ , \tag{12}$$

where $k\in\mathbb{Z}$ is fixed and where a is a random parameter on $[0,1]$ with density (11).

The spectral density of the induced aggregated field Z is unbounded on the line $\lambda_2 = k\lambda_1$ since

$$f(\lambda_1,\lambda_2) \sim c\left|\lambda_2 - k\lambda_1\right|^{\alpha-1} , \quad \text{as} \quad \lambda_2 - k\lambda_1 \to 0$$

where c is a positive constant. Hence the long-memory is non-isotropic.

We present two 2-dimensional models produced by aggregating $N = 1000$ autoregressive fields with random parameters. The first one (figure 1) is constructed according to the scheme of example 3, the parameters a and b having the same density $\frac{3}{2}\sqrt{1-x}$.

The second model (figure 2) is constructed as in example 4 with $k = -1$ and where a has the same density as above. For both models, an image of size 100×100 has been obtained where, at each point, the realization of the random variable is represented by a level of gray.

Anisotropy clearly appears in figure 2. The strong dependence only occurs in one direction and the long memory is non-isotropic. Its periodogram is unbounded all over the line $\lambda_2 + \lambda_1 = 0$ and fails to follow (2). Moreover, its covariance function decays slowly in only one direction and is not of the form (1). In contrast, in the first model, strong dependence occurs along two directions, with the same intensity. This is the reason why the phenomena of anysotropy is less visible in figure 1.

2.2 Long memory in statistical mechanics

Statistical mechanics explains the macroscopic behaviour of systems of particles by their microscopic properties and provides interpretations of thermodynamic or magnetic phenomena like phase transition. There is phase transition when a system is unstable. For instance, it is the case during the liquid-vapour transition of a gas or when a magnetic material is in transition between the ferromagnetic and the paramagnetic phase. A rigorous mathematical formalism of statistical mechanics can be found for example in [Geo88]. Our aim is to underline the strong dependence properties of some systems in phase transition by focusing on the Ising model and on systems based on quadratic interactions.

Let us consider a system of particles on the lattice $\mathbb{Z}^d$. The state of a particle located on $j \in \mathbb{Z}^d$ is described by the spin x_j, a random variable with values in a polish space X. The pair potential $\Phi = (\Phi_{i,j})_{i,j\in\mathbb{Z}^d}$ gives the interactions between the pairs of particles.

A system configuration is an element $\omega = (x_i)_{i\in\mathbb{Z}^d}$ of the space $\Omega = X^{\mathbb{Z}^d}$. The energy on each finite set Λ of $\mathbb{Z}^d$ involves not only the energy quantity inside the set Λ but also the edges interactions:

$$E_\Lambda(\omega) = \sum_{\{i,j\}\subset\Lambda} \Phi_{i,j}(x_i, x_j) + \sum_{\substack{i\in\Lambda \\ j\in\Lambda^c}} \Phi_{i,j}(x_i, x_j) \ . \tag{13}$$

Now, consider on Ω an a priori measure $\rho = \otimes_{i\in\mathbb{Z}^d}\rho_i$ (typically ρ_i is the Lebesgue measure when $X = \mathbb{R}$ or a Bernoulli measure when $X = \{\pm 1\}$). A measure μ on Ω is called a Gibbs measure associated with the potential Φ with respect to ρ if, for every finite set Λ, ω_Λ and ω_{Λ^c} denoting the restriction of ω to Λ and to its complementary set,

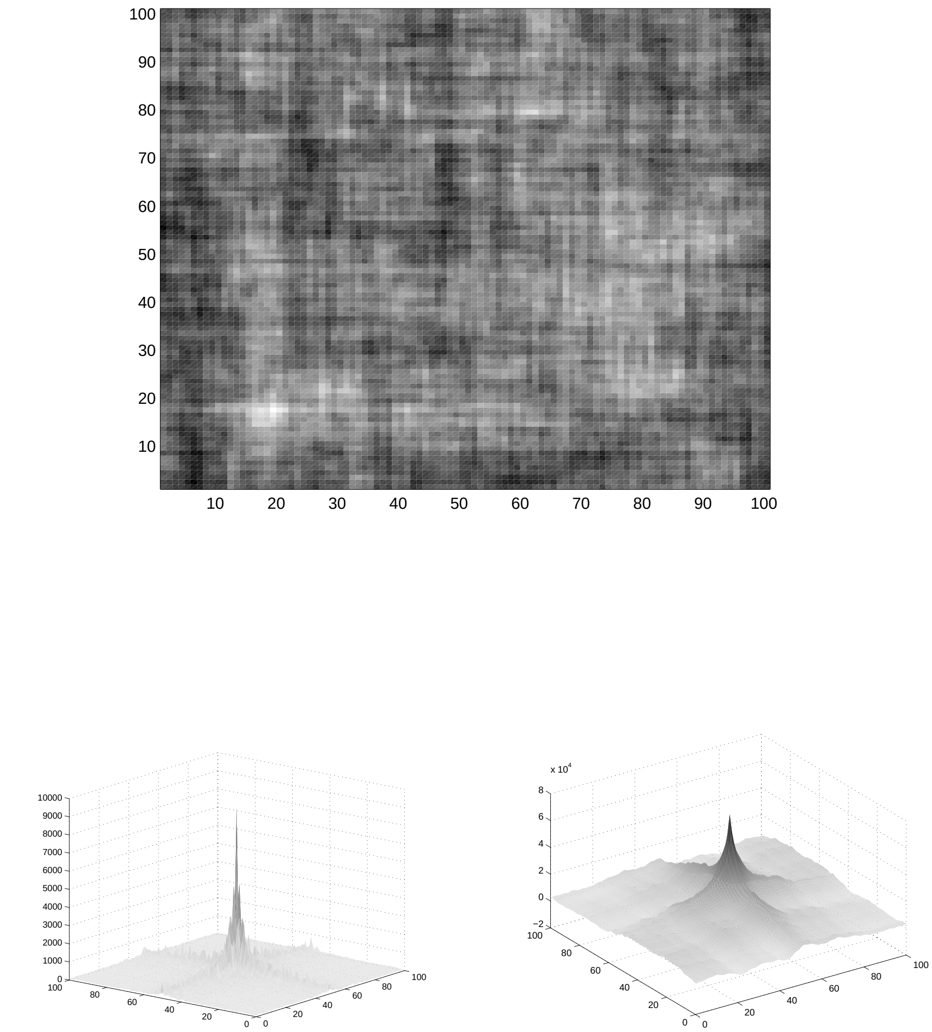

Fig. 1. *[top]* Long memory random field of a product form obtained by aggregating random parameters AR fields of the form (10) with $\alpha = \beta = 0.5$ *[bottom-left]* Its periodogram *[bottom-right]* Its covariance function

