TWISTED NOVIKOV HOMOLOGY AND CIRCLE-VALUED MORSE THEORY FOR KNOTS AND LINKS

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1. Introduction

Let L be a link, that is, a C^∞ embedding of disjoint union of the oriented circles in S^3. Recall that L is called *fibred* if there is a fibration $\phi : C_L = S^3 \setminus L \to S^1$ behaving "nicely" in a neighborhood of L. If L is not fibred, it is still possible to construct a Morse map $f : C_L = S^3 \setminus L \to S^1$ behaving nicely in a neighborhood of L; such a map has necessarily a finite number of critical points. The minimal number of critical points of such map is an invariant of the link, called *Morse-Novikov number* of L and denoted by $\mathcal{MN}(L)$; it was first introduced and studied in [18]. This invariant can be studied via the methods of the *Morse-Novikov theory*, in particular the Novikov inequalities [15] provide the lower estimates for the number $\mathcal{MN}(L)$. This inequalities can be considered as *fibering obstructions* for the link. The Alexander polynomials provide another fibering obstruction: if a link is fibred then its Alexander polynomial is monic (see [19], 10.G.9). The invariants arising from the Novikov theory are in a sense stronger than the Alexander polynomial, since they allow in certain cases to give lower bounds for $\mathcal{MN}(L)$, see [18] for examples of knots with arbitrarily large Morse-Novikov number. Both the Alexander polynomial and the Novikov homology above are *abelian invariants*, that is, they are calculated from the homology of the infinite cyclic covering of C_L. More information is provided by the *non-abelian* coverings, although the corresponding invariants are more difficult to deal with. Several non-abelian versions of the Alexander polynomial were developed in 90s (see the papers by X.S.Lin [14], M.Wada [20], T.Kitano [13], and Kirk-Livingston [12]). In the recent preprint [9] by H.Goda, T.Kitano, and T.Morifuji, it was shown that if a knot is fibred, then the twisted Alexander invariant is monic.

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In the present paper we shall develop the twisted version of the Novikov homology. This part (Section 3) may be of independent interest for the Morse-Novikov theory. The existing versions of the non-abelian Novikov homology for a CW complex X are either modules over the Novikov completion of the $\mathbf{Z}\pi_1(X)$, or the vector spaces associated to some representation of $\pi_1(X)$. Our version is in a sense intermediate between these two: the twisted Novikov homology which we define is a module over the ring $\mathbf{Z}((t))$, thus it is computable (as we shall show in this paper) and on the other hand it allows to keep track of the non-abelian homological algebra associated to $\mathbf{Z}\pi_1(X)$. Theorem 4.2 gives a lower bound for the Morse-Novikov number of a link in terms of the twisted Novikov homology.

We show that the twisted Novikov homology is additive with respect to the connected sum of knots. We apply these techniques to determine the Morse-Novikov numbers of the knots $n\mathfrak{KT}, n\mathfrak{C}$, where $\mathfrak{KT}$ is the Kinoshita-Terasaka knot [11], $\mathfrak{C}$ is the Conway knot [3], and nK stands for the connected sum of the knot K with itself. We show that

$$\mathcal{MN}(n\mathfrak{KT}) = \mathcal{MN}(n\mathfrak{C}) \geqslant 2n/5.$$

The computations of the invariants arising from the twisted Novikov homology for these knots was done with the help of the Kodama's KNOT program (available at http://www.math.kobe-u.ac.jp/ kodama /knot.html). Applying the techniques of the papers [6], [7] we prove that

$$\mathcal{MN}(n\mathfrak{KT}) = \mathcal{MN}(n\mathfrak{C}) \leqslant 2n.$$

We recall the corresponding notions and results in Section 2 from [4], [6] and [7].

We show that the twisted Novikov homology is related to twisted Alexander polynomials in the same way as the abelian Novikov homology is related to the classical Alexander polynomial.

The final section of the paper contains a discussion about necessary and sufficient condition for a link to be fibred.

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2. Heegaard splitting for sutured manifold

The basic Morse theory gives a relationship of a Morse map and a handle decomposition for a manifold. In this section, we review the notion of Heegaard splitting for sutured manifold introduced in [6] and [7], and reveal the relationship with circle-valued Morse maps. Moreover, we present some properties which are used to determine Morse-Novikov numbers.

In this section, we assume that a link is always non-split.

Sutured manifold was defined by D.Gabai [4]. We use here a special class of sutured manifolds, namely, complementary sutured manifolds:

Definition 2.1. Let L be an oriented link in S^3, and R a Seifert surface of L. Set $R_E = R \cap E(L)$ ($E(L) = \mathrm{cl}(S^3 - N(L))$), and $(P, \delta) = (N(R_E, E(L)), N(\partial R_E, \partial E(L))$. We call (P, δ) a *product sutured manifold* for R. Thus P is homeomorphic to $R_E \times [0,1]$, and then we denote by $R_+(\delta)$ ($R_-(\delta)$ resp.) the surface $R_E \times \{1\}$ ($R_E \times \{0\}$ resp.). Let

$$(M, \gamma) = (\mathrm{cl}(E(L) - P), \mathrm{cl}(\partial E(L) - \delta))$$

with $R_\pm(\gamma) = R_\mp(\delta)$. We call (M, γ) a *complementary sutured manifold* for R. In this paper, we call this a *sutured manifold* for short.

