ON THE ASYMPTOTICS OF MORSE NUMBERS OF FINITE COVERS OF MANIFOLDS

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Abstract. Let M be a closed connected manifold. We denote by $\mathcal{M}(M)$ the Morse number of M, that is, the minimal possible number of critical points of a Morse function f on M. M.Gromov posed the following question: Let $N_k, k \in \mathbf{N}$ be a sequence of manifolds, such that each N_k is an a_k-fold cover of M where $a_k \to \infty$ as $k \to \infty$. What are the asymptotic properties of the sequence $\mathcal{M}(N_k)$ as $k \to \infty$?

In this paper we study the case $\pi_1(M) \approx \mathbf{Z}^m, \dim M \geqslant 6$. Let $\xi \in H^1(M, \mathbf{Z}), \xi \neq 0$. Let $M(\xi)$ be the infinite cyclic cover corresponding to ξ, with generating covering translation $t : M(\xi) \to M(\xi)$. Let $M(\xi, k)$ be the quotient $M(\xi)/t^k$. We prove that $\lim_{k\to\infty} \mathcal{M}(M(\xi, k))/k$ exists. For ξ outside a subset $\mathfrak{M} \subset H^1(M)$ which is the union of a finite family of hyperplanes, we obtain the asymptotics of $\mathcal{M}(M(\xi, k))$ as $k \to \infty$ in terms of homotopy invariants of M related to the Novikov homology of M. It turns out that the limit above does not depend on ξ (if $\xi \notin \mathfrak{M}$). Similar results hold for the stable Morse numbers. Generalizations for the case of non-cyclic coverings are obtained.

Introduction and the statement of the result

Let M be a closed connected smooth manifold. Denote by $\mathcal{M}(M)$ the Morse number of M, that is, the minimal possible number of critical points of a Morse function on M. In the case $\pi_1(M) = 0, \dim M \geqslant 6$, this number is easily computable in terms of homology of M (see [14]). In the case of arbitrary fundamental group (even for $\dim M \geqslant 6$), the number $\mathcal{M}(M)$ is very difficult to compute: it depends on the simple homotopy type of M, the relevant algebraic constructions are rather complicated, and it is not easy to extract the needed numerical invariant (see [15], or [16], Ch. 7).

M.Gromov posed the following question:

Let $N_k, k \in \mathbf{N}$ be a sequence of manifolds, such that each N_k is an a_k-fold cover of the manifold M where $a_k \to \infty$ as $k \to \infty$. What are the asymptotic properties of the sequence $\mathcal{M}(N_k)$ as $k \to \infty$?

In the present article we study the problem for $\pi_1(M)$ free abelian and $\dim M \geqslant 6$. To formulate our results, we need some terminology from algebra. Denote $\mathbf{Z}[\mathbf{Z}^m]$ by Λ. Let

$$C_* = \{0 \leftarrow C_0 \overset{\partial_1}{\leftarrow} C_1 ... \overset{\partial_k}{\leftarrow} C_k \leftarrow 0\}$$

be a free finitely generated Λ-complex. Denote by $B_i(C_*)$ the rank of the module $H_i(C_*) \underset{\Lambda}{\otimes} \{\Lambda\}$ over the field of fractions $\{\Lambda\}$. Denote by $B(C_*)$ the sum of all $B_i(C_*)$. Consider now the homomorphism $\partial_{i+1} : C_{i+1} \to C_i$, and let $d = \text{rk } C_i$. Recall that the Fitting invariant $\mathcal{F}_t$ of the homomorphism ∂_{i+1} (see e.g. [4], p.278) is the ideal of Λ generated by the $(d-t) \times (d-t)$ subdeterminants of the matrix of ∂_{i+1} (for $t \geqslant d$ one sets $\mathcal{F}_t = \Lambda$ by

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definition). We shall denote the sequence $\mathcal{F}_0 \subset \mathcal{F}_1 \subset .. \subset \mathcal{F}_d$ of the Fitting invariants by $F(i)$.

Define the *reduced Fitting sequence* for ∂_{i+1} to be the sequence

$$FR(i) \qquad \mathcal{F}_s \subset ... \subset \mathcal{F}_r$$

where $\mathcal{F}_s$, respectively $\mathcal{F}_r$, is the first, respectively the last, term of the Fitting sequence $F(i)$, not equal to 0, respectively to Λ. The sequence $F(i)$ is not a homotopy invariant of C_*, but the sequence $FR(i)$ is (see e.g. [16], Ch.4,§2). We say that an ideal J of Λ is *numerically prime* if there is no number $l \in \mathbf{Z}, l \neq \pm 1$, such that every $R \in J$ is divisible by l, and we denote by $Q_i(C_*)$ the number of ideals in the sequence $FR(i)$ which are *not* numerically prime. Denote by $Q(C_*)$ the sum of all $Q_i(C_*)$.

A subgroup $G \subset \mathbf{Z}^m$ will be called an *integral hyperplane* if it is a direct summand of $\mathbf{Z}^m$ of rank $m-1$.

Now let M be a closed connected manifold, $\pi_1(M) \approx \mathbf{Z}^m, m \geqslant 1$. It is convenient to set $m = n+1, n \geqslant 0$. For every non-zero $\xi \in H^1(M)$ there is a unique connected infinite cyclic covering $\mathcal{P}_\xi : M(\xi) \to M$ such that $\mathcal{P}_\xi^*(\xi) = 0$. Denote by $M(\xi, k) \to M$ the k-fold cyclic covering of M obtained from $\mathcal{P}_\xi$. Let $C_*(\widetilde{M})$ be the cellular chain complex of the universal cover $\widetilde{M}$. We shall abbreviate $B(C_*(\widetilde{M}))$ to $B(M)$ and $Q(C_*(\widetilde{M}))$ to $Q(M)$.

Main Theorem. *Let* $\dim M \geqslant 6, \pi_1(M) \approx \mathbf{Z}^{n+1}, n \geqslant 0$. *Then:*

(1) *For any non-zero* $\xi \in H^1(M)$ *the limit* $\lim_{k\to\infty} \mathcal{M}(M(\xi,k))/k$ *exists.*
(2) *There is a subset* $\mathfrak{M} \subset H^1(M)$ *which is a finite union of integral hyperplanes in* $H^1(M)$, *and for every non-zero* $\xi \notin \mathfrak{M}$ *there is a real number* a *such that for every* $k \in \mathbf{N}$ *we have*
$$k(B(M) + 2Q(M)) - a \leqslant \mathcal{M}(M(\xi,k)) \leqslant k(B(M) + 2Q(M)) + a$$

Remarks: 1) A similar result holds for the stable Morse numbers of M, see §5.

2) The limit $\lim_{k\to\infty} \mathcal{M}(M(\xi,k))/k$ will be denoted by $\mu(M,\xi)$. The second point of the Main Theorem implies that for a "generic" cohomology class ξ we have $\mu(M,\xi) = B(M) + 2Q(M)$.

3) Denote by $\mathcal{M}_i(M)$ the minimal number of critical points of index i of a Morse function on M. The methods of the present paper allow also to prove that (under the assumptions of the Main Theorem) the limit $\lim_{k\to\infty} \mathcal{M}_i(M(\xi,k))/k$ exists, and that for all ξ, except those belonging to a finite union of integral hyperplanes, there is a real number a such that for every natural k we have

$$k(B_i(M) + Q_i(M) + Q_{i-1}(M)) - a \leqslant \mathcal{M}_i(M(\xi,k)) \leqslant k(B_i(M) + Q_i(M) + Q_{i-1}(M)) + a.$$

4) The numbers $B_s(M), Q_s(M)$ are closely related to the Novikov homology of M. Namely, $B_s(M)$ is equal to the Novikov Betti number $b_s(M,\xi)$ [7] for every non-zero class $\xi \in H^1(M)$ (note that $B_s(M)$ is also equal to the L^2-Betti number $b_s^{(2)}(M)$). Further, for every non-zero $\xi \notin \mathfrak{M}$ we have $Q_s(M) \leqslant q_s(M,\xi)$ where $q_s(M,\xi)$ is the Novikov torsion number [7](that follows from Remark 2.6 and Proposition 3.3 of the present paper).