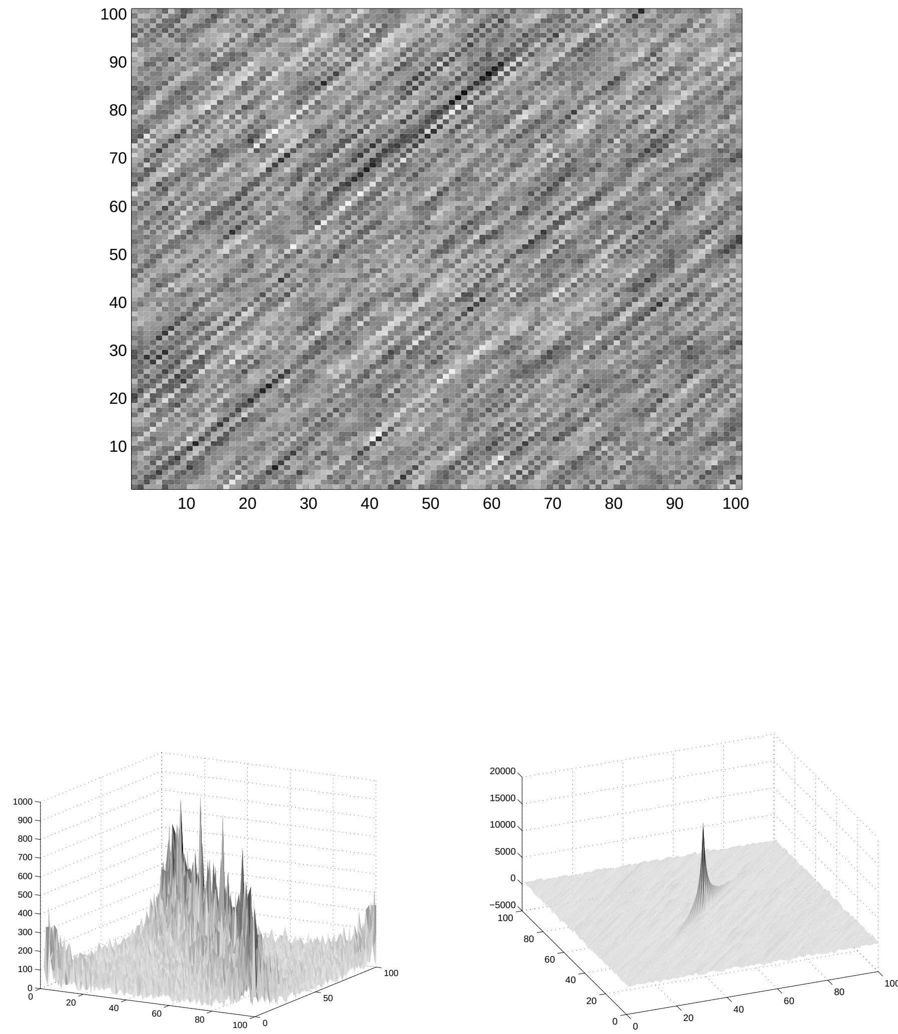

Fig. 2. *[top]* Non-isotropic long memory random field obtained by aggregating random parameters AR fields of the form (12) with $k = -1$ *[bottom-left]* Its periodogram *[bottom-right]* Its covariance function

$$\mu\left(d\omega_{\Lambda}|\omega_{\Lambda^c}\right) = \frac{1}{Z_{\Lambda}(\omega_{\Lambda^c})}\mathrm{e}^{-E_{\Lambda}(\omega)}\rho(d\omega)\ , \tag{14}$$

where $Z_{\Lambda}(\omega_{\Lambda^c})$ is a normalizing constant.

A Gibbs measure is locally characterized by (14). This formalism, attributed to Dobrushin, Landford and Ruelle, guarantees the coherence of the conditional distributions.

For a given system, a fundamental question is whether a Gibbs measure exists or not. Phase transition occurs when there exists several Gibbs measures. The set of all Gibbs measures is a convex set whose extreme elements are the pure phases, the other Gibbs measures being mixtures of the pure phases.

Now, consider the spins' system equipped with the Gibbs measure as a random field. When the second order moments exist, we can measure the memory of the spins' system via the covariance between two sites i and j, $r(i,j) = \mathrm{cov}(x_i, x_j)$. In the following examples the field is stationary ($r(h) = \mathrm{cov}(x_i, x_{i+h})$) and presents long-range dependence properties.

The Ising model

The well known Ising model has been introduced to study magnetism and fluid dynamic. The state space is $X = \{-1, 1\}$, the a priori measure is the Bernoulli measure $1/2(\delta_{-1} + \delta_1)$ and the potential is restricted to the nearest neighbors:

$$\Phi_{i,j}(x_i, x_j) = \begin{cases} \beta x_i x_j & \text{if } |\mathrm{i} - \mathrm{j}| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta > 0$ is a constant representing the inverse temperature.

In dimension $d = 1$, there exists a unique Gibbs measure for any β, therefore the system is never in phase transition. In dimension $d \geq 2$, Gibbs measures exist and phase transition takes place if β is greater than a critical value β_c depending on the dimension d (see [Ons44] in dimension $d = 2$ and [Dob65] in any dimension). When $d = 2$, $\beta_c = \frac{1}{2}\ln(1 + \sqrt{2}) \approx 0.441$.

Let us consider the covariance function. In their physical approach of the Ising model, [KO49] and [Fis64] obtain the asymptotic behaviour of r. When $\beta \neq \beta_c$, the covariance function decays exponentially but when $\beta = \beta_c$ the rate of decay is slow and the covariance is not summable. We have

$$r(h) \underset{h\to\infty}{\sim} \begin{cases} |h|^{-1}\mathrm{e}^{-\kappa|h|} & \text{if } \beta \neq \beta_{\mathrm{c}} \\ |h|^{-(d-2+\mu)} & \text{if} \beta = \beta_{\mathrm{c}}\ , \end{cases}$$

where $\kappa > 0$ is the Boltzmann's constant and $\mu \in [0, 2]$ is a critical parameter which is $1/4$ in case $d = 2$. The strong dependence at the critical point is isotropic.

Remark 1. The long-range dependence structure of the Ising model at the critical point was pointed out in [CJL78] where one can also find others models exhibiting long memory.

Remark 2. There exist some models, slightly more complex than the Ising model, which exhibit long-range dependence without being in phase transition. This is the case for the XY model and for the Heiseinberg model : they are never in phase transition when $d \leq 2$ but their covariance function in dimension $d = 2$ is not summable all over an interval of low temperatures (see [KT78]).

