# Inverse scattering for Stark Hamiltonians with short-range potentials.

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**Abstract**

We consider the Stark Hamiltonian $H = \frac{1}{2}\ p^2 - x_1 + V(x)$ which describes the scattering of a quantum mechanical particle in $I\!R^n$ by a short-range potential in the presence of a constant electric field. We show that the electric potential $V$ is uniquely determined by the high energy limit of the scattering operator, if the dimension $n \geq 3$. We prove our results using the Enss-Weder's time-dependent method.

# 1 Introduction.

We study a short-range quantum mechanical scattering in the presence of a constant electric field. For the sake of simplicity, we assume that the electric field acts in the $e_1$-direction, where $e_1 = (1, 0, ..., 0) \in I\!R^n,\ n \geq 2$.

The corresponding Stark Hamiltonian defined on $L^2(I\!R^n)$ is given by :

$$H_0 \ = \ \frac{1}{2}\ p^2 - x_1\ , \tag{1.1}$$

where $p = -i\nabla$. It is well-known that $H_0$ is essentially self-adjoint on $C_0^\infty(I\!R^n)$, (see [11] for example). We denote also by $H_0$ the self-adjoint realization with domain $D(H_0)$.

Let us recall the Avron-Herbst formula [2] which describes the free time evolution :

$$e^{-itH_0} = e^{-i\frac{t^3}{6}}\ e^{itx_1}\ e^{-i\frac{t^2}{2}p_1}\ e^{-i\frac{t}{2}p^2}\ , \tag{1.2}$$

where $p_1$ is the first component of $p = (p_1, p')$. Equation (1.2) shows that, up to a phase factor, the evolution $e^{-itH_0}$ consists of a translation by $\frac{t^2}{2}$ units to the right in the $e_1$-direction, followed by the usual free time evolution without electric field.

Now, let $H$ be a second Hamiltonian considered as a perturbation of $H_0$ : $H = H_0 + V(x)$. We assume that $V$ is a short-range potential, i.e $V \in C^\infty(I\!R^n)$ and it satisfies $\forall \alpha \in I\!N^n$ :

$$(H_1) \qquad \mid \partial_x^\alpha V(x) \mid \leq C_\alpha < x >^{-\rho-|\alpha|}\ ,\ \rho > \frac{1}{2}\ ,$$

where $< x > = (1+x^2)^{\frac{1}{2}}$. Actually, let us remark that in our paper we only need $(H_1)$ for $\alpha$ with finite order, (for example $\mid \alpha \mid \leq n$). It is well-known that under the asumption $(H_1)$, $H$ is essentially self-adjoint with domain $D(H) = D(H_0)$. Moreover, $H$ has no eigenvalues [2] and $\sigma_{sc}\ (H) = \emptyset$, where $\sigma_{sc}\ (H)$ is the singular continuous spectrum of $H$, [10].

Under the asumption $(H_1)$ the wave operators :

$$W^{\pm}\ =\ s - \lim_{t \to \pm\infty}\ e^{itH}\ e^{-itH_0} \tag{1.3}$$

exist and are complete, (i.e $Ran\ W^{\pm} = \mathcal{H}^{(ac)}(H)$, the later being the subspace of absolute continuity of $H$), [2]. Actually, we can prove existence and completeness of $W^{\pm}$ with weaker asumptions on the potential [2], but we need $(H_1)$ to solve the inverse scattering problem.

We denote $S = S(V) = W^{+*}W^{-}$ the scattering operator. The inverse scattering problem consists of reconstructing the perturbation $V$ from the scattering operator.

In this paper, we prove that in dimensions $n \geq 3$, the $S$-operator determines uniquely the potential $V$. More precisely, it suffices to know the high energy limit of $S$, (cf Proposition 2). We need $n \geq 3$ in order to use the inversion of the Radon transform (or the Fourier transform) on a hyperplane, (see section 2.3 for details).

Our main result is :

**Theorem 1**
*Let $V_1,\ V_2$ be potentials satisfying $(H_1)$ and assume that $n \geq 3$. Then :*

$$S(V_1) = S(V_2) \iff V_1 = V_2\ .$$

A similar problem has been studied by Weder [12] by a time-dependent method; he obtained the same result for $n \geq 2$ (with weaker conditions on the derivatives of $V$), but he needed a stronger decay condition on $V$ : he assumed $\rho > \frac{3}{4}$. In the last section, we prove that our method allows to recover this result.

