SIMPLE HOMOTOPY TYPE OF THE NOVIKOV COMPLEX AND LEFSCHETZ ζ-FUNCTION OF THE GRADIENT FLOW

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To my teacher S.P.Novikov for his 60th birthday

ABSTRACT. Let $f : M \to S^1$ be a Morse map from a closed connected manifold to a circle. S.P.Novikov constructed an analog of the Morse complex for f. The Novikov complex is a chain complex defined over the ring of Laurent power series with integral coefficients and finite negative part. As its classical predecessor this complex depends on the choice of a gradient-like vector field. The homotopy type of the Novikov complex is the same as the homotopy type of the completed complex of the simplicial chains of the cyclic covering associated to f.

In the present paper we prove that for every C^0-generic f-gradient there is a homotopy equivalence between these two chain complexes, such that its torsion equals the Lefschetz zeta-function of the gradient flow. For these gradients the Novikov complex is defined over the ring of rational functions and the Lefschetz zeta-function is also rational. The main theorem of the paper contains a more general statement concerning the Lefschetz zeta function with twisted coefficients and the version of the Novikov complex defined, respectively, over the completion of the group ring of $H_1(M)$.

The paper contains also a survey of Morse-Novikov theory and of previous results of the author on the C^0-generic properties of the Novikov complex and the Novikov exponential growth conjecture.

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1. Introduction and statement of results

We begin with Subsection 1.1 which presents to the reader the basic ideas of the paper. The next three subsections contain the minimal set of definitions necessary for the statement of the main theorem, given in Subsection 1.6. In the text of this section the reader will find references to other parts of the text which contain further explanations, details etc.

1.1. **Introduction.** In this paper we continiue the study of C^0-generic properties of Novikov complex, began in the papers [Pa4], [Pa3]. Let M be a closed connected manifold. Recall that for a Morse map $f : M \to S^1$ and an f-gradient v, satisfying Transversality Condition, an analog of the classical Morse complex – the Novikov complex $\mathcal{C}_*(v)$ – can be associated. It is a free finitely generated chain complex over the ring $\mathbf{Z}[[t]][t^{-1}]$ of the formal power series with finite negative part. In the cited papers we constructed for a given Morse map $f : M \to S^1$ a special class of f-gradients which form a subset, C^0-open-and-dense in the subset of all f-gradients satisfying Transversality Condition. For any f-gradient v in this class the boundary operators in Novikov complex, associated to v, are not merely power series, but rational functions. The main construction in the proof is a certain group homomorphism $h(v)$, introduced in §4 of [Pa3], which we call here *homological gradient descent.* (To explain the origins of this notion, consider a regular value $\lambda \in S^1$ of f, and consider the integral curve γ of $(-v)$ starting at $x \in V = f^{-1}(\lambda)$. If this curve does not converge to a critical point of f, then it will intersect again the submanifold V at some point, say, $\rho(x)$. The map ρ is thus a (not everywhere defined) smooth map of V to V, and the *homological gradient descent operator* is a substitute for the homomorphism induced by ρ in homology. This operator is defined for a C^0-generic v-gradient satisfying Transversality Condition.) In the present paper we develop further the properties of the homological gradient descent. We show that this homomorphism carries enough information to compute the Lefschetz zeta-function of the gradient flow. This leads to the main theorem of the present paper. See Subsection 1.6 for the precise statement. To explain intuitively the idea of the theorem, recall ([Pa1]) that there is a chain homotopy equivalence between the Novikov complex and the completed chain simplicial complex of the corresponding cyclic covering. The main theorem of the present paper says that for C^0-generic f-gradient v this chain homotopy equivalence can be chosen so that its torsion equals to Lefschetz zeta-finction of the flow generated by $(-v)$.

1.2. **Algebraic notions: K_1, Novikov rings etc.** Let R be a commutative ring with a unit. Recall from [Mi2] the group $\overline{K}_1 R = K_1 R/\{0, [-1]\}$ where $[-1]$ denotes the element of order 2 corresponding to the unit $(-1) \in GL(R,1) \subset GL(R)$. Further, $K_1 R = R^\bullet \oplus SK_1(R)$ where $R^\bullet$ is the full group of units of R. Let U be a subgroup in $R^\bullet$. Set $K_1(R|U) = \overline{K}_1 R/U = R^\bullet/(\pm U) \oplus SK_1(R)$ (here $\pm U = \{\pm u | u \in U\}$ and we identify the units in $R^\bullet$ with their images in $\overline{K}_1 R$). For $S \in GL(R)$ the image of S in $\overline{K}_1 R$ or in $K_1(R|U)$ will be denoted by $[S]$.

A free based finitely generated chain complex C_* of right R-modules (see[Pa2], def. 1.3) is called R-*complex.*

Recall from [Mi2], §3 that for an acyclic R-complex C_* the torsion $\tau(C_*) \in \overline{K}_1 R$ is defined.

If $\phi : F_* \to D_*$ is a homotopy equivalence of R-complexes, then its *torsion* $\tau(\phi) \in \overline{K}_1 R$ is defined to be the torsion of the chain cone $C_*(\phi)$. The image of $\tau(\phi)$ in the group $K_1(R|U)$ will be denoted by $\tau(\phi|U)$.

Proceeding to the definitions related to Novikov completions, let G be an abelian group. Let $\xi : G \to \mathbf{R}$ be a homomorphism. Set $\Lambda = \mathbf{Z}G$ and denote by $\widehat{\widehat{\Lambda}}$ the abelian group of all functions $G \to \mathbf{Z}$. Equivalently, $\widehat{\widehat{\Lambda}}$ is the set of all formal linear combinations $\sum_{g\in G} n_g g$ (not necessarily finite) of the elements of G with integral coefficients. For $\lambda \in \Lambda$ set supp $\lambda = \{g \in G \mid n_g \neq 0\}$. Denote by $\widehat{\Lambda}_\xi$ the subgroup of $\widehat{\widehat{\Lambda}}$ formed by all the λ such that for every $C > 0$ the set $\xi^{-1}([C, \infty[) \cap \operatorname{supp} \lambda$ is finite. Then $\widehat{\Lambda}_\xi$ has a natural structure of a ring. We shall also need an analog of this ring with $\mathbf{Z}$ replaced by $\mathbf{Q}$. That is, let $\widehat{\widehat{\Lambda}}_{\mathbf{Q}}$ be the set of all the functions $G \to \mathbf{Q}$ and set

$$\widehat{\Lambda}_{\xi,\mathbf{Q}} = \{\lambda \in \widehat{\widehat{\Lambda}}_{\mathbf{Q}} \mid \forall C \in \mathbf{R} \quad \operatorname{supp} \lambda \cap \xi^{-1}([C, \infty[) \text{ is finite }\}$$

In the sequel we shall work with two subgroups of units of $\widehat{\Lambda}_\xi$. First is the group G itself, the second is introduced in the following formula:

$$U_\xi = \{\lambda \in \widehat{\Lambda}_\xi \mid \lambda = \pm g(1+\mu), \text{ where } g \in G \text{ and } \operatorname{supp} \mu \subset \xi^{-1}(]-\infty, 0[)\}$$

In [Pa1] the group $K_1(\widehat{\Lambda}_\xi | U_\xi)$ was denoted by $Wh(G, \xi)$.

In the present paper we shall also use a ring which is smaller than $\widehat{\Lambda}_\xi$ but still has a lot of essential properties of $\widehat{\Lambda}_\xi$. Set $S_\xi = \{\lambda \in \mathbf{Z}G \mid \lambda = 1 + \mu, \operatorname{supp} \mu \subset \xi^{-1}(]-\infty, 0])$. Denote $S_\xi^{-1}\Lambda$ also by $\Lambda_{(\xi)}$, note that we have natural inclusions $\Lambda \hookrightarrow \Lambda_{(\xi)} \hookrightarrow \widehat{\Lambda}_\xi$.

1.3. **Novikov complex.** Here we just say *what it is*, leaving the details of definition and the construction together with motivations for the section 2.4.

Let $f : M \to S^1$ be a Morse map. We assume that $f_* : H_1(M) \to H_1(S^1) = \mathbf{Z}$ is epimorphic. Let $\mathcal{P} : \widetilde{M} \to M$ be a connected regular covering with structure group G, such that $f \circ \mathcal{P}$ is homotopy to zero. [1] In this paper we consider only the case when G is abelian. Then we obtain the natural epimorphism $\pi : H_1(M) \twoheadrightarrow G$ and $f_* : H_1(M) \to \mathbf{Z}$ factors through a homomorphism $\xi : G \to \mathbf{Z}$. Set $\Lambda = \mathbf{Z}G$, then the corresponding Novikov ring $\widehat{\Lambda}_\xi$ is defined. Let v be an f-gradient satisfying Transversality Condition (see Subsection 2.1 for definition). Choose for every critical point p of f an orientation of the stable manifold of p and a lifting of p to $\widetilde{M}$. To this data one associates a $\widehat{\Lambda}_\xi$-complex $\widetilde{\mathcal{C}}_*(v)$ such that the number of free generators of $\widetilde{\mathcal{C}}_k(v)$ equals to the number of critical points of f of index k.

1.4. **Terminology: flows and orbits.** Let w be a C^∞ vector field on a closed manifold M. *A closed orbit* of w is a non constant trajectory γ of w, defined on a closed interval $[a, b] \subset \mathbf{R}$ and satisfying $\gamma(b) = \gamma(a)$. We shall identify two such trajectories $\gamma : [a, b] \to \mathbf{R}, \gamma' : [a', b'] \to \mathbf{R}$ if there is C, such that $a' = a + C, b' = b + C$ and $\gamma(t) = \gamma'(t + C)$.

The set of all closed orbits is denoted by $Cl(w)$. For $\gamma \in Cl(w), \gamma : [a, b] \to M$ and $m \in \mathbf{N}$ define $\gamma^m \in Cl(w)$ as the result of successive gluing of m copies of γ to each other. *The multiplicity* $m(\gamma) \in \mathbf{N}$ is the maximal number $k \in \mathbf{N}$ such that

[1] Thus we use the symbol $\widetilde{\ }$, normally reserved for the universal covering, for another purpose, but there will be no possibility for confusion.

$\gamma = \theta^m$ for some $\theta \in Cl(w)$. For every $\gamma \in Cl(w)$ its homology class $[\gamma] \in H_1(M)$ is well defined. A vector field w will be called *Kupka-Smale* if every zero and every closed orbit is hyperbolic, and w satisfies the Transversality Condition. For every Kupka-Smale vector field w and every $\gamma \in Cl(w)$ the index $\varepsilon(\gamma) \in \{-1, 1\}$ is defined as the index of the corresponding Poincare map.

1.5. **Lefschetz zeta-functions.** We shall consider zeta-functions only for a very particular class of flows: flows on closed maniflods generated by gradients of Morse maps $f : M \to S^1$.

The set of all Kupka-Smale gradients of f is denoted by $\mathcal{GKS}(f)$. Let $v \in \mathcal{GKS}(f)$. Consider $\pi([\gamma])$ as an element of the group ring $\Lambda = \mathbf{Q}G$, multiply it by the rational number $\frac{\varepsilon(\gamma)}{m(\gamma)}$ and consider an infinite series

$$\eta_L(-v) = \sum_{\gamma \in Cl(-v)} \frac{\varepsilon(\gamma)}{m(\gamma)} \pi([\gamma])$$

The Kupka-Smale property implies that there is at most finite set of $\gamma \in Cl(-v)$ with given $[\gamma]$. Thus the above series is a well-defined element of the abelian group $\widehat{\widehat{\Lambda}}_{\mathbf{Q}}$. It is not difficult to show that $\eta_L(-v)$ belongs to the Novikov ring $\widehat{\Lambda}_{\xi,\mathbf{Q}}$. Note also that supp $\eta_L(-v) \subset \xi^{-1}(]-\infty, -1])$. Therefore supp $(\eta_L(-v))^k \subset \xi^{-1}(]-\infty, -k])$ and the power series

$$\zeta_L(-v) = \exp(\eta_L(-v))$$

is well defined and belongs again to $\widehat{\Lambda}_{\xi,\mathbf{Q}}$. This element will be called *Lefschetz zeta-function.*

1.6. **Statement of the main theorem.** Let M be a closed connected manifold, $f : M \to S^1$ be a Morse map. We assume here the terminology of Sect. 1.3. For a unit λ of the ring $\Lambda_{(\xi)}$ we shall denote the image of λ in $\overline{K}_1(\Lambda_{(\xi)} \mid G)$ by $\bar{\lambda}$. We denote by $C_*^{\Delta}(\widetilde{M})$ the simplicial complex of $\widetilde{M}$ associated with a smooth triangulation of M, so that $C_*^{\Delta}(\widetilde{M})$ is a $\mathbf{Z}G$-complex.

Theorem

There is a subset $\mathcal{GKS}\mathfrak{C}(f) \subset \mathcal{GKS}(f)$ with the following properties:

1. *$\mathcal{GKS}\mathfrak{C}(f)$ is open-and-dense subset of $\mathcal{GKS}(f)$ with respect to C^0-topology.*
2. *For every $v \in \mathcal{GKS}\mathfrak{C}(f)$ the Novikov complex $\widetilde{\mathcal{C}}_*(v)$ is defined over $\Lambda_{(\xi)}$, and $\zeta_L(-v) \in \Lambda_{(\xi)}$.*
3. *For every $v \in \mathcal{GKS}\mathfrak{C}(f)$ there is a homotopy equivalence*

$$\phi : \widetilde{\mathcal{C}}_*(v) \xrightarrow{\sim} C_*^{\Delta}(\widetilde{M}) \underset{\Lambda}{\otimes} \Lambda_{(\xi)}$$

such that $\tau(\phi|G) = \overline{\zeta_L(-v)}$.

1.7. **Contents of the paper section by section and further remarks.** The results of this paper first appeared in the e-print [Pa5]. The present paper contains the detailed exposition of the main part of the results of [Pa5]; due to the lack of time we left aside the results of [Pa5] concerning the irrational Morse forms, as well as the relations to Seiberg-Witten invariants of 3-manifolds.

The subsections 2.1 – 2.3 of Section 2 establish the terminology we need. In the subsections 2.4 and 2.5 the reader will find an introduction to Morse-Novikov theory and an exposition of the author's results on the Novikov Exponential Growth Conjecture ([Pa4], [Pa3]). This part is written with the emphasis on the ideas rather than on the technical details.

The section 3 contains a compressed survey of the techniques and results of [Pa3] used in the sequel. We included this part in order to make the exposition self-contained in a sense (still I did not include the proofs, for which the reader is invited to consult [Pa3]). The contents of the section 4 is a version of the Section 4 of [Pa3]; the condition ($\mathfrak{C}$) which we introduce here is of the same origin as the condition (RP) of the §4 of [Pa3], but better suited for our present purposes.

The section §5 is one of the central parts of the paper. See the beginning of §5 for an introduction to the main ideas of this section.

§6 contains some simple algebraic computations. The section 7 contains the most important step of the proof of the main theorem: we compute here the simple homotopy type of the Novikov complex in terms of the Morse filtrations of cobordisms (introduced in §5) and related invariants. The section 8 contains the end of the proof of the main theorem.

We end the present introduction by a remark on the origins of our paper. Using the methods of [Pa3] it is not difficult to prove that the Lefschetz zeta-function of a C^0-generic Kupka-Smale gradient is rational, but one does not see immediately how to compute it in terms of usual invariants of the manifold. The formula of Hutchings-Lee ([HL1], Theorem 1.1) was the first example of such computation. It suggested that in general case there should be a formula relating the simple homotopy type of the Novikov complex, the simple homotopy type of the manifold itself and the zeta-function of the gradient flow.

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2. Morse-Novikov theory

2.1. **Morse functions and their gradients: basic terminology.** Before going into the details of Morse-Novikov theory and Novikov Exponential Growth Conjecture, we first list some basic notions in order to establish the Morse-theoretic language which we use.

Let v be a C^1 vector field on a manifold M. The value of the integral curve of v passing by x at $t=0$ will be denoted by $\gamma(x,t;v)$.

We call *cobordism* a compact manifold W together with a presentation $\partial W = \partial_0 W \sqcup \partial_1 W$ where $\partial_1 W$ and $\partial_0 W$ are compact manifolds without boundary of dimension $\dim W - 1$ (one or both of them can be empty). The manifold $W \setminus \partial W$ will be denoted by W°.

A *Morse function* $f : W \to [a,b]$ on a cobordism W is a C^∞ map $f : W \to \mathbf{R}$, such that $f(W) \subset [a,b], f^{-1}(b) = \partial_1 W, f^{-1}(a) = \partial_0 W$, all the critical points of f are non-degenerate and belong to W°.

The set of all critical points of a Morse function f will be denoted by $S(f)$; the set of all critical points of f of index k will be denoted by $S_k(f)$.

Let f be a Morse function on a cobordism $W, \dim W = n$. A vector field v is called *gradient* of f, or f-gradient, if for every critical point p of f there is a chart $\phi : U \to V$ around p, such that $\phi_*(v)$ is a standard vector field $(-x_1, ..., -x_k, x_{k+1}, ..., x_n)$ on $\mathbf{R}^n$ (where $k = \mathrm{ind} p, n = \dim M$), and $f \circ \phi^{-1}$ is a quadratic form $\sum_i \alpha_i x_i^2$ with $\alpha_i > 0$ for $i > k$ and $\alpha_i < 0$ for $i \leqslant k$.

The set of all f-gradients will be denoted by $\mathcal{G}(f)$.

Remark 2.1.

1. The reader will check easily that if we demand $||a_i|| = 1$ in the definition above, we obtain an equivalent definition.
2. Riemannian gradient of f (with respect to some riemannian metric) is not included in this definition. There is a natural way to extend the definition so as to cover the case of Riemannian gradients and to carry over our results to this general framework (see [Pa6]).

$\triangle$

From now on up to the end of this subsection $f : W \to [a,b]$ is a Morse function on a cobordism of dimension n, and v is an f-gradient.

Denote by K_1 the set of all $x \in \partial_1 W$, such that the $(-v)$-trajectory starting at x converges to a critical point of f. Similarly, K_0 denotes the set of all $x \in \partial_0 W$, such that $\gamma(x,t;v)$ converges to a critical point of f. Note that K_0, K_1 are compacts. The shift along the trajectories of $(-v)$ defines a diffeomorphism $\partial_1 W \setminus K_1 \to \partial_0 W \setminus K_0$, which will be denoted by $(-v)^{\leadsto}$. The set $(-v)^{\leadsto}(A \setminus K_1)$ will be denoted (by the abuse of notation) by $(-v)^{\leadsto}(A)$.

Let $\lambda < \mu$ be regular values of f. Set $W' = f^{-1}([\lambda,\mu])$, and $w = v \mid W'$. The diffeomorphism $(-w)^{\leadsto}$ will be denoted by $(-v)^{\leadsto}_{[\mu,\lambda]}$.

We shall need a construction which is quite standard in Morse theory (see [Mi1], p.62). Let $\Psi : \partial_0 W \times [0,l] \to W$ be a map, given by $\Psi(x,t) = \gamma(x,t;v)$, and choose l small enough so that Ψ is a diffeomorphism onto its image. Let h be a

C^∞ positive function on $[0,l]$, supp $h \subset]0,l[$. Set $T = \int_0^l h(\tau)d\tau$. Let u be a vector field on $\partial_0 W$. Define a vector field w on $\partial_0 W \times [0,l]$ by $w(x,t) = h(t)u(x)$. Define a vector field w' on W setting $w' = \Psi_*(w)$ in Im Ψ and $w' = 0$ in $W \setminus \mathrm{Im}\ \Psi$. Set $v' = v + w'$. If h and u are sufficiently small then $\|v - v'\|$ is small, and v' is still an f-gradient. Note that $(-v')^{\leadsto} = \Phi(-u,T) \circ (-v)^{\leadsto}$. We shall call this construction *adding to v a horizontal component u nearby $\partial_0 W$.*

We say that v satisfies *Transversality Condition* , if

$$\big(x, y \in S(f)\big) \Rightarrow \big(D(x,v) \cap W^\circ \pitchfork D(y,-v) \cap W^\circ\big)$$

We say that v satisfies *Almost Transversality Condition*, if

$$\big(x, y \in S(f)\ \&\ \mathrm{ind} x \leqslant \mathrm{ind} y\big) \Rightarrow \big(D(x,v) \cap W^\circ \pitchfork D(y,-v) \cap W^\circ\big)$$

The set of all f-gradients satisfying Transversality Condition, resp. Almost Transversality Condition will be denoted by $\mathcal{GT}(f)$, resp. by $\mathcal{GA}(f)$.

A Morse function $\phi : W \to [\alpha, \beta]$ is called *adjusted to* (f, v), if:

1) $S(\phi) = S(f)$, and v is also a ϕ-gradient.

2) The function $f - \phi$ is constant in a neighborhood of $\partial_0 W$, in a neighborhood of $\partial_1 W$, and in a neighborhood of each point of $S(f)$.

We say that f is *ordered* Morse function with an *ordering sequence* $(a_0, ..., a_{n+1})$, if $a = a_0 < a_1 < ... < a_{n+1} = b$ are regular values of f such that $S_i(f) \subset f^{-1}(]a_i, a_{i+1}[)$.

2.2. **More terminological conventions.** We adopt the convention that structure groups of regular coverings act on the cover *from the right.* All the regular coverings considered in this paper will have abelian structure groups (a large part of our results can be carried over to the non-abelian case without problems).

2.3. **Morse complex for functions: recollections.** Before proceeding to Morse-Novikov theory, we shall recall briefly the classical notion of Morse complex of a Morse function. Let $g : M \to \mathbf{R}$ be a Morse function on a closed manifold M. The classical Morse theory [Mo] implies that $m_p(g) \geqslant b_p(M)$, where $m_p(g)$ is the number of critical points of index p, and $b_p(M)$ is the p-th Betti number of M. This can be refined as to construct a chain complex C_* of free finitely generated abelian groups, such that the number $\mu(C_p)$ of free generators of C_p equals to $m_p(g)$ and $H_*(C_*) \approx H_*(M)$. The construction of this complex historically was done in several steps, the main progress being done in the papers of R.Thom [Th] and S.Smale [Sm1]. See the book [Mi1] of J.Milnor for systematic exposition of this step. (In this book one does not find the definition of the complex, but it can be extracted from the §6 of this book.) Later this construction was reconsidered by Witten [Wi] from the entirely new point of view, involving DeRham cohomology.

Here is the main idea of the construction. Choose a g-gradient v satisfying Transversality Condition. Define C_s to be a free abelian group generated by the critical points of g of index s. Let p, q be critical points of g of indices k, resp. $k-1$. Since v satisfies Transversality Condition, the set $L(p,q)$ of all orbits, joining p to q is finite (exercise for the reader). For each critical point of g choose an orientation of the stable manifold of this point (with respect to the vector field v). Then to each orbit $\gamma \in L(p,q)$ a sign $\varepsilon(\gamma)$ can be attributed in a natural way. Set $n(p,q) = \sum_{\gamma\in L(p,q)} \varepsilon(\gamma)$. The boundary $\partial_k : C_k \to C_{k-1}$ is defined by $\partial_k p = \sum_q n(p,q)q$ (where q ranges over the set of critical points of g of index $k-1$).

It turns out that $\partial_k \circ \partial_{k+1} = 0$ for every k and that the homology of C_* with respect to ∂_* is isomorphic to $H_*(M)$ (the details can be found in [Pa1], Appendix). This complex is called *Morse complex.*

2.4. **Short introduction to Morse-Novikov theory.** In the paper [No1] Novikov have laid the foundation of Morse theory for closed 1-forms. One of the basic concepts of this theory is that of *Novikov complex.* We shall give here a brief outline of the construction (see [Pa1] for more details).

Consider a Morse map $f : M \to S^1$. The construction from 2.3 can not be carried over to this case as it is, since for two critical points x, y of adjacent indices the number of the trajectories of a gradient v joining x to y can well be infinite. Here is a way round this difficulty: split the set of all trajectories joining x and y into some number of disjoint finite subsets indexed by some family I, count separately the trajectories in each class and obtain then an incidence coefficient as a function $I \to \mathbf{Z}$. Thus the base ring of the resulting complex will be much larger than $\mathbf{Z}$. We shall describe the most natural way of such splitting (introduced in [No1]) which leads to the base ring $\mathbf{Z}((t)))$.

Definition 2.2.

The ring $\mathbf{Z}((t))$ consists of power series in t with integral coefficients and finite negative part. That is, $\lambda = \sum_{-\infty}^{\infty} a_i t^i$ is in $\mathbf{Z}((t))$, if $a_i \in \mathbf{Z}$ for all i and there is $N = N(\lambda)$, such that $a_i = 0$ if $i < N(\lambda)$.

$\triangle$

We shall assume that our Morse map f is not homotopic to zero, otherwise v is a gradient of an $\mathbf{R}$-valued Morse function, and the construction from Subsection 2.3 applies. Let $\mathcal{C}_p(f)$ be the free $\mathbf{Z}((t))$-module generated by critical points of f of index p. Consider the infinite cyclic cover $\mathcal{C} : \bar{M} \to M$ such that $f \circ \mathcal{C}$ is homotopic to zero. Lift the function $f : M \to S^1$ to a function $F : \bar{M} \to \mathbf{R}$. Let t be the generator of the structure group of $\mathcal{C}$ such that $F(xt) < F(x)$. Choose a gradient $v \in \mathcal{GT}(f)$ and lift it to $\bar{M}$ (we shall keep for this lifting the same notation v). As above choose for every critical point $x \in S(f)$ an orientation of the stable manifold of x with respect to v. Moreover, for each critical point x of $f : M \to S^1$ choose a lifting $\bar{x}$ of x to $\bar{M}$. Let x, y be a pair of critical points of f with $\mathrm{ind}x = \mathrm{ind}y + 1$. It is not difficult to show that for any $k \in \mathbf{Z}$ the set of all v-orbits joining $\bar{x}$ and $\bar{y}t^k$ is finite. Sum them up with the corresponding signs (as above) and obtain an integer $n_k(x,y)$. Set $n(x,y) = \sum_k n_k(x,y)t^k \in \mathbf{Z}((t))$ and set $\partial_k x = \sum_y y \cdot n(x,y)$. (The power series $n(x,y)$ are called *novikov incidence coefficients.* Note that $n(x,y)$ and $n_k(x,y)$ depend on v and sometimes we shall write $n(x,y;v)$, resp. $n_k(x,y;v)$ to stress this dependance.) Note the Novikov complex is for us a complex of *right modules* (see Subsection 2.2).

