A STATIONARY APPROACH TO INVERSE SCATTERING FOR SCHRODINGER OPERATORS WITH FIRST ORDER PERTURBATION

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Abstract

In this paper, we study the inverse scattering of Schrödinger operators with short-range (resp. long-range) electric and magnetic potentials. We develop a stationary approach to determine the high energy asymptotics of the scattering operator (resp. modified scattering operator). As a corollary, we show that the electric potential and the magnetic field are uniquely determined by the first two terms of this asymptotic expansion.

1 Introduction.

In quantum scattering theory, given a pair of Hamiltonians (H, H_0), where H is a perturbation of the free operator H_0, one of the main objectives is to show existence and asymptotic completeness of the Moeller wave operators in order to define the scattering operator S. Given this operator S, there is a very natural question :

can one determine and reconstruct the perturbation from the scattering operator ?

In the case of two-body Schrödinger Hamiltonians $H = H_0 + V$, $H_0 = -\Delta$ on $I\!R^n$, V short range, many papers give an affirmative answer. For a long-range potential V, this problem was solved by Isozaki-Kitada, ([8]), using stationary modified wave operators and microlocal techniques. In the N-body case, the resolution of this problem was given by Wang, ([16]), using high energy asymptotics.

Recently, Enss and Weder have proposed a geometrical method : they show that the potential is uniquely determined by the high velocity limit of the usual scattering operator and they give a reconstruction formula. This approach is rather simple and intuitive, and can be extended to the long-range case (Dollard potentials), to the N-body case, ([5]) and to Schrödinger operators with Stark effect ([17]).

In this paper, we use a different method to study Hamiltonians with electric and magnetic potentials. It is a stationary approach based on the construction of suitable modified wave operators; to do this, we use microlocal techniques to define Fourier integral operators.

A similar problem was studied by Shiota, ([15]), for the wave equation and for electric and magnetic potentials with compact supports. In ([6]), for exponentially decreasing potentials, Eskin-Ralston proved that the scattering amplitude at a fixed energy determines the electric potential and the magnetic field in all dimension greater than two. Using these results, they study in ([7]) this inverse scattering problem, in dimension greater than two, without the hypothesis of exponential decay. Our results in this paper are close to theirs, (for more details, see ([7])).

Finally, in ([1]), Arians studies the same problem : using the geometrical approach developed by Enss and Weder, he studies the high velocity limit of the scattering operator for a class of short-range magnetic fields, (singularities are allowed). His results are similar to our Theorems 2-3.

Notation

H denotes the quantum Schrödinger operator, considered as a perturbation of $H_0 = -\frac{1}{2}\Delta$ on $I\!R^n$, $n \geq 2$, which describes the interaction of a charged particle in electric and magnetic fields :

$$(1.1) \qquad H \;=\; \frac{1}{2}\sum_{j=1}^{n}(D_j - A_j(x))^2 + V(x) \;,$$

where

$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$.

$V(x)$ is the electric potential.

$A \;= \sum_{j=1}^{n} A_j \; dx_j$ is the magnetic potential.

$B = dA$ is the magnetic field identified with the antisymmetric matrix (b_{jk}), $b_{jk}(x) = \partial_{x_j} A_k(x) - \partial_{x_k} A_j(x)$, in the canonical basis of $I\!R^n$.

Let us recall some well-known results. Suppose $A, V \in C^\infty(I\!R^n)$ and satisfy : $\forall \alpha \in I\!N^n$,

$$(H_1) \qquad | \partial_x^\alpha V(x) | \leq C_\alpha < x >^{-\delta-|\alpha|} \quad , \quad \delta > 0 \ ,$$

$$(H_2) \qquad | \partial_x^\alpha A(x) | \leq C_\alpha < x >^{-\rho-|\alpha|} \quad , \quad \rho > 0 \ ,$$

where $< x > = (1 + || x ||^2)^{\frac{1}{2}}$.

When $\delta, \rho > 1$, H is a classical short-range perturbation of the Laplace operator H_0, and in particular the Moeller wave operators

$$(1.2) \qquad W^\pm \ = \ s - \lim_{t \to \pm\infty} e^{itH} \ e^{-itH_0} \quad \text{exist and are complete.}$$

It is very important to remark that a magnetic field with compact support in $I\!R^2$ and with nonzero flux generates a magnetic potential which cannot decay faster than the Coulomb potential ; nevertheless, intuitively, such magnetic fields are of short range. For Schrödinger operators with magnetic fields, the short-range result is given by the following theorem, using a special gauge. This result was proven by Perry, for magnetic fields with compact support, ([13]), and by Loss and Thaller for the general case, ([10]).

Theorem 1 : the short-range case.
Suppose (H_1), (H_2) with $\delta > 1$, $\rho > \frac{1}{2}$. Suppose in addition that A is the transversal gauge :

$$(H_3) \qquad A(x).x = 0 \qquad \forall x \in I\!R^n.$$

Then the wave operators $W^\pm$ exist and are complete.

Remark
(i) This gauge determines uniquely the potential A. More precisely,

$$(1.3) \qquad A(x) \ = \ - \int_0^1 s \ B(sx).x \ ds \ .$$

In particular, the hypotheses (H_2) and (H_3) can be satisfied under the following condition :

$$(1.4) \qquad | \, \partial_x^\alpha B(x) \, | \leq \; C_\alpha \; < x >^{-1-\rho-|\alpha|} \; ,$$

where $\rho > \frac{1}{2}$.

(ii) Theorem 2 was improved by Enss in the case of long-range electric potentials ([4]). Also, one can find in ([12]) a stationary proof of Theorem 1, and in ([11]), one studies the scattering matrices for the pair (H, H_0).

In this paper, we shall give the asymptotic expansion of the scattering operator at high energies ; then, we shall show that the S-operator determines the electric and magnetic potentials. Finally, in the last section, we study the long-range case, using suitable modified wave operators ; we obtain the same results as in the short-range case.

2 The short-range case.

2.1 Results.

Let us define :

$$(2.1) \qquad \mathcal{V}_{SR} \; = \; \{ \; V \;\; \text{satisfying } (H_1) \text{ with } \delta > 1 \; \} \;\; .$$

$$(2.2) \qquad \mathcal{A}_{SR} \; = \; \{ \; A \;\; \text{satisfying } (H_2) \;\; \text{with } \rho > \frac{1}{2} \; , \; \text{and} \;\; (H_3) \; \} \;\; .$$

Using Theorem 1, we can define the scattering operator $S = W^{+*}W^- = S(A, V)$. Let $\mathcal{L}(L^2(I\!R^n))$ be the set of bounded operators on $L^2(I\!R^n)$.

The main result of this section is :

Theorem 2
The map $S(.,.) : \mathcal{A}_{SR} \times \mathcal{V}_{SR} \to \mathcal{L}(L^2(I\!R^n))$ *is injective.*

Remark
Theorem 2 is false without the hypothesis (H_3) in the definition of $\mathcal{A}_{SR}$, because changing the gauge does not change the scattering matrix.

To show Theorem 2, we will see that the first two terms of the high energy asymptotics of the scattering operator S permit to reconstruct the potentials A and V.

Notation

As in ([11]), we use the maps :

$$c_A^{\pm}(x,\xi) \ = \ -\int_0^{\pm\infty} \ A(x+t\xi).\xi \ dt \quad , \tag{2.3}$$

The above maps are well-defined since one has by (2.3) the following relation :

$$A(x+t\xi)\ .\ \xi \ = \ -A(x+t\xi)\ .\ \frac{x}{t} \quad , \quad \forall t \neq 0 \ . \tag{2.4}$$

An easy calculation gives :

$$\partial_x \ c_A^{\pm}(x,\xi) \ = \ A(x) + R_{\pm}(x,\xi) \quad , \tag{2.5}$$

where

$$R_{\pm}(x,\xi) \ = \ -\int_0^{\pm\infty} \ B(x+t\xi).\xi \ dt \quad . \tag{2.6}$$

The main property of $R_{\pm}(x,\xi)$ is the geometrical relation :

$$\forall \ (x,\xi) \in I\!R^{2n} \ , \ R_{\pm}(x,\xi).\xi \ = \ 0 \quad , \tag{2.7}$$

since B is antisymmetric. Finally, we define

$$c_A(x,\xi) \ = \ c_A^{-}(x,\xi) - c_A^{+}(x,\xi) \ = \ \int_{-\infty}^{+\infty} \ A(x+t\xi).\xi \ dt, \tag{2.8}$$

$$f^{\pm}(x,\xi) \ = \frac{1}{2}\ (R_{\pm}^2(x,\xi) - i \ \mathrm{div} \ R_{\pm}(x,\xi)) \quad , \tag{2.9}$$

where div $f = \sum \partial_{x_j} f_j$, if $f = (f_1, \ ... \ , \ f_n)$.

Now, let us recall some basic definitions for pseudo-differential operators :

Definition

(i) For $m, p \in I\!R$, we define :

$$S(m,p) \ = \ \{a(x,\xi) \in C^{\infty}(I\!R^n_x \times I\!R^n_\xi), \ , \ \forall \alpha,\beta \in I\!N^n, \ \exists \ C_{\alpha,\beta} > 0, \ \forall (x,\xi) \in I\!R^{2n},$$
$$| \ \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \ | \leq \ C_{\alpha,\beta} \ < x >^{(m-|\alpha|)+} < \xi >^{p-|\beta|} \}.$$

(ii) For $a \in S(m,p)$, $\Phi \in \mathcal{S}(I\!R^n)$, let

$$Op\ (a)\ \Phi(x) \ = \ (2\pi)^{-n} \int \ e^{ix.\xi} \ a(x,\xi) \ \hat{\Phi}(\xi) \ d\xi \quad ,$$

where $\hat{\Phi}$ is the usual Fourier transform of Φ.

Now, we can give the high energy asymptotics of the scattering operator. Let $<,>$ the usual scalar product in $L^2(I\!R^n)$, $\omega \in S^{n-1}$, (S^{n-1} is the unit sphere of $I\!R^n$), and Φ, Ψ are the Fourier transforms of functions in $C_0^\infty(I\!R^n)$.

Theorem 3
Under the hypotheses of Theorem 1, we have the asymptotic expansion for $\lambda \to +\infty$ *:*

$$(2.10)\quad < e^{i\sqrt{\lambda}x.\omega}\Phi \ , \ (S(A,V)-1)\ e^{i\sqrt{\lambda}x.\omega}\Psi > \sim \sum_{j=0}^{+\infty} \lambda^{-\frac{j}{2}} \ < \Phi, Op\,(a_{j,\omega}) \ \Psi >$$

where $a_{j,\omega} \in S(j,j)$. *In particular,*

$$(2.11)\qquad a_{0,\omega}(x) \ = \ e^{ic_A(x,\omega)} - 1 \ ,$$

$$(2.12)\qquad a_{1,\omega}(x,\xi) \ = \ -i \ e^{ic_A(x,\omega)} \left(\int_{-\infty}^{+\infty} V(x+t\omega)\ dt \ + a_{B,\omega}(x,\xi) \right) \ ,$$

where

$$(2.13)\qquad a_{B,\omega}(x,\xi) \ = \ \int_0^{+\infty} \overline{f}^{+}(x+t\omega,\omega)\ dt + \int_{-\infty}^{0} f^{-}(x+t\omega,\omega)\ dt$$

$$-i \sum_{k=1}^{n} \ \partial^2_{x_k\xi_k} c_A^{+}(x,\omega) + \partial_\xi c_A^{+}(x,\omega).\partial_x c_A(x,\omega) - \partial_\xi c_A(x,\omega).\xi \ .$$

The above theorem allows to reconstruct all the potentials and to etablish Theorem 2.

In particular, when $A \equiv 0, \ V \in \mathcal{V}_{SR}$, we recover Enss-Weder's result ([5]) :

$$(2.14)\qquad < \ e^{i\sqrt{\lambda}x.\omega}\Phi \ , \ (S(0,V)-1)\ e^{i\sqrt{\lambda}x.\omega}\Psi \ > =$$

$$i\ \lambda^{-\frac{1}{2}} \ < \ \Phi \ , \ (\int_{-\infty}^{+\infty} V(x+t\omega)\ dt \ )\ \Psi \ > + O(\lambda^{-1}) \ .$$

2.2 Proof of Theorem 2.

2.2.1 Reconstruction of the magnetic potential.

Let $V_1, V_2 \in \mathcal{V}_{SR}$ and $A_1, A_2 \in \mathcal{A}_{SR}$ such that $S(A_1,V_1) = S(A_2,V_2)$.
By Theorem 3, $\forall \Phi, \ \Psi$ such that $\widehat{\Phi} \ , \ \widehat{\Psi} \in C_0^\infty(I\!R^n)$, we have :

$$(2.15)\qquad < \ \Phi, (e^{ic_{A_1}(x,\omega)} - e^{ic_{A_2}(x,\omega)})\Psi \ > = \ 0 \ .$$

By density of states Φ, Ψ, since the maps $x \to c_{A_i}(x,\omega)$ are smooth, we deduce

$$(2.16) \qquad \forall x \in I\!R^n \ , \ \forall \omega \in S^{n-1} \ , \ e^{ic_A(x,\omega)} - 1 \ = \ 0 \ ,$$

where $A = A_1 - A_2$.
So, $\forall \omega \in S^{n-1}, \forall x \in I\!R^n$, $c_A(x,\omega) = 2k(x,\omega)\pi$ where $k(x,\omega) \in Z\!\!Z$. By the standard continuity argument, $k(x,\omega) = k(0,\omega)$ and using (H_3), we obtain $k(0,\omega) = 0$.

Uniqueness of the magnetic potential comes from the next lemma :

Lemma 4
Let A be a function satisfying (H_2), (H_3) such that

$$(2.17) \qquad \forall \omega \in S^{n-1} \ , \ \forall x \in I\!R^n \ , \ \int_{-\infty}^{+\infty} A(x+t\omega).\omega \ dt \ = \ 0 \ .$$

Then : $A \equiv 0$.

Proof
First, we observe that

$$(2.18) \ \partial_x \left(\int_{-\infty}^{+\infty} A(x+t\omega) \ . \ \omega \ dt \right) \ = \ \int_{-\infty}^{+\infty} B(x+t\omega) \ . \ \omega \ dt \ = \ 0 \ .$$

Differentiating the second integral, we obtain :

$$(2.19) \qquad \forall \alpha \in I\!N^n \ , \ \int_{-\infty}^{+\infty} \partial_x^\alpha B(x+t\omega) \ . \ \omega \ dt \ = \ 0 \ .$$

Now, let us introduce new coordinates in $I\!R^n$:
for fixed $\omega \in S^{n-1}$, we write $x \in I\!R^n$ as : $x = t\omega + x'$ where $t \in I\!R$, $x' \in \Pi_\omega$ = orthogonal hyperplane to ω.

Using (H_2), for $|\alpha| \geq n-1$, $\partial_x^\alpha B(x) \in L^1(I\!R^n)$. So, for $\xi' \in \Pi_\omega$,

$$\begin{aligned} \widehat{\partial_x^\alpha B}(\xi').\omega \ &= \ \int_{\mathbf{R}^n} e^{-ix.\xi'} \ \partial_x^\alpha B(x).\omega \ dx \\ &= \int_{\Pi_\omega} e^{-ix'.\xi'} \ \left(\int_{-\infty}^{+\infty} \partial_x^\alpha B(x'+t\omega).\omega \ dt \right) \ dx' \ , \end{aligned}$$

so, by (2.19),

$$\forall \omega \in S^{n-1} \ , \ \forall \xi' \in \Pi_\omega \ , \ \widehat{\partial_x^\alpha B}(\xi').\omega = 0 \ ,$$

or equivalently,

$$\forall \xi \neq 0 \ , \ \forall \omega \in \Pi_\xi \ , \ \widehat{\partial_x^\alpha B}(\xi).\omega = 0 \ .$$

Thus, for fixed $\xi \neq 0$, $\widehat{\partial_x^\alpha B}(\xi)$ is an antisymmetric matrix vanishing on a hyperplane.
This easily implies : $\forall \xi \neq 0$, $\widehat{\partial_x^\alpha B}(\xi) = 0$. By injectivity of Fourier transform on $L^1(I\!R^n)$, one has : $\forall \alpha \in I\!N^n$, $|\alpha| \geq n-1$, $\partial_x^\alpha B \equiv 0$.
So B is polynomial and goes to zero at infinity, ; thus $B \equiv 0$, and by (1.3), we obtain the lemma. $\square$

2.2.2 Reconstruction of the electric potential.

In the last section, we showed

$$(2.20) \qquad S(A_1, V_1) \ = \ S(A_2, V_2) \ \Rightarrow \ A_1 = A_2 \ .$$

Using Theorem 3 again, since $a_{B,\omega}(x,\xi)$ depends only on magnetic field, it suffices to prove the following lemma :

Lemma 5
Let V be a function satisfying (H_1) such that

$$(2.21) \qquad \forall \omega \in S^{n-1} \ , \ \forall x \in I\!R^n \ , \ \int_{-\infty}^{+\infty} V(x+t\omega)\, dt \ = \ 0 \ .$$

Then : $V \equiv 0$.

Proof
We follow the same strategy as in the preceding lemma :

$$(2.22) \ \forall \alpha \in I\!N^n, \forall \omega \in S^{n-1} \ , \ \forall x \in I\!R^n \ , \ \int_{-\infty}^{+\infty} \partial_x^\alpha V(x+t\omega)\, dt \ = \ 0 \ .$$

As in Lemma 4, $\forall \ |\alpha| \geq n$, $\forall \xi' \in \Pi_\omega$, $\widehat{\partial_x^\alpha V}(\xi') = 0$. Varying ω, we have $\widehat{\partial_x^\alpha V} \equiv 0$, so $V \equiv 0$ as in Lemma 4. $\square$

Remark
In ([5]), to show uniqueness of the electric potential, (under less restrictive hypotheses), Enss and Weder give a different proof using the inversion of the Radon transform.

2.3 Asymptotic expansion for the $S-$operator at high energies.

Notation and method

- Φ, Ψ are the Fourier transforms of functions in $C_0^\infty(I\!R^n)$.

- $\omega \in S^{n-1}$ is fixed.

To determine the asymptotic expansion at high energies of

$$(2.23) \qquad F(\lambda) \ = < \ e^{i\sqrt{\lambda}x.\omega}\Phi \ , \ (S(A,V)-1) \ e^{i\sqrt{\lambda}x.\omega}\Psi \ > \ ,$$

it is natural to introduce two operators :

$$(2.24) \qquad \Omega^{\pm}(\lambda,\omega) \ = \ e^{-i\sqrt{\lambda}x.\omega} \ W^{\pm} \ e^{i\sqrt{\lambda}x.\omega} \ ,$$

since

$$(2.25) \qquad F(\lambda) \ = < \Omega^{+}(\lambda,\omega) \ \Phi, \ \Omega^{-}(\lambda,\omega) \ \Psi > - < \Phi, \ \Psi > \ .$$

Obviously,

$$(2.26) \qquad \Omega^{\pm}(\lambda,\omega) \ = \ s - \lim_{t\to\pm\infty} \ e^{itH(\lambda,\omega)} \ e^{-itH_0(\lambda,\omega)} \ ,$$

where

(2.27)

$$H(\lambda,\omega) \ = \ \frac{1}{2}(D+\sqrt{\lambda}\omega - A(x))^2 + V(x) \ , \ H_0(\lambda,\omega) \ = \ \frac{1}{2}(D+\sqrt{\lambda}\omega)^2 \ .$$

Our stationary approach is rather simple and close to Isozaki - Kitada's method ([8]). We construct two Fourier integral operators, (F.I.O), $J_N^{\pm}(\lambda,\omega)$, called energy modifier, (see [14] for definition of Fourier integral operators), such that for $\lambda >> 1$,

$$(2.28) \qquad \Omega^{\pm}(\lambda,\omega) \ \Phi = \lim_{t\to\pm\infty} \ e^{itH(\lambda,\omega)} \ J_N^{\pm}(\lambda,\omega) \ e^{-itH_0(\lambda,\omega)} \ \Phi \ ,$$

$$(2.29) \qquad || \ (\Omega^{\pm}(\lambda,\omega) - J_N^{\pm}(\lambda,\omega))\Phi \ || \ = \ O(\lambda^{-\frac{N}{2}}) \ .$$

So, using (2.25), we shall obtain :

$$F(\lambda) \ = < J_N^{+}(\lambda,\omega)\Phi, \ J_N^{-}(\lambda,\omega)\Psi > - < \Phi \ , \ \Psi > \ + O \ (\lambda^{-\frac{N}{2}}) \ ,$$

and thus, we can calculate the asymptotic expansion easily.

2.3.1 Construction of the phase of the energy modifier.

In the first time, we have to recall some auxiliary results.

Notation

$$\chi \in C^\infty(IR^n) \quad , \quad \chi(x) = 0 \ \text{ for } \ | x | \leq \frac{1}{2} \quad , \quad \chi(x) = 1 \ \text{ for } \ | x | \geq 1 \ ,$$

$$\begin{aligned} \Psi_+ \in C^\infty([-1,1]) \quad , \quad & \Psi_+(\sigma) = 1 \ \text{ if } \sigma \in [\sigma_+, 1] \ , \\ & \Psi_+(\sigma) = 0 \ \text{ if } \ \sigma \in [-1, \frac{\sigma_- + \sigma_+}{2}] \ , \\ \Psi_- \in C^\infty([-1,1]) \quad , \quad & \Psi_-(\sigma) = 1 \ \text{ if } \sigma \in [-1, \sigma_-] \ , \\ & \Psi_-(\sigma) = 0 \ \text{ if } \ \sigma \in [\frac{\sigma_- + \sigma_+}{2}, 1] \ , \end{aligned}$$

for some $-1 < \sigma_- < \sigma_+ < 1$.

In ([11]), we define modifiers $J^\pm$, (F. I. O), with amplitude 1 and phase

$$(2.30) \qquad \varphi^\pm(x,\xi) \ = \ x.\xi + c_A^\pm(x,\xi) \ \Psi_\pm(\cos(x,\xi)) \ \chi(\frac{x}{R}) \ \chi(\frac{\xi}{\theta})$$

where $R, \theta > 0$, and $\cos(x,\xi) \ = \ \dfrac{x.\xi}{| x || \xi |}$.

Remark
For $(x,\xi) \in \Gamma^\pm = \Gamma^\pm(R,\theta,\sigma)$, it is clear that

$$\varphi^\pm(x,\xi) \ = \ x.\xi + c_A^\pm(x,\xi) \quad ,$$

where

$$\Gamma^\pm \ = \ \{ \ (x,\xi) \in IR^{2n} \ : | \ x \ | \geq R \ , | \ \xi \ | \geq \theta \ , \pm \ x.\xi \geq \pm \ \sigma_\pm \ | \ x \ | \ | \ \xi \ | \ \} \ .$$

(Γ^- is called an incoming zone and Γ^+ an outgoing zone).

We have the following result :

Lemma 6
Let $a = \theta^2/2$ and $E_{H_0}(I)$ be the spectral projection on I, $I \subset IR$. Then :

$$(2.31) \qquad W^\pm \ E_{H_0}(a,\infty) \ = \ s - \lim_{t \to \pm\infty} \ e^{itH} \ J^\pm \ e^{-itH_0} \ .$$

Proof
We only sketch the proof, (for more details see ([12])). To prove (2.31), it suffices to establish that for Φ in some dense subset of $Ran \ E_{H_0}(a,\infty)$, one has

$$(2.32) \qquad \lim_{t \to \pm\infty} (J^\pm - 1) \ e^{-itH_0}\Phi \ = 0 \quad .$$

We introduce

$$(2.33)\qquad L_{jk}(x,\xi) = x_j\xi_k - x_k\xi_j \quad ,$$

$$(2.34)\qquad a_{jk}(x) \ = \ -\int_0^1 \ s b_{jk}(sx)\ ds \quad .$$

Using (1.3), it is easy to see that

$$(2.35)\qquad c_A^{\pm}(x,\xi) \ = \ -\frac{1}{2}\sum_{j,k} \int_0^{\pm\infty} \ a_{jk}(x+t\xi)\ dt \quad L_{jk}(x,\xi)$$

So, for $(x,\xi) \in \Gamma^{\pm}$ such that $\sum\limits_{j,k} L_{jk}^2(x,\xi) \leq M$, one has

$$(2.3\Lambda 6)\qquad \mid \varphi^{\pm}(x,\xi) - x.\xi \mid \leq \ C < x >^{-min(1,\rho)}$$

Let $f(x,\xi) = \eta_1(\sum\limits_{j,k} L_{jk}^2(x,\xi))\ \eta_2(\xi^2)$ where $\eta_1 \in C_0^{\infty}([0,M])$, $\eta_2 \in C_0^{\infty}([\theta^2,\infty[)$.
We define $F = Op\ (f)$. Using (2.36), we can prove that $(J^{\pm}-1)F$ is a pseudo-differential operator with symbol $s(x,\xi)$ which goes to zero as $\mid x \mid + \mid \xi \mid$ goes to infinity. Hence $(J^{\pm}-1)F$ is a compact operator.
Thus, for Φ such that $F\Phi = \Phi$ for some F, since F commutes with e^{-itH_0} and $e^{-itH_0}\Phi$ tends weakly to zero, (2.32) is proven. Since the Φ's form a dense subset of $Ran\ E_{H_0}(a,\infty)$, we obtain the lemma. □

Lemma 6 suggests to construct $J_N^{\pm}(\lambda,\omega)$ close to $e^{-i\sqrt{\lambda}x.\omega}\ J^{\pm}\ e^{i\sqrt{\lambda}x.\omega}$.
In incoming (outgoing) zone, the phase of $e^{-i\sqrt{\lambda}x.\omega}\ J^{\pm}\ e^{i\sqrt{\lambda}x.\omega}$ is given by :

$$x.\xi + c_A^{\pm}(x,\xi+\sqrt{\lambda}\omega).$$

In order to obtain the asymptotic expansion of the scattering operator, we have to define a phase without a significant cut-off. Since $Supp\ \widehat{\Phi}$ is bounded, the basic idea is to develop $c_A^{\pm}(x,\xi+\sqrt{\lambda}\omega)$ near $\sqrt{\lambda}\omega$. Using Taylor expansion and 0-homogeneity of the maps $\xi \to c_A^{\pm}(x,\xi)$, one has :

$$(2.37)\quad c_A^{\pm}(x,\xi+\sqrt{\lambda}\omega) \ = \ \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!}\,\lambda^{-\frac{|\alpha|}{2}}\ \xi^{\alpha}\ \partial_{\xi}^{\alpha} c_A^{\pm}(x,\omega) + g_N^{\pm}(x,\xi,\lambda,\omega)\ .$$

Now, we can define the phase of the F.I.O. $J_N^{\pm}(\lambda,\omega)$, for $\lambda >> 1$.