Homogeneous Gaussian models

The state space is $X = \mathbb{R}$, the a priori measure ρ is the Lebesgue measure and the potential is

$$\Phi_{i,j}(x_i, x_j) = \begin{cases} \beta\left(\frac{1}{2}J(0)x_i^2 + ex_i\right) & \text{if i = j} \\ \beta J(i-j)x_i x_j & \text{if i} \neq \text{j,} \end{cases}$$

where β and e are constants representing respectively the inverse temperature and an external magnetic field and where $(J(i))_{i\in\mathbb{Z}^d}$ is a positive definite real sequence with $J(i) = J(-i)$ for every i and $\sum_{i\in\mathbb{Z}^d} J(i) < \infty$. We suppose for simplicity that $e = 0$. Contrary to the Ising model, the temperature has no influence on the appearance of phase transition. The main parameter is the sequence J, improperly named potential.

This system was studied by [Kn80] and [Dob80]. All the results can be found in [Geo88]. The pure phases are Gaussian and their characteristics are directly linked to the potential J via its Fourier transform

$$\hat{J}(\lambda) = \sum_{n\in\mathbb{Z}^d} J(n)\mathrm{e}^{\mathrm{i}<n,\lambda>} \ , \quad \lambda \in [-\pi, \pi]^d \ .$$

Theorem 1 (Künsch, Dobrushin). *Under the above hypotheses on J and in the case $\mathrm{e} = 0$, the set of Gibbs measures is non empty if and only if*

$$\int_{[-\pi,\pi]^d} \hat{J}^{-1}(\lambda)\mathrm{d}\lambda < \infty \ .$$

In this case, the pure phases are the Gaussian measures with covariance function

$$r(h) = \int_{[-\pi,\pi]^d} \hat{J}^{-1}(\lambda)\mathrm{e}^{\mathrm{i}<h,\lambda>}\mathrm{d}\lambda \tag{15}$$

and with mean vector a sequence $(u_n)_{n\in\mathbb{Z}^d}$ such that, for all $k \in \mathbb{Z}^d$,

$$\sum J(n)u_{k+n} = 0 \ .$$

Remark 3. In the case $e \neq 0$, further hypotheses are needed for the existence of a Gibbs measure.

The occurrence of phase transition in the particular case $e = 0$ can be deduced from Theorem 1 and is given in the following corollary. Note that, despite the fact that the pure phases are Gaussian, all Gibbs measures are not necessarily so. Phase transition can take place with one or several measures without second moment. Insofar as we are interested in the covariance function, the corollary is stated in the L^2 setting:

Corollary 1 (Künsch). *Under the hypotheses of Theorem 1, there exist several Gibbs measures with finite second moments if and only if* $\hat{J}$ *has at least one root in* $[-\pi, \pi]^d$.

Therefore, when the system is in phase transition, every Gibbs measure having a finite second moment is strongly dependent. Indeed $\hat{J}^{-1}$, which is the spectral density of the pure phases, according to (15), is unbounded if there is phase transition.

Example 5. In dimension $d \geq 3$, the harmonic potential is a simple example of finite range interaction leading to long-memory random fields. The potential is defined by:

$$J(n) = \begin{cases} -\frac{1}{2d} & \text{if } |\mathrm{n}| = 1 \\ 1 & \text{if } \mathrm{n} = 0 \\ 0 & \text{otherwise} \end{cases}$$

and we have

$$\hat{J}(\lambda) = 1 - \sum_{|n|=1} \frac{1}{2d} \mathrm{e}^{\mathrm{i}<n,\lambda>} = 1 - \frac{1}{d} \sum_{k=1}^{d} \cos(\lambda_k)$$

whose inverse is integrable on $[-\pi, \pi]^d$ since $d \geq 3$. Hence, Theorem 1 guarantees the existence of a Gibbs measure associated with this potential. Moreover $\hat{J}(0) = 0$ and according to Corollary 1, the system is in phase transition and the second order Gibbs measures exhibit long memory. The long-range dependence is isotropic in the sense of definition 1.

Example 6. In dimension $d = 2$, consider the potential :

$$J(k, l) = \begin{cases} \prod_{0<j\leq k} \frac{j-1-\alpha}{j+\alpha} & \text{if } \mathrm{l} = \mathrm{pk},\ |\mathrm{k}| > 1 \\ 1 & \text{if } \mathrm{k} = \mathrm{l} = 0 \\ 0 & \text{otherwise} \end{cases}$$

where p is a non null fixed integer and $\alpha \in]0, 1/2[$.

The sequence $J(k, pk)$ corresponds to the autocorrelation function of an integrated stationary process of order α, from which (see [BD91])

$$J(k, pk) \sim \frac{\Gamma(1+\alpha)}{\Gamma(-\alpha)} k^{-2\alpha-1}, \quad \text{when} \quad k \to \infty\ .$$

This shows the summability of J. Moreover, using the well known properties of the FARIMA processes,

$$\sum_{k\in\mathbb{Z}} J(k,pk)\mathrm{e}^{\mathrm{i}k\lambda}\frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}\left|2\sin\left(\frac{\lambda}{2}\right)\right|^{2\alpha} .$$

Finally

$$\hat{J}(\lambda_1,\lambda_2)=\sum_{k,l\in\mathbb{Z}^2} J(k,l)\mathrm{e}^{\mathrm{i}(k\lambda_1+l\lambda_2)}=\sum_{k\in\mathbb{Z}} J(k,pk)\mathrm{e}^{\mathrm{i}k(\lambda_1+p\lambda_2)}$$
$$=\frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}\left|2\sin\left(\frac{\lambda_1+p\lambda_2}{2}\right)\right|^{2\alpha} .$$

Since $\alpha\in]0,1/2[$, $\hat{J}^{-1}$ is integrable on $[-\pi,\pi]^2$ and the existence of a Gibbs measure is guaranteed by Theorem 1. In addition, $\hat{J}$ vanishes all along the line $\lambda_1+p\lambda_2=0$ which shows that the system is in phase transition according to Corollary 1 and that the Gibbs measures exhibit non-isotropic long memory.

3 Limit theorems under isotropic and non-isotropic strong dependence

We present some limit theorems for the partial sums process and the doubly indexed empirical process of long memory random fields.

3.1 Partial sums of long memory random fields

Since the results for isotropic long-memory fields are nearly classical while those related to non-isotropic long memory are newer and still incomplete, we split this section in two parts according to the regularity of the strong dependence. The first one is devoted to isotropic long memory: available results concern Gaussian subordinated fields and some particular functionals of linear fields. In the second part, related to non-isotropic long memory, we first present the spectral convergence theorem on which is based the convergence of the partial sums. Then we apply it to some non-isotropic long memory fields.

In a third part, we give a tightness criterion for partial sums and we apply it to situations needed for the doubly-indexed empirical process treated in the next section.