One can show that that $\partial_k \circ \partial_{k+1} = 0$. Thus we obtain a chain complex of free finitely generated $\mathbf{Z}((t))$-modules. We shall denote it by $\mathcal{C}_*(v)$ in order to stress its dependance on v, which is essential for our purposes.[2] (This complex actually depends also on the choice of orientations of stable manifolds of critical points of

[2] We have chosen the italic $\mathcal{C}$ in order to distinguish between the Novikov complex and the ordinary Morse complex.

f, and on the liftings of the critical points to $\bar{M}$, but the influence of these choices on the result is less important: the ambiguity is reduced to the multiplication of the lines and columns of the matrices of the boundary operators ∂_k by some units of of the form $\pm t^m$ of the base ring. Note, on the other hand that this complex does not depend on the particular choice of a function f for which v is a gradient.) To describe the homology of the complex, recall that $H_*(\bar{M})$ is a $\mathbf{Z}[t,t^{-1}]$-module, thus the tensor product $\widehat{H}_*(\bar{M}) = H_*(\bar{M}) \underset{\mathbf{Z}[t,t^{-1}]}{\otimes} \mathbf{Z}((t))$ makes sense. One can prove that $H_*(\mathcal{C}_*(v)) \approx \widehat{H}_*(\bar{M})$.

All the cited results on the Novikov complex are stated in [No1]. The detailed proofs can be found in [Pa1].

2.5. **Novikov exponential growth conjecture and the operator of homological gradient descent.** Let $f : M \to S^1$ be a Morse map, and v be a generic f-gradient, satisfying Transversality Condition. We have seen in the previous subsection that to this data the Novikov complex $\mathcal{C}_*(v)$ is associated. It is a free based chain complex over $\mathbf{Z}((t))$, therefore each boundary operator ∂_k is represented by a matrix $D^{(k)}$ with the entries $D^{(k)}_{ij}$ are in $\mathbf{Z}((t))$ (these entries are nothing else than the Novikov incidence coefficients). Each of $D^{(k)}_{ij}$ is then a power series $a(t) = \sum_k a_k t^k$ with a finite negative part. The Novikov exponential growth conjecture says that the coefficients a_k grow at most exponentially when $k \to \infty$ (maybe with some restrictions on v of the type analyticity or general position imposed). See the Introduction of [Pa3] for more details on this conjecture. This conjecture was proved in [Pa3] for any Morse map $f : M \to S^1$ and a set $\mathcal{GT}_0(f)$ of f-gradients, which is C^0 open and dense in $\mathcal{GT}(f)$. For the gradients in $\mathcal{GT}_0(f)$ the Novikov incidence coefficients are actually Taylor series of some rational functions. The proof of the rationality of the incidence coefficients for $v \in \mathcal{GT}_0(f)$ is based on the construction of what we call *operator of homological gradient descent.* This construction is essential for our work so we shall spend the rest of this subsection explaining the main idea of the construction. See the detailed exposition in Subsections 3.5, 4.3.

Assume that the homotopy class of f in $[M, S^1] \approx H^1(M, \mathbf{Z})$ is indivisible. Let p, q be critical points of f, $\mathrm{ind} p = \mathrm{ind} q + 1$; we shall work with the Novikov incidence coefficient $n(p, q; v)$. Assume for simplicity of notation that $1 \in S^1$ is a regular value of f, that the first critical level after $f(q)$ in the counter-clockwise direction is $f(p)$ and that 1 is between $f(q)$ and $f(p)$. Cut M along $f^{-1}(1)$ to obtain a cobordism W with two parts of the boundary: $\partial_1 W$ and $\partial_0 W$. There is the identification diffeomorphism $\Phi : \partial_0 W \to \partial_1 W$. We obtain also a Morse function $f_0 : W \to [0, 1]$ and its gradient, which will be denoted by the same letter v. Note that p is the lowest critical point of f_0, and q is the highest critical point of f_0. The descending disc of p intersects $\partial_0 W$ by an embedded sphere $S(p)$. The ascending disc of q intersects $\partial_1 W$ by an embedded sphere $S(q)$. The Novikov incidence coefficient $n_k = n_k(p, q; v)$ is then the algebraic intersection number of $((-v)^{\leadsto} \circ \Phi)^k(S(p))$ with $\Phi^{-1}(S(q))$. [3] Denote $(-v)^{\leadsto} \circ \Phi$ by ϕ. If ϕ were an everywhere defined diffeomorphism, then, denoting the homology classes of $S(p)$ and of $\Phi^{-1}(S(q))$ by,

[3] See page 6 for the definition of $(-v)^{\leadsto}$.

respectively, $[p]$ and $[q]$, we would obtain the following formula

$$n_k(p,q;v) = (\phi_*)^k([p]) \# [q] \tag{1}$$

(where $\#$ stands for the algebraic intersection index of homology classes; we assume for the moment that $\partial_0 W$ is oriented, although actually the orientability is not necessary). Then the rationality of the power series $\sum_k n_k t^k$ would be a simple fact of linear algebra. (For the sake of completeness we shall indicate here this linear-algebraic argument. Let L be a free finitely generated abelian group, $A : L \to L$ be a homomorphism. We shall identify A with its matrix. Then the power series $\sum_{k=0}^{\infty} A^k t^k$ represents the matrix $(1-At)^{-1}$. It is easy to deduce from this that

(2) for each $x \in L$ and each $\xi \in \mathrm{Hom}(L, \mathbf{Z})$ the power series $\sum_{k=0}^{\infty} \xi(A^k x) t^k$ is a rational function with denominator $1 - t \det A$.

By the way, the similar argument proves that the homological Lefschetz ζ-function of a diffeomorphism is always rational see [Sm2], p.767 – 768). Unfortunately ϕ is not everywhere defined, the soles of the ascending discs form exactly the set of indetermination of ϕ. [4] The iterations of ϕ only accumulate this indetermination. But it is *generically* possible to construct some objects which will serve as the substitutes of $[p]$, $[q]$ and ϕ_* in the formula (1).

Now I must prevent the reader that the exposition in the following paragraphs up to the end of the section 2 is very informal. One should consider it as a *program* which was realized in [Pa3]. Nevertheless I think that the understanding of these few lines is rather helpful for understanding of the proofs in [Pa3] and of the present paper.

Start with a "nice" cellular decomposition of $\partial_1 W$. By "nice" I mean in particular that every cell e is represented by a C^∞ map ρ of D^k to $\partial_1 W$ such that $\rho \mid \mathrm{Int}\ D^k$ is an embedding onto a submanifold of $\partial_1 W$ (k is the dimension of the cell). One expects that such nice decomposition can be constructed from the stratification of $\partial_1 W$ by the stable manifolds of critical points of some Morse function. Carry over this decomposition to $\partial_0 W$ by means of the diffeomorphism Φ^{-1}. The map $(-v)^{\leadsto}$ is not a continuous map of $\partial_1 W$ to $\partial_0 W$ but nevertheless one can try to perturb v so that the map $(-v)^{\leadsto}$ corresponding to the perturbed gradient be cellular. To explain why it should work, let p be a critical point of f of index k. Denote $D(p,v) \cap \partial_0 W$ by $S_-(p,v)$ and $D(p,-v) \cap \partial_1 W$ by $S_+(p,v)$. Perturbing v we can assume that all the manifolds $S_+(p,v)$ are transversal to all the cells of $\partial_1 W$. In particular the cells of $\partial_1 W$ of dimension k will intersect only the manifolds $S_+(p,v)$ with $\mathrm{ind} p \leqslant k$. Let e be a cell of $\partial_1 W$ of dimension k. The part of e where $(-v)^{\leadsto}$ is *not* defined is then the union of $S_+(p,v) \cap e$, where p ranges over critical points of f of index $\leqslant k$. The geometric picture of behavior of the descending trajectories of the gradient is as follows. The gradient descent shrinks the set $S_+(p,v) \cap e$ down to the point p (during infinite time). If we try to descend still lower to $\partial_0 W$, then we shall obtain a map which is multivalued in the sense that to every point of

[4] Novikov writes in [No2]: *In our case we do not have a mapping, but a cobordism with two equal boundaries generating the* $\mathbf{Z}$*-covering over the compact manifold.*

$S_+(p,v)\cap e$ corresponds the *set* $S_-(p,v)$. Note however that $\dim S_-(p,v)\leqslant k-1$. Assume for a moment that for every s all the sets $S_-(p,v)$ with $\mathrm{ind} p\leqslant s+1$ were in the s-skeleton of $\partial_0 W^{[s]}$ (which is plausible in view of the cellular approximation procedure). Then our "map" $(-v)^{\leadsto}$ will be defined as a continuous map

$$W_k : C_k^{(1)} = \partial_1 W^{[k]}/\partial_1 W^{[k-1]} \to C_k^{(0)} = \partial_0 W^{[k]}/\partial_0 W^{[k-1]}$$

or, equivalently, as a map from $C_k^{(0)}$ to itself, since Φ identifies $\partial_0 W$ with $\partial_1 W$. The space $C_k^{(1)}$ is of course a wedge of spheres of dimension k. Denote its homology by H_k; it is a free abelian group, and the map W_k induces an endomorphism ξ of H_k. Let r,q be critical points of f of indices resp. $k+1,k$. By the argument similar to the above the manifold $S_-(r,v)$ defines a relative cycle in $\big(\partial_0 W^{[k]},\partial_0 W^{[k-1]}\big)$, hence an element $[r]\in H_k\big(\partial_0 W^{[k]},\partial_0 W^{[k-1]}\big)$. Intersection index with $S_-(q,v)$ defines a homomorphism $\lambda(q) : H_k\to \mathbf{Z}$ and we have: $n_k(r,q;v)=\lambda(q)\big(\xi^k([r])\big)$. By the argument above from linear algebra (page 10), the Novikov incidence coefficient $n(r,q;v)$ is a rational function.

Now I shall make two remarks concerning the above program.

1. The resulting perturbation of our vector field v can be chosen C^0-small, but in general *not* C^∞-small.
2. We have consciously simplified the exposition, working with cellular decompositions of $\partial_0 W$ and $\partial_1 W$. In fact this does not work this way; instead of cellular decompositions we should consider *handle decompositions* of these manifolds (the sets $D_\delta(\mathrm{ind}\leqslant k\ ;v)$ in the notation of [Pa3]).

3. Brief survey of some results of [Pa3]

We shall use the techniques and the results of [Pa3] in the following sections; so we collected in this section some basic ideas and results of [Pa3].

3.1. **δ-thin handle decompositions.** In this subsection W is a riemannian cobordism of dimension n, $f : W \to [a,b]$ is a Morse function on W, and v is an f-gradient.

Let $x \in W$. Let $\delta > 0$. Assume that for some $\delta_0 > \delta$ the restriction of the exponential map $\exp_q : T_qW \to W$ to the disc $B^n(0,\delta_0)$ is a diffeomorphism on its image. Denote by $B_\delta(p)$ (resp. $D_\delta(p)$) the riemannian open ball (resp. closed ball) of radius δ centered in p. We shall use the notation $B_\delta(p), D_\delta(p)$ only when the assumption above on δ holds.

Set

$$B_\delta(p,v) = \{x \in W \mid \exists t \geqslant 0 : \gamma(x,t;v) \subset B_\delta(p)\}$$
$$D_\delta(p,v) = \{x \in W \mid \exists t \geqslant 0 : \gamma(x,t;v) \subset D_\delta(p)\},$$
$$D(p,v) = \{x \in W \mid \lim_{t\to\infty} \gamma(x,t;v) = p\}$$

We denote by $D(\mathrm{ind}_{\leqslant s} ; v)$ the union of $D(p,v)$ where p ranges over critical points of f of index $\leqslant s$. We denote by $B_\delta(\mathrm{ind}_{\leqslant s} ; v)$, resp. by $D_\delta(\mathrm{ind}_{\leqslant s} ; v)$ the union of $B_\delta(p,v)$, resp. of $D_\delta(p,v)$, where p ranges over critical points of f of index $\leqslant s$. We shall use similar notation like $D_\delta(\mathrm{ind}_{=s} ; v)$ or $B_\delta(\mathrm{ind}_{\geqslant s} ; v)$, which is now clear without special definition. The union of all $D(p,v)$ is denoted by $D(v)$, the union of all $D_\delta(p,v)$ is denoted by $D_\delta(v)$. For the uniformity of notation we shall also denote $D(p,v)$ by $D_0(p,v)$, and $D(v)$ by $D_0(v)$ etc.

Let $\phi : W \to [a,b]$ be an ordered Morse function with an ordering sequence $(a_0 < a_1 ... < a_{n+1})$. Let w be a ϕ-gradient. Denote $\phi^{-1}([a_i, a_{i+1}])$ by W_i.

Definition 3.1.

We say that v is *δ-separated with respect to ϕ* (and the ordering sequence $(a_0, ..., a_{n+1})$), if

i) for every i and every $p \in S_i(f)$ we have $D_\delta(p) \subset W_i^\circ$;

ii) for every i and every $p \in S_i(f)$ there is a Morse function $\psi : W_i \to [a_i, a_{i+1}]$, adjusted to $(\phi \mid W_i, w)$ and a regular value λ of ψ such that

$$D_\delta(p) \subset \psi^{-1}(]a_i, \lambda[)$$

and for every $q \in S_i(f), q \neq p$ we have

$$D_\delta(q) \subset \psi^{-1}(]\lambda, a_{i+1}[)$$

$\triangle$

We say that v is *δ-separated* if it is δ-separated with respect to some ordered Morse function $\phi : W \to [a,b]$, adjusted to (f,v).

It is obvious that each f-gradient satisfying Almost Transversality Condition is δ-separated for some $\delta > 0$.

Proposition 3.2.

If v is δ_0-separated, then $\forall \delta \in [0,\delta_0]$ and $\forall s : 0 \leqslant s \leqslant n$

1. $D_\delta(\mathrm{ind}_{\leqslant s} ; v)$ *is compact.*
2. $\bigcap_{\theta > \delta} B_\theta(\mathrm{ind}_{\leqslant s} ; v) = D_\delta(\mathrm{ind}_{\leqslant s} ; v)$

3. *If* $\delta > 0$ *then* $\overline{B_\delta(\mathrm{ind}_{\leqslant s}\ ; v)} = D_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$. $\square$

Thus the collection of descending discs $D(p, v)$ form a sort of stratified manifold, and the δ-thickened descending discs $B_\delta(v)$ form a collection of neighborhoods of this manifold. In the preceding proposition we have listed some natural properties of these objects. These properties deserve to be formulated and studied separately. Thus we are lead to the notions of s-submanifold and of ts-submanifolds, which will be introduced in the following subsection.

3.2. **s-submanifolds and ts-submanifolds.** Let $\mathbb{A} = \{A_0, ..., A_k\}$ be a finite sequence of subsets of a topological space X. We denote A_s also by $\mathbb{A}_{(s)}$, and we denote $A_0 \cup ... \cup A_s$ by $\mathbb{A}_{(\leqslant s)}$. We say that $\mathbb{A}$ is a *compact family* if $\mathbb{A}_{(\leqslant s)}$ is compact for every s.

Definition 3.3.
Let M be a manifold without boundary. A finite sequence $\mathbb{X} = \{X_0, ..., X_k\}$ of subsets of M is called *s-submanifold of M* (s for stratified) if

1. Each X_i is a submanifold of M of dimension i with the trivial normal bundle.
2. $\mathbb{X}$ is a compact family.
3. For $i \neq j$ we have: $X_i \cap X_j = \emptyset$

$\triangle$

For a diffeomorphism $\Phi : M \to N$ and an s-submanifold $\mathbb{X}$ of M we denote by $\Phi(\mathbb{X})$ the s-submanifold of N defined by $\Phi(\mathbb{X})_{(i)} = \Phi(\mathbb{X}_{(i)})$.

Let $\mathbb{X}, \mathbb{Y}$ be two s-submanifolds of M. We say, that $\mathbb{X}$ is *transversal to* $\mathbb{Y}$ (notation: $\mathbb{X} \pitchfork \mathbb{Y}$) if $\mathbb{X}_{(i)} \pitchfork \mathbb{Y}_{(j)}$ for every i, j; we say that $\mathbb{X}$ is *almost transversal to* $\mathbb{Y}$ (notation: $\mathbb{X} \nmid \mathbb{Y}$) if $\mathbb{X}_{(i)} \pitchfork \mathbb{Y}_{(j)}$ for $i + j < \dim M$. Note, that $\mathbb{X} \nmid \mathbb{Y}$ if and only if $\mathbb{X}_{(\leqslant i)} \cap \mathbb{Y}_{(\leqslant j)} = \emptyset$ whenever $i + j < \dim M$.
There is an analog of Thom transversality theorem: given two s-submanifolds $\mathbb{X}, \mathbb{Y}$ of M, there is a small isotopy Φ_t of M, such that $\Phi_1(\mathbb{X}) \nmid \mathbb{Y}$ (see [Pa3], Theorem 2.3.

Definition 3.4.
Let X be a topological space, $\mathbb{A} = \{A_0, ..., A_k\}$ be a compact family of subsets of X , I be an open interval $]0, \delta_0[$. A *good fundamental system of neighborhoods of* $\mathbb{A}$ (abbreviation: gfn-system for $\mathbb{A}$) is a family $\mathbf{A} = \{A_s(\delta)\}_{\delta \in I, 0 \leqslant s \leqslant k}$ of open subsets of X, satisfying the following conditions:

1. For every s and every $\delta_1 < \delta_2$ we have $\overline{A_s} \subset A_s(\delta_1) \subset A_s(\delta_2)$
2. For every δ and every j we have $\overline{A_{\leqslant j}(\delta)} = \bigcap_{\theta > \delta} (A_{\leqslant j}(\theta))$
3. For every j we have $A_{(\leqslant j)} = \bigcap_{\theta > 0} (\mathbf{A}_{(\leqslant i)}(\theta))$.

I is called *interval of definition* of the system. $\mathbb{A}$ is called the *core* of $\mathbf{A}$, and $\mathbf{A}$ is called *thickening* of $\mathbb{A}$. We shall denote $A_s(\delta)$ also by $\mathbf{A}_{(s)}(\delta)$ and $A_{\leqslant i}(\delta)$ also by $\mathbf{A}_{(\leqslant i)}(\delta)$. Sometimes we shall denote A_s by $\mathbf{A}_s(0)$ for uniformity of notation. $\triangle$ $\triangle$

The term "good fundamental system" is justified by the following easy consequence of the properties $1 - 3$: if X is compact, then for every $\delta \geqslant 0$ and every s the family $\{A_{\leqslant s}(\theta)\}_{\theta > \delta}$ is a fundamental system of neighborhoods of $\overline{\mathbf{A}_{(\leqslant s)}(\delta)}$.

If M is a manifold without boundary, $\mathbb{X}$ is an s-submanifold of M, and $\mathbf{X}$ is a gfn-system for $\mathbb{X}$, then we say that $\mathbf{X}$ is a *ts-submanifold with the core* $\mathbb{X}$. We say also that $\mathbf{X}$ is the *thickening* of $\mathbb{X}$. For a ts-submanifold $\mathbf{X} = \{X_s(\delta)\}_{\delta \in I, 0 \leqslant s \leqslant k}$ we shall denote $X_i(\delta)$ by $\mathbf{X}_{(i)}(\delta)$, and $X_{\leqslant i}(\delta)$ by $\mathbf{X}_{(\leqslant k)}(\delta)$.

Here is the basic example of gfn-system. Let $f : W \to [a, b]$ be a Morse function on a cobordism W, and v be an f-gradient. Assume that v is ϵ-separated. Then the family $\{B_\delta(\mathrm{ind}{=}s\ ; v)\}_{\delta \in]0,\epsilon[, 0 \leqslant s \leqslant n}$ is a gfn-system for $\mathbb{D}(v)$. If W is a closed manifold, this family is a ts-submanifold with the core $\mathbb{D}(v)$. This gfn-system will be denoted by $\mathbf{D}(v)$. (If $\partial W \neq \emptyset$ then $\mathbb{D}(v)$ is a family of manifolds with boundary, but we shall neither use it, nor formulate the corresponding generalization of the notion of ts-submanifold.)

Let $f : W \to [a, b]$ be a Morse function on a cobordism W, and v be an f-gradient. For a subset $X \subset \partial_1 W$ the set $T(X, v)$ formed by all the $(-v)$-trajectories starting in X will be called *track* of X. The set $T(X, v)$ is not necessarily compact even if X is. But one can show that if X is compact, then $T(X, v) \cup D_\delta(v)$ is compact for every δ (see [Pa3], Lemma 2.7). Next we shall extend the notion of track to s-submanifolds of $\partial_1 W$. Let $\mathbb{A}$ be an s-submanifold of $\partial_1 W$. Intuitively, the k-th component of the track-family $\mathbb{T}(\mathbb{A}, v)$ is the track $T(A_k, v)$ plus all the discs $D(p, v)$ with $\mathrm{ind} p \leqslant k$. Proceeding to precise definitions, we shall assume from here to the end of the section that v satisfies the Almost Transversality Condition, and that $\mathbb{A} \pitchfork \mathbb{D}_b(-v)$. (The condition $\mathbb{A} \pitchfork \mathbb{D}_b(-v)$ is not very restrictive, since given $\mathbb{A}$ and v one can always find a C^∞-small perturbation of v such that this condition holds, see [Pa3], Lemma 4.2.)

Definition 3.5.
Set $TA_i(v) = T(A_{i-1}, v) \cup D(\mathrm{ind}{=}i\ ; v)$ $(A_{-1} = \emptyset$ by definition). Denote by $\mathbb{T}(\mathbb{A}, v)$ the family $\{TA_i(v)\}_{0 \leqslant i \leqslant k+1}$ of subsets of W and by $(-v)^{\rightsquigarrow}_{[b,\lambda]}(\mathbb{A})$ the family $\{TA_{i+1}(v) \cap f^{-1}(\lambda)\}_{0 \leqslant i \leqslant k}$ of subsets of $f^{-1}(\lambda)$. If the values b, λ are clear from the context we shall abbreviate $(-v)^{\rightsquigarrow}_{[b,\lambda]}(\mathbb{A})$ to $\overset{\rightsquigarrow}{(-v)}(\mathbb{A})$. The family $\mathbb{T}(\mathbb{A}, v)$ will be called *track* of $\mathbb{A}$, and the family $(-v)^{\rightsquigarrow}_{[b,\lambda]}(\mathbb{A})$ will be called $\overset{\rightsquigarrow}{(-v)}$*-image* of $\mathbb{A}$. △

One can prove that $\mathbb{T}(\mathbb{A}, v)$ and $(-v)^{\rightsquigarrow}_{[b,\lambda]}(\mathbb{A})$ are compact families; further, if λ is a regular value of f, then $(-v)^{\rightsquigarrow}_{[b,\lambda]}(\mathbb{A})$ is an s-submanifold of $f^{-1}(\lambda)$ (see [Pa3], Lemma 2.9).

Now we can introduce tracks of ts-submanifolds. If $\mathbf{A}$ is a ts-submanifold of $\partial_0 W$ with the core $\mathbb{A}$, then the track of $\mathbf{A}$ will be a gfn-system with the core $\mathbb{T}(\mathbb{A}, v)$.

Definition 3.6.
Let $\mathbf{A} = \{A_s(\delta)\}_{\delta \in]0,\delta_0[, 0 \leqslant s \leqslant k}$ be a ts-submanifold of $\partial_1 W$ with the core $\mathbb{A}$. Assume that v is δ_1-separated. For $0 < \delta < \min(\delta_0, \delta_1)$ set $TA_s(\delta, v) = T(A_{s-1}(\delta), v) \cup B_\delta(\mathrm{ind}{=}s\ ; v)$ $(A_{-1}(\delta) = \emptyset$ by definition). △ △

Proposition 3.7. **([Pa3], Prop. 2.12)**
There is $\epsilon \in]0, \min(\delta_0, \delta_1)[$ such that $\{TA_s(\delta, v)\}_{\delta \in]0,\epsilon[,\ 0 \leqslant s \leqslant k+1}$ is a gfn-system for $\mathbb{T}(\mathbb{A}, v)$.

The gfn-system, introduced in the Proposition 3.7 will be denoted by $\mathbf{T}(\mathbf{A}, v)$ and called *track of* $\mathbf{A}$.

3.3. **Rapid flows.** Let M be a closed manifold of dimension m, f be a Morse function on M, v be an f-gradient satisfying Almost Transversality Condition. Let k be an integer and U be an open neighborhood of $D(\mathrm{ind}_{\leqslant k}\ ; -v)$, V be an open neighborhood of $D(\mathrm{ind}_{\leqslant m-k-1}\ ; v)$. It is clear that the flow Φ_t, generated by v will carry $M\setminus V$ to U if only we wait sufficiently long. (In precise terms: denote the diffeomorphism $t\mapsto \gamma(x,t;v)$ of M by $\Phi(v,t)$, then for T large enough we have: $\Phi(v,T)(M\setminus V)\subset U$.) If the flow Φ_t does this operation in a small time and moreover if v has small norm, we shall say that Φ_t is *rapid.* Here is the precise definition.