Here we denote by $N(X, Y)$ a regular neighborhood of X in Y.

In [2], the notion of compression body was intorduced by A. Casson and C. Gordon. It is a generalization of a handlebody, and important to define a Heegaard splitting for 3-manifolds with boundaries.

Definition 2.2. A *compression body* W is a cobordism rel ∂ between surfaces $\partial_+ W$ and $\partial_- W$ such that $W \cong \partial_+ W \times [0,1] \cup$ 2-handles $\cup$ 3-handles and $\partial_- W$ has no 2-sphere components. We can see that if $\partial_- W \neq \emptyset$ and W is connected, W is obtained from $\partial_- W \times [0,1]$ by attaching a number of 1-handles along the disks on $\partial_- W \times \{1\}$ where $\partial_- W$ corresponds to $\partial_- W \times \{0\}$.

We denote the number of these 1-handles by $h(W)$.

These notions enable us to define a Heegaard splitting for sutured manifold.

Definition 2.3. ([6]) (W, W') is a Heegaard splitting for (M, γ) if

(i) W, W' are compression bodies,
(ii) $W \cup W' = M$,
(iii) $W \cap W' = \partial_+ W = \partial_+ W', \partial_- W = R_+(\gamma)$, and $\partial_- W' = R_-(\gamma)$.

Definition 2.4. ([7]) Set $h(R) = \min\{h(W)(= h(W'))|(W, W')$ is a Heegaard splitting for the sutured manifold of $R\}$. We call $h(R)$ the *handle number* of R.

The behaviors of the handle number are studied in [6] and [7].

In order to state the relationship between the handle number and the Morse-Novikov number, we recall some definitions on circle-valued Morse map according to [18].

Definition 2.5. ([18]) Let L be a link. A Morse map $f : C_L \to S^1$ is said to be *regular* if the link L has a neighborhood framed as $L \times D^2$ with $L \approx L \times 0$, in such a way that the restriction $f | L \times (D^2 \setminus \{0\}) \to S^1$ is given by $(x, y) \mapsto y/|y|$.

For a regular Morse map we denote by $S_i(f)$ the set of all critical points of index i and by $m_i(f)$ the cardinality of $S_i(f)$. We say that a Morse map $f : C_L \to S^1$ is *minimal* if it is regular and $m_i(f)$ are minimal possible among all regular maps homotopic to f. We define $\mathcal{MN}(L)$ as the number of critical points of the minimal Morse map.

Definition 2.6. A regular Morse map $f : C_L \to S^1$ is said to be *moderate* if it satisfies the all of the following:

(i) $m_0(f) = m_3(f) = 0$;
(ii) all critical values corresponding to critical points of the same index coincide;
(iii) $f^{-1}(x)$ is a connected Seifert surface for any regular value $x \in S^1$.

Theorem 2.7. ([18]) *Every link has a minimal Morse map which is moderate.*

Corollary 2.8. (1) *Let f be a moderate map, then $m_1(f) = m_2(f)$.*
(2) *Let f be a regular Morse map realizing $\mathcal{MN}(L)$, then $\mathcal{MN}(L) = m_1(f) + m_2(f)$.*
(3) $\mathcal{MN}(L) = 2 \times \min\{h(R) | R$ *is a Seifert surface for* $L\}$.

We denote by $h(L)$ the minimum handle number among all Seifert surfaces of L. Note that L is a fibred if and only if $h(L) = 0$.

Thus we know that the handle number and Morse-Novikov number are same essentially, that is,

$$\mathcal{MN}(L) = 2 \times h(L).$$

We shall finish this section with some remarks on behaviors of the invariant introduced above with respect to connected sum and plumbing. Let us denote $\natural$ the operation of plumbing. For a Seifert surface R of a link L, we set $\mathcal{MN}(R) = 2 \times h(R)$. Conjecture 6.3 in [18] says that

$$\mathcal{MN}(R_1 \natural R_2) \leq \mathcal{MN}(R_1) + \mathcal{MN}(R_2).$$

This conjecture follows from Theorem A in [7] in the case of non-split links. In the recent paper [10], they prove it in general case. Let K_1

and K_2 be a knot in S^3, and we denote by $K_1 \sharp K_2$ their connected sum. As far as we know the question in [18]

$$\mathcal{MN}(K_1 \sharp K_1) = \mathcal{MN}(K_1) + \mathcal{MN}(K_2)?$$

is still unproved, so we know only

$$\mathcal{MN}(K_1 \sharp K_2) \leq \mathcal{MN}(K_1) + \mathcal{MN}(K_2). \tag{1}$$

3. Twisted Novikov homology

Let R be a commutative ring, put

$$Q = R[t, t^{-1}], \quad \widehat{Q} = R((t)) = R[[t]][t^{-1}].$$

The ring Q is isomorphic to the group ring $R[\mathbf{Z}]$, via the isomorphism sending $t \in Q$ to the element $-1 \in \mathbf{Z}$. The ring $\widehat{Q}$ is then identified with *the Novikov completion* of $R[\mathbf{Z}]$. In the case when R is a field, $\widehat{Q}$ is also a field. When $R = \mathbf{Z}$, the ring $\widehat{Q}$ is PID. We shall need in this paper only the particular cases when $R = \mathbf{Z}$, or R is a field. Let X be a CW complex; let $G = \pi_1 X$, and let $\xi : G \to \mathbf{Z}$ be a homomorphism. Let $\rho : G \to GL(n, R)$ be a right representation of G, that is, $\rho(g_1 g_2) = \rho(g_2)\rho(g_1)$. The homomorphism ξ extends to a ring homomorphism $\mathbf{Z}[G] \to Q$, which will be denoted by the same symbol ξ. The tensor product $\rho \otimes \xi$ (where ξ is considered as a representation $G \to GL(1, Q)$) induces a representation $\rho_\xi : G \to GL(n, Q)$. The composition of this representation with the natural inclusion $Q \hookrightarrow \widehat{Q}$ gives a representation $\widehat{\rho}_\xi : G \to GL(n, \widehat{Q})$. Let us form a chain complex

$$\widehat{C}_*(\widetilde{X}; \xi, \rho) = \widehat{Q}^n \underset{\widehat{\rho}_\xi}{\otimes} C_*(\widetilde{X}). \tag{2}$$

Here $\widetilde{X}$ is the universal cover of X, $C_*(\widetilde{X})$ is a module over $\mathbf{Z}\pi_1 X$, and $\widehat{Q}^n$ is a $\mathbf{Z}G$-module via the representation $\widehat{\rho}$. Then (2) is a chain complex of free left modules over $\widehat{Q}$, and the same is true for its homology. The modules

$$\widehat{H}_*(X; \xi, \rho) = H_*(\widehat{C}_*(\widetilde{X}; \xi, \rho)),$$

will be called *ρ-twisted Novikov homology* or simply *twisted Novikov homology* if no confusion is possible. When these modules are finitely generated (this is the case for example for any X homotopy equivalent to a finite CW complex) we set

$$\widehat{b}_i(X; \xi, \rho) = \operatorname{rk}_{\widehat{Q}}(\widehat{H}_i(X; \xi, \rho)), \quad \widehat{q}_i(X; \xi, \rho) = \text{t.n.}_{\widehat{Q}}(\widehat{H}_i(X; \xi, \rho)).$$

where t.n. stands for the *torsion number* of the $\widehat{Q}$-module, that is the minimal possible number of generators over $\widehat{Q}$.

The numbers $\widehat{b}_i(X;\xi,\rho)$ and $\widehat{q}_i(X;\xi,\rho)$ can be recovered from the canonical decomposition of $\widehat{H}_i(X;\xi,\rho)$ into a direct sum of cyclic modules. Namely, let

$$\widehat{H}_i(X;\xi,\rho) = \Big(\bigoplus_{j=1}^{\alpha_i}\widehat{Q}\Big)\bigoplus\Big(\bigoplus_{j=1}^{\beta_i}\widehat{Q}/\lambda_j^{(i)}\widehat{Q}\Big)$$

where $\lambda_j^{(i)}$ are non-zero non-invertible elements of $\widehat{Q}$ and $\lambda_{j+1}^{(i)}|\lambda_j^{(i)}\ \forall j$. (Such decomposition exists since $\widehat{Q}$ is a PID.) Then $\alpha_i = \widehat{b}_i(X;\xi,\rho)$ and $\beta_i = \widehat{q}_i(X;\xi,\rho)$. It is not difficult to show that we can always choose $\lambda_j^{(i)} \in Q\ \ \forall i, \forall j$.

When ρ is a trivial 1-dimensional representation, we obtain the usual Novikov homology, which can be also calculated from the infinite cyclic covering $\bar{X}$ associated to ξ, namely

$$\widehat{H}_*(X;\xi,\rho) = \widehat{Q}\underset{Q}{\otimes} H_*(\bar{X}) \quad \text{for} \quad \rho = 1 : G \to GL(1,R).$$

If R is a field the numbers $\widehat{q}_i(X;\xi,\rho)$ vanish (for every representation ρ), and the module $\widehat{H}_*(X;\xi,\rho)$ is a vector space over the field $\widehat{Q}$.