In the sequel we adopt the notation $A_n=\{1,\dots,n\}^d$ and $\overset{fidi}{\Longrightarrow}$ for the convergence of the finite dimensional distributions.

Convergence of partial sums under isotropic long memory

The first study of partial sums is due to [DM79] who considered Gaussian subordinated fields presenting isotropic long memory. Then the same results for some functional of linear fields are obtained in [Sur82] and [AT87].

Let us first introduce the so-called Hermite process Z_m of order m which is the limiting process we shall encounter here.

$$Z_m(t) = \int_{\mathbb{R}^{md}} \prod_{j=1}^{d} \frac{\mathrm{e}^{\mathrm{i}t_j\left(x_j^{(1)}+\cdots+x_j^{(m)}\right)}-1}{\mathrm{i}\left(x_j^{(1)}+\cdots+x_j^{(m)}\right)} Z_{G_0}(\mathrm{d}x^{(1)})\ldots Z_{G_0}(\mathrm{d}x^{(m)}) \qquad (16)$$

where Z_{G_0} is the random Gaussian spectral field with control measure G_0. The spectral measure G_0 depends on a parameter α and a function b continuous on the unit sphere in $\mathbb{R}^d$ and it is given by

$$2^d \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}<t,x>} \prod_{j=1}^{d} \frac{1-\cos(x_j)}{x_j^2} G_0(\mathrm{d}x) = \int_{[-1,1]^d} \frac{b\left(\frac{x+t}{|x+t|}\right)}{|x+t|^\alpha} \prod_{j=1}^{d}(1-|x_j|)\mathrm{d}x \ . \qquad (17)$$

When $d=1$ (16) simplifies because G_0 admits a density proportional to $|x|^{\alpha-1}$ and in this case

$$Z_m(t) = \kappa^{-k/2} \int_{\mathbb{R}^m} \frac{\mathrm{e}^{\mathrm{i}t(x^{(1)}+\cdots+x^{(m)})}-1}{\mathrm{i}(x^{(1)}+\cdots+x^{(m)})} \prod_{k=1}^{m} \left|x^{(k)}\right|^{\frac{\alpha-1}{2}} \mathrm{d}W(x^{(k)}) \ ,$$

where W is the Gaussian white noise spectral field and where $\kappa = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x}|x|^{\alpha-1}$.

Let us now summarize the convergence results.

Theorem 2. *[[DM79]] Let $(X_n)_{n\in\mathbb{Z}^d}$ be a zero-mean, stationary, Gaussian random field. Let H be a measurable function such that*

$$\int_{\mathbb{R}} H(x)\mathrm{e}^{\frac{-x^2}{2}}\mathrm{d}x = 0 \quad \text{and} \quad \int_{\mathbb{R}} H^2(x)\mathrm{e}^{\frac{-x^2}{2}}\mathrm{d}x < \infty \ .$$

Denote by m its Hermite rank.

We suppose that (X_n) admits the following covariance function

$$r(k) = |k|^{-\alpha} L(|k|) b\left(\frac{k}{|k|}\right) \ ,$$

with $r(0)=1$, where $0 < m\alpha < d$ and where L is a slowly varying function at infinity and b is a continuous function on the unit sphere in $\mathbb{R}^d$.

Then

$$\frac{1}{N^{d-m\alpha/2}(L(N))^{m/2}} \sum_{k\in A_{[Nt]}} H(X_k) \stackrel{fidi}{\Longrightarrow} c_m Z_m(t) \ , \qquad (18)$$

where Z_m is the Hermite process of order m defined in (16) and where c_m is the coefficient of rank m in the Hermite expansion of H.

The following theorem concerns linear fields. The class of functions H is restricted to the Appell polynomials.

Theorem 3. *[[Sur82] and [AT87]] Let $(\epsilon_n)_{n\in\mathbb{Z}^d}$ be a sequence of zero-mean i.i.d random fields with variance 1 and finite moments of any order. Let $(X_n)_{n\in\mathbb{Z}^d}$ be the linear field*

$$X_n = \sum_{k\in\mathbb{Z}^d} a_k \epsilon_{n-k} \ ,$$

where

$$a_k = |k|^{-\beta} L(|k|) a\left(\frac{k}{|k|}\right) \ , \qquad d < 2\beta < d\,(1 + 1/m) \ , \tag{19}$$

where L is a slowly varying function at infinity and a is a continuous function on the unit sphere in $\mathbb{R}^d$.

Let P_m be the m^{th} Appell polynomial associated with the distribution of X_0. Then

$$\frac{1}{N^{d-m(\beta-\frac{d}{2})}} \sum_{k\in A_{[Nt]}} P_m(X_k) \overset{fidi}{\Longrightarrow} Z_m(t) \ ,$$

where Z_m is the Hermite process of order m defined by (16) and (17) in which $\alpha = 2\beta - d$ and

$$b(t) = \int_{\mathbb{R}^d} a\left(\frac{s}{|s|}\right) a\left(\frac{s-t}{|s-t|}\right) |s|^{-\beta} |t-s|^{-\beta} \mathrm{d}s \ .$$

Remark 4. Theorem 3 relates to isotropic long memory since condition (19) implies that the covariance function of X has asymptotically the form (1).

Remark 5. One can find a presentation of the tools for proving Theorems 2 and 3 in [DOT03].

Convergence of partial sums under non-isotropic long memory

The proofs of Theorem 2 and 3 rely on the convergence of multiple stochastic integrals. This method fails to work under non-isotropic long memory. So we turn to a method based on convergence of spectral measures.

Starting from a filter $a \in L^2([-\pi,\pi]^d)$ and a zero-mean random field ξ having a spectral density f_ξ, we consider the linear field

$$X_n = \sum_{k\in\mathbb{Z}^d} a_k \xi_{n-k} \ , \quad n \in \mathbb{Z}^d \tag{20}$$

where a_k are the Fourier coefficients of a:

$$a(\lambda) = \sum_{k\in\mathbb{Z}^d} a_k \mathrm{e}^{-\mathrm{i}<k,\lambda>} \ .$$

The filter a is directly linked to the spectral density f_X of X by the relation :

$$f_X(\lambda) = f_\xi(\lambda)|a(\lambda)|^2 \ .$$

First, the partial sums are rewritten using the spectral field W of ξ. Since

$$\xi_k = \int_{[-\pi,\ \pi]^d} \mathrm{e}^{\mathrm{i}<k,\lambda>} \mathrm{d}W(\lambda) \ , \tag{21}$$

if the random measure W_n on $[-n\pi;\ n\pi]^d$ is defined for all Borel set A by

$$W_n(A) = n^{d/2} W(n^{-1}A) \ ,$$

we have

$$n^{-d/2} \sum_{k \in A_{[nt]}} X_k = \int_{[-n\pi, n\pi]^d} a\left(\frac{\lambda}{n}\right) \prod_{j=1}^{d} \frac{\mathrm{e}^{\mathrm{i}\lambda_j [t_j n]/n} - 1}{n(\mathrm{e}^{\mathrm{i}\lambda_j/n} - 1)} \mathrm{d}W_n(\lambda) \ , \tag{22}$$

where $[nt] = ([nt_1], \ldots, [nt_d])$.