Definition 3.8.

Let v be an f-gradient, satisfying Almost Transversality Condition. Let $\epsilon > 0, t\geqslant 0$. We say that the flow, generated by v is (t,ϵ)-*rapid*, if for every $s: 0\leqslant s\leqslant m$ we have:

$$\Phi(v,t)\big(M\setminus B_\epsilon(\mathrm{ind}_{\leqslant s}\ ; v)\big)\subset B_\epsilon\big(\mathrm{ind}_{\leqslant m-1-s}\ ; -v\big) \tag{3}$$

$$\Phi(-v,t)\big(M\setminus B_\epsilon(\mathrm{ind}_{\leqslant s}\ ; -v)\big)\subset B_\epsilon\big(\mathrm{ind}_{\leqslant m-1-s}\ ; v\big) \tag{4}$$

(Sometimes for the sake of brevity we shall say that v is (t,ϵ)-rapid.) $\triangle$ $\triangle$

It is not difficult to check that each flow v satisfying Almost Transversality Condition is (t,ϵ)-rapid for *some* numbers t,ϵ.

Remark on the Terminology. The reader will not find the notion of rapid flow in [Pa3]. In [Pa3] we used the notion of *quick flows* which is better suited for the proof of the theorem 3.10 below.

Definition 3.9.

Let $f: W\to[a,b]$ be a Morse function on a cobordism, and u,v be f-gradients. We say that u and v are *equivalent*, if $u(x)=\phi(x)v(x)$, where $\phi: W\to\mathbf{R}$ is a C^∞ function, such that $\phi(x)>0$ for every x and $\phi(x)=1$ for x in a neighborhood of $S(f)$. $\triangle$

Theorem 3.10.

Let M be a closed riemannian manifold. Let $C>0, t>0$.

There is a Morse function $f: M\to\mathbf{R}$ and an f-gradient v, satisfying Almost Transversality Condition such that for every $\epsilon>0$ there is a (t,ϵ)-rapid f-gradient u equivalent to v, and satisfying $\|u\|\leqslant C$.

This theorem is proved in [Pa3] (Corollary 1.14).

3.4. **Condition (RP).** In this subsection W is a riemannian cobordism, $f: W\to[a,b]$ a Morse function, v an f-gradient, satisfying Almost Transversality Condition, $n=\dim W$.

Definition 3.11.

We shall say that v has a ranging pair (or, equivalently, that v satisfies condition (RP)) if there are Morse functions $\phi_0:\partial_0 W\to\mathbf{R}, \phi_1:\partial_1 W\to\mathbf{R}$, and their gradients u_0, resp. u_1, satisfying Almost Transversality Condition, and a number

$\delta > 0$, such that for every s the following conditions hold:

(RP1)

δ is in the interval of the definition of $\mathbf{T}(\mathbf{D}(u_1), v)$ and of $\mathbf{T}(\mathbf{D}(-u_0), -v)$

(RP2) The gradients v, u_0 and u_1 are $\delta -$ separated

(RP3) $$\overline{\mathbf{T}(\mathbf{D}(u_1), v)_{(\leqslant s+1)}(\delta)} \cap \partial_0 W \subset \mathbf{D}(u_0)_{(\leqslant s)}(\delta)$$

(RP4) $$\overline{\mathbf{T}(\mathbf{D}(-u_0), -v)_{(\leqslant s+1)}(\delta)} \cap \partial_1 W \subset \mathbf{D}(-u_1)_{(\leqslant s)}(\delta)$$

$\triangle$

The set of f-gradients of v satisfying (RP) will be denoted by $\mathcal{GRP}(f)$.

Theorem 3.12.
The set $\mathcal{GRP}(f)$ is C^0 dense in $\mathcal{G}(f)$.

Proof. Let $v \in \mathcal{GA}(f)$. Let $\phi_1 : \partial_1 W \to \mathbf{R}$, $\phi_0 : \partial_0 W \to \mathbf{R}$ be Morse functions, u_1, u_0 be their gradients, satisfying Almost Transversality Condition. Here if the plan of our proof. To obtain an f-gradient $w \in \mathcal{GRP}(f)$ we shall modify v by adding the horizontal components: ξ_1 nearby $\partial_1 W$ and ξ_0 nearby $\partial_0 W$. The vector field ξ_1 will be equivalent to u_1 and ξ_0 will be equivalent to u_0. We shall first do this modification without making any assumptions on u_1, u_0, and we shall obtain the condition (RP) for the modified gradient. In the end we shall show how to choose u_1, u_0 so that the resulting f-gradient w be C^0 close to v.

Proceed now to the proof. Perturbing v if necessary we can assume that $(-v)^{\leadsto}(\mathbb{D}(u_1))$ is almost transversal to $\mathbb{D}(-u_0)$, that is, $(-v)^{\leadsto}(\mathbb{D}_{(\leqslant k)}(u_1))$ does not intersect $\mathbb{D}_{(\leqslant m)}(-u_0)$ if $k + m < n - 1$. Therefore by the properties of gfn-systems we have for small δ:

$$(-v)^{\leadsto}(D_\delta(\mathrm{ind}\leqslant k\ ; u_1)) \cap D_\delta(\mathrm{ind}\leqslant m\ ; -u_0) = \emptyset \text{ if } k + m < n - 1$$

To make our notation more brief, set:

$$A_k^-(\delta) = D_\delta(\mathrm{ind}\leqslant k\ ; u_1); \qquad A_k^+(\delta) = D_\delta(\mathrm{ind}\leqslant k\ ; -u_1);$$
$$B_k^-(\delta) = D_\delta(\mathrm{ind}\leqslant k\ ; u_0); \qquad B_k^+(\delta) = D_\delta(\mathrm{ind}\leqslant k\ ; -u_0);$$

(Thus $A_k^-(\delta)$ is "the δ-thickened k-skeleton of $\partial_1 W$", and $A_k^+(\delta)$ is "the δ-thickened k-skeleton of the dual cell decomposition of $\partial_1 W$". Similar interpretation is valid for $B_k(\delta)^\pm$.)

Let ξ_0 be a ϕ_0-gradient, equivalent to u_0. Let ξ_1 be a ϕ_1-gradient equivalent to u_1. For $T > 0$ sufficiently large and any m we have:

(5) $$\Phi(\pm\xi_0, T)(\partial_0 W \setminus B_k^\mp(\delta)) \subset B_{n-k-2}^\pm(\delta)$$

(6) $$\Phi(\pm\xi_1, T)(\partial_1 W \setminus A_k^\mp(\delta)) \subset A_{n-2-k}^\pm(\delta)$$

Set $\Phi_0 = \Phi(\xi_0, T)$, $\Phi_1 = \Phi(\xi_1, T)$. We shall add to v two horizontal components: ξ_0 nearby $\partial_0 W$, and ξ_1 nearby $\partial_1 W$. If we choose the parameter function h with $\int_0^l h(\tau) d\tau = T$, then for the resulting vector field v' we have:

(7) $$(-v')^{\leadsto} = \Phi_0 \circ (-v)^{\leadsto} \circ \Phi_1^{-1}$$

It is obvious now that (RP3) and (RP4) hold for v'. Indeed, the set $A_m^-(\delta)$ is sent to itself by Φ_1^{-1}, then the result is sent to $\partial_0 W \setminus B^+_{n-2-m}(\delta)$ by $(-v)^{\rightsquigarrow}$ and Φ_0 pushes the $\partial_0 W \setminus B^+_{n-2-m}(\delta)$ to $B_m^-(\delta)$. This proves (RP3), similar for (RP4). (Of course we should take δ small enough in order to satisfy (RP1) and (RP2). One must also make sure that v' be an f-gradient. We refer to [Pa3], §4 for details.) Now I shall indicate how to make the norm $\|v' - v\|$ small. If we choose u_0, u_1 as to satisfy the conclusions of the theorem on the rapid flows, we can obtain vector fields ξ_0, ξ_1 with small norm for which the properties (6), (5) hold already for T sufficiently small. Therefore the modified vector field v' will be C^0 close to v. $\square$

3.5. **Ranging systems and homological gradient descent (first version).** As we have already mentioned, the map $(-v)^{\rightsquigarrow}$ is not everywhere defined. Still it is sometimes possible to define an analog of "the homomorphism, induced by $(-v)^{\rightsquigarrow}$ in homology". More precisely, in some cases (for example, when v satisfies (RP)) one can define a homomorphism $H(v) : H_*(\partial_1 W \setminus B_1, A_1) \to H_*(\partial_0 W \setminus B_0, A_0)$, where A_i, B_i are some disjoint subsets of $\partial_i W$. The construction of $H(v)$ was given in [Pa3], §4, and we shall briefly expose it here. We would like to stress that this homomorphism is *not* induced by a continuous map of pairs.

Definition 3.13.

Let $\Lambda = \{\lambda_0, ..., \lambda_k\}$ be a finite set of regular values of f, such that $\lambda_0 = a, \lambda_k = b$, and for every i we have $\lambda_i < \lambda_{i+1}$ and there is exactly one critical value of f in $[\lambda_i, \lambda_{i+1}]$. The values λ_i, λ_{i+1} will be called *adjacent*. The set of pairs $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ is called *ranging system for* (f, v) if

(RS1) $\forall \lambda \in \Lambda$ we have: A_λ and B_λ are disjoint compacts in $f^{-1}(\lambda)$.

(RS2) Let $\lambda, \mu \in \Lambda$ be adjacent. Then for every $p \in S(f) \cap f^{-1}([\lambda, \mu])$ either
i) $D(p, v) \cap f^{-1}(\lambda) \subset \operatorname{Int} A_\lambda$
or
ii) $D(p, -v) \cap f^{-1}(\mu) \subset \operatorname{Int} B_\mu$.

(RS3) Let $\lambda, \mu \in \Lambda$ be adjacent. Then $(-v)^{\rightsquigarrow}_{[\mu,\lambda]}(A_\mu) \subset \operatorname{Int} A_\lambda$ and $v^{\rightsquigarrow}_{[\lambda,\mu]}(B_\lambda) \subset \operatorname{Int} B_\mu$. $\triangle$

$\triangle$

Ranging systems have the following properties (see [Pa3], §4.3).

(1) If $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ is a ranging system for (f, v), then for every f-gradient w, sufficiently C^0-close to v, $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ is a ranging system for (f, w).

(2) Let N be a submanifold of $\partial_1 W \setminus B_b$ such that $N \setminus \operatorname{Int} A_b$ is compact.
Then $N' = v^{\rightsquigarrow}_{[b,a]}(N)$ is a submanifold of $\partial_0 W \setminus B_a$ such that $N' \setminus \operatorname{Int} A_a$ is compact.

(3) There is a homomorphism $H(v) : H_*(\partial_1 W \setminus B_b, A_b) \longrightarrow H_*(\partial_0 W \setminus B_a, A_a)$, such that
  (a) If N is an oriented submanifold of $\partial_1 W$, satisfying the hypotheses of (2), then $H(v)([N]) = [v^{\rightsquigarrow}_{[b,a]}(N)]$.
  (b) There is an $\epsilon > 0$ such that for every f-gradient w with $\|w - v\| < \epsilon$ we have $H(v) = H(w)$.

We shall explain here the main idea of the construction of $H(v)$ in the following simple case: f has only one critical point p, and $D(p, v) \cap \partial_0 W \subset \operatorname{Int} A_a$. Moreover, we shall assume $B_a = \emptyset, B_b = \emptyset$. Let U be a small compact neighborhood of

$D(p,-v)\cap\partial_1 W$, such that $(-v)^{\leadsto}(U)\subset \mathrm{Int}\ A_a$. Let $x\in H_*(\partial_1 W\setminus B_b, A_b)$. Consider the image of x in $H_*(\partial_1 W, A_1)$ and reduce it further to $x'\in H_*(\partial_1 W, A_b\cup U)$. Applying excision, we obtain an element $\bar{x}\in H_*\big(\partial_1 W\setminus D(p,-v), A_b\cup U\setminus D(p,-v)\big)$. Any singular chain representing $\bar{x}$ is in the domain of definition of $(-v)^{\leadsto}$. Apply $(-v)^{\leadsto}_*$ and obtain an element in $H_*(\partial_0 W, A_0)$, which is by definition $H(v)(x)$. (The reader will recognize here the basic idea of the construction of homological gradient descent from Subsection 2.5.)

4. Condition ($\mathfrak{C}$) and operator of homological gradient descent

We need some more terminology, we introduce it in Subsection 4.1. In the second part of this Subsection we recall the classical construction of Morse complex. The material of this subsection is not necessary for understanding of the rest of the present Chapter, we placed it here just because it is natural from the logical point of view.

In Subsection 4.2 we introduce the condition ($\mathfrak{C}$), which is a close analog of (RP), but it has some technical advantages over (RP). In particular, the set of all f-gradients satisfying ($\mathfrak{C}$) is C^0-open in the set of all gradients. In Subsection 4.3 we introduce a new version of the operator of homological gradient descent.

Terminology: In the rest of §4 W is a riemannian cobordism of dimension n.

4.1. Handle-like filtrations of cobordisms and Morse complexes. In this subsection $\phi : W \to [a,b]$ is an ordered Morse function with ordering sequence $a_0 = a < \ldots < a_{n+1} = b$, and v is a ϕ-gradient. Assume that v is δ-separated. Let $\nu \in]0,\delta]$ and s be an integer between 0 and $n+1$. Set

$$W^{\{\leqslant s\}} = \phi^{-1}([a_0, a_{s+1}]); \quad W^{\{\geqslant s\}} = \phi^{-1}([a_s, a_{n+1}])$$
$$W^{[\leqslant s]}(\nu) = D_\nu(\mathrm{ind}_{\leqslant s}\ ; v) \cup \partial_0 W;$$
$$W^{\lceil s \rceil}(\nu) = \phi^{-1}\big([a_0, a_s]\big) \cup D_\nu(\mathrm{ind}_{=s}\ ; v)$$

We shall reserve for all these filtrations of W the generic name "handle-like filtrations". Of course these definitions are valid with larger assumptions on v and ϕ, but only with our assumptions these filtrations have the nice homological properties. These properties are collected in the following proposition. For a critical point $p \in S_k(\phi)$ we denote by $d(p)$ the embedded disc $D(p,v) \cap \phi^{-1}([a_k, a_{k+1}])$. For each critical point $p \in S(\phi)$ choose an orientation of the descending disc $D(p,v)$. Let $[p]$ denote the image in the group $H_k(W^{\{\leqslant k\}}, W^{\{\leqslant k-1\}})$ of the fundamental class of the pair $(d(p), \partial d(p))$.

Proposition 4.1.
Let $s \in \mathbf{N}$ and $\nu \in]0,\delta]$.

1. *The inclusions $W^{[\leqslant s]}(\nu) \subset W^{\lceil s \rceil}(\nu) \subset W^{\{\leqslant s\}}$ are homotopy equivalences.*
2. *The inclusions of pairs*
$$\Big(W^{[\leqslant s]}(\nu), W^{[\leqslant s-1]}(\nu)\Big) \subset \Big(W^{\lceil s \rceil}, W^{\{\leqslant s-1\}}\Big) \subset \Big(W^{\{\leqslant s\}}, W^{\{\leqslant s-1\}}\Big)$$
are homotopy equivalences.
3. *$H_*\big(W^{\{\leqslant s\}}, W^{\{\leqslant s-1\}}\big) = 0$ if $* \neq s$. The group $H_s\big(W^{\{\leqslant s\}}, W^{\{\leqslant s-1\}}\big)$ is a free abelian group generated by the elements $[p], p \in S_s(\phi)$.*

Denote $H_s(W^{\{\leqslant s\}}, W^{\{\leqslant s-1\}})$ by C_s. The boundary operator of the exact sequence of the triple $(W^{\{\leqslant s\}}, W^{\{\leqslant s-1\}}, W^{\{\leqslant s-2\}})$ gives a homomorphism $\partial_s : C_s \to C_{s-1}$; we have obviously $\partial_s \circ \partial_{s+1} = 0$ for every s. Thus the graded group C_* endowed with the boundary operator ∂_* is a free chain complex. In the case when v satisfies the Transversality Condition one can give another construction of this complex, which is known as *Morse complex.* Namely, let k be a positive integer, and let $p \in S_k(\phi), q \in S_{k-1}(\phi)$. Denote by $\Gamma(p,q;v)$ the set of the integral curves of $(-v)$ joining p with q, where each integral curve is considered up to a reparametrization. It is not difficult to deduce from Transversality Condition, that $\Gamma(p,q;v)$ is a finite set. The chosen orientations of the descending discs allow to attribute to each $\gamma \in \Gamma(p,q;v)$ a sign $\varepsilon(\gamma) \in \{-1,1\}$. Now let $C_k(v)$ be the free abelian group freely generated by critical points of ϕ of index k. Define a homomorphism $\partial'_k(v) : C_k(v) \to C_{k-1}(v)$ setting $\partial'_k(v)(p) = \sum_{q\in S_{k-1}(\phi)} n(p,q;v)q$. It turns out that the homomorphism $J_k : C_k(v) \to C_k$ sending p to $[p]$ satisfies $J_k \circ \partial'_{k+1}(v) = \partial_{k+1} \circ J_{k+1}$; thus $(C_*(v), \partial_k(v))$ is a chain complex, isomorphic to C_*.

To define an "equivariant version" of the Morse complex, consider a regular covering $\mathcal{P} : \widetilde{W} \to W$ with structure group G. The inverse image of a subset $A \subset W$ in $\widetilde{W}$ will be denoted by $\widetilde{A}$. In addition to the orientation of descending discs, choose for every point $p \in S(\phi)$ a lifting $\widetilde{p}$ of p to $\widetilde{W}$, that is a point $\widetilde{p}$ with $\mathcal{P}(\widetilde{p}) = p$. Then the lifting of the disc $d(p)$ to $\widetilde{W}$ is defined. Denote by $[\widetilde{p}]$ the image in the group $H_k(\widetilde{W}^{\{\leqslant k\}}, \widetilde{W}^{\{\leqslant k-1\}})$ of the fundamental class of the pair $(d(\widetilde{p}), \partial d(\widetilde{p}))$.

Proposition 4.2.
$H_*(\widetilde{W}^{\{\leqslant s\}}, \widetilde{W}^{\{\leqslant s-1\}}) = 0$, if $* \neq s$. The group $H_s(\widetilde{W}^{\{\leqslant s\}}, \widetilde{W}^{\{\leqslant s-1\}})$ is a free $\mathbf{Z}G$-module freely generated by the elements $[\widetilde{p}], p \in S_s(\phi)$.

Similarly to the above one gives a construction of chain complex $(\widetilde{C}_*(v), \widetilde{\partial}_*(v))$ of free $\mathbf{Z}G$-modules, which is freely generated in degree p by the critical points of ϕ of index p, and the boundary operator is defined via counting the v-trajectories joining the critical points. The reader will find the details and the proofs of the cited statements in [Pa1], Appendix.

4.2. Condition ($\mathfrak{C}$). *Terminology*

Let $g : W \to [a,b]$ be a Morse function on a cobordism W of dimension n, v be a g-gradient.

Definition 4.3.
Let $\lambda \in [a,b]$ be a regular value of f. Let $A \subset \partial_1 W$. We say, that v *descends A to $f^{-1}(\lambda)$*, if every $(-v)$-trajectory starting in A reaches $f^{-1}(\lambda)$. Let $X \subset W$. We say that v *descends A to $f^{-1}(\lambda)$ without intersecting X*, if v descends A to $f^{-1}(\lambda)$ and $T(A,v) \cap X = \emptyset$.

We leave to the reader to the meaning of *"v lifts B to $f^{-1}(\lambda)$"* and *"v lifts B to $f^{-1}(\lambda)$ without intersecting Y"*, where $B \subset \partial_0 W, Y \subset W$). △

Recall the sets $B_\delta(\mathrm{ind}_{\leqslant s}\ ; v), D_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$ from Subsection 3.1 and introduce one more set with similar properties: set $C_\delta(\mathrm{ind}_{\leqslant s}\ ; v) = W \setminus B_\delta(\mathrm{ind}_{\geqslant n-s-1}\ ; v)$. This terminology is justified by the following lemma which is proved by standard Morse-theoretical arguments.

Lemma 4.4.
Assume that v is δ-separated with respect to ϕ and the ordering sequence $(a_0, ..., a_{n+1})$. Then for every s we have:

$$\partial_0 W \cup B_\delta(\mathrm{ind}_{\leqslant s}\ ; v) \subset \phi^{-1}\left([a_0, a_{s+1}]\right) \subset C_\delta(\mathrm{ind}_{\leqslant s}\ ; v) \cup \partial_0 W$$

and these inclusions are homotopy equivalences. □

Now we can formulate the condition $(\mathfrak{C})$.

Definition 4.5.

We say, that v *satisfies condition* $(\mathfrak{C})$ if there are objects 1) - 4), listed below, with the properties (A), (B) below.

Objects:

1) An ordered Morse function ϕ_1 on $\partial_1 W$ with ordering sequence $(\alpha_0, ..., \alpha_n)$, and a ϕ_1-gradient u_1.
2) An ordered Morse function ϕ_0 on $\partial_0 W$ with ordering sequence $(\beta_0, ..., \beta_n)$, and a ϕ_0-gradient u_0.
3) An ordered Morse function ϕ on W with ordering sequence $(a_0, ..., a_{n+1})$, adjusted to (f, v).
4) A number $\delta > 0$.

Properties:

(A) u_0 is δ-separated with respect to ϕ_0, u_1 is δ-separated with respect to ϕ_1, v is δ-separated with respect to ϕ.

(B1)

$$(-v)^{\leadsto}\Big(C_\delta\big(\mathrm{ind}{\leqslant}j\ ; u_1\big)\Big) \cup \Big(D_\delta\big(\mathrm{ind}{\leqslant}j{+}1\ ; v\big) \cap \partial_0 W\Big) \subset B_\delta\big(\mathrm{ind}{\leqslant}j\ , u_0\big) \text{ for every } j$$

(B0)

$$\overset{\leadsto}{v}\Big(C_\delta\big(\mathrm{ind}{\leqslant}j\ ; -u_0\big)\Big) \cup \Big(D_\delta\big(\mathrm{ind}{\leqslant}j{+}1\ ; -v\big) \cap \partial_1 W\Big) \subset B_\delta\big(\mathrm{ind}{\leqslant}j\ ; -u_1\big) \text{ for every } j$$

$\triangle$ $\triangle$

The set of all f gradients satisfying $(\mathfrak{C})$ will be denoted by $\mathcal{G}\mathfrak{C}(f)$. Recall that the set of all f-gradients satisfying Transversality Condition is denoted by $\mathcal{GT}(f)$. The intersection $\mathcal{G}\mathfrak{C}(f) \cap \mathcal{GT}(f)$ will be denoted by $\mathcal{G}\mathfrak{C}\mathcal{T}(f)$.

Comments:

1. As we have already mentioned in the introduction, the condition (B) is an analog of cellular approximation condition. Indeed, the sets $C_\delta(\mathrm{ind}{\leqslant}j\ ; u_1)$ and $B_\delta(\mathrm{ind}{\leqslant}j\ ; u_0)$ are "thickenings" of $D(\mathrm{ind}{\leqslant}j\ ; u_1)$, resp. of $D(\mathrm{ind}{\leqslant}j\ ; u_0)$, the first one being "thicker" than the second. Thus the condition (B1) requires that $(-v)^{\leadsto}$ sends a certain thickening of a j-skeleton of $\partial_1 W$ to a certain thickening of a j-skeleton of $\partial_0 W$. Warning: (B1) actually requires more than that: for every j the soles of δ-handles of W of indices $\leqslant j+1$ must belong to the set $B_\delta(\mathrm{ind}{\leqslant}j\ ; u_0)$.
2. The condition $(\mathfrak{C})$ is stronger than the condition $(\mathcal{C})$ from [Pa5], §1.2. This is obvious, since $(\mathfrak{C})$ is formulated similarly to $(\mathcal{C})$, except that in $(\mathcal{C})$ we have demanded $(-v)^{\leadsto}(\phi_1^{-1}([a_0, a_{j+1}]) \subset B_\delta(\mathrm{ind}{\leqslant}j\ ; u_0)$. The condition $(\mathcal{C})$ is in turn stronger, than the condition (RP) from the previous Section, this is a bit less obvious, and we shall not prove it here.

3. If we are given a ϕ_0-gradient u_0, a ϕ_1-gradient u_1 a ϕ-gradient v, which satisfy Almost Transversality Condition, then the condition (A) is always true if δ is sufficiently small.

Theorem 4.6.
$\mathcal{G}\mathfrak{C}(f)$ is open and dense in $\mathcal{G}(f)$ with respect to C^0 topology. Moreover, if v_0 is any f-gradient then one can choose a C^0 small perturbation v of v_0 such that $v \in \mathcal{G}\mathfrak{C}(f)$ and $v = v_0$ in a neighborhood of ∂W.

The proof occupies the rest of the section and is subdivided into 2 parts: openness and density.

C^0-openness of $\mathcal{G}\mathfrak{C}(f)$

Let v be an f-gradient satisfying $(\mathfrak{C})$. Choose the corresponding functions ϕ_0, ϕ_1, ϕ, their gradients u_0, u_1, and a number $\delta > 0$ satisfying (A) and (B0), (B1). Fix now the string $\mathcal{S} = (\delta, u_0, u_1, \phi_0, \phi_1, \phi)$. For a ϕ-gradient w we shall denote the condition (A) with respect to w and $\mathcal{S}$ by $(A)(w)$. Similarly, we shall denote the conditions (B1), (B0) with respect to w and $\mathcal{S}$ by $(B1)(w)$, $(B0)(w)$. We know that $(A)(v)\&$ $(B1)(v)$ & $(B0)(v)$ holds; we shall prove that $(A)(w)\&$ $(B1)(w)$ & $(B0)(w)$ holds for every w sufficiently C^0 close to v.