The proof is outlined as follows. Assume that $\xi \in H^1(M)$ is indivisible. Let $f : M \to S^1$ be a Morse map, representing ξ, and let $V = f^{-1}(\lambda)$ be a regular level surface of f. We can assume that V is connected and that $\pi_1(V) \to \pi_1(M)$ is an isomorphism onto Ker ξ. Cut M along V, and obtain a cobordism W, such that the boundary ∂W has two connected components $\partial_0 W$ and $\partial_1 W$, each diffeomorphic to V. The cyclic cover $M(\xi)$ is the union of a countable family of copies of W glued successively to each other. The union W_k of k successive copies is a cobordism. Its boundary ∂W_k has two connected components $\partial_0 W_k$

and $\partial_1 W_k$, each diffeomorphic to V (see §4 for details). We show that $\mathcal{M}(M(\xi,k))$ and $\mathcal{M}(W_k,\partial_0 W_k)$ have the same asymptotics as $k\to\infty$ (see §4). Further, $\mathcal{M}(W_k,\partial_0 W_k)$ is equal to the Morse number of the $\mathbf{Z}[\mathbf{Z}^n]$-complex $C_*(\widetilde{W_k},\widetilde{\partial_0 W_k})$, see §1 for definitions. It turns out that the asymptotic behaviour of this Morse number (as $k\to\infty$) depends only on the chain homotopy type of $C_*(\widetilde{M(\xi)})$ (moreover it depends only on the Novikov completion of this complex). The definition and the properties of the corresponding invariant of chain complexes are the subject of §§1 - 3 of the paper. These sections are purely algebraic. It follows from the author's earlier result [9], that for ξ outside a finite union of integral hyperplanes in $H^1(M)$, the Novikov-completed chain homotopy type of $C_*(\widetilde{M(\xi)})$ is easily computable. (This is the subject of the second half of §2 and of §3.) This leads to the effective computation of the asymptotics presented in the main theorem.

I am grateful to M.Gromov for a stimulating discussion on the subject. He suggested in particular, that asymptotically the numbers $\mathcal{M}(N)$ above should be related to Novikov numbers. He indicated also that the Morse number $\mathcal{M}(M(\xi,k))$ should have the same asymptotics as the Morse number of the pair $(W_k,\partial_0 W_k)$ (see Prop. 4.1 of the present paper).

1. Morse numbers of chain complexes

In this section we define the notion of the Morse number for arbitrary chain complexes over $\mathbf{Z}[\mathbf{Z}^n]$ and we develop some basic properties of these numbers. We assume that the reader is familiar with §3 of [15] and with §1 of [11]. We denote $\mathbf{Z}[\mathbf{Z}^n]$ by R.

Terminological remark. Let A_*, B_* be chain complexes. We shall denote the chain maps from A_* to B_* as follows: $f_*: A_*\to B_*$, so that f_k is a homomorphism $A_k\to B_k$.

Definition 1.1. An *R-complex* is a chain complex $\{0\leftarrow C_0\leftarrow C_1 ... \leftarrow C_k\leftarrow 0\}$ of finitely generated R-modules. The *length* $l(C_*)$ of an R-complex C_* is the maximal number l such that $C_l\neq 0$. An R-complex C_* is called a *free R-complex* (or simply *f-complex*) if every C_i is a free finitely generated module over R.

Definition 1.2 [15]. Let C_* be an f-complex over R. The minimal possible number of free generators of an f-complex D_*, having the same homotopy type as C_*, is called the *Morse number of* C_* and denoted by $\mathcal{M}(C_*)$ (or by $\mathcal{M}_R(C_*)$, if we want to stress the base ring).

One of the consequences of the Quillen-Suslin theorem ([13],[17]) is that R is an s-ring, that is, every projective R-module is free (see [4], Ch.5, §4). R is also an IBN-ring, that is, the number of free generators of a free module is uniquely determined. Therefore, in the homotopy type of every f-complex over R there exists a minimal chain complex, that is, a complex D_* such that the number of free generators of D_* in each dimension is minimal over all the free complexes in this homotopy type (see [15], Th.3.7).

Definition 1.3. Let A_* be an R-complex. We call a *free model* of A_* a free R-complex A'_* together with a chain map $\alpha_*: A'_*\to A_*$ which is epimorphic and induces an isomorphism in homology. [1] If $\alpha_*: A'_*\to A_*$, $\beta_*: B'_*\to B_*$ are free models, and $f_*: A_*\to B_*$ is a chain map, then a chain map $F_*: A'_*\to B'_*$ is called *covering* of f if $\beta_* F_* = f_*\alpha_*$. Similar terminology is accepted for chain homotopies.

[1] Sometimes we shall say (by abuse of terminology) that the complex A'_* itself is a free model of A_*.

Lemma 1.4. *Let A_* be an R-complex. Then there is a free model of A_* and*

(1) *Every chain map $A_* \to B_*$ admits a covering with respect to any free models of A_* and B_*.*
(2) *Let $h_* : A_* \to B_{*+1}$ be a chain homotopy from f_* to g_*, and F_*, G_* be coverings of f_*, respectively g_*, with respect to some free models of A_*, B_*. Then there is a chain homotopy H_* from F_* to G_*, covering h_*.*
(3) *Two free models of a complex A_* are homotopy equivalent.*

Proof. To prove the existence of a free model, we proceed by induction in the length of A_*. If $l(A_*) = 0$, then it follows from the fact that every finitely generated module over R has a free finite resolution of finite length. To make the induction step, it suffices to construct a free model for a complex of the type $C_* = \{0 \leftarrow A_0 \stackrel{\partial_1}{\leftarrow} C_1 \stackrel{\partial_2}{\leftarrow} C_2 ... \stackrel{\partial_n}{\leftarrow} C_n \leftarrow 0\}$, where C_i are free finitely generated modules and A_0 is a finitely generated module. Let $B_* = \{0 \leftarrow A_0 \stackrel{\epsilon}{\leftarrow} E_0 \stackrel{d_1}{\leftarrow} E_1 \stackrel{d_2}{\leftarrow} ...\}$ be a finite free resolution of A_0. There is a chain map $\phi_* : C_* \to B_*$, such that $\phi_0 = \mathrm{id}$. Define now an R-complex

$$F_* = \{0 \leftarrow E_0 \stackrel{D_1}{\leftarrow} C_1 \oplus E_1 \stackrel{D_2}{\leftarrow} C_2 \oplus E_2 ...\}$$

setting $D_1(c_1, e_1) = \phi_1(c_1) + d_1(e_1)$ and $D_i(c_i, e_i) = (\partial_i(c_i), d_i(e_i) + (-1)^{i+1}\phi_i(c_i))$ for $i \geqslant 2$.

Define further a map $\gamma_* : F_* \to C_*$ to be the projection $(x, y) \mapsto x$ when $* \geqslant 1$ and set $\gamma_0 = \epsilon$. It is easy to check that F_* is indeed an f-complex, and that γ_* is a free model. The points (1) and (2) of our lemma are proved by a standard homological algebra argument; (3) follows from (2). □

Definition 1.5. The *Morse number* $\mathcal{M}(C_*)$ of a complex C_* is the Morse number of any of its free models.

Proposition 1.6. *Let $0 \leftarrow A_* \leftarrow B_* \leftarrow C_* \leftarrow 0$ be an exact sequence of R-complexes. Then 1) $\mathcal{M}(B_*) \leqslant \mathcal{M}(A_*) + \mathcal{M}(C_*)$, and 2) $\mathcal{M}(A_*) \leqslant \mathcal{M}(C_*) + \mathcal{M}(B_*)$.*

Proof. 1) The following lemma reduces the assertion to the case of free R-complexes.