Definition of the phase

$$(2.38) \qquad \varphi_N^{\pm}(x,\xi,\lambda,\omega) \;=\; x.\xi + \sum_{|\alpha|\le N-1} \frac{1}{\alpha!}\,\lambda^{-\frac{|\alpha|}{2}}\xi^{\alpha}\;\partial_\xi^\alpha c_A^{\pm}(x,\omega)$$
$$+\; r_N^{\pm}(x,\xi,\lambda,\omega)\;,$$

where

$$(2.39) \qquad r_N^{\pm}(x,\xi,\lambda,\omega) \;=\; g_N^{\pm}(x,\xi,\lambda,\omega)\;\Psi_{\pm}(cos(x,\xi+\sqrt{\lambda}\omega))\;\chi(\frac{x}{R}).$$

Remark
By construction, it is clear that, for suitable σ :

$$(2.40)\;\; \varphi_N^{\pm}(x,\xi,\lambda,\omega) \;=\; x.\xi + c_A^{\pm}(x,\xi+\sqrt{\lambda}\omega) \;\text{ if }\; (x,\xi+\sqrt{\lambda}\omega)\in\Gamma^{\pm}(R,\theta,\sigma)$$

2.3.2 Construction of the amplitude of the energy modifier.

Following the construction of $\varphi_N^{\pm}(x,\xi,\lambda,\omega)$, we look for the amplitude in the form :

$$(2.41) \qquad e_N^{\pm}(x,\xi,\lambda,\omega) \;=\; \sum_{m=0}^{N-1} \lambda^{-\frac{m}{2}}\; d_m^{\pm}(x,\xi,\omega)\;\chi_1(\xi)\;,$$

with $d_0^{\pm}\equiv 1$, and $\chi_1 \in C_0^\infty(I\!R^n)$.

Assuming (2.28) , we have :

$$(2.42)\;\; (\Omega^{\pm}(\lambda,\omega)-J_N^{\pm}(\lambda,\omega))\Phi = i\int_0^{\pm\infty} e^{itH(\lambda,\omega)}\;T_N^{\pm}(\lambda,\omega)\,e^{-itH_0(\lambda,\omega)}\;\Phi\;\;dt\;,$$

where

$$(2.43) \qquad T_N^{\pm}(\lambda,\omega) \;=\; H(\lambda,\omega)\;J_N^{\pm}(\lambda,\omega) \;-\; J_N^{\pm}(\lambda,\omega)\;H_0(\lambda,\omega)\;.$$

It is easy to verify that $T_N^{\pm}(\lambda,\omega)$ is a F.I.O with phase $\varphi_N^{\pm}(x,\xi,\lambda,\omega)$ and with amplitude $c_N^{\pm}(x,\xi,\lambda,\omega)$, where :

$$(2.44) \qquad c_N^{\pm}(x,\xi,\lambda,\omega) \;=\; \frac{1}{2}\;\Big\{\;[\;|\;\nabla\varphi_N^{\pm}(x,\xi,\lambda,\omega)+\sqrt{\lambda}\omega - A(x)\;|^2$$

$$+2V(x) - \ (\xi + \sqrt{\lambda}\omega)^2 - i\ \Delta\varphi_N^{\pm}(x,\xi,\lambda,\omega) + i \text{ div } A(x)\ ]\ e_N^{\pm}(x,\xi,\lambda,\omega)$$
$$-2i\ [\ \nabla\varphi_N^{\pm}(x,\xi,\lambda,\omega)+\sqrt{\lambda}\omega - A(x)\ ]\ .\ \nabla e_N^{\pm}(x,\xi,\lambda,\omega)\ -\Delta e_N^{\pm}(x,\xi,\lambda,\omega)\ \Big\}\ \chi_1(\xi).$$

where ∇, Δ, div are x-derivatives.

In order to satisfy (2.29), we are going to make $c_N^{\pm}$ small.

We treat only the case $(+)$, the case $(-)$ is similar. Using (2.3) – (2.9), and (2.40), we obtain easily if $(x, \xi + \sqrt{\lambda}\omega) \in \Gamma^+(R,\theta,\sigma)$:

$$(2.45) \qquad c_N^+(x,\xi,\lambda,\omega) \ = \ \Big\{\ [f^+(x,\xi+\sqrt{\lambda}\omega) + V(x)\ ]\ e_N^+(x,\xi,\lambda,\omega)$$
$$-i\ (\xi + \sqrt{\lambda}\omega + R_+(x, \xi+\sqrt{\lambda}\omega))\ .\ \nabla e_N^+(x,\xi,\lambda,\omega)$$
$$-\frac{1}{2}\ \Delta e_N^+(x,\xi,\lambda,\omega)\ \Big\}\ \chi_1(\xi)\ .$$

It is important to remark that $f^+(x,\xi)$ is a short range perturbation on Γ^+. Using Taylor expansion again, and the fact that $\xi \to f^+(x,\xi)$, and $\xi \to R_+(x,\xi)$ are 0-homogeneous, one has :

$$(2.46)\ \ f^+(x,\xi+\sqrt{\lambda}\omega) \ = \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!}\ \lambda^{-\frac{|\alpha|}{2}}\ \xi^\alpha\ \partial_\xi^\alpha f^+(x,\omega) \ + \ f_N^+(x,\xi,\lambda,\omega)$$

$$(2.47) \qquad R_+(x,\xi+\sqrt{\lambda}\omega) \ = \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!}\ \lambda^{-\frac{|\alpha|}{2}}\ \xi^\alpha\ \partial_\xi^\alpha R_+(x,\omega)$$
$$+\ R_N^+(x,\xi,\lambda,\omega)\ .$$

So, in order to prove (2.29), substituting (2.46), (2.47) into (2.45), we obtain transport equations :

For $p \geq 0$,

$$(2.48) \qquad \omega.\nabla d_{p+1} \ = \ -i\ V(x)\ d_p^+ - i \sum_{|\alpha|+m=p} \frac{1}{\alpha!}\ \ \xi^\alpha\ \partial_\xi^\alpha f^+(x,\omega)\ d_m^+$$
$$- \sum_{|\alpha|+m=p} \frac{1}{\alpha!}\ \xi^\alpha\ \partial_\xi^\alpha R^+(x,\omega)\ .\ \nabla d_m^+ + \frac{i}{2}\ \Delta d_p^+ - \xi\ .\ \nabla d_p^+\ ,$$

where d_p^+ is written for $d_p^+(x,\xi,\omega)$.

It is easy to solve these equations. In particular, $d_1^+(x,\xi,\omega)$ does not depend on ξ, and is given by :

$$(2.49) \qquad d_1^+(x,\omega) \ = \ i\ \int_0^{+\infty}\ [\ f^+(x+t\omega,\omega) + V(x+t\omega)\ ]\ dt$$

We need some estimates on $d_p^+(x,\xi,\omega)$:

For $(x,\xi+\sqrt{\lambda}\omega)\in\Gamma^+(R,\theta,\sigma)$ and $\xi\in Supp\ \chi_1$, it is easy to verify that for $\lambda>>1$, $\cos(x,\omega)\geq -1+\epsilon$ for some $\epsilon>0$, $\lambda-$independent. So, one has for such (x,ξ) :

$$(2.50)\qquad \forall p\geq 1,\ \mid\ \partial_x^\alpha\partial_\xi^\beta d_p^+(x,\omega,\xi)\ \mid\leq\ C_{p\alpha}\ <x>^{1-\mu-|\alpha|}<\xi>^{p-1-|\beta|}$$

where $\mu=min\ (\delta,\rho+1,2\rho)>1$.

Remark
We will obtain in the same way $d_p^-(x,\xi,\omega)$ integrating over $(-\infty,0)$ with same estimates if $(x,\xi+\sqrt{\lambda}\omega)\in\Gamma^-(R,\theta,\sigma)$, and replacing $(+)$ by $(-)$.

Notation
Let $J_N^\pm(\lambda,\omega)$ the F.I.O with phase $\varphi_N^\pm(x,\xi,\lambda,\omega)$ given by (2.38) $-$ (2.39) and with amplitude $e_N^\pm(x,\xi,\lambda,\omega)$ given by (2.41) with $d_m^\pm(x,\xi,\omega)$ satisfying transport equations.

