COUNTING CLOSED ORBITS OF GRADIENTS OF CIRCLE-VALUED MAPS

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Abstract. Let M be a closed connected manifold, $f : M \to S^1$ be a Morse map, belonging to an indivisible integral class $\xi \in H^1(M)$, v be an f-gradient satisfying the transversality condition. The Novikov construction associates to these data a chain complex $C_* = C_*(f, v)$. The first main result of the paper is the construction of a *functorial* chain homotopy equivalence from C_* to the completed simplicial chain complex of the infinite cyclic covering of M, corresponding to ξ. The second main result states that the torsion of this chain homotopy equivalence is equal to the Lefschetz zeta function of the gradient flow for any gradient-like vector field v satisfying the transversality condition and having only hyperbolic closed orbits.

1. Introduction

The subject of the present paper belongs to the intersection of two domains of topology: Morse-Novikov theory of circle-valued functions, and the theory of dynamical zeta functions. We start by a quick recollection of the basics of the Morse-Novikov theory, and continue with the background up to Section 1.8 which contains the statement of the results of the paper.

1.1. **Morse complex.** One of the basic constructions in the classical Morse theory is that of *Morse complex*. Starting with a Morse function $g : M \to \mathbf{R}$ on a closed manifold M, and a gradient-like vector field v for g, this construction gives a chain complex $C_*(g, v)$, which computes the homology of M (here the vector field v must satisfy the transversality condition, that is the stable and unstable manifolds of all critical points intersect transversally). The group $C_k(g, v)$ is a free abelian group, freely generated by the critical points of g of index k, and the boundary operator is defined via counting flow lines of v joining critical points of g.

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1.2. **Novikov complex.** In the beginning of 80s this construction was generalized by S.P.Novikov [20] to the case of circle-valued Morse functions. The input data for this construction is a circle-valued Morse function $f : M \to S^1$ on a closed connected manifold M and a gradient-like vector field v for f. As before, we assume that v satisfies the transversality condition. The result is a chain complex $C_*(f,v)$ of free modules over a ring $\widehat{L} = \mathbf{Z}[[t]][t^{-1}]$ of Laurent series in one variable with integral coefficients and finite negative part. The construction of the Novikov complex is recalled in Section 3.2 in details, here we just mention that $C_k(f,v)$ is freely generated over $\widehat{L}$ by the critical points of f. The homology of $C_*(f,v)$ is described by the following isomorphism:

$$H_*(C_*(f,v)) \approx H_*(\bar{M}) \underset{L}{\otimes} \widehat{L} \tag{1}$$

Here $\bar{M}$ is the infinite cyclic cover of $\bar{M}$ induced by f from $\mathbf{R} \to S^1$, and $L = \mathbf{Z}[t, t^{-1}]$ (we assume that the homotopy class of f is indivisible in $H_1(M, \mathbf{Z})$). The proof of the isomorphism (1) is contained in [22]. This paper contains actually some more than the proof of the above isomorphism. Namely, we gave there an explicit construction of a chain homotopy equivalence

$$\phi : C_*(f,v) \to C_*^{\Delta}(\bar{M}) \underset{L}{\otimes} \widehat{L} \tag{2}$$

(here $C^{\Delta}(\bar{M})$ stands for the simplicial chain complex of $\bar{M}$).

1.3. **Dynamical zeta functions.** Now we turn to dynamical systems. Let us start with a continuous map $h : X \to X$. In order to investigate the sets of periodic points of h of period n for $n \to \infty$ Artin and Mazur [1] introduced a zeta function which encodes the information about all the periodic points into a single power series in one variable. This zeta function and its generalizations have been intensively studied. The Artin-Mazur zeta function has a homotopy counterpart – the *Lefschetz zeta function* introduced by S.Smale [29]. Here is the definition. Let Fix h denote the set of fixed points of h. Assume that for any n the set Fix h^n is finite. Set

$$L_k(h) = \sum_{a \in \mathrm{Fix}\ h^k} \nu(a) \tag{3}$$

where $\nu(a) \in \mathbf{Z}$ is the index of the fixed point a. Define the Lefschetz zeta function by the following formula:

$$\zeta_L(t) = \exp\left(\sum_{k=1}^{\infty} \frac{L_k(h)}{k} t^k\right) \tag{4}$$

Here is the formula for ζ_L in terms of the homology invariants of h:

$$\zeta_L(t) = \prod_i \det(I - th_i)^{(-1)^{i+1}} \tag{5}$$

where h_i stands for the homomorphism induced in $H_i(X)$ by h.

Proceeding to the dynamical systems generated by flows on manifolds, one would expect that the theory described above generalizes to this setting. For every flow on a smooth manifold there should be some power series of the type (4), computable in homotopical terms. Such results were obtained in 80s in many important cases (see [8]) but only for non-singular flows. We shall cite here only the simplest result in this direction, which goes back to [16]. We shall reformulate it in terms which we use later, so we start by introducing the necessary definitions.

1.4. **Eta function and zeta function.** Let M be a closed connected manifold, and $f : M \to S^1$ be a Morse map such that $[f] \in H_1(M, \mathbf{Z})$ is indivisible. Let v be a gradient-like vector field for f. We shall study the dynamics of the flow generated by $(-v)$. As before, we assume that v satisfies transversality condition. Assume moreover, that all closed orbits of $(-v)$ are hyperbolic. A natural numerical invariant associated to a non-degenerate closed orbit γ of $(-v)$ is the *Fuller index* ([11]) $\iota_F(\gamma) = \frac{\varepsilon(\gamma)}{m(\gamma)}$ where $\varepsilon(\gamma) \in \{1, -1\}$ is the Poincaré index of γ, and $m(\gamma) \in \mathbf{N}$ is the multiplicity of γ. For each closed orbit γ of $(-v)$ define $n(\gamma)$ to be the integer, opposite to the winding number of $f \circ \gamma$. In another words, if $[\gamma] \in H_1(M)$ stands for the homology class of γ, then $f(\gamma) = -n(\gamma) \cdot \mathbf{1}$, where $\mathbf{1}$ is the generator of $H_1(S^1)$ corresponding to the positive orientation. Introduce the *Lefschetz eta function* by the following formula:

$$\eta_L(-v) = \sum_{\gamma \in Cl(-v)} \iota_F(\gamma) t^{n(\gamma)} \tag{6}$$

where $Cl(-v)$ denotes the set of all the closed orbits of $(-v)$ (we identify closed orbits which are obtained one from another by reparametrization). Using our conditions on v it is not difficult to check that the expression in the right hand side of (6) is a well-defined power series in t with rational coefficients and vanishing constant term. Therefore the power series

$$\zeta_L(-v) = \exp(\eta_L(-v)) \tag{7}$$

is well defined and is again in $\widehat{L}_{\mathbf{Q}}$. This power series is called *Lefschetz zeta function (of the flow generated by $(-v)$).*

Lemma 1.1. *The power series $\zeta_L(-v)$ is in $\widehat{L}$.*

Proof. A well-known computation (see [7], Prop. 5.19) gives

$$\zeta_L(-v) = \prod_{\gamma \in ClPr(-v)} \left(1 + \varepsilon_1(\gamma) t^{n(\gamma)}\right)^{\varepsilon_2(\gamma)} \tag{8}$$

where $ClPr(-v)$ stands for the set of *prime* closed orbits, and $\varepsilon_1(\gamma), \varepsilon_2(\gamma) \in \{1, -1\}$. The righthand side of (8) is obviously in $\widehat{L}$. □

1.5. **Counting closed orbits of a non-singular map $M \to S^1$.** Assume now that f has no critical points, so that f is a fibration over S^1. In this case $H_*(\bar{M}) \underset{L}{\otimes} \widehat{L}$ vanishes and we can associate to M a (suitably defined) Whitehead torsion. We shall work with the following versions of K_1-groups:

$$\overline{K}_1(\widehat{L}) = K_1(\widehat{L})/\{0, [-1]\},$$
$$\mathrm{Wh}\ (\widehat{L}) = K_1(\widehat{L})/T, \quad \text{where } T = \{\pm t^n \mid n \in \mathbf{Z}\}.$$

(We denote by $\widehat{L}^\bullet$ the multiplicative group of all units in $\widehat{L}$, and by $[x]$ the image in $K_1(\widehat{L})$ of the element $x \in \widehat{L}^\bullet$.) The ring $\widehat{L}$ is euclidean, and therefore the determinant map defines an isomorphism $\det : K_1(\widehat{L}) \to \widehat{L}^\bullet$ and also an isomorphism

$$\mathrm{Wh}\ (\widehat{L}) = K_1(\widehat{L})/T \xrightarrow{\det} W$$

where

$$W = \{1 + \sum_{i>0} a_i t^i \mid a_i \in \mathbf{Z}\}$$

is the multiplicative group of power series with the constant term 1 (such power series is called *Witt vector*). Let Δ be any C^1 triangulation of M, then it lifts to a $\mathbf{Z}$-invariant triangulation of $\bar{M}$, and the simplicial chain complex $C_*^\Delta(\bar{M})$ obtains a natural base, well defined up to the action of the elements $\pm t^n, n \in \mathbf{Z}$. Therefore the torsion of the acyclic complex $C_*^\Delta(\bar{M})$ is well defined as an element of Wh $(\widehat{L})$. It will be denoted by $\tau(M)$. The theorem proved in [16] says that

$$\det(\tau(M)) = (\zeta_L(-v))^{-1} \tag{9}$$

Note that the left hand side depends only on M and $[f] \in H^1(M, \mathbf{Z})$ but not on the particular choice of v. Generalizations of this formula to the case when $f : M \to S^1$ has critical points were obtained only very recently in [13], [24]. These two papers consider various particular cases of the problem; we shall explain the results in the next two subsections.

1.6. **The acyclic case: the Hutchings-Lee formula.** The first generalization of the formula (9) to the case of Morse maps with critical points was obtained by M.Hutchings and Y-J.Lee [13]. Their formula for $\zeta_L(-v)$ contains an additional term depending on the Novikov complex. Let $\mathcal{G}(f)$ denote the set of gradient-like vector fields v, such that that the transversality condition holds and every closed orbit of v is hyperbolic (such gradient-like vector fields will be called *Kupka-Smale gradients*). Set

$$L_{\mathbf{Q}} = \mathbf{Q}[t, t^{-1}], \quad \widehat{L}_{\mathbf{Q}} = \mathbf{Q}[[t]][t^{-1}]$$

Assume that $C_*(f, v) \underset{L}{\otimes} \widehat{L}_{\mathbf{Q}}$ and $H_*(\bar{M}) \underset{L}{\otimes} \widehat{L}_{\mathbf{Q}}$ are acyclic, and denote by $\tau_{Nov} \in \overline{K}_1(\widehat{L}_{\mathbf{Q}})/T$, resp. by $\tau_M \in \overline{K}_1(\widehat{L}_{\mathbf{Q}})/T$ the torsions of these complexes. Then ([13], Theorem 1.12) for every $v \in \mathcal{G}(f)$

$$\det(\tau_{Nov}/\tau_M) = \zeta_L(-v) \tag{10}$$

1.7. **The non-acyclic case.** A formula for $\zeta_L(-v)$ without any acyclicity assumption was obtained in [24]. The methods of this work are different from these of [13]; they were introduced earlier [23] to prove the generic rationality of the boundary operators in the Novikov complex. In [24] I proved that there is a chain homotopy equivalence

$$\psi : C_*(f, v) \to C_*^{\Delta}(\bar{M}) \underset{L}{\otimes} \widehat{L} \tag{11}$$

such that

$$\det \tau(\psi) = (\zeta_L(-v))^{-1}. \tag{12}$$

However the class of gradients v for which the formula (11) was proved is smaller than the class of all Kupka-Smale gradients. Namely the formula (11) holds for every gradient-like vector field v from a C^0-open-and-dense subset of $\mathcal{G}(f)$.

It is natural to ask if the formula (12) is true for *every* f-gradient in $\mathcal{G}(f)$. This question was posed in my paper [25] (in a more general non-abelian setting) and was the starting point for the present work. The theorem B below (8) provides the positive answer, at least for the abelian case.

The other question suggested by the formula (12) is whether the chain homotopy equivalence ψ is homotopic to the chain equivalence (2) (the constructions of the two equivalences are different from one another). This is related to the question, whether there is a *canonical* homotopy equivalence $C_*(f, v) \to C_*^{\Delta}(\bar{M}) \underset{L}{\otimes} \widehat{L}$, such that the formula (12) holds. This question was posed to me in 1998 by M.Kervaire and V.Turaev. Their question is answered by Theorem A below, and this precise form of the answer was conjectured by V.Turaev. The methods developed in the present paper for the proof of the functoriality of the equivalence (2) lead also to a quick proof that ϕ and ψ are homotopic.

1.8. **Statement of results.** The main aim of the present paper is to obtain an analog of the formula (12) for *any* $v \in \mathcal{G}(f)$. The formula which we obtain (Theorem B) generalizes both the results of [13] and [24], and provides the answer in the general case. We start by studying in details the chain homotopy equivalences between $C_*(f,v)$ and $C^\Delta_*(\bar M) \underset{L}{\otimes} \widehat{L}$. It is clear from the preceding that these chain homotopy equivalences are important geometric objects, related to both the Novikov complex and the dynamics of the gradient flow. In the first part of the present paper we prove that the construction of the chain homotopy equivalence $C_*(f,v) \to C^\Delta_*(\bar M) \underset{L}{\otimes} \widehat{L}$ given in [22] is *functorial* (see Theorem A). In the second part (Theorem B) we prove that the torsion of this chain homotopy equivalence equals to the Lefschetz zeta function of the gradient flow for any $v \in \mathcal{G}(f)$. In the course of the proof we show also that the chain homotopy equivalences $C_*(f,v) \to C^\Delta_*(\bar M) \underset{L}{\otimes} \widehat{L}$ constructed in [22] and [24] are chain homotopic (this is done in Subsection 5.1). Now we proceed to precise statements. The next definition is convenient to state the functoriality property.

Definition 1.2. *A Morse-Novikov triple* (or *MN-triple* for short) is a triple (M,f,v) where M is a closed connected manifold, $f : M \to S^1$ a Morse map, such that $\xi(f) = f_* : H_1(M) \to H_1(S^1) \approx \mathbf{Z}$ is indivisible, and v a gradient-like vector field, satisfying transversality condition.

Let (M,f,v) be an MN-triple. The Novikov construction, which is standard by now (we recall it in Section 3.2) associates to each Morse-Novikov triple a chain complex $C_*(f,v)$ of $\widehat{L}$-modules, such that $C_k(f,v)$ is freely generated over $\widehat{L}$ by the set $S_k(f)$ of the critical points of f of index k.

Let (M_1,f_1,v_1), (M_2,f_2,v_2) be two MN-triples. Denote $\xi(f_1)$ by ξ_1, and $\xi(f_2)$ by ξ_2. Let $\bar M_1, \bar M_2$ be the infinite cyclic covers corresponding to ξ_1, resp. ξ_2. Let $g : M_1 \to M_2$ be a diffeomorphism, satisfying the following condition:

$$\alpha)\ g_*(v_1) = v_2; \quad \beta)\ g^*(\xi_2) = \xi_1 \tag{13}$$

Let $\bar g : \bar M_1 \to \bar M_2$ be a lift of g. Then $\bar g$ is a diffeomorphism which commutes with the action of $\mathbf{Z}$, and we obtain an $\widehat{L}$-isomorphism

$$\bar g_\sharp : C^s_*(\bar M_1) \underset{L}{\otimes} \widehat{L} \to C^s_*(\bar M_2) \underset{L}{\otimes} \widehat{L}$$

(here and elsewhere we denote by $C^s(X)$ the singular chain complex of X). Also $\bar g$ induces an isomorphism $\bar g_! : C_*(f_1, v_1) \to C_*(f_2, v_2)$ of Novikov complexes (see 3.6).

Theorem A. *To each Morse-Novikov triple (M, f, v) is associated a chain homotopy equivalence*

$$\Phi = \Phi(M, f, v) : C_*(f, v) \to C^s_*(\bar M) \underset{L}{\otimes} \widehat L \tag{14}$$

which is functorial in the following sense:

For any two Morse-Novikov triples (M_1, f_1, v_1), (M_2, f_2, v_2) and a diffeomorphism $g : M_1 \to M_2$ satisfying the condition (13) the following diagram is chain homotopy commutative:

$$\begin{tikzcd} C_*(f_1, v_1) \arrow[r, "\bar g_!"] \arrow[d, "\Phi_1"] & C_*(f_2, v_2) \arrow[d, "\Phi_2"] \\ C^s_*(\bar M_1) \underset{L}{\otimes} \widehat L \arrow[r, "\bar g_\sharp"] & C^s_*(\bar M_2) \underset{L}{\otimes} \widehat L \end{tikzcd} \tag{15}$$

(here $\Phi_i = \Phi(M_i, f_i, v_i)$ for $i = 1, 2$, and $\bar g : \bar M_1 \to \bar M_2$ is any lift of g).

Remark 1.3. The diffeomorphism $\bar g$ (and the maps $\bar g_!, \bar g_\sharp$) are defined by g uniquely up to multiplication by $t^n, n \in \mathbf{Z}$.

If we replace the complex of singular chains $C^s_*(\bar M)$ in Theorem A by the simplicial chain complex $C^\Delta_*(\bar M)$ it will be possible to consider the torsion of the resulting chain homotopy equivalence. Let us recall the corresponding notions. Let X be a topological space. Let $\pi : \widetilde X \to X$ be a regular covering with structure group H. The singular chain complex $C^s_*(\widetilde X)$ is free over $\mathbf{Z}H$. If X is a simplicial complex then $\widetilde X$ inherits a natural H-invariant triangulation from X, and the simplicial chain complex $C^\Delta_*(\widetilde X)$ is a chain complex of free finitely generated $\mathbf{Z}H$-modules (here Δ stands for the triangulation of X). There is a chain homotopy equivalence

$$\chi_\Delta : C^\Delta_*(\widetilde X) \to C^s_*(\widetilde X) \tag{16}$$

of chain complexes over $\mathbf{Z}H$, functorial up to chain homotopy (see [30], Ch. 4, §4.) The next lemma is a corollary of the well known result on the combinatorial invariance of the Whitehead torsion (see [17]).

Lemma 1.4. *Let X be a C^∞ closed manifold. Let Δ_1, Δ_2 be C^1-triangulations of X. Then*

$$\chi_{\Delta_1}^{-1} \circ \chi_{\Delta_2} : C_*^{\Delta_2}(\widetilde{X}) \to C_*^{\Delta_1}(\widetilde{X}) \tag{17}$$

is a simple homotopy equivalence of finitely generated free based chain complexes over $\mathbf{Z}H$ *(that is, its torsion* $\tau(\chi_{\Delta_1}^{-1} \circ \chi_{\Delta_2})$ *vanishes in* $\mathrm{Wh}\ (H) = K_1(\mathbf{Z}H)/(\pm H)$*).* □

Return now to the Morse-Novikov theory. Let (M, f, v) be a MN-triple. In the previous discussion let us set $X = M$ and consider the infinite cyclic covering $\bar{M} \to M$ corresponding to $\xi = \xi(f)$. Let Δ be a triangulation of M and define a chain map

$$\Phi_\Delta = \chi_\Delta^{-1} \circ \Phi(M, f, v) : C_*(f, v) \to C_*^\Delta(\bar{M}) \underset{L}{\otimes} \widehat{L} \tag{18}$$

The chain complexes $C_*(f, v)$ and $C_*^\Delta(\bar{M}) \underset{L}{\otimes} \widehat{L}$ are finitely generated free complexes over $\widehat{L}$. Choosing a lift of a subset $S(f) \subset M$ to $\bar{M}$ and lifts of all the simplices of Δ to $\bar{M}$ we obtain free bases in both complexes. With respect to any such choice of bases we obtain the torsion $\tau(\Phi_\Delta) \in \overline{K}_1(\widehat{L})$. The ambiguity in the choice of bases leads to multiplication by $\pm t^n$, so that the image of $\tau(\Phi_\Delta)$ in $\overline{K}_1(\widehat{L})/T$ does not depend on these choices and, moreover, does not depend on the choice of Δ by Lemma 1.4.

Definition 1.5. The element $\det(\tau(\Phi_\Delta)) \in W$ will be denoted by $w(M, f, v)$.

Our second main result – Theorem B – says that $w(M, f, v)$ is equal to the inverse Lefschetz zeta function (Subsection 1.4) of the flow generated by $(-v)$. For convenience of notation we shall abbreviate $(\zeta_L(-v))^{-1}$ as $\zeta(v)$.

Theorem B. *For every $v \in \mathcal{G}(f)$ we have*

$$w(M, f, v) = \zeta(v). \tag{19}$$

Now we shall say some words about the proofs. Theorem A is proved by analysing step by step the construction from [22]. The arguments which we use here to prove the functoriality are based on noetherian commutative algebra. The proof of Theorem B is based on our work [23]. In this paper we have introduced a class $\mathcal{GT}_0(f)$ of f-gradients (where $f : M \to S^1$ is a given Morse function) which is C^0-open-and-dense in the set $\mathcal{GT}(f)$ of all f-gradients satisfying transversality condition. For every $v \in \mathcal{GT}_0(f)$ the dynamics of the gradient flow generated by v has many remarkable properties; in particular the Novikov incidence coefficients are rational functions. A slight modification of the definition of the class $\mathcal{GT}_0(f)$ ([24]) leads to the subset $\mathcal{G}_0(f) \subset \mathcal{G}(f)$ which is C^0-open-and-dense in $\mathcal{G}(f)$; for every $v \in \mathcal{GT}_0(f)$ the Novikov incidence coefficients are rational functions and the formula (12) holds. The case of general gradient v as required by Theorem B is done by a C^0-perturbation argument which makes the reduction to the C^0-generic case. Actually we do not know if both sides of (12) are invariant under such perturbation. But note that both these expressions are power series in t. We show that each *finite part* of such series is invariant with respect to such perturbation, and this finishes the proof.

1.9. **Related work.** In the paper [6] M.Farber and A.Ranicki gave an alternative construction of a chain complex generated by critical points of a circle-valued Morse map $f : M \to S^1$, and proved that this complex computes the completed homology of the infinite cyclic covering. (The paper [6] contains actually a more general construction of a chain complex over Novikov completions of fundamental group.) In the preprint [26] A.Ranicki identified this chain complex in the particular case of the infinite cyclic covering with the chain complex from [23]. The results of the paper [24] were generalized in [25] to the non-abelian case. The work of D.Schütz [27] contains a generalization of the results of [25] to the case of irrational forms. One of the ingredients in the work of D.Schütz, which is quite new to the Morse-Novikov theory is the application of Hochshild homology techniques, which is related to the work [12] by R.Geoghegan and A.Nicas. The results of [27] pertained to the C^0-generic case; in a recent preprint [28] D.Schütz considers the general case. He constructs, in particular, a new chain equivalence between Novikov complex and the completed simplicial chain complex.

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2. Preliminaries on chain complexes

In this section we work in the category of chain complexes of left R-modules, where R is a ring. We consider in this paper only chain complexes, concentrated in positive degrees. We shall often omit the adjective "chain", so that *complex* means "chain complex", *map* means "chain map", *homotopy* means "chain homotopy" etc. A chain map $f : C_* \to D_*$ is called *homology equivalence* if it induces an isomorphism in homology. We shall wrtie $f \sim g$ if f is homotopic to g.

2.1. **Models.** Let R be a ring (not necessarily commutative).

Definition 2.1. Let X_*, Y_* be complexes over R. A map $f : X_* \to Y_*$ is called *model for* Y_* if f is a homology equivalence and X_* is free.

The next proposition states that for a given Y_* its model is essentially unique.

Proposition 2.2. *Let $f : X_* \to Y_*$, $f' : X'_* \to Y_*$ be models for Y_*. Then there is a homotopy equivalence $\mu : X'_* \to X_*$, such that $f \circ \mu \sim f'$. The homotopy class of μ is uniquely determined by the homotopy classes of f and f'.*

This proposition follows from the next one.

Proposition 2.3. *Let $\alpha : X_* \to Y_*$, $\beta : Z_* \to Y_*$ be chain maps, and assume that X_* is free and β is a homology equivalence. Then there is a map $\gamma : X_* \to Z_*$, such that $\beta \circ \gamma \sim \alpha$. The homotopy class of γ is uniquely determined by the homotopy classes of α and β.*

Proof. We shall prove the existence of γ, the homotopy uniqueness is proved similarly. Adding to Z_* a contractible chain complex if necessary, we can assume that the map β is epimorphic. Using induction on degree we shall construct a map γ satisfying $\beta\gamma(x) = \alpha(x)$. Assume that γ is already defined on every X_i with $i < k$ and satisfies the equation $\beta\gamma(x) = \alpha(x)$ for $\deg x < k$. It suffices to construct for every free generator e_k of X_k an element $x \in Z_k$, such that $\beta(x) = \alpha(e_k)$ and $\partial x = \gamma(\partial e_k)$. Choose any $x_0 \in Z_k$, such that $\beta(x_0) = \alpha(e_k)$. Consider the element $y = \partial x_0 - \gamma(\partial e_k)$. It is a cycle of the complex $Z'_* = \operatorname{Ker} \beta$. Since β is a homology equivalence, the complex Z'_* is acyclic, therefore, there is $z \in Z'_k$, such that $\partial z = y$. Now set $x = x_0 - z$ and the proof is over. □

2.2. **Filtrations and adjoint complexes.** We recall here briefly the material of [22], §3.A. We shall call *filtration* of a chain complex C_* a sequence of subcomplexes $C_*^{(i)}$, $-1 \leqslant i$, such that

$$0 = C_*^{(-1)} \subset C_*^{(0)} \subset \ldots \quad \text{and} \quad \cup_i C_*^{(i)} = C_* \tag{20}$$

A filtration $C_*^{(i)}$ is *good* if $H_k(C_*^{(i)}/C_*^{(i-1)}) = 0$ for $k \neq i$. For a filtration $\{C_*^{(i)}\}$ of a complex C_* set $C_n^{gr} = H_n(C_*^{(n)}/C_*^{(n-1)})$, and let $\partial_n : C_n^{gr} \to C_{n-1}^{gr}$ be the boundary operator of the exact sequence of the triple

$(C_*^{(n)}, C_*^{(n-1)}, C_*^{(n-2)})$. Then C_*^{gr} endowed with the boundary operator ∂_n is a complex, which will be called *adjoint* to C_*.

Example. Let D_* be any complex. The filtration

$$D_*^{(i)} = \{0 \leftarrow D_0 \leftarrow ... \leftarrow D_i \leftarrow 0 \leftarrow ...\} \tag{21}$$

is called *trivial*. This is obviously a good filtration and $D_*^{gr} = D_*$. The proof of the next lemma is in standard diagram chasing.

Lemma 2.4 ([22], Lemma 3.2). *Let C_*, D_* be complexes. Assume that C_* is endowed with a good filtration, and that D_* is a free complex, endowed with the trivial filtration. Let $\phi : D_* \to C_*^{gr}$ be a map. Then there exists a map $f : D_* \to C_*$ preserving filtrations and inducing the map ϕ in the adjoint complexes. The map f is unique up to a homotopy preserving filtrations.* □

Definition 2.5. A good filtration $\{C_*^{(i)}\}$ of a complex C_* is called *nice* if every module $H_n(C_*^{(n)}, C_*^{(n-1)})$ is a free R-module.

Corollary 2.6. *For a nice filtration $\{C_*^{(i)}\}$ of a complex C_* there exists a homology equivalence $C_*^{gr} \to C_*$ functorial up to homotopy in the category of nicely filtered complexes. If C_* is a complex of free R-modules, this homology equivalence is a homotopy equivalence.* □

2.3. **Strings and inverse limits.** An infinite sequence

$$\mathcal{C} = \{C_*^0 \leftarrow C_*^1 \leftarrow ...\} \tag{22}$$

of chain epimorphisms is called *string*.

A *map of strings* $\mathcal{C} \xrightarrow{h} \mathcal{D}$ is a diagram of the following type:

$$\begin{tikzcd}
C_*^0 \arrow[d, "h_0"] & C_*^1 \arrow[l] \arrow[d, "h_1"] & ..... \arrow[l] & C_*^n \arrow[l] \arrow[d, "h_n"] & C_*^{n+1}... \arrow[l, "p_n"'] \arrow[d, "h_{n+1}"] \\
D_*^0 & D_*^1 \arrow[l] & ..... \arrow[l] & D_*^n \arrow[l] & D_*^{n+1}... \arrow[l, "q_n"']
\end{tikzcd} \tag{23}$$

where h_i are chain maps and all the squares are homotopy commutative.

A *strict map of strings* is a map of strings where all the squares in (23) are commutative. For a string $\mathcal{C}$ the chain complex $\varprojlim C_*^i$ will be denoted by $|\mathcal{C}|_*$ and called *inverse limit of* $\mathcal{C}$.

A strict map of strings induces obviously a chain map of their inverse limits. The aim of the next proposition is to generalize this property to the case of arbitrary maps of strings. This proposition (and its proof) is quite close to Proposition 3.7 of [22], so we shall give only a sketch of the proof.

Proposition 2.7. *Let $h : \mathcal{A} \to \mathcal{B}$ be a map of strings, $h = \{h_i\}$. Assume that $|\mathcal{A}|_*$ is a chain complex of free R-modules. Then there is a chain map $\mathcal{H} : |\mathcal{A}|_* \to |\mathcal{B}|_*$, such that for every k the following diagram is homotopy commutative:*

$$\begin{array}{ccc} |\mathcal{A}|_* & \xrightarrow{\mathcal{H}} & |\mathcal{B}|_* \\ \downarrow & & \downarrow \\ A^k_* & \xrightarrow{h_k} & B^k_* \end{array} \tag{24}$$

(where the vertical arrows are natural projections)

Sketch of proof. For $k \geqslant 0$ let Z^k_* be the chain cylinder of the map $h_k : A^k_* \to B^k_*$. Using homotopy commutativity of the squares in (23) one defines a string $\mathcal{Z} = \{Z^0_* \leftarrow Z^1_* \leftarrow ...\}$ together with two strict maps of strings

$$\mathcal{A} \xrightarrow{\alpha} \mathcal{Z}, \quad \mathcal{B} \xrightarrow{\beta} \mathcal{Z} \tag{25}$$

such that the corresponding maps $\alpha_k : A^k_* \to Z^k_*, \beta_k : B^k_* \to Z^k_*$ are the standard inclusions of the source, resp. the target to the cylinder. Form the corresponding maps of the inverse limits:

$$|\mathcal{A}|_* \xrightarrow{\widetilde{\alpha}} |\mathcal{Z}|_*, \quad |\mathcal{B}|_* \xrightarrow{\widetilde{\beta}} |\mathcal{Z}|_* \tag{26}$$

The map $\widetilde{\beta}$ is a homology equivalence. (Indeed, by the very definition every string satisfies the Mittag-Leffler condition ([14], definition A16). Therefore for every n we have $\lim^1 B^i_n = 0$, $\lim^1 Z^i_n = 0$. In view of [14], Th. A.19 it suffices to recall that each β_k is a homology equivalence.) Apply now Proposition 2.3 to obtain a chain map $\mathcal{H} : |\mathcal{A}|_* \to |\mathcal{B}|_*$, such that $\widetilde{\beta} \circ \mathcal{H} \sim \widetilde{\alpha}$. The commutativity of the diagram (24) is now easy to check using Proposition 2.3. $\square$

It is natural to ask whether the chain homotopy class of $\mathcal{H}$ is uniquely determined by the condition (24). The next two subsections provide an answer to this question for the particular case of the commutative noetherian base ring.

2.4. **A bit of noetherian homological algebra.** We interrupt for a moment our study of strings and their inverse limits in order to prove a proposition about chain complexes over commutative power series rings. In this section A is a commutative noetherian ring, $R = A[[t]]$, and $R_n = A[[t]]/t^n$ (where $n \geqslant 0$). A finite chain complex of free R-modules will be called *homotopy finitely generated* if it is chain equivalent to a finite chain complex of finitely generated free R-modules.