Lemma 1.7. *Let $0 \leftarrow A_* \leftarrow B_* \leftarrow C_* \leftarrow 0$ be an exact sequence of R-complexes. Then there is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longleftarrow & A'_* & \longleftarrow & B'_* & \longleftarrow & C'_* & \longleftarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle\alpha_*} & & \downarrow{\scriptstyle\beta_*} & & \downarrow{\scriptstyle\gamma_*} & & \downarrow \\
0 & \longleftarrow & A_* & \longleftarrow & B_* & \longleftarrow & C_* & \longleftarrow & 0
\end{array}$$

where $\alpha_*, \beta_*, \gamma_*$ are free models.

Proof of the lemma. Let $g'_* : C'_* \to B'_*$ be a covering of $C_* \to B_*$ with respect to some free models B'_*, C'_*. We can assume that g'_* is a monomorphism onto a direct summand (the proof repeats almost verbally the proof of Lemma 1.8 from [11] and will be omitted). Now, setting $A'_* = B'_*/\mathrm{Im}\ C'_*$, we obtain the first line of the commutative diagram above. □

For the case of free complexes the assertion follows from the next one.

Lemma 1.8. *Let $0 \leftarrow A_* \leftarrow B_* \leftarrow C_* \leftarrow 0$ be an exact sequence of free R-complexes. Then there is an exact sequence $0 \leftarrow A'_* \leftarrow B'_* \leftarrow C'_* \leftarrow 0$ of free R-complexes such that $A'_* \sim A_*$, $B'_* \sim B_*$, $C'_* \sim C_*$, and A'_*, C'_* are minimal.*

The proof of this lemma is an exercise in the theory of minimal complexes ([16],§4), and will be left to the reader. □

To prove 2) observe that there is an exact sequence $\mathcal{Z} = \{0 \leftarrow \Sigma C_* \leftarrow D_* \leftarrow B_* \leftarrow 0\}$ where D_* is the mapping cone of j_*, and ΣC_* is the suspension of C_*. Now apply the point 1) to the sequence $\mathcal{Z}$. □

In some cases the first inequality of the preceding proposition turns to equality. We shall say that a complex C_* *is concentrated in dimensions* $[k, r]$ if $C_i = 0$ for $i < k$ and for $i > r$. We denote by $F(i, s)_*$ the chain complex $\{0 \leftarrow R^s \leftarrow 0\}$ concentrated in dimensions $[i, i]$.

Lemma 1.9. *1) For every f-complex C_* we have $\mathcal{M}(C_* \oplus F(i, s)_*) = \mathcal{M}(C_*) + s$. 2) Let C_*, D_* be f-complexes, concentrated respectively in dimensions $[a, b]$, and $[b, c]$. Then $\mathcal{M}(C_* \oplus D_*) = \mathcal{M}(C_*) + \mathcal{M}(D_*)$.*

The proof of this lemma is easily obtained from V.V.Sharko's criterion of minimality of chain complexes (see [15], Lemma 3.6). □

2. A numerical invariant of free chain $R((t))$-complexes

We denote $\mathbf{Z}[\mathbf{Z}^n]$ by R (as in the previous section). Let us start with a free $R[[t]]$-complex A_*. For $k \in \mathbf{N}$ denote by $A[k]_*$ the free R-complex $A_*/t^k A_*$, and denote its Morse number by $\mu_k(A_*)$, so $\mu_k(A_*) = \mathcal{M}_R(A[k]_*)$. Note that $\mu_k(A_*) + \mu_l(A_*) \geqslant \mu_{k+l}(A_*)$. (Indeed, consider the short exact sequence $0 \leftarrow A[k]_* \leftarrow A[k+l]_* \leftarrow A[l]_* \leftarrow 0$ and apply Proposition 1.6.) Therefore the sequence $\{\mu_k/k\}_{k \in \mathbf{N}}$ has a limit (see [12], ex. 98) which will be denoted by $\sigma(A_*)$. It is clear that $\sigma(A_*)$ is a chain homotopy invariant of A_*.

Now we shall consider free complexes over the ring $R((t)) = \sigma^{-1}R[[t]]$ where σ is the multiplicative set $\{t^l \mid l \in \mathbf{N}\}$. Let C_* be such a complex. We say that a chain subcomplex $D_* \subset C_*$ is a *basic subcomplex* if 1) D_* is a free $R[[t]]$-complex, and 2) $\sigma^{-1}D_* = C_*$. It is clear that each free complex C_* over $R((t))$ has basic subcomplexes.

Proposition 2.1. *Let C_* be a free $R((t))$-complex. Then the number $\sigma(D_*)$ is the same for every basic subcomplex $D_* \subset C_*$.*

Proof. Let D_*, F_* be basic subcomplexes. The Noetherian property of $R[[t]]$ and the condition (2) in the definition of a basic subcomplex imply immediately that there is $s \in \mathbf{N}$ such that $t^s F_* \subset D_*$. Since $\sigma(D_*) = \sigma(t^k D_*)$ we can assume that $t^s F_* \subset D_* \subset F_*$. Now for every $l \in \mathbf{N}$ we obtain two exact sequences of finitely generated chain complexes over R.

$$0 \leftarrow F_*/D_* \leftarrow F_*/t^l D_* \leftarrow D_*/t^l D_* \leftarrow 0 \tag{1}$$

$$0 \leftarrow F_*/t^l D_* \leftarrow F_*/t^{l+s} F_* \leftarrow t^l D_*/t^{l+s} F_* \leftarrow 0 \tag{2}$$

Applying Prop. 1.6 we deduce from (1) and (2) that $\mu_{l+s}(F_*) \leqslant C + \mu_l(D_*)$ where C does not depend on l. This implies easily that $\sigma(F_*) \leqslant \sigma(D_*)$; by symmetry we obtain $\sigma(F_*) = \sigma(D_*)$. □

Now we can define an invariant of $R((t))$-complexes. Namely, if C_* is a free $R((t))$-complex, we set $s(C_*) = \sigma(D_*)$ where D_* is any basic subcomplex of C_*. The number $s(C_*)$ depends only on the homotopy type of the $R((t))$-complex C_*. Indeed, a version of the Cockroft-Swan theorem ([11], Prop. 1.7) shows that it is sufficient to check that $\sigma(C_*)$ does not change when we add to C_* a complex of the form $\{0 \leftarrow R((t)) \xleftarrow{\mathrm{id}} R((t)) \leftarrow 0\}$. But this is obvious.

For some free $R((t))$-complexes the asymptotic properties of the Morse numbers are still better. We shall say that a sequence a_k of real numbers is *asymptotically linear* if $\exists C, \alpha, \forall k : \alpha k - C \leqslant a_k \leqslant \alpha k + C$. We shall say that a free $R((t))$-complex C_* is of *asymptotically linear growth* (abbreviation: *aslg*) if for some basic subcomplex $D_* \subset C_*$ the sequence $\mu_k(D_*)$ is asymptotically linear. Similarly to the proof of Proposition 2.1, one can show that in an *aslg*-complex *every* basic subcomplex D'_* has an asymptotically linear sequence $\mu_k(D'_*)$. Note also that the property of being *aslg* is homotopy invariant. We do not know if every $R((t))$-complex is *aslg*, but we shall prove that every complex of a certain class appearing in our geometrical setting is *aslg*, and we shall calculate its s-invariant. We need some definitions. *A monomial* of R is an element of the form ag where $a \in \mathbf{Z}$, and $g \in \mathbf{Z}^n$. Let $Z = z_k t^k + \ldots + z_l t^l \in R[t, t^{-1}]$ where $l, k \in \mathbf{Z}, k \leqslant l$, and $z_k, z_l \neq 0$. We shall say that Z is:

- *monic* if $z_k = \pm g, g \in \mathbf{Z}^n$ (Our terminology differs here from the standard one.)
- *numerically prime* if it is not divisible by an integer not equal to ± 1.
- *special* if each z_i is a monomial in R.