The properties of $J_N^\pm(\lambda,\omega)$ constructed above are summarized in the following two lemmas :

Lemma 7
Let $N\geq 1$. $J_N^\pm(\lambda,\omega)$ is a F.I.O with :
1 - a phase $\varphi_N^\pm(x,\xi,\lambda,\omega)$ such that :

$$(i)\ \varphi_N^\pm(x,\xi,\lambda,\omega)\ =\ x.\xi+c_A^\pm(x,\xi+\sqrt{\lambda}\omega)\ \ if\ \ (x,\xi+\sqrt{\lambda}\omega)\in\Gamma^\pm(R,\theta,\sigma)\ .$$

$$(ii)\qquad \forall\mid\alpha\mid\geq 1\ ,\ \exists C_{\alpha N}>0\ s.t\ \forall\ (x,\xi)\in I\!R^{2n},\forall\lambda>>1\ ,$$

$$\mid\partial_x^\alpha\ [\ \varphi_N^\pm(x,\xi,\lambda,\omega)-(x.\xi+c_A^\pm(x,\xi+\sqrt{\lambda}\omega))\ ]\ \mid\leq\ C_{\alpha,N}\ \lambda^{-\frac{N}{2}}\ ,$$

and
2 - an amplitude $e_N^\pm(x,\xi,\lambda,\omega)$ such that

$$(i)\ \forall L>0,\ \forall\alpha,\beta,\ \exists C_{\alpha\beta NL}>0,\ \forall\lambda>>1,\ \forall\ (x,\xi+\sqrt{\lambda}\omega)\in\Gamma^\pm(R,\theta,\sigma)\ ,$$

$$\mid\partial_x^\alpha\partial_\xi^\beta\ (e_N^\pm(x,\xi,\omega,\lambda)-\chi_1(\xi))\mid\leq\ C_{\alpha\beta NL}\ <x>^{1-\mu-|\alpha|}<\xi>^{-L}\ \lambda^{-\frac{1}{2}}$$

$$(ii)\ \ \forall L>0,\ \forall\alpha,\beta,\ \exists C_{\alpha\beta NL}>0,\ \forall\lambda>>1,\ \forall(x,\xi)\in I\!R^{2n}\ ,$$

$$\mid\partial_x^\alpha\partial_\xi^\beta e_N^\pm(x,\xi,\omega,\lambda)\mid\leq\ C_{\alpha\beta NL}\ <\xi>^{-L}\ .$$

Lemma 8
Let $N \geq 1$. $T_N^{\pm}(\lambda,\omega) = H(\lambda,\omega)J_N^{\pm}(\lambda,\omega) - J_N^{\pm}(\lambda,\omega)H_0(\lambda,\omega)$ *is a F.I.O with phase* $\varphi_N^{\pm}(x,\xi,\lambda,\omega)$ *and amplitude* $c_N^{\pm}(x,\xi,\lambda,\omega)$ *such that :*

(i) $\forall L > 0,\ \forall \alpha, \beta,\ \exists C_{\alpha\beta NL} > 0, \forall \lambda >> 1,\quad \forall (x, \xi + \sqrt{\lambda}\omega) \in \Gamma^{\pm}(R,\theta,\sigma),$

$$| \ \partial_x^{\alpha} \partial_{\xi}^{\beta} c_N^{\pm}(x,\xi,\omega,\lambda) \ | \leq \ C_{\alpha\beta NL} \ < x >^{-\mu-|\alpha|} \ \lambda^{-\frac{N-1}{2}} \ < \xi >^{-L} \ .$$

(ii) $\forall L > 0,\ \forall \alpha, \beta,\ \exists C_{\alpha\beta NL} > 0, \forall \lambda >> 1, \forall (x,\xi) \in I\!R^{2n},$

$$| \ \partial_x^{\alpha} \partial_{\xi}^{\beta} c_N^{\pm}(x,\xi,\omega,\lambda) \ | \leq \ C_{\alpha\beta NL} \ < \xi >^{-L} \ \lambda^{-\frac{N-1}{2}} \ .$$

Now we can prove (2.28), (2.29).

Notation
Φ is the Fourier transform of a function in $C_0^{\infty}(I\!R^n)$ and we choose $\chi_1 \in C_0^{\infty}(I\!R^n)$ such that $\chi_1 \equiv 1$ on $Supp\ \widehat{\Phi}$.

Proposition 9
For $\lambda >> 1$,

$$(2.51) \qquad \Omega^{\pm}(\lambda,\omega)\ \Phi \ = \ \lim_{t \to \pm\infty} \ e^{itH(\lambda,\omega)} \ J_N^{\pm}(\lambda,\omega) \ e^{-itH_0(\lambda,\omega)} \ \Phi \ .$$

Proof
To simplify, we treat only the case $(+)$. Let $K_N^{+}(\lambda,\omega) = e^{i\sqrt{\lambda}x.\omega} \ J_N^{+}(\lambda,\omega) \ e^{-i\sqrt{\lambda}x.\omega}$. Obviously,

$$(2.52) \ \ e^{itH(\lambda,\omega)} \ J_N^{\pm}(\lambda,\omega) \ e^{-itH_0(\lambda,\omega)} \ \Phi = e^{-i\sqrt{\lambda}x.\omega} \ e^{itH} \ K_N^{+}(\lambda,\omega) \ e^{-itH_0} \ e^{i\sqrt{\lambda}x.\omega}\Phi \ .$$

When $t \to +\infty$, it is well-known it suffices to investigate $K_N^{+}(\lambda,\omega)$ on outgoing zones. By construction and by Lemma 7, $K_N^{+}(\lambda,\omega)$ is a F. I. O with phase $x.\xi + c_A^{+}(x,\xi)$ on Γ^{+}, and with amplitude $1 + O(< x >^{1-\mu})$, $\mu > 1$ on Γ^{+}. Hence, using the same arguments as in (2.32), we obtain

$$\lim_{t \to +\infty} e^{itH} \ K_N^{+}(\lambda,\omega) \, e^{-itH_0} \ e^{i\sqrt{\lambda}x.\omega}\Phi \ = \ \lim_{t \to +\infty} e^{itH} \ e^{-itH_0} \ e^{i\sqrt{\lambda}x.\omega}\Phi \ = \ W^{+}e^{i\sqrt{\lambda}x.\omega}\Phi \ .$$

$\square$

Proposition 10
For $\lambda \to +\infty$, one has the estimate :

$$(2.53) \qquad || \ (\Omega^{\pm}(\lambda,\omega) - J_N^{\pm}(\lambda,\omega))\Phi \ || \ = \ O(\lambda^{-\frac{N}{2}}) \ .$$

Proof
As usual, we treat the case (+) and we suppose $\lambda >> 1$. Using (2.42),

$$|| \ (\Omega^{+}(\lambda,\omega) - J_N^{+}(\lambda,\omega))\Phi \ || \ \leq \ \int_0^{+\infty} \ || \ T_N^{+}(\lambda,\omega) \ e^{-itH_0(\lambda,\omega)} \ \Phi \ || \ \ dt \ .$$

It is very easy to see that, (see Lemma 8),

$$(2.54) \qquad T_N^{+}(\lambda,\omega) \ e^{-itH_0(\lambda,\omega)} \ \Phi \ = \ e^{-it\lambda} \ \ e^{-i\sqrt{\lambda}\omega.D} \ U_N^{+}(t,\lambda,\omega) \ e^{-itH_0}\Phi \ ,$$

where $D=-i\nabla$ and $U_N^{+}(t,\lambda,\omega)$ is a F.I.O with amplitude $c_N^{+}(x+t\sqrt{\lambda}\omega,\xi,\lambda,\omega)$ and with phase

$$(2.55) \qquad \Psi_N^{+}(t,x,\xi,\lambda,\omega) \ = \ x.\xi + \theta_N^{+}(x+t\sqrt{\lambda}\omega,\xi,\lambda,\omega) \ ,$$

where

$$\theta_N^{+}(x,\xi,\lambda,\omega) = \varphi_N^{+}(x,\xi,\lambda,\omega) - x.\xi \ ,$$

So,

$$(2.56) \ \ || \ (\Omega^{+}(\lambda,\omega) - J_N^{\pm}(\lambda,\omega))\Phi \ || \leq \ \int_0^{+\infty} \ || \ U_N^{+}(t,\lambda,\omega) \, e^{-itH_0} \ \Phi \ || \ \ dt \ .$$

In order to estimate the integrand, one introduces a cut-off :

$$\chi \in C_0^{\infty}(I\!R^n) \ \text{ such that } \ \chi(x) = 1 \text{ if } \ | \ x \ | \leq \frac{1}{2} \ \ , \ \ \chi(x) = 0 \text{ if } \ | \ x \ | \geq 1,$$

$$\zeta = 1-\chi \ , \ g \in C_0^{\infty}(I\!R^n) \ \ s.t \ \ g \equiv 1 \text{ on } Supp \ \widehat{\Phi} \ .$$

One has :

$$|| \ U_N^{+}(t,\lambda,\omega) \ e^{-itH_0} \ \Phi \ || \leq || \ U_N^{+}(t,\lambda,\omega) \ e^{-itH_0} g(D)\zeta(\frac{8x}{t\sqrt{\lambda}}) \ \Phi \ ||$$

$$+ \ || \ U_N^{+}(t,\lambda,\omega) \ e^{-itH_0} g(D)\chi(\frac{8x}{t\sqrt{\lambda}}) \ \Phi \ || \ \ .$$

Denote by $g_1(t)+g_2(t)$ the right hand side of this inequality.