Proposition 2.8. *Let C_*, D_* be homotopy finitely generated complexes over R. Let $f, g : C_* \to D_*$ be chain maps. Let $C_*^{(n)} = C_*/t^n C_*$, $D_*^{(n)} = D_*/t^n D_*$, and $f_n = f/t^n, g_n = g/t^n : C_*^{(n)} \to D_*^{(n)}$ be the corresponding quotient maps. Assume that $f_n \sim g_n$ for every $n \in \mathbf{N}$. Then $f \sim g$.*

Proof. It suffices to prove the proposition for the particular case when both C_*, D_* are finitely generated. In this assumption, let H be the set of *homotopy classes* of maps $C_* \to D_*$; then H is a finitely generated R-module. Note that the maps f_n, g_n are homotopic if and only if the map $f - g : C_* \to D_*$ is homotopic to a map divisible by t^n. The proof is finished by the following lemma.

Lemma 2.9. *Let H be a finitely generated R-module. Let $x \in H$. Assume that x is divisible by t^n for every n. Then $x = 0$.*

Proof. Let us first consider the case, when H has no t-torsion (that is, $tx = 0$ implies $x = 0$ for $x \in H$). Let $N \subset H$ be the submodule of elements, divisible by t^n for any n. Then N satisfies $tN = N$. Indeed, let $x \in N$. Then for every $n \in \mathbf{N}$ we have $x = t^n y_n$. Moreover, there is only one y_n satisfying this relation, and $y_n = t y_{n+1}$ (this follows since N has no t-torsion). Now y_1 is obviously divisible by all powers of t. Since t is in the Jacobson radical of R, the Nakayama's lemma implies $N = 0$.

Let us now consider the case of arbitrary H. Let x be an element divisible by every power of t. Let $T \subset H$ be the submodule of all t-torsion elements, i.e. $T = \{x \mid \exists n \in \mathbf{N}, t^n x = 0\}$. Set $H' = H/T$. Applying to H' the reasoning above, we conclude that $x \in T$. Choose some $k \in \mathbf{N}$, such that $t^k T = 0$. Since $x = t^k u$ (with u necessarily in T), we have $x = 0$. □

2.5. **Strings and inverse limits: part 2.** As in the previous section let A be a noetherian commutative ring and $R = A[[t]]$. Let C_* be a free finitely generated complex of R-modules. The string

$$\mathcal{C} = \{C_*/tC_* \longleftarrow C_*/t^2 C_* \longleftarrow \ldots \longleftarrow C_*/t^n C_* \longleftarrow \ldots\} \tag{27}$$

will be called *the special string corresponding to C_**. Note that since C_* is free and finitely generated, there is a natural isomorphism $|\mathcal{C}|_* \approx C_*$.

Proposition 2.10. *In the hypotheses of Proposition 2.7 assume moreover that $\mathcal{A}$ and $\mathcal{B}$ are special strings. There is only one (up to homotopy) map $\mathcal{H} : |\mathcal{A}|_* \to |\mathcal{B}|_*$ such that all the squares (24) are homotopy commutative.*

Proof. Let A_*, B_* be the free finitely generated complexes over P, generating $\mathcal{A}$, resp. $\mathcal{B}$. If $\mathcal{H}_1, \mathcal{H}_2 : |\mathcal{A}|_* \to |\mathcal{B}|_*$ both make all the squares (24) commutative, then the maps

$$\mathcal{H}_1/t^k, \mathcal{H}_2/t^k : |\mathcal{A}|_*/t^k|\mathcal{A}|_* \to |\mathcal{B}|_*/t^k|\mathcal{B}|_* \tag{28}$$

are homotopic for every k. Apply now Proposition 2.8 and the proof is over. □

Therefore any map $h : \mathcal{A} \to \mathcal{B}$ of special strings determines a chain map $|\mathcal{A}|_* \to |\mathcal{B}|_*$ which is well defined up to homotopy; this map will be denoted $|h|$. The next corollary is immediate.

Corollary 2.11. *Let $\mathcal{A} = (A^i_*)$, $\mathcal{B} = (B^i_*)$, $\mathcal{C} = (C^i_*)$, $\mathcal{D} = (D^i_*)$ be special strings, and*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\alpha"] \arrow[d, "\psi"] & \mathcal{B} \arrow[d, "\phi"] \\ \mathcal{C} \arrow[r, "\beta"] & \mathcal{D} \end{tikzcd} \tag{29}$$

be a square of strings, which is chain homotopy commutative on each finite level, i.e. for every n the chain maps $\phi_n \circ \alpha_n$, $\beta_n \circ \psi_n : A^n_* \to D^n_*$ are homotopic. Then the following square is chain homotopy commutative.

$$\begin{tikzcd} |\mathcal{A}_*| \arrow[r, "|\alpha|"] \arrow[d, "|\psi|"] & |\mathcal{B}_*| \arrow[d, "|\phi|"] \\ |\mathcal{C}_*| \arrow[r, "|\beta|"] & |\mathcal{D}_*| \end{tikzcd} \tag{30}$$

□

3. Proof of theorem A

We start by giving more details on the Morse complex and the Morse-Novikov complex (Subsections 3.1, 3.2). In Subsection 3.3 we recall some points from [22] which will be useful in the sequel. The construction of a functorial chain equivalence Φ is done in Subsection 3.5 after a technical Subsection 3.4. The end of the proof of Theorem A is in Subsection 3.6.

Now some words about terminology. The term *f-gradient* means *a gradient-like vector field for f* (see [15], §3 for definition). For a C^1 vector field v on a manifold M the symbol $\gamma(x,t;v)$ denote the value at t of the integral curve of v which passes through x at $t=0$.

3.1. **About Morse complexes.** Let $f: W \to [a,b]$ be a Morse function on a cobordism W, v be an f-gradient. The set of critical points of f will be denoted by $S(f)$, the set of critical points of f of index k is denoted by $S_k(f)$. For $p \in S(f)$ let $D(p,v)$ denote the stable manifold of p with respect to v:

$$D(p,v) = \{x \in W \mid \gamma(x,t;v) \xrightarrow[t\to\infty]{} p\}$$

We assume that v satisfies *transeversality condition* that is for every $p,q \in S(f)$ we have

$$D(p,v) \pitchfork D(q,-v).$$

For every $p \in S(f)$ choose an orientation of $D(p,v)$. To this data one associates a chain complex C_* of free abelian groups as follows. By definition C_k is a free abelian group generated by $S_k(f)$. To define the boundary operator, let $p \in S_k(f), q \in S_{k-1}(f)$, and let $\Gamma(p,q;v)$ be the set of all orbits of v, joining p to q. The choice of orientations allows to attribute to each orbit $\gamma \in \Gamma(p,q;v)$ a sign $\varepsilon(\gamma) \in \{-1,1\}$. Set $n(p,q;v) = \sum_{\gamma\in\Gamma(p,q;v)} \varepsilon(\gamma)$ and define a homomorphism $\partial_k : C_k \to C_{k-1}$ by

$$\partial_k p = \sum_{q\in C_{k-1}(f)} n(p,q;v)\cdot q \tag{31}$$

One can check that $\partial_{k-1}\circ\partial_k = 0$. The resulting chain complex is called the *Morse complex* or the *Morse-Thom-Smale-Witten complex.* We shall denote it $C_*^{\mathcal{M}}(f,v)$, or simply $C_*(f,v)$ if there is no possibility of confusion. We shall now outline the construction of a chain homotopy equivalence

$$C_*(f,v) \to C_*^s(W,\partial_0 W) \tag{32}$$

(following [22], Appendix, where the reader will find the details). A Morse function $\phi: W \to [a,b]$ is called *ordered* if $\phi(x) < \phi(y)$ whenever $x,y \in S(\phi)$ and $\mathrm{ind}x < \mathrm{ind}y$. By a standard application of the rearrangement procedure, see [15], §4) there is an ordered Morse function $\phi: W \to [a,b]$ such that v is also a ϕ-gradient. Let $a = a_0 < a_1 < \ldots < a_m < a_{m+1} = b$ be an *ordering sequence* for ϕ, that is, every a_i is a regular value for ϕ and $S_i(\phi) \subset \phi^{-1}([a_i,a_{i+1}])$ for every i (here m stands for the dimension of M). Let $W^{(i)} = \phi^{-1}([a_0,a_{i+1}])$ and consider the filtration of the pair $(W,\partial_0 W)$ by the pairs $(W^{(i)},\partial_0 W)$. The standard Morse theory (see [18]) implies that (up to homotopy equivalence) $W^{(i)}$ is obtained from $W^{(i-1)}$ by attaching cells of dimension i, therefore the corresponding filtration in the singular chain complex $C^s_*(W, \partial_0 W)$ is *nice*. The corresponding adjoint complex D_* (see 2.2) is freely generated (as abelian group) in dimension k by $S_k(f)$. One can show that the boundary operator in D_* is given by the formula (31) (the argument is the same as in [15], Corollary 7.3). Thus the Morse complex $C^{\mathcal{M}}_*(f, v)$ is identified with D_*. The existence of the natural homotopy equivalence (32) follows now from 2.6.

3.2. **Basics of Morse-Novikov theory.** Let (M, f, v) be a Morse-Novikov triple, as defined in 1.8. As in the case of real-valued Morse functions we denote by $D(p, v)$ the stable manifold of p with respect to v. Choose an orientation of $D(p, v)$ for every $p \in S(f)$. To this data one associates a chain complex C_* of free finitely generated $\widehat{L}$-modules as follows. Let $F : \bar{M} \to \mathbf{R}$ be a lift of $f : M \to S^1$. Let $S_k(F)$denote the set of critical points of F of index k. Consider the set C_k of all formal linear combinations λ of elements of $S_k(F)$, such that for any $c \in \mathbf{R}$ there is only a finite set of points in supp λ which lie above c. It is easy to see that C_* is a free $\widehat{L}$-module (any lift of the set $S_k(f)$ to $\bar{M}$ provides a family of free $\widehat{L}$-generators for C_k). To define the boundary operators, let $p \in S_k(F), q \in S_{k-1}(F)$ and denote by $\Gamma(p, q; v)$ the set of all orbits of v in $\bar{M}$, joining p to q. The transversality condition implies that $\Gamma(p, q; v)$ is finite. The choice of orientations allows to attribute to each flow line $\gamma \in \Gamma(p, q; v)$ a sign $\varepsilon(\gamma) \in \{-1, 1\}$. Set $n(p, q; v) = \Sigma_{\gamma \in \Gamma(p,q;v)} \varepsilon(\gamma)$ and define a homomorphism $\partial_k : C_k \to C_{k-1}$ by

$$\partial_k p = \sum_{q \in S_{k-1}(f)} n(p, q; v) \cdot q \tag{33}$$

One can check that $\partial_{k-1} \circ \partial_k = 0$. The resulting chain complex is called the *Novikov complex.* We denote it $C^{\mathcal{N}}_*(M, f, v)$ or simply $C_*(f, v)$ if there is no possibility of confusion.

3.3. **Morse complexes of finite pieces of the cyclic covering.** Working with the terminology of 3.2, choose a regular value λ of $F : \bar{M} \to \mathbf{R}$ and set $V = F^{-1}(\lambda)$. For $\alpha \in \mathbf{R}$ set $V_\alpha = F^{-1}(\alpha)$. Set

$$W = F^{-1}([\lambda - 1, \lambda]), \quad V^- = F^{-1}(] - \infty, \lambda]).$$

Thus the cobordism W is the result of cutting of M along V. The structure group of the covering $\bar{M} \to M$ is isomorphic to $\mathbf{Z}$ and we choose the generator t of this group so that $tV_\alpha = V_{\alpha-1}$.

We have

$$\bar{M} = \bigcup_{s\in\mathbf{Z}} t^s W, \qquad \text{with} \qquad t^{s+1}W \cap t^s W = V_{\lambda-s-1}.$$

For any $k \in \mathbf{Z}$ the map $t^k : V_{\lambda+k} \to V$ is a diffeomorphism. For $n \in \mathbf{Z}, n \geqslant 1$ let

$$W_n = \bigcup_{0\leqslant s\leqslant n-1} t^s W = F^{-1}([\lambda - n, \lambda]) \tag{34}$$

so that in particular $W = W_1$. The Morse complex

$$\mathcal{M}_*(n) = C_*^{\mathcal{M}}(F|W_n, v|W_n) \tag{35}$$

of the function $F|W_n : W_n \to [\lambda-n, \lambda]$ is a chain complex of free abelian groups. But it has some more structure, coming from the $\mathbf{Z}$-action on $\bar{M}$. Set

$$\widehat{L}_- = \mathbf{Z}[[t]], \quad L_n = \mathbf{Z}[t]/t^n, \quad L_- = \mathbf{Z}[t]$$

It is easy to show, using the $\mathbf{Z}$-invariance of v, that $\mathcal{M}_*(n)$ is a free chain complex over L_n. In particular, $\mathcal{M}_*(n)$ is a chain complex of $\widehat{L}_-$-modules. It is clear that there is a natural isomorphism

$$\mathcal{M}_*(n+1) \underset{\widehat{L}}{\otimes} L_n \approx \mathcal{M}_*(n) \tag{36}$$

Any lift to W of $S(f)$ provides a free L_n-base for $\mathcal{M}_*(n)$ and the isomorphsism (36) preserves these bases. Introduce now the string

$$\mathcal{M} = \{\mathcal{M}_*(1) \xleftarrow{\pi_2} \mathcal{M}_*(2) \xleftarrow{\pi_3} \dots \xleftarrow{\pi_n} \mathcal{M}_*(n) \longleftarrow \dots\} \tag{37}$$

where π_n is the projection $\mathcal{M}_*(n) \to \mathcal{M}_*(n) \underset{L}{\otimes} L_{n-1}$. The inverse limit $|\mathcal{M}|_*$ is a free $\widehat{L}_-$-complex, which will be denoted by $C_*^-(f, v; \lambda)$. It is clear that there is a base preserving isomorphism

$$C_*^{\mathcal{N}}(M, f, v) \approx C_*^-(f, v; \lambda) \underset{\widehat{L}_-}{\otimes} \widehat{L}. \tag{38}$$

Therefore the Novikov complex can be reconstructed from the Morse complexes $\mathcal{M}_*(n)$ if we take into account the structure of L_n-modules on these complexes.

The natural chain equivalence $\mathcal{M}_*(n) \to C_*^s(W_n, \partial_0 W_n)$ can be refined so as to respect the L_n-structure. Now we shall recall this construction (from [22], §5) in more details. The chain equivalence $\mathcal{M}_*(n) \to C_*^s(W_n, \partial_0 W_n)$ is constructed using an ordered Morse function on W_n. It turns out that if we choose an ordered Morse function on W_n which is well fitted to the action of $\mathbf{Z}$ on $\bar{M}$, the chain homotopy equivalence will respect the L_n-structure.

Definition 3.1. An ordered Morse function $\phi : W_n \to [\alpha,\beta]$ is called t-ordered if for every $x \in W_n$ we have

$$tx \in W_n \Rightarrow \phi(tx) < \phi(x). \tag{39}$$

Proposition 3.2 ([22], Lemma 5.1). *There is a t-ordered Morse function ϕ on W_n, such that v is a ϕ-gradient.* □

Let us see how the proposition applies. Let ϕ be any t-ordered Morse function on W_n such that v is a ϕ-gradient. Then ϕ induces a filtration $W^{(i)}$ on W as explained in 3.1. Set

$$X^{(i)} = W^{(i)} \cup t^n V^-. \tag{40}$$

The chain complexes $C_*^s(X^{(i)}, t^n V^-)$, $0 \leqslant i \leqslant \dim M$ form a nice filtration of $C_*^s(V^-, t^n V^-)$ (with L_n as the base ring). The adjoint complex is $\mathcal{M}_*(n)$ and thus we obtain a chain homotopy equivalence

$$J_n : \mathcal{M}_*(n) \to C_*^s(V^-, t^n V^-) \tag{41}$$

of chain complexes over $\widehat{L}_n$. One can show that the chain homotopy class of J_n does not depend on the particular choice of ϕ ([22], p. 324).