4. Novikov-type inequalities for knots and links

Now we shall apply the algebraic techniques developed in the previous section to the topology of knots and links. Let $L \subset S^3$ be an oriented link and put $C_L = S^3 \setminus L$. Let G denote $\pi_1(C_L)$. For every positively oriented meridian μ_i of a component of L, there is a unique element $\xi \in H^1(M,\mathbf{Z})$ such that for each i we have $\xi(\mu_i) = 1$. We shall identify this element with the corresponding homomorphism $G \to \mathbf{Z}$.

Let $\rho : G \to GL(n,R)$ be any representation of G (where $R = \mathbf{Z}$ or R is a field). The next theorem follows from the main theorem in [16]. See also Theorem 8.1 of the present paper.

Theorem 4.1. *There is a free chain complex $\mathcal{N}_*$ over $\widehat{Q}$ such that*

(1) *for every i the number of free generators of $\mathcal{N}_*$ in degree k equals $n \times m_i(f)$;*
(2) *$H_*(\mathcal{N}_*) \approx \widehat{H}_*(C_L;\xi,\rho)$.*

In what follows, we use the following terminology. The cohomology class ξ is determined by the orientation of the link, so we shall omit it in the notation. Further,

$$\widehat{H}_*(L,\rho) = \widehat{H}_*(C_L,\rho),$$

$$\widehat{b}_i(L,\rho) = \widehat{b}_i(C_L,\rho),$$

$$\widehat{q}_i(L,\rho) = \widehat{q}_i(C_L,\rho).$$

The next theorem follows from Theorem 4.1 by a simple algebraic argument.

Theorem 4.2. *Let $f : C_L \to S^1$ be any regular map. Then*

$$m_i(f) \geqslant \frac{1}{n}\big(\widehat{b}_i(L,\rho) + \widehat{q}_i(L,\rho) + \widehat{q}_{i-1}(L,\rho)\big) \quad \text{for every} \quad i. \tag{3}$$

□

Corollary 4.3. *If L is fibred, then $\widehat{H}_*(L,\rho) = 0$, and $\widehat{b}_i(L,\rho) = \widehat{q}_i(L,\rho) = 0$ for every representation ρ and every i.*

Proposition 4.4. *The twisted Novikov numbers of the space C_L satisfy the following relations:*

$$\widehat{b}_i(L,\rho) = \widehat{q}_i(L,\rho) = \widehat{q}_2(L,\rho) = 0 \quad \text{for} \quad i = 0, i \geqslant 3, \tag{4}$$

$$\widehat{b}_1(L,\rho) = \widehat{b}_2(L,\rho). \tag{5}$$

Proof. According to Theorem 3.3 of [18] there is a regular map $f : C_L \to S^1$ such that f has only critical points of indices 1 and 2 and $m_1(f) = m_2(f)$. Using Theorem 4.2 we deduce (4). As for the point (5) it follows from the fact that the Euler characteristics of the chain complex $\mathcal{N}_*$ is equal to 0. □

In view of the preceding theorem the non-trivial part of the Novikov inequalities is as follows:

$$m_1(f) \geqslant \frac{1}{n}\big(\widehat{b}_1(L,\rho) + \widehat{q}_1(L,\rho)\big); \tag{6}$$

$$m_2(f) \geqslant \frac{1}{n}\big(\widehat{b}_1(L,\rho) + \widehat{q}_1(L,\rho)\big). \tag{7}$$

Let us consider some examples and particular cases.

1. $R = \mathbf{Z}$ and ρ is the trivial 1-dimensional representation. The chain complex $\widehat{C}_*(\widetilde{C}_L;\xi,\rho)$ is equal to the chain complex $C_*(\bar{C}_L)$, where $\bar{C}_L$ is the infinite cyclic covering associated to the cohomology class ξ. Thus the twisted Novikov homology in this case coincides with the Novikov homology for links introduced and studied in [18]. Theorem 4.2 and Proposition 4.4 in this case are reduced to Proposition 2.1 and the formulas (2) – (6) of [18].

2. The ring R is a field. In this case the Novikov ring $R((t))$ is also a field, and the torsion numbers $\widehat{q}_i(L,\rho)$ vanish for every i and every representation ρ. The Novikov inequalities have the simplest possible form:

$$m_1(f) \geqslant \frac{1}{n}\widehat{b}_1(L,\rho) \leqslant m_2(f).$$

3. Now we shall investigate the twisted Novikov homology for the connected sum of knots. Let K_1, K_2 be oriented knots in S^3, and put $K = K_1 \sharp K_2$. We have:

$$\pi_1(K) = \pi_1(K_1) \underset{Z}{*} \pi_1(K_2),$$

where Z is the infinite cyclic group generated by a meridian μ of K (see [1], Ch.7, Prop. 7.10). In particular the groups $\pi_1 K_1, \pi_1 K_2$ are naturally embedded into $\pi_1 K$, and the meridian element $\mu \in \pi_1 K$ is the image of some meridian elements μ_1, μ_2 of K_1, resp. K_2. Now let $\rho_1 : \pi_1 K_1 \to GL(m, R)$, $\rho_2 : \pi_1 K_2 \to GL(m, R)$ be two representations. Assume that $\rho_1(\mu_1) = \rho_2(\mu_2)$. In this case we obtain the product representation $\rho = \rho_1 * \rho_2 : \pi_1 K \to GL(m, R)$.

Theorem 4.5. $\widehat{H}_*(K, \rho) \approx \widehat{H}_*(K_1, \rho_1) \oplus \widehat{H}_*(K_2, \rho_2)$.

Proof. The complement C_K is the unions of two subspaces C_1, C_2 with C_i having the homotopy type of C_{K_i} (for $i = 1, 2$). The intersection $C_1 \cap C_2$ is homeomorphic to the twice punctured sphere $\Delta' = S^2 \setminus \{*, *\}$. The universal covering is therefore the union of two subspaces, which have the Novikov homology respectively equal to $\widehat{H}_*(K_1, \rho_1)$ and $\widehat{H}_*(K_2, \rho_2)$. The intersection of these two subspaces have the same Novikov homology as Δ', and this module vanishes. Then the standard application of the Mayer-Vietoris sequence proves the result sought. $\square$

Corollary 4.6. *Denote by nK the connected sum of n copies of the knot K. Let $\rho : \pi_1 K \to GL(m, R)$ be a representation (where $R = \mathbf{Z}$ or R is a field). Let $\rho^n : \pi_1(nK) \to GL(m, R)$ be the product of n copies of representations ρ. Then*

$$\widehat{q}_1(nK, \rho^n) = n \cdot \widehat{q}_1(K, \rho).$$

Proof. This follows from the purely algebraic equality:

$$\text{t.n.}(nN) = n \cdot \big(\text{t.n.}(N)\big)$$

where N is any finitely generated module over a principal ideal domain, and nN stands for the direct sum of n copies of N. $\square$

5. The Kinoshita-Terasaka and Conway knots: lower bounds for the Morse-Novikov numbers

The Kinoshita-Terasaka knot $\mathfrak{KT}$ was introduced in the paper [11], and the Conway knot $\mathfrak{C}$ was discovered by J.Conway much later [3]. The resemblance of these two knots is obvious on the pictures as in

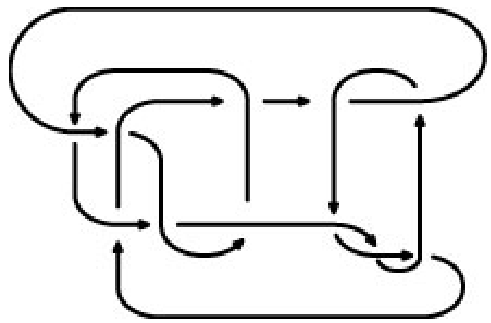

FIGURE 1. The Kinoshita-Terasaka knot $\mathfrak{KT}$

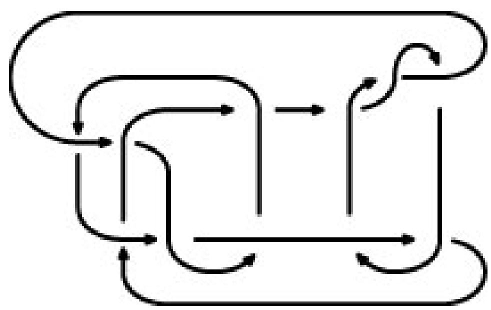

FIGURE 2. The Conway knot $\mathfrak{C}$

Figures 1 & 2 and is proved by coincidence of many algebraic invariants (still these knots are different, as was proved by M.Wada in [20]).

These knots are not fibred. Indeed, for a fibred knot the degree of its Alexander polynomial equals to two times of the genus of the knot, and the Alexander polynomial is trivial for both knots.

In this section, we prove the following:

Theorem 5.1. *There is a representation* $\rho : \pi_1\mathfrak{C} \to SL(5, \mathbf{Z})$ *such that* $\widehat{q}_1(\mathfrak{C}, \rho) \neq 0$.