## 2 Proof of Theorem 1.

In this section, we study the high energy limit of the scattering operator using the Enss-Weder's time-dependent method : see [5] where they study the case of two-body Schrödinger Hamiltonians $H = \frac{1}{2}\, p^2 + V$ on $L^2(I\!R^n)$. This method can be used to study Hamiltonians with electric and magnetic potentials on $L^2(I\!R^n)$ [1], the Dirac equation [7], the N-body case [5] and the Aharonov-Bohm effect [13].

In [8], [9], a stationary approach is proposed to solve scattering inverse problems for Schrödinger operators with magnetic fields or with the Aharonov-Bohm effect. Unfortunately, for the Stark effect, this method is not easily applicable.

The method proposed below is very close to [5], [12].The main steps in the proof are :

a) we define an auxiliary wave operator $\Omega^{\pm}$ (a Dollard's modified wave operator) which coincides with $W^{\pm}$ up to an energy phase.

b) we study the high energy asymptotics of $S$ (by means of $\Omega^{\pm}$) in a direction orthogonal to $e_1$.

c) using the high energy asymptotics of $S$ and the fact that $dim\ \Pi_{e_1} \geq 2$, where $\Pi_{e_1}$ is the orthogonal hyperplane to $e_1$, we solve the inverse scattering problem.

### 2.1 Construction of an auxiliary wave operator.

In this section, we construct a modified wave operator $\Omega^{\pm}$ which is close to the canonical one $W^{\pm}$. The advantage of using $\Omega^{\pm}$ unstead of $W^{\pm}$ is that $\Omega^{\pm}$ admits sharper estimations when the energy goes to infinity, (see Lemma 3 below, and [12] Lemma 2.2, Corollary 2.3).

First, let us define a free-modified dynamic $U_D(t)$ by :

$$U_D(t) = e^{-itH_0}\ e^{-i\int_0^t\ V(sp'+\frac{1}{2}s^2 e_1)\ ds}\ . \tag{2.1}$$

This dynamic is close to the dynamic introduced in [14] for the long-range case. Note that we take $p'$ unstead of $p$ in the integral since $H_0$ commutes with $p'$.

Now, we can define the modified wave operators ("Dollard's modified wave operators") :

$$\Omega^{\pm} = s - \lim_{t\to\pm\infty}\ e^{itH}\ U_D(t)\ . \tag{2.2}$$

Since $\rho > \frac{1}{2}$, it is clear that this limit exists and we obtain by (1.3) :

$$\Omega^{\pm} = W^{\pm}\ e^{-ig^{\pm}(p')}\ , \tag{2.3}$$

where

$$g^{\pm}(p')\ =\ \int_0^{\pm\infty}\ V(sp' + \frac{1}{2}s^2 e_1)\ ds\ . \tag{2.4}$$

We set $T = \Omega^{+*}\Omega^{-}$, and by (2.3) we deduce :

$$S = e^{-ig^{+}(p')}\ T\ e^{ig^{-}(p')} \tag{2.5}$$

## 2.2 High energy asymptotics of the scattering operator.

In order to formulate the main result of this section, we need additional notation.

- $\Phi, \Psi$ are the Fourier transforms of functions in $C_0^\infty(I\!R^n)$.
- $\omega \in S^{n-1} \cap \Pi_{e_1}$ is fixed.
- $\Phi_{\lambda,\omega} = e^{i\sqrt{\lambda}x.\omega}\Phi,\ \Psi_{\lambda,\omega} = e^{i\sqrt{\lambda}x.\omega}\Psi$.

We have the following high energy asymptotics where $<\ ,\ >$ is the usual scalar product in $L^2(I\!R^n)$ :

**Proposition 2**

$$< [S,p]\ \Phi_{\lambda,\omega}\ ,\ \Psi_{\lambda,\omega} > = \lambda^{-\frac{1}{2}}\ < \left(\int_{-\infty}^{+\infty}\ \nabla V(x+t\omega)\ dt\right)\ \Phi\ ,\ \Psi > + o\ (\lambda^{-\frac{1}{2}})\ .$$

In [12], Weder studied the same asymptotics; he also obtained Proposition 2 but with a stronger decay condition on the potential $V$ : $\rho > \frac{3}{4}$.