3.4. **Special t-ordered functions.** In view of later applications it will be useful to choose the t-ordered function ϕ so that the terms of the corresponding filtration of W_n are very close to the descending discs of v. To make this precise, we introduce first a definition. Set

$$D(\mathrm{ind}\leqslant i\ ; v) = \bigcup_{\mathrm{ind} p \leqslant i} D(p, v) \tag{42}$$

where $D(p,v)$ is the stable manifold in W_n of the point $p \in S(F)$, and let U_i be any neighborhood in W_n of $D(\mathrm{ind}\leqslant i\ ; v) \cup \partial_0 W_n$. Let $m = \dim M$.

Proposition 3.3. *There is a t-ordered Morse function $g : W_n \to \mathbf{R}$, such that v is a g-gradient, and an ordering sequence $a_0 < \dots < a_{m+1}$ for g, such that the corresponding filtration $\{W^{(i)}\}$ satisfies $W^{(i)} \subset U_i$ for every i.*

Proof. Start with any t-ordered Morse function $h : W_n \to [\alpha,\beta]$. We can assume that for an $\epsilon > 0$ the function $dh(x)(v(x))$ is constant in $h^{-1}([\beta-\epsilon,\beta]) \cup h^{-1}([\alpha,\alpha+\epsilon])$. Choose ϵ so small that $h^{-1}([\alpha,\alpha+\epsilon]) \subset U_i$ for every i. Let $\mu : [\alpha,\beta] \to [0,1]$ be a C^∞ function, such that

$$\begin{cases} \mu(y) = 0 & \text{for} \quad y \in [\alpha, \alpha+\epsilon/2] \quad \text{or} \quad y \in [\beta-\epsilon/2, \beta] \\ \mu(y) = 1 & \text{for} \quad y \in [\alpha+\epsilon, \beta-\epsilon] \end{cases} \tag{43}$$

Set $w(x) = \mu(h(x))v(x)$ and let Φ_τ denote the one-parametric group of diffeomorphisms of W_n, induced by w. I claim that for $T > 0$ sufficiently large the function $g = h \circ \Phi_T$ satisfies the conclusion of our Proposition. It is quite obvious that v is an g-gradient. Next we shall check that g is t-ordered. Let $\gamma_1(\tau) = \gamma(x,\tau,w), \gamma_2(\tau) = \gamma(tx,\tau,w)$. We have to prove

$$h(\gamma_1(T)) > h(\gamma_2(T)) \tag{44}$$

We shall show that (44) holds for x satisfying the following condition: both x and tx are not in supp $(v - w)$, and both γ_1, γ_2 reach $\partial_1 W$ at some moment. (The case of arbitrary x is similar.) Let t_1 (resp. t_2) be the moment when γ_1, (resp. γ_2) reaches $h^{-1}(\beta - \epsilon)$. Note that $\gamma_2(\tau) = t\gamma_1(\tau)$ for $\tau \in [0, t_1]$. Therefore $t_2 > t_1$ and (44) holds if $T \leqslant t_1$. It holds also for $T \in [t_1, t_2]$, since for these values of T we have $h(\gamma_2(T)) \leqslant \beta - \epsilon \leqslant h(\gamma_1(T))$. Finally, for $T \geqslant t_2$ both $\gamma_1(T), \gamma_2(T)$ are in $h^{-1}([\beta - \epsilon, \beta])$ and $h(\gamma_1(T)) = h(\gamma_2(T + t_2 - t_1)) > h(\gamma_2(T))$.

Any ordering sequence $\alpha_0 < \alpha_1 < ... < \alpha_{m+1}$ for h is also an ordering sequence for g. A standard "gradient descent" argument shows that $g^{-1}([\alpha_0, \alpha_{i+1}]) \subset U_i$ if only T is sufficiently large and $\alpha_{m+1} < \beta - \epsilon$. $\square$

3.5. **Construction of the chain homotopy equivalence Φ.** Introduce the following notation:

$$S_* = C^s_*(V^-) \underset{L_-}{\otimes} \widehat{L}_-, \quad S_*(n) = C^s_*(V^-, t^n V^-) \tag{45}$$

Then S_* is a free complex over $\widehat{L}_-$, and $S_*(n)$ is free over L_n. The chain complex S_* is infinitely generated, but it is homotopy finitely generated (in the sense of the definition on the page 12). Indeed, choose a C^1-triangulation Δ of M, such that V is a simplicial subcomplex. Then V^- obtains a triangulation, invariant with respect to the action of t, so that the simplicial chain complex $C^\Delta_*(V^-)$ is a free finitely generated complex over L_-. Let $D_* = C^\Delta_*(V^-) \underset{L_-}{\otimes} \widehat{L}_-$. the natural chain equivalence $C^\Delta_*(V^-) \to C^s_*(V^-)$ being tensored with $\widehat{L}_-$ gives then the homotopy equivalence sought.

Proposition 3.4. *There is a unique (up to homotopy) chain homotopy equivalence $C^-_*(f, v; \lambda) \xrightarrow{\mu} S_*$, such that for every $n \geqslant 1$ the following diagram is homotopy commutative:*

$$\begin{tikzcd} C^-_*(f,v;\lambda) \arrow[r, "\mu"] \arrow[d] & S_* \arrow[d] \\ \mathcal{M}_*(n) \arrow[r, "J_n"] & S_*(n) \end{tikzcd} \tag{46}$$

(where the vertical arrows are natural projections, and J_n is the chain homotopy equivalence from (41)).

Proof. The uniqueness part follows immediately from Proposition 2.8, since both $C_*^-(f,v;\lambda)$ and S_* are homotopy finitely generated. Now to the existence of μ. Recall that $C_*^-(f,v;\lambda)=|\mathcal{M}|_*$ (see (37)). Introduce an auxiliary string

(47)

$$\mathcal{Z}=\{S_*(1)\xleftarrow{p_2}S_*(2)\longleftarrow\ldots\longleftarrow S_*(n-1)\xleftarrow{p_n}S_*(n)\longleftarrow\ldots\}$$

where the map $p_n : S_*(n)\to S_*(n-1)$ is induced by the inclusion of the pairs $(V^-,t^nV^-)\subset(V^-,t^{n-1}V^-)$. As the first step we shall prove $|\mathcal{M}|_*\sim|\mathcal{Z}|_*$. See [22], p.325 for the proof of the next lemma.

Lemma 3.5. *For every n the following square is homotopy commutative*

$$\begin{tikzcd}
\mathcal{M}_*(n) \arrow[d, "J_n"] & \mathcal{M}_*(n+1) \arrow[l, "\pi_{n+1}"'] \arrow[d, "J_{n+1}"] \\
S_*(n) & S_*(n+1) \arrow[l, "p_{n+1}"']
\end{tikzcd} \tag{48}$$

□

Therefore we have a map of strings $J:\mathcal{M}\to\mathcal{Z}$ which induces by 2.7 a homology equivalence $|J|:C_*^-(f,v;\lambda)\to|\mathcal{Z}|_*$ of chain complexes over $\widehat{L}_-$. Now we are going to construct a homology equivalence $S_*\to|\mathcal{Z}|_*$. The quotient chain maps $S_*\to S_*(n)=S_*\underset{\widehat{L}_-}{\otimes}L_n$ are compatible with each other, so they induce a chain map

$$\xi : S_*\to\varprojlim S_*(n)=|\mathcal{Z}|_* \tag{49}$$

Lemma 3.6. *The map ξ is a homology equivalence.*

Proof. We have the following commutative diagram

$$\begin{tikzcd}
S_* \arrow[r, "\xi"] & \varprojlim S_*(n) \\
D_* \arrow[u] \arrow[r, phantom, "="] & \varprojlim C_*^{\Delta}(V^-,t^nV^-) \arrow[u]
\end{tikzcd} \tag{50}$$

Both vertical arrows are homology equivalences and the lemma follows. □

Apply now Proposition 2.2 to obtain a chain homology equivalence $\mu : C_*^-(f,v;\lambda)\to S_*$, such that $\xi\circ\mu\sim|J|$. Since both $C_*^-(f,v;\lambda)$ and S_*

are free chain complexes over $\widehat{L}_-$, the map μ is a chain homotopy equivalence . The property $\mu_n \sim J_n$ goes by construction, and the existence of μ is proved. Now the construction of the homotopy equivalence Φ from the Theorem A is straightforward. Recall that $\widehat{L} = S^{-1}\widehat{L}_-$ where S is the multiplicative subset $S = \{t^n \mid n \in \mathbf{N}\}$. Also

$$C_*(f,v) = S^{-1}C_*^-(f,v;\lambda)$$

$$C_*^s(\bar{M}) \underset{L}{\otimes} \widehat{L} = C_*^s(V^-) \underset{L_-}{\otimes} \widehat{L} = S^{-1}\Big(C_*^s(V^-) \underset{L_-}{\otimes} \widehat{L}_-\Big)$$

Therefore localizing μ we obtain a chain homotopy equivalence

$$S^{-1}\mu : C_*(f,v) \xrightarrow{\sim} C_*^s(\bar{M}) \underset{L}{\otimes} \widehat{L}$$

as required in Theorem A. It remains to check that the chain homotopy equivalence thus constructed depends only on the triple (M,f,v) and not on the auxiliary choices we have made during the construction. To this end, note first of all that for given (M,f,v) the chain homotopy class of $S^{-1}\mu$ depends apriori only on the choice of the lift $F:\bar{M}\to\mathbf{R}$ and of the regular value λ (by Prop. 3.4). We shall therefore denote the homotopy class of $S^{-1}\mu$ by $\Phi(F,\lambda)$. Our next aim is to show that actually $\Phi(F,\lambda)$ does not depend on the choice of F and λ neither. It is clear that $\Phi(F-1,\lambda) = \Phi(F,\lambda+1)$, thus it suffices to prove the next proposition.

Proposition 3.7. *Let λ_1,λ_2 be regular values of F, then $\Phi(F,\lambda_1)\sim\Phi(F,\lambda_2)$.*

Proof. Assuming $\lambda_1<\lambda_2$, set $V_{(1)} = F^{-1}(\lambda_1)$, $V_{(2)} = F^{-1}(\lambda_2)$ and

$$V_{(1)}^- = F^{-1}(]-\infty,\lambda_1]),\quad V_{(2)}^- = F^{-1}(]-\infty,\lambda_2]). \tag{51}$$

Consider the inclusions

$$I : V_{(1)}^- \hookrightarrow V_{(2)}^-,\qquad I_n : (V_{(1)}^-, t^nV_{(1)}^-) \hookrightarrow (V_{(2)}^-, t^nV_{(2)}^-) \tag{52}$$

We have also a chain map of the corresponding Morse complexes

$$(I_n)_! : \mathcal{M}_*^{(1)}(n) \to \mathcal{M}_*^{(2)}(n) \tag{53}$$

Lemma 3.8. *The following diagram is homotopy commutative:*

$$\begin{tikzcd} \mathcal{M}_*^{(1)}(n) \arrow[r, "(I_n)_!"] \arrow[d, "J_n^{(1)}"'] & \mathcal{M}_*^{(2)}(n) \arrow[d, "J_n^{(2)}"'] \\ C_*^s\Big(V_{(1)}^-, t^nV_{(1)}^-\Big) \arrow[r, "(I_n)_*"] & C_*^s\Big(V_{(2)}^-, t^nV_{(2)}^-\Big) \end{tikzcd} \tag{54}$$

where $J_n^{(1)}, J_n^{(2)}$, *are the chain homotopy equivalences from (41).*

Proof. Let $W_{1,n} = F^{-1}([\lambda_1 - n, \lambda_1])$, $W_{2,n} = F^{-1}([\lambda_2 - n, \lambda_2])$. The maps $J_n^{(1)}, J_n^{(2)}$ are constructed with the help of t-ordered Morse functions say g_1, g_2 on $W_{1,n}$, $W_{2,n}$. The function g_1 defines a filtration in the pair $(V^-_{(1)}, t^n V^-_{(1)})$; let $(X_1^{(k)}, t^n V^-_{(1)})$ denote the k-th term of this filtration. Similarly, we obtain a filtration $(X_2^{(k)}, t^n V^-_{(2)})$ in $(V^-_{(2)}, t^n V^-_{(2)})$. Proposition 3.3 implies that we can choose g_1 so that $X_1^{(k)}$ be arbitrary close to

$$t^n V^-_{(1)} \cup \bigcup_{\substack{p \in W_n, \\ \mathrm{ind} p \leqslant k}} D(p; v).$$

In particular, we can assume that $X_1^{(k)} \subset X_2^{(k)}$. Then the map I_n preserves the corresponding filtrations in singular homology, and our result follows from Lemma 2.4. □

Corollary 3.9. *The following diagram is homotopy commutative*

$$\begin{array}{ccc} C^-_*(f, v; \lambda_1) & \xrightarrow{I} & C^-_*(f, v; \lambda_2) \\ \downarrow \mu_1 & & \downarrow \mu_2 \\ C^s_*(V^-_{(1)}) \underset{L_-}{\otimes} \widehat{L}_- & \xrightarrow{I_*} & C^s_*(V^-_{(2)}) \underset{L_-}{\otimes} \widehat{L}_- \end{array} \tag{55}$$

(here μ_1, μ_2 are the homotopy equivalences corresponding to λ_1, resp. λ_2 by 3.4).

Proof. The reduction modulo t^n of this diagram coincides with (54) and, therefore, our Corollary follows from 2.11. □

Now Proposition 3.7 follows immediately since after localizing the square (55) both horizontal arrows become identity maps, and localized vertical arrows are exactly $\Phi(\lambda_1)$ and $\Phi(\lambda_2)$. □

3.6. **Functoriality of $\Phi(M, f, v)$.** We proceed to the proof of the second part of Theorem A. Let $F_1 : \bar{M}_1 \to \mathbf{R}, F_2 : \bar{M}_2 \to \mathbf{R}$ be lifts of f_1, f_2. The diffeomorphism $\bar{g}$ sends $S_k(F_1)$ to $S_k(F_2)$, therefore for every k it determines an isomorphism of $\widehat{L}$-modules $\bar{g}_! : C_k(f_1, v_1) \to C_k(f_2, v_2)$, which commutes with boundary operators, since $\bar{g}$ sends the orbits of v_1 to those of v_2 (by condition β). Choose a regular value λ_1 for F_1, and a regular value λ_2 for F_2, set $V_{(1)} = F_1^{-1}(\lambda_1)$, $V_{(2)} = F_2^{-1}(\lambda_2)$ and

$$V^-_{(1)} = F_1^{-1}(]-\infty, \lambda_1]), \quad V_{(2)} = F_2^{-1}(]-\infty, \lambda_2]). \tag{56}$$

Assume that λ_1, λ_2 are chosen in such a way, that

$$\bar{g}(V^-_{(1)}) \subset V^-_{(2)} \tag{57}$$

For $i = 1, 2$ let $\mu_i : C^-_*(f_i, v_i; \lambda_i) \to C^s_*(V^-_{(i)}) \underset{L_-}{\otimes} \widehat{L}_-$ be the corresponding chain homotopy equivalences (see Proposition 3.4). The commutativity of the diagram (15) follows from the next proposition.

Proposition 3.10. *The following diagram is homotopy commutative:*

$$\begin{tikzcd} C^-_*(f_1, v_1; \lambda_1) \arrow[r, "\bar{g}_!"] \arrow[d, "\mu_1"] & C^-_*(f_2, v_2; \lambda_2) \arrow[d, "\mu_2"] \\ C^s_*(V^-_{(1)}) \underset{L_-}{\otimes} \widehat{L}_- \arrow[r, "\bar{g}_*"] & C^s_*(V^-_{(2)}) \underset{L_-}{\otimes} \widehat{L}_- \end{tikzcd} \tag{58}$$

The proof of this proposition is completely similar to that of Corollary 3.9 and will be omitted. Now, applying the S-localization to the square (58) we obtain the second part of Theorem A. □

4. C^0-perturbations of gradient-like vector fields

This section contains recollections of results from [24], which are crucial for the proof of Theorem B. In Subsections 4.1 – 4.6 we work with real-valued functions of cobordisms and their gradients. The first and second subsections contain the results about behaviour of stable manifolds and similar objects under C^0-small perturbations of the gradient. In Subsection 4.3 we recall the condition ($\mathfrak{C}$) on the gradient-like vector field and the properties of gradient-like vector fields satisfying this condition. Subsections 4.4 – 4.6 contain an account of the properties of f-gradients satisfying ($\mathfrak{C}$). In Subsection 4.7 we proceed to the circle-valued Morse maps and we recall the condition ($\mathfrak{C}'$) which is the appropriate analog of ($\mathfrak{C}$) in this framework. In Subsection 4.9 we give the construction of the chain equivalence ψ from [24] between the Novikov complex and the completed singular chain complex of $\bar{M}$.

4.1. **δ-thin handle decompositions.** Through Subsections 4.1 – 4.6 $f : W \to [a, b]$ is a Morse function on a riemannian cobordism W of dimension m, and v is an f-gradient. We shall need some more of Morse-theoretic terminology. Set

$$U_1 = \{x \in \partial_1 W \mid \gamma(x, \cdot; v) \text{ reaches } \partial_0 W\}.$$

Then U_1 is an open subset of $\partial_1 W$ and the gradient descent along the trajectories of v determines a diffeomorphism of U_1 onto an open subset U_0 of $\partial_0 W$. This diffeomorphism will be denoted by $(-v)^{\leadsto}$ and we shall abbreviate $(-v)^{\rightsquigarrow}(X \cap U_1)$ to $(-v)^{\rightsquigarrow}(X)$. We say that v satisfies *Almost Transversality Condition* if

$$\big(x, y \in S(f)\ \&\ \mathrm{ind} x \leqslant \mathrm{ind} y\big) \Rightarrow \big(D(x,v) \pitchfork D(y,-v)\big)$$

A Morse function $\phi : W \to [\alpha, \beta]$ is called *adjusted to* (f, v), if:

1) $S(\phi) = S(f)$, and v is also a ϕ-gradient.

2) The function $f - \phi$ is constant in a neighborhood of $\partial_0 W$, in a neighborhood of $\partial_1 W$, and in a neighborhood of each point of $S(f)$.

Using the standard techniques of rearrangement of critical points (see for example [15], §4, or an exposition which uses our present terminology in [21], §2) it is not difficult to show that for an arbitrary Morse function f and an f-gradient satisfying the the almost transversality assumption, there is an ordered Morse function g, adjusted to (f, v). Let $p \in W \setminus \partial W$, and $\delta > 0$. Assume that for some $\delta_0 > \delta$ the restriction of the exponential map $\exp_p : T_p W \to W$ to the disc $B^m(0, \delta_0)$ is a diffeomorphism on its image. Denote by $B_\delta(p)$ (resp. $D_\delta(p)$) the riemannian open ball (resp. closed ball) of radius δ centered in p. We shall use the notation $B_\delta(p), D_\delta(p)$ only when the assumption above on δ holds. Set

$$B_\delta(p, v) = \{x \in W \mid \exists t \geqslant 0 : \gamma(x, t; v) \in B_\delta(p)\}$$
$$D_\delta(p, v) = \{x \in W \mid \exists t \geqslant 0 : \gamma(x, t; v) \in D_\delta(p)\}$$

We denote by $B_\delta(\mathrm{ind}{\leqslant s}\ ; v)$, the union of $B_\delta(p, v)$, where p ranges over critical points of f of index $\leqslant s$. We shall also use similar notation like $B_\delta(\mathrm{ind}{\leqslant s}\ ; v)$, $D_\delta(\mathrm{ind}{=s}\ ; v)$ or $B_\delta(\mathrm{ind}{\geqslant s}\ ; v)$, which is now clear without special definition. (Actually the symbol $D(\mathrm{ind}{\leqslant s}\ ; v)$ was already introduced in (42).) Set

$$C_\delta(\mathrm{ind}{\leqslant s}\ ; v) = W \setminus B_\delta(\mathrm{ind}{\leqslant m-s-1}\ ; -v)$$

Let $\phi : W \to [a, b]$ be an ordered Morse function with an ordering sequence $a_0 < a_1 < \cdots < a_{m+1}$. Let w be a ϕ-gradient. Denote $\phi^{-1}([a_i, a_{i+1}])$ by W_i.

Definition 4.1. We say that w is *δ-separated with respect to ϕ* (and the ordering sequence $(a_0, \ldots, a_{m+1})$), if

i) for every i and every $p \in S_i(f)$ we have $D_\delta(p) \subset W_i \setminus \partial W_i$;

ii) for every i and every $p \in S_i(f)$ there is a Morse function $\psi : W_i \to [a_i, a_{i+1}]$, adjusted to $(\phi \mid W_i, w)$ and a regular value λ of ψ such that

$$D_\delta(p) \subset \psi^{-1}(]a_i, \lambda[)$$

and for every $q \in S_i(f), q \neq p$ we have

$$D_\delta(q) \subset \psi^{-1}(]\lambda, a_{i+1}[)$$

We say that w is *δ-separated* if it is δ-separated with respect to some ordered Morse function $\phi : W \to [a,b]$, adjusted to (f,v). Each f-gradient satisfying almost transversality assumption is δ-separated for some $\delta > 0$.

Proposition 4.2 ([24], Prop. 3.2, 4.1). *If v is ϵ-separated, then $\forall \delta \in [0,\epsilon]$ and $\forall s : 0 \leqslant s \leqslant m$*

1) *$D_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$ is compact.*
2) *The family $\bigcap_{\theta>\delta} B_\theta(\mathrm{ind}_{\leqslant s}\ ; v)$ form a fundamental system of neighborhoods of $D_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$. The family $\bigcap_{\theta>0} B_\theta(\mathrm{ind}_{\leqslant s}\ ; v)$ form a fundamental system of neighborhoods of $D(\mathrm{ind}_{\leqslant s}\ ; v)$.*
3) *$B_\delta(\mathrm{ind}_{\leqslant s}\ ; v) = \mathrm{Int}\ D_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$ and $D_\delta(\mathrm{ind}_{\leqslant s}\ ; v) \subset C_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$.*
4) *The group*

$$H_*\Big(D_\delta(\mathrm{ind}_{\leqslant s}\ ; v) \cup \partial_0 W,\ D_\delta(\mathrm{ind}_{\leqslant s-1}\ ; v) \cup \partial_0 W\Big)$$

vanishes if $* \neq s$ and is a free abelian group generated by the classes of the descending discs $D(p,v)$ with $p \in S_s(f)$.

We shall often denote $D_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$ by $W^{\{\leqslant s\}}$ if the values of v, f, δ are clear from the context.

4.2. **C^0-continuity properties of descending discs.** In this subsection we study the behaviour of descending discs under C^0-small perturbations of the gradient. let $G(f)$ denote the set of all f-gradients.

Lemma 4.3. *Let $\delta > 0$ and $K \subset B_\delta(v)$ a compact set. There exists $\eta > 0$, such that for every $w \in G(f)$ with $||w - v|| < \eta$ we have $K \subset B_\delta(w)$.* [1]

Proof. Set $B = \cup_{p \in S(f)} B_\delta(p)$. For a subset $Q \subset W$ let us denote by $\mathcal{R}(Q,w)$ the following condition:

For every $x \in Q$ the trajectory $\gamma(x,t;w)$ intersects B.

Note that $\mathcal{R}(K,v)$ holds. For $x \in K$ choose $t(x) \geqslant 0$ such that $\gamma(x,t(x);v) \in B$. By Corollary 5.5 of [23] there is a neighborhood $S(x)$ of x and $\delta(x) > 0$ such that for every f-gradient w with $||w - v|| < \delta(x)$ the property $\mathcal{R}(S(x),w)$ holds. Choosing a finite covering of K by the subsets $S(x)$ finishes the proof. ☐

Lemma 4.4. *Let $\delta > 0$ and assume that v is δ-separated. Let $U \subset W$ be an open set and assume $D_\delta(\mathrm{ind}_{\leqslant k}\ ; v) \subset U$. Then there is $\epsilon > 0$ such that for every $w \in G(f)$ with $||w-v|| < \epsilon$ we have $D_\delta(\mathrm{ind}_{\leqslant k}\ ; v) \subset U$.*

[1] In the present paper the symbol $||\cdot||$ denotes the C^0-norm.

Proof. First we prove the lemma for $k = \dim M$, so that $D_\delta(\mathrm{ind}_{\leqslant k}\ ; v)$ is the union of all δ-thickened descending discs. Let $K = W \setminus U$. Let us denote by $Q(w)$ the following condition:

For every $x \in K$ the trajectory $\gamma(x, t; w)$ reaches $\partial_1 W$, without intersecting the set $\bigcup_{p\in S(f)} D_\delta(p)$.

We know that $Q(v)$ holds and we have to prove that $Q(w)$ holds for every gradient-like vector field w sufficiently C^0-close to v. This follows directly from Corollary 5.6 of [23]. To prove the lemma for any k we consider an ordered Morse function $\phi : W \to [a, b]$ adjusted to (f, v) and we set $W' = \phi^{-1}([a_0, a_{k+1}])$. Then $D_\delta(\mathrm{ind}_{\leqslant k}\ ; v) = D_\delta(\mathrm{ind}_{\leqslant m}\ ; v|W')$ and applying the previous argument to W' we are over. $\square$

4.3. **Condition** $(\mathfrak{C})$. In this subsection we recall the condition $(\mathfrak{C})$ on the gradient v. If this condition holds, the gradient descent map, corresponding to v can be endowed with a structure, resembling closely the cellular maps between CW-complexes. Also the cobordism W can be endowed with certain handle-like filtration which encompasses the handle-like filtrations on $\partial_0 W$ and $\partial_1 W$.

Definition 4.5. ([24], Def. 4.5) We say that v *satisfies condition* $(\mathfrak{C})$ if there are objects 1) - 4), listed below, with the properties (1 - 3) below.

Objects:

1) An ordered Morse function ϕ_1 on $\partial_1 W$ and a ϕ_1-gradient u_1.
2) An ordered Morse function ϕ_0 on $\partial_0 W$ and a ϕ_0-gradient u_0.
3) An ordered Morse function ϕ on W adjusted to (f, v).
4) A number $\delta > 0$.

Properties:

(1) u_0 is δ-separated with respect to ϕ_0, u_1 is δ-separated with respect to ϕ_1, v is δ-separated with respect to ϕ.

$$(2) \qquad (-v)^{\leadsto}\Big(C_\delta(\mathrm{ind}_{\leqslant j}\ ; u_1)\Big) \cup \Big(D_\delta(\mathrm{ind}_{\leqslant j+1}\ ; v) \cap \partial_0 W\Big) \subset B_\delta(\mathrm{ind}_{\leqslant j}\ , u_0) \text{ for every } j$$

$$(3) \qquad \overset{\leadsto}{v}\Big(C_\delta(\mathrm{ind}_{\leqslant j}\ ; -u_0)\Big) \cup \Big(D_\delta(\mathrm{ind}_{\leqslant j+1}\ ; -v) \cap \partial_1 W\Big) \subset B_\delta(\mathrm{ind}_{\leqslant j}\ ; -u_1) \text{ for every } j$$

△

The set of all f gradients satisfying $(\mathfrak{C})$ will be denoted by $GC(f)$.

Theorem 4.6. *([24], Th. 4.6) $GC(f)$ is open and dense in $G(f)$ with respect to C^0 topology. Moreover, let $v_0 \in G(f)$, let U be a neighborhood of ∂W, and $\delta > 0$. Then there is $v \in GC(f)$ with $||v - v_0|| < \delta$ and* supp $(v - v_0) \subset U$.

4.4. **Condition $(\mathfrak{C})$ and regularization of gradient descent map.** As we have already mentioned the application $(-v)^{\leadsto}$ is not everywhere defined. But if the gradient v satisfies the condition $(\mathfrak{C})$, we can associate to v some family of continuous maps which plays the role of "cellular approximation" of $(-v)^{\leadsto}$, and a homomorphism (*homological gradient descent*) which is a substitute for the homomorphism induced by $(-v)^{\leadsto}$ in homology. Let v be an f-gradient satisfying $(\mathfrak{C})$. We have two ordered Morse functions ϕ_1, ϕ_0 on $\partial_1 W$, resp. $\partial_0 W$, and their gradients u_1, u_0, which give rise to filtrations on $\partial_1 W$, resp. $\partial_0 W$. Namely, let $\beta_0 < \beta_1 < ... < \beta_m$ be an ordering sequence for ϕ_1, $\alpha_0 < \alpha_1 < ... < \alpha_m$ be an ordering sequence for ϕ_0. Let

$$\partial_0 W^{(k)} = \phi_1^{-1}([\beta_0, \beta_{k+1}]), \quad \partial_1 W^{(k)} = \phi_0^{-1}([\alpha_0, \alpha_{k+1}]) \tag{59}$$

For $k \geqslant 0$ consider the set Q_k of all $x \in \partial_1 W^{(k)}$ where $(-v)^{\leadsto}$ is *not* defined. Equivalently,

$$Q_k = \{x \in \partial_1 W^{(k)} \mid \gamma(x, t; -v) \text{ converge to point in } S(f) \text{ as } t \to \infty\}.$$

This is a compact set, and the condition $(\mathfrak{C})$ implies that $Q_k \subset D(\mathrm{ind}{\geqslant m-k}\ ; -v)$. Therefore there is a neighborhood U of Q_k in $\partial_1 W^{(k)}$ such that $(-v)^{\leadsto}(U)$ is in $D_\delta(\mathrm{ind}{\leqslant k}\ ; v) \cap \partial_0 W$ and this last set is in Int $\partial_1 W^{(k-1)}$ (again by $(\mathfrak{C})$). It follows that the map $(-v)^{\leadsto}$ gives rise to a well-defined continuous map

$$v{\downarrow}\colon \partial_1 W^{(k)}/\partial_1 W^{(k-1)} \to \partial_0 W^{(k)}/\partial_0 W^{(k-1)} \tag{60}$$

The homomorhism, induced by this map in homology, is called *homological gradient descent*. It will be denoted by $\mathcal{H}_k(-v)$ or simply $\mathcal{H}_k$ if the gradient v is clear from the context.

4.5. **Morse type decomposition of cobordisms: preliminary discussion.** For any Morse function on a closed manifold the stable manifolds of the critical points (with respect to to a gradient-like vector field satisfying the transversality assumption) form a cellular-like decomposition of the manifold. This construction generalizes directly to Morse functions on cobordisms, but we lose some of its properties.

Namely, the union of descending discs of critical points is not necessarily equal to W (consider for example the second coordinate projection $V \times [0,1] \to [0,1]$; this is a Morse function without critical points). In this subsection we discuss a natural approach to overcome this difficulty.

Definition 4.7. For $X \subset \partial_1 W$ let us denote by $T(X, -v)$ and call *track of* X the union of all $(-v)$-trajectories starting at a point of the set X. Similarly, for $Y \subset \partial_0 W$ let $T(Y, v)$ be union of all v-trajectories starting at a point of the set Y.

A nice cellular-like decomposition fitted to f and v should contain:

1) The cells of $\partial_0 W$,
2) the cells of $\partial_1 W$,
3) the descending discs $D(p, v)$ for $p \in S(f)$,
4) the tracks of the cells of $\partial_1 W$.

By "cells of $\partial_1 W$" we mean descending discs corresponding to some Morse function $\phi_1 : \partial_1 W \to \mathbf{R}$ and its gradient u_1 (similarly for $\partial_0 W$). In order to get a cellular-like decomposition, some requirements must be met. Namely, for $p \in S(f)$ the sole $D(p, v) \cap \partial_0 W$ of the descending disc $D(p, v)$ with $p \in S(f)$ must belong to the $(\mathrm{ind} p - 1)$-skeleton of $\partial_0 W$. Let us write down this condition:

$$D(p, v) \cap \partial_0 W \subset \partial_0 W^{(\mathrm{ind} p - 1)} \tag{61}$$

Similar requirement must hold for the tracks of the cells of $\partial_1 W$. Although it is plausible that such requirement could be met (see the discussion in [24], end of Section 2) I do not know whether it is true in general. The method which we apply to realize this idea uses the thickenings of descending discs. This is how the conditions of the type $(\mathfrak{C})$ appeared first, in [23]. We introduced in this paper (§4) the conditions (RP) on the f-gradient, which allow to regularize the gradient descent map and to give it a cellular-like structure. The condition $(\mathfrak{C})$, introduced in [24], being quite similar to (RP), has some technical advantages over it, for example $(\mathfrak{C})$ is an open and dense condition with respect to C^0-topology (while (RP) was only proved to be C^0-dense).

If we look now at the condition $(\mathfrak{C})$ from the point of view of (61), we notice that the part (2) of $(\mathfrak{C})$ is exactly the "δ-thickened" analog of this requirement. In the next subsection we shall see that the condition $(\mathfrak{C})$ gives rise indeed to a filtration of the cobordism W, which has the properties of handle filtrations of closed manifolds.

Our main aim is in the applications of these methods to the circle-valued maps. To this end we note that in the framework of circle-valued Morse maps the corresponding analog of the requirement (61) can not be met, while the methods based on the condition ($\mathfrak{C}$) work also for this generalization. These methods will be our main tools for the construction of the chain equivalence from Theorem B.

4.6. **Handle-like filtrations of cobordisms.** In this subsection we associate to each f-gradient v satisfying the condition ($\mathfrak{C}$) a handle-like filtration of the cobordism. Let ϕ be the ordered Morse function on W, from the definition 4.5, and $(a_0, ..., a_{m+1})$ be an ordering sequence for ϕ.

Let Z_k be the set of all $z \in \phi^{-1}([a_0, a_{k+1}])$ such that the $(-v)$-trajectory $\gamma(z, t; -v)$ converges to a critical point of ϕ as $t \to \infty$ in $\phi^{-1}([a_0, a_{k+1}])$ or reaches $\partial_0 W$ and intersects it at a point in $\partial_0 W^{(k)}$. In other words

$$Z_k = \Big(T(\partial_0 W^{(k)}, v) \cup D(-v)\Big) \cap \phi^{-1}([a_0, a_{k+1}]) \tag{62}$$

(where $D(-v)$ stands for the union of all ascending discs of v). Note in particular that Z_k contains the set $\partial_0 W^{(k)}$ and the descending discs $D(p, v)$ for $\mathrm{ind} p \leqslant k$.

Let T_k be the set of all points $y \in \phi^{-1}([a_k, b])$ such that the v-trajectory starting at y reaches $\partial_1 W$ and intersects it at $\partial_1 W^{(k-1)}$. In other words

$$T_k = T(\partial_1 W^{(k-1)}; -v) \cap \phi^{-1}([a_k, b]) \tag{63}$$

Now we can define the filtration $\{W^{\langle k\rangle}\}$ of W; set

$$W^{\langle k\rangle} = \partial_1 W^{(k)} \cup Z_k \cup T_k. \tag{64}$$

Thus $W^{\langle k\rangle}$ contain the handles of index $\leqslant k$ of the manifolds $\partial_0 W$, $\partial_1 W$, W, and also the tracks of the handles of index $\leqslant k-1$ of $\partial_1 W$. (The sets $Z_k, T_k, W^{\langle k\rangle}, W^{\langle k-1\rangle}$ are depicted on the page 44.) Consider the corresponding filtration in the singular homology and denote by E_* the corresponding adjoint complex, so that

$$E_k = H_k(W^{\langle k\rangle}, W^{\langle k-1\rangle}).$$

In [24] we proved that the filtration induced by $W^{\langle k\rangle}$ in the singular homology is nice, and computed the boundary operator in E_* in terms of Morse complexes of ϕ_0, ϕ_1 and ϕ and the gradient descent homomorphism. We shall now recall these results. Consider the Morse complexes $C_*(\phi_0, u_0)$, $C_*(\phi_1, u_1)$, $C_*(\phi, u)$ associated to Morse functions ϕ_0, ϕ_1, ϕ. The obvious inclusions

(65)
$$\Big(\partial_0 W^{(k)}, \partial_0 W^{(k-1)}\Big) \hookrightarrow (W^{\langle k\rangle}, W^{\langle k-1\rangle}) \hookleftarrow \Big(\partial_1 W^{(k)}, \partial_1 W^{(k-1)}\Big)$$

induce chain maps

$$C_*(\phi_0, u_0) \xrightarrow{\lambda_0} E_* \xleftarrow{\lambda_1} C_*(\phi_1, u_1) \tag{66}$$

It follows from the condition ($\mathfrak{C}$) that for every $p \in S_k(f)$ the descending disc $D(p, v)$ has a well defined fundamental class $[p]$ in $H_k(W^{\langle k\rangle}, W^{\langle k-1\rangle})$ and the corresponding map $p \mapsto [p]$ defines an inclusion

$$C_k(\phi, v) \xrightarrow{\mu} H_k(W^{\langle k\rangle}, W^{\langle k-1\rangle}) \tag{67}$$

The images of λ_1, λ_0 and μ correspond to the components 1) – 3) of the hypothetical cell-like decomposition from the page 28. Now to the fourth component. The condition ($\mathfrak{C}$) implies that no $(-v)$-trajectory starting at $x \in \partial_1 W^{(k-1)}$ converges to a critical point of ϕ in $\phi^{-1}([a_k, b])$, therefore we have a homeomorphism

$$T_k \approx \partial_1 W^{(k-1)} \times I \tag{68}$$

where $I = [a_k, b]$, and the pair $(T_k, T_k \cap W^{\langle k-1\rangle})$ is therefore homeomorphic to

$$\Big(\partial_1 W^{(k-1)}, \partial_1 W^{(k-2)}\Big) \times (I, \partial I). \tag{69}$$

The multiplication with the fundamental class of $(I, \partial I)$ defines the map

$$\tau : H_{k-1}\Big(\partial_1 W^{(k-1)}, \partial_1 W^{(k-2)}\Big) = C_{k-1}(\phi_1, u_1) \to H_k(W^{\langle k\rangle}, W^{\langle k-1\rangle}) = E_k$$

Proposition 4.8 ([24], Theorem 5.5). *$H_*(W^{\langle k\rangle}, W^{\langle k-1\rangle}) = 0$ for $* \neq k$, and the map*

$$\begin{aligned} \mathcal{L} = (\lambda_1, \lambda_0, \mu, \tau) :& \\ C_k(\phi_1, u_1) \oplus C_k(\phi_0, u_0) \oplus C_k(\phi, v) \oplus C_{k-1}(\phi_1, u_1) &\longrightarrow \\ \longrightarrow H_k(W^{\langle k\rangle}, W^{\langle k-1\rangle})& \end{aligned} \tag{70}$$

is an isomorphism.

Proposition 4.9 ([24], Proposition 5.9). *The matrix of the boundary operator $\partial : E_k \to E_{k-1}$ with respect to the decomposition (70) is*

$$\begin{pmatrix} \partial_k^{(1)} & 0 & 0 & 1 \\ 0 & \partial_k^{(0)} & * & -\mathcal{H}_{k-1} \\ 0 & 0 & \partial_k & * \\ 0 & 0 & 0 & -\partial_{k-1}^{(1)} \end{pmatrix} \tag{71}$$

Here $\mathcal{H}_{k-1}$ is the homological gradient descent homomorphism (Subsection 4.4), and $\partial_*^{(1)}$, $\partial_*^{(0)}$, ∂_* are the boundary operators in the Morse complexes $C_*(\phi_1, u_1)$, resp. $C_*(\phi_0, u_0)$, and $C_*(\phi, u)$. The terms denoted by $*$ are not important for us.

4.7. **Filtration of $\bar{M}$ associated with C^0-generic gradient.** We proceed now to circle-valued Morse functions. Let $f : M \to S^1$ be a Morse map, v be an f-gradient. We shall work with the terminology of Section 3.3. Assume that M is endowed with a riemannian metric; then $\bar{M}$ and $W = F^{-1}([\lambda - 1, \lambda])$ inherit the riemannian structure. The generator t of the structure group of the covering $\bar{M} \to M$ is an isometry of $\bar{M}$ and induces an isometry $\partial_1 W \to \partial_0 W$.

We shall say that v satisfies condition $(\mathfrak{C}')$ if the $(F|W)$-gradient v satisfies the condition $(\mathfrak{C})$ from subsection 4.3, and, moreover, the Morse functions ϕ_0, ϕ_1 and their gradients u_0, u_1 can be chosen so as to satisfy $\phi_0(tx) = \phi_1(x), t_*(u_1) = u_0$. The set of Kupka-Smale f-gradients v satisfying $(\mathfrak{C}')$ will be denoted by $\mathcal{G}_0(f)$. The set $\mathcal{G}_0(f)$ is C^0-open and dense in $\mathcal{G}(f)$ (this is a version of the theorem 4.6, see [24], §8). Let $v \in \mathcal{G}_0(f)$. Consider the corresponding filtration $\{W^{\langle k\rangle}\}$ of $W = F^{-1}([\lambda - 1, \lambda])$ from the section 4.6. Define the filtration $\{V^-_{\langle k\rangle}\}$ of V^- by

$$V^-_{\langle k\rangle} = \bigcup_{s\geqslant 0} t^s W^{\langle k\rangle} \tag{72}$$

We shall now describe the homology $H_*(V^-_{\langle k\rangle}, V^-_{\langle k-1\rangle})$. Note first of all the splitting

$$V^-_{\langle k\rangle}/V^-_{\langle k-1\rangle} \approx \bigvee_{s\geqslant 0} t^s \Big(W^{\langle k\rangle}/W^{\langle k-1\rangle}\Big) \tag{73}$$

It follows immediately from 4.8 and (70) that the filtration $V^-_{\langle k\rangle}$ of V^- is nice. The homology $H_k(V^-_{\langle k\rangle}, V^-_{\langle k-1\rangle})$ can be described explicitly. Set

$$\mathcal{R}_* = C_*(\phi_1, u_1) \underset{\mathbf{Z}}{\otimes} L_-, \quad \mathcal{N}_* = C_*(\phi, v) \underset{\mathbf{Z}}{\otimes} L_- \tag{74}$$

Then $\mathcal{R}_*$ and $\mathcal{N}_*$ are free finitely generated Λ_--complexes. Tensoring with L_- the maps λ_1, μ, τ we obtain homomorphisms

$$\widetilde{\lambda}_1 : \mathcal{R}_k \to H_k(V^-_{\langle k\rangle}, V^-_{\langle k-1\rangle}), \quad \widetilde{\mu} : \mathcal{N}_k \to H_k(V^-_{\langle k\rangle}, V^-_{\langle k-1\rangle}),$$
$$\widetilde{\tau} : \mathcal{R}_{k-1} \to H_k(V^-_{\langle k\rangle}, V^-_{\langle k-1\rangle}) \tag{75}$$

The module $C_k(\phi_0, u_0)$ is identified with $tC_k(\phi_1, u_1)$ and therefore the homological gradient descent $\mathcal{H}_k$ can be considered after tensoring by

L_-, as a homomorphism $\mathcal{R}_k \to \mathcal{R}_k$. Note that its image is in $t\mathcal{R}_*$. The following two propositions are proved in [24], Prop. 7.4.

Proposition 4.10. 1) $H_*(V^-_{\langle k\rangle}, V^-_{\langle k-1\rangle}) = 0$ *if* $* \neq k$

2) *The map*

$$L = (\widetilde{\lambda}_1, \widetilde{\mu}, \widetilde{\tau}) : \mathcal{R}_k \oplus \mathcal{N}_k \oplus \mathcal{R}_{k-1} \to H_k(V^-_{\langle k\rangle}, V^-_{\langle k-1\rangle}) \tag{76}$$

is an isomorphism.

Thus the filtration induced by $\{V^-_{\langle k\rangle}\}$ in the singular chain complex of V^- is nice; the corresponding adjoint chain complex will be denoted by $\mathcal{E}_*$.

Proposition 4.11. *The matrix of the boundary operator* $\partial : \mathcal{E}_k \to \mathcal{E}_{k-1}$ *with respect to the direct sum decomposition (76) is*

$$\begin{pmatrix} \partial_k^{(1)} & * & 1 - \mathcal{H}_{k-1} \\ 0 & \partial_k & * \\ 0 & 0 & -\partial_{k-1}^{(1)} \end{pmatrix} \tag{77}$$

□

The direct sum decomposition (76) gives rize to a natural free base in $\mathcal{E}_*$, namely the family of the free generators of $\mathcal{E}_k$ is is

$$S_k(\phi_1) \sqcup S_k(\phi) \sqcup S_{k-1}(\phi_1).$$

Corollary 2.6 implies the existence of a chain equivalence

$$\mathcal{E}_* \to C^s_*(V^-).$$

Composing it with the inverse of a natural chain equivalence

$$\iota : C^\Delta_*(V^-) \to C^s_*(V^-)$$

(where Δ is the triangulation of M chosen so that V be a simplicial subcomplex, see the beginning of Subsection 3.5) we obtain a chain equivalence

$$(\iota)^{-1} \circ \rho : \mathcal{E}_* \to C^\Delta_*(V^-) \tag{78}$$

of finitely generated free based L_--complexes. Note, that the torsion of this chain equivalence vanishes, since it belongs to the group $\overline{K}_1(\mathbf{Z}[t])$ and this group is trivial by [3].

4.8. **Homological gradient descent and zeta function.** As in the preceding subsection we assume here that v satisfies the condition $(\mathfrak{C}')$. We assume moreover that $v \in \mathcal{G}(f)$ so that the eta and zeta functions are defined. We have

Proposition 4.12 ([24], §8). [2]

$$\zeta_L(-v) = \prod_{i=0}^{m} \Big(\det(1 - \mathcal{H}_m) \Big)^{(-1)^{m+1}} \tag{79}$$

Without reproducing the full proof of this proposition we wish to indicate here the main idea. Let γ be a closed orbit of $(-v)$, and x be the intersection of γ with $V = \partial_1 W$. There is a unique k satisfying $x \in V^{\langle k\rangle} \setminus V^{\langle k-1\rangle}$. It follows from the condition $(\mathfrak{C})$ that every time the trajectory γ intersects V, the point of intersection belongs again to $x \in V^{\langle k\rangle} \setminus V^{\langle k-1\rangle}$. Therefore the set of closed orbits of $(-v)$ falls into the disjoint union of subsets $\mathcal{F}_k$, indexed by $k, 0 \leqslant k \leqslant m-1$ and the subset $\mathcal{F}_k$ is identified with the set of periodic orbits of the gradient descent map

$$v\!\downarrow\colon \partial_1 W^{(k)}/\partial_1 W^{(k-1)} \to \partial_0 W^{(k)}/\partial_0 W^{(k-1)} \approx \partial_1 W^{(k)}/\partial_1 W^{(k-1)} \tag{80}$$

Now recall that $\mathcal{H}_k$ stands for the homomorphsim induced by $v\!\downarrow$ in homology, and apply the Lefschetz fixed point formula to finish the proof.

4.9. **Homotopy equivalence** ψ**.** Now everything is ready to describe the chain equivalence $\psi : C_*(f,v) \to C_*^s(\bar{M}) \underset{L}{\otimes} \widehat{L}$. As it was the case for the equivalence Φ from Theorem A, we construct first an equivalence of chain complexes $C_*^-(f,v;\lambda)$ and $C_*(V^-) \underset{L_-}{\otimes} \widehat{L}_-$. The formula (88) defining this chain equivalence, requires some preliminary computations with descending discs and their homology classes. Let $p \in S_k(F) \cap W$. For $r \geqslant 1$ set

$$\Sigma_r(p) = D(p,v) \cap t^r \cdot V,$$

so that $\Sigma_r(p)$ is an oriented $(k-1)$-dimensional submanifold of $t^r \cdot V$.

Proposition 4.13. 1) $D(p,v) \subset \operatorname{Int} V^-_{\langle k\rangle}$.

2) *For every* $r \geqslant 1$ *we have*

$$\Sigma_r(p) \subset t^r \cdot \operatorname{Int} \partial_1 W^{(k-1)}. \tag{81}$$

3) *The set* $\Sigma_r(p) \setminus t^r \cdot \operatorname{Int} \partial_1 W^{(k-2)}$ *is compact.*

Proof. From the very definition of the set Z_k we have $D(p,v) \cap W \subset Z_k$. The condition $(\mathfrak{C}')$ implies also

$$\Sigma_1(p) \subset \operatorname{Int} \partial_0 W^{(k-1)} = t \cdot \operatorname{Int} \partial_1 W^{(k-1)}.$$

[2] In [24] there is a sign error in this formula corrected in [25].

Thus (81) is proved for $r = 1$. It is now easy to prove the points 2) and 3) by induction in r, using the condition $(\mathfrak{C}')$, and deduce from it the inclusion

$$D(p, v) \cap t^r \cdot W \subset t^r \cdot \mathrm{Int}\ V^-_{\langle k\rangle}. \qquad \square \tag{82}$$

It follows that the fundamental class of $\Sigma_r(p)$ is well defined modulo $t^r \cdot \partial_1 W^{(k-2)}$. This element will be denoted

$$\sigma_r(p) \in H_{k-1}\Big(t^r \cdot \partial_1 W^{(k-1)},\ t^r \cdot \partial_1 W^{(k-2)}\Big)$$

The next formula is easy to prove by induction in r:

$$\sigma_r(p) = \mathcal{H}^{r-1}_{k-1}(\sigma_1(p)). \tag{83}$$

Now let $n \in \mathbf{N}$ and consider the chain complex $\mathcal{E}_* \underset{L_-}{\otimes} L_n = \mathcal{E}_*/t^n\mathcal{E}_*$.

Using (73) it is not difficult to obtain a natural isomorphism:

(84)

$$\mathcal{E}_k/t^n\mathcal{E}_k \approx H_k(V^-_{\langle k\rangle}, V^-_{\langle k-1\rangle} \cup t^n V^-_{\langle k\rangle}) \approx H_k(V^-_{\langle k\rangle} \cup t^n V^-, V^-_{\langle k-1\rangle} \cup t^n V^-)$$

The critical points of indices $\leqslant k-1$ and their descending discs are contained in Int $V^-_{\langle k-1\rangle}$ by 4.13, therefore the set $D(p, v) \setminus \mathrm{Int}\ (V^-_{\langle k-1\rangle} \cup t^n V^-)$ is compact. This implies in particular, that the chosen orientation of $D(p, v)$ defines a homology class

$$\Delta_p \in H_k(V^-_{\langle k\rangle} \cup t^n V^-, V^-_{\langle k-1\rangle} \cup t^n V^-) = \mathcal{E}_k/t^n\mathcal{E}_k. \tag{85}$$

Our next aim is to compute the components of this element with respect to the direct sum decomposition (76).

Proposition 4.14. *Let $p \in S_k(F) \cap W$. Then*

$$\Delta_p = [p] - \sum_{r=1}^{n-1} \widetilde{\tau}(\mathcal{H}^{r-1}_{k-1}(\sigma_1(p))) \tag{86}$$

Proof. Let $\mathcal{Z} = \cup_{0\leqslant s\leqslant n-1} t^s W^{\langle k-1\rangle}$. The set $Q = D(p, v) \setminus \mathrm{Int}\ \mathcal{Z}$ falls into the disjoint union:

$$Q = \sqcup_i Q_i \quad \text{with} \quad Q_i = Q \cap F^{-1}([\lambda - i - 1, \lambda - i])$$

(where i is a natural number in $[0, n-1]$). The fundamental class of Q_0 modulo $W^{\langle k-1\rangle}$ equals to $[p]$ by definition. For $i > 0$ we have:

$$Q_i = T(\Sigma_i(p), -v) \cap F^{-1}([\lambda - i - 1 + a_k, \lambda - i])$$

Thus the homology class of Q_i modulo its boundary equals to the element $\tau(\sigma_i(p))$ and the application of (83) finishes the proof. $\square$

Now we can define a map of $\widehat{L}_-$-modules

$$C_*^-(f,v) \xrightarrow{\xi} \mathcal{E}_* \underset{L_-}{\otimes} \widehat{L}_- \tag{87}$$

setting for the generators $p \in S_k(F) \cap W$

$$\xi(p) = [p] - \sum_{r=0}^{\infty} \widetilde{\tau}(\mathcal{H}_{k-1}^r(\sigma_1(p))) \tag{88}$$

By 2.6 there is a chain homotopy equivalence

$$\lambda : \mathcal{E}_* \underset{L_-}{\otimes} \widehat{L}_- \to C_*^s(V^-) \underset{L_-}{\otimes} \widehat{L}_- \tag{89}$$

By definition ([24]) the chain homotopy equivalence ψ is the composition $\lambda \circ \xi$. In [24] the fact that ξ is a chain map is verified by a direct computation. Using the methods developed in this paper we shall give a simpler proof, which constitutes the first part of the Proposition 5.1 in the next section. The second part of this proposition implies Theorem B.

5. Proof of Theorem B

In the first subsection we prove Theorem B for C^0-generic gradients in $\mathcal{G}(f)$. The subsections 5.2, 5.3 contain the proof of the general case, which will be deduced from the C^0-generic case by a perturbation argument. Here is the schema of this argument. Let v be an arbitrary Kupka-Smale gradient, and

$$\tau(v) = w(M,f,v), \quad \zeta(v) = (\zeta_L(-v))^{-1} \in \mathbf{Z}[[t]]$$

be the corresponding invariants. It suffices to prove that for every $n \geqslant 0$ the images $\tau_n(v), \zeta_n(v)$ of these elements in the quotient ring $L_n = \mathbf{Z}[[t]]/t^n$ are equal. Using Theorem 4.6 pick a C^0-small perturbation w of v such that $w \in \mathcal{G}_0(f)$. We show that $\tau_n(v) = \tau_n(w)$ (Section 5.2 below), and $\zeta_n(v) = \zeta_n(w)$ (Section 5.3 below). Theorem B follows, since $\zeta(w) = \tau(w)$ by the results of Subsection 5.1.

5.1. **The case of C^0-generic f-gradient.** In this and the next subsection we work with the terminology of Subsection 4.9. Thus v is an f-gradient satisfying the condition $(\mathfrak{C})$, $F : \bar{M} \to \mathbf{R}$ is a lift of $f : M \to S^1$ and λ is a regular value of F. We fix λ up to the end of the section, and the chain complex $C_*^-(f,v;\lambda)$ (see Subsection 3.3 for definition) will be denoted $C_*^-(f,v)$ for brevity. Recall the chain equivalence $\mu : C_*^-(f,v) \to C_*^s(V^-) \underset{L_-}{\otimes} \widehat{L}_-$ from Proposition 3.4 and the homomorphism (88) of $\widehat{L}$-modules $\xi : C_*^-(f,v) \to \mathcal{E}_* \underset{L_-}{\otimes} \widehat{L}_-$.

Proposition 5.1. *1) ξ is a chain map.*

2) The following diagram is chain homotopy commutative

$$
\begin{tikzcd}
C_*^-(f,v) \arrow[d, "\mu"] \arrow[r, "\xi"] & \mathcal{E}_* \underset{L_-}{\otimes} \widehat{L}_- \arrow[dl, "\lambda"] \\
C_*^s(V^-) \underset{L_-}{\otimes} \widehat{L}_- &
\end{tikzcd}
\tag{90}
$$

Proof. Consider the map

$$\xi_n = \xi \otimes L_n : C_*^-(f,v)\Big/ t^n C_*^-(f,v) \longrightarrow \mathcal{E}_*/t^n\mathcal{E}_* \tag{91}$$

Both the target and the source of ξ_n are adjoint chain complexes corresponding to two different filtrations of the pair (V^-, t^nV^-). The first filtration is associated with a t-ordered Morse function χ on W_n (see Subsection 3.3), the second is the filtration

$$\Big\{ \Big(V_{\langle k \rangle}^- \cup t^nV^-, t^nV^- \Big) \Big\} \tag{92}$$

constructed in 4.7. According to 3.3 we can choose a t-ordered function $\chi : W_n \to \mathbf{R}$ and an ordered sequence $(a_0, ..., a_{m+1})$ for χ in such a way that for every k we have

$$\chi^{-1}([a_0, a_{k+1}]) \cup t^nV^- \subset V_{\langle k \rangle}^- \cup t^nV^- \tag{93}$$

Thus the identity map of (V^-, t^nV^-) sends the terms of the first filtration to the respective terms of the second filtration. It follows from the definition of ξ and Proposition 4.14 that the map ξ_n is exactly the map induced in the adjoint complexes by the identity map. Thus ξ_n is the chain map for every n, and therefore ξ is also a chain map. Moreover the diagram obtained from (90) by tensoring with L_n is homotopy commutative by 2.6. Therefore (90) is homotopy commutative by Proposition 2.8. □

The case of C^0-generic gradients will be over with the next proposition.

Proposition 5.2. *Let $v \in \mathcal{G}(f)$. Then*

$$\tau(v) = \zeta(v). \tag{94}$$

Proof. In order to compute the torsion of the chain map

$$\Phi_\Delta = \chi_\Delta^{-1} \circ \Phi(M, f, v)$$

it suffices to compute the torsion of the chain equivalence

$$C_*^-(f,v) \xrightarrow{\xi} \mathcal{E}_* \underset{L_-}{\otimes} \widehat{L}_-$$

since the torsion of $\chi_\Delta^{-1} \circ \lambda$ vanishes (see the remark just before Subsection 4.9). To this end note that for every generator $\{p\}$ of $C_*^-(f,v)$ corresponding to a critical point $p \in S(f)$ its image $\xi(p)$ differs from the free generator $[p]$ of $\widehat{\mathcal{E}}_* = \mathcal{E}_* \underset{L_-}{\otimes} \widehat{L}_-$ by an element in $t \cdot \widehat{\mathcal{E}}_*$. Therefore for every k the family

$$\{[p]_{p\in S_k(\phi_1)},\ \xi([q])_{q\in S_k(\phi_1)},\ \widetilde{\tau}([r])_{r\in S_{k-1}(\phi_1)}\}$$

is a free base of $\widehat{\mathcal{E}}_k$, and the torsion of ξ is equal to the torsion of the acyclic complex

$$\widehat{\mathcal{E}}'_* = \widehat{\mathcal{E}}_*/\mathrm{Im}\ \xi$$

The module $\widehat{\mathcal{E}}'_k$ is isomorphic to $\widehat{\mathcal{R}}_k \oplus \widehat{\mathcal{R}}_{k-1}$ where

$$\widehat{\mathcal{R}}_k = \mathcal{R}_k \underset{L_-}{\otimes} \widehat{L}_- = C_k(\phi_1, u_1) \underset{\mathbf{Z}}{\otimes} \widehat{L}_-.$$

Therefore the $\widehat{L}_-$-base of $\widehat{\mathcal{E}}'_*$ is identified with $S_k(\phi_1) \sqcup S_{k-1}(\phi_1)$ and the matrix of the boundary operator

$$\partial'_k : \widehat{\mathcal{E}}'_k \to \widehat{\mathcal{E}}'_{k-1}$$

with respect to this base is obtained immediately from (77):

$$\partial'_k = \begin{pmatrix} \partial_k^{(1)} & 1 - \mathcal{H}_{k-1} \\ 0 & \delta_{k-1} \end{pmatrix} \tag{95}$$

where

$$\delta_k : \widehat{\mathcal{R}}_k \to \widehat{\mathcal{R}}_{k-1}$$

is some homomorphism with $\delta_{k-1} \circ \delta_k = 0$. Recall that for erevy k we have $\mathrm{Im}\ \mathcal{H}_k \subset t \cdot \mathcal{R}_k$ so that the homomorphism

$$\big(1 - \mathcal{H}_k\big) \otimes \mathrm{id} : \mathcal{R}_k \underset{L_-}{\otimes} \widehat{L}_- \to \mathcal{R}_k \underset{L_-}{\otimes} \widehat{L}_-$$

is invertible, and determines an element $[1 - \mathcal{H}_k]$ in $\overline{K}_1(\widehat{L}_-)$. The following lemma is purely algebraic, its proof uses only the algebraic information on the complex $\widehat{\mathcal{E}}'_*$, cited above.

Lemma 5.3 ([24], §6). *The torsion of the chain complex $\widehat{\mathcal{E}}'_*$ is equal to*

$$\sum_k (-1)^k [1 - \mathcal{H}_k]. \qquad \square \tag{96}$$

Now recall the formula (79) and the proposition is proved. □

5.2. **Invariance of $\tau_n(v)$ under small C^0-perturbations of v.** Let $\phi : W_n \to [a,b]$ be any t-ordered Morse function on W_n, such that v is also a ϕ-gradient. Choose an ordering sequence $(a_0, ..., a_{m+1})$ for ϕ. For each p choose a neighborhood U_p of p in M, so small that $\pi^{-1}(U_p) \cap W_n \subset \phi^{-1}(]a_k, a_{k+1}[)$ where $\pi : \bar{M} \to M$ is the infinite cyclic covering and $k = \mathrm{ind} p$. Let $U = \cup_p U_p$. The aim of the present subsection is the following proposition.

Proposition 5.4. *There is $\epsilon > 0$ such that for any $w \in G(f)$ with $||w - v|| < \epsilon$* [3] *and $w|U = v|U$ we have*

$$\tau_n(w) = \tau_n(v).$$

Proof. Let $(X^{(i)}, t^n V^-)$ be the filtration of the pair $(V^-, t^n V^-)$ corresponding to the function ϕ (see (40)). There is an isomorphism of the truncated Novikov complex $C_*^-(f,v)/t^n C_*^-(f,v)$ to $C_*^{gr}(V^-, t^n V^-)$. This isomorphism is base preserving, if we choose in $C_*^{gr}(V^-, t^n V^-)$ the base formed by the homology classes of the descending discs of the critical points. Choose a triangulation Δ in such a way that V is a simplicial subcomplex. Then W_n is a simplicial subcomplex of $\bar{M}$ and the simplicial chain complex $C_*^{\Delta}(V^-, t^n V^-)$ obtains a structure of a free based $\mathbf{Z}[t]/t^n$-module. The element $\tau_n(v) \in \mathbf{Z}[t]/t^n$ (modulo ± 1) is the determinant of the torsion of the following composition:

$$C_*^{gr}(V^-, t^n V^-) \xrightarrow{J_n(v)} C_*^s(V^-, t^n V^-) \longrightarrow C_*^{\Delta}(V^-, t^n V^-) \tag{97}$$

where the base in the left hand side complex is formed by the homotopy classes of the descending discs of v. It is easy to prove that there is $\epsilon > 0$ such that every f-gradient w with $||v - w|| < \epsilon$ and with $w|U = v|U$ satisfies the two following conditions

1) w is still a ϕ-gradient.
2) the homotopy classes of the descending discs corresponding to v coincide with those corresponding to w.

Thus the chain equivalences $J(v)$ and $J(w)$ have the same domain and are chain homotopic. Moreover, the bases in their domains are the same. The proposition follows. □

5.3. **Invariance of $\zeta_n(v)$ under small C^0-perturbations of v.** This part is more delicate. Let $k \in \mathbf{N}$ and recall the (not everywhere defined) map $((-v)^{\leadsto})^k : V \to t^k V$. The map $t^{-k}((-v)^{\leadsto})^k$ is a diffeomorphism of an open subset in V to another open subset of V. For $v \in \mathcal{G}(f)$ the set $\mathcal{F}_k(v)$ of the fixed points of this diffeomorphism is finite. Let $L_k(v)$ be the algebraic number of these fixed points, i.e. $L_k(v) = \sum \varepsilon(p)$

[3] We recall that the symbol $||\cdot||$ denotes the C^0-norm.

where $\varepsilon(p) = \pm 1$ stands for the index of the fixed point p. Then we have the following formula (which follows easily from the definition):

$$\eta_L(-v) = \sum_{k \geqslant 1} \frac{L_k(v)}{k} t^k \tag{98}$$

The next proposition asserts the invariance of $\zeta_n(v)$ with respect to small C^0-perturbations.