Let w be another f-gradient. Lemma 1.6 and Corollary 1.7 of [Pa3] imply that if $\|w - v\|$ is sufficiently small, then w is a ϕ-gradient and is also δ-separated, so the condition $A(w)$ is satisfied.

It is convenient to reformulate the condition $(B1)(w)\&(B0)(w)$.

Introduce three new conditions:

$\beta_1(w)$: For every j we have: w descends $C_\delta(\mathrm{ind}{\leqslant}j\ ; u_1)$ to $\phi^{-1}(a_{j+1})$ without intersecting $\cup_{\mathrm{ind}p \geqslant j+1} D_\delta(p)$.

$\beta_2(w)$: For every j we have: w lifts $C_\delta(\mathrm{ind}{\leqslant}n-j-2\ ; -u_0)$ to $\phi^{-1}(a_{j+2})$ without intersecting $\cup_{\mathrm{ind}p \leqslant j+1} D_\delta(p)$.

$\beta_3(w)$: For every j the sets $R_j^+(w) = (-v)^{\rightsquigarrow}_{[b,a_{j+2}]}(C_\delta(\mathrm{ind}{\leqslant}j\ ; u_1))$
and
$R_j^-(w) = v^{\rightsquigarrow}_{[a,a_{j+2}]}(C_\delta(\mathrm{ind}{\leqslant}n-j-2\ ; -u_0))$
are disjoint.

Note that $\beta_1(w)$ and $\beta_2(w)$ are in a sense dual to each other, that is: $\beta_2(w) = \beta_1(-w)$.

Lemma 4.7.
Assume that $A(w)$ holds. Then $(B1)(w)\&(B0)(w) \Leftrightarrow \beta_1(w)\&\beta_1(w)\&\beta_3(w)$

Proof. $\Rightarrow$

The condition β_1 follows from (B0), β_2 follows from (B1). To prove β_3, note that if $R_j^+(w) \cap R_j^-(w) \neq \emptyset$, then there is a $(-v)$-trajectory joining $x \in C_\delta(\mathrm{ind}{\leqslant}j\ ; u_1)$ with $y \in C_\delta(\mathrm{ind}{\leqslant}n-j-2\ ; -u_0) = \partial_0 W \setminus B_\delta(\mathrm{ind}{\leqslant}j\ ; u_0)$. And this contradicts (B1).

$\Leftarrow$

β_1 says that for every j we have:

$$\left(\bigcup_{\mathrm{ind}p \geqslant j+1} D_\delta(p, -v) \right) \cap \partial_1 W \subset \partial_1 W \setminus C_\delta(\mathrm{ind}{\leqslant}j\ ; u_1) = B_\delta(\mathrm{ind}{\leqslant}n-2-j\ ; -u_1).$$

That is, $D_\delta(\mathrm{ind}\leqslant n-1-j\ ; -v) \cap \partial_1 W \subset B_\delta(\mathrm{ind}\leqslant n-2-j\ ; -u_1)$. Further, β_3 implies that for every j every v-trajectory starting in $C_\delta(\mathrm{ind}\leqslant n-j-2\ ; -u_0)$ and reaching $\partial_1 W$, cannot intersect $\partial_1 W$ at a point of $C_\delta(\mathrm{ind}\leqslant j\ ; u_1)$. Therefore it intersects $\partial_1 W$ at a point of $B_\delta(\mathrm{ind}\leqslant n-2-j\ ; -u_1)$. Therefore (B0)($w$) holds. The proof of (B1)(w) is similar. □

Returning to the proof of C^0-openness of ($\mathfrak{C}$), note that the conditions $\beta_1(w)$ and $\beta_2(w)$ are obviously open. We know that $\beta_1(v)$ and $\beta_2(v)$ hold, therefore for every j the sets $R_j^+(v), R_j^-(v)$ are compact. Since they are disjoint (by $\beta_3(v)$), we can choose two disjoint open subsets U_j, V_j of $\phi^{-1}(a_{j+2})$ such that $R_j^+(v) \subset U_j, R_j^-(v) \subset V_j$. Then for every w sufficiently C^0-close to v and for every j we have (see [Pa3], Corollary 5.6):

$$(-w)^{\leadsto}_{[b,a_{j+2}]}\big(C_\delta(\mathrm{ind}\leqslant j\ ; u_1)\big) \subset U_j, \quad w^{\leadsto}_{[a,a_{j+2}]}\big(C_\delta(\mathrm{ind}\leqslant n-2-j\ ; -u_0)\big) \subset V_j$$

and $\beta_3(w)$ holds. Therefore $\beta_1(w)\&\beta_2(w)\&\beta_3(w)$ is C^0 open and we have proved C^0 openness of ($\mathfrak{C}$).

C^0 density.

The proof is very close to that of C^0 density of (RP). We shall not give it here in details; we just indicate that during the proof of C^0 density of (RP) we have constructed a perturbation v' of a given gradient v, which satisfies actually the condition ($\mathfrak{C}$). (Indeed, return to the paragraph just after (7). The diffeomorphism Φ_1^{-1} sends the whole of $\partial_1 W \setminus A^+_{n-2-m}(\delta)$ to the set $A^-_m(\delta)$. Since u_0 and u_1 are δ-separated, the set $(\partial_1 W)^{\{\leqslant m\}}$ is in $\partial_1 W \setminus A^+_{n-2-m}(\delta)$ and we have checked the property (B1).) □

4.3. **Homological gradient descent(second version).** In this subsection we define a version of the homological gradient descent operator from Subsection 3.5. This version is better suited for the study of Morse-type filtrations on cobordisms, which is the subject of the next section. Assume that v satisfies condition ($\mathcal{C}$) with respect to $\delta, \phi_0, \phi_1, u_0, u_1$. We have the corresponding filtrations $(\partial_1 W)^{\{\leqslant s\}}, (\partial_0 W)^{\{\leqslant s\}}$. Denote $\phi^{-1}(\alpha)$ by V_α.

Theorem 4.8.
For every s there is a homomorphism

$$\mathcal{H}_s(-v) : H_*(V_b^{\{\leqslant s\}}, V_b^{\{\leqslant s-1\}}) \to H_*(V_a^{\{\leqslant s\}}, V_a^{\{\leqslant s-1\}})$$

with the following properties:

1) *Let N be an oriented submanifold of V_b, such that $N \subset V_b^{\{\leqslant s\}}$ and $N \setminus \mathrm{Int}\, V_b^{\{\leqslant s-1\}}$ is compact. Then the manifold $N' = (-v)^{\leadsto}_{[b,a]}(N)$ is in $V_a^{\{\leqslant s\}}$ and $N' \setminus \mathrm{Int}\, V_a^{\{\leqslant s-1\}}$ is compact and the fundamental class of N' modulo $V_a^{\{\leqslant s-1\}}$ equals to $\mathcal{H}_s(-v)([N])$.*
2) *There is an $\epsilon > 0$ such that for every f-gradient w with $\|w - v\| \leqslant \epsilon$ and every s we have: $\mathcal{H}_s(-v) = \mathcal{H}_s(-w)$.*

Proof. 1) Construction of $\mathcal{H}_s(-v)$.

The homomorphism $\mathcal{H}_s(-v)$ will be defined as the composition of two homomorphisms: $\mathcal{H}1_s(-v)$ and $\mathcal{H}0_s(-v)$. (Intuitively, $\mathcal{H}1$ corresponds to the descent from the level b to the level a_{s+1}, and $\mathcal{H}0$ corresponds to the descent from a_{s+1} to

a.) We shall denote the maps $(-v)^{\leadsto}_{[b,a_{s+1}]}$ and $(-v)^{\leadsto}_{[a_{s+1},a]}$ by $(-v1)^{\leadsto}$, and respectively $(-v0)^{\leadsto}$. Every $(-v)$- trajectory starting in $V_b^{\{\leqslant s\}}$ reaches $V_{a_{s+1}}$. Therefore $V_b^{\{\leqslant s\}}$ is in the domain of $(-v1)^{\leadsto}$ and $(-v1)^{\leadsto}$ defines a homeomorphism of the pair $(V_b^{\{\leqslant s\}}, V_b^{\{\leqslant s-1\}})$ to its image, which is a pair of compact subsets of $V_{a_{s+1}}$.

Definition 4.9.

Let k be an integer, $0 \leqslant k \leqslant n$. Denote by $Y_k(v)$ the set of all $y \in \phi^{-1}([a, a_{k+1}])$ such that either

i) $\gamma(y,\cdot;-v)$ reaches V_a and intersects it at a point $z \in V_a^{\{\leqslant k-1\}}$ or

ii) $\lim_{t\to\infty} \gamma(y,t;-v) = p$, where $p \in S(f)$. △ △

In other words, $Y_k(v) = T(V_a^{\{\leqslant k-1\}}, -v \mid W') \cup D(-v \mid W')$, where $W' = \phi^{-1}([a, a_{k+1}])$. The set $Y_k(v)$ is compact (by [Pa3], Lemma 2.7). Set

(8) $A = Y_{s+1}(v) \cap V_{a_{s+1}}, \quad B = Y_s(v) \cap V_{a_{s+1}}, \quad C = \big(\cup_{\mathrm{ind} p \leqslant s} D(p,-v)\big) \cap V_{a_{s+1}}$

Then C is closed and $\bar{C} \subset \mathrm{Int}\, B$ and $A \supset B \supset \mathrm{Int}\, B \supset C$. Further, the map $(-v1)^{\leadsto}$ defines a continuous map of pairs

$$\big(V_b^{\{\leqslant s\}}, V_b^{\{\leqslant s-1\}}\big) \to (A, B)$$

and the map $(-v0)^{\leadsto}$ defines a continuous map of pairs

$$(A \setminus C, B \setminus C) \to (V_a^{\{\leqslant s\}}, V_a^{\{\leqslant s-1\}})$$

Now denote by Exc the excision isomorphism $H_*(A\setminus C, B\setminus C) \to H_*(A, B)$ and set

$$\mathcal{H}_s(-v) = \big((-v0)^{\leadsto} \circ Exc^{-1} \circ (-v1)^{\leadsto}\big)_*$$

2) Proof of the properties of $\mathcal{H}_s(-v)$.

The proof is reduced to the properties of the operator $H(v)$ associated to ranging systems (see Subsection 3.5).

Namely we construct a ranging system $\mathcal{R} = \{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ for (ϕ, v), such that the homomorphism $H(v)$, associated to $\mathcal{R}$ is isomorphic to $\mathcal{H}(-v)$. To explain the construction of $\mathcal{R}$ we make a simplifying assumption: ϕ has only two critical values: one in $[a, a_{s+1}]$ and the other in $[a_{s+1}, b]$. (In the end of the proof we indicate how to generalize it to the general situation.) We set

$$A_b = V_b^{\{\leqslant s-1\}}, A_a = V_a^{\{\leqslant s-1\}},$$
$$B_a = V_a^{\{\geqslant s+1\}}, B_b = V_b^{\{\geqslant s+1\}}$$

Denote a_{s+1} by τ for brevity. To define A_τ, B_τ, let $(\alpha_i), (\beta_i)$ be ordering sequences for ϕ_1, resp. ϕ_0, and choose $\epsilon > 0$ so small that:

$$D_\delta(\mathrm{ind}_{\leqslant n-s-2}\,, -u_1) \subset \phi_1^{-1}(]\alpha_{s+1} + \epsilon, \alpha_n]),$$
$$\alpha_s + \epsilon < \alpha_{s+1},$$
$$(-v)^{\leadsto}(\phi_1^{-1}([\alpha_0, \alpha_s + \epsilon])) \subset \phi_0^{-1}([\beta_0, \beta_s[)$$
$$\overset{\leadsto}{v}(\phi_0^{-1}([\beta_{s+1}, \beta_n])) \subset \phi_1^{-1}(]\alpha_{s+1} + \epsilon, \alpha_n])$$

Denote by $L_s(v)$ the set of all points $y \in \phi^{-1}([a_{s+1}, b])$, such that either

i) $\gamma(y,\cdot;v)$ reaches V_b and intersects it at a point $z \in \phi^{-1}([\alpha_{s+1} + \epsilon, \alpha_n])$, or

ii) $y \in D(p, v)$, where $p \in S(f)$.

Now set

$$A_\tau = (-v1)^{\leadsto}(\phi_1^{-1}([\alpha_0, \alpha_0 + \epsilon])), \quad B_\tau = L_s(v) \cap V_\tau$$

It follows from $(\mathfrak{C})$ that $\{(A_\lambda, B_\lambda)\}_{\lambda \in \{a,\tau,b\}}$ is a ranging system for (ϕ, v).

There are inclusions of pairs

$$j_b : (V_b^{\{\leqslant s\}}, V_b^{\{\leqslant s-1\}}) \subset (V_b \setminus B_b, A_b)$$
$$j_a : (V_a^{\{\leqslant s\}}, V_a^{\{\leqslant s-1\}}) \subset (V_a \setminus B_a, A_a)$$

and these inclusions induce isomorphisms in homology.

Moreover, checking through the definition of the homomorphism $H(v)$, associated with $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$, it is easy to see, that we have $H(v)\circ(j_b)_* = (j_a)_*\circ\mathcal{H}(-v)$. Now the properties of $\mathcal{H}(-v)$ follow immediately from those of $H(-v)$.

In the case when there is more than one critical value in $[a_{s+1}, b]$ and in $[a, a_{s+1}]$, the argument is similar: one constructs a ranging system $\mathcal{R} = \{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$, such that the operator $H(v)$, associated to $\mathcal{R}$ is isomorphic to $\mathcal{H}(-v)$. In this ranging system we can define A_λ, B_λ by the same formulas as above for $\lambda = a, b, \tau$. To define A_λ, B_λ for other values of λ, choose first a set $\Lambda = \{\lambda_i\}$ of regular values such that $\lambda_0 = a, \lambda_N = b, \lambda_i = \tau$ and for every i there is only one critical value of ϕ in $[\lambda_i, \lambda_{i+1}]$. For $i \leqslant j \leqslant N$ we define A_{λ_j} as some thickening of $(-v)^{\leadsto}_{[b,\lambda_j]}(V_b^{\{\leqslant s-1\}})$. The set B_{λ_j} for $i \leqslant j \leqslant N$ is defined similarly to $L_s(v)$ (see the definition above). The definition of the sets A_λ, B_λ for the other values of λ, as well as the proof that $H(v)$ is isomorphic to $\mathcal{H}(-v)$ will be left to the reader. $\square$

4.4. **Equivariant homological gradient descent.** Let $p : \widehat{W} \xrightarrow{H} W$ be a regular covering of W with the structure group H. For $A \subset W$ we denote $p^{-1}(A)$ also by $\widehat{A}$. The shift along the $(-v)$-trajectories defines a diffeomorphism $(-\widehat{v})^{\leadsto}$: $(\widehat{\partial_0 W \setminus K_1}) \to (\widehat{\partial_0 W \setminus K_0})$ (see the page 6 for the definition of K_1, K_0) which commutes with the right action of G. As before we abbreviate $(-\widehat{v})^{\leadsto}(X \setminus \widehat{K}_1)$ to $(-\widehat{v})^{\leadsto}(X)$. Assume that v satisfies condition $(\mathfrak{C})$ with respect to $\delta, \phi_0, \phi_1, u_1, u_0$. Denote $\phi^{-1}(\alpha)$ by V_α. Then we have a corresponding equivariant version of homological gradient descent operator. The properties of this operator are the contents of the next theorem. The proof is completely similar to the proof of Theorem 4.8 and will be omitted.

Theorem 4.10.
For every s there is a homomorphism of right $\mathbf{Z}H$-modules

$$\widehat{\mathcal{H}}_s(-v) : H_*(\widehat{V}_b^{\{\leqslant s\}}, \widehat{V}_b^{\{\leqslant s-1\}}) \to H_*(\widehat{V}_a^{\{\leqslant s\}}, \widehat{V}_a^{\{\leqslant s-1\}})$$

having the following property:

Let N be an oriented submanifold of $\widehat{V}_b$, such that $N \subset \widehat{V}_b^{\{\leqslant s\}}$ and $N \setminus \mathrm{Int}\, \widehat{V}_b^{\{\leqslant s-1\}}$ is compact.

Then $N' = (-\widehat{v})^{\leadsto}_{[b,a]}(N)$ is a submanifold of $\partial_0 W \setminus B_a$ such that $N' \subset \widehat{V}_a^{\{\leqslant s\}}$ and $N' \setminus V_a^{\{\leqslant s-1\}}$ is compact, and the fundamental class of N' modulo $V_a^{\{\leqslant s-1\}}$ equals to $\widehat{\mathcal{H}}_s(-v)([N])$. $\square$

4.5. **Cyclic cobordisms and the condition ($\mathfrak{C}\mathcal{Y}$).** In this section we develop some techniques which will be used in §§5-7.

Definition 4.11.

A *cyclic cobordism* is a riemannian cobordism W together with an isometry $\Phi : \partial_0 W \to \partial_1 W$. △

Let $f : W \to [a, b]$ be a Morse function on a cyclic cobordism W of dimension n, v be an f-gradient. We say that v satisfies ($\mathfrak{C}\mathcal{Y}$) if v satisfies condition ($\mathfrak{C}$) (see Subsection 4.2), and, moreover, the Morse functions ϕ_1, ϕ_0 and their gradients u_1, u_0 from the definition 4.5 can be chosen so that $\phi_1 \circ \Phi = \phi_0, u_1 = (\Phi)_*(u_0)$. The set of all f-gradients satisfying ($\mathfrak{C}\mathcal{Y}$) will be denoted by $\mathcal{G}\mathfrak{C}\mathcal{Y}(f)$. The intersection $\mathcal{G}\mathfrak{C}\mathcal{Y}(f) \cap \mathcal{G}\mathcal{T}(f)$ will be denoted by $\mathcal{G}\mathfrak{C}\mathcal{Y}\mathcal{T}(f)$.

Theorem 4.12.

$\mathcal{G}\mathfrak{C}\mathcal{Y}(f)$ is open and dense in $\Gamma(f)$ with respect to C^0 topology. Moreover, if v_0 is any f-gradient then one can choose a C^0 small perturbation v of v_0 such that $v \in \mathcal{G}\mathfrak{C}\mathcal{Y}(f)$ and $v = v_0$ in a neighborhood of ∂W.

Proof. It goes exactly as the proof of Theorem 4.6 . To prove C^0-openness of the set $\mathcal{G}\mathfrak{C}\mathcal{Y}(f)$ we do not need to change whatever it be in the first part of the proof of Theorem 4.6 . For the proof of C^0 density return again to the proof of C^0 density of (RP). (Theorem 3.12). All what we have to notice is that we can choose $u_1 = \Phi_*(u_0), \xi_1 = \Phi_*(\xi_0), \phi_0 = \phi_1 \circ \Phi$. □

For an f-gradient v satisfying ($\mathfrak{C}\mathcal{Y}$) we can turn the operator of homological gradient descent to an endomorphism of an abelian group. Namely, set $h_s(-v) = \mathcal{H}_s(-v) \circ \Phi_*$; then $h_s(-v)$ is an endomorphism of $H_*((\partial_0 W)^{\{\leqslant s\}}, (\partial_0 W)^{\{\leqslant s-1\}})$.

5. Morse-type filtrations of cobordisms

Let W be a cobordism, $\phi : W \to [a,b]$ be an ordered Morse function on W, v be a ϕ-gradient. Assume that W is riemannian, and that v is δ-separated. We have considered in the subsection 4.1 the handle-like filtrations of W, associated to v. One can reconstruct the homology of $(W, \partial_0 W)$ from the homology of the successive factors of each of these filtrations. But we can not reconstruct the absolute homology of W from these factors. (Indeed, consider the following example: $W = V \times [0,1]$. Here all the terms of the filtration are homotopy equivalent to V, and the factors are contractible.) In this section we introduce another filtration of W, which is in a sense adjusted to given filtrations of $\partial_0 W$ and $\partial_1 W$, and from which one can reconstruct the homology of W. In order to construct such filtration one must impose the condition $(\mathfrak{C})$ on v.

5.1. **Definition of Morse-type filtration.** Let W be a riemannian cobordism of dimension n, $f : W \to [a,b]$ be a Morse function on W and v be an f-gradient. Assume that v satisfies condition $(\mathfrak{C})$. Recall from Subsection 4.2 that the condition $(\mathfrak{C})$ requires the existence of ordered Morse functions $\phi : W \to [a,b], \phi_1 : \partial_1 W \to \mathbf{R}, \phi_0 : \partial_0 W \to \mathbf{R}$, a δ-separated ϕ_0-gradient u_0 and a δ-separated ϕ_1-gradient u_1, satisfying the conditions (A), (B1), (B0) from Subsection 4.2. We obtain therefore three handle-like filtrations associated with the ordered Morse functions ϕ_0, ϕ_1 and ϕ: the filtration $(\partial_1 W)^{\{\leqslant s\}}$ of $\partial_1 W$, the filtration $(\partial_0 W)^{\{\leqslant s\}}$ of $\partial_0 W$ and the filtration $W^{\{\leqslant s\}}$ of W.

Recall the set

$$Y_k(v) = \Big(T((\partial_0 W)^{\{\leqslant k-1\}}, -v) \cup D(-v) \Big) \cap \phi^{-1}([a, a_{k+1}])$$

from the previous section (definition 4.9). In the present section we deal only with one f-gradient, so we shall abbreviate $Y_k(v)$ to Y_k.

Definition 5.1.
Let $A \subset W$ and $c, d \in [a,b]$. We denote $A \cap \phi^{-1}([c,d])$ by $A|_{[c,d]}$. For $\mu \in [a,b]$ we denote $A \cap \phi^{-1}(\mu)$ by$A|_\mu$. $\triangle$ $\triangle$

Set

$$Z_k = (T((\partial_1 W)^{\{\leqslant k-1\}}, v))|_{[a_{k+1}, b]} \tag{9}$$

Since $(-v)$ descends $(\partial_1 W)^{\{\leqslant k-1\}}$ to $\phi^{-1}(a_k)$ (this follows from $(\mathfrak{C})$) the set Z_k is homeomorphic to the product $(\partial_1 W)^{\{\leqslant k-1\}} \times [0,1]$.

Set for $k \geqslant 0$:

$$\begin{aligned} W^{\prec k \succ} &= (\partial_0 W)^{\{\leqslant k\}} \cup (\partial_1 W)^{\{\leqslant k\}} \cup Y_k \cup Z_k \\ W^{\langle k \rangle} &= (\partial_1 W)^{\{\leqslant k\}} \cup \big(Y_{k+1} \cap \phi^{-1}([a, a_{k+1}]) \big) \cup Z_k \end{aligned} \tag{10}$$

Both $\{W^{\langle k \rangle}\}_{k \in \mathbf{N}}$ and $\{W^{\prec k \succ}\}_{k \in \mathbf{N}}$ are filtrations of W, and $W^{\prec k \succ} \subset W^{\langle k \rangle}$. The set $W^{\langle k \rangle}$ is obtained from $W^{\prec k \succ}$ by adding all the points in $W_{[a, a_{k+1}]}$, lying on v-trajectories starting in $(\partial_0 W)^{\{\leqslant k\}} \setminus (\partial_0 W)^{\{\leqslant k-1\}}$.

(To visualize $W^{\langle k \rangle}$ and $W^{\langle k-1 \rangle}$ look at Figures 1 and 2 below.)