By Corollary 4.6, this theorem implies

$$\mathcal{MN}(n\mathfrak{C}) \geqslant 2n/5 \quad \text{for every} \quad n. \tag{8}$$

Proof. The Wirtinger presentation for the group $\pi_1\mathfrak{C}$ has 11 generators s_i and 11 relations:

$$s_1 = s_{10}s_2s_{10}^{-1}, \quad s_2 = s_9^{-1}s_3s_9, \quad s_3 = s_6^{-1}s_4s_6, \quad s_4 = s_7^{-1}s_5s_7, \tag{9}$$

$$s_5 = s_{11}s_6s_{11}^{-1}, \quad s_6 = s_4^{-1}s_7s_4, \quad s_7 = s_2^{-1}s_8s_2, \quad s_8 = s_{11}s_9s_{11}^{-1}, \tag{10}$$

$$s_9 = s_7^{-1}s_{10}s_7, \quad s_{10} = s_8s_{11}s_8^{-1}, \quad s_{11} = s_5s_1s_5^{-1}. \tag{11}$$

There is a homomorphism $h : \pi_1(\mathfrak{C}) \to S(5)$ given by the following formulas:

$$h(s_1) = h(s_5) = h(s_6) = h(s_{11}) = (253), \quad h(s_2) = (234), \tag{12}$$

$$h(s_3) = h(s_7) = (123), \quad h(s_4) = (135), \; h(s_8) = (142), \tag{13}$$

$$h(s_9) = (145), \; h(s_{10}) = (345). \tag{14}$$

The image of h is contained in the subgroup $A(5)$. The group $S(5)$ acts by permutation of coordinates on the free $\mathbf{Z}$-module $\mathbf{Z}^5$ and we obtain therefore a representation $\rho : \pi_1\mathfrak{C} \to SL(n, \mathbf{Z})$. The twisted Novikov homology $H_1(C_{\mathfrak{K}}; \rho^*)$, with respect to the conjugate representation ρ^*, can be computed from the free $\mathbf{Z}((t))$-chain complex

$$C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2$$

where rk $C_0 = 5$, rk $C_1 = 55 =$ rk C_2. The generators of C_1 correspond to $s_i, 1 \leqslant i \leqslant 11$, the generators of C_2 correspond to the eleven relations (9) - (11), and the matrix of ∂_2 is obtained by the Fox calculus using these relations. The homomorphism ∂_1 is epimorphic, since the 0-dimensional twisted Novikov homology of a knot always vanishes. Thus the rank of the 55×55-matrix of ∂_2 is not more than 50. The determinant of the 50×50-minor of the matrix of ∂_2 obtained from the matrix by omitting the last five columns and the last five rows is equal to

$$\begin{gathered}-5t^{-29} + 14t^{-28} - 15t^{-27} + 16t^{-26} - 19t^{-25} + 10t^{-24} + 5t^{-23} - 24t^{-22} \\ +34t^{-21} - 32t^{-20} + 34t^{-19} \\ -24t^{-18} + 5t^{-17} + 10t^{-16} - 19t^{-15} + 16t^{-14} - 15t^{-13} + 14t^{-12} - 5t^{-11}.\end{gathered}$$

This polynomial is a non-invertible element of $\mathbf{Z}((t))$, since the leading coefficient is $-5 \neq \pm 1$. Therefore the torsion part of the twisted Novikov homology in dimension 1 is not zero, and $q_1(\mathfrak{C}; \rho) > 0$. By Corollary 4.6 we deduce the inequality (8). □

By using the Kodama's KNOT program, we can show that the Kinoshita-Terasaka knot $\mathfrak{KT}$ has also a representation $\rho : \pi_1\mathfrak{KT} \to SL(5, \mathbf{Z})$ such that $\widehat{q}_1(\mathfrak{KT}, \rho) \neq 0$.

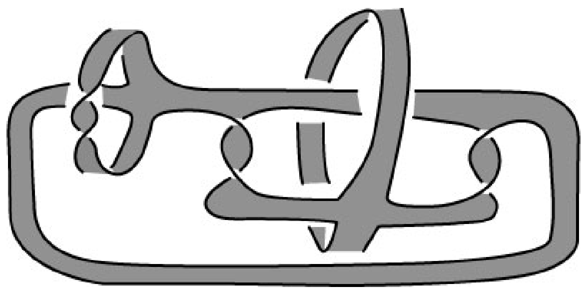

FIGURE 3. The Kinoshita-Terasaka knot $\mathfrak{KT}$

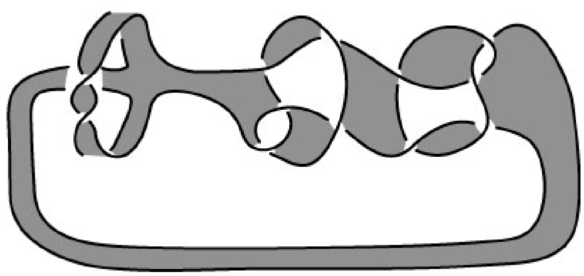

FIGURE 4. The Conway knot $\mathfrak{C}$

6. The Kinoshita-Terasaka and Conway knots: upper bounds for the Morse-Novikov numbers

In this section, we show that both $\mathcal{MN}(\mathfrak{KT})$ and $\mathcal{MN}(\mathfrak{C})$ are less than or equal to 2. Therefore $\mathcal{MN}(n\mathfrak{KT}) = \mathcal{MN}(n\mathfrak{C}) \leqslant 2n$ by the inequality (1).

