Inverse scattering for a Schrödinger operator with a repulsive potential.

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Abstract

We consider a pair of Hamiltonians (H, H_0) on $L^2(I\!\!R^n)$ where $H_0 = p^2 - x^2$ is a Schrödinger operator with a repulsive potential, and $H = H_0 + V(x)$. We show that, under suitable assumptions on the decay of the electric potential, V is uniquely determined by the high energy limit of the scattering operator.

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1 Introduction.

The aim of this paper is to study an inverse scattering problem for a Hamiltonian with a repulsive potential and for a class of short-range potentials.

The free Hamiltonian defined on $L^2(I\!R^n)$, $n \geq 2$, is given by :

$$H_0 \ = \ p^2 - x^2 \ , \tag{1.1}$$

where $p = -i\nabla$. It is well-known that H_0 is essentially self-adjoint on $C_0^\infty(I\!R^n)$, (see [1] for example). We denote also by H_0 the self-adjoint realization with domain $D(H_0)$.

Moreover, if we denote by $A = x \cdot p + p \cdot x$ the generator of dilations, we have :

$$H_0 \ U = U \ A \ , \tag{1.2}$$

where U is the unitary operator on $L^2(I\!R^n)$ defined by :

$$U\Phi(x) = (\sqrt{2}\pi)^{-\frac{n}{2}} \ e^{-i\frac{x^2}{2}} \ \int_{I\!R^n} \ e^{i\sqrt{2}x\cdot y} \ e^{-i\frac{y^2}{2}} \Phi(y) \ dy. \tag{1.3}$$

From (1.3), we deduce that the spectrum of H_0 : $\sigma(H_0) = \sigma_{ac}(H_0) = I\!R$.

Now, let us recall Mehler's formula [2] which describes the free time evolution :

$$\forall t \neq 0 \ , \ e^{-itH_0} = M_t \ D_t \ \mathcal{F} \ M_t \ , \tag{1.4}$$

where M_t is the multiplication operator :

$$M_t\Phi(x) = e^{\frac{i}{2}\coth(2t) \ x^2} \ \Phi(x) \ , \tag{1.5}$$

D_t is the dilation operator :

$$D_t\Phi(x) = (i\sinh(2t))^{-\frac{n}{2}} \ \Phi\left(\frac{x}{\sinh(2t)}\right) \ , \tag{1.6}$$

and $\mathcal{F}$ is the usual Fourier transform on $L^2(I\!R^n)$.

Intuitively, since the classical flow of the system is given by : $\forall x, \xi \in I\!R^n, \ \forall t \in I\!R$,

$$\Phi_0^t(x,\xi) = (\cosh(2t)x + \sinh(2t)\xi, \ \sinh(2t)x + \cosh(2t)\xi) \ , \tag{1.7}$$

a real-valued measurable electric potential V satisfying :

$$(H_1) \qquad \mid V(x) \mid \leq C \ (\ln < x >)^{-1-\epsilon}, \ \epsilon > 0 \ ,$$

where $< x > = (1+x^2)^{\frac{1}{2}}$, is of short-range for H_0.

With this hypothesis, $H = H_0 + V(x)$ is essentially self-adjoint with domain $D(H) = D(H_0)$ and $\sigma_{ess}(H) = I\!R$. Moreover, H has no eigenvalues and $\sigma_{sc}(H) = \emptyset$, [3].

Under the asumption (H_1), it is shown in [3] that the wave operators :

$$W^{\pm} \ = \ s - \lim_{t \to \pm\infty} \ e^{itH} \ e^{-itH_0} \tag{1.8}$$

exist and are complete, (i.e $Ran\ W^{\pm} = \mathcal{H}^{(ac)}(H)$, the later being the subspace of absolute continuity of H).

We denote $S = S(V) = W^{+*}W^{-}$ the scattering operator. The inverse scattering problem consists of reconstructing the perturbation V from the scattering operator.

In this paper, in order to solve the inverse problem, we need stronger hypotheses on the electric potential V.

We assume that $V \in C^{\infty}(I\!R^n)$ and it satisfies $\forall \alpha \in I\!N^n$:

$$(H_2) \qquad\qquad | \ \partial_x^{\alpha} V(x) \ | \leq \ C_{\alpha} < x >^{-\rho-|\alpha|} \ , \ \rho \in]\frac{1}{2}, 1[\ .$$

Actually, let us remark that in our paper we only need (H_2) for α with finite order, (for example $| \ \alpha \ | \leq n$).

We prove that the S-operator determines uniquely the potential V. More precisely, it suffices to know the high energy limit of S, (cf Proposition 2).

Our main result is :

Theorem 1
Let $V_1, \ V_2$ be potentials satisfying (H_2). Then :

$$S(V_1) = S(V_2) \iff V_1 = V_2 \ .$$

2 Proof of Theorem 1.

In this section, we study the high energy limit of the scattering operator using the Enss-Weder's time-dependent method : see [4] where they study the case of two-body Schrödinger Hamiltonians $H = \frac{1}{2} \ p^2 + V$ on $L^2(I\!R^n)$. This method can be used to study Hamiltonians with electric and magnetic potentials on $L^2(I\!R^n)$ [5], the Dirac equation [6], the N-body case [4], the Stark effect [7], [8], the Aharonov-Bohm effect [9].

In [10], [11], a stationary approach is proposed to solve scattering inverse problems for Schrödinger operators with magnetic fields or with the Aharonov-Bohm effect. In [12], the same approach is used to reconstruct the asymptotics of short or long-range potentials from the scattering operator localized in energy. Unfortunately, for repulsive potentials, this method is not easily applicable.

2.1 High energy asymptotics of the scattering operator.

In order to formulate the main result of this section, we need additional notation.

- $\Phi, \Psi \in \mathcal{S}(I\!R^n)$ the set of the Schwartz functions.
- $\omega \in S^{n-1}$ is fixed.
- $\Phi_{\lambda,\omega} = e^{i\sqrt{\lambda}x\cdot\omega}\Phi,\ \Psi_{\lambda,\omega} = e^{i\sqrt{\lambda}x\cdot\omega}\Psi.$

We have the following high energy asymptotics when $\lambda \to +\infty$, where $< \ , \ >$ is the usual scalar product in $L^2(I\!R^n)$:

Proposition 2

$$< [S,p]\ \Phi_{\lambda,\omega}\ ,\ \Psi_{\lambda,\omega} > = \lambda^{-\frac{1}{2}}\ < \left(\int_{-\infty}^{+\infty}\ \nabla V(x+t\omega)\ dt \right)\ \Phi\ ,\ \Psi > + o\ (\lambda^{-\frac{1}{2}})\ .$$

2.1.1 Preliminary results.

First, we have to precise the free evolution at high energies. Denote by τ_a the translation operator of vector a on $L^2(I\!R^n)$: $\tau_a f(x) = f(x-a)$.

Lemma 3
For $t \in I\!R$, we have :

$$e^{-itH_0}\Phi_{\lambda,\omega} = \tau_{\sinh(2t)\sqrt{\lambda}\omega}\ e^{\frac{i\lambda}{2}\cosh(2t)\ \sinh(2t)}\ e^{i\sqrt{\lambda}\cosh(2t)\ x\cdot\omega}\ e^{-itH_0}\Phi. \tag{2.1}$$

Proof :
For $t=0$, the result is clear. For $t \neq 0$, we use Mehler's formula :

$$\begin{aligned} e^{-itH_0}\Phi_{\lambda,\omega} &= M_t\ D_t\ \mathcal{F}\ M_t\ \Phi_{\lambda,\omega} = M_t\ D_t\ \mathcal{F}\ e^{i\sqrt{\lambda}x\cdot\omega}\ M_t\ \Phi \\ &= M_t\ D_t\ \tau_{\sqrt{\lambda}\omega}\ \mathcal{F}\ M_t\ \Phi = M_t\ \tau_{\sinh(2t)\sqrt{\lambda}\omega}\ D_t\ \mathcal{F}\ M_t\ \Phi \\ &= \tau_{\sinh(2t)\sqrt{\lambda}\omega}\ e^{\frac{i\lambda}{2}\cosh(2t)\ \sinh(2t)}\ e^{i\sqrt{\lambda}\cosh(2t)\ x\cdot\omega}\ e^{-itH_0}\ \Phi.\ \ \square \end{aligned}$$

The main tool to prove Proposition 2 is the following technical lemma :

Lemma 4
For $\lambda \gg 1$, $N \gg 1$, $\delta > 0$ and $p > \frac{n}{\delta}$, there exists $C_{Np} > 0$ such that for $t \neq 0$,

$$\begin{aligned} || < x >^{-\delta}\ e^{-itH_0}\Phi_{\lambda,\omega}\ || \leq\ C_{Np}\ \Big(&< \sqrt{\lambda}\sinh(2t) >^{-\delta} + |\sinh(2t)|^{-\frac{n}{p}}\ [< \sqrt{\lambda}\tanh(2t) >^{-N} \\ &+ \lambda^{\frac{n-Nq}{2q}}\ ((\sqrt{\lambda}\ |\tanh(2t)|)^{-1} + 1)^N] \Big), \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

Proof :

Let $\theta \in C_0^\infty(I\!R^n)$ such that $\theta(x) = 1$ if $|\, x \,| \leq 1$ and $\theta(x) = 0$ if $|\, x \,| \geq 2$. For $t \neq 0$, using Lemma 3 and Mehler's formula, we have :

$$|| < x >^{-\delta} e^{-itH_0} \Phi_{\lambda,\omega} \ || = || < \sinh(2t)\ (x + \sqrt{\lambda}\omega) >^{-\delta} \ \mathcal{F} M_t \Phi \ ||$$

$$\leq || \ \theta\left(\frac{4x}{\sqrt{\lambda}}\right) \ < \sinh(2t)\ (x + \sqrt{\lambda}\omega) >^{-\delta} \ \mathcal{F} M_t \Phi \ ||$$

$$+ \, || \, (1-\theta)\left(\frac{4x}{\sqrt{\lambda}}\right) \ < \sinh(2t)\ (x + \sqrt{\lambda}\omega) >^{-\delta} \ \mathcal{F} M_t \Phi \ ||$$

$$:= (1) + (2).$$

• Step 1 :

On $Supp\ \theta\left(\frac{4x}{\sqrt{\lambda}}\right)$, $|\, x \,| \leq \frac{\sqrt{\lambda}}{2}$, so $|\, x + \sqrt{\lambda}\omega \,| \geq \frac{\sqrt{\lambda}}{2}$. Thus,

$$(1) \ \leq \ C \ < \sqrt{\lambda} \sinh(2t) >^{-\delta},$$

since $\mathcal{F}$ and M_t are unitary operators on L^2.

• Step 2 :

By Hölder inequality,

$$(2) \ \leq ||< \sinh(2t)\ (x + \sqrt{\lambda}\omega) >^{-\delta} ||_p \ || \, (1-\theta)\left(\frac{4x}{\sqrt{\lambda}}\right) \ \mathcal{F} \ M_t \Phi \ ||_q \ ,$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

First, we give an estimation of the L^p-norm for a suitable p : for $p > \frac{n}{\delta}$, it is easy to obtain :

$$||< \sinh(2t)\ (x + \sqrt{\lambda}\omega) >^{-\delta} ||_p = ||< \sinh(2t)\ x) >^{-\delta} ||_p \leq C \ \ |\sinh(2t)\,|^{-\frac{n}{p}} \ .$$

Now, we estimate the L^q-norm :

$$|| \, (1-\theta)\left(\frac{4x}{\sqrt{\lambda}}\right) \ \mathcal{F} \ M_t \ \Phi \ ||_q \leq || \, (1-\theta)\left(\frac{4x}{\sqrt{\lambda}}\right) \ \mathcal{F} \ M_t \ (1-\theta)\left(\frac{16x}{\sqrt{\lambda}\tanh(2t)}\right) \ \Phi \ ||_q$$

$$+ \, || \, (1-\theta)\left(\frac{4x}{\sqrt{\lambda}}\right) \ \mathcal{F} \ M_t \ \theta\left(\frac{16x}{\sqrt{\lambda}\tanh(2t)}\right) \ \Phi \ ||_q$$

$$:= (a) + (b).$$

• Step 2a :

One has :

$$(a) \ \leq || \ \mathcal{F} \ M_t \ (1-\theta)\left(\frac{16x}{\sqrt{\lambda}\tanh(2t)}\right) \ \Phi \ ||_q,$$

and by Hausdorff-Young inequality [10] if $\frac{1}{q}+\frac{1}{r}=1$,

$$(a) \ \leq (2\pi)^{\frac{n}{2}-\frac{n}{r}} \ || \ M_t \, (1-\theta) \left(\frac{16x}{\sqrt{\lambda}\tanh(2t)} \right) \ \Phi \ ||_r = \ (2\pi)^{\frac{n}{2}-\frac{n}{r}} \ || \, (1-\theta) \left(\frac{16x}{\sqrt{\lambda}\tanh(2t)} \right) \ \Phi \ ||_r \ .$$

On $Supp \ (1-\theta) \left(\frac{16x}{\sqrt{\lambda}\tanh(2t)} \right)$, $\mid x \mid \geq \frac{\sqrt{\lambda}}{16} \mid \tanh(2t) \mid$. Since $\Phi \in \mathcal{S}(I\!R^n)$, we have for all $N \geq 1$:

$$(a) \ \leq \ C_N \ < \sqrt{\lambda}\tanh(2t) >^{-N} .$$

• **Step 2b :**

This term describes the free propagation into the classical forbidden region, and one can evaluate this contribution by using a standard non-stationary phase estimate :

$$\mathcal{F} \ M_t \ \theta \left(\frac{16x}{\sqrt{\lambda}\tanh(2t)} \right) \ \Phi(x) \ = \ (2\pi)^{-\frac{n}{2}} \ \int \ e^{-i \ S(t,x,y)} \ \theta \left(\frac{16y}{\sqrt{\lambda}\tanh(2t)} \right) \ \Phi(y) \ dy, \qquad (2.2)$$

where the phase $S(t,x,y)$ is given by :

$$S(t,x,y) = x \cdot y - \frac{1}{2} \ \coth(2t) \ y^2. \qquad (2.3)$$

On $Supp \ (1-\theta) \left(\frac{4x}{\sqrt{\lambda}} \right)$, $\mid x \mid \geq \frac{\sqrt{\lambda}}{4}$ and on $Supp \ \theta \left(\frac{16y}{\sqrt{\lambda}\tanh(2t)} \right)$, $\mid y \mid \leq \frac{\sqrt{\lambda}}{8} \ \mid \tanh(2t) \mid$. Thus,

$$\mid \partial_y S(t,x,y) \mid = \mid x - \coth(2t) \ y \mid \geq \ C \ (\mid x \mid + \sqrt{\lambda}).$$