• Step 1 :
Using Lemma 8 and continuity of F.I.O, ([14]), we have

$$|| \ U_N^{+}(t,\lambda,\omega) \ e^{-itH_0} \ g(D) \ || \leq \ C_N \ \lambda^{-\frac{N-1}{2}} \ .$$

Since on $Supp\ \zeta$, $\mid x \mid \geq \frac{t\sqrt{\lambda}}{16}$, and $\Phi \in S(I\!R^n)$,

$$\forall L > 0 \ , \quad \| \ \zeta(\frac{8x}{t\sqrt{\lambda}}) \ \Phi \ \| \leq \ C_L \ < \sqrt{\lambda} t >^{-L} \quad ,$$

then $g_1(t) \ \leq \ C_{NL} \ \lambda^{-\frac{N-1}{2}} < \sqrt{\lambda} t >^{-L}$.

• Step 2 :
We have

$$g_2(t) \ \leq \| \ \chi(\frac{2x}{t\sqrt{\lambda}}) U_N^+(t,\lambda,\omega) \ e^{-itH_0} g(D) \chi(\frac{8x}{t\sqrt{\lambda}}) \ \Phi \ \|$$

$$+ \| \ \zeta(\frac{2x}{t\sqrt{\lambda}}) U_N^+(t,\lambda,\omega) \ e^{-itH_0} g(D) \chi(\frac{8x}{t\sqrt{\lambda}}) \ \Phi \ \| \ ,$$

denoted $h_1(t) + h_2(t)$.

• Step 2a :
On $Supp\ \chi(\frac{2x}{t\sqrt{\lambda}})$, $\mid x \mid \leq \frac{t\sqrt{\lambda}}{2}$. So, it is easy to verify for $t \geq 0$,

$$(i) \qquad\qquad (x + t\sqrt{\lambda}\omega, \xi + \sqrt{\lambda}\omega) \ \in \Gamma^+(R,\theta,\sigma) \ ,$$

$$(ii) \qquad\qquad \mid x + t\sqrt{\lambda}\omega \mid \geq \frac{t\sqrt{\lambda}}{2} \ ,$$

so by Lemma 8 and continuity of F.I.O, one has

$$\| \ \chi(\frac{2x}{t\sqrt{\lambda}}) U_N^+(t,\lambda,\omega) \ g(D) \ \| \leq C_N \ \lambda^{-\frac{N-1}{2}} \ < t\sqrt{\lambda} >^{-\mu} \ ,$$

and likewise for $h_1(t)$.

• Step 2b :
This term describes the free propagation into the classical forbidden region, ([3]), and one can evaluate this contribution by using a standard non-stationary phase estimate :

$$(2.57) \qquad \zeta(\frac{2x}{t\sqrt{\lambda}}) U_N^+(t,\lambda,\omega) \ e^{-itH_0} g(D) \chi(\frac{8x}{t\sqrt{\lambda}}) \ \Phi(x) \ =$$

$$(2\pi)^{-n} \int\!\!\int \ e^{i[\Psi_n^+(t,x,\xi,\lambda,\omega) - y\xi - \frac{t}{2}\xi^2]} \ \chi(\frac{8y}{t\sqrt{\lambda}}) \ \zeta(\frac{2x}{t\sqrt{\lambda}})$$

$$c_N^+(x + t\sqrt{\omega}, \xi, \lambda, \omega) \ g(\xi) \Phi(y) \ dy \ d\xi$$

In order to investigate possible critical points, we calculate :

$$\partial_\xi \ [\Psi_n^+(t,x,\xi,\lambda,\omega) - y\xi - \frac{t}{2}\xi^2] \ = \ x - y - t\xi +$$

$$\partial_\xi \ \left(\sum_{1 \leq |\alpha| \leq N-1} \lambda^{-\frac{|\alpha|}{2}} \ \xi^\alpha \ \partial_\xi^\alpha c_A^+(x + t\sqrt{\lambda}\omega, \omega) + r_N^+(x + t\sqrt{\lambda}\omega, \xi, \lambda, \omega) \right)$$

Using the properties :

$$(i) \qquad \forall \mid \alpha \mid \geq 1, \ \mid \partial_\xi^\alpha c_A^+(x,\xi) \mid \leq \ C_\alpha \ < x > , \ \forall (x,\xi) \in I\!R^{2n} \ ,$$

$$(ii) \qquad \mid x \mid \geq \frac{t\sqrt{\lambda}}{4} \ \ , \ \ \mid \xi \mid \leq C \ ,$$

we easily show :

$$\mid \partial_\xi \ [\Psi_n^+(t,x,\xi,\lambda,\omega) - y\xi - \frac{t}{2}\xi^2] \ \mid = \ x - y - t\xi + O(\lambda^{-\frac{1}{2}} \mid x \mid) + O(\mid x \mid^{1-\rho} \lambda^{-\frac{N}{2}}).$$

Since $\mid y \mid \leq \frac{t\sqrt{\lambda}}{8}$, $\exists c > 0$ such that

$$\mid \partial_\xi \ [\Psi_n^+(t,x,\xi,\lambda,\omega) - y\xi - \frac{t}{2}\xi^2] \ \mid \geq c \ \sqrt{\lambda} \ t \ ,$$

so, we conclude by a standard argument of non-stationary phase that $\forall L > 0$,

$$h_2(t) \ \leq \ C_L \ \lambda^{-L} \ < t >^{-L} .$$

Integrating over $(0, +\infty)$ and using (2.56), we obtain the lemma. $\square$

2.3.3 High energy asymptotics of the scattering operator.

By Proposition 10 and (2.25), one has :

$$(2.58) \qquad F(\lambda) \ = < J_N^+(\lambda,\omega)\Phi, \ J_N^-(\lambda,\omega)\Psi > - < \Phi, \Psi > + O(\lambda^{-\frac{N}{2}}) \ .$$

First, we give the asymptotic expansion of $J_N^\pm(\lambda,\omega)\Phi$ at high energies.

Lemma 11
For $\lambda \to +\infty$, $\exists \ a_{m,\omega}^\pm \in S(m,m)$,

$$J_N^\pm(\lambda,\omega)\Phi \ = \ e^{ic_A^\pm(x,\omega)} \ \sum_{m=0}^{N-1} \lambda^{-\frac{m}{2}} \ Op \ (a_{m,\omega}^\pm)\Phi + O(\lambda^{-\frac{N}{2}}) \ .$$

In particular,

$$a_{0,\omega}^\pm \equiv 1 \quad , \quad a_{1,\omega}^\pm(x,\xi) = i\xi.\partial_\xi c_A^\pm(x,\omega) + d_1^\pm(x,\omega).$$

Proof
We only consider the case $(+)$. One has :

$$(2.59)\ J_N^+(\lambda,\omega)\Phi(x) \ = \ (2\pi)^{-n} \sum_{m=0}^{N-1} \lambda^{-\frac{m}{2}} \int e^{i\varphi_N^+(x,\xi,\lambda,\omega)}\, d_m^+(x,\xi,\omega)\ \hat{\Phi}(\xi)\ d\xi\ ,$$

$$= \ e^{ic_A^+(x,\omega)}\,(2\pi)^{-n} \sum_{m=0}^{N-1} \lambda^{-\frac{m}{2}} \int exp \left(i\ [x.\xi + \sum_{1\leq|\alpha|\leq N-1} \frac{1}{\alpha!}\ \lambda^{-\frac{|\alpha|}{2}} \xi^\alpha \partial_\xi^\alpha c_A^+(x,\omega) \right.$$
$$\left. + r_N^+(x,\xi,\lambda,\omega)] \right)\ d_m^+(x,\xi,\omega)\ \hat{\Phi}(\xi)\ d\xi\ .$$

So, using Taylor expansion of e^t near $t=0$, we obtain the lemma. □

Applications :

By (2.59) and Lemma 11, one has :

$$F(\lambda) \ = < e^{ic_A^+(x,\omega)} \sum_{m=0}^{N-1} \lambda^{-\frac{m}{2}}\ Op\ (a_{m,\omega}^+)\ \Phi,\ e^{ic_A^-(x,\omega)} \sum_{p=0}^{N-1} \lambda^{-\frac{p}{2}}\ Op\ (a_{p,\omega}^-)\ \Psi >$$
$$- < \Phi, \Psi > + O(\lambda^{-\frac{N}{2}})\ .$$

So,

$$F(\lambda) \ = < \Phi, (e^{ic_A(x,\omega)} - 1)\Psi >$$
$$+ \sum_{j=1}^{N-1}\ \lambda^{-\frac{j}{2}}\ < \Phi, \sum_{m+p=j} [Op^*(a_{m,\omega}^+)\ e^{ic_A(x,\omega)}\ Op\ (a_{p,\omega}^-)]\ \Psi > + O(\lambda^{-\frac{N}{2}})\ .$$

Using symbolic calculus, we obtain for $j \geq 1$,

$$\sum_{m+p=j} [Op^*\ (a_{m,\omega}^+)\ e^{ic_A(x,\omega)}\ Op\ (a_{p,\omega}^-)] \ = \ Op\ (a_{j,\omega})\ .$$