Here we use the minimal genus Seifert surfaces for $\mathfrak{KT}$ and $\mathfrak{C}$, which were found in [5]. See Figures 3 and 4. Since the proofs are same, we consider only the Conway knot $\mathfrak{C}$.

Since the Hopf band is a fiber surface, we may calculate the handle number of the Seifert surface illustrated in left-hand Figure 5 by Theorem B in [7]. We call R this Seifert surface, and denote by (M, γ) the sutured manifold for R. Further, we name $R_+(\gamma)$ and $R_-(\gamma)$, and let α be an arc properly embedded in M as in Figure 5. (We abbreviate $R_\pm(\gamma)$ to $R_\pm$, and γ to its core circles.) Then, $N(R_+(\gamma) \cup \alpha, M)$ becomes a compression body W with $h(W) = 1$. By using Lemma 2.4 in [6], we can observe that $\mathrm{cl}(M - W)$ is a compression body W' such

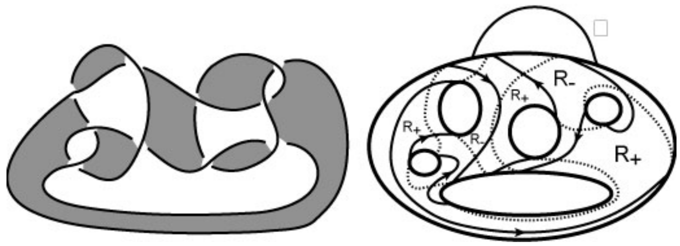

FIGURE 5

that $\partial_- W' = R_-(\gamma)$ and $h(W') = 1$. Hence we have $h(R) \leq 1$, namely, $\mathcal{MN}(\mathfrak{C}) \leq 2$. This completes the proof.

7. Relation with the twisted Alexander invariants

In this section we investigate the relations between the invariants arising from the Morse-Novikov theory and the twisted Alexander invariant. We shall use the version of twisted Alexander polynomials via the Reidemeister torsion as suggested by Kitano in [13]. It is proved in [13] that this version is equivalent to the initial Wada's definition. Let K be an oriented knot in S^3. Let $\xi : G = \pi_1 K \to \mathbf{Z}$ the homomorphism introduced in the previous section; let $\rho : G \to GL(n, F)$ be a right representation of the knot group, where F is a field. Let $S = F(t)$; the representation $\rho_\xi : G \to SL(n, Q)$ determines also the representation $\widetilde{\rho}_\xi : G \to GL(n, S)$. Form the chain complex

$$C_*(K;\rho) = S^n \underset{\widetilde{\rho}_\xi}{\otimes} C_*(\widetilde{C}_K). \tag{15}$$

This is a chain complex of vector spaces over S and the same is true for its homology. We shall limit ourselves to the case when the chain complex (15) is acyclic. In this case the *twisted Alexander invariant* is defined as the Reidemeister torsion of the chain complex $C_*(K;\rho)$. It is an element of S, well-defined modulo multiplication by $\pm t^n, n \in \mathbf{Z}$. The field S can be considered as a subfield of the Novikov ring $F((t))$, and therefore the twisted Alexander invariant can be also considered as an element of $F((t))$, that is, as a Laurent series, well-defined up to multiplication by $\pm t^n$. This Laurent series will be denoted by $\tau(K;\rho)$.

Definition 7.1. An element $x \in F((t))$ will be called *monic*, if the coefficient corresponding to the term of the lowest degree of t equals 1 or -1.

Proposition 7.2. *Assume that the representation $\rho : \pi_1 K \to SL(n, F)$ is obtained from a representation $\rho_0 : \pi_1 K \to SL(n, \mathbf{Z})$ via the tautological homomorphism $\mathbf{Z} \to F$, $1 \to 1$. Assume that the Novikov homology $\widehat{H}_*(K, \rho_0)$ vanishes. Then the twisted Alexander polynomial $\tau(K, \rho)$ is $\pm$-monic.*

Proof. The chain complex $\widehat{C}_*(C_K, \rho_0)$ is acyclic. The Whitehead torsion of this complex is an element

$$\tau_0 \in \overline{K}_1\left(\widehat{\mathbf{Z}}((t)) \right) \approx \mathbf{Z}((t))^\bullet,$$

defined up to multiplication by $\pm t^n$. The multiplicative group $\mathbf{Z}((t))^\bullet$ of all invertible elements of $\mathbf{Z}((t))$ consists of the Laurent series λ such that the lowest degree term equals $\pm t^n$. The image of τ_0 in $F((t))$ coincides with the image of the twisted Alexander polynomial $\tau(K, \rho)$, therefore the latter is monic. $\square$

8. A conjecture

The examples of the previous sections show that the lower bounds provided by the twisted Novikov homology are often optimal. Every such estimate arises from a linear representation of a fundamental group. In the present section we explore an approach which starts from the most general form of the Novikov homology and which should give the best possible lower bounds.