Integrating by part with respect to y, we easily obtain, $\forall N \geq 1$:

$$(b) \ \leq \ C_N \ \lambda^{\frac{n-Nq}{2q}} \ [\left(\sqrt{\lambda} \ \mid \tanh(2t) \mid \right)^{-1} + 1]^N,$$

and Lemma 4 is proved. □

Corollary 5
For $\lambda \gg 1$, one has :

$$|| \ (W^{\pm} - 1) e^{-itH_0} \Phi_{\lambda,\omega} \ || = \ O \ (\lambda^{-\frac{\rho}{2}}) \ , \ \text{uniformly for } t \in I\!R \ .$$

Proof :

We only consider the case $(+)$:

$$(W^+ - 1) \ e^{-itH_0} \ \Phi_{\lambda,\omega} \ = \ i \ \int_0^{+\infty} \ e^{isH} \ V \ e^{-i(s+t)H_0} \ \Phi_{\lambda,\omega} \ ds.$$

So,

$$|| \, (W^+ - 1) \ e^{-itH_0} \ \Phi_{\lambda,\omega} \, || \leq \ \int_{\boldsymbol{R}} \ || \, V \ e^{-isH_0} \ \Phi_{\lambda,\omega} \, || \ \ ds$$

$$\leq C \int_{\mathbf{R}} \| < x >^{-\rho} \ e^{-isH_0} \ \Phi_{\lambda,\omega} \| \ \ ds \text{ by } (H_2)$$

$$\leq C \int_{|s| \geq \lambda^{-\epsilon}} \| < x >^{-\rho} \ e^{-isH_0} \ \Phi_{\lambda,\omega} \| \ \ ds \ + \ O(\lambda^{-\epsilon}),$$

where $0 < \epsilon < \frac{1}{2}$. Now, let us study the integrand : for $| \ s \ | \geq \lambda^{-\epsilon}$, by Lemma 4, one has :

$$\| < x >^{-\rho} \ e^{-isH_0} \ \Phi_{\lambda,\omega} \| \leq \ C_{Np} \left(< \sinh(2s)\sqrt{\lambda} >^{-\rho} + | \sinh(2s) \ |^{-\frac{n}{p}} \ \ \lambda^{-M} \right), \qquad (2.4)$$

where $M = min \ ((\frac{1}{2} - \epsilon)N, \ \frac{Nq-n}{2q})$ and $p > \frac{n}{\rho} > n$. Thus, integrating with respect to s, we obtain :

$$\int_{|s| \geq \lambda^{-\epsilon}} \| < x >^{-\rho} \ e^{-isH_0} \ \Phi_{\lambda,\omega} \| \ ds \leq C_{Np} \int_{\mathbf{R}} < \sinh(2s)\sqrt{\lambda} >^{-\rho} \ \ ds + O\left(\lambda^{-M}\right). \quad (2.5)$$

Now, we estimate the integral in the (R.H.S) of (2.5). By a change of variables, we have :

$$\int_{\mathbf{R}} < \sinh(2s)\sqrt{\lambda} >^{-\rho} \ \ ds = \frac{1}{\sqrt{\lambda}} \int_0^{+\infty} \ f(t,\lambda) \ dt, \qquad (2.6)$$

where

$$f(t,\lambda) = < t >^{-\rho} \ \ (1 + \frac{t^2}{\lambda})^{-\frac{1}{2}}. \qquad (2.7)$$

So,

$$\int_{\mathbf{R}} < \sinh(2s)\sqrt{\lambda} >^{-\rho} \ \ ds = \frac{1}{\sqrt{\lambda}} \left(\int_0^{\sqrt{\lambda}} \ f(t,\lambda) \ dt + \int_{\sqrt{\lambda}}^{+\infty} \ f(t,\lambda) \ dt \right)$$

$$\leq \ \frac{1}{\sqrt{\lambda}} \left(\int_0^{\sqrt{\lambda}} \ < t >^{-\rho} \ \ dt + \sqrt{\lambda} \int_{\sqrt{\lambda}}^{+\infty} \ t^{-(\rho+1)} \ dt \right)$$

$$= O \ (\lambda^{-\frac{\rho}{2}}).$$

Taking $\epsilon = \frac{\rho}{2}$, Corollary 5 is proved. □

Now, in order to prove Proposition 2, we give an elementary result which is the quantum version of (1.7).