We denote $R((t))$ by $\mathcal{L}$.

Definition 2.2. Let C_* be a complex over $\mathcal{L}$. We shall say that C_* is *of principal type* if for every i an isomorphism

$$H_i(C_*) \approx \Big(\bigoplus_{j=1}^{b_i} \mathcal{L}\Big) \oplus \Big(\bigoplus_{s=1}^{q_i} \mathcal{L}/a_s^{(i)}\mathcal{L}\Big) \tag{3}$$

is fixed, and for every i, s: 1) $a_s^{(i)} \in R[t, t^{-1}]$ and $a_s^{(i)}$ is special, non-zero and not monic 2) $a_s^{(i)} \mid a_{s+1}^{(i)}$.

For a complex C_* of principal type we denote by $\varkappa_i$ the number of those polynomials $a_s^{(i)}$ which are not numerically prime.

Theorem 2.3. *Let C_* be a free $\mathcal{L}$-complex of principal type. Then C_* is of asymptotically linear growth, and $s(C_*) = \sum_i b_i + 2\sum_i \varkappa_i$.*

Proof. We can assume that all the elements $a_s^{(i)}$ in (3) are of the form $z_0 + \ldots + z_k t^k$ where $z_0 \in \mathbf{Z}, z_0 \neq 0$. Denote by $\mathcal{F}(i)_*$ the free $\mathcal{L}$-complex $\{0 \leftarrow \mathcal{L}^{b_i} \leftarrow 0\}$ concentrated in dimensions $[i, i]$. For $\rho \in \mathcal{L}$ and $i \in \mathbf{N}$, denote by $\tau(\rho, i)_*$ the free complex $\{0 \leftarrow \mathcal{L} \xleftarrow{\rho} \mathcal{L} \leftarrow 0\}$ concentrated in dimensions $[i, i+1]$. Note that if $\rho \in R[[t]]$ then $\tau(\rho, i)_*$ has a standard basic subcomplex $\{0 \leftarrow R[[t]] \xleftarrow{\rho} R[[t]] \leftarrow 0\}$ which will be denoted by $\tau'(\rho, i)_*$.

For a given i denote by π (resp. by ν) the set of all s such that $a_s^{(i)}$ is numerically prime (resp. *not* numerically prime). Set

$$\mathcal{TP}(i)_* = \bigoplus_{s\in\pi} \tau(a_s^{(i)}, i)_*, \quad \mathcal{TN}(i)_* = \bigoplus_{s\in\nu} \tau(a_s^{(i)}, i)_*, \quad \mathcal{T}(i)_* = \mathcal{TP}(i)_* \oplus \mathcal{TN}(i)_*.$$

(Morally, $\mathcal{F}(i)_*$ corresponds to the free part of the homology $H_i(C_*)$, and $\mathcal{T}(i)_*$ to the torsion part.) The complexes $\mathcal{T}(i)_*, \mathcal{TP}(i)_*, \mathcal{TN}(i)_*$ have basic subcomplexes $\mathcal{T}'(i)_*, \mathcal{TP}'(i)_*, \mathcal{TN}'(i)_*$ which are obtained as direct sums of the corresponding complexes $\tau'(\rho, i)_*$.

Lemma 5.1 of [8] implies that C_* is homotopy equivalent to the direct sum (over all i) of the complexes $\mathcal{F}(i)_* \oplus \mathcal{T}(i)_*$. We call this direct sum *principal model* for C_*. Applying successively Lemma 1.9, it is easy to deduce our theorem from the next lemma.

Lemma 2.4. *For every i, k we have: (1) $\mu_k(\mathcal{TN}'(i)_*) = 2k\varkappa_i$, (2) $\mu_k(\mathcal{T}'(i)_*) \geqslant 2k\varkappa_i$. (3) For every i the sequence $\{\mu_k(\mathcal{TP}'(i)_*)\}_{k\in\mathbf{N}}$ is bounded.*

Proof. 1) Fix some i. The condition 2) from the definition 2.2 implies that there is a prime number p such that every polynomial $a_s^{(i)}$ which is not numerically prime is divisible by p. Abbreviate $\mathcal{TN}'(i)_*$ to L_*; the inequality $\mu_k(L_*) \leqslant 2k\varkappa_i$ is immediate. To prove the inverse inequality consider an $\mathbf{F}_p$-complex $L[k]_* \underset{R}{\otimes} \mathbf{F}_p$ (where $\mathbf{F}_p$ is considered as $R = \mathbf{Z}[\mathbf{Z}^n]$-module via the trivial $\mathbf{F}_p$-representation of $\mathbf{Z}^n$). It is obvious that $\mathcal{M}(L[k]_*)$ is not less than $\dim H_*(L[k]_* \otimes \mathbf{F}_p)$ which equals $2k\varkappa_i$. A similar argument proves the point 2). To prove 3) it suffices to show that if $\rho = a_0 + a_1 t + \ldots + a_r t^r \in R[t]$ is special, numerically prime, and has $a_0 \neq 0$, then $\mu_k(\tau'(\rho, i)_*)$ is bounded. Write $a_j = A_j g_j$ with $g_j \in \mathbf{Z}^n$ and $A_j, a_0 \in \mathbf{Z}$, and note that $a_0, A_1, \ldots, A_r$ are relatively prime. Abbreviate $\tau'(\rho, i)_*$ to S_*. If $k \geqslant 1$ then $S[k+r]_*$ is a free R-complex of the form $\{0 \leftarrow R^{k+r} \overset{\mathcal{A}_k}{\leftarrow} R^{k+r} \leftarrow 0\}$ where $\mathcal{A}_k$ is the following $(k+r) \times (k+r)$-matrix

$$\mathcal{A}_k = \begin{pmatrix} a_0 & 0 & \ldots & 0 & 0 \\ a_1 & a_0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_r & a_{r-1} & \ldots & 0 & 0 \\ 0 & a_r & \ldots & a_0 & 0 \\ \ldots & \ldots & \ldots & a_1 & a_0 \end{pmatrix} \tag{4}$$

Denote by $\mathcal{B}_k$ the matrix formed by the first k columns of the matrix $\mathcal{A}_k$, and denote by I_k the ideal of R generated by all $k \times k$ subdeterminants of $\mathcal{B}_k$. The point 2) of the next lemma implies our assertion.

Lemma 2.5. *1) $I_k = R$; 2) The submodule $\mathcal{A}_k(R^{k+r})$ of R^{k+r} contains a direct summand of R^{k+r}, which is a free module of rank k.*

Proof. The point 2) follows from the point 1) by a standard argument based on the Quillen-Suslin theorem (we leave the details to the reader). Proceeding to the proof of the point 1), note that $a_0^k \in I_k$. Therefore we can assume that $a_0 \neq \pm 1$. It suffices to show that for every prime number p from the prime decomposition of a_0 there is an element $C \in R$ such that $1 + pC \in I_k$. To show this, recall that the numbers A_j are relatively prime. So there is i such that $p \mid A_j$ for $j < i$, and $p \nmid A_i$. Consider the $k \times k$ subdeterminant of the matrix $\mathcal{B}_k$ formed by all the columns and by the lines from $i+1$ to $i+k$. The terms of the principal diagonal are all equal to A_i, and the terms above the diagonal are divisible by p. Therefore this subdeterminant equals $Q + pC$ where Q is a monomial of the form $Q = qg$ with $(q, p) = 1$, and Lemma 2.5 follows. This finishes the proof of Theorem 2.3. $\square$

Remark 2.6. Let $L = \mathbf{Z}[t, t^{-1}], \hat{L} = \mathbf{Z}((t))$. The homomorphism $\epsilon : \mathbf{Z}^n \to \{1\}$ extends to ring homomorphisms $e : R[t, t^{-1}] \to L$ and $\hat{e} : \mathcal{L} \to \hat{L}$. Therefore for every $\mathcal{L}$-complex C_* we can form an $\hat{L}$-complex $\overline{C_*} = C_* \underset{\mathcal{L}}{\otimes} \hat{L}$. Assume that C_* is of principal type. Using the homotopy equivalence $C_* \sim \oplus_i(\mathcal{F}(i)_* \oplus \mathcal{T}(i)_*)$ from the proof of Theorem 2.3, it is easy to see that $\overline{C_*}$ is also of principal type and

$$H_i(\overline{C_*}) \approx \Big(\bigoplus_{j=1}^{b_i} \hat{L}\Big) \oplus \Big(\bigoplus_{s=1}^{q_i} \hat{L}/\alpha_s^{(i)}\hat{L}\Big) \tag{5}$$

with $\alpha_s^{(i)} = e(a_s^{(i)})$. Since $a_s^{(i)}$ are special and not monic, b_i and q_i are equal respectively to the rank and to the torsion number of $H_i(\overline{C_*})$ over the principal ring $\hat{L}$. It is easy to see that the above decomposition satisfies Definition 2.2, therefore $\overline{C_*}$ is *aslg*. Further, $a_s^{(i)}$ is numerically prime if and only if $\alpha_s^{(i)}$ is, and this implies $s(C_*) = s(\overline{C_*})$.

3. A numerical invariant $S(C_*, \xi)$

In this section $\Lambda = \mathbf{Z}[\mathbf{Z}^{n+1}]$, C_* is a free Λ-complex, and $\xi : \mathbf{Z}^{n+1} \to \mathbf{Z}$ is a non-zero homomorphism. We define a numerical invariant $S(C_*, \xi)$. For the cohomology classes ξ outside a finite union of integer hyperplanes we calculate $S(C_*, \xi)$ in terms of the reduced Fitting sequences of the boundary operators of C_* (the mentioned finite union of integer hyperplanes depends on C_*). An element $z \in \Lambda$ is called *ξ-monic* if $z = \pm g + z_0$ where $g \in \mathbf{Z}^{n+1}$ and supp $z_0 \subset \{h \in \mathbf{Z}^{n+1} \mid \xi(h) < \xi(g)\}$. An element z is called *ξ-special* if any two different elements $a, b \in$ supp z satisfy $\xi(a) \neq \xi(b)$. We denote by S_ξ the multiplicative subset of all ξ-monic polynomials, and we denote by $\Lambda_{(\xi)}$ the localization $S_\xi^{-1}\Lambda$.

Definition 3.1. A subset $X \subset \mathbf{Z}^k$ will be called *small* if it is a finite union of integer hyperplanes.

Theorem 3.2 ([9], Th. 0.1). *There is a small subset $\mathfrak{N} \subset \mathrm{Hom}(\mathbf{Z}^{n+1}, \mathbf{Z})$ such that for every $\xi \notin \mathfrak{N}$ and every p we have:*

$$S_\xi^{-1} H_p(C_*) \approx \left(\bigoplus_{i=1}^{b_p(C_*,\xi)} \Lambda_{(\xi)} \right) \oplus \left(\bigoplus_{j=1}^{q_p(C_*,\xi)} \Lambda_{(\xi)}/a_j^{(p)}\Lambda_{(\xi)} \right) \tag{6}$$

where $a_j^{(p)} \in \Lambda$ are non-zero and not ξ-monic elements of Λ (depending on ξ), and $a_j^{(p)} \mid a_{j+1}^{(p)}$. $\square$

Sketch of the proof of Theorem 3.2. We shall recall here the basic idea of the proof of 3.2 following [8] and [9], see [9] for the full proof. Let $\xi : \mathbf{Z}^{n+1} \to \mathbf{R}$ be a non-zero homomorphism. Similarly to the above, we define the notion of ξ-monic polynomial, and we introduce the ring $\Lambda_{(\xi)} = S_\xi^{-1}\Lambda$. (We take here the occasion to note that for the first time the localization technique was applied to Novikov rings and Novikov inequalities in the paper [2] of M.Farber. In this paper M.Farber considers the ring $S_\xi^{-1}\Lambda$, where $\Lambda = \mathbf{Z}[\mathbf{Z}]$, and ξ is the inclusion of $\mathbf{Z}$ to $\mathbf{R}$.) Recall next the definition of the Novikov ring Λ_ξ^- (see, e.g. [11], p. 326). Denote by $\hat{\hat{\Lambda}}$ the abelian group of all the linear combinations of the form $\lambda = \sum_{g \in \mathbf{Z}^{n+1}} n_g g$ where $n_g \in \mathbf{Z}$ and the sum may be infinite. Let Λ_ξ^- be the subset of $\hat{\hat{\Lambda}}$ consisting of $\lambda \in \hat{\hat{\Lambda}}$ such that for every $c \in \mathbf{R}$ the set supp $\lambda \cap \xi^{-1}([c, \infty[)$ is finite. This subset is called *Novikov ring* (it is not difficult to see that Λ_ξ^- has a natural ring structure).

Proceeding to Theorem 3.2, recall that Theorem 1.4 of [8] asserts that if ξ is injective then $\Lambda_{(\xi)}$ is euclidean. (The proof is based on a theorem by J.Cl.Sikorav, which asserts that if ξ is injective then Λ_ξ^- is euclidean, see [8], Th. 1.1.)

Therefore we obtain the decomposition (6) for any monomorphism ξ. This implies that (6) is true for every homomorphism η belonging to an open conical set containing ξ (see [8], the beginning of §7). Since the monomorphisms are dense in $\mathrm{Hom}(\mathbf{Z}^{n+1}, \mathbf{R})$, we obtain the decomposition (6) for every ξ belonging to some open and dense conical subset U in $\mathrm{Hom}(\mathbf{Z}^{n+1}, \mathbf{R})$.

Analyzing further the algebraic structure of the rings $\Lambda_\xi^-, \Lambda_{(\xi)}$ (it is done in [9]), one can prove that U can be chosen in such a way that the complement $\mathrm{Hom}(\mathbf{Z}^{n+1}, \mathbf{R}) \setminus U$ is a finite union $\mathfrak{R} = \cup_i L_i$ of hyperplanes L_i. Moreover, each L_i is of the form $l_i \otimes \mathbf{R}$ where $l_i \in \mathrm{Hom}(\mathbf{Z}^{n+1}, \mathbf{Z})$ is an integer hyperplane. That proves Theorem 3.2. (See [9] for more information about the numbers $b_p(C_*, \xi), q_p(C_*, \xi)$.) $\square$

The following proposition relates the above numbers and the elements $a_j^{(p)}$ to the Fitting invariants of the boundary operators of C_*. We need some definitions. Let $A : F_1 \to F_2$ be a homomorphism of free finitely generated Λ-modules. Let $J_0 \subset ... \subset J_r$ be the reduced Fitting sequence of A, let $\rho_i \in \Lambda$ be the g.c.d. of the elements of J_i, and denote ρ_i/ρ_{i+1} by $\zeta_i(A)$. Let $\xi : \mathbf{Z}^{n+1} \to \mathbf{Z}$ be a non-zero homomorphism. Denote by $k(A, \xi)$ the number of those ρ_i which are not ξ-monic. Set $R_j(A, \xi) = \zeta_{k-j}(A, \xi)$ where $k = k(A, \xi)$. Now let $C_* = \{0 \leftarrow ... \leftarrow C_{i-1} \overset{\partial_i}{\leftarrow} C_i \leftarrow ...\}$ be a free Λ-complex.