Hence, in order to investigate the convergence of the partial sums (22), it suffices to handle stochastic integrals of the form $\int \Phi_n \mathrm{d}W_n$ where $\Phi_n \in L^2(\mathbb{R}^d)$. This is made possible by the spectral convergence theorem.

The spectral convergence theorem

Let $(\xi_k)_{k \in \mathbb{Z}^d}$ be a real stationary random field. We work under the following assumptions :

H1 : The zero-mean stationary random field $(\xi_k)_{k\in\mathbb{Z}^d}$ has a spectral density f_ξ bounded above by $M > 0$. Moreover, the sequence of partial sums of the noise

$$S_n^\xi(t) = n^{-d/2} \sum_{k \in A_{[nt]}} \xi_k, \quad t \in [0,1]^d \ , \tag{23}$$

converges in the finite dimensional distributions sense to a field B.

Theorem 4. *Under* **H1**, *there exists a linear application* I_0 *from* $L^2(\mathbb{R}^d)$ *into* $L^2(\Omega, \mathcal{A}, \mathbb{P})$ *which has the following properties :*

(i) $\forall \Phi \in L^2(\mathbb{R}^d) \quad \mathbb{E}\left(I_0\left(\Phi\right)\right)^2 \leq (2\pi)^d M ||\Phi||_2^2$

(ii) $I_0\left(\prod_{j=1}^d \frac{\mathrm{e}^{\mathrm{i}t_j\lambda_j}-1}{\mathrm{i}\lambda_j}\right) = B(t_1, \ldots, t_d)$

(iii) If the sequence Φ_n *converges in* $L^2(\mathbb{R}^d)$ *to* Φ*, then* $\int \Phi_n(x)\mathrm{d}W_n(x)$ *converges in law to* $I_0\left(\Phi\right)$*.*

(iv) If ξ *is i.i.d, then* $\forall \Phi \in L^2(\mathbb{R}^d)$ $I_0(\Phi) = \int \Phi \mathrm{d}W_0$*, where* W_0 *is the Gaussian white noise spectral field.*

Remark 6. When ξ is i.i.d, B is the Brownian sheet, property (ii) corresponding to its harmonisable representation

$$B(t) = \int \prod_{j=1}^{d} \frac{\mathrm{e}^{\mathrm{i}t_j\lambda_j} - 1}{\mathrm{i}\lambda_j} \mathrm{d}W_0(\lambda) \ ,$$

and I_0 becomes in this case an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\Omega, \mathcal{A}, \mathbb{P})$ which can then be considered as the stochastic integral with respect to W_0.

In the general case, point (i) shows that I_0 might not be an isometry so that I_0 cannot be always viewed as a stochastic integral.

Remark 7. Although our purpose is only to investigate the convergence of the partial sums, Theorem 4 appears to be a useful tool to obtain the asymptotic properties of any linear statistic writable in the form $\int \Phi_n \mathrm{d}W_n$.

Proof. The theorem is proved in [LS00] in dimension $d = 1$. The details of the generalization to the context of random fields can be found in [Lav05a], so we only give a sketch of the proof. Let us consider the field

$$B_n(t) = \int_{[-n\pi,\ n\pi]^d} \prod_{j=1}^{d} \frac{\mathrm{e}^{\mathrm{i}t_j\lambda_j} - 1}{\mathrm{i}\lambda_j} \mathrm{d}W_n(\lambda) \ . \tag{24}$$

Denoting $\hat{\Phi}$ the Fourier transform of Φ, we prove after some integrations by parts that

$$\int_{[-n\pi,\ n\pi]^d} \hat{\Phi}(x)\mathrm{d}W_n(x) = \frac{(-1)^d}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\partial\Phi(t_1,\dots,t_d)}{\partial t_1\dots\partial t_d} B_n(t_1,\dots,t_d)\mathrm{d}t_1\dots\mathrm{d}t_d \ . \tag{25}$$

Besides, $B_n - S_n^\xi$ converges to 0 in L^2, which leads to the finite dimensional convergence of B_n to B. Then, extending to $d > 1$ a theorem of [Gri76] leads to the convergence in law of (25) to

$$I_B(\Phi) = (-1)^d \int_{\mathbb{R}^d} \frac{\partial\Phi(t)}{\partial t_1\dots\partial t_d} B(t)\mathrm{d}t \ .$$

Finally the linear application I_0 of the theorem is defined by

$$I_0(\Phi) = I_B(\check{\Phi}) \ , \tag{26}$$

where $\check{\Phi}$ is the inverse Fourier transform of Φ in $L^2(\mathbb{R}^d)$ and we have

$$\begin{aligned}\mathbb{E}\left(I_0(\Phi)\right)^2 &= \mathbb{E}\left(I_B(\check{\Phi})\right)^2 \\ &\le \underline{\lim}\ \mathbb{E}\left((2\pi)^{d/2}\int_{[-n\pi,\ n\pi]^d} \hat{\Phi}\mathrm{d}W_n\right)^2 \le (2\pi)^d M||\Phi||_2^2 \ ,\end{aligned}$$

which is (i) of Theorem 4.

Theorem 4.2 in [Bil68] implies that $\int \Phi \mathrm{d}W_n$ converges to $I_0(\Phi)$. Hence $\int \Phi_n \mathrm{d}W_n$ converges to $I_0(\Phi)$ as soon as Φ_n goes to Φ in $L^2(\mathbb{R}^d)$. This proves (iii).

The particular choice $\check{\Phi} = \mathbb{1}_{[0,t_1]\times\dots\times[0,t_d]}$ in (26) leads to (ii).

Convergence of partial sums

In view of the spectral representation (22) and of Theorem 4, for proving the convergence of the partial sums it is sufficient to check the L^2-convergence of $a(x/n)$. This leads to several types of proofs according to the form of a.

The following propositions focus on filters which lead to non-isotropic long memory random fields. Their proofs can be found in [Lav05a].

The first result concerns the simplest situation of a tensorial product.