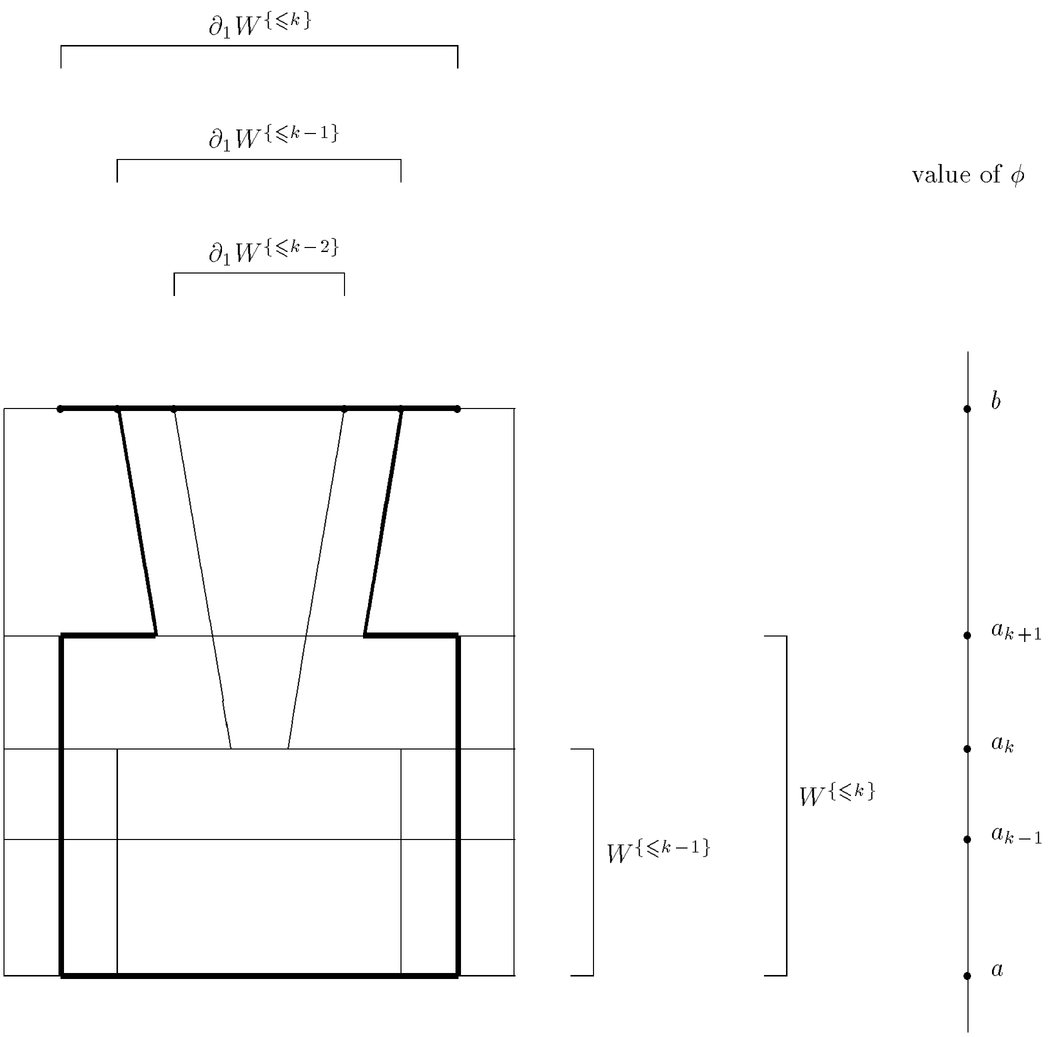

FIGURE 1: $W^{\langle k \rangle}$

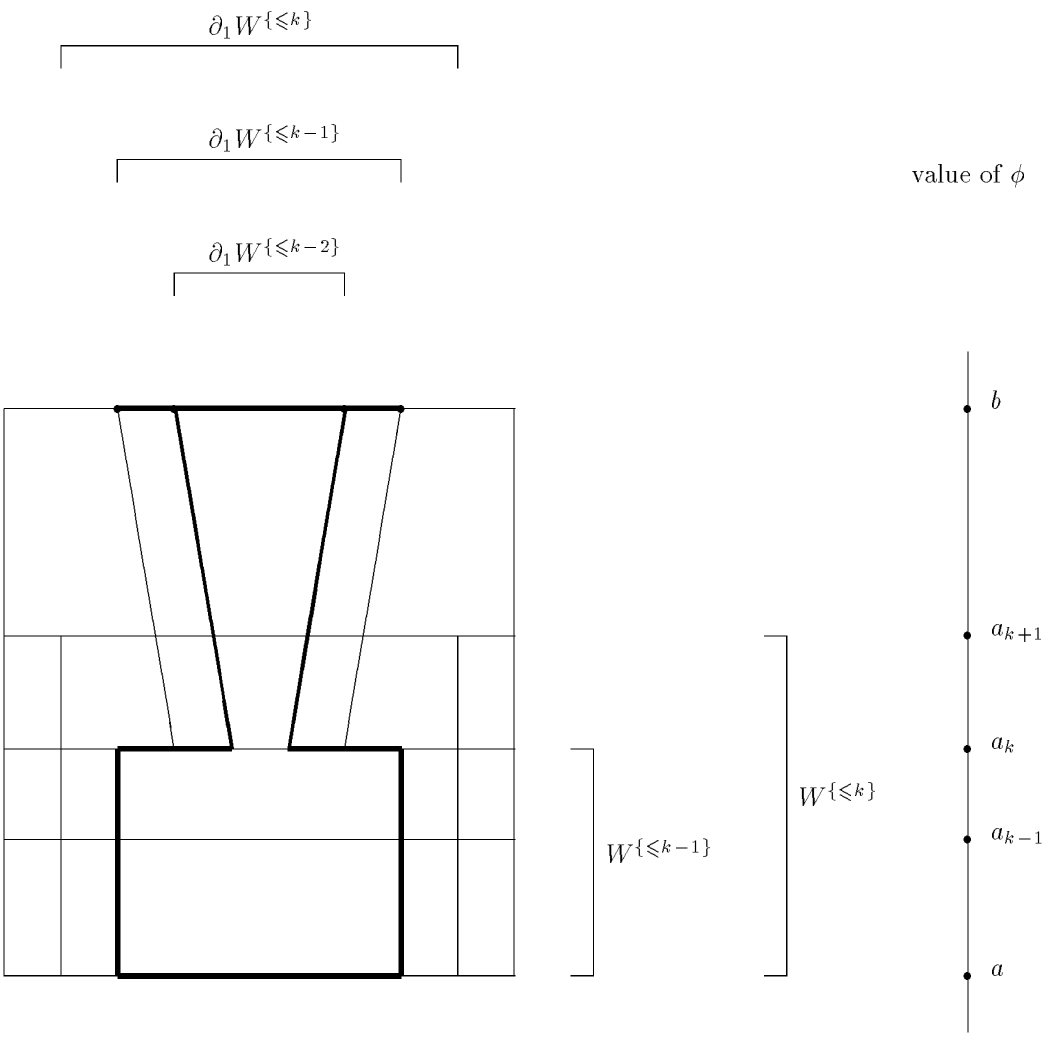

FIGURE 2: $W^{\langle k-1\rangle}$

Remark 5.2.

1. Here is an informal description of these filtrations. If we consider the descending discs $D(p,v)$ of critical points of f of index k as the "k-cells" of W, then $W^{\prec k\succ}$ contains all the cells of $\partial_0 W, \partial_1 W$ and W of dimensions $\leqslant k$, and the tracks of the cells of $\partial_1 W$ of dimension $\leqslant k-1$. The sets $W^{\prec k\succ}, W^{\langle k\rangle}$ are thickenings of this "skeleton". The condition ($\mathfrak{C}$) guarantees that the boundary of the "cells" corresponding to $D(p,v), p\in S(\phi)$ belongs to the union of the thickened cells of dimension $<$ ind$p$.
2. The two filtrations are homotopy equivalent, see Lemma 5.3 below. The set $W^{\prec k\succ}$ was introduced in [Pa5], and denoted there by $W^{\langle k\rangle}$. The filtration $\{W^{\langle k\rangle}\}_{k\in\mathbf{N}}$ has some technical advantages over $W^{\prec k\succ}$; that is why we have changed our notation and why we shall consider mainly the filtration $W^{\langle k\rangle}$ in the present paper.

△

We set $W^{\prec k\succ} = W^{\langle k\rangle} = \emptyset$ for $k<0$.

Lemma 5.3.
The inclusion $(W^{\prec k\succ}, W^{\prec k-1\succ}) \hookrightarrow (W^{\langle k\rangle}, W^{\langle k-1\rangle})$ *is a homotopy equivalence.*

Proof. We have

$$W^{\langle k\rangle}\setminus W^{\prec k\succ} = (Y_{k+1}\cap W^{\{\leqslant k\}})\setminus (Y_k\cup(\partial_0 W)^{\{\leqslant k\}}) \tag{11}$$

The function ϕ has no critical points in the domain $A=\phi^{-1}([a,a_{k+1}])\setminus Y_k$. Therefore there is $T>0$, such that for every $x\in A$ we have: $\gamma(x,T;-v)\in\partial_0 W\cup Y_k$. For $x\in W$ denote by $\tau(x)$ the moment when $\gamma(x,\cdot;-v)$ reaches $\partial_0 W$ (if the trajectory never reaches $\partial_0 W$, set $\tau(x)=\infty$.) Set

$$W_0^{\langle k\rangle} = W^{\langle k\rangle}\cap\phi^{-1}([a,a_{k+1}]) \tag{12}$$

$$W_0^{\prec k\succ} = W^{\prec k\succ}\cap\phi^{-1}([a,a_{k+1}]) \tag{13}$$

Define now a deformation H_t of the space $W_0^{\langle k\rangle}$ to itself (where $t\in[0,T]$) setting

$$H_t(x)=\begin{cases}\gamma(x,t;-v) & \text{if } t\leqslant\min(T,\tau(x))\\ \gamma(x,\min(T,\tau(x));-v) & \text{if } t\geqslant\min(T,\tau(x))\end{cases} \tag{14}$$

It is easy to see that $H_0=\mathrm{id}$, $H_T(W_0^{\langle k\rangle})\subset W_0^{\prec k\succ}$ and that $W_0^{\prec k\succ}$ is H_t-invariant for all $t\geqslant 0$. Therefore the inclusion $W_0^{\prec k\succ}\hookrightarrow W_0^{\langle k\rangle}$ is a homotopy equivalence.

Now it is not difficult to extend the deformation H_t to the whole of $W^{\langle k\rangle}$ (set $H_t(x)=x$ for $x\in\phi^{-1}([a_{k+1}+\epsilon,b])$ and glue appropriately two maps in the band $\phi^{-1}([a_{k+1},a_{k+1}+\epsilon])$. The extended deformation will provide the homotopy inverse to the inclusion $W^{\prec k\succ}\hookrightarrow W^{\langle k\rangle}$.

A homotopy inverse to the inclusion cited in the statement of the lemma is constructed in the same way (only one may need to choose T larger). We leave the details to the reader. □

Remark 5.4.
The method used in the proof will be referred to as $(-v)$-*shift*. We shall use it several times in this section. In particular, in the proof of the Lemma above we have applied the $(-v)$-shift in the domain $\phi^{-1}([a,a_{k+1}])\setminus Y_k$. △

5.2. **Homology of $\big(W^{\langle k\rangle}, W^{\langle k-1\rangle}\big)$.** Here we show that $W^{\langle k\rangle}$ form a cellular filtration of W, and compute the homology of $\big(W^{\langle k\rangle}, W^{\langle k-1\rangle}\big)$. The remark 5.2 suggests that for $*\neq k$ the group $H_*(W^{\langle k\rangle}, W^{\langle k-1\rangle})$ vanishes and that for $*=k$ this group is free abelian with the base formed by the homology classes of the corresponding cells. In this subsection we show that it is indeed so.

To compute $H_*(W^{\langle k\rangle}, W^{\langle k-1\rangle})$ we shall first define four homomorphisms with values in $H_k(W^{\langle k\rangle}, W^{\langle k-1\rangle})$. We shall prove in Theorem 5.5 that their images form a direct sum decomposition of this group.

1. For every s the sets $(\partial_0 W)^{\{\leqslant s\}}, (\partial_1 W)^{\{\leqslant s\}}$ are subsets of $W^{\langle s\rangle}$. Therefore there are homomorphisms, induced by the corresponding inclusions:

$$C_k(u_0) = H_k\Big((\partial_0 W)^{\{\leqslant k\}}, (\partial_0 W)^{\{\leqslant k-1\}}\Big) \xrightarrow{\mathcal{J}_0} H_k(W^{\langle k\rangle}, W^{\langle k-1\rangle}) \tag{15}$$

$$C_k(u_1) = H_k\Big((\partial_1 W)^{\{\leqslant k\}}, (\partial_1 W)^{\{\leqslant k-1\}}\Big) \xrightarrow{\mathcal{J}_1} H_k(W^{\langle k\rangle}, W^{\langle k-1\rangle}) \tag{16}$$

2. Every $(-v)$-trajectory starting in $(\partial_1 W)^{\{\leqslant k-1\}}$ reaches $\phi^{-1}(a_k)$. Therefore there is a continuous map

$$G_k : (\partial_1 W)^{\{\leqslant k-1\}} \times [a_k, b] \to \phi^{-1}\big([a_k, b]\big)$$

which is a homeomorphism onto its image and for every x the curve $t\mapsto G_k(x,t)$ is a reparameterized v-trajectory, and $\phi(G_k(x,t)) = t$. Set $I_k = [a_k, b]$. We obtain a map of pairs

$$G_k : \Big((\partial_1 W)^{\{\leqslant k-1\}}, (\partial_1 W)^{\{\leqslant k-2\}}\Big) \times (I_k, \partial I_k) \to (W^{\langle k\rangle}, W^{\langle k-1\rangle})$$

Define a homomorphism

$$\mathcal{S} : C_{k-1}(u_1) \to H_*(W^{\langle k\rangle}, W^{\langle k-1\rangle}) \tag{17}$$

by the formula $\mathcal{S}(x) = (G_k)_*(x\otimes\iota)$ (where $\otimes$ is the Künneth product and ι is the fundamental class of $(I_k, \partial I_k)$).

3. Let $p\in S_k(\phi)$. Recall that we denote by $d(p)$ the embedded disc $D(p,v)\cap \phi^{-1}\big([a_k, b]\big)$. The condition $(\mathfrak{C})$ implies that $\partial d(p)\subset W^{\langle k-1\rangle}$. We obtain thus a homomorphism

$$C_k(v) = H_k\big(W^{\{\leqslant k\}}, W^{\{\leqslant k-1\}}\big) \xrightarrow{\mathcal{I}} H_k(W^{\langle k\rangle}, W^{\langle k-1\rangle}) \tag{18}$$

Theorem 5.5.

1. $H_*(W^{\langle k\rangle}, W^{\langle k-1\rangle}) = 0$ *if* $*\neq k$
2. *The homomorphism*

$$L_k = (\mathcal{J}_1, \mathcal{J}_0, \mathcal{I}, \mathcal{S}) : C_k(u_1)\oplus C_k(u_0)\oplus C_k(v)\oplus C_{k-1}(u_1) \to H_k(W^{\langle k\rangle}, W^{\langle k-1\rangle}) \tag{19}$$

is an isomorphism.

Remark 5.6.

This theorem implies in particular that $H_k(W^{\langle k\rangle}, W^{\langle k-1\rangle})$ is a free abelian group of rank $\sharp S_k(\phi_1) + \sharp S_k(\phi_0) + \sharp S_k(\phi) + \sharp S_{k-1}(\phi_1)$

$\triangle$

Proof. We fix the value of k for the proof. Set

$$a_k = \lambda, a_{k+1} = \mu, Z = T((\partial_1 W)^{\{\leqslant k-1\}}, v)|_{[\lambda,b]}$$

Further, set

$$\Delta = \cup_{p\in S_k(\phi)} d(p), \tag{20}$$

$$R = Z \cup \Delta \cup (\partial_1 W)^{\{\leqslant k\}} \cup Y_{k+1}|_\lambda, \tag{21}$$

$$S = W^{\langle k-1\rangle}|_{[\lambda,b]} \tag{22}$$

so that $S \subset R \subset \phi^{-1}([\lambda, b])$. The next lemma reduces the study of the homology of $(W^{\langle k\rangle}, W^{\langle k-1\rangle})$ to the study of the intersection of this pair with $\phi^{-1}([\lambda, b])$.

Lemma 5.7.
The inclusions of pairs

$$(R, S) \hookrightarrow (W^{\langle k\rangle}|_{[\lambda,b]}, W^{\langle k-1\rangle}|_{[\lambda,b]}) \hookrightarrow (W^{\langle k\rangle}, W^{\langle k-1\rangle}) \tag{23}$$

$$(\Delta \cup (Y_{k+1}|_{[a_k]}), Y_k|_{[a_k]}) \hookrightarrow (W^{\langle k\rangle}|_{[\lambda,\mu]}, W^{\langle k-1\rangle}|_{[\lambda,\mu]}) \tag{24}$$

induce isomorphisms in homology.

Proof. We shall prove that (24) induces an isomorphism in homology . The proof for the (23) is similar. Using the $(-v)$-shift in the domain $W^{\langle k\rangle}|_{[\lambda,\mu]} \setminus W^{\langle k-1\rangle}|_{[\lambda,\mu]}$ it is easy to prove that the inclusion

$$(Y_k|_{[\lambda,\mu]} \cup Y_{k+1}|_\lambda, Y_k|_\lambda) \hookrightarrow (W^{\langle k\rangle}|_{[\lambda,\mu]}, W^{\langle k-1\rangle}|_{[\lambda,\mu]})$$

is a homology equivalence. The intersection of the sets $Y_k|_{[\lambda,\mu]}$ and $Y_{k+1}|_\lambda$ is in $Y_k|_\lambda$, thus the quotient $(Y_k|_{[\lambda,\mu]} \cup Y_{k+1}|_\lambda)/Y_k|_\lambda$ is the wedge

$$Y_k|_{[\lambda,\mu]}/Y_k|_\lambda \bigvee Y_{k+1}|_\lambda/Y_k|_\lambda$$

Use again the $(-v)$-shift in the domain $\phi^{-1}([\lambda,\mu]) \setminus Y_k|_{[\lambda,\mu]}$ and then excision to see that $Y_k|_{[\lambda,\mu]}/Y_k|_\lambda \hookrightarrow \phi^{-1}([\lambda,\mu])/\phi^{-1}(\lambda)$ is a homology equivalence. The quotient $(\Delta \cup Y_{k+1}|_\lambda)/Y_k|_\lambda$ is homeomorphic to the wedge

$$(\Delta/\partial\Delta) \vee (Y_{k+1}|_\lambda)/Y_k|_\lambda)$$

and the standard Morse-theoretic argument identifying the homotopy type of $(\phi^{-1}([\lambda,\mu]), \phi^{-1}(\lambda))$ with that of $(\Delta, \partial\Delta)$ finishes the proof. $\square$

So it remains to compute the homotopy type of R/S. Set

$$\rho = R|_\lambda, \quad \sigma = S|_\lambda$$

Note that R is obtained from S by attaching four compact subsets: $\phi_1^{-1}([\alpha_k, \alpha_{k+1}]), G_k(\phi_1{}^{-1}([\alpha_{k-1}, \alpha_k]))\times [\lambda, b]), \Delta, \rho$. Every two of these subsets intersect by a subset of S, therefore

$$R/S = ((\partial_1 W)^{\{\leqslant k\}}/(\partial_1 W)^{\{\leqslant k-1\}}) \vee \Sigma((\partial_1 W)^{\{\leqslant k-1\}}/(\partial_1 W)^{\{\leqslant k-2\}}) \vee ((\Delta\cup\rho)/\sigma)$$

where Σ is suspension. Now we shall describe $(\Delta \cup \rho)/\sigma$. Since $\Delta \cap \rho \subset \sigma$, we have a homeomorphism

$$(\Delta/\partial\Delta) \vee (\rho/\sigma) \xrightarrow[\approx]{(i_1,i_2)} (\Delta\cup\rho)/\sigma \tag{25}$$

The space $\Delta/\partial\Delta$ is homeomorphic to the wedge of all the spaces $d(p)/\partial d(p)$ and ρ/σ is homeomorphic (via the map $(-v)^{\leadsto}_{[\lambda,a]}$) to $\big((\partial_0 W)^{\{\leqslant k\}}/(\partial_0 W)^{\{\leqslant k-1\}}\big)$. Thus we obtain the following homeomorphism:

$$\begin{aligned} R/S \approx \big((\partial_1 W)^{\{\leqslant k\}}/(\partial_1 W)^{\{\leqslant k-1\}}\big) \vee \Sigma\big((\partial_1 W)^{\{\leqslant k-1\}}/(\partial_1 W)^{\{\leqslant k-2\}}\big) \vee \\ \Big(\bigvee_{p\in S_k(\phi)} d(p)/\partial d(p)\Big) \vee \big((\partial_0 W)^{\{\leqslant k\}}/(\partial_0 W)^{\{\leqslant k-1\}}\big) \end{aligned} \tag{26}$$

The homology of this last wedge is isomorphic to

$$C_k(u_1)\oplus C_k(u_0)\oplus C_k(v)\oplus C_{k-1}(u_1)$$

It is clear that the images of the direct summands of this group in $H_k(W^{\langle k\rangle}, W^{\langle k-1\rangle})$ equal to $\operatorname{Im}\mathcal{J}_1$, resp. $\operatorname{Im}\mathcal{J}_0, \operatorname{Im}\mathcal{I}, \operatorname{Im}\mathcal{S}$. □

Remark 5.8.
Note here for the further use a byproduct of our proof: the inclusion

$$(\Delta/\partial\Delta)\vee(\rho/\sigma) \xhookrightarrow{j} W^{\langle k\rangle}|_{[\lambda,\mu]}/W^{\langle k-1\rangle}|_{[\lambda,\mu]} \tag{27}$$

is a homology equivalence, and therefore it induces an isomorphism

$$J : C_*(v)\oplus C_*(u_0) \xrightarrow{\approx} H_*(W^{\langle k\rangle}|_{[\lambda,\mu]}, W^{\langle k-1\rangle}|_{[\lambda,\mu]}) \tag{28}$$

△

5.3. **Boundary operators, associated with Morse-type filtrations.** We have seen in the previous subsection, that $W^{\langle k\rangle}$ form a cellular filtration of W. Since $W^{\langle k\rangle}=\emptyset$ for $k<0$, the homology of W can be reconstructed from the chain complex, associated with the filtration. Namely, let $E_k = H_k(W^{\langle k\rangle}, W^{\langle k-1\rangle})$. The Proposition 1.3 of [Do], Ch. 5 says that $H_*(W)$ is isomorphic to the homology of the complex (E_*, d_*), where $d_k : E_k\to E_{k-1}$ is the boundary operator of the exact sequence of the triple $\big(W^{\langle k\rangle}, W^{\langle k-1\rangle}, W^{\langle k-2\rangle}\big)$. In this subsection we compute the matrix of this boundary operator with respect to the direct sum decomposition of E_* provided by Theorem 5.5 . Set $d'_s = L_s^{-1}\circ D_s\circ L_{s+1}$. Then d'_s is given by its (4×4)-matrix of group homomorphisms. To describe the components of this matrix, recall that $C_*(u_1), C_*(u_0), C_*(v)$ are themselves chain complexes, and denote the corresponding boundary operators by $\partial^{(1)}_*, \partial^{(0)}_*, \partial_*$.

Proposition 5.9.
The matrix of d'_{k+1} is

$$\begin{pmatrix} \partial^{(1)}_{k+1} & 0 & 0 & \mathrm{Id} \\ 0 & \partial^{(0)}_{k+1} & P_k & -\mathcal{H}_k(-v) \\ 0 & 0 & \partial_{k+1} & -N_k \\ 0 & 0 & 0 & -\partial^{(1)}_k \end{pmatrix} \tag{29}$$

Here $\mathcal{H}_k(-v)$ is the operator of homological gradient descent, (see Theorem 4.8) and P_k, N_k are some homomorphisms, which will be defined during the proof.

Proof. We shall use in the proof the terminology of Subsections 5.1 and 5.2.

The first two columns

They are obtained from the definition of $\mathcal{J}_0, \mathcal{J}_1$ by the functoriality of the boundary operators associated with filtrations.

The third column

By Lemma 5.7 it suffices to calculate the homology class of $\partial d(p)$ in $\big(W^{\langle k\rangle}|_{[\lambda,b]}, W^{\langle k-1\rangle}|_{[\lambda,b]}\big)$. Since $\partial d(p)$ belongs to $\phi^{-1}(\mu)$, it suffices to calculate the homology class of $\partial d(p)$ in $\big(W^{\langle k\rangle}|_{[\lambda,\mu]}, W^{\langle k-1\rangle}|_{[\lambda,\mu]}\big)$. The homology of this pair is isomorphic to $C_k(v)\oplus C_k(u_0)$ (by Remark 5.8). The projection of $[\partial d(p)]$ to the first component of this direct sum is easily identified with $\partial_{k+1}([p])$. Denote by P_k the projection onto the second component, and the calculation of the third column is finished.

Homomorphism N_k

Let $x\in C_k(u_1) = H_*((\partial_1 W)^{\{\leqslant k\}}, (\partial_1 W)^{\{\leqslant k-1\}})$.
Choose a singular chain $\bar x$, representing x. The condition $(\mathfrak{C})$ implies that $(-v)$ descends $\bar x$ to $\phi^{-1}(\mu)$; consider the descended singular chain $\bar{\bar x}$ as a cycle in the pair

$$\big(W^{\langle k\rangle}|_{[\lambda,\mu]}, W^{\langle k-1\rangle}|_{[\lambda,\mu]}\big)$$

We denote by $N_k(x)$ the projection of $J^{-1}([x])$ to the first direct summand $C_*(v)$ of (28).

Recall that $C_k(v)$ is the free abelian group generated by $[p], p\in S_k(\phi)$. Write $N_k(x) = \sum_{p\in S_k(\phi)}\langle x,p\rangle[p]$. Then the map $x\mapsto\langle x,p\rangle$ is a homomorphism $C_k(u_1)\to\mathbf{Z}$. This homomorphism will be useful in the sequel, and now we shall give an interpretation of the number $\langle x,p\rangle$ when the following restriction on x holds:

x is represented by an oriented submanifold X of $\partial_1 W$ belonging to Int $(\partial_1 W)^{\{\leqslant k\}}$ and transversal to $D(p,-v)\cap\partial_1 W$.