Proposition 3.3. *Assume that for $\xi \in \mathrm{Hom}(\mathbf{Z}^{n+1}, \mathbf{Z})$ and every p the decomposition (6) holds. Then: 1) $b_p(C_*, \xi) = B_p(C_*)$. 2) $q_p(C_*, \xi) = k(\partial_{p+1}, \xi)$. 3) For every p, s the elements $a_s^{(p)}$ and $R_s(\partial_{p+1}, \xi)$ are equal up to multiplication by a ξ-monic element. 4) $Q_p(C_*)$ equals to the number of not numerically prime $a_s^{(p)}$.*

Proof. Recall that the reduced Fitting sequences are homotopy invariants of C_*. This implies that $k(\partial_i, \xi)$ and $\varkappa(\partial_i, \xi)$ are homotopy invariants of C_* for fixed ξ. 1) is obvious. Further, let $0 \leqslant p \leqslant n$ and let $J_0 \subset ... \subset J_r$ be the reduced Fitting sequence for $\partial_{p+1} : C_{p+1} \to C_p$. Then the reduced Fitting sequence $FR(p)$ of the localized complex is a part of the sequence $S_\xi^{-1} J_0 \subset ... \subset S_\xi^{-1} J_r$, and the g.c.d. of $S_\xi^{-1} J_i$ is still ρ_i. Therefore the sequence $FR(p)$ has $k(\partial_{p+1}, \xi)$ terms. Using the principal model for C_*, it is easy to prove that $FR(p)$ equals to the sequence of principal ideals $(a_1^{(p)} \cdot ... \cdot a_N^{(p)}), (a_1^{(p)} \cdot ... \cdot a_{N-1}^{(p)}), ..., (a_1^{(p)})$ where $N = q_p(C_*, \xi)$. 2), 3) and 4) follow easily. $\square$

Let $\xi = l\overline{\xi}$ where $\overline{\xi} : \mathbf{Z}^{n+1} \to \mathbf{Z}$ is an epimorphism. Choose an isomorphism $\mathrm{Ker}\ \xi \approx \mathbf{Z}^n$, and an element $t \in \mathbf{Z}^{n+1}$ such that $\overline{\xi}(t) = -1$. We obtain a decomposition $\mathbf{Z}^{n+1} = \mathrm{Ker}\ \xi \oplus \mathbf{Z}$ and an isomorphism $I(\xi) : \Lambda \approx R[t, t^{-1}]$. Consider the free $\mathcal{L}$-complex $\widehat{C}_*(\xi) = C_* \underset{\Lambda}{\otimes} \mathcal{L}$, where C_* is an $R[t, t^{-1}]$-module via the isomorphism $I(\xi)^{-1}$. Set $S(C_*, \xi) = s(\widehat{C}_*(\xi))$ (it is easy to check that $S(C_*, \xi)$ depends indeed only on ξ and C_*).

Theorem 3.4. *There is a small subset $\mathfrak{M} \subset \mathrm{Hom}(\mathbf{Z}^{n+1}, \mathbf{Z})$ such that for every $\xi \notin \mathfrak{M}$ the complex $\widehat{C}_*(\xi)$ is of asymptotically linear growth and $S(C_*, \xi) = B(C_*) + 2Q(C_*)$.*

Proof. The $I(\xi)$-image of a ξ-monic polynomial is obviously invertible in $\mathcal{L}$, therefore the homomorphism $\Lambda \to R[t, t^{-1}] \to R((t))$ factors through $\Lambda_{(\xi)}$. Therefore, for every ξ outside a small subset $\mathfrak{N}$ the formula (3) holds with $C_* = \widehat{C}_*(\xi)$. The complex $\widehat{C}_*(\xi)$ is not necessarily of principal type, since the polynomials $a_s^{(i)}$ in the decomposition (3) are not necessarily special. But Proposition 3.3 implies that the elements $a_s^{(i)}$ in the decomposition (3) can be chosen between the elements of the finite set $\{\zeta_j(\partial_{i+1})\}$. Therefore, adding to $\mathfrak{N}$ some integer hyperplanes if necessary, we can assume that all $a_s^{(i)}$ are special. Now our theorem follows from 2.3. $\square$

4. Proof of the main theorem

Let M be a closed connected manifold and $\xi \in H^1(M, \mathbf{Z})$ be an indivisible cohomology class. Denote by $\mathcal{P}_\xi : M(\xi) \to M$ the infinite cyclic covering such that $\mathcal{P}_\xi^*(\xi) = 0$. Choose a generator $t \in \mathbf{Z} \approx \pi_1(M)/\mathrm{Ker}\ \xi$ of the structure group of $\mathcal{P}_\xi$ such that $\xi(t) = -1$. Let $f : M \to S^1$ be a Morse map representing ξ, and let V be its regular level surface, say, $V = f^{-1}(\lambda)$. Then f lifts to a Morse function $F : M(\xi) \to \mathbf{R}$ and V lifts to $F^{-1}(\lambda') \subset M(\xi)$. Denote by V^- the subset $F^{-1}(]-\infty, \lambda'])$. For $k \geqslant 1$ denote by W_k the cobordism $F^{-1}([\lambda' - k, \lambda'])$, $\partial W_k \approx V \sqcup t^k V$, and denote by $\alpha(k, V)$ its Morse number, that is, the minimal number of critical points of a Morse function on the cobordism W_k. Note that $\alpha(k + n, V) \leqslant \alpha(k, V) + \alpha(n, V)$. Therefore the sequence $\alpha(k, V)/k$ has a limit as $k \to \infty$. Denote this limit by $\alpha(V)$. It is easy to see that $\alpha(V)$ depends only on M and ξ, so we denote it by $\alpha(M, \xi)$. An elementary construction, using the gluing of the upper part V of ∂W_k to the lower part $t^k V$, allows to obtain the inequality $\mathcal{M}(M(\xi, k)) \leqslant \alpha(k, V) + 2\mathcal{M}(V)$. In particular, if ξ is represented by a fibration over S^1, the sequence $\mathcal{M}(M(\xi, k))$ is bounded.

In general, it is all what we can say about the numbers $\alpha(k, V)$ and their relation to the asymptotics of the Morse numbers of cyclic covers. However, if the fundamental group of M is free abelian and $\dim M \geqslant 6$, one can say much more.

Proposition 4.1. *Let M be a closed connected manifold with a free abelian fundamental group. Assume that $\dim M \geqslant 6$. Let $\xi \in H^1(M), \xi \neq 0$. Then the sequence $\alpha(k, V) - \mathcal{M}(M(\xi, k))$ is bounded.*

Proof. Let $\pi_1(M) \approx \mathbf{Z}^{n+1}, n \geqslant 0$. An argument similar to the one used in ([3], p.325) shows that one can choose V above such that the embedding $V \hookrightarrow M$ induces an isomorphism $\pi_1(V) \to \mathrm{Ker}\ \xi$ (such V will be called *admissible ξ-splittings*, see [8], p. 371). In this case all the embeddings $V \subset W_k \subset M(\xi) \supset t^k V$ induce isomorphisms of π_0 and of π_1. Choose an element $T \in \mathbf{Z}^{n+1}$, such that $\xi(T) = -1$. Let $\overline{M(\xi, k)} = \widetilde{M}/T^k$, then there is a $\mathbf{Z}^n$-covering $\overline{M(\xi, k)} \to M(\xi, k)$. Choose a triangulation of M such that V is a subcomplex of M; then we obtain a t-invariant triangulation of $M(\xi)$ and the corresponding triangulations of all the covers. There are two exact sequences of corresponding $\mathbf{Z}[\mathbf{Z}^n]$-complexes:

$$0 \to C_*(\widetilde{V}) \to C_*(\overline{M(\xi, k)}) \to C_*(\overline{M(\xi, k)}, \widetilde{V}) \to 0 \tag{7}$$

$$0 \to C_*(\widetilde{V}) \to C_*(\widetilde{W_k}, \widetilde{t^k V}) \to C_*(\overline{M(\xi, k)}, \widetilde{V}) \to 0 \tag{8}$$