Proposition 5.5. *Let $v \in \mathcal{G}(f)$, and $k \in \mathbf{N}$. There is $\epsilon > 0$ such that for every $w \in \mathcal{G}(f)$ with $||w - v|| < \epsilon$ we have*

$$L_k(v) = L_k(w). \tag{99}$$

The proof occupies the rest of this section. We shall work in the cobordism W_k. Pick a number δ sufficiently small so that $v \mid W_k$ is δ-separated. Let

$$K^+(\delta) = \bigcup_{p \in S(F) \cap W_k} D_\delta(p, -v), \qquad K^-(\delta) = \bigcup_{p \in S(F) \cap W_k} D_\delta(p, v)$$

Proposition 5.6. *For any sufficiently small $\delta > 0$ we have:*

$$\mathcal{F}_k(v) \cap K^+(\delta) = \emptyset. \tag{100}$$

Proof. Let K_r^+ be the union of all ascending discs of v corresponding to the critical points of v of indices $\geqslant r$, i.e.

$$K_r^+ = \bigcup_{\substack{p \in S(F) \cap W_k, \\ \operatorname{ind} p \geqslant r}} D(p, -v)$$

The set K_r^+ is compact, as well as its δ-thickening $K_r^+(\delta)$ introduced in the next formula

$$K_r^+(\delta) = \bigcup_{\substack{p \in S(F) \cap W_k, \\ \operatorname{ind} p \geqslant r}} D_\delta(p, -v), \tag{101}$$

Similary, set

$$K_r^- = \bigcup_{\substack{p \in S(F) \cap W_k, \\ \operatorname{ind} p \leqslant r}} D(p, v), \qquad K_r^-(\delta) = \bigcup_{\substack{p \in S(F) \cap W_k, \\ \operatorname{ind} p \leqslant r}} D_\delta(p, v) \tag{102}$$

The transversality condition implies

$$t^k(K_r^+) \cap K_r^- = \emptyset. \tag{103}$$

Since $K_r^+(\delta)$, resp $K_r^-(\delta)$ form a fundamental system of neighborhoods of K_r^+, resp. K_r^-, (see Proposition 4.2) there is $\delta > 0$ such that

$$t^k(K_r^+(\delta)) \cap K_r^-(\delta) = \emptyset. \tag{104}$$

Take δ so small that (104) holds for all r. Note that with this δ we have $\mathcal{F}_k(v) \cap K^+(\delta) = \emptyset$. Indeed, let $x \in \mathcal{F}_k(v)$ and assume $x \in D_\delta(p, -v)$ with $\operatorname{ind} p = r$. Then $x \in K_r^+(\delta)$ and thus $t^k x \notin K_r^-(\delta)$. On the other hand the $(-v)$-trajectory $\gamma(x, \cdot; -v)$ reaches $t^k V$ and intersects it at the point

$$t^k x \in D_\delta(p, v) \cap t^k V \subset K_r^-(\delta),$$

contradiction. $\square$

Our next aim is to show that the formula (100) still holds if we replace v by its C^0-small perturbation, i.e. an f-gradient w, which is close to v in C^0-topology. Choose $\delta > 0$ such that the conclusion of Proposition 5.6 hold. Let $\delta > \eta > \mu > 0$. Let w be an f-gradient. Similarly to the definitions above set

(105)
$$K^+(\delta, w) = \bigcup_{p \in S(F) \cap W_k} D_\delta(p, -w), \qquad K^-(\delta, w) = \bigcup_{p \in S(F) \cap W_k} D_\delta(p, w)$$

Proposition 5.7. *There is $\epsilon > 0$, such that for every $w \in \mathcal{G}(f)$ with $||w - v|| < \epsilon$ we have:*

$$\mathcal{F}_k(w) \cap K^+(\eta, w) = \emptyset. \tag{106}$$

Proof. As in the proof of Proposition 5.6 it suffices to show that for any given $r \in \mathbf{N}, 1 \leqslant r \leqslant n$ and a C^0-small perturbation w of v we have:

$$t^k(K_r^+(\eta, w)) \cap K_r^-(\eta, w) = \emptyset.$$

Here the sets $K_r^+(\eta, w)$, $K_r^-(\eta, w)$ are defined similarly to (101), (102), namely

$$K_r^+(\eta, w) = \bigcup_{\substack{p \in S(F) \cap W_k, \\ \operatorname{ind} p \geqslant r}} D_\eta(p, -w), \qquad K_r^-(\eta, w) = \bigcup_{\substack{p \in S(F) \cap W_k, \\ \operatorname{ind} p \leqslant r}} D_\eta(p, w).$$

These sets are the η-thickenings of the following unions of descending, respectively ascending discs:

$$K_r^+(w) = \bigcup_{\substack{p \in S(F) \cap W_k, \\ \operatorname{ind} p \geqslant r}} D(p, -w), \qquad K_r^-(w) = \bigcup_{\substack{p \in S(F) \cap W_k, \\ \operatorname{ind} p \leqslant r}} D(p, w),$$

so that we have in particular

$$K_r^+(w) \subset \operatorname{Int}\,(K_r^+(\eta, w)), \quad K_r^-(w) \subset \operatorname{Int}\,(K_r^-(\eta, w)).$$

By Lemma 4.4 we have

$$K_r^+(\eta, w) \subset \text{Int } (K_r^+(\delta, v)), \qquad K_r^-(\eta, w) \subset \text{Int } (K_r^-(\delta, v)).$$

for every sufficiently C^0-small perturbation w of v. Now apply (104) and the proof is over. □

Next we shall interpret the number $L_k(w)$ for C^0-small perturbations w of the vector field v in terms of fixed point indices of some continuous maps. For this we need some preliminaries. The next lemma follows from Lemma 4.4.

Lemma 5.8. *There is $\epsilon > 0$, such that for every $w \in G(f)$ with $||w - v|| < \epsilon$ we have*

$$K_r^+(w) \subset \text{Int } K_r^+(\mu, v). \qquad \square \tag{107}$$

Pick any $(m-1)$-dimensional compact submanifold B with boundary of V, such that

$$K^+(\mu, v) \subset \text{Int } B \subset B \subset \text{Int } K^+(\eta, v). \tag{108}$$

Lemma 5.9. *There is $\epsilon > 0$, such that for every $w \in G(f)$ with $||w - v|| < \epsilon$ we have*

$$K^+(w) \subset \text{Int } B \subset B \subset \text{Int } K^+(\eta, w) \tag{109}$$

Proof. The first inclusion follows from Lemma 5.8. The last one follows from Lemma 4.3. □

The set

$$C = V \setminus \text{Int } B$$

is a compact $(m-1)$-dimensional submanifold with boundary of V. It follows from (109) that every $(-w)$-trajectory starting at a point of C reaches $t^k V$.

Proposition 5.10. *There is $\epsilon > 0$, such that for every $w \in G(f)$ with $||w - v|| < \epsilon$ we have*

$$\mathcal{F}_k(w) \subset \text{Int } C.$$

Proof. Indeed $\mathcal{F}_k(w) \cap K^+(\eta, w) = \emptyset$ by Proposition 5.7 therefore $\mathcal{F}_k(w) \subset V \setminus B$. □

Now let w be an f-gradient such that (109) holds. The set C is then in the domain of definition of the map $((-w)^{\leadsto})^k$ and the restriction to C of the map $t^{-k}((-w)^{\leadsto})^k$ is a C^∞ embedding $S_w : C \to V$. Assuming further that $w \in \mathcal{G}(f)$ and that (106) holds, we deduce that the set Fix $(S_w) = \mathcal{F}_k(w)$ is finite and does not intersect with ∂C. Denote the sum of the indices of all the fixed points of S_w by $\mathcal{J}(S_w)$, then we have by definition

$$L_k(w) = \mathcal{J}(S_w). \tag{110}$$

Thus for such f-gradients w the computation of the number $L_k(w)$ is reduced to the computation of fixed point indices of the continuous map $S_w : C \to V$ associated to w. Actually the map $S_w : C \to V$ is defined for every C^1 vector field w sufficiently close to v in C^0 topology and the correspondance $w \mapsto S_w$ is continuous (with respect to the C^0-topology on the set of vector fields and on the set of maps $C \to V$). This is the contents of the next proposition, which follows directly from [21], Prop. 2.66.

Proposition 5.11. *Let $K \subset \partial_1 W$ be a compact set, and u be a C^1 vector field on W, such that $u|\partial_1 W$ points outward W, and $u|\partial_0 W$ points inward W. Assume that every $(-u)$-trajectory starting at K reaches $\partial_0 W$, and denote the restriction of $(-u)^{\rightsquigarrow}$ to K by S_u. Then there is $\epsilon > 0$ such that for every C^1 vector field $\widetilde{u}$ on W with $||\widetilde{u} - u|| < \epsilon$, every $(-\widetilde{u})$-trajectory starting at K reaches $\partial_0 W$ and the map $\widetilde{u} \mapsto S(\widetilde{u})$ is continuous.* □

A continuous map $g : C \to V$ will be called *of class $\mathcal{F}$* if the fixed point set Fix (g) is finite and Fix $(g) \cap \partial C = \emptyset$. For a map g of class $\mathcal{F}$ we denote the sum of indices of its fixed points by $\mathcal{J}(g)$. Our main example of maps of class $\mathcal{F}$ is any map S_w with $w \in \mathcal{G}(f)$ and $||w - v|| < \epsilon$, where $\epsilon > 0$ is chosen so small that Proposition 5.7, and Lemmas 5.8, 109 hold with this ϵ. Now the proposition 5.5 follows from (110) and the next lemma.

Lemma 5.12. *Let $g : C \to V$ be a continuous map of class $\mathcal{F}$. There is $\delta > 0$ such that for every map $h : C \to V$ of class $\mathcal{F}$ with $d(h, g) < \delta$ we have*

$$\mathcal{J}(h) = \mathcal{J}(g).$$

Proof. Let $p \in$ Fix (g). Choose a chart $\Psi : U \to \mathbf{R}^n$ of the manifold V with $p \in U$ and $\Psi(p) = 0$. The inverse image of the closed disc $D(0, R) \subset \mathbf{R}^n$ with respect to Ψ will be denoted $D_R(p)$. Similarly the inverse image of the open disc $B(0, R) \subset \mathbf{R}^n$ with respect to Ψ will be denoted $B_R(p)$. Choose R so small that for every p the only fixed point of g in $D_R(p)$ is p, and the discs $D_R(p)$ are disjoint for different points p. Pick a number $\lambda \in]0, R[$ so small that for each $p \in$ Fix (g) we have $g(D_\lambda(p)) \subset B_R(p)$. The next lemma is proved by a standard continuity argument.

Lemma 5.13. *There is $\nu > 0$ such that for every map $h : C \to V$ of class $\mathcal{F}$, such that $d(h,g) < \nu$ we have:*

$$\begin{cases} h(D_\lambda(p)) \subset B_R(p) & \text{for every } p \in \text{Fix }(g) \\ \text{Fix }(h) \subset \cup_p D_{\lambda/2}(p). \end{cases} \tag{111}$$ □

For every h satisfying (111) the set Fix (h) falls into the disjoint union of the sets of the fixed points of the maps

$$h[p] = h \mid D_\lambda(p) : D_\lambda(p) \to B_R(p).$$

Thus it remains to prove that there is ρ such that for every h with $d(h,g) < \rho$ we have

$$\mathcal{J}(h[p]) = \mathcal{J}(g[p]) \quad \text{for every} \quad p \tag{112}$$

Let μ be the injectivity radius of the riemannian manifold V, so that for every $\mu' < \mu$ every two maps $F_1, F_2 : C \to V$ satisfying

$$d(F_1, F_2) \leqslant \mu' \tag{113}$$

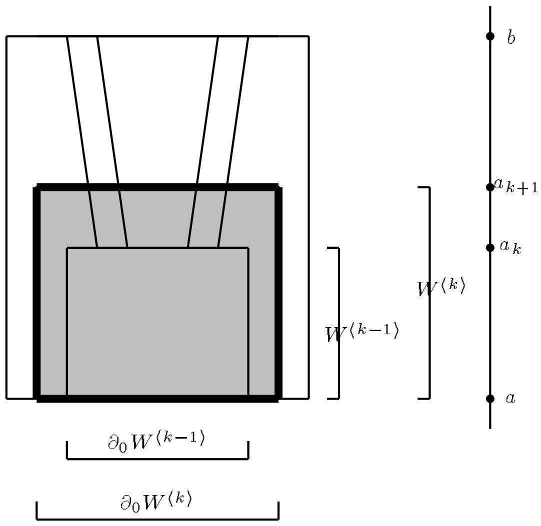
are homotopic in the class of maps satisfying (113). Then for every $h : C \to V$ satisfying

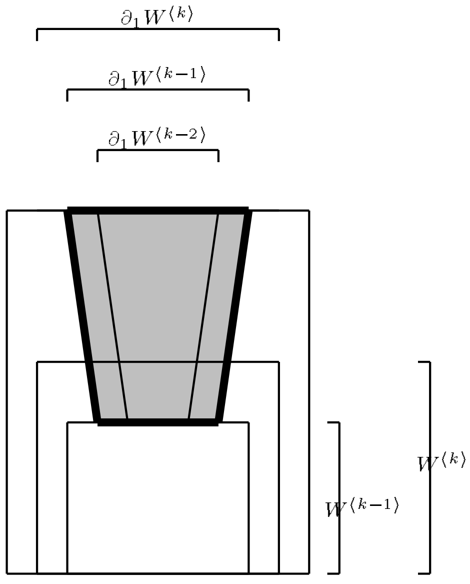
$$d(h,g) \leqslant \min(\mu/2, \nu)$$

the maps $g[p], h[p] : D_\lambda(p) \to B_R(p)$ are homotopic via a homotopy $H_t : D_\lambda(p) \to B_R(p)$, such that for every t the set of fixed points of H_t is compact and contained in $D_{\lambda/2}(p)$. Then the equality (112) holds by the homotopy invariance of the fixed point index ([5], 5.8), and the proof of the lemma is over. □

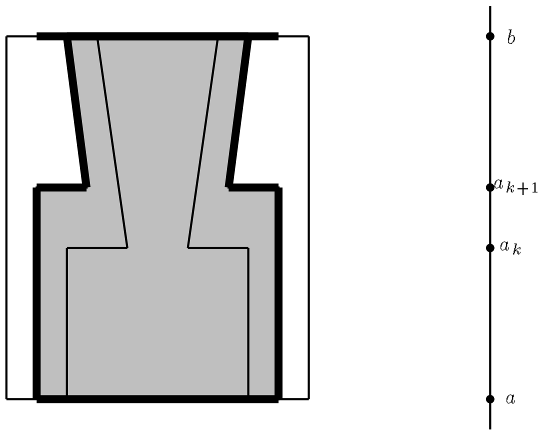
Now we can finish the proof of Theorem B. As we have mentioned already it suffices to prove that for every $n \in \mathbf{N}$ we have $\zeta_n(v) = \tau_n(v)$. For each $p \in S(f)$ choose a neighborhood U_p of p as in the beginning of Subsection 5.2. Choose $\epsilon > 0$ so small that for any $w \in G(f)$ with $||w - v|| < \epsilon$ the conclusions of propositions 5.4, 5.5 hold. Proposition 4.6 implies the existence of an f-gradient $w \in GC(f)$ such that $||v - w|| < \epsilon/3$. By a standard Kupka-Smale argument there is an f-gradient $u \in \mathcal{G}(f)$ with $||u - w|| < \epsilon/3$. Moreover the gradient u is still in $GC(f)$ if only $||u - w||$ is sufficiently small (since $GC(f)$ is C^0-open). Proposition 5.2 implies $\tau(u) = \zeta(u)$; in particular we have $\tau_n(u) = \zeta_n(u)$ for every n. Note finally that Proposition 5.5 implies $\zeta_n(u) = \zeta_n(v)$ and Proposition 5.4 implies $\tau_n(u) = \tau_n(v)$. The proof of Theorem B is over. □

The set Z_k (shaded).

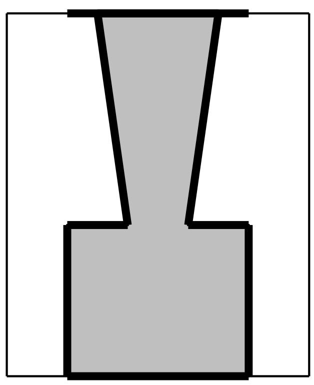




The set T_k (shaded).



The set $W^{\langle k\rangle}$ is shaded
and the set $W^{\langle k-1\rangle}$ is shown by thin lines.



The set $W^{\langle k-1\rangle}$ (shaded).

Handle-like filtration of the cobordism W (Subsection 4.6).

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