Consider the cooriented $(n-k-1)$-dimensional submanifold $B_p = D(p,-v)\cap\partial_1 W$ of $\partial_1 W$. The condition $(\mathfrak{C})$ implies that $B_p\subset\phi_1^{-1}([\alpha_k,\alpha_n])$. We have the following formula for the algebraic intersection index of X and B_p:

$$\langle x,p\rangle = X\#B_p \tag{30}$$

Thus

$$N_k(x) = \sum_{p\in S_k(\phi)} (X\#B_p)[p] \tag{31}$$

The fourth column

Let $x\in C_k(u_1) = H_k((\partial_1 W)^{\{\leqslant k\}}, (\partial_1 W)^{\{\leqslant k-1\}})$. For notational simplicity we shall assume that x is represented by an oriented submanifold X of $\partial_1 W$, $\dim X = k$, such that $X\subset(\partial_1 W)^{\{\leqslant k\}}, \partial X\subset(\partial_1 W)^{\{\leqslant k-1\}}$. Then it follows from the definition of $\mathcal{S}(x)$, see Subsection 5.2, that $d_{k+1}(\mathcal{S}(x))$ is represented by the following sum of oriented singular manifolds with boundary :

$$X - G_k(\partial X\times[a_{k+1},b]) - G_k(X\times\{a_{k+1}\}) \tag{32}$$

The first term corresponds to the top of the 4th column. The second and the third terms of (32) do not represent cycles of $(W^{\langle k\rangle}, W^{\langle k-1\rangle})$. We can repair this as follows. Every $(-v)$- trajectory starting in ∂X reaches $\phi^{-1}(a_k)$, and we can write

$$G_k(\partial X\times[a_{k+1},b]) = G_k\big(\partial X\times[a_k,b]\big) - G_k\big(\partial X\times[a_k,a_{k+1}]\big) \tag{33}$$

and then $d_{k+}(\mathcal{S}(x))$ is represented by the sum

$$X - G_k\big(\partial X \times [a_k, b]\big) + Y \tag{34}$$

where $Y = G_k\big(\partial X \times [a_k, a_{k+1}]\big) - G_k\big(\partial X \times \{a_{k+1}\}\big)$. Now the two last terms in the righthand side of (34) represent cycles in $(W^{\langle k\rangle}, W^{\langle k-1\rangle})$. The homology class of $G_k\big(\partial X \times [a_k, b]\big)$ equals obviously to $\mathcal{S}(\partial_k^{(1)} x)$. It is not difficult to check that $\mathcal{L}_k^{-1}(Y)$ equals to the element $-N_k(x) - \mathcal{H}_k(-v)(x)$ and that finishes the computation of the fourth column. □

5.4. **Morse-type filtrations of regular coverings of cobordisms.** In the preceding subsection we have seen that it is possible to calculate the homology $H_*(W)$ from a chain complex E_*, associated to the filtration $W^{\langle k\rangle}$. In the present subsection we refine this result and show how to recover the simple homotopy type of W from the homological data associated to the filtration.

Let $p : \widetilde{W} \to W$ be a regular covering of W with structure group H. For $A \subset W$ we shall denote $p^{-1}(A)$ by $\widetilde{A}$.

The Morse type filtration $W^{\langle k\rangle}$ constructed in the previous subsection gives rise to the filtration $\widetilde{W}^{\langle k\rangle} = p^{-1}(W^{\langle k\rangle})$ of $\widetilde{W}$. An immediate generalization of the results of the previous subsection leads to the following result, describing the structure of relative homology $H_*(\widetilde{W}^{\langle k\rangle}, \widetilde{W}^{\langle k-1\rangle})$.

Proposition 5.10.

1. *$H_s(\widetilde{W}^{\langle k\rangle}, \widetilde{W}^{\langle k-1\rangle}) = 0$, if $s \neq k$.*
2. *Set $\widetilde{E}_k = H_k(\widetilde{W}^{\langle k\rangle}, \widetilde{W}^{\langle k-1\rangle})$. Then $\widetilde{E}_k$ is a free $\mathbf{Z}H$-module and there is an isomorphism of $\mathbf{Z}H$-modules*

$$\widetilde{L}_k : \widetilde{C}_k(u_1) \oplus \widetilde{C}_k(u_0) \oplus \widetilde{C}_k(v) \oplus \widetilde{C}_{k-1}(u_1) \to \widetilde{E}_k \tag{35}$$

3. *Let $\widetilde{d}_{k+1} : \widetilde{E}_{k+1} \to \widetilde{E}_k$ be the boundary operator in the exact sequence of the triple $(\widetilde{W}^{\langle k+1\rangle}, \widetilde{W}^{\langle k\rangle}, \widetilde{W}^{\langle k-1\rangle})$. Then the matrix of the homomorphism $\widetilde{d}'_{k+1} = (\widetilde{L}_k)^{-1} \circ \widetilde{d}_{k+1} \circ \widetilde{L}_{k+1}$ is*

$$\begin{pmatrix} \widetilde{\partial}^{(1)}_{k+1} & 0 & 0 & \mathrm{Id} \\ 0 & \widetilde{\partial}^{(0)}_{k+1} & \widetilde{P}_k & -\widetilde{\mathcal{H}}_k(-v) \\ 0 & 0 & \widetilde{\partial}_{k+1} & -\widetilde{N}_k \\ 0 & 0 & 0 & -\widetilde{\partial}^{(1)}_k \end{pmatrix} \tag{36}$$

(Here $\widetilde{\mathcal{H}}_k(-v) : \widetilde{C}_k(u_1) \to \widetilde{C}_k(u_0)$ is the operator of homological gradient descent associated to the covering $\widetilde{W} \to W$, see Subsection 4.4.)

Proof. The proof runs parallel to the proof of Proposition 5.9 . We shall develop here only the points which are of use in the sequel, namely the construction of chains representing generators of $\widetilde{E}_k$, and the construction and properties of the homomorphism $\widetilde{N}_k$.

Generators of $\widetilde{E}_k$

To start, we choose for every point $x \in S(\phi) \cup S(\phi_0) \cup S(\phi_1)$ a lifting $\widetilde{x}$ of x to $\widetilde{W}$. Then for every $r \in S_k(\phi)$ a lifting $d(\widetilde{r})$ of the disc $d(r)$ to $\widetilde{W}$ is defined. We have obviously $d(\widetilde{r}) \subset \widetilde{W}^{\langle k\rangle}, \partial d(\widetilde{r}) \subset \widetilde{W}^{\langle k-1\rangle}$. We define $\widetilde{L}_k([\widetilde{r}])$ to be the image in $H_*(\widetilde{W}^{\langle k\rangle}, \widetilde{W}^{\langle k-1\rangle})$ of the fundamental class of the pair $(d(\widetilde{r}), \partial d(\widetilde{r}))$.

Similarly we define $\widetilde{L}_k$ on the elements $[\widetilde{q}], [\widetilde{t}]$, where $q \in S_k(\phi_1), t \in S_k(\phi_0)$.

Proceeding to the image of the generators of the fourth direct summand, note that the lifting $\widetilde{q}$ defines also a lifting T_k of the manifold $G_k(d(q) \times I_k)$ to $\widetilde{W}$. The space T_k is an oriented topological manifold with boundary, thus the fundamental class in $H_k(T_k, \partial T_k)$ is defined. Denote its image in $H_k(\widetilde{W}^{\langle k\rangle}, \widetilde{W}^{\langle k-1\rangle})$ by $[[\widetilde{q}]]$ and define $\widetilde{L}_k$ on the component $\widetilde{C}_{k-1}(u_1)$ of the lefthand side of (35) setting $\widetilde{L}_k([\widetilde{q}]) = [[\widetilde{q}]]$.

Homomorphism $\widetilde{N}_k$

Let $x \in \widetilde{C}_k(u_1) = H_k((\partial_1 \widetilde{W})^{\{\leqslant k\}}, (\partial_1 \widetilde{W})^{\{\leqslant k-1\}})$. Choose a singular chain $\bar{x}$ representing x. The condition $(\mathfrak{C})$ implies that $(-v)$ descends $\bar{x}$ to $\phi^{-1}(a_{k+1})\widetilde{}$. Consider the descended singular chain $\bar{\bar{x}}$ as a cycle in the pair

$$\left(\widetilde{W}^{\langle k\rangle}|_{[\lambda,\mu]}, \widetilde{W}^{\langle k-1\rangle}|_{[\lambda,\mu]}\right)$$

Similarly to Lemma 5.7 the inclusion

$$((\Delta \cup Y_{k+1}|_{a_k})\widetilde{}, \widetilde{Y}_k|_{a_k}) \hookrightarrow (\widetilde{W}^{\langle k\rangle}|_{[\lambda,\mu]}, \widetilde{W}^{\langle k-1\rangle}|_{[\lambda,\mu]})$$

induces an isomorphism in homology:

$$\widetilde{J} : \widetilde{C}_k(v) \oplus \widetilde{C}_{k-1}(u_0) \longrightarrow H_*(\widetilde{W}^{\langle k\rangle}|_{[\lambda,\mu]}, \widetilde{W}^{\langle k-1\rangle}|_{[\lambda,\mu]})$$

Denote by $\widetilde{N}_k(x)$ the projection of $\widetilde{J}^{-1}([\bar{x}])$ to the first summand. We have an equivariant analog of the formulas (30) and (31). Namely, for every $p \in S_k(\phi)$ choose a lifting $\widetilde{p} \in \widetilde{M}$. Recall that the homology classes $[\widetilde{p}]$ of the pairs $(d(\widetilde{p}), \partial d(\widetilde{p}))$ form a base of the module $\widetilde{C}_k(v)$. Write

$$\widetilde{N}_k(x) = \sum_{p \in S_k(\phi)} [\widetilde{p}]\langle\langle x, p\rangle\rangle$$

with $\langle\langle x, p\rangle\rangle \in \mathbf{Z}H$; then the map $x \to \langle\langle x, p\rangle\rangle$ is a homomorphism of $\mathbf{Z}H$-modules $\widetilde{C}_k(u_1) \to \mathbf{Z}H$. Assume now that x is represented by a lifting $\widetilde{X}$ to $\partial_1 \widetilde{W}$ of an oriented submanifold X of $\partial_1 W$ belonging to Int $(\partial_1 W)^{\{\leqslant k\}}$ and transversal to $B_p = D(p, -v) \cap \partial_1 W$. We have a lifting $\widetilde{B}_p$ of the submanifold $B_p \subset \partial_1 W$, and for every $g \in H$ the intersection index $\widetilde{X} \# \widetilde{B}_p g \in \mathbf{Z}$ is defined. We have:

$$\langle\langle x, p\rangle\rangle = \sum_{g \in H} \left(\widetilde{X} \# \widetilde{B}_p g\right) g \tag{37}$$

and thus

$$\widetilde{N}_k(x) = \sum_{p \in S_k(\phi), g \in H} \left(\widetilde{X} \# \widetilde{B}_p g\right) [\widetilde{p}] g \qquad \square \tag{38}$$

Thus the chain complex $\{... \to \widetilde{E}_k \xrightarrow{\widetilde{d}_k} \widetilde{E}_{k-1} \to ...\}$ is a free $\mathbf{Z}H$-complex. The point (1) of the above proposition shows that the filtration $\widetilde{W}^{\langle k\rangle}$ is cellular, therefore $H_*(\widetilde{E}_*) \approx H_*(\widetilde{W})$ (and it is not difficult to show that this isomorphism can be chosen as to commute with the $\mathbf{Z}H$-action).

We can say more. Choose a C^1 triangulation Δ of W and lift it to $\widetilde{W}$ to obtain a H-invariant triangulation of $\widetilde{W}$. Then the chain simplicial complex $C_*^\Delta(\widetilde{W})$ of $\widetilde{W}$ is a free chain complex of finitely generated $\mathbf{Z}H$-modules. Moreover, choose some liftings of simplices of W to $\widetilde{W}$, and $C_*^\Delta(\widetilde{W})$ becomes a based chain complex.

On the other hand, if we choose orientations of descending discs, and liftings to $\widetilde{W}$ of all the critical points of ϕ_0, ϕ_1, ϕ, then we obtain a $\mathbf{Z}H$-base in each of the chain complexes $\widetilde{C}(u_1), \widetilde{C}_*(u_0), \widetilde{C}_*(v)$ and (via the isomorphism $\widetilde{L}_k$) a free $\mathbf{Z}H$-base in $\widetilde{E}_*$.

To state the next proposition we need a definition.

Definition 5.11.

1. Let R be a ring. An *R-complex* is a finite chain complex of finitely generated free based right R-modules.
2. Two $\mathbf{Z}H$-complexes C_*, D_* are said to be simply homotopy equivalent if there is a homotopy equivalence $\phi : C_* \to D_*$ such that the torsion $\tau(\phi) \in Wh(\mathbf{Z}H)$ vanishes.

$\triangle$

Proposition 5.12.
The $\mathbf{Z}H$-complexes $C_*(\widetilde{W})$ and $\widetilde{E}_*$ are simply homotopy equivalent.

Proof. For the case of closed manifolds (that is $\partial W = \emptyset$) this proposition is proved in [Pa1], Appendix, Theorem A.5. In the general case the proof is similar. We shall just give the outline of the proof. Choose a C^1 triangulation of W so that it satisfies the following condition $(\mathcal{T})$:

$(\mathcal{T})$:
All the subspaces $\partial_0 W, \partial_1 W$ are simplicial subspaces of W. All the level surfaces $\phi^{-1}(a_i), \phi_1^{-1}(\alpha_i), \phi_0^{-1}(\beta_i)$ and the discs $d(p), d(q), d(r)$, where $p \in S(\phi), q \in S(\phi_0), r \in S(\phi_1)$ are simplicial subcomplexes of W. For every k the subspaces $W^{\langle k \rangle}$ and the subspaces R, S defined in (20) and (21) are simplicial subcomplexes of W.

(The existence of such triangulation is a standard exercise in the theory of triangulating of manifolds, see [Pa1], p.330 – 332 for a detailed proof of a similar result).

In what follows we assume that the reader is familiar with the paper [Pa1]. To adjust our present terminology to that of of §3 of [Pa1], abbreviate $C_*^\Delta(\widetilde{W})$ to C_*, and denote $C_*^\Delta(\widetilde{W}^{\langle k \rangle})$ by $C_*^{(k)}$; then the $\mathbf{Z}H$-complex $\widetilde{E}_*$ defined above is just C_*^{gr} in the terminology of [Pa1]. The proposition 5.10 implies that the filtration $C_*^{(k)}$ is *nice* (see [Pa1], page 310 for the definition of nice filtration). Therefore ([Pa1], Corollary 3.4), there is a canonical chain homotopy equivalence $\xi : C_*^{gr} \to C_*$. The homotopy class of this chain homotopy equivalence is uniquely determined by the following requirement:

- F1 $\xi(C_k^{gr}) \subset C_*^{(k)}$ for every k,
- F2 The homomorphism $C_k^{gr} \to H_k(C_*^{(k)}, C_*^{(k-1)})$ induced by ξ equals to the identity.

Define a $\mathbf{Z}G$-complex $\mu(k)_*$ setting

$$\begin{cases} \mu(k)_j = 0 & \text{if } j \neq k \\ \mu(k)_k = H_k(C_*^{(k)}, C_*^{(k-1)}) \end{cases} \tag{39}$$

The chain map ξ gives rise to chain homotopy equivalences

$$\xi_k : \mu(k)_* \to C_*^{(k)}/C_*^{(k-1)} \tag{40}$$

One proves by an easy induction argument that if all maps ξ_k are simple homotopy equivalences, then ξ is also a simple homotopy equivalence.

Thus it remains to show that every ξ_k is a simple homotopy equivalence. This is done in three steps.

First step: one shows that the inclusion $(R, S) \hookrightarrow (W^{\langle k\rangle}, W^{\langle k-1\rangle})$ induces a simple homotopy equivalence of $\mathbf{Z}H$-complexes

$$j : C_*(\widetilde{R}, \widetilde{S}) \to C_*(\widetilde{W^{\langle k\rangle}}, \widetilde{W^{\langle k-1\rangle}}) = C_*^{(k)}/C_*^{(k-1)} \tag{41}$$

(the argument is parallel to the proof of Lemma 5.7).

Second step: The map ξ_k factors through a homotopy equivalence $\xi'_k : \mu(k)_* \to C_*(\widetilde{R}, \widetilde{S})$. The homology of $L_* = C_*(\widetilde{R}, \widetilde{S})$ vanishes except in the dimension k and the module $H_k(\widetilde{R}, \widetilde{S}) \approx \widetilde{E}_k$ is free. Thus, introducing the trivial filtration $\{0, L_*\}$ in L_* we shall obtain a chain homotopy equivalence $\xi''_k : \mu(k)_* \to L_*$ (by [Pa1], Corollary 3.4) and it is not difficult to see that ξ''_k is homotopic to ξ'_k.

Third step: One proves that ξ''_k splits as a sum of four homotopy equivalences corresponding to the splitting of R/S into the wedge of four summands of (26), and one proves that each of these equivalences is simple. $\square$

6. Algebraic lemmas.

We shall use here the terminology from Subsection 1.2 and we introduce some more. We shall say that ϕ is an *U-simple homotopy equivalence* if $\tau(\phi|U)$ vanishes. If the value of U is clear from the context we abbreviate *U-simple homotopy equivalence* to *simple homotopy equivalence.*

Note that if each $\phi_i : F_i \to D_i$ is an isomorphism, then $\tau(\phi) = \sum_i (-1)^i [\phi_i]$ [5]

6.1. **A lemma on cone-like chain complexes.** Let C_* be an R-complex, let $d_k : C_k \to C_{k-1}$ be its boundary homomorphisms. Set $E_k = C_k \oplus C_{k-1}$. Assume that for each k an isomorphism $A_k : C_k \to C_k$ and a homomorphism $d'_k : C_k \to C_{k-1}$ are given. Let $\mathcal{D}_k : E_k \to E_{k-1}$ be the homomorphism, with the matrix $\begin{pmatrix} d_k & A_{k-1} \\ 0 & d'_{k-1} \end{pmatrix}$

Assume that $\mathcal{D}_{k-1} \circ \mathcal{D}_k = 0$, so that $(E_*, \mathcal{D}_*)$ is a chain complex.

Lemma 6.1.

1. *E_* is chain homotopy equivalent to 0.*
2. $\tau(E_*) = \sum_i (-1)^{i+1} [A_i]$

Proof. Consider the cone L_* of the identity map $\mathrm{Id} : C_* \to C_*$; we have: $L_k = C_k \oplus C_{k-1}$, and the matrix of the boundary operator $\Delta_k : L_k \to L_{k-1}$ is $\begin{pmatrix} d_k & \mathrm{Id} \\ 0 & -d_{k-1} \end{pmatrix}$.

We have of course $\tau(L_*) = 0$. Now the condition for $\mathcal{D}_k$ to be boundary operator is equivalent to the conjunction of two following conditions:

1. d'_k is boundary operator
2. $d_k A_k + A_{k-1} d'_k = 0$ for every k.

Define a map $\lambda : E_* \to L_*$ by $(c_k, c_{k-1}) \mapsto (c_k, A_{k-1} c_{k-1})$ A simple computation shows that λ is an isomorphism of complexes. Therefore E_* is chain homotopy equivalent to 0, and $\tau(L_*) = \tau(E_*) + \tau(\lambda) = \tau(\lambda)$. But $\tau(\lambda) = \sum_i (-1)^{i+1} [A_i]$ $\square$

6.2. **The ring $A[t]$.** Let A be an arbitrary commutative ring with a unit, let $R = A[t]$. Consider the embedding $j : A \hookrightarrow A[t]$ and the projection $\pi : A[t] \twoheadrightarrow A[t]/tA[t] = A$. We have $\pi j = \mathrm{id}$. If C_* is an R-complex, then the complex $C_*^- = C_* \underset{A}{\otimes} R$ is called *R-extended from C_** (or just *extended* if there is no possibility of confusion) [6] If $f : C_* \to D_*$ is a map of A-complexes, then the map $f \otimes \mathrm{id} : C_* \otimes R \to D_* \otimes R$ is called *R-extended from f*. We shall keep for $f \otimes \mathrm{id}$ the same notation f, if there is no possibility of confusion . Note that $C_* = C_*^- / tC_*^-$, thus $H_*(C_*^-) = 0$ if and only if $H_*(C_*) = 0$. If U is a subgroup of $A^\bullet$, then the map $j_* : K_1(A|U) \to K_1(R|U)$ is a split injection, the right inverse being given by $\pi_* : K_1(R|U) \to K_1(A|U)$. The following lemma is now obvious.

Lemma 6.2.

The complex C_*^- is U-simply homotopy equivalent to zero if and only if C_* is U-simply homotopy to zero. $\square$

Corollary 6.3.

[5] We adopt here the sign convention of [Mi2], so that our torsion $\tau(C_*)$ differs by sign from that of Turaev's one ([Tu]).

[6] The reason for the notation C_*^- will be clear in the next section, see (50).

Let

$$0 \longrightarrow M_* \xrightarrow{\phi} N_* \longrightarrow P_* \longrightarrow 0 \tag{42}$$

be an exact sequence of R-complexes such that P_* is extended. Then ϕ is a U-simple homotopy equivalence if and only if ϕ/t is an U-simple homotopy equivalence.

Proof. The exact sequence (42) splits as an exact sequence of R-modules, therefore it remains exact after tensor multiplication by R/tR. Further, the homology of P_* vanishes if and only if the homology of P_*/tP_* vanishes, and thus ϕ/t is a homotopy equivalence if and only if ϕ is. Moreover, the torsion of ϕ, resp. of ϕ/t equals to the torsion of P_* (resp. P_*/tP_*). But $\tau(P_*) = j_*(\tau(P_*/tP_*))$ since P_* is extended. $\square$

7. Condition $(\mathfrak{C}\mathcal{C})$, Morse-type filtrations for Morse maps $M \to S^1$, and C^0-generic rationality of Novikov incidence coefficients

7.1. **Condition $(\mathfrak{C}\mathcal{C})$ and C^0-generic rationality of Novikov incidence coefficients.** In this section we proceed at last to Morse maps $M \xrightarrow{f} S^1 = \mathbf{R}/\mathbf{Z}$. We shall use here the terminology of 2.4 and we introduce still some more.

Assume that f is primitive, that is $f_* : H_1(M) \to H_1(S^1) = \mathbf{Z}$ is epimorphic. To simplify the notation we shall assume that $1 \in S^1$ is a regular value for f. Denote $f^{-1}(1)$ by V. Recall that $\mathcal{C} : \bar{M} \to M$ is the infinite cyclic covering, associated to f, and $F : \bar{M} \to \mathbf{R}$ is a lifting of f. Denote $F^{-1}(\alpha)$ by V_α, and $F^{-1}([0,1])$ by W, and $F^{-1}(]-\infty,1])$ by V^-. The cobordism W can be thought of as the result of cutting M along V. Our choice of the generator t of the structure group of $\mathcal{C}$ implies that $V_\alpha t = V_{\alpha-1}$. Denote Wt^s by W_s; then $\bar{M}$ is the union $\cup_{s\in\mathbf{Z}} W_s$, the neighbor copies W_{s+1} and W_s intersecting by V_{-s}. For any $k \in \mathbf{Z}$ the restriction of $\mathcal{C}$ to V_k is a diffeomorphism $V_k \to V$. Endow M with an arbitrary riemannian metric and lift it to a t-invariant riemannian metric on $\bar{M}$. Now W is a riemannian cobordism, and actually it is a cyclic cobordism with respect to the isometry $t^{-1} : \partial_0 W = V_0 \to \partial_1 W = V_1$. We shall say that v satisfies condition $(\mathfrak{C}\mathcal{C})$ ($\mathfrak{C}$ for *cellular* and $\mathcal{C}$ for *circle*) if the $(F|W)$-gradient v satisfies the condition $(\mathfrak{C}\mathcal{Y})$ from Subsection 4.5. The set of f-gradients v satisfying $(\mathfrak{C}\mathcal{C})$ will be denoted by $\mathcal{G}\mathfrak{C}\mathcal{C}(f)$. It follows from the theorem 4.12 that the set $\mathcal{G}\mathfrak{C}\mathcal{C}\mathcal{T}(f) = \mathcal{G}\mathfrak{C}\mathcal{C}(f) \cap \mathcal{G}\mathcal{T}(f)$ is C^0-open-and-dense in $\mathcal{G}\mathcal{T}(f)$.

Let $v \in \mathcal{G}\mathfrak{C}\mathcal{C}(f)$. The condition $(\mathfrak{C}\mathcal{Y})$ provides a Morse function ϕ_1 on V_1 together with its gradient u_1, and a Morse function $\phi_0 = \phi_1 \circ t^{-1}$ on V_0 together with its gradient $u_0 = (t)_*(u_1)$. For every $k \in \mathbf{Z}$ we obtain also an ordered Morse function $\phi_k = \phi_0 \circ t^k : V_k \to \mathbf{R}$ and a ϕ_k-gradient $u_k = (t^{-k})_*(u_0)$.

Theorem 7.1.
Let $v \in \mathcal{G}\mathfrak{C}\mathcal{C}\mathcal{T}(f)$. Then every Novikov incidence coefficient is a rational function of t.

Proof. Let $r, s \in S(f)$, $\operatorname{ind} r = l + 1 = \operatorname{ind} s + 1$. We can assume that the liftings $\bar{r}, \bar{s}$ of the critical points r, s to $\bar{M}$ are in W. Then $n_k(r, s; v) = 0$ for $k < 0$.

Consider the oriented submanifold $S_{\bar{r}} = D(\bar{r}, v) \cap V_0$ of V_0. The condition $(\mathfrak{C})$ implies that $S_{\bar{r}} \subset (V_0)^{\{\leqslant l\}}$ and $S_{\bar{r}} \setminus \operatorname{Int}(V_0^{\{\leqslant l-1\}})$ is compact, so there is the fundamental class $[S_{\bar{r}}]$ of $S_{\bar{r}}$ in the group $H_l(V_0^{\{\leqslant l\}}, V_0^{\{\leqslant l-1\}})$. Then applying Theorem 4.8 1) we deduce by induction that for every k the submanifold $X_k = (-v)^{\rightsquigarrow}_{[0,-k]}(S_{\bar{r}})$ of V_{-k} is in $V_{-k}^{\{\leqslant l\}}$ and $X_k \setminus \operatorname{Int}(V_{-k}^{\{\leqslant l-1\}})$ is compact. Moreover the homology class of X_k in $H_*(V_{-k}^{\{\leqslant l\}}, V_{-k}^{\{\leqslant l-1\}})$ equals to $\mathcal{H}_l^k(-v)([S_{\bar{r}}])$.

Set $Y_k = B_{\bar{s}t^{k+1}} = D(\bar{s}t^{k+1}, -v) \cap V_{-k}$. Then Y_k is a cooriented embedded $(n-l-1)$-dimensional submanifold of V_{-k} (where $n = \dim M$) and $Y_k \subset \phi_{-k}^{-1}([\alpha_l, \alpha_{n+1}])$. Since v satisfies Transversality Condition, we have: $X_k \pitchfork Y_k$ and $n_{k+1}(r, s; v) = X_k \# Y_k$. Applying the formula (30) to the cobordism W_{k+1} we obtain:

$$n_{k+1}(r, s; v) = \langle [X_k], \bar{s}t^{k+1} \rangle \tag{43}$$

Denote $t^{-1}\mathcal{H}_l(-v)$ by h; then we have

$$n_{k+1}(r, s; v) = \langle h^k([S_{\bar{r}}])t^k, \bar{s}t^{k+1} \rangle = \langle h^k([\bar{S}_r]), \bar{s}t \rangle \tag{44}$$

Thus we have the following formula for the total Novikov incidence coefficient:

$$n(r,s;v) = n_0(r,s;v) + t\sum_k \xi(A^k x)t^k$$

with $A = h, x = [S_{\bar{F}}], \xi = \langle \cdot, \bar{s}t\rangle$. Now return to (2) to see that $n(r,s;v)$ is rational. $\square$

7.2. **Novikov incidence coefficients associated to a regular covering.** Here we deal with a version of Novikov complex associated to a regular covering of the manifold. We assume the terminology of Subsection 2.4 and Subsection 1.2. Let $f : M \to S^1$ be a Morse map, non homotopic to zero. Let $\mathcal{P} : \widetilde{M} \to M$ be a regular covering with structure group G. As always in this paper we assume that G is abelian. Assume that $f \circ \mathcal{P}$ is homotopic to zero; then the covering $\mathcal{P}$ is factored through $\mathcal{C}$ as follows:

$$\begin{tikzcd} \widetilde{M} \arrow[r, "p"] \arrow[dr, "\mathcal{P}"'] & \bar{M} \arrow[d, "\mathcal{C}"] \\ & M \end{tikzcd} \tag{45}$$

Therefore there is a lifting of $f : M \to S^1$ to a Morse function $\widetilde{F} : \widetilde{M} \to \mathbf{R}$. The homomorphism $f_* : \pi_1 M \to \mathbf{Z}$ can be factored through a homomorphism $\xi : G \to \mathbf{Z}$. Denote Ker ξ by H and $\xi^{-1}(]-\infty,0])$ by G_-. Recall that to define the ordinary version of the Novikov complex (Subsection 2.4) we need to choose orientations of the descending discs of critical points of f. Choose in addition for each critical point $x \in S(f)$ a lifting $\widetilde{x}$ of x to $\widetilde{M}$. Let $v \in \mathcal{GT}(f)$. Lift v to a G-invariant vector field on $\widetilde{M}$ (which will be denoted by the same letter v). Let $x, y \in S(f), \mathrm{ind} x = \mathrm{ind} y + 1$. Denote by $\Gamma(x,y,g)$ the set of $(-v)$-orbits, joining $\widetilde{x}$ to $\widetilde{y}g$. Let $\gamma \in \Gamma(x,y,g)$. Then as usual a sign $\varepsilon(\gamma) \in \{-1,1\}$ is defined. Set $\widetilde{n}(x,y;v) = \sum_{g\in G}\left(\sum_{\gamma\in\Gamma(x,y,g)}\varepsilon(\gamma)\cdot g\right)$. Then $\widetilde{n}(x,y;v)$ is an element of $\widehat{\widehat{\Lambda}}$, andit is not difficult to prove that $\widetilde{n}(x,y;v) \in \widehat{\Lambda}_\xi$. We shall keep the same name "Novikov incidence coefficient" for this element, since there is no possibility of confusion . Let $\widetilde{\mathcal{C}}_p(v)$ be the free right $\widehat{\Lambda}_\xi$-module, freely generated by $S_p(f)$. Define a $\widehat{\Lambda}_\xi$-homomorphism $\widetilde{\mathcal{D}}_p : \widetilde{\mathcal{C}}_p(v) \to \widetilde{\mathcal{C}}_{p-1}(v)$ setting

$$\widetilde{\mathcal{D}}_p(x) = \sum_{y\in S_{p-1}(f)} y \cdot \widetilde{n}(x,y;v)$$

It turns out that $\widetilde{\mathcal{D}}_p \circ \widetilde{\mathcal{D}}_{p+1} = 0$; therefore we obtain a based chain complex $\widetilde{\mathcal{C}}_*(v)$, called *Novikov Complex*. Moreover, there is a chain homotopy equivalence $\phi : \widetilde{\mathcal{C}}_*(v) \to C_*^\Delta(\widetilde{M}) \underset{\Lambda}{\otimes} \widehat{\Lambda}_\xi$, such that the image of $\tau(\phi)$ in $K_1(\widehat{\Lambda}_\xi | U_\xi)$ vanishes ([Pa1], Th. 2.2).

In view of Theorem 7.1 it is natural to suppose that if $v \in \mathcal{G}\mathfrak{C}\mathcal{T}(f)$, then the Novikov incidence coefficients are not merely power series, but "rational functions of several variables". That is, one expects $\widetilde{n}(x,y;v) \in \Lambda_{(\xi)}$. It is really the case:

Theorem 7.2.

Let $v \in \mathcal{G}\mathfrak{CC}\mathcal{T}(f)$. Then all the Novikov incidence coefficients are in $\Lambda_{(\xi)}$.

Proof. It is an equivariant version of the proof of Theorem 7.1 . We shall just indicate the main changes to make. Let $r, s \in S(f), \operatorname{ind} r = \operatorname{ind} s + 1 = l + 1$. We can assume that all the liftings of the critical points of f to $\widetilde{M}$ are in $\widetilde{W}$. Thus we obtain a lifting X' of the manifold X to $\widetilde{V}_1$. For $a \in S(f)$ set $\bar{a} = p(\widetilde{a})$. Choose an element $\theta \in G$, such that $\xi(\theta) = -1$. Every critical point of $F : \bar{M} \to \mathbf{R}$ is of the form $\bar{b}t^k$ with $k \in \mathbf{Z}, b \in S(f)$. Lift the critical point $\bar{b}t^k \in \bar{M}$ to the point $\widetilde{b}\theta^k \in \widetilde{M}$. Then we obtain for every k a lifting Y_k' of Y_k to $\widetilde{V}_{-k} \subset \widetilde{M}$, such that $Y_k' = Y_0'\theta^k$. Similarly to the end of Subsection 4.5 define an endomorphism $\widetilde{h}_l(-v)$ of the free right $\mathbf{Z}H$-module $H_*(\widetilde{V}_0^{\{\leqslant l\}}, \widetilde{V}_0^{\{\leqslant l-1\}})$ by $\widetilde{h}_l(-v) = \widetilde{\mathcal{H}}_l(-v) \circ (\theta^{-1})_*$. Since we consider only one f-gradient, we shall abbreviate $\widetilde{h}_s(-v)$ to $\widetilde{h}_s$.

Set

$$\widetilde{n}_k(r, s; v) = \sum_{g : \xi(g) = -k} \left(\sum_{\gamma \in \Gamma(r,s;g)} \varepsilon(\gamma) g \right)$$

Then $\widetilde{n}(r, s; v) = \sum_{k \geqslant 0} \widetilde{n}_k(r, s; v)$. We have:

$$\widetilde{n}_{k+1}(r, s; v) = \sum_{g \in H} (\widetilde{X}_k \#\!\!\# \widetilde{B}_{\widetilde{s} g \theta^{k+1}}) g \theta^{k+1} = \tag{46}$$

$$\sum_{g \in H} (\widetilde{X}_k \theta^{-k} \#\!\!\# \widetilde{B}_s g \theta) g \theta^{k+1} = \langle\langle [\widetilde{X}_k \theta^{-k}], \widetilde{s}\theta \rangle\rangle \theta^{k+1} \tag{47}$$

Applying the theorem 4.10 we obtain by induction that $[\widetilde{X}_k \theta^{-k}] = \widetilde{h}^k([\widetilde{X}_0])$. Thus

$$\widetilde{n}(r, s; v) = \widetilde{n}_0(r, s; v) + \sum_{k \geqslant 0} \langle\langle \widetilde{h}^k([\widetilde{X}_0]), \widetilde{s}\theta \rangle\rangle \theta^{k+1} \tag{48}$$

and this last element belongs to $\Lambda_{(\xi)}$ which can be proved by the suitable generalization of (2) □

Remark 7.3.

Here is one more formula for Novikov incidence coefficients, which will be useful in the sequel. It follows from (38).

$$\sum_s \widetilde{n}_{k+1}(r, s; v)[\widetilde{s}] = \widetilde{N}_l(\widetilde{h}^k([\widetilde{X}_0]) \theta^{k+1} \tag{49}$$

△

7.3. **Morse-type filtration, associated with a Morse map $M \to S^1$.** We keep here the terminology of two previous sections. Let $v \in \mathcal{G}\mathfrak{CC}\mathcal{T}(f)$. We associate to v the corresponding filtration $W^{\langle k \rangle}$ of W, and we introduce a t-invariant filtration in V^- setting $V^-_{\langle k \rangle} = \bigcup_{s \geqslant 0} t^s W^{\langle k \rangle}$. It induces in turn a G_--invariant filtration $\widetilde{V}^-_{\langle k \rangle}$ of $\widetilde{V}^- = p^{-1}(V^-)$ (see the beginning of Subsection 7.2 for the definition of p, G_- etc.) In this subsection we shall reconstruct the simple homotopy type of $\widetilde{M}$ from the data associated to the filtration $V^-_{\langle k \rangle}$. (Of course there is already a Morse-theoretic procedure of this kind: just take a Morse function $g : M \to \mathbf{R}$ and associate to g and to the covering $\mathcal{P}$ the corresponding Morse complex. The procedure which we propose here is better suited to Novikov homology of M) The next proposition continues the lines of Theorem 5.5 , Proposition 5.9 and Proposition 5.10 . We need some more terminology. Set

$$\widetilde{C}^-_*(u_1) = \widetilde{C}_*(u_1) \underset{\mathbf{Z}H}{\otimes} \mathbf{Z}G_-, \quad \widetilde{C}^-_*(v) = \widetilde{C}_*(v) \underset{\mathbf{Z}H}{\otimes} \mathbf{Z}G_- \tag{50}$$

These are $\mathbf{Z}G_-$-complexes. Extend the $\mathbf{Z}H$-homomorphisms $\widetilde{\partial}_* : \widetilde{C}_*(v) \to \widetilde{C}_{*-1}(v), \widetilde{\partial}^{(1)}_* : \widetilde{C}_*(u_1) \to \widetilde{C}_{*-1}(u_1)$ by $\mathbf{Z}G_-$-linearity to $\mathbf{Z}G_-$-homomorphisms $\widetilde{C}^-_*(v) \to \widetilde{C}^-_{*-1}(v)$, resp. $\widetilde{C}^-_*(u_1) \to \widetilde{C}^-_{*-1}(u_1)$. We keep for the extended homomorphisms the same notation $\widetilde{\partial}^{(1)}_*, \widetilde{\partial}_*$. Choose any $\theta \in G_{(-1)}$. We have an identification $\mathbf{Z}G_- = \mathbf{Z}H[\theta]$. We shall identify the $\mathbf{Z}H$-submodule $\widetilde{C}_*(u_1)\theta$ with $\widetilde{C}_*(u_0)$. The homomorphisms $\widetilde{\mathcal{H}}_k(-v) : \widetilde{C}_*(u_1) \to \widetilde{C}_*(u_0)$ and $\widetilde{P}_k : \widetilde{C}_*(v) \to \widetilde{C}_*(u_0)$ can be considered as homomorphisms with values in $\widetilde{C}^-_*(u_1)$. Extend them by $\mathbf{Z}G_-$-linearity to the whole of $\widetilde{C}_*(u_1), \widetilde{C}_*(v)$ (keeping the same notation for the extensions). Set $\widetilde{h}_k = \mathcal{H}_k(-v)\theta^{-1} : \widetilde{C}_*(u_1) \to \widetilde{C}_*(u_1)$

Proposition 7.4.

1. *$H_s(\widetilde{V}^-_{\langle k\rangle}, \widetilde{V}^-_{\langle k-1\rangle}) = 0$, if $s \neq k$.*
2. *Set $\widetilde{\mathcal{E}}_k = H_k(\widetilde{V}^-_{\langle k\rangle}, \widetilde{V}^-_{\langle k-1\rangle})$. Then $\widetilde{\mathcal{E}}_k$ is a free $\mathbf{Z}G_-$-module and there is an isomorphism of $\mathbf{Z}G_-$-modules*

$$\widetilde{\mathcal{L}}_k : \widetilde{C}^-_k(u_1) \oplus \widetilde{C}^-_k(v) \oplus \widetilde{C}^-_{k-1}(u_1) \xrightarrow{\approx} \widetilde{\mathcal{E}}_k \tag{51}$$

3. *Let $\widetilde{D}_{k+1} : \widetilde{\mathcal{E}}_{k+1} \to \widetilde{\mathcal{E}}_k$ be the boundary operator in the exact sequence of the triple $(\widetilde{V}^-_{\langle k+1\rangle}, \widetilde{V}^-_{\langle k\rangle}, \widetilde{V}^-_{\langle k-1\rangle})$. Then the matrix of $\widetilde{D}'_{k+1} = \widetilde{\mathcal{L}}^{-1}_k \circ \widetilde{D}_{k+1} \circ \widetilde{\mathcal{L}}_{k+1}$ with respect to the decomposition of the left hand side of (51) is*

$$\begin{pmatrix} \widetilde{\partial}^{(1)}_{k+1} & \widetilde{P}_{k+1} & \mathrm{Id} - \theta\widetilde{h}_k(-v) \\ 0 & \widetilde{\partial}_{k+1} & \widetilde{N}_k \\ 0 & 0 & -\widetilde{\partial}^{(1)}_k \end{pmatrix} \tag{52}$$

Proof. We shall give the proof for a particular case when $\widetilde{M} = \bar{M}$ and thus p is the identity map. The general case differs from this particular one only by notational complications: add tildas and replace t by θ.

Recall the filtration $W^{\prec k\succ}$ of W introduced in (10). Set $V^-_{\prec k\succ} = \cup_{s\geqslant 0} t^s W^{\prec k\succ}$. It is not difficult to prove that the inclusions

$$(V^-_{\prec k\succ}, V^-_{\prec k-1\succ}) \hookrightarrow (V^-_{\prec k\succ}, V^-_{\langle k-1\rangle}) \hookrightarrow (V^-_{\langle k\rangle}, V^-_{\langle k-1\rangle}) \tag{53}$$

are homotopy equivalences (the argument used for the proof of the Lemma 5.3 applies here also). Consider the subspace

$$W^{[k]} = (\partial_1 W)^{\{\leqslant k\}} \cup Y_k \cup Z_k = W^{\prec k\succ} \setminus ((\partial_0 W)^{\{\leqslant k\}} \setminus (\partial_0 W)^{\{\leqslant k-1\}}) \tag{54}$$

of W. We have obviously

$$W^{\langle k-1\rangle} \subset W^{[k]} \subset W^{\prec k\succ}$$

therefore there is an inclusion of pairs

$$(W^{[k]}, W^{\langle k-1\rangle}) \overset{\lambda}{\hookrightarrow} (V^-_{\prec k\succ}, V^-_{\langle k-1\rangle}) \tag{55}$$

It is clear that

$$H_k(W^{[k]}, W^{\langle k-1\rangle}) \approx C_k(u_1) \oplus C_k(v) \oplus C_{k-1}(u_1) \tag{56}$$

$$H_*(W^{[k]}, W^{\langle k-1\rangle}) = 0 \text{ if } * \neq k \tag{57}$$

Lemma 7.5.
The homomorphism

$$H_*(W^{[k]}, W^{\langle k-1\rangle}) \xrightarrow{\lambda_*} H_*(V^-_{\prec k\succ}, V^-_{\langle k-1\rangle}) \tag{58}$$

induced by the inclusion λ is injective and $H_*(V^-_{\prec k\succ}, V^-_{\langle k-1\rangle})$ is the $\mathbf{Z}[t]$-module, extended from Im λ_*.

Proof. Note that

$$W^{[k]} \cap \phi^{-1}(]0, a_k[) = W^{\langle k-1\rangle} \cap \phi^{-1}(]0, a_k[)$$

Thus by excision we can get away the set $\phi^{-1}(]0, a_k[)$ from both $W^{[k]}$ and $W^{\langle k-1\rangle}$. Applying this procedure to t^sW for every s, it is easy to see that $H_*(V^-_{\prec k\succ}, V^-_{\langle k-1\rangle})$ is a $\mathbf{Z}[t]$-extended module from its $\mathbf{Z}$-submodule

$$H_*\Big(W^{[k]} \cap \phi^{-1}([a_k, b]), W^{\langle k-1\rangle} \cap \phi^{-1}([a_k, b])\Big)$$

This last module is isomorphic to $H_*(W^{[k]}, W^{\langle k-1\rangle})$ and the lemma is proven. $\square$

Now the expression (52) is obtained directly from (29); just note that in the space V^- the subspace $\partial_0 W$ coincides with $\partial_1 W \cdot t$. $\square$

Thus $\widetilde{\mathcal{E}}_*$ is a free chain complex over $\mathbf{Z}G_-$. The filtration $\widetilde{V}^-_{\langle k\rangle}$ of $\widetilde{V}^-$ induces the corresponding filtration of the singular chain complexes. This filtration is nice (by points 1) and 2) of Proposition 7.4), therefore the homology $H_*(\widetilde{V}^-)$ is isomorphic to that of $\widetilde{\mathcal{E}}_*$. We can also partially recover the simple homotopy type of the $\mathbf{Z}G$-complex $C_*(\widetilde{M})$ from the complex $\widetilde{\mathcal{E}}_*$. To explain what it means choose any smooth triangulation of M, then $\widetilde{M}$ obtains a smooth G-invariant triangulation. Choose some liftings of simplices of M to $\widetilde{M}$ and obtain a base of $C^\Delta_*(\widetilde{M})$. Thus the simplicial chain complex $C^\Delta_*(\widetilde{M})$ is a $\mathbf{Z}G$-complex. Note further that $\widetilde{\mathcal{E}}_*$ obtains a natural base via the isomorphism $\widetilde{\mathcal{L}}_k$ from (51) thus $\widetilde{\mathcal{E}}_*$ is a $\mathbf{Z}G_-$-complex.

Proposition 7.6.
The $\mathbf{Z}G$-complexes $C^\Delta_*(\widetilde{M})$ and $\widetilde{\mathcal{E}}_* \underset{\mathbf{Z}G_-}{\otimes} \mathbf{Z}G$ are simply homotopy equivalent (that is, there is a homotopy equivalence ϕ between them with torsion $\tau(\phi) \in Wh(G) = \overline{K}_1(\mathbf{Z}G|G)$ equal to zero).

Proof. Choose a triangulation of M so that V is a simplicial subcomplex, and so that the resulting triangulation of W satisfies the condition $(\mathcal{T})$ from the page 37.

Then

$$C^\Delta_*(\widetilde{M}) = C^\Delta_*(\widetilde{V}^-) \underset{\mathbf{Z}G_-}{\otimes} \mathbf{Z}G$$

Thus it suffices to prove that there is a chain homotopy equivalence of $\mathbf{Z}G_-$-complexes

$$\psi : \widetilde{\mathcal{E}}_* \longrightarrow C^\Delta_*(\widetilde{V}^-) \tag{59}$$

such that that $\tau(\psi)$ vanishes in $K_1(\mathbf{Z}G_-|H)$ (recall from that such chain homotopy equivalences ψ are called H-simple). Note that each $\widetilde{V}^-_{\langle k\rangle}$ is a $\mathbf{Z}G_-$-invariant simplicial subcomplex of $\widetilde{V}^-$. Thus $C^\Delta_*(\widetilde{V}^-_{\langle k\rangle})$ form a filtration of $C^\Delta_*(\widetilde{V}^-)$. The part 2 of Proposition 7.4 implies that this filtration is nice. Thus there is a canonical chain homotopy equivalence

$$\psi : \widetilde{\mathcal{E}}_* \to C^\Delta_*(\widetilde{V}^-) \tag{60}$$

uniquely determined by the conditions F1) and F2) (page 37). Define a chain complex $\widetilde{\mathcal{E}}(k)_*$ by

$$\widetilde{\mathcal{E}}(k)_* = \begin{cases} 0 & \text{if } * \neq k \\ \widetilde{\mathcal{E}}_k & \text{if } * = k \end{cases} \tag{61}$$

Arguing by induction, it suffices to prove that every chain map

$$\psi_k : \widetilde{\mathcal{E}}(k)_* \longrightarrow C^\Delta_*(\widetilde{V}^-_{\langle k\rangle}/\widetilde{V}^-_{\langle k-1\rangle}) \tag{62}$$

is an H-simple homotopy equivalence.

Lemma 7.7.
The chain maps

$$C^\Delta_*(\widetilde{V}^-_{\prec k\succ}, \widetilde{V}^-_{\prec k-1\succ}) \twoheadrightarrow C^\Delta_*(\widetilde{V}^-_{\prec k\succ}, \widetilde{V}^-_{\langle k-1\rangle}) \hookrightarrow C^\Delta_*(\widetilde{V}^-_{\langle k\rangle}, \widetilde{V}^-_{\langle k-1\rangle}) \tag{63}$$

induced by the inclusions from (53) are H-simple homotopy equivalences.

Proof. It suffices to prove that for every k the inclusion

$$J : C^\Delta_*(\widetilde{V}^-_{\prec k\succ}) \hookrightarrow C^\Delta_*(\widetilde{V}^-_{\langle k\rangle})$$

is an H-simple homotopy equivalence. The quotient $\mathbf{Z}G_-$-complex $C_*(\widetilde{V}^-_{\langle k\rangle}/\widetilde{V}^-_{\prec k\succ})$ is $\mathbf{Z}G_-$-extended. (Note that $\mathbf{Z}G_- = (\mathbf{Z}H)[\theta]$.) Thus it suffices (by Corollary 6.3) to prove that J/θ is an H-simple homotopy equivalence. The chain map J/θ coincides with the following chain homotopy equivalence, induced by the inclusion:

$$C^\Delta_*(\widetilde{W}^{\prec k\succ}, (\partial_0\widetilde{W})^{\{\leqslant k\}}) \xhookrightarrow{J'} C^\Delta_*(\widetilde{W}^{\langle k\rangle}, (\partial_0\widetilde{W})^{\{\leqslant k\}}) \tag{64}$$

The proof that J' is a simple homotopy equivalence is an exercise in the theory of simple homotopy types and is left to the reader (Indication: note that $W^{\langle k\rangle}$ is obtained from $W^{\prec k\succ}$ by adding a subset homeomorphic to the cylinder $\phi_0^{-1}([\alpha_k, \alpha_{k+1}]) \times [0,1]$.) □

The map ψ_k factors obviously through a chain map

$$\psi'_k : \widetilde{\mathcal{E}}(k)_* \to C^\Delta_*(\widetilde{V}^-_{\prec k\succ}/\widetilde{V}^-_{\langle k-1\rangle}) \tag{65}$$

and it suffices to prove that ψ'_k is a simple homotopy equivalence. The map ψ_k is determined up to chain homotopy by the condition that for every $x \in \widetilde{\mathcal{E}}(k)_*$ the homology class of $\psi_k(x)$ is x itself. Thus we can consider that for every $p \in S_k(\phi)$ the element $\psi'_k([\widetilde{p}])$ is equal to the fundamental class of $[d(\widetilde{p}), \partial d(\widetilde{p})]$ in the pair

$$(W^{\prec k\succ}, W^{\langle k-1\rangle}) \subset (V^-_{\prec k\succ}, \widetilde{V}^-_{\langle k-1\rangle})$$

So we can assume that ψ'_k is an extended chain map and in order to check the vanishing of $\tau(\psi'_k|H)$ it suffices to check the vanishing of $\tau(\psi'_k/\theta|H)$.

The proof of this last assertion is similar to the proof of Proposition 5.12 and will be omitted. □

7.4. **Change of base.** In this subsection we continue the study of the chain complex $\widetilde{\mathcal{E}}_*$ introduced in the previous subsection. We are going to compare the three chain complexes: $\widetilde{\mathcal{E}}_*, C_*^\Delta(\widetilde{M}) \underset{\Lambda}{\otimes} \Lambda_{(\xi)}$ and $\widetilde{\mathcal{C}}_*(v)$. We proved in [Pa1] that for every f-gradient v satisfying Transversality Condition the $\widehat{\Lambda}_\xi$-complexes $C_*^\Delta(\widetilde{M})\underset{\Lambda}{\otimes}\widehat{\Lambda}_\xi$ and $\widetilde{\mathcal{C}}_*(v)$ are homotopy equivalent, and one can choose this homotopy equivalence ϕ in such a way that $\tau(\phi|U_\xi) \in K_1(\widehat{\Lambda}_\xi|U_\xi)$ vanishes. (The torsion of ϕ in the group $\overline{K_1}(\widehat{\Lambda}_\xi)$ does not vanish in general, as we shall see in the sequel.) Now we shall refine this result assuming that $v \in \mathcal{GCCT}(f)$. Using the formula (51) we shall obtain in this and the following subsection a chain homotopy equivalence

$$\widetilde{\mathcal{C}}_*(v) \xrightarrow{\psi} \widetilde{\mathcal{E}}_* \underset{\mathbf{Z}G_-}{\otimes} \Lambda_{(\xi)}$$

of $\Lambda_{(\xi)}$-complexes (recall that $\widetilde{\mathcal{C}}_*(v)$ is defined over $\Lambda_{(\xi)}$ by Theorem 7.2) such that its torsion $\tau(\psi|G)$ is explicitly computable in terms of the homomorphisms $\widetilde{h}_k$.