Lemma 6

(i)
$$e^{itH_0} \ x \ e^{-itH_0} = \cosh(2t) \ x + \sinh(2t) \ p$$

(ii)
$$e^{itH_0} \ p \ e^{-itH_0} = \sinh(2t) \ x + \cosh(2t) \ p$$

Proof :
Let $f(t) = e^{itH_0} \ x \ e^{-itH_0}$. Using $i[H_0, x] = 2p$ and $i[H_0, p] = 2x$, we see that f solves the differential equation $f''(t) = 4f(t)$ which implies (i). The proof of (ii) is similar.□

2.1.2 Proof of Proposition 2.

First, we begin by giving a different expression for $[S, p]$. Since $[S, p] = [S-1, p]$ and

$$S - 1 = (W^+ - W^-)^* \ W^- = -i \int_{\boldsymbol{R}} e^{itH_0} \ V \ W^- \ e^{-itH_0} \ dt, \tag{2.8}$$

by the intertwining property, we have :

$$[S, p] = [T, p] + [U, p], \tag{2.9}$$

where

$$T \ = \ -i \int_{\boldsymbol{R}} e^{itH_0} \ V \ \ e^{-itH_0} \ dt, \tag{2.10}$$

and

$$U \ = \ -i \int_{\boldsymbol{R}} e^{itH_0} \ V \ (W^- - 1) \ e^{-itH_0} \ dt. \tag{2.11}$$

Thus, if we denote $F(\lambda, \omega) \ = < [S, p] \ \Phi_{\lambda,\omega}, \ \Psi_{\lambda,\omega} >$, we have :

$$\begin{aligned} F(\lambda, \omega) \ &= < [T, p] \ \Phi_{\lambda,\omega}, \ \Psi_{\lambda,\omega} > + < [U, p] \ \Phi_{\lambda,\omega}, \ \Psi_{\lambda,\omega} > \\ &:= F_1(\lambda, \omega) + F_2(\lambda, \omega). \end{aligned} \tag{2.12}$$

Let us prove that $F_2(\lambda, \omega)$ gives a negligible contribution. One has :

$$\begin{aligned} F_2(\lambda, \omega) \ &= < [U, p - \sqrt{\lambda}\omega] \ \Phi_{\lambda,\omega}, \ \Psi_{\lambda,\omega} > \\ &= < U \ (p\Phi)_{\lambda,\omega}, \Psi_{\lambda,\omega} > - < U \ \Phi_{\lambda,\omega}, (p\Psi)_{\lambda,\omega} > \\ &:= (a) - (b). \end{aligned} \tag{2.13}$$

Using (2.11), one has :

$$(a) \ = \ -i \int_{\boldsymbol{R}} < (W^- - 1) \ e^{-itH_0} \ (p\Phi)_{\lambda,\omega}, \ Ve^{-itH_0} \ \Psi_{\lambda,\omega} > \ dt. \tag{2.14}$$

So,

$$\begin{aligned} |\, (a) \,| &\leq \int_{\boldsymbol{R}} || \, (W^- - 1) \ e^{-itH_0} \ (p\Phi)_{\lambda,\omega} \, || \ || \, Ve^{-itH_0} \ \Psi_{\lambda,\omega} \, || \ \ dt \\ &\leq \ C \ \lambda^{-\frac{\rho}{2}} \int_{\boldsymbol{R}} || \, Ve^{-itH_0} \ \Psi_{\lambda,\omega} \, || \ \ dt, \end{aligned} \tag{2.15}$$

where we have used Corollary 5. Following the proof of Corollary 5, we have also :

$$\int_{\boldsymbol{R}} || \, Ve^{-itH_0} \ \Psi_{\lambda,\omega} \, || \ \ dt \ = O \ \left(\lambda^{-\frac{\rho}{2}}\right).$$

So, we have proved that :

$$|\, (a) \,| = \ O \ \left(\lambda^{-\rho}\right),$$

and this result is also true for (b). Thus,

$$F_2(\lambda, \omega) \ = \ O \ \left(\lambda^{-\rho}\right) = o \ (\lambda^{-\frac{1}{2}}). \tag{2.16}$$

Now, we study the leading term $F_1(\lambda,\omega)$. Using Lemma 6 (ii), one has :

$$[T,p] \;=\; \int_{\mathbb{R}} e^{itH_0}\ \nabla V\ e^{-itH_0}\ \cosh(2t)\ dt. \tag{2.17}$$

Then,

$$\begin{aligned} F_1(\lambda,\omega) \;&=\; \int_{\mathbb{R}} <\nabla V\ e^{-itH_0}\ \Phi_{\lambda,\omega},\ e^{-itH_0}\ \Psi_{\lambda,\omega}>\ \cosh(2t)\ dt \qquad (2.18)\\ &=\; \int_{\mathbb{R}} <\nabla V(x+\sqrt{\lambda}\sinh(2t)\omega)\ e^{-itH_0}\ \Phi,\ e^{-itH_0}\ \Psi>\ \cosh(2t)\ dt \end{aligned}$$

by Lemma 3. By a change of variables,

$$\begin{aligned} F_1(\lambda,\omega) \;&=\; \frac{1}{2\sqrt{\lambda}} \int_{\mathbb{R}} <\nabla V(x+s\omega)\ e^{-\frac{i}{2}\arg\sinh(\frac{s}{\sqrt{\lambda}})H_0}\Phi,\ e^{-\frac{i}{2}\arg\sinh(\frac{s}{\sqrt{\lambda}})H_0}\Psi>\ ds \qquad (2.19)\\ &=\; \frac{1}{2\sqrt{\lambda}}\left(\int_{|s|\le\lambda^{\epsilon}} \ldots\ ds\ +\ \int_{|s|>\lambda^{\epsilon}} \ldots\ ds\right) := \frac{1}{2\sqrt{\lambda}}\ ((a)+(b)), \end{aligned}$$

for a suitable $\epsilon>0$ defined later.

First, we study (a); obviously:

$$\begin{aligned} (a) \;&=\; \int_{|s|\le\lambda^{\epsilon}} <\nabla V(x+s\omega)\ \Phi,\Psi>\ ds \qquad (2.20)\\ &+\int_{|s|\le\lambda^{\epsilon}} <\nabla V(x+s\omega)\left(e^{-\frac{i}{2}\arg\sinh(\frac{s}{\sqrt{\lambda}})H_0}-1\right)\Phi, e^{-\frac{i}{2}\arg\sinh(\frac{s}{\sqrt{\lambda}})H_0}\Psi>\ ds\\ &+\int_{|s|\le\lambda^{\epsilon}} <\nabla V(x+s\omega)\Phi,\left(e^{-\frac{i}{2}\arg\sinh(\frac{s}{\sqrt{\lambda}})H_0}-1\right)\Psi>\ ds\\ &:= (1)+(2)+(3). \end{aligned}$$

Choosing $\epsilon<\frac{1}{4}$, since for $\mid s\mid\le\lambda^{\epsilon}$,

$$||\left(e^{-\frac{i}{2}\arg\sinh(\frac{s}{\sqrt{\lambda}})H_0}-1\right)\Phi\ || = \ O\ (\lambda^{\epsilon-\frac{1}{2}}), \tag{2.21}$$

we have $(2)+(3)=O\ (\lambda^{2\epsilon-\frac{1}{2}})$. Thus, since $f(s):=<\nabla V(x+s\omega)\Phi,\Psi>\ \in L^1(\mathbb{R},\ ds)$, one has :

$$(a) \;=\; \int_{\mathbb{R}} <\nabla V(x+s\omega)\Phi,\Psi>\ ds\ +o(1). \tag{2.22}$$

Now, we study (b); using Lemma 3 and Lemma 4 for $\delta=\rho+1,\ t=\frac{1}{2}\ \arg\sinh(\frac{s}{\sqrt{\lambda}})$ and $p>\frac{n}{1+\rho}$,

$$\begin{aligned} ||\ \nabla V(x+s\omega)e^{-\frac{i}{2}\arg\sinh(\frac{s}{\sqrt{\lambda}})H_0}\Phi\ || \le\ C_{Np}\ &\Bigg(<s>^{-(\rho+1)}+\mid\frac{s}{\sqrt{\lambda}}\mid^{-\frac{n}{p}}\ [<s(1+\frac{s^2}{\lambda})^{-\frac{1}{2}}>^{-N}\\ &+\lambda^{\frac{n-Nq}{2q}}\left((\mid s\mid(1+\frac{s^2}{\lambda})^{-\frac{1}{2}})^{-1}+1\right)^{N}]\Bigg). \end{aligned}$$

Since for $| s |> \lambda^\epsilon$, $| s | (1+\frac{s^2}{\lambda})^{-\frac{1}{2}} \geq \lambda^\epsilon \ (1+\lambda^{2\epsilon-1})^{-\frac{1}{2}} \ \geq C \ \lambda^\epsilon$, choosing $N \gg 1$ in the last inequality, one has :

$$|| \nabla V(x+s\omega) e^{-\frac{i}{2} \arg\sinh(\frac{s}{\sqrt{\lambda}}) H_0} \Phi \ || = O \ (< s >^{-\frac{n}{p}}). \tag{2.23}$$

Taking p such that $1 < \frac{n}{p} < 1+\rho$ and integrating with respect to s, we obtain :

$$(b) \quad \leq C \ \int_{|s|>\lambda^\epsilon} < s >^{-\frac{n}{p}} \ ds \ = \ O \ \left(\lambda^{(1-\frac{n}{p})\epsilon} \right). \tag{2.24}$$