The main tool to prove Proposition 2 is the following lemma :

**Lemma 3**
*For $\lambda \gg 1$, we have :*

$$(i) \qquad \|\ \left(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\right)\ U_D(t)\ e^{ig^{\pm}(p')}\Phi_{\lambda,\omega}\ \| \leq\ C\ (1+ \mid t\sqrt{\lambda}\mid)^{-\frac{1}{2}-\rho}\ .$$

$$(ii) \qquad \|\ (e^{-itH}\Omega^{\pm} - U_D(t))e^{ig^{\pm}(p')}\Phi_{\lambda,\omega}\ \| =\ O\ (\lambda^{-\frac{1}{2}})\ ,\ \text{uniformly for } t \in I\!R\ .$$

**Proof of Lemma 3 (the case +).**
$(i)$ First, we need some notation :

Let $\chi \in C^\infty(I\!R^{n-1})$ such that $\chi(x') = 1$ if $\mid x' \mid \geq \frac{1}{2}$, $\chi(x') = 0$ if $\mid x' \mid \leq \frac{1}{4}$. Denote $V_{\lambda,t}(x) = V(x)\ \chi(\dfrac{x'}{t\sqrt{\lambda}})$ where $x = (x_1, x') \in I\!R^n$, and choose $\theta \in C_0^\infty(I\!R^{n-1})$ such that $\theta(p')\ \Phi = \Phi$, $f \in C_0^\infty(I\!R)$ such that $f(p_1)\ \Phi = \Phi$.

At last , denote $U^{\pm}(t,p') = e^{i\int_t^{\pm\infty}\ V(sp' + \frac{1}{2}s^2 e_1)\ ds}$.

Our strategy is close to Lemma 3.3 [5]. We have :

$$\begin{aligned}A(t,\lambda) &:= \| \ \Big(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\Big) \ U_D(t) \ e^{ig^+(p')}\Phi_{\lambda,\omega} \ \| \\ &= \| \ \Big(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\Big) \ e^{-itH_0} \ U^+(t,p')\Phi_{\lambda,\omega} \ \| \\ &= \| \ \Big(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\Big) \ e^{-itH_0} \ \theta(p' - \sqrt{\lambda}\omega) \ U^+(t,p')\Phi_{\lambda,\omega} \ \| ,\end{aligned}$$

since $\omega \in \Pi_{e_1}$ and $\theta(p') \ \Phi = \Phi$. Thus,

$$\begin{aligned}A(t,\lambda) &\leq \| \ (V(x) - V_{\lambda,t}(x)) \ e^{-itH_0} \ \theta(p' - \sqrt{\lambda}\omega) \ U^+(t,p')\Phi_{\lambda,\omega} \ \| \\ &+ \| \ \Big(V_{\lambda,t}(x) - V(tp' + \frac{1}{2}t^2 e_1)\Big) \ e^{-itH_0} \ \theta(p' - \sqrt{\lambda}\omega) \ U^+(t,p')\Phi_{\lambda,\omega} \ \| \\ &:= (1) + (2).\end{aligned}$$

**Step 1 :**

We remark that $Supp \ (V - V_{\lambda,t}) \subset \{ \ x \in I\!R^n \ : | \ x' \ | \leq \ \frac{|t\sqrt{\lambda}|}{2} \ \}$, so we have :

$$(1) \ \leq \ C \ \| \ \ F(| \ x' - t\sqrt{\lambda}\omega \ | \geq \frac{| \ t\sqrt{\lambda} \ |}{2}) \ e^{-itH_0} \ \theta(p' - \sqrt{\lambda}\omega) \ U^+(t,p')\Phi_{\lambda,\omega} \ \| ,$$

where $F(x \in O)$ denotes the multiplication operator with the characteristic function on the set $O$. Thus,

$$\begin{aligned}(1) \ &\leq \ C \ \| \ \ F(| \ x' - t\sqrt{\lambda}\omega \ | \geq \frac{| \ t\sqrt{\lambda} \ |}{2}) \ e^{-itH_0} \ \theta(p' - \sqrt{\lambda}\omega) \ F(| \ x' \ | < \frac{| \ t\sqrt{\lambda} \ |}{8}) \ \| \\ &+ C \ \| \ \ F(| \ x' \ | \geq \frac{| \ t\sqrt{\lambda} \ |}{8}) \ U^+(t,p')\Phi_{\lambda,\omega} \ \| \\ &:= (a) + (b).\end{aligned}$$

First, we estimate $(a)$. Using the Avron-Herbst formula (1.2) we obtain :

$$(a) \ \leq \| \ \ F(| \ x' - t\sqrt{\lambda}\omega \ | \geq \frac{| \ t\sqrt{\lambda} \ |}{2}) \ e^{-i\frac{t}{2}p'^2} \ \theta(p' - \sqrt{\lambda}\omega) \ F(| \ x' \ | < \frac{| \ t\sqrt{\lambda} \ |}{8}) \ \| \ .$$