Set $\widetilde{\widetilde{\mathcal{E}}}_* = \widetilde{\mathcal{E}}_* \underset{\mathbf{Z}G_-}{\otimes} \Lambda_{(\xi)}$. We shall make a change of base in $\widetilde{\widetilde{\mathcal{E}}}_*$ so that the Novikov complex $\widetilde{\mathcal{C}}_*(v)$ will appear as a free subcomplex generated by some base elements. It turns out that the quotient complex is contractible.

The elements of the new base are introduced in the definition 7.8 ; for each critical point $p \in S_k(\phi)$ there is a base element $\{\widetilde{p}\}$. The geometric sense of these elements can be described as follows: The generator $[\widetilde{p}]$ corresponds to the part $D(\widetilde{p}, v) \cap (\phi \circ p)^{-1}([a_k, a_{k+1}])$ of the descending disc, and the generator $\{\widetilde{p}\}$ corresponds to the totality of the descending disc $D(\widetilde{p}, v)$.

Terminological remarks

1. In order not to overburden the notation we shall *identify* $\widetilde{\mathcal{E}}_k$ with $\widetilde{C}_k^-(u_1) \oplus \widetilde{C}_k^-(v) \oplus \widetilde{C}_{k-1}^-(u_1)$. The base elements in these three direct summands will be denoted as $[\widetilde{p}]$, resp. $[\widetilde{r}]$, resp. $[[\widetilde{q}]]$, where $p \in S_k(\phi_1), r \in S_k(f), q \in S_{k-1}(\phi_1)$. So these base elements (in this order) form a free base in $\widetilde{\mathcal{E}}_k$ denoted by $\mathcal{B}_k$. The identification of the $\mathbf{Z}G_-$-submodule $\widetilde{C}_k^-(u_1) \oplus 0 \oplus 0 \subset \widetilde{\mathcal{E}}_k$ with the submodule $0 \oplus 0 \oplus \widetilde{C}_k^-(u_1) \subset \widetilde{\mathcal{E}}_{k+1}$ will be denoted by σ_k.
2. We shall keep the notation $\widetilde{D}_*$ for the boundary operator as well as for the boundary operator in $\widetilde{\widetilde{\mathcal{E}}}_*$, and in $\widetilde{\mathcal{E}}_* \underset{\Lambda}{\otimes} \widehat{\Lambda}_\xi$
3. We shall often suppress the indices $k+1, k$ from the notation whenever there is no possibility of confusion . Thus we write for example $\widetilde{\partial}^{(1)}$ instead of $\widetilde{\partial}^{(1)}_{k+1}$ an $\widetilde{\partial}^{(1)}_k$. Similarly, we shall often suppress the symbol v in the notation since in rest of the section we consider only one f-gradient. The definition below is the first example of the abbreviated notation.
4. Set

$$\widetilde{\widetilde{C}}_*(u_1) = \widetilde{C}_*(u_1) \underset{\mathbf{Z}H}{\otimes} \Lambda_{(\xi)}$$

$$\widetilde{\widetilde{C}}_*(v) = \widetilde{C}_*(v) \underset{\mathbf{Z}H}{\otimes} \Lambda_{(\xi)}$$

these modules are direct summands of $\widetilde{\widetilde{\mathcal{E}}}_*$.

5. We identify $\Lambda_{(\xi)}$ with its image in $\widehat{\Lambda}_\xi$ under the canonical inclusion. Similarly we identify $\widetilde{\widetilde{\mathcal{E}}}_*$ with its image in $\widehat{\mathcal{E}}_* = \widetilde{\mathcal{E}}_* \underset{\Lambda}{\otimes} \widehat{\Lambda}_\xi$

Definition 7.8.
Let $r \in S_k(f)$. Set

$$\{\widetilde{r}\} = [\widetilde{r}] - \sum_{j=0}^{\infty} \sigma \widetilde{h}^j(\widetilde{P}([\widetilde{r}]))\theta^j \tag{66}$$

△

Note first of all that the element $\sum_{j=0}^{\infty} \widetilde{h}^j(a)\theta^j$ belongs to $\widetilde{\widetilde{C}}_*(u_1)$ for every $a \in \widetilde{C}_*(u_1)$. Thus $\{\widetilde{r}\}$ is a well defined element in $\widetilde{\widetilde{\mathcal{E}}}_*$. Note further that the element $\{\widetilde{r}\} - [\widetilde{r}]$ is a linear combination (with $\Lambda_{(\xi)}$-coefficients) of elements $[[\widetilde{q}]]$, therefore the elements $[\widetilde{p}], \{\widetilde{r}\}, [[\widetilde{q}]]$ (here $r \in S_k(\phi), q \in S_{k-1}(\phi_1), p \in S_k(\phi)$) form a base $\mathcal{B}'_k$ in $\widetilde{\widetilde{\mathcal{E}}}_k$, and the matrix of passage from $\mathcal{B}_k$ to $\mathcal{B}'_k$ is a product of elementary matrices.

Denote by $\widetilde{\mathcal{C}}'_k(v)$ the free $\Lambda_{(\xi)}$-submodule of $\widetilde{\widetilde{\mathcal{E}}}_k$, generated by the elements $\{\widetilde{p}\}, p \in S_k(\phi)$. Define a homomorphism $\delta : \widetilde{\mathcal{C}}'_k(v) \to \widetilde{\mathcal{C}}'_{k-1}(v)$ setting

$$\delta_k(\{\widetilde{r}\}) = \sum_{r' \in S_{k-1}(\phi)} \{\widetilde{r}'\} \cdot \widetilde{n}(r, r'; v)$$

(Thus the graded module $\widetilde{\mathcal{C}}'_*(v)$, endowed with the differential δ_* is isomorphic to Novikov complex $\widetilde{\mathcal{C}}_*(v)$.)

Proposition 7.9.
The matrix of the boundary operator $\widetilde{D}_{k+1} : \widetilde{\widetilde{\mathcal{E}}}_{k+1} \to \widetilde{\widetilde{\mathcal{E}}}_k$ with respect to the bases $\mathcal{B}'_{k+1}, \mathcal{B}'_k$ is:

$$\begin{pmatrix} \widetilde{\partial}^{(1)}_{k+1} & 0 & Id - \widetilde{h}_{k+1}\theta \\ 0 & \delta_{k+1} & \eta_{k+1} \\ 0 & 0 & \Delta_k \end{pmatrix} \tag{67}$$

(Here η_{k+1} and Δ_k are some homomorphisms, we do not compute them for the moment.)

Proof. The first and the third columns are easy. To proceed further, we need more definitions. For an element $x \in \widehat{\mathcal{E}}_*$ we shall need to consider "the part of x between the levels a and b of the function $\widetilde{F}$." This and similar notions are introduced in the following definition.

Definition 7.10.

1. *A monomial* of $\widehat{\Lambda}_\xi$ is an element of the form $ng, n \in \mathbf{Z}, g \in G$. *A monomial of* $\widehat{\mathcal{E}}_*$ is an element of $\widehat{\mathcal{E}}_*$ of the form $x \cdot ng$, where $n \in \mathbf{Z}, g \in G$, and x is one of the generators of the base $\mathcal{B}_*$
2. The *height* $ht(\lambda)$ of a monomial $\lambda = ng$ of $\widehat{\Lambda}_\xi$ where $n \neq 0$ is by definition $\xi(g)$. The height $ht(l)$ of a monomial l of $\widehat{\mathcal{E}}_*$ is by definition the subset of

$\mathbf{R}$ defined by

$$ht(l) = \begin{cases} \xi(g) + [0,1], \text{ if } l = [\widetilde{r}] \cdot ng, n \neq 0, g \in G \\ \xi(g) + [0,1], \text{ if } l = [[\widetilde{r}]] \cdot ng, n \neq 0, g \in G \\ \xi(g) + 1, \text{ if } l = [\widetilde{p}] \cdot ng, n \neq 0, g \in G \end{cases} \tag{68}$$

3. Let $A \subset \mathbf{R}$. For $\lambda \in \widehat{\Lambda}_\xi, \lambda = \sum_{g \in G} n_g g$ set $\lambda_A = \sum_{n_g \neq 0, \xi(g) \in A} n_g g$

   For $x \in \widehat{\Lambda}_\xi$ define x_A as the sum of all the monomials μ from the decomposition of x with respect to the base $\mathcal{B}_k$, satisfying $ht(\mu) \subset A$.
4. For $A = \{a\}$ we shall also write x_a instead of x_A.
5. For $n \in \mathbf{Z}, \lambda \in \widehat{\mathcal{E}}_*$ we denote by $\Sigma_n \lambda$ the element $(\widetilde{D}(\lambda_{[n,\infty[}))_n$

$\triangle$

Lemma 7.11.
Let $n \geqslant 0, r \in S_k(\phi)$. Then

$$\Sigma_n(\{\widetilde{r}\}) = \widetilde{h}^n(\widetilde{P}([\widetilde{r}])\theta^n \tag{69}$$

$$\{\widetilde{r}\}_{[-n-1,-n]} = -\sigma(\Sigma_{-n}(\{\widetilde{r}\})) \tag{70}$$

Proof. Obvious $\square$

Lemma 7.12.
Let $A \in \mathbf{Z}G, \operatorname{supp} A \subset \xi^{-1}([n,\infty[)$, where $n \in \mathbf{Z}$. Let $R = \{\widetilde{r}\}A$. Let $m \leqslant n$. Then

$$R_{[m-1,\infty[} = R_{[m,\infty[} - \sigma(\Sigma_m(R)) \tag{71}$$

Proof. The assertion is reduced by linearity to the case $A = g$ with $g \in G$, and this case is obtained easily from Lemma 7.11 . $\square$

Lemma 7.13.
For every $n \geqslant 0$:

$$(\delta\{\widetilde{r}\})_{[-n-1,\infty[} = (\delta\{\widetilde{r}\})_{[-n,\infty[} - \sigma(\Sigma_{-n}(\delta\{\widetilde{r}\})) + \widetilde{N}\widetilde{h}^n\widetilde{(}P([\widetilde{r}]))\theta^n \tag{72}$$

Proof. We have

(73)
$$(\delta\{\widetilde{r}\})_{[-n-1,\infty[} = \sum_{r' \in S_{k-1}(\phi)} \{\widetilde{r}\} \cdot (\widetilde{n}(r,r'))_{[-n,\infty[} + \sum_{r' \in S_{k-1}(\phi)} \{\widetilde{r}\} \cdot (\widetilde{n}(r,r'))_{-n-1}$$

By Lemma 7.12 the first term of the righthand side (73) equals to the sum of the two first terms in the righthandside of (72). The second term equals to $\widetilde{N}\widetilde{h}^n(\widetilde{P}([\widetilde{r}]))\theta^n$ by (49). $\square$

Lemma 7.14.
Let $n \geqslant 0$. Then

$$\widetilde{D}(\{\widetilde{r}\}_{[-n,\infty[}) = (\delta\{\widetilde{r}\})_{[-n,\infty[} + \widetilde{h}^n(\widetilde{P}([\widetilde{r}]))\theta^n \tag{74}$$

Proof. Induction in n. The case $n=0$ is obvious. Assume that the formula (74) holds for some integer n. Then we have in particular

$$\widetilde{D}(\delta\{\widetilde{r}\}_{[-n,\infty]}) = -\widetilde{\partial}^{(1)}\widetilde{h}^n(\widetilde{P}([\widetilde{r}]))\theta^n = \Sigma_{-n}(\delta\{\widetilde{r}\}) \tag{75}$$

We have:

$$\widetilde{D}(\{\widetilde{r}\}_{[-n-1,-n]}) = \Big(-\widetilde{h}^n\widetilde{P}([\widetilde{r}]) + \widetilde{h}^{n+1}\widetilde{P}([\widetilde{r}])\theta + \widetilde{N}\widetilde{h}^n\widetilde{P}([\widetilde{r}]) + \sigma\widetilde{\partial}^{(1)}\widetilde{h}^n\widetilde{P}([\widetilde{r}])\Big)\theta^n \tag{76}$$

and (by 7.13):

$$(\delta\{\widetilde{r}\})_{[-n-1,\infty[} - (\delta\{\widetilde{r}\})_{[-n,\infty[} = \widetilde{N}\widetilde{h}^n\widetilde{P}([\widetilde{r}])\theta^n + \sigma\widetilde{\partial}^{(1)}\widetilde{h}^n\widetilde{P}([\widetilde{r}])\theta^n \tag{77}$$

Now just combine the above formulas to get the assertion of the Lemma at the rank $n+1$. □

Now the computation of the second column of the matrix of $\widetilde{D}_{k+1}$ is finished, since Lemma 7.14 implies $\widetilde{D}\{\widetilde{r}\} = \delta\{\widetilde{r}\}$. □

7.5. **Homotopy equivalence $\widetilde{\mathcal{C}}_*(v) \to \widetilde{\widetilde{\mathcal{E}}}_*$ and its torsion.** Thus we have an inclusion $I : \widetilde{\mathcal{C}}_*(v) \hookrightarrow \widetilde{\widetilde{\mathcal{E}}}_*$ and the quotient is a free $\Lambda_{(\xi)}$-complex $\mathcal{Q}_*$ where $\mathcal{Q}_k = \widetilde{\widetilde{C}}_k(u_1) \oplus \widetilde{\widetilde{C}}_{k-1}(u_1)$. The matrix of the boundary operator $\nabla_k : \mathcal{Q}_k \to \mathcal{Q}_{k-1}$ is

$$\begin{pmatrix} \widetilde{\partial}^{(1)}_k & 1-\widetilde{h}_k\theta \\ 0 & \Delta_{k-1} \end{pmatrix} \tag{78}$$

Note that the homomorphism $1-\widetilde{h}_k\theta : \widetilde{\widetilde{C}}_k(u_1) \to \widetilde{\widetilde{C}}_k(u_1)$ is an isomorphism since $\det(1-\widetilde{h}_k\theta)$ is of the form $1+\theta\xi$ with $\xi \in \mathbf{Z}H$, and such elements are invertible in $\Lambda_{(\xi)}$. Thus we can apply Lemma 6.1 to deduce that I is a chain homotopy equivalence and that its torsion $\tau(I) \in \overline{K_1}(\Lambda_{(\xi)})$ equals to $\sum_i (-1)^{i+1}\alpha_i$ where α_i is the class of the isomorphism $(1-\widetilde{h}_i\theta)$. Due to the special form of the homomorphism $(1-\widetilde{h}_k\theta)$ we can simplify still more. In the next lemma A is a commutative ring with unit, $R = A[t]$, $S = \{P \in R \mid P = 1+tQ(t)\}$, $\widetilde{R} = S^{-1}R$, L is a free finitely generated R-module, and $\widetilde{L}$ stands for $S^{-1}L$.

Lemma 7.15.
Let $\xi : L \to L$ be a homomorphism. Then the class of the isomorphism $1+\xi\theta : \widetilde{L} \to \widetilde{L}$ in the group $\overline{K}_1(\widetilde{R})$ equals to the unit $\det(1+\xi\theta)$ of the ring $\widetilde{R}$.

Proof. Let $n = \mathrm{rk}\, L$. Consider the class of invertible $(n\times n)$-matrices M over $\widetilde{R}$, satisfying the property that $[M] = [\det M]$ in $\overline{K_1}(\widetilde{R})$. The upper triangular matrices are obviously in this class. Note also that this class is closed under the elementary operations. Thus it suffices to prove that our matrix $(1+\xi\theta)$ can be reduced to an upper triangular matrix in $\widetilde{R}$ by elementary operations. This is easy and will be left to the reader. □

Thus we have constructed an inclusion $I : \widetilde{C}_*(v) \hookrightarrow \widetilde{\widetilde{\mathcal{E}}}_*$ such that it is a chain homotopy equivalence and its torsion $\tau(I) \in \overline{K}_1(\Lambda_{(\xi)})$ satisfies

$$\tau(I) = \Big[\prod_k \Big(\det(1-\widetilde{h}_k\theta)\Big)^{(-1)^{k+1}}\Big] \tag{79}$$

Note that we have chosen the bases in $\widetilde{\mathcal{C}}_*(v)$ and in $\widetilde{\widetilde{\mathcal{E}}}_*$ in a special way: we constructed them from one and the same family of liftings of the critical points of f to $\widetilde{M}$. Moreover, we required that the liftings of the critical points of f to $\widetilde{M}$ belong to $\widetilde{W}$.

If we impose no restrictions on choosing the liftings of the critical points to $\widetilde{M}$, we can keep track only of the torsions in $\overline{K_1}(\Lambda_{(\xi)}|G)$ and we arrive (with the help of Proposition 7.6) at the following corollary.

Corollary 7.16.
There is a chain homotopy equivalence $\phi : \widetilde{\mathcal{C}}_*(v) \to \mathcal{C}^{\Delta}_*(\widetilde{M}) \underset{\Lambda}{\otimes} \Lambda_{(\xi)}$ *with*

$$\tau(\phi|G) = \left[\prod_k \det(1 - \widetilde{h}_k\theta)^{(-1)^{k+1}}\right] \tag{80}$$

□

8. Proof of the main theorem

We keep here the terminology of the previous section. The set of all Kupka-Smale f-gradients v, satisfying $(\mathfrak{C}\mathcal{Y})$ will be denoted by $\mathcal{GKSC}(f)$, and we shall show that this set satisfies the conclusions of the theorem. Indeed, the first item follows since $(\mathfrak{C}\mathcal{Y})$ is an C^0-open-and-dense condition, and since $\mathcal{GKSC}(f)$ is dense in $\mathcal{GKS}(f)$. The first part of the item 2) follows from Theorem 7.2 . It remains to prove that $\zeta_L(-v) \in \Lambda_{(\xi)}$ and that the image of $\zeta_L(-v)$ in $\overline{K}_1(\Lambda_{(\xi)}|G)$ equals to the element

$$\left[\prod_m \det(1 - \widetilde{h}_m\theta)^{(-1)^{m+1}}\right]$$

To do this we need one more description of the operator $\widetilde{h}_m$.

8.1. **Homological gradient descent (third version).** Recall that we have a Morse function $\phi_1 : V_1 \to \mathbf{R}$ and its δ-separated gradient u_1. For $p \in S_m(\phi_1)$ set

$$R(p) = \big(D_\delta(p, u_1) \cup V_1^{\{\leqslant m-1\}}\big)/V_1^{\{\leqslant m-1\}}$$

This space is homotopy equivalent to the sphere S^m. For $k \in \mathbf{Z}$ set

$$Q_k^{\lceil m\rceil} = \widetilde{V}_k^{\lceil m\rceil}(\delta)/\widetilde{V}_k^{\{\leqslant m-1\}}$$

(where as usual $\widetilde{A}$ stands for $\mathcal{P}^{-1}(A)$). The group H acts on $Q_k^{\lceil m\rceil}$ leaving invariant the point $\alpha_k = [\widetilde{V}_k^{\{\leqslant m-1\}}]$. Denote by $R(\widetilde{p})$ the lifting of $R(p)$ to $Q_0^{\lceil m\rceil}$ corresponding to the lifting $\widetilde{p}\theta$ of p. Then

$$Q_k^{\lceil m\rceil} = \bigvee_{\substack{p\in S_m(\phi_1)\\ h\in H, \xi(h)=k}} R(\widetilde{p})\cdot h$$

The generator of the mth homology group of $R(\widetilde{p})$ corresponding to the chosen orientation of $D(p,v)$ will be denoted by $\rho(\widetilde{p})$. The $\mathbf{Z}H$-module $H_*(Q_k^{\lceil m\rceil})$ is then free with the base $\{\rho(\widetilde{p})\}_{p\in S_m(\phi_1)}$. It is not difficult to show that the gradient descent map $(-v)^{\leadsto}$ defines a continuous map $v\downarrow : Q_k^{\lceil m\rceil} \to Q_{k-1}^{\lceil m\rceil}$ which is H-equivariant, sends α_k to α_{k-1}, and such that the following diagram is commutative

$$
\begin{tikzcd}
H_*(\widetilde{V}_k^{\{\leqslant m\}}, \widetilde{V}_k^{\{\leqslant m-1\}}) \arrow[r, "\widetilde{\mathcal{H}}(-v)"] & H_*(\widetilde{V}_{k-1}^{\{\leqslant m\}}, \widetilde{V}_{k-1}^{\{\leqslant m-1\}}) \\
H_*(Q_k^{\lceil m\rceil}) \arrow[u, "\approx"'] \arrow[r, "(v\downarrow)_*"] & H_*(Q_{k-1}^{\lceil m\rceil}) \arrow[u, "\approx"']
\end{tikzcd} \tag{81}
$$

We leave to the reader the proof of the existence of $v\downarrow$ as an exercice in homological gradient descent theory.

8.2. **Computation of the ζ-function.** Let $Cl^{[s]}(-v)$ be the subset of all the closed orbits of $(-v)$ passing through a point of $V^{\lceil\leqslant s\rceil}(\delta)\setminus V^{\{\leqslant s-1\}}$. It follows from the property $(\mathfrak{CY})$ that the set $Cl(-v)$ is the disjoint union of its subsets $Cl^{[s]}(-v)$. Set

$$
\eta_s(-v)=\sum_{\gamma\in Cl^{[s]}(-v)}\varepsilon(\gamma)\frac{\pi([\gamma])}{m(\gamma)}, \tag{82}
$$

Since

$$
\ln\det(1-\theta\widetilde{h}_m)=-\sum_{k>0}\frac{\mathrm{Tr}(\theta\widetilde{h}_m)^k}{k}
$$

it suffices to prove that for every s we have:

$$
(-1)^s\sum_{k>0}\frac{\mathrm{Tr}(\theta\widetilde{h}_s)^k}{k}=\eta_s(-v) \tag{83}
$$

So we fix s up to the end of the proof. We shall abbreviate $Cl^{[s]}(-v)$ to Cl.

The equality (83) will be proved by translating our data to the language of the fixed point theory.

We shall say that a point $a\in\beta\setminus\{\omega\}$ is a *G-fixed-point* of $(v\downarrow)^k$, if $(v\downarrow)^k(a)=a\cdot g, g\in G$. The element $g\in G_{(-k)}$ is uniquely determined by a and will be denoted by $g(a)$. The set of all G-fixed points of $(v\downarrow)^k$ will be denoted by $GF(k)$. The set of all G-fixed points of $(v\downarrow)^k$ with given $g(a)=g$ will be denoted by $GF(k,g)$. Thus $GF(k)=\sqcup_{g\in G_{(-k)}}GF(k,g)$. By analogy with the standard fixed point theory we define the multiplicity $\mu(a)$ and the index $\mathrm{ind}a$ for every $a\in GF(k)$.

Let $a\in GF(k)$. Let $a_i=(v\downarrow)^i(a)$ and let $\widehat{a}_i$ be (the unique) point in β belonging to the G-orbit of a_i. The set of all $\widehat{a}_i, i\in\mathbf{N}$ will be called *quasiorbit of* a, and denoted by $\mathcal{Q}(a)$; it is a finite set of cardinality $\frac{k}{\mu(a)}$.

Now let $a\in GF(k)$. Consider the integral curve γ of $(-v)$ in $\widetilde{M}$, such that $\gamma(0)=a$; then for some $T>0$ we have

$$
\gamma(T)=(v\downarrow)^k(a)=a\cdot g(a)\in\widetilde{V}_{-k}
$$

The map $\mathcal{P}\circ\gamma:[0,T]\to M$ is then a closed orbit. Thus we obtain a map $\Gamma:GF(k)\to Cl$ whose image is exactly the subset $Cl_k\subset Cl$, consisting of all $\gamma\in Cl$ with $f_*([\gamma])=-k$.

For every $\gamma\in Cl_k$ the set $\Gamma^{-1}(\gamma)$ is the quasiorbit of some $a\in GF_k$. (Note that the set $\mathcal{P}(\Gamma^{-1}(\gamma))$ is the intersection of the orbit γ with V.) Further, for every $a\in\Gamma^{-1}(\gamma)$ we have: $g(a)=\pi([\gamma]),\varepsilon(\gamma)=\mathrm{ind}a$ and $\mu(\gamma)=\mu(a)$.

Thus the following power series

$$\nu(-v) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{a \in GF_k} (\mathrm{ind} a) g(a) \tag{84}$$

equals to $\eta_s(-v)$.

To prove our theorem it remains to show that $\nu(-v) = (-1)^s \sum_{k>0} \frac{\mathrm{Tr}(\theta \widetilde{h}_s)^k}{k}$. This follows obviously form the next lemma.

Lemma 8.1.

$$Tr(\theta \widetilde{h}_s)^k = (-1)^s \sum_{a \in GF(k)} (\mathrm{ind} a) g(a) \tag{85}$$

Proof. Set $\beta = Q_0^{\lceil s \rceil}$. The set $GF(k, g)$ is exactly the fixed point set of the following composition:

$$\beta \xrightarrow{(v\downarrow)^k} B_{(-k)} = \bigvee_{h \in G_{(-k)}} \beta \cdot h \xrightarrow{\pi_g} \beta \tag{86}$$

where π_g is the map which sends every component of the wedge to ω, except the component βg, and this one is sent to β via the map $x \mapsto x g^{-1}$.

The point $\omega \in \beta$ is a fixed point of this map, and its index equals to 1. Apply now the Lefschetz-Dold fixed point formula and the proof of the Lemma and of the Main theorem is over. $\square$

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