Thus,

$$F_1(\lambda, \omega) \ = \ \frac{1}{2\sqrt{\lambda}} \ \left(\int_{\mathbb{R}} \ < \nabla V(x+s\omega)\Phi, \Psi > \ ds + o(1) \right), \tag{2.25}$$

and Proposition 2 is proved. $\square$

2.2 Uniqueness of the potential.

Let V_1 and V_2 be potentials satisfying (H_2) such that $S(V_1) = S(V_2)$. By Proposition 2, we have :

$$\int_{-\infty}^{+\infty} \nabla V(x+t\omega) \ dt \ = \ 0 \ , \ \forall x \in \mathbb{R}^n. \tag{2.26}$$

where $V = V_1 - V_2$. This integral represents the X-ray transform of ∇V and it is well-known that (2.26) implies that $V=0$, (see for example [10], Lemma 5).

3 Generalizations.

In this section, we generalize our results in the case where the free Hamiltonian H_0 is given by :

$$H_0 = p^2 - \sum_{k=1}^{n_-} a_k^2 \ x_k^2 + \sum_{k=n_-+1}^{n} \ a_k^2 \ x_k^2 \ , \tag{3.1}$$

with $a_k > 0, \ n_- \geq 1$ and with the convention $\sum_{j=a}^b = 0$ if $b < a$.

From Faris-Levine Theorem [1], it is well-known that H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. It is important to remark that the repulsive effects due to $-x_1^2$ overwhelm the confinement due to $+x_{n_-+1}^2$ and moreover, at high energies, the classical trajectories become straight lines, (see [13] for a similar problem with constant magnetic fields).

Now, let us consider a real potential $V \in L^\infty(\mathbb{R}^n)$ satisfying for some $\epsilon > 0$:

(H_3) $$| V(x) | \leq \ C \ \ < \ln < x_- >>^{-1-\epsilon} \ ,$$

where $x = (x_-, x_+) \in \mathbb{R}^{n_-} \times \mathbb{R}^{n-n_-}$. We denote by $H = H_0 + V(x)$ the full Hamiltonian.

The scattering theory for the pair (H, H_0) has been studied in [3] where the authors show that, under assumption (H_3), the wave operators $W^{\pm}$ exist and are complete. As in section 1, we define the scattering operator $S = S(V)$ and in order to investigate the inverse scattering problem, we assume stronger hypotheses on V; we consider smooth potentials such that :

$$(H_4) \qquad | \partial_x^\alpha V(x) | \leq C_\alpha < x >^{-\rho-|\alpha|} ,$$

with $\rho > \frac{1}{2}$, (and for α with finite order). We emphasize that we need a decay assumption on variable x in order to define the X-ray transform of ∇V for all incident directions $\omega \in S^{n-1}$.

The main result is the following :

Theorem 7
Let V_1, V_2 be potentials satisfying (H_4). Then :

$$S(V_1) = S(V_2) \iff V_1 = V_2 .$$

To prove Theorem 7, we follow the same strategy as in Theorem 1; it suffices to show that Proposition 2 is valid for test functions $\Phi(x) = \Phi_1(x_-)\Phi_2(x_+)$, (resp. $\Psi(x) = \Psi_1(x_-)\Psi_2(x_+)$), with Φ_j and Ψ_j in the set of the Schwartz functions, and for $\omega = (\omega_-, \omega_+) \in S^{n-1}$ with $\omega_- \neq 0$. The proof of Proposition 2 is based on generalized Mehler's formula [2].

Proof of Theorem 7 :

We only sketch the proof and we leave the details to the reader. For the sake of simplicity, we give the main arguments with a free Hamiltonian given on $L^2(I\!R^2)$ by :

$$H_0 = p^2 - a_1^2\ x_1^2 + a_2^2\ x_2^2\ , \tag{3.2}$$

with $a_j > 0$. In this case, the free evolution is given by generalized Mehler's formula :

$$\forall t \neq 0\ ,\ e^{-itH_0} = M_t\ D_t\ \mathcal{F}\ M_t\ , \tag{3.3}$$

where M_t is the multiplication operator :

$$M_t\Phi(x) = e^{\frac{i}{2a_1}\coth(2a_1 t)\ x_1^2}\ e^{\frac{i}{2a_2}\cot(2a_2 t)\ x_2^2}\ \Phi(x), \tag{3.4}$$

D_t is the dilation operator :

$$D_t\Phi(x) = \left(\frac{a_1}{i\sinh(2a_1 t)}\right)^{\frac{1}{2}} \left(\frac{a_2}{i\sin(2a_2 t)}\right)^{\frac{1}{2}} \Phi\left(\frac{a_1 x_1}{\sinh(2a_1 t)}, \frac{a_2 x_2}{\sin(2a_2 t)}\right)\ , \tag{3.5}$$

and $\mathcal{F}$ is the usual Fourier transform on $L^2(I\!R^2)$.

As in Lemma 6, one has :

$$e^{itH_0}\ p_1\ e^{-itH_0} = \sinh(2a_1 t)\ x_1 + \cosh(2a_1 t)\ p_1\ . \tag{3.6}$$

$$e^{itH_0}\ p_2\ e^{-itH_0} = -\sin(2a_2 t)\ x_2 + \cos(2a_2 t)\ p_2\ . \tag{3.7}$$

Following the proof of Proposition 2, we can show that the leading term of $< [S, p_1]\Phi_{\lambda,\omega}, \Psi_{\lambda,\omega} >$, when $\lambda \to +\infty$, is given by :

$$I_1 = \int_{\mathbb{R}} \partial_1 V \left(x_1 + \frac{\sqrt{\lambda}}{a_1} \sinh(2a_1 t)\ \omega_1,\ x_2 + \frac{\sqrt{\lambda}}{a_2} \sin(2a_2 t)\ \omega_2 \right)\ \Phi, \Psi >\ \cosh(2a_1 t)\ dt, \quad (3.8)$$

where $\omega = (\omega_1, \omega_2)$, $\omega_1 \neq 0$. Using the change of variables $t = \frac{1}{2}\ \arg\sinh(\frac{s}{\sqrt{\lambda}})$, we easily obtain :

$$I_1 \sim \frac{1}{2\sqrt{\lambda}} < \int_{\mathbb{R}} \partial_1 V(x + s\omega)\ ds\ \Phi, \Psi > . \quad (3.9)$$

In the same way, the leading term of $< [S, p_2]\Phi_{\lambda,\omega}, \Psi_{\lambda,\omega} >$ is given by :

$$I_2 = \int_{\mathbb{R}} \partial_2 V \left(x_1 + \frac{\sqrt{\lambda}}{a_1} \sinh(2a_1 t)\ \omega_1,\ x_2 + \frac{\sqrt{\lambda}}{a_2} \sin(2a_2 t)\ \omega_2 \right)\ \Phi, \Psi >\ \cos(2a_1 t)\ dt, \quad (3.10)$$

$$\sim \frac{1}{2\sqrt{\lambda}} < \int_{\mathbb{R}} \partial_2 V(x + s\omega)\ ds\ \Phi, \Psi > . \quad (3.11)$$

Thus, the knowlegde of $S(V)$ at high energies determines the X-ray transform of ∇V for incident directions $\omega = (\omega_1, \omega_2)$ with $\omega_1 \neq 0$. Using a standard continuity argument in the variable ω, we conclude as in section 2.2. $\Box$

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References

[1] M. Reed - R. Simon, Methods of mathematical physics, Vol. 2, *Academic Press*, (1978).

[2] L. Hörmander, Symplectic classification of quadratic forms, and generalized Mehler formulas, *Math. Z.* 219 (3), 413-449, (1995).

[3] J. F. Bony - R. Carles - D. Häfner - L. Michel, Scattering theory for the Schrödinger equation with repulsive potential, *Arxiv : math.AP/0402170*, (2004), submitted.

[4] V. Enss - R. Weder, The geometrical approach to multidimensional inverse scattering, *J. Math. Phys*, Vol. 36 (8), 3902-3921, (1995).

[5] S. Arians, Geometric approach to inverse scattering for the Schrödinger equation with magnetic and electric potentials, *J. Math. Phys.* 38 (6), 2761-2773, (1997).

[6] W. Jung, Geometric approach to inverse scattering for Dirac equation, *J. Math. Phys.* 36 (8), 3902-3921, (1995).

[7] R. Weder, Multidimensional inverse scattering in an electric field, *Journal of Functional Analysis*, Vol. 139 (2), 441-465, (1996).

[8] F. Nicoleau, Inverse scattering for Stark Hamiltonians with short-range potentials, *Asymptotic Analysis*, 35 (3-4), 349-359, (2003).

[9] R. Weder, The Aharonov-Bohm effect and time-dependent inverse scattering theory, *Inverse Problems*, Vol. 18 (4), 1041-1056, (2002).

[10] F. Nicoleau, A stationary approach to inverse scattering for Schrödinger operators with first order perturbation, *Communication in P.D.E*, Vol 22 (3-4), 527-553, (1997).

[11] F. Nicoleau, An inverse scattering problem with the Aharonov-Bohm effect, *Journal of Mathematical Physics*, Issue 8, 5223-5237, (2000).

[12] F. Nicoleau, A constructive procedure to recover asymptotics of short-range or long-range potentials, *Rapport de recherche Université de Nantes*, 03/05-1, (2003), submitted.

[13] S. Arians, Geometric approach to inverse scattering for hydrogen-like systems in a homogeneous magnetic field, *Journal of Mathematical Physics 39*, no. 4, 1730-1743, (1998).