Proposition 1.6 implies that there is $C = C(V)$ such that for every $k > 0$ we have: $\mathcal{M}(C_*(\overline{M(\xi, k)})) \geqslant \mathcal{M}(C_*(\widetilde{W_k}, \widetilde{t^k V})) - C$. Since $\mathcal{M}(C_*(\widetilde{W_k}, \widetilde{t^k V})) = \alpha(k, V)$ (see the proof of Corollary 6.3 in [15]), our Proposition is proved. $\square$

Proof of the Main theorem. The point 1) follows immediately from 4.1 (with $\alpha(M, \xi) = \lim_{k\to\infty} \mathcal{M}(M(\xi, k))/k$). To prove 2) note that Theorem 3.4 implies that for all $\xi \in H^1(M)$ outside a small subset $\mathfrak{M} \subset H^1(M)$ the complex $(C_*(\widetilde{M}))\widehat{\ }(\xi)$ is *aslg*, and $S(C_*(\widetilde{M}), \xi) = B(M) + 2Q(M)$. Note further that for every admissible ξ-splitting V the complex $D_* = C_*(\widetilde{V^-}) \underset{\Lambda}{\otimes} R[[t]]$ is a basic subcomplex of $C_*(\widetilde{M}) \underset{\Lambda}{\otimes} R((t))$, and that $\mu_k(D_*) = \mathcal{M}(C_*(\widetilde{W_k}, \widetilde{t^k V})) = \alpha(k, V)$. Now just apply Proposition 4.1. $\square$

Remark 4.2. A similar argument, together with Remark 2.6, shows that for ξ outside a small subset of $H^1(M)$ the sequence $(B(M) + 2Q(M))k - \mathcal{M}_{\mathbf{Z}}(C_*(M(\xi, k)))$ is bounded where $C_*(M(\xi,k))$ is the chain complex of $M(\xi,k)$, defined over $\mathbf{Z}$ (see 1.2 for the definition of $\mathcal{M}_{\mathbf{Z}}(\cdot)$).

5. Further results and conjectures

5.1. **Stable Morse numbers.** Let M be a closed connected manifold. Recall that a *stable Morse function* on M is a Morse function $f : M \times \mathbf{R}^N \to \mathbf{R}$ such that there is a compact $K \subset M \times \mathbf{R}^N$, and a non-degenerate quadratic form Q of index 0 on $\mathbf{R}^N$ such that $f(x,y) = Q(y)$ outside K. Let $f : M \times \mathbf{R}^N \to \mathbf{R}$ be a stable Morse function. Denote by $\widetilde{m}_p(f)$ the number of critical points of f of index $p + N/2$. The Morse-Pitcher inequalities hold: $\widetilde{m}_p(f) \geqslant b_p(M) + q_p(M) + q_{p-1}(M)$.

Denote by $\mathcal{MS}(M)$ the minimal possible number of critical points of a stable Morse function on M; we have $\mathcal{MS}(M) \leqslant \mathcal{M}(M)$.

Theorem 5.1. *Let* $\dim M \geqslant 6, \pi_1(M) \approx \mathbf{Z}^{n+1}, n \geqslant 0$. *There is a subset* $\mathfrak{M} \in H^1(M)$ *which is a finite union of integral hyperplanes in* $H^1(M)$, *and for every* $\xi \notin \mathfrak{M}$ *there is a real number* a *such that for every* $k \in \mathbf{N}$ *we have*

$$k(B(M) + 2Q(M)) - a \leqslant \mathcal{MS}(M(\xi,k)) \leqslant k(B(M) + 2Q(M)) + a$$

For the proof just recall that (by Remark 4.2) for every ξ outside a small subset of $H^1(M)$ we have $k(B(M) + 2Q(M)) \leqslant \mathcal{M}_{\mathbf{Z}}(C_*(M(\xi,k))) + C$. $\square$

We refer to [1] for a systematic exposition of the theory of stable Morse functions and its applications to Lagrangian intersection theory.

5.2. **Non generic cohomology classes $\xi \in H^1(M)$.** Here we construct a manifold M with $\pi_1(M) \approx \mathbf{Z}^2$ and $\dim M \geqslant 6$, and a class $\xi \in H^1(M)$ such that $\mu(M,\xi) \neq B(M) + 2Q(M)$. Let N be a closed connected manifold with $\pi_1(N) \approx \mathbf{Z}, \dim N \geqslant 5$ and $B(N) \neq 0$. Set $M = N \times S^1$. Let $\lambda : \pi_1(M) \to \mathbf{Z}$, resp. $\xi : \pi_1(M) \to \mathbf{Z}$, be epimorphisms with $\operatorname{Ker} \lambda = \pi_1(N)$, resp. $\operatorname{Ker} \xi = \pi_1(S^1)$. Then λ is represented by the fibration $N \times S^1 \to S^1$. Therefore there is an open cone $C \subset H^1(M,\mathbf{R})$ containing λ, such that every integral non-divisible $\lambda' \in C$ can be represented by a fibration, and so $\mu(M,\lambda') = B(M) + 2Q(M) = 0$.

Now we shall show that $\mu(M,\xi) \neq 0$. Note that $M(\xi,k) = N_k \times S^1$ where N_k is the k-fold cyclic cover of N, and therefore $\mathcal{M}(M(\xi,k)) \geqslant \mathcal{M}_{\mathbf{Z}}(C_*(N_k \times S^1))$. We shall obtain a lower estimate for $\mathcal{M}_{\mathbf{Z}}(C_*(N_k \times S^1))$. Let $\xi_0 : \pi_1 N \to \mathbf{Z}$ be the restriction $\xi \mid \pi_1 N$. Let $\overline{N} \to N$ be the infinite cyclic covering and $V \subset N$ be an admissible ξ_0-splitting. Let W_k be the corresponding cobordism in $\overline{N}$. Using exact sequences similar to the exact sequences (7, 8) from §4, it is easy to prove that $\mathcal{M}_{\mathbf{Z}}(C_*(N_k \times S^1)) - \mathcal{M}_{\mathbf{Z}}(C_*(W_k \times S^1, t^kV \times S^1))$ is bounded. Let $X_k = W_k/t^kV, \quad Y_k = (W_k/t^kV) \times S^1$. Then $\mathcal{M}_{\mathbf{Z}}(C_*(Y_k)) = \sum_p (b_p(Y_k) + q_p(Y_k) + q_{p-1}(Y_k))$. Since $H_*(X_k \times S^1) = H_*(X_k) \oplus H_{*-1}(X_k)$ we have $q_p(X_k \times S^1) \geqslant q_p(X_k)$, and $\mathcal{M}_{\mathbf{Z}}(C_*(Y_k)) \geqslant \mathcal{M}_{\mathbf{Z}}(C_*(X_k))$. Recall from Proposition 4.1 that the sequence $\mathcal{M}(N_k) - \mathcal{M}_{\mathbf{Z}}(C_*(X_k))$ is bounded. Therefore $\mathcal{M}_{\mathbf{Z}}(C_*(N_k \times S^1)) \geqslant \mathcal{M}(N_k) + C \geqslant kB(N) + C'$ (where C and C' do not depend on k), and, finally, $\mu(M,\xi) \geqslant B(N)$.

5.3. **Non cyclic finite coverings.** The Main Theorem of the present paper allows also to deal with some non cyclic finite coverings.

Proposition 5.2. *Let M be a closed connected manifold,* $\dim M \geqslant 6$, *and* $\pi_1(M) \approx \mathbf{Z}^{n+1}, n \geqslant 0$. *Let* $M_k \to M$ *be the finite covering corresponding to the subgroup* $k\mathbf{Z}^{n+1}$.