Proposition 1. *Let $(\xi_k)_{k\in\mathbb{Z}^d}$ be a noise satisfying* $\mathbf{H\,1}$. *Let $(X_k)_{k\in\mathbb{Z}^d}$ be the random field defined by (20), constructed by filtering ξ through a filter of the form :*

$$a(\lambda_1,\ldots,\lambda_d)=\prod_{j=1}^{d}a_j(\lambda_j)\ , \tag{27}$$

where the a_j's satisfy:

$$a_j(\lambda_j)\sim|\lambda_j|^{-\alpha_j}\quad \text{when } \lambda_j\to 0\ ,$$

with $0<\alpha_j<1/2$. Then

$$\frac{1}{n^{d/2-(\sum_{j=1}^{d}\alpha_j)}}\sum_{k\in A_{[nt]}}X_k\overset{fidi}{\Longrightarrow}I_0\left(\prod_{j=1}^{d}\frac{\mathrm{e}^{\mathrm{i}t_j\lambda_j}-1}{\mathrm{i}\lambda_j|\lambda_j|^{\alpha_j}}\right)\ , \tag{28}$$

where I_0 is the linear application defined in Theorem 4.

Remark 8. When ξ is i.i.d, the limiting field (28) is the Fractional Brownian sheet with parameters $(\alpha_j,\quad j=1,\ldots,d)$.

It is well known that, in dimension $d=1$, only the spectral behaviour at 0 determines the asymptotic of the partial sums. This result still holds for $d=2$, as stated in the next proposition.

Proposition 2. *Let $(\xi_k)_{k\in\mathbb{Z}^d}$ be a stationary random field satisfying* $\mathbf{H\,1}$. *Let $(X_k)_{k\in\mathbb{Z}^d}$ be defined by (20), constructed by filtering ξ through a.*

(i) If the filter $a\in L^2([-\pi,\pi]^d)$ is continuous at the origin with $a(0)\neq 0$, then, for $d\le 2$,

$$\frac{1}{n^{d/2}}\sum_{k\in A_{[nt]}}X_k\overset{fidi}{\Longrightarrow}a(0)B(t)\ , \tag{29}$$

where B is the limit of the partial sums of ξ introduced in hypotheses $\mathbf{H\,1}$.

(ii) If the filter a is equivalent at 0 to a homogeneous function $\tilde{a}$, i.e. for all c, $\tilde{a}(c\lambda)=|c|^{-\alpha}\,\tilde{a}(\lambda)$, with degree $\alpha\in]0,1[$ such that $a\in L^2([-\pi,\pi]^d)$, then, for $d\le 2$,

$$\frac{1}{n^{d/2+\alpha}}\sum_{k\in A_{[nt]}}X_k\overset{fidi}{\Longrightarrow}I_0\left(\tilde{a}(\lambda)\prod_{j=1}^{d}\frac{\mathrm{e}^{\mathrm{i}t_j\lambda_j}-1}{\mathrm{i}\lambda_j}\right)\ , \tag{30}$$

where I_0 is the linear application defined in Theorem 4.

Remark 9. When ξ is i.i.d, the limiting process can be written as a stochastic integral with respect to a Gaussian white noise measure (cf Remark 6).

Remark 10. Filtering a white noise through a filter satisfying the hypotheses in (i) can produce a weakly dependent random field, for instance if a is continuous on $[-\pi,\pi]^d$. It produces non-isotropic long memory when a is unbounded since the covariance function is then not absolutely summable. This memory involves only non-zero singularities of the spectral density and, as expected, does not modify the limit obtained under weak dependence.

Condition (ii) of Theorem 2 can be satisfied with isotropic as well as with non isotropic long-memory. The memory is non-isotropic for instance when the filter is $a(\lambda_1,\lambda_2)=|\lambda_1+\theta\lambda_2|^{-\alpha}$, where $0<\alpha<1/2$ and $\theta\in\mathbb{R}$, $\theta\neq 0$.

Unfortunately, probably due to the spectral method, these results cannot be extended in dimension $d\geq 3$ without further assumptions. We only give an example of filters unbounded all over a linear subspace of $[-\pi,\pi]^d$.

Proposition 3. *Let $(\xi_k)_{k\in\mathbb{Z}^d}$ be a stationary random field satisfying* $\mathbf{H\,1}$*. Let $(X_k)_{k\in\mathbb{Z}^d}$ be the random field defined by (20).*

Suppose that a has the following form :

$$a(\lambda)=\left|\sum_{i=1}^{d}c_i\lambda_i\right|^{-\alpha},$$

where $0<\alpha<1/2$ and the c_i's are real constants.

Then, as long as

$$0<2\alpha<\frac{1}{(d-2)\vee 1}, \tag{31}$$

we have

$$\frac{1}{n^{d/2+\alpha}}\sum_{k\in A_{[nt]}}X_k\overset{fidi}{\Longrightarrow}I_0\left(a(\lambda)\prod_{j=1}^{d}\frac{\mathrm{e}^{\mathrm{i}t_j\lambda_j}-1}{\mathrm{i}\lambda_j}\right), \tag{32}$$

where I_0 is the linear application defined in Theorem 4.

Remark 11. The condition (31) on α is a restriction only when $d\geq 4$.

Tightness criteria for partial sums

So far, only the convergence of the finite-dimensional distributions of the partial sums has been stated. In dimension $d=1$, a convenient criterion for tightness is given in [Taq75] from which the convergence in $\mathcal{D}([0,1])$ follows easily.

General conditions for tightness in $\mathcal{D}([0,1]^d)$ of a sequence of random fields are given in [BW71]. The following lemma, a corollary of Theorems 2 and 3 in [BW71], is very useful for proving tightness of the partial sums of strongly dependent fields.

Lemma 1. *Let us consider a stationary random field* $(X_k)_{k\in\mathbb{Z}^d}$ *and its normalized partial sum process*

$$S_n(t) = d_n^{-1} \sum_{k_1=0}^{[nt_1]} \cdots \sum_{k_d=0}^{[nt_d]} X_{k_1,\dots,k_d} \ , \quad t \in [0,1]^d \ .$$

If the finite-dimensional distributions of S_n *converge to those of* X *and if there exist* $c > 0$ *and* $\beta > 1$ *such that for all* $p_1, \dots, p_d \in \{1, \dots, n\}$

$$\mathbb{E}\left(d_n^{-1} \sum_{k_1=0}^{p_1} \cdots \sum_{k_d=0}^{p_d} X_{k_1,\dots,k_d}\right)^2 \le c \left(\prod_{i=1}^{d} \frac{p_i}{n}\right)^{\beta} \ , \tag{33}$$

then

$$S_n \overset{\mathcal{D}([0,1]^d)}{\Longrightarrow} X \ .$$

Moreover the field X *admits a continuous version.*

The details of the proof can be found in [Lav05b].

In the next section, we study the doubly-indexed empirical process of long memory random fields and we investigate its asymptotic behaviour for the long memory Gaussian subordinated fields of Theorem 2 and for the non-isotropic long memory situation of Proposition 3. For this, we need the convergence of the partial sums in $\mathcal{D}([0,1]^d)$ in both settings. Since the convergence of their finite-dimensional distributions has already been stated, only tightness is missing, which is the subject of the next results. Their proofs, based on the tightness criterion presented in Lemma 1, can be found in [Lav05b].

Proposition 4. *Under the hypothesis of Theorem 2, the partial sums process*

$$\frac{1}{N^{d-m\alpha/2}(L(N))^{m/2}} \sum_{k\in A_{[Nt]}} H(X_k)$$

is tight and convergence (18) takes place in $\mathcal{D}([0,1]^d)$.

Proposition 5. *Under the hypothesis of Proposition 3, the partial sums process*

$$\frac{1}{N^{d/2+\alpha}} \sum_{k\in A_{[Nt]}} X_k$$

is tight and convergence (32) takes place in $\mathcal{D}([0,1]^d)$.

3.2 Empirical Process of long memory random fields

We study the asymptotic behaviour of the empirical process

$$\sum_{j \in A_{[nt]}} \left[\mathbb{1}_{\{G(X_j) \leq x\}} - F(x) \right] \ , \tag{34}$$

where G is a measurable function and where F is the cumulative distribution function of $G(X_1)$, $(X_k)_{k \in \mathbb{Z}^d}$ being a long-range dependent stationary random field.

Our presentation relates to Gaussian subordinated random fields and to (non necessarily Gaussian) linear random fields.

In the first situation, we prove a uniform weak reduction principle and apply it to different situations of strong dependence. We present the convergence of (34) in $\mathcal{D}(\overline{\mathbb{R}} \times [0,1]^d)$ when X is Gaussian with isotropic long-range dependence, generalizing in dimension $d > 1$ the result of [DT89]. In the non-isotropic long memory setting, we give the convergence of (34) in $\mathcal{D}(\overline{\mathbb{R}} \times [0,1]^d)$ when the random field X is linear, Gaussian, and when the Hermite rank of $\mathbb{1}_{\{G(X_j) \leq x\}} - F(x)$ is 1.

In the situation of (non necessarily Gaussian) linear random fields a uniform weak reduction principle is more difficult to obtain. The only available results are those proved in [DLS02] where the authors obtain the convergence of (34) for $t = 1$, when G is the identity function, and in the situation of isotropic long-memory.

In each situation described above, the limiting process is degenerated insofar as it has the form $f(x)Z(t)$ where f is a deterministic function and Z a random field. This asymptotic behaviour of the empirical process is a characteristic property of strong dependence in dimension $d = 1$. It seems to be also the case with random fields even if the strong dependence is anisotropic such as in Corollary 3 below.

Empirical process of Gaussian subordinated fields

The main tool to obtain the convergence of the empirical process is the uniform weak reduction principle introduced in [DT89] which allows to replace in most cases the empirical process by the first term in its expansion on the Hermite basis. We present an inequality generalizing this principle to dimension $d > 1$. Then we specify the dependence structure of the random field in two corollaries. The first one refers to the isotropic long-range dependent Gaussian fields of Theorem 2. The second one relates to non-isotropic long memory. It focuses on the random field of Proposition 3 which is in addition supposed here to be Gaussian. The proofs of this section are detailed in [Lav05b].

Let $(X_n)_{n \in \mathbb{Z}^d}$ be a stationary Gaussian random field with covariance function r such that $r(0) = 1$.

Let G be a measurable function. We consider the following expansion on the Hermite basis :

$$\mathbb{1}_{\{G(X_j) \leq x\}} - F(x) = \sum_{q=m}^{\infty} \frac{J_q(x)}{q!} H_q(X_j) \ ,$$

where $F(x) = \mathbb{P}(G(X_1) \leq x)$. H_q is the Hermite polynomial of degree q and

$$J_q(x) = \mathbb{E}\left[\mathbb{1}_{\{G(X_1)\leq x\}} H_q(X_1)\right] \ .$$

Let

$$S_n(x) = \sum_{j\in A_n} \left[\mathbb{1}_{\{G(X_j)\leq x\}} - F(x) - \frac{J_m(x)}{m!} H_m(X_j)\right] \ .$$

Now, we formulate the inequality leading to the uniform weak reduction principle. Its proof follows the same lines as in [DT89].

Theorem 5. *Let*

$$d_N^2 = \mathrm{var}\left(\sum_{j\in A_N} H_m(X_j)\right) = m! \sum_{j,k\in A_N^2} r^m(k-j) \ .$$

If $d_N \longrightarrow \infty$, we have, for all η, $\delta > 0$ and for all $n \leq N$,

$$\mathbb{P}\left(\sup_x d_N^{-1} |S_n(x)| > \eta\right) \leq C N^\delta d_N^{-2} \sum_{j,k\in A_N^2} |r(k-j)|^{m+1} + \frac{d_n^2}{N^{2d}} \ , \qquad (35)$$

where C is a positive constant depending only on η.

If the limit of $d_N^{-1} \sum_{j\in A_{[Nt]}} H_m(X_j)$ is known, inequality (35) provides the asymptotic behaviour of the empirical process (34) if the upper bound in (35) vanishes when N goes to infinity.

The first corollary below relates to the Gaussian subordinated fields of Theorem 2.

Corollary 2. *Under the above notations, we suppose that the Gaussian field $(X_n)_{n\in\mathbb{Z}^d}$ admits the covariance function*

$$r(k) = |k|^{-\alpha} L(|k|) b\left(\frac{k}{|k|}\right) \ , \quad r(0) = 1 \ , \qquad (36)$$

where $0 < m\alpha < d$, where L is slowly varying at infinity and where b is continuous on the unit sphere in $\mathbb{R}^d$.

Then

$$\frac{1}{N^{d-m\alpha/2}(L(N))^{m/2}} \sum_{j\in A_{[Nt]}} \left[\mathbb{1}_{\{G(X_j)\leq x\}} - F(x)\right] \overset{\mathcal{D}(\bar{\mathbb{R}}\times[0,1]^d)}{\Longrightarrow} \frac{J_m(x)}{m!} Z_m(t) \ ,$$

where the convergence takes place in $\mathcal{D}(\bar{\mathbb{R}} \times [0,1]^d)$ endowed with the uniform topology and the σ-field generated by the open balls and where Z_m, defined in (16), is the Hermite process of order m.

Proof (Sketch of proof). From (36), as $N \to \infty$

$$d_N^2 \sim N^{2d-m\alpha}(L(N))^m \ ,$$

and

$$\sum_{j,k\in A_N} |r(k-j)|^{m+1} = O(N^{2d-(m+1)\alpha}L(N)^{m+1}) + O(N^d) \ .$$

Hence the upper bound in (35) goes to zero for small values of δ.

Moreover Theorem 2 gives the convergence of $d_N^{-1}\sum_{j\in A_{[Nt]}} H_m(X_j)$ to the Hermite process, this convergence taking place in $\mathcal{D}([0,1]^d)$ from Proposition 4. Now, J_m is bounded and so :

$$J_m(x)d_N^{-1}\sum_{j\in A_{[Nt]}} H_m(X_j) \stackrel{\mathcal{D}(\bar{\mathbb{R}}\times[0,1]^d)}{\Longrightarrow} J_m(x)Z_m(t) \ . \tag{37}$$

The measurability of the empirical process is obtained if $\mathcal{D}(\bar{\mathbb{R}}\times[0,1]^d)$, endowed with the uniform topology, is equipped with the σ-field generated by the open balls. Finally (37) and (35) give the convergence claimed in the corollary.

The next corollary focuses on the non-isotropic random field of Proposition 3 based on Gaussian noise. Since this Proposition only gives the limit distribution of $d_N^{-1}\sum_{j\in A_{[Nt]}} X_j$, we restrict ourselves to functions G such that the Hermite rank of (34) is 1.

Corollary 3. *Let $(\epsilon_n)_{n\in\mathbb{Z}^d}$ be a stationary Gaussian field with a bounded spectral density. We consider the linear field*

$$X_n = \sum_{k\in\mathbb{Z}^d} a_k\epsilon_{n-k} \ , \tag{38}$$

where the (a_k)'s are, up to a normalisation providing $\mathrm{var}(X_1)=1$*, the Fourier coefficients of*

$$a(\lambda) = \left|\sum_{i=1}^d c_i\lambda_i\right|^{-\alpha} \ , \quad 0<\alpha<1/2 \ , \tag{39}$$

where $(c_1,\ldots,c_d)$ are real valued parameters.

We suppose that the Hermite rank of $\mathbb{1}_{\{G(X_n)\le x\}} - F(x)$ is 1.

If

$$0 < 2\alpha < \frac{1}{(d-2)\vee 1} \ , \tag{40}$$

then

$$\frac{1}{n^{d/2+\alpha}}\sum_{j\in A_{[nt]}}\left(\mathbb{1}_{\{G(X_j)\le x\}} - F(x)\right) \stackrel{\mathcal{D}(\bar{\mathbb{R}}\times[0,1]^d)}{\Longrightarrow} J_1(x)R(t) \ ,$$

where $J_1(x) = \mathbb{E}[\mathbb{1}_{\{G(X_1) \le x\}} X_1]$*, and where the convergence takes place in* $\mathcal{D}(\bar{\mathbb{R}} \times [0,1]^d)$ *endowed with the uniform topology and the* σ*-field generated by the open balls.*

When ϵ *is a white noise, the limiting field is defined by*

$$R(t) = \int_{\mathbb{R}^d} a(u) \prod_{j=1}^{d} \frac{\mathrm{e}^{\mathrm{i}t_j u_j} - 1}{\mathrm{i}u_j} \mathrm{d}W_0(u) \ ,$$

where W_0 *is the Gaussian white noise spectral field.*

Remark 12. As in Proposition 3, the condition (40) is not a restriction when $d \le 3$.

Proof (Sketch of proof). From (39), $d_n^2 \sim n^{d+2\alpha}$ when $n \to \infty$ and

$$\text{if } 0 < 2\alpha < 1/2 \ , \quad \sum_{\mathrm{j,k} \in \mathrm{A}_\mathrm{n}^2} \mathrm{r}^2(\mathrm{k}-\mathrm{j}) = \mathrm{O}(\mathrm{n}^\mathrm{d}) \ ,$$
$$\text{if } 1/2 < 2\alpha < 1 \ , \quad \sum_{\mathrm{j,k} \in \mathrm{A}_\mathrm{n}^2} \mathrm{r}^2(\mathrm{k}-\mathrm{j}) = \mathrm{O}(\mathrm{n}^{\mathrm{d}-1+4\alpha}) \ .$$

Therefore the upper bound in (35) tends to zero if δ is small enough. Since Proposition 3 and Proposition 5 prove the convergence of the partial sums of X in $\mathcal{D}([0,1]^d)$, the convergence of the empirical process follows.

Empirical process of long memory linear fields

Without the Gaussian assumption, a general uniform weak reduction principle as in Theorem 5 is not yet available. This has been done in [DLS02] in the particular case of the isotropic long memory linear random fields of Theorem 3. These authors obtain the convergence of the empirical process (34) for $t = 1$ and when G is the identity function.

Theorem 6 ([DLS02]). *Let* ϵ *be a zero-mean i.i.d random field with variance 1. Assume that there exist positive constants* C *and* δ *such that*

$$\left|\mathbb{E}\mathrm{e}^{\mathrm{i}a\epsilon_0}\right| \le C(1+|a|)^{-\delta} \ , \quad a \in \mathbb{R} \ ,$$

and

$$\mathbb{E}|\epsilon_0|^{2+\delta} < \infty \ .$$

Let X *be the linear field defined by*

$$X_n = \sum_{k \in \mathbb{Z}^d} a_k \epsilon_{n-k} \ , \quad n \in \mathbb{Z}^d$$

with

$$a_k = |k|^{-\alpha} b\left(\frac{k}{|k|}\right) \ , \quad k \in \mathbb{Z}^d \ ,$$

where $d/2 < \alpha < d$ and where b is continuous on the unit sphere in $\mathbb{R}^d$.

Then, with $Z \sim \mathcal{N}(0,1)$ a standard Gaussian variable,

$$\frac{1}{n^{3d/2-\alpha}} \sum_{k \in A_n} \left[\mathbb{1}_{\{X_k \leq x\}} - F(x)\right] \stackrel{\mathcal{D}(\overline{\mathbb{R}})}{\Longrightarrow} cf(x)Z \ ,$$

where c is a positive constant, F denoting the cumulative distribution function of X_1 and $f = F'$.

Remark 13. In [DLS02], the authors actually studied the convergence of the weighted empirical process

$$\sum_{k \in A_n} \gamma_{n,k} \mathbb{1}_{\{X_k \leq x + \xi_{n,k}\}} \ ,$$

where $\sup_n \max_{k \in A_n} (|\xi_{n,k}| + |\gamma_{n,k}|) = O(1)$. They obtain the same result.

4 Conclusion

All the above results confirm some specificities of the long memory compared with the short one : particularly a non standard normalisation and a degenerated limit for the empirical process. However, the study is far from being complete and should be extended for instance in the direction of seasonal phenomena, as it is done in dimension $d = 1$ ([OH02]), where the correct approximation of the empirical process might not be based on the first term of the Hermite expansion.

Finally, all results on the empirical process are a first step towards the study of U-statistics, Cramer Von Mises or Kolmogorov Smirnov statistics, and of M and L-statistics. They are the object of a current work.

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