In particular, we have :

$$a_{1,\omega}(x,\xi) \ = \ -i\ e^{ic_A(x,\omega)} \left(\int_{-\infty}^{+\infty}\ V(x+t\omega)\ dt\ \ + a_{B,\omega}(x,\xi) \right)$$

where

$$(2.60) \qquad a_{B,\omega}(x,\xi) \ = \ \int_0^{+\infty}\ \overline{f}^+(x+t\omega,\omega)\ dt + \int_{-\infty}^0\ f^-(x+t\omega,\omega)\ dt$$
$$-i \sum_{k=1}^n\ \partial_{x_k\xi_k}^2 c_A^+(x,\omega) + \partial_\xi c_A^+(x,\omega).\partial_x c_A(x,\omega) - \partial_\xi c_A(x,\omega).\xi$$

Thus, Theorem 3 is proven. □

3 The long-range case.

In this section, we generalize to the long-range case the results obtained in the preceding section for short-range interactions.
We consider generic long-range potentials A and V, (i.e $\delta > 0,\ \rho > 0$), satisfying (H_3). Obviously, under such hypotheses, in general, the usual wave operators do not exist. So, we have to define modified wave operators.
For $A \equiv 0$ and $\delta > \frac{1}{2}$, we can use the well-known Dollard wave operators. In ([5]), using such a modification, Enss and Weder give the leading term of the high energy asymptotics of the modified Dollard scattering operator, and prove that this term determines uniquely the potential V.
Since we consider general long-range interactions, we have to define other modified wave operators. We choose time-independent modifiers, close to Isozaki-Kitada's method ([8]). Briefly, the basic idea is to construct an appropriate approximate of the phase $\varphi(x,\xi)$ which is solution of the eikonal equation :

$$(3.1) \qquad \frac{1}{2} \mid \partial_x \varphi(x,\xi) - A(x) \mid^2 + V(x) \ = \ \frac{1}{2}\xi^2 \ .$$

In fact, we can solve explicitly (3.1) using Hamilton-Jacobi theory, but this phase is rather complicated and difficult to control.

Our plan in this section is as follows : first, we construct modified wave operators, then we give the main results, and finally we prove the theorems.

3.1 Construction of modified wave operators.

In this section, we define modified wave operators, with time-independent modifiers. We construct two Fourier integral operators, (F.I.O), $J_p^{\pm}$, $p \in I\!N$, with phase $\varphi_p^{\pm}(x,\xi)$ and amplitude 1, such that

$$(3.2) \qquad W_p^{\pm} \ = \ s - \lim_{t \to \pm\infty} \ e^{itH} J_p^{\pm} e^{-itH_0} \quad ,$$

exist and are complete. This approach is based on Isozaki-Kitada's idea, ([8]), for Schrödinger operators with long-range electric potential, (see also [11], [12] for Schrödinger operators with long-range magnetic potentials).
When A, V are short range, (in the sense of Theorem 1), we shall see that $W_p^{\pm}$ coincides with $W^{\pm}$ up to an energy phase.

3.1.1 Definition of the modifier.

To define the F.I.O $J_p^{\pm}$, we construct two phases $\varphi_p^{\pm}(x,\xi)$ such that

$$(3.3) \qquad \frac{1}{2} \mid \partial_x \varphi_p^{\pm}(x,\xi) - A(x) \mid^2 + V(x) \ \approx \ \frac{1}{2}\xi^2$$

modulo a short-range perturbation on Γ^- (incoming zone) and Γ^+ (outgoing zone). To solve (3.3), we use an induction. First, we define on $\Gamma^\pm$:

$$\varphi_0^\pm(x,\xi) \ = \ x.\xi + c_A^\pm(x,\xi) \ , \tag{3.4}$$

where $c_A^\pm(x,\xi)$ is given by (2.3). Using (2.3) – (2.7), we obtain

$$\frac{1}{2} \mid \partial_x \varphi_0^\pm(x,\xi) - A(x) \mid^2 + V(x) \ = \ \frac{1}{2}\xi^2 \ + q_0^\pm(x,\xi) \ , \tag{3.5}$$

where

$$q_0^\pm(x,\xi) \ = \ \frac{1}{2} R_\pm^2(x,\xi) + V(x) \tag{3.6}$$

Thus, on $\Gamma^\pm$, $\xi \to q_0^\pm(x,\xi)$ is 0-homogeneous and satisfies for $\mu_1 = min\ (2\rho, \delta)$,

$$\forall \alpha, \beta \in I\!N^n \ , \ \mid \partial_x^\alpha \partial_\xi^\beta q_0^\pm(x,\xi) \mid \leq \ C_{\alpha,\beta} \ < x >^{-\mu_1 - |\alpha|} < \xi >^{-|\beta|} \ . \tag{3.7}$$

Now, we can begin the inductive process. For $p \geq 1$, let

$$\varphi_p^\pm(x,\xi) \ = \ \varphi_{p-1}^\pm(x,\xi) + \int_0^{\pm\infty} [q_{p-1}^\pm(x+t\xi,\xi) - q_{p-1}^\pm(t\xi,\xi)] \ dt \ , \tag{3.8}$$

so

$$\frac{1}{2} \mid \partial_x \varphi_p^\pm(x,\xi) - A(x) \mid^2 + V(x) \ = \ \frac{1}{2}\xi^2 \ + q_p^\pm(x,\xi) \ , \tag{3.9}$$

with

$$q_p^\pm(x,\xi) \ = \ \sum_{j=p}^{u_p} q_{p,j}^\pm(x,\xi) \tag{3.10}$$

where (u_p) is defined by $u_0 = 0$ and $u_p = 2(u_{p-1}+1)$.

It is easy to see that $\xi \to q_{p,j}^\pm(x,\xi)$ is $-j$-homogeneous and for $(x,\xi) \in \Gamma^\pm$, one has the estimation for $\mu_2 = min\ (\rho, \delta)$, and for all $\alpha, \beta \in I\!N^n$,

$$\mid \partial_x^\alpha \partial_\xi^\beta q_p^\pm(x,\xi) \mid \leq \ C_{\alpha,\beta} \ < x >^{-(\mu_1 + p\mu_2) - |\alpha|} < \xi >^{-p-|\beta|} \ . \tag{3.11}$$

So, for $p \in I\!N$ such that $\mu_1 + p\mu_2 > 1$, the assertion (3.3) holds.
In particular, if $\delta > 1, \ \rho > \frac{1}{2}$, (hypotheses of Theorem 1), we can take $p = 0$.

Remark
We can define in the same way $\varphi_p^-(x,\xi)$, replacing $(+)$ by $(-)$ and integrating over $(-\infty, 0)$ in the above definitions.

Definition
(i) For $p \in I\!N$ such that $\mu_1 + p\mu_2 > 1$, let

$$(3.12) \quad \Phi_p^{\pm}(x,\xi) \ = \ x.\xi + \left(\varphi_p^{\pm}(x,\xi) - x.\xi\right) \ \Psi_{\pm}(\cos(x,\xi)) \ \chi(\frac{x}{R}) \ \chi(\frac{\xi}{\theta}) \ .$$

where $\Psi_{\pm}$, χ are cut-off functions defined in section 2.
(ii) Let $J_p^{\pm}$ = F.I.O with phase $\Phi_p^{\pm}(x,\xi)$ and with amplitude 1.

3.1.2 The modified wave operators.

In order to state the next theorem, we recall the notation :
$a = \theta^2/2$, $E_{H_0}([a,\infty[)$, (resp. $E_H([a,\infty[)$), is the spectral projection for H_0, (resp. H), on $[a,\infty[$ and $\mathcal{H}_{ac}(H)$ is the absolutely continuous subspace for H.

Theorem 12 : the long-range case.
Assume that A, V satisfy $(H_1) - (H_3)$. Let $\mu_1 = min\ (2\rho,\delta)$, $\mu_2 = min\ (\rho,\delta)$. Then,

$\forall p \in I\!N$ such that $\mu_1 + p\mu_2 > 1$, the modified wave operators :

$$(3.13) \qquad W_p^{\pm} \ = \ s - \lim_{t\to\pm\infty} \ e^{itH} \ J_p^{\pm} \ e^{-itH_0} \ E_{H_0}([a,\infty[) \ ,$$

exist and are complete, i.e Ran $W_p^{\pm} = E_H[a,\infty[\ \mathcal{H}_{ac}(H)$.

Proof
We only sketch the proof and we consider the case (+), (for more details, see [8], [12]).
Existence of the modified wave operators is easy to obtain using the well-known Cook's argument; one has to show that for Ψ in some dense subset in $L^2(I\!R^n)$,

$$(HJ_p^+ - J_p^+ H_0) \ e^{-itH_0} \ \Psi \in L^1([1,+\infty[, \ dt) \ .$$

By construction, $(HJ_p^+ - J_p^+ H_0)$ is a F.I.O with phase $\Phi_p^+(x,\xi)$ and with amplitude given on Γ^+ by :

$$q_p^+(x,\xi) - \frac{i}{2} \ [\Delta_x \Phi_p^+(x,\xi) - \ \mathrm{div} \ A(x)] \ ,$$

which is a short range perturbation by (3.9) – (3.11).
Using Isozaki-Kitada's argument, ([8]), we show completeness in the same way.
□

In order to see that these modified wave operators are a suitable generalization of the ordinary ones, in the next corollary, we compare $W_p^{\pm}$ with $W^{\pm}$ in the short range case, and when $A \equiv 0, \delta > \frac{1}{2}$, we compare $W_p^{\pm}$ with the well-known Dollard modified wave operators defined by :

$$(3.14) \qquad W_D^{\pm} \ = \ s - \lim_{t \to \pm\infty} \ e^{itH} \ e^{-i(tH_0 + \int_0^t \ V(sD) \ ds)}$$

where H is written for $H_0 + V(x)$ and $D = -i\nabla$. It is well known that $W_D^{\pm}$ exist and are complete, ([2]). Finally, we give the relation between $W_{p_1}^{\pm}$ and $W_{p_2}^{\pm}$ for $p_1 \neq p_2$.