Then

$$\lim_{k\to\infty}\frac{\mathcal{M}(M_k)}{k^{n+1}} = B(M) + 2Q(M)$$

Proof. We shall give only the main idea of the proof. Let $\xi : \pi_1(M) \to \mathbf{Z}$ be an epimorphism not belonging to the small set $\mathfrak{M}$ of the Main Theorem. Then $M_k \to M$ factors through $M(\xi, k) \to M$, and therefore $\mathcal{M}(M_k)/k^n \leqslant k(B(M) + 2Q(M)) + C$. To obtain the lower estimate, note that the $\mathbf{Z}^n$-covering $\overline{M(\xi,k)} \to M(\xi, k)$ factors through $\overline{M(\xi,k)} \to M_k$ which corresponds to the subgroup $G_k = k\mathbf{Z}^n \subset \mathbf{Z}^n$. Therefore $\mathcal{M}(M_k) \geqslant \mathcal{M}_{\mathbf{Z}[G_k]}(C_*(\overline{M(\xi,k)}))$. To obtain the lower estimate for $\mathcal{M}_{\mathbf{Z}[G_k]}(C_*(\overline{M(\xi,k)}))$, use (7) and (8) to reduce the question to finding the corresponding lower estimate for $\mathcal{M}_{\mathbf{Z}[G_k]}(C_*(\widetilde{W}_k, \widetilde{t^kV}))$. Then proceed similarly to the proof of the Main Theorem. $\square$

It seems that a similar result must hold for more general systems of non-cyclic finite coverings. To discuss a more general setting we need some definitions. Let G be a group. A sequence of subgroups $G = G_0 \supset G_1 \supset ...$ will be called a *tower* if for every i the index of G_i in G is finite. It will be called a *nested tower* if, moreover, $\cap_n G_n = \{0\}$.

If M is a closed connected manifold with $\pi_1(M) = G$ and $\mathbb{G} = \{G_n\}$ is a tower of subgroups of $\pi_1(M)$, then there is the corresponding tower of finite coverings $M = M_0 \leftarrow ... \leftarrow M_n ...$ of M. The sequence $\mathcal{M}(M_k)/ \mid G/G_k \mid$ is decreasing, therefore it has a limit which will be denoted by $\mu(\mathbb{G})$. Recall a theorem of W.Lück [5], saying that if $\mathbb{G}$ is nested, then the limit of the sequence $b_p(M_k)/ \mid G/G_k \mid$ exists and is equal to $b_p^{(2)}(M)$.

Problem. Is it true in general (at least for $\dim M \geqslant 6$) that $\mu(\mathbb{G})$ does not depend on the choice of the nested tower $\mathbb{G}$?

We believe that $\mu(\mathbb{G})$ does not depend on $\mathbb{G}$ for the case of free abelian fundamental group. Here is a result in this direction. Let $G = \mathbf{Z}^{n+1}$. For a tower $\mathbb{G} = \{G = G_0 \supset G_1 \supset ...\}$ we denote $\max_i m(G/G_i)$ by $r(\mathbb{G})$ (here $m(H)$ stands for the minimal number of generators of H). Denote by $\mathbb{G}^{[i]}$ the tower $G_i \supset G_{i+1} \supset ...$ The sequence $r(\mathbb{G}^{[i]})$ is decreasing, denote its limit by $\rho(\mathbb{G})$; then $\rho(\mathbb{G}) \leqslant \mathrm{rk}\, G$.

Proposition 5.3. *Let M be a closed connected manifold, $\dim M \geqslant 6$, $\pi_1(M) \approx \mathbf{Z}^{n+1}, n \geqslant 0$. Let $\mathbb{G} = \{G_k\}$ be a tower, and let $M_k \to M$ be the finite covering corresponding to $G_k \subset \pi_1(M)$. Assume that $\rho(\mathbb{G}) = n + 1$. Then*

$$\lim_{k\to\infty}\frac{\mathcal{M}(M_k)}{\mid G/G_k \mid} = B(M) + 2Q(M) \tag{9}$$

Proof. It is not difficult to show that if $\rho(\mathbb{G}) = n + 1$, then for each k there are $a_k, b_k \in \mathbf{N}$ such that $a_kG \supset G_k \supset b_kG$ with $a_k, b_k \to \infty$ as $k \to \infty$. Now our Proposition follows from Proposition 5.2. $\square$

Conjecture. Equality (9) is true for every nested tower in $\mathbf{Z}^{n+1}$.

References

[1] Y.Eliashberg, M.Gromov, *Lagrangian intersections theory. Finite-dimensional approach*, Advances in Mathematical Sciences (to appear)

[2] M.Farber, Tochnost neravenstv Novikova, Funktsionaljnyi analiz i ego prilozheniya, **19** (1985), 49–59 (in Russian). English translation:M.Farber, *Exactness of Novikov inequalities* , Functional Analysis and Applications **19**, 1985.

[3] F.T.Farrell, *The obstruction to fibering a manifold over a circle*, Indiana Univ. J. **10**, (1971) 315–346.

[4] T.Y.Lam, *Serre's Conjecture,* Lecture Notes in Mathematics **635**, (1978) 227 p.

[5] W.Lück, *Approximating L^2-invariants by their finite-dimensional analogs*, Geometric and Functional Analysis, **4**, (1994) 455–481.

[6] B.McDonald, *Linear Algebra over commutative rings*, Marcel Dekker, 1984

[7] S.P.Novikov, Mnogoznachnye funktsii i funktsionaly. Analog teorii Morsa, Dokl. Akad. Nauk SSSR **260** (1981), 31-35. English translation: S.P.Novikov. *Many-valued functions and functionals. An analogue of Morse theory*, Sov.Math.Dokl. **24** (1981), 222-226.

[8] A.V.Pazhitnov, O tochnosti neravenstv tipa Novikova dlya mnogoobrazii so svobodnoi abelevoi fundamentaljnoi gruppoi, Mat. Sbornik **180**, (1989) no. 11 (in Russian). English translation: A.V.Pazhitnov, *On the sharpness of Novikov-type inequalities for manifolds with free abelian fundamental group.*, Math. USSR Sbornik, **68** (1991), 351 - 389.

[9] A.V.Pazhitnov, O modulyah nad nekotorymi lokalizatsiyami koljtsa Loranovskih polinomov, Mat. Zametki ANSSSR **46**, (1989) no. 5 (in Russian). English translation: A.V.Pazhitnov, *On modules over some localizations of Laurent polynomial rings* , Math. USSR Notices, **46** (1989).

[10] A.V.Pazhitnov, *On the Novikov complex for rational Morse forms*, Annales de la Faculté de Sciences de Toulouse, **4** (1995), 297 – 338.

[11] A.V.Pajitnov, *Surgery on the Novikov Complex*, K-theory **10** (1996), 323-412.

[12] G.Polya, G.Szegö, *Aufgaben und Lehrsätze aus der Analysis* Springer, 1964

[13] D.Quillen, *Projective modules over polynomial rings*, Invent. Math. **36**, (1976) 167–171.

[14] S. Smale, *On the structure of manifolds*, Am. J. Math., **84** (1962) 387–399.

[15] V.V.Sharko, Stabiljnaya algebra teorii Morsa. Izvestiya Akad.Nauk SSSR, ser. matem. T. 54 $N°3$, 1990 (in russian). English translation: V.V.Sharko, *Stable algebra in Morse theory.* Math.USSR Izvestia, v.36 No 3, 629-653. (1991).

[16] V.V.Sharko, Funktsii na mnogoobraziyah, Naukova Dumka, Kiev (in russian). English translation: V.V.Sharko, *Functions on Manifolds. Algebraic and Topological aspects.* Translations of Mathematical Monographs. Vol. 131. Amer. Math. Soc. Providence, Rhode Island, 1993.

[17] A.A.Suslin, Proektivnye moduli nad koljtsami mnogochlenov svobodny. Doklady Akad.Nauk SSSR, T. 229 $N°5$, 1976 (in russian). English translation: A.A.Suslin, *Projective modules over a polynomial ring are free.* Soviet Math. Doklady v.17 (1976), 1160 - 1164.

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