Let us recall first the construction of the Novikov completion of a given group G with respect to a homomorphism $\xi : G \to \mathbf{Z}$. Set $\Lambda = \mathbf{Z}G$ and denote by $\widehat{\widehat{\Lambda}}$ the abelian group of all functions $G \to \mathbf{Z}$. Equivalently, $\widehat{\widehat{\Lambda}}$ is the set of all formal linear combinations $\lambda = \sum_{g \in G} n_g g$ (not necessarily finite) of the elements of G with integral coefficients. For $\lambda \in \widehat{\widehat{\Lambda}}$ set $\operatorname{supp} \lambda = \{g \in G \mid n_g \neq 0\}$. Set

$$\widehat{\Lambda}_\xi = \{\lambda \in \widehat{\widehat{\Lambda}} \mid \forall C \in \mathbf{R} \quad \operatorname{supp} \lambda \cap \xi^{-1}([C, \infty[) \text{ is finite } \} \tag{16}$$

Then $\widehat{\Lambda}_\xi$ has a natural structure of a ring.

The following theorem was first proved in [16].

Theorem 8.1. ([16]) *Let $f : M \to S^1$ be a Morse map such that $f_* = \xi : \pi_1 M \to \mathbf{Z}$. Then there is a chain complex $\mathcal{N}_*$ of free finitely generated $\widehat{\Lambda}_\xi$-modules (the Novikov complex), such that*

1. *the number of free generators in each degree i equals $m_i(f)$,*
2. *there is a chain homotopy equivalence*

$$\phi : \mathcal{N}_* \longrightarrow \widehat{\Lambda}_\xi \underset{\Lambda}{\otimes} C^\Delta_*(\widetilde{M}).$$

This theorem was proved in [16] for the case of closed manifolds. A similar result holds for the regular Morse maps $f : C_L \to S^1$.

Remark 8.2. In the paper [16], we worked with the convention that the fundamental group acts on the universal covering on the right. Theorem 8.1 above is the translation of the results of [16] to the language of the left modules.

Corollary 8.3. *If the link L is fibred, then*

$$H_*\big(\widehat{\Lambda}_\xi \underset{\Lambda}{\otimes} C_*^\Delta(\widetilde{C}_L)\big) = 0.$$

In general the vanishing of the Novikov homology is not sufficient for existence of a fibration. Even in the case when the dimension of the manifold is $\geqslant 6$ there is a secondary obstruction in the Whitehead group of the Novikov ring of the fundamental group. Let us give the definition. An element $\lambda \in \widehat{\Lambda}_\xi$ is called *trivial unit* if

$$\lambda = \varepsilon \cdot g \cdot (1+\mu) \quad \text{with} \quad \varepsilon = \pm 1,\ g \in G,\ \operatorname{supp} \mu \in \xi^{-1}(]-\infty, 0[).$$

The set of all trivial units is a multiplicative subgroup U of the group of all units of $\widehat{\Lambda}_\xi$. The group $\overline{K}_1(\widehat{\Lambda}_\xi)/U'$ where U' if the image of U, is denoted Wh (G, ξ) and called *the Whitehead group of the Novikov ring.* The next theorem follows immediately by combining the classical theorem of Waldhausen [21] (the Whitehead group of the link group vanishes) and the main theorem of [17].

Theorem 8.4. *We have* Wh $(\pi_1 L, \xi) = 0$ *where* $\xi : \pi_1 L \to \mathbf{Z}$ *is the homomorphism which sends each meridian generator to* $1 \in \mathbf{Z}$. □

Thus in the case of links the total obstruction to fibering provided by the Novikov complex is reduced to the Novikov homology.

Conjecture 8.5. *A link L is fibred if and only if*

$$H_*\big(\widehat{\Lambda}_\xi \underset{\Lambda}{\otimes} C_*^\Delta(\widetilde{C}_L)\big) = 0. \tag{17}$$

The Novikov ring $\widehat{\Lambda}_\xi$ is a complicated algebraic object, and the verification of the condition (17) is certainly a difficult algebraic task. The twisted Novikov homology as introduced and studied in the previous sections provides an effectively computable tools for evaluating the Novikov homology, and as we have seen in many examples, the twisted Novikov homology is often sufficient to compute the Morse-Novikov number. Thus we are led to the following problem.

Problem 8.6. *Is it true that vanishing of the ρ-twisted Novikov homology for every representation implies the condition (17) ?*

A natural and a very interesting question would be to investigate the relations between the Problem 8.6 and the Problem 1.1 of [8], which asks whether the information contained in the twisted Alexander polynomials for all $SL(2k, \mathbf{F})$-representations is sufficient to decide whether a link is fibred.

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