Corollary 13
(i) When $\delta > 1, \ \rho > \frac{1}{2}$, one has :

$$\forall \Psi \in \ Ran \ E_{H_0}[a, \infty[\quad , \quad W_0^{\pm}\Psi \ = \ W^{\pm}\Psi$$

(ii) When $A \equiv 0$ and $\delta > \frac{1}{2}$, one has :

$$\forall \Psi \in \ Ran \ E_{H_0}[a, \infty[\quad , \quad W_1^{\pm}\Psi \ = \ W_D^{\pm}\Psi$$

(iii) For $p_2 > p_1$ such that $\mu_1 + p_1\mu_2 > 1$, $\exists \ g_{p_2,p_1}^{\pm} \in C^{\infty}(I\!R^n \backslash B(0,\theta))$,

$$W_{p_2}^{\pm} \ = \ W_{p_1}^{\pm} \ e^{-ig_{p_2,p_1}^{\pm}(D)}$$

In particular, if $\delta > 1$ and $\rho > \frac{1}{2}$, $\forall \mid \xi \mid \geq \theta$,

$$g_{1,0}^{\pm}(\xi) \ = \ \int_0^{\pm\infty} \left(V(t\xi) + \frac{1}{2} R_{\pm}^2(t\xi, \xi) \right) \ dt$$

Proof
(i) Under these hypotheses, one can take $p = 0$. So the result comes from Lemma 6.

(ii) To compare the Dollard modified wave operators with $W_1^{\pm}$, it suffices to establish : $\forall \Psi \in L^2(I\!R^n)$ such that $\widehat{\Psi} \in C_0^{\infty}([a, \infty[)$,

$$(3.15) \qquad W_D^{\pm *} \ W_1^{\pm}\Psi \ = \ \Psi$$

or equivalently,

$$(3.16) \qquad e^{i\int_0^t \ V(sD) \ ds} \ e^{itH_0} \ J_1^{\pm} \ e^{-itH_0} \ \Psi \ = \ \Psi \ + o(1) \ , \ t \to \pm\infty \ ,$$

We only sketch the proof ; for more details, see ([9]). As usual, we consider the case $(+)$.
It is easy to see that $e^{itH_0}\ J_1^{\pm}\ e^{-itH_0}$ is a F.I.O with amplitude 1 and with phase $\varphi^+(t,x,\xi)$ given on Γ^+ by :

$$\begin{aligned}\varphi^+(t,x,\xi) &= x.\xi + \int_0^{+\infty} (V(x+t\xi+s\xi) - V(s\xi))\ ds \\ &= x.\xi + \int_t^{+\infty} [V(x+s\xi) - V(s\xi)]\ ds - \int_0^t V(s\xi)\ ds \\ &= x.\xi - \int_0^t V(s\xi)\ ds + o(1)\ \ ,\ \ t \to +\infty\end{aligned}$$

So, using the same arguments as in ([9]), one obtains (3.16).

(iii) This assertion is obvious, since on Γ^+, $\exists\ g^+_{p_1,p_2} \in C^\infty(I\!R^n \backslash B(0,\theta))$ such that :

$$\varphi^+_{p_2}(x,\xi) - \varphi^+_{p_1}(x,\xi) = g^+_{p_1,p_2}(\xi) + O\left(< x >^{1-(\mu_1+p_1\mu_2)}\right) . \qquad \square$$

3.2 Results.

Let us define

$$(3.17) \qquad \mathcal{V}_{LR}(\delta) = \{V \text{ satisfying } (H_1) \text{ with } \delta > 0\ \}$$

$$(3.18) \qquad \mathcal{A}_{LR}(\rho) = \{A \text{ satisfying } (H_2) \text{ with } \rho > 0, \text{ and } (H_3)\ \}$$

Using Theorem 12, we can define the modified scattering operator $S_p = S_p(A,V) = W_p^{+*} W_p^-$. The main result of this section is :

Theorem 14
Let $\delta, \rho > 0$, $p \in I\!N$ such that $\mu_1 + p\mu_2 > 1$ with the notations of Theorem 12. Then, the map $S_p(.,.) : \mathcal{A}_{LR}(\rho) \times \mathcal{V}_{LR}(\delta) \to \mathcal{L}(L^2(I\!R^n))$ is injective.

This theorem contains the short-range case, $(p = 0)$ which is proven in section 2. For the general long-range case, $(p \geq 1)$, to show Theorem 14, as in the short-range case, we determine the asymptotic expansion at high energies of the modified scattering operator S_p, and we will see that the first two terms of this asymptotic expansion determine the potentials A, V.

Now, we can give the high energy asymptotics of the scattering operator, with the same notation as in the short-range case.

Theorem 15
Let $A \in \mathcal{A}_{LR}(\rho)$, $V \in \mathcal{V}_{LR}(\delta)$, for $p \geq 1$ such that $\mu_1 + p\mu_2 > 1$. We have the asymptotic expansion for $\lambda \to +\infty$:

$$(3.19) \quad < e^{i\sqrt{\lambda}x.\omega}\Phi \ , \ (S_p(A,V)-1)\, e^{i\sqrt{\lambda}x.\omega}\Psi > \sim \sum_{j=0}^{+\infty} \lambda^{-\frac{j}{2}} \ < \Phi, Op\ (a_{j,\omega,p})\ \Psi >$$

where $a_{j,\omega,p} \in S(j,j)$. In particular, $a_{0,\omega,p}$, $a_{1,\omega,p}$ do not depend on p and are given by :

$$(3.20) \qquad a_{0,\omega,p}(x) \ = \ e^{ic_A(x,\omega)} - 1 \ ,$$

$$(3.21) \qquad a_{1,\omega,p}(x,\xi) \ = \ -i\ e^{ic_A(x,\omega)} \left(\int_{-\infty}^{+\infty} [V(x+t\omega) - V(t\omega)]\ dt \right. \\ \left. + b_{B,\omega}(x,\xi) \ \right)$$

where

$$(3.22) \qquad b_{B,\omega}(x,\xi) = -i \sum_{k=1}^{n} \ \partial^2_{x_k\xi_k} c_A^+(x,\omega) + \partial_\xi c_A^+(x,\omega).\partial_x c_A(x,\omega) \\ + \frac{i}{2} \left(\int_0^{+\infty} \text{div}\ R_+(x+t\omega,\omega)\ dt - \int_{-\infty}^{0} \text{div}\ R_-(x+t\omega,\omega)\ dt \right) \\ + \frac{1}{2} \int_0^{+\infty} [R_+^2(x+t\omega,\omega) - R_+^2(t\omega,\omega)]\ dt - \partial_\xi c_A(x,\omega).\xi \\ \left. + \frac{1}{2} \int_{-\infty}^{0} [R_-^2(x+t\omega,\omega) - R_-^2(t\omega,\omega)]\ dt \right)$$

Remarks
(i) For $A \equiv 0$, $\delta > \frac{1}{2}$, (Dollard potentials), using Corollary 13 (ii), we recover the results obtained by Enss-Weder ([5]).

(ii) In fact, Theorem 15 gives also the asymptotic expansion for the short-range case, $(p = 0)$, because using Corollary 12 (iii), one has far from the 0-energy :

$$S(A,V) \ = e^{i\int_0^{+\infty} \ V(tD)+\frac{1}{2}R_+^2(tD,D)\ dt} S_1(A,V)\ e^{i\int_{-\infty}^{0} \ V(tD)+\frac{1}{2}R_-^2(tD,D)\ dt}$$

Since these phases are -1-homogeneous, one recovers the results obtained in section 2.

3.3 Proof of theorems.

3.3.1 Proof of Theorem 14.

Let $V_1,\ V_2 \in \mathcal{V}_{LR}(\delta)$ and $A_1, A_2 \in \mathcal{A}_{LR}(\rho)$ such that $S_p(A_1, V_1) = S_p(A_2, V_2)$ for a suitable fixed p. By Theorem 14, since the leading term in the asymptotic expansion is the same as in the short-range case, we have $A_1 = A_2$ by Lemma 4.
Using Theorem 14 again and the same strategy as in Lemma 5, we obtain $V_1 = V_2$. $\Box$

3.3.2 Proof of Theorem 15.

Since the phase of the F.I.O $J_p^{\pm}$ is the sum of homogeneous functions, (see $(3.8)-(3.10)$), everything done in section 2 works for the sequence of modified operators. We leave the details to the reader. $\Box$

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