This term describes the free propagation for the free Hamiltonian $\frac{1}{2} \ p'^2$ into the classical forbidden region. So, we have for all $N$ and $\lambda \gg 1$ (see [3]) :

$$(2.6) \qquad (a) \ \leq C_N \ (1 + | \ t\sqrt{\lambda} \ |)^{-N} \ .$$

Now, we estimate $(b)$ :

$$(b) \ = \ C \ \| < x' >^{-N} \ \ F(| \ x' \ | \geq \frac{| \ t\sqrt{\lambda} \ |}{8}) < x' >^{N} \ U^+(t,p')\Phi_{\lambda,\omega} \ \|$$

$$\leq C\ (1+\mid t\sqrt{\lambda}\mid)^{-N}\ \ || < x' >^N\ U^+(t,p') < x' >^{-N} ||\ \ .$$

Writing $< x' >^N\ U^+(t,p') < x' >^{-N} = U^+(t,p') + [< x' >^N, U^+(t,p')] < x' >^{-N}$ and using standard pseudo-differential calculus, it is easy to show that

$$|| < x' >^N\ U^+(t,p') < x' >^{-N} || \leq\ C_N \text{ uniformly for } t \in I\!R\ .$$

So, we obtain :

$$(b) \leq C_N\ (1+\mid t\sqrt{\lambda}\mid)^{-N}\ . \tag{2.7}$$

**Step 2 :**
Since $\theta(p'-\sqrt{\lambda}\omega)$ commutes with $H_0$, we have for $\lambda \gg 1$,

$$(2)\ = ||\ \left(V_{\lambda,t}(x) - V_{\lambda,t}(tp' + \frac{1}{2}t^2 e_1)\right)\ e^{-itH_0}\ \theta(p'-\sqrt{\lambda}\omega)\ U^+(t,p')\Phi_{\lambda,\omega}\ \ ||\ \ .$$

The Avron-Herst formula (1.2) implies :

$$(2)\ = ||\ \left(V_{\lambda,t}(x + tp + \frac{1}{2}t^2 e_1) - V_{\lambda,t}(tp' + \frac{1}{2}t^2 e_1)\right)\ \theta(p'-\sqrt{\lambda}\omega)\ U^+(t,p')\Phi_{\lambda,\omega}\ \ ||\ ,$$

where by definition $g(x+tp)$ is equal to $e^{i\frac{t}{2}p^2}\ g(x)\ e^{-i\frac{t}{2}p^2}$ for any borelian function $g$ .
Thus,

$$\begin{aligned}(2)\ \leq &\ ||\ \left(V_{\lambda,t}(x + tp + \frac{1}{2}t^2 e_1) - V_{\lambda,t}(tp + \frac{1}{2}t^2 e_1)\right)\ \theta(p'-\sqrt{\lambda}\omega)\ U^+(t,p')\Phi_{\lambda,\omega}\ \ || \\ &+\ ||\ \left(V_{\lambda,t}(tp + \frac{1}{2}t^2 e_1) - V_{\lambda,t}(tp' + \frac{1}{2}t^2 e_1)\right)\ \theta(p'-\sqrt{\lambda}\omega)\ U^+(t,p')\Phi_{\lambda,\omega}\ \ ||\ \ . \\ &:= (a) + (b).\end{aligned}$$

First, we estimate $(a)$. Using the following formula given in [4] :

$$\begin{aligned}V_{\lambda,t}(x + tp + \frac{1}{2}t^2 e_1) - V_{\lambda,t}(tp + \frac{1}{2}t^2 e_1)\ =\ & \int_0^1 (\nabla V_{\lambda,t})(sx + tp + \frac{1}{2}t^2 e_1).x \\ & -\frac{it}{2}\ (\Delta V_{\lambda,t})(sx + tp + \frac{1}{2}t^2 e_1)\ ds\ ,\end{aligned} \tag{2.8}$$

as well as the estimate $\mid \partial_x^\alpha\ V_{\lambda,t}(x) \mid \leq C_\alpha\ (1+\mid t\sqrt{\lambda}\mid)^{-\rho-|\alpha|}$, we obtain :

$$(a)\ \leq\ C\ (1+t\sqrt{\lambda})^{-\rho-1}\ \ ||< x > U^+(t,p') < x >^{-1}||$$

Thus, since $||< x > U^+(t,p') < x >^{-1}|| = O(1)$ uniformly for $t \in I\!R$ by using the standard pseudo-differential calculus (see Step 1), we have :

$$(a)\ \leq\ C\ (1+\mid t\sqrt{\lambda}\mid)^{-\rho-1}. \tag{2.9}$$

Now, we estimate $(b)$. We use again that $\omega$ is orthononal to $e_1$. First, we see that :

$$V_{\lambda,t}(tp+\frac{1}{2}t^2e_1)-V_{\lambda,t}(tp'+\frac{1}{2}t^2e_1) \;=\; \left(\int_0^1 t\ (\partial_1 V_{\lambda,t})\ ((stp_1+\frac{1}{2}t^2)e_1+tp')\ ds\right)\ p_1\ .$$

So, since $\omega\in\Pi_{e_1}$,

$$(b)\ =\ ||\ \left(\int_0^1 t\ (\partial_1 V_{\lambda,t})\ ((stp_1+\frac{1}{2}t^2)e_1+tp')\ ds\right)\ p_1 f(p_1)\ \theta(p'-\sqrt{\lambda}\omega)\ U^+(t,p')\Phi_{\lambda,\omega}\ \ ||\ .$$

On the other hand $(\partial_1 V_{\lambda,t})\ (x)=\partial_1 V(x)\ \chi(\dfrac{x'}{t\sqrt{\lambda}})$, hence $|\ (\partial_1 V_{\lambda,t})\ (x)\ |\leq C\ \ <x>^{-1-\rho}$. Using that, in Fourier representation, $|\ (stp_1+\frac{1}{2}t^2)e_1+tp'\ |\geq C(t^2+\mid t\sqrt{\lambda}\mid)$, we obtain :

$$(b)\leq C\ \ |\ t\ |\ \ (t^2+\mid t\sqrt{\lambda}\mid+1)^{-1-\rho}\leq C\ \ |\ t\ |\ \ (t^2+\mid t\sqrt{\lambda}\mid+1)^{-\frac{1}{2}}\ (t^2+\mid t\sqrt{\lambda}\mid+1)^{-\frac{1}{2}-\rho}$$

thus

$$(b)\ \leq C\ (\mid t\sqrt{\lambda}\mid+1)^{-\frac{1}{2}-\rho}\ . \tag{2.10}$$

Then, $(i)$ follows from $(2.6)-(2.10)$.

$(ii)$ Arguing as in Corollary 3.4, [5], we have :

$$\begin{aligned} e^{-itH}\ \Omega^+ \ &=\ s-\lim_{\tau\to+\infty}\ e^{-itH}\ e^{i\tau H}\ U_D(\tau) \\ &=\ s-\lim_{T\to+\infty}\ \ e^{iTH}\ U_D(t+T)\ . \end{aligned}$$

So,

$$e^{-itH}\ \Omega^+-U_D(t)\ \ =\ i\ \int_0^{+\infty}\ \ e^{iTH}\ \left(V(x)-V((T+t)p'+\frac{1}{2}((T+t)^2e_1)\right)\ U_D(T+t)\ dt\ .$$

Thus,

$$||\ (e^{-itH}\ \Omega^+ - U_D(t))\ e^{ig^+(p')}\ \Phi_{\lambda,\omega}\ ||\leq\ \int_{-\infty}^{+\infty}\ ||\ \left(V(x)-V(sp'+\frac{1}{2}s^2e_1)\right)\ U_D(s)\ e^{ig^+(p')}\ \Phi_{\lambda,\omega}\ ||\ \ ds\ .$$

Using $(i)$, we obtain :

$$||\ (e^{-itH}\ \Omega^+-U_D(t))\ e^{ig^+(p')}\ \Phi_{\lambda,\omega}\ ||\ =\ \ O(\lambda^{-\frac{1}{2}})\ \text{uniformly for}\ t\in I\!R\ .\ \ \Box$$

**Proof of Proposition 2.**
We denote $F(\lambda,\omega)=<[S,p]\ \Phi_{\lambda,\omega}\ ,\ \Psi_{\lambda,\omega}>$. Using (2.5), we have :

$$F(\lambda,\omega)\ \ =<[e^{-ig^+(p')}\ T\ \ e^{ig^-(p')},p]\ \Phi_{\lambda,\omega}\ ,\ \ \Psi_{\lambda,\omega}>$$

$$
\begin{aligned}
&= < [T,p]\ e^{ig^-(p')}\Phi_{\lambda,\omega}\ ,\ e^{ig^+(p')}\Psi_{\lambda,\omega} > \\
&= < [T-1, p-\sqrt{\lambda}\omega]\ e^{ig^-(p')}\Phi_{\lambda,\omega}\ ,\ e^{ig^+(p')}\Psi_{\lambda,\omega} > \\
&= < (T-1)\ e^{ig^-(p')}(p\Phi)_{\lambda,\omega}\ ,\ e^{ig^+(p')}\Psi_{\lambda,\omega} > \\
&-\ < (T-1)\ e^{ig^-(p')}\Phi_{\lambda,\omega}\ ,\ e^{ig^+(p')}(p\Psi)_{\lambda,\omega} > \\
&:= F_1(\lambda,\omega) - F_2(\lambda,\omega).
\end{aligned}
$$

We study $F_1(\lambda,\omega)$ at first. Writing $T-1 = (\Omega^+ - \Omega^-)^*\Omega^-$ and using

$$
\Omega^+ - \Omega^- \ =\ i \int_{-\infty}^{+\infty} e^{itH}\ \left(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\right)\ U_D(t)\ dt\ , \tag{2.11}
$$

we obtain :

$$
T - 1 = -i \int_{-\infty}^{+\infty} U_D(t)^*\ \left(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\right)\ e^{-itH}\ \Omega^-\ dt\ . \tag{2.12}
$$

Thus,

$$
\begin{aligned}
F_1(\lambda,\omega) &= -i \int_{-\infty}^{+\infty} < e^{-itH}\ \Omega^-\ e^{ig^-(p')}(p\Phi)_{\lambda,\omega}\ ,\ \left(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\right)\ U_D(t)\, e^{ig^+(p')}\Psi_{\lambda,\omega} >\ dt \\
&= -i \int_{-\infty}^{+\infty} < U_D(t)\, e^{ig^-(p')}(p\Phi)_{\lambda,\omega}\ ,\ \left(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\right)\ U_D(t)\, e^{ig^+(p')}\Psi_{\lambda,\omega} >\ dt \\
&\quad +\ R_1(\lambda,\omega)\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
R_1(\lambda,\omega)\ =\ -i \int_{-\infty}^{+\infty} &< (e^{-itH}\ \Omega^- - U_D(t))\ e^{ig^-(p')}(p\Phi)_{\lambda,\omega}\ , \\
&\left(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\right)\ U_D(t)\ e^{ig^+(p')}\Psi_{\lambda,\omega} >\ dt\ .
\end{aligned} \tag{2.13}
$$

Now, by Lemma 3, we have easily $R_1(\lambda,\omega) = O\ (\lambda^{-1})$. Thus,

$$
\begin{aligned}
F_1(\lambda,\omega)\ =& -\frac{i}{\sqrt{\lambda}} \int_{-\infty}^{+\infty} < U_D(\frac{t}{\sqrt{\lambda}})\ e^{ig^-(p')}\ (p\Phi)_{\lambda,\omega}\ , \\
&\left(V(x) - V(\frac{t}{\sqrt{\lambda}}p' + \frac{1}{2\lambda}t^2 e_1)\right)\ U_D(\frac{t}{\sqrt{\lambda}})\, e^{ig^+(p')}\Psi_{\lambda,\omega} >\ dt + O\ (\lambda^{-1})\, .
\end{aligned} \tag{2.14}
$$

Denote by $f_1(t,\lambda,\omega)$ the integrand of the (R.H.S) of (2.14). By Lemma 3 (i),

$$
\mid f_1(t,\lambda,\omega) \mid \leq C\ (1+ \mid t \mid)^{-\frac{1}{2}-\rho}\ .
$$

So, by Lebesgue's theorem, to obtain the asymptotics of $F_1(\lambda,\omega)$, it suffices to determine $\lim\limits_{\lambda\to+\infty} f_1(t,\lambda,\omega)\ ,\ \forall t \in I\!R$.

We have :

$$(2.15) \qquad f_1(t,\lambda,\omega) \ = < e^{-i\frac{t}{\sqrt{\lambda}}H_0} \ U^-(\frac{t}{\sqrt{\lambda}},p') \ (p\Phi)_{\lambda,\omega} \ ,$$

$$\left(V(x) - V(\frac{t}{\sqrt{\lambda}}p' + \frac{1}{2\lambda}t^2 e_1)\right) \ e^{-i\frac{t}{\sqrt{\lambda}}H_0} \ U^+(\frac{t}{\sqrt{\lambda}},p') \ \Psi_{\lambda,\omega} > \ .$$

Using the Avron-Herbst formula (1.2), we deduce that :

$$(2.16) \qquad f_1(t,\lambda,\omega) \ = < e^{-i\frac{t}{2\sqrt{\lambda}}p^2} \ U^-(\frac{t}{\sqrt{\lambda}},p') \ (p\Phi)_{\lambda,\omega} \ ,$$

$$\left(V(x+\frac{1}{2\lambda}t^2 e_1) - V(\frac{t}{\sqrt{\lambda}}p' + \frac{1}{2\lambda}t^2 e_1)\right) \ e^{-i\frac{t}{2\sqrt{\lambda}}p^2} \ U^+(\frac{t}{\sqrt{\lambda}},p') \ \Psi_{\lambda,\omega} > \ .$$

Then, we obtain :

$$(2.17) \qquad f_1(t,\lambda,\omega) \ = < e^{-i\frac{t}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} \ U^-(\frac{t}{\sqrt{\lambda}},p'+\sqrt{\lambda}\omega) \ p\Phi \ ,$$

$$\left(V(x+\frac{1}{2\lambda}t^2 e_1) - V(\frac{t}{\sqrt{\lambda}}(p'+\sqrt{\lambda}\omega) + \frac{1}{2\lambda}t^2 e_1)\right) \ e^{-i\frac{t}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} \ U^+(\frac{t}{\sqrt{\lambda}},p'+\sqrt{\lambda}\omega) \ \Psi > \ .$$

Since

$$(2.18) \qquad e^{-i\frac{t}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} \ = \ e^{-i\frac{t\sqrt{\lambda}}{2}} \ e^{-it\omega.p} \ e^{-i\frac{t}{2\sqrt{\lambda}}p^2} \ ,$$

we have

$$(2.19) \qquad f_1(t,\lambda,\omega) \ = < e^{-i\frac{t}{2\sqrt{\lambda}}p^2} \ U^-(\frac{t}{\sqrt{\lambda}},p'+\sqrt{\lambda}\omega) \ \ p\Phi \ ,$$

$$\left(V(x+t\omega+\frac{1}{2\lambda}t^2 e_1) - V(t\omega+\frac{t}{\sqrt{\lambda}}p' + \frac{1}{2\lambda}t^2 e_1)\right) \ e^{-i\frac{t}{2\sqrt{\lambda}}p^2} \ U^+(\frac{t}{\sqrt{\lambda}},p'+\sqrt{\lambda}\omega) \ \Psi > \ .$$

Since $\mid V(s(p'+\sqrt{\lambda}\omega) + \frac{1}{2}s^2 e_1)) \mid \ \leq \ C \ (s^2+1)^{-\rho} \in L^1(I\!R, \ ds)$, it is easy to show by Lebesgue's theorem that :

$$(2.20) \qquad s - \lim_{\lambda \to +\infty} \ U^{\pm}(\frac{t}{\sqrt{\lambda}},p'+\sqrt{\lambda}\omega) \ = 1 \ .$$

Then,

$$(2.21) \qquad \lim_{\lambda \to +\infty} \ f_1(t,\lambda,\omega) \ = < p\Phi \ , \ (V(x+t\omega) - V(t\omega)) \ \Psi > \ .$$

So, we have obtained

$$(2.22) \qquad F_1(\lambda,\omega) \ = \ -\frac{i}{\sqrt{\lambda}} \ < p\Phi, \ \left(\int_{-\infty}^{+\infty} \ (V(x+t\omega) - V(t\omega)) \ dt\right) \ \Psi > + o\,(\frac{1}{\sqrt{\lambda}}) \ ,$$

In the same way, we obtain

$$(2.23) \qquad F_2(\lambda,\omega) \;=\; -\frac{i}{\sqrt{\lambda}} \;<\Phi,\; \Big(\int_{-\infty}^{+\infty} (V(x+t\omega)-V(t\omega))\;dt\Big)\; p\Psi> + o\,(\frac{1}{\sqrt{\lambda}})\;,$$

so

$$(2.24) \qquad \begin{aligned} F(\lambda,\omega) \;&=\; F_1(\lambda,\omega) \;-\; F_2(\lambda,\omega) \\ &=\; \frac{1}{\sqrt{\lambda}} \;<\Phi,\; \Big(\int_{-\infty}^{+\infty} \nabla V(x+t\omega)\;dt\Big))\;\Psi> + o\,(\frac{1}{\sqrt{\lambda}})\;. \quad \square \end{aligned}$$

## 2.3 Uniqueness of the potential.

In this section, we use Proposition 2 to prove Theorem 1.

Let $V_1$ and $V_2$ be potentials satisfying $(H_1)$ such that $S(V_1) = S(V_2)$. By proposition 2, we have :

$$(2.25) \qquad \int_{-\infty}^{+\infty} \nabla V(x+t\omega)\;dt \;=\; 0\;,\; \forall x \in I\!R^n\;,\; \forall \omega \in \Pi_{e_1} \cap S^{n-1}\;,$$

where $V = V_1 - V_2$. Now, fix $a \in I\!R$ and define for $x' \in \Pi_{e_1}$, $V_a(x') = V(ae_1 + x')$. Using (2.25), we have $\forall \alpha \in I\!N^{n-1}$ with $\mid \alpha \mid \geq 1$ :

$$(2.26) \qquad \int_{-\infty}^{+\infty} \partial_{x'}^{\alpha} V_a(x'+t\omega)\;dt \;=\; 0\;,\; \forall x' \in\; \Pi_{e_1}\;,\; \forall \omega \in \Pi_{e_1} \cap S^{n-1}\;.$$

Now, we can easily prove that $V_a = 0$.

First, remark that for $n \geq 3$, $dim\ \Pi_{e_1} \cap \Pi_\omega = n-2 \geq 1$. Let $\xi' \in \Pi_{e_1} \cap \Pi_\omega$ and consider for $\mid \alpha \mid \geq n-1$, the Fourier transform in $L^1(\Pi_{e_1})$ :

$$(2.27) \qquad \widehat{\partial_{x'}^{\alpha} V_a}(\xi') \;=\; \int_{\Pi_{e_1}} e^{-ix'.\xi'}\;\partial_{x'}^{\alpha'} V_a(x')\;dx'\;.$$

Writing $x' = y + t\omega$ where $y \in \Pi_{e_1} \cap \Pi_\omega$, we have by (2.26)

$$(2.28) \qquad \widehat{\partial_{x'}^{\alpha'} V_a}(\xi') \;=\; \int_{\Pi_{e_1}\cap\Pi_\omega} e^{-iy.\xi'}\;\Big(\int_{-\infty}^{+\infty} \partial_{x'}^{\alpha'} V_a(y+t\omega)\;dt\Big)\;dy \;=\; 0\;.$$

Varying $\omega$, by the uniqueness of the Fourier transform in $L^1(\Pi_{e_1})$, we have $\partial_{x'}^{\alpha} V_a = 0$. So $V_a$ is polynomial and goes to zero at infinity. We deduce that $V_a = 0$, $\forall a \in I\!R$. Then $V = 0$. $\square$

**Remark.**
In [12], Weder proved uniqueness of the potential by using the inversion for the Radon transform (see [6]).

## 3 Comments.

With the method proposed above, we can recover Weder's result [12]. Actually, in [12], he showed this result with weaker conditions on the derivatives of the potentials.

**Theorem 4** ([12], Theorem 2.4 )
*Let $V_1$, $V_2$ be potentials satisfying $(H_1)$ with $\rho > \frac{3}{4}$ and assume that $n \geq 2$. Then :*

$$S(V_1) = S(V_2) \iff V_1 = V_2 \ .$$

**Sketch of proof.**
Let $\omega \in S^{n-1}$ fixed such that $| \omega.e_1 | < 1$. We easily show that if $| p - \sqrt{\lambda}\omega | \leq C$, (in Fourier representation) , there exists $C_1 > 0$ such that :

$$\forall t \in I\!R \ , \ \forall s \in [0,1] \ , \ \forall \lambda \gg 1 \ , \ | \ (stp_1 + \frac{1}{2}t^2)e_1 + tp' \ | \geq C_1 \ (t^2 + | \ t\sqrt{\lambda} \ |) \ . \tag{3.1}$$

Now, following the proof of Lemma 3 and using (3.1) in the Step 2 (b), we obtain :

$$|| \ \left( V(x) - V(tp' + \frac{1}{2}t^2 e_1) \right) \ U_D(t) \ e^{ig^{\pm}(p')} \Phi_{\lambda,\omega} \ || \leq \ C \ (1 + | \ t\sqrt{\lambda} \ |)^{-1-\rho} + C \ | \ t\sqrt{\lambda} \ | \ (1 + | \ t\sqrt{\lambda} \ | + t^2)^{-1-\rho}. \tag{3.2}$$

Then, using the same arguments as in Lemma 3 (ii), we have for $\delta < min \ (\rho, 1)$ :

$$|| \ (e^{-itH}\Omega^{\pm} - U_D(t))e^{ig^{\pm}(p')}\Phi_{\lambda,\omega} \ || = \ O \ (\lambda^{\frac{1}{2}-\delta}) \ , \ \text{uniformly for } t \in I\!R \ . \tag{3.3}$$

With the notation of the proof of Proposition 2, we obtain easily :

$$R_1(\lambda, \omega) = O(\lambda^{1-2\delta}) \ . \tag{3.4}$$

So, if $\rho > \frac{3}{4}$, Proposition 2 is valid. By a standard continuity argument, we have (2.25) for all $\omega \in S^{n-1}$ and using the same arguments as in the section 2.3, we obtain Theorem 4. □

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