C^0-generic properties of boundary operators in Novikov Complex

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CHAPTER 1

Introduction

1.1. Introduction to Novikov Complex and Novikov exponential growth conjecture

1.1.1. Morse Complex of a Morse function. We start with a recollection of one of the basic constructions of the Morse theory – Morse complex. Let M be a closed manifold, and $g : M \to \mathbf{R}$ be a Morse function. Let $S(g)$ be the set of critical points of g, and $S_k(g)$ be the set of critical points of g of index k. Consider the free abelian group $C_k(g)$, freely generated by $S_k(g)$. We shall introduce homomorphisms $\partial_k : C_k(g) \to C_{k-1}(g)$ which will endow $C_*(g)$ with the structure of a chain complex, called *Morse complex*. The homomorphisms depend on some additional choices. Namely one must choose:

1. A gradient-like vector field v for g, satisfying Transversality Condition.[1]
2. Orientations of stable manifolds of zeros of v

Having made these choices, we proceed as follows. Let $p, q \in S(f), \mathrm{ind} p = \mathrm{ind} q + 1$. Let $L(p,q)$ be the set of orbits of v joining p with q. The transversality condition for v implies that $L(p,q)$ is finite. The choice of orientation of stable manifolds allows to associate to each $\gamma \in L(p,q)$ a sign $\varepsilon(\gamma) \in \{-1, +1\}$. Set $n(p,q) = \sum_{\gamma \in L(p,q)} \varepsilon(\gamma)$. The integer $n(p,q)$ is called *incidence coefficient* corresponding to p, q. Now define the homomorphism ∂_k on the generators $p \in S_k(f)$ by the following formula

$$\partial_k(p) = \sum_{r \in S_{k-1}(f)} n(p,r) r$$

One can prove that $\partial_k \circ \partial_{k+1} = 0$ and that the homology of the resulting complex is isomorphic to $H_*(M)$. (See for example [**28**], Appendix, for the details.)

1.1.2. Novikov Complex of a Morse map to the circle. In the early 80s S.P.Novikov generalized the construction of Morse complex to the case of Morse maps $f : M \to S^1$. We shall give a construction in details a bit later (see Subsection 5.1.2), now we just sum up the properties of the resulting chain complex.

So let $f : M \to S^1$ be a Morse map, where M is a closed connected manifold. We assume that f is not homotopic to zero. As before, let $S(f)$ be the set of critical points of f, and $S_k(f)$ be the set of critical points of f of index k. As the classical Morse complex, the Novikov Complex is a free complex generated in dimension k by the set $S_k(f)$, but the base ring here is much larger. Namely, consider the ring of all Laurent power series in one variable with integral coefficients and finite negative

[1] See [**19**], Def 3.1 for the definition of gradient-like vector field.

part, that is

$$\mathbf{Z}((t)) = \{\lambda = \sum a_k t^k \mid \forall k a_k \in \mathbf{Z} \\ \text{and } \exists N = N(\lambda) \text{ such that } a_k = 0 \text{ if } k < N\}$$

Let $\mathcal{C} : \bar{M} \to M$ be the infinite connected cyclic cover, such that $f \circ \mathcal{C}$ is homotopic to zero (note that $\mathcal{C}$ is uniquely determined by this last property). To define Novikov Complex one must make some additional choices. Namely, we must choose:

1. A gradient-like vector field v for f, satisfying Transversality Condition(the definition of gradient-like vector field for the maps to the circle copies that for Morse functions).
2. Orientation of stable manifolds of critical points of f.
3. For every $p \in S(f)$ a lifting $\bar{p}$ of p to $\bar{M}$ (that is, a point $\bar{p} \in \bar{M}$, satisfying $\mathcal{C}(\bar{p}) = p$).

Having made these choices, one applies the procedure, exposed in Subsection 1.1.1 (with corresponding modifications, see Subsection 5.1.3 for details) and gets for every pair p, q of critical points of f, satisfying $\mathrm{ind}p = \mathrm{ind}q + 1$ an element $n(\bar{p}, \bar{q}; v) \in \mathbf{Z}((t))$, called *Novikov incidence coefficient* of p, q. (This coefficient depends on the choice of liftings $\bar{p}, \bar{q} \in \bar{M}$, see the formula 46). We shall also write $n(\bar{p}, \bar{q}; v) = \sum_i n_k(\bar{p}, \bar{q}; v)t^k$.

One can prove that $\partial_{k+1} \circ \partial_k = 0$ for every k. Thus we obtain a chain complex $C_*(f, v)$ such that $C_p(f, v)$ is a free $\mathbf{Z}((t))$-complex freely generated by $S_p(f)$. One can prove that

$$H_*(C_*(f, v)) \approx H_*(\bar{M}) \underset{\mathbf{Z}[t,t^{-1}]}{\otimes} \mathbf{Z}((t))$$

(See **[28]** for details.)

Remark 1.1.

In the present paper we introduce a notion of f-gradient which is a generalization of the notion of gradient-like vector field for f (definition 2.3). The gradient of f with respect to an arbitrary Riemannian metric is an f-gradient, as well as a gradient-like vector field for f. One defines Novikov incidence coefficient and Novikov Complex for f-gradients in a way similar to the exposed above (see Subsection 5.1.3). We shall use this in the following sections of the Introduction. $\triangle$

1.1.3. Morse-Novikov theory. The Novikov Complex was introduced by S.P.Novikov in **[23]**. In this paper the analogs of the classical Morse inequalities were obtained. These inequalities (called *Novikov inequalities*) were then studied in a series of papers[2].

M. Farber **[10]** proved that these inequalities are sharp in the case where $\pi_1 M = \mathbf{Z}$, $\dim M \geqslant 6$. J.-C. Sikorav [25] applied these inequalities to the theory of Lagrangian intersections. In **[36]**, **[37]** the author obtained some results concerning the sharpness of these inequalities for $\pi_1 M = \mathbf{Z}^m$, $\dim M \geqslant 6$. The study of the Novikov complex itself advanced slower. In **[43]** yet another version of the Novikov complex, defined over a completion of the group ring of $\pi_1(M)$, was given. In **[32]**, **[28]** I proved that the simple homotopy type of this chain complex equals the simple homotopy type of the completed chain simplicial complex of the

[2] We make no attempt to give here a complete survey.

universal covering of M. In [**33**], [**29**] I proved a theorem on realization of the abstract chain complexes as the Novikov complexes of Morse maps $M \to S^1$. This theorem implies the results of [**10**] and of [**37**]. The algebraic result by Ranicki [**42**] shows that the main theorem of [**33**], [**29**] implies also the Farrell fibration theorem [**9**]. In [**16**] F.Latour has developed an approach to Morse-Novikov theory, based on the Cerf's method [**6**].

1.1.4. Novikov exponential growth conjecture. A priori the Novikov incidence coefficients $n(\bar p, \bar q; v)$ are elements of $\mathbf{Z}((t))$, that is, power series of the form $\sum_{i=-N}^{\infty} n_k(\bar p, \bar q; v)t^k$, where $n_k(\bar p, \bar q; v)$ are integers. From the outset S.P.Novikov conjectured that these powers series had some nice analytic properties. Namely, he conjectured that the positive part of the power series $n(\bar p, \bar q; v)$ converges in a disc of a non-zero radius with the center 0 (equivalently: the coefficients $n_k(\bar p, \bar q; v)$ grow at most exponentially with k). During several years the conjecture has been discussed and adjusted. Now there are several versions of the conjecture. To discuss these versions it is convenient to introduce a definition.

Definition 1.2.

1. Let $f : M \to S^1$ be a Morse map, v be an f-gradient, satisfying Transversality Condition. The pair (f, v) will be called *Morse-Novikov pair.*
2. Let (f, v) be a *Morse-Novikov pair.* We say that the *exponential growth property* holds for the pair (f, v) if for every $p, q \in S(f)$ with $\operatorname{ind} p = \operatorname{ind} q + 1$ there are $C, D > 0$ such that for every $k \in \mathbf{Z}$:
$$|n_k(\bar p, \bar q; v)| \leqslant Ce^{kD}$$

$\triangle$

The initial version of Novikov exponential growth conjecture said that the exponential growth property holds for every Morse-Novikov pair (f, v).

As far as I know, the first published version of the conjecture appeared in the paper of V. I. Arnold "Dynamics of Intersections", 1989. Arnold wrote in [**2**], p.83:

> "The author is indebted to S. P. Novikov who has communicated the following conjecture, which was the starting point of the present paper. Let $p : \widetilde M \to M$ be a covering of a compact manifold M with fiber $\mathbf{Z}^n$, and let α be a closed 1-form on M such that $p^*\alpha = df$, where $f : M \to \mathbf{R}$ is a Morse function. The Novikov conjecture states that, "generically" the number of the trajectories of the vector field $-\operatorname{grad} f$ on M starting at a critical point x of the function f of index k and connecting it with the critical points y having index $k-1$ and satisfying $f(y) \geqslant f(x) - n$, grows in n more slowly than some exponential, e^{an}."

Thus the version of Novikov exponential growth conjecture, cited in [**2**], says in particular, that exponential growth property holds for every generic pair (f, v), where v is a *riemannian gradient* of f. Note that this version is stated in somewhat more general setting than our present one: it deals with Morse 1-forms, and not only with Morse maps to the circle. Note also that Arnold has conjectured something a bit stronger that the original Novikov exponential growth conjecture. Namely, this version deals with the *absolute number* of trajectories joining p with q.

Here is the version of the conjecture, published in 1993 by S. P. Novikov [**27**], p.229:

"CONJECTURE. For any closed quantized analytic 1-form ω, the boundary operator ∂ in the Morse complex has all coefficients $a_{pq}^{(i)} \in K$ with positive part convergent in some region $|t| \leqslant r_1^*(\omega) \leqslant r_1$. (If we replace S by $(-S)$ and t by t^{-1}, we have to replace r_1 by r_1^{-1}.) In particular, $r_1^* \neq 0$ and there will be coefficients $a_{pq}^{(i)}$ with radius of convergence not greater than $r_1(M, [\omega])$ if at least one jumping point really exists." [3]

Thus the version of **[27]** says that exponential growth property holds for every Morse pair (f, v) where f is analytic and v is a Riemannian gradient of f.

We proceed to versions of Novikov exponential growth conjecture which are in a sense stronger than Novikov exponential growth conjecture.

Definition 1.3.

We say that the *rationality property* holds for the Morse-Novikov pair (f, v), if for every $p, q \in S(f)$ with ind$p$ = ind$q + 1$ the power series $n(\bar{p}, \bar{q}; v)$ is the Laurent series of a rational function. $\triangle$

It is obvious that rationality property for a Morse-Novikov pair(f, v) implies the exponential growth property for (f, v).

M.Farber has conjectured that the rationality property holds for every Morse-Novikov pair (f, v). P.Vogel conjectured, that the rationality property holds for every Morse-Novikov pair(f, v). He motivated it by the fact, that the Novikov incidence coefficients are results of one and the same procedure applied to the fundamental cobordism of f (see the definition of the fundamental cobordism of f below in the Subsection 1.2.2).

[3]For convenience of the reader, we explain the terminology in this citation. S. P. Novikov denotes by K the ring denoted by $\mathbf{Z}((t))$ in the present paper; p, q correspond to critical points with ind$p$ = ind$q + 1$, and $a_{pq}^{(i)}$ is $n(p, q; v)$ in our notation. A closed 1-form w is called *quantized* if its cohomology class belongs to $H^1(M, \mathbf{Z})$.

1.2. C^0-generic rationality of Novikov incidence coefficients. The contents of the paper

1.2.1. The contents of the paper. In this subsection we give a brief outline of the contents of the present paper; the reader will find more information on the principal ideas of the work in their relation to the main theorem in Subsection 1.2.4 and in the comments and introductions to different chapters.

About 1995 I had proved that the Novikov incidence coefficients are C^0-generically rational functions. For the first time the results appeared in the e-print dg-ga 9603006 (14 March 1996). They were announced in [**30**] and published in the paper [**38**], which is a revised for publication version of [**34**].

The aim of the present paper is to generalize this result and to provide a systematic treatment of the subject. For a Morse map $f : M \to S^1$ we define a new class of vector fields, which contains the gradient-like vector fields for f and riemannian gradients for f. These vector fields are called f-gradients (see Definition 2.3). We give (Chapter 2) the systematic and self-contained presentation of all the necessary results and constructions from Morse theory; so this chapter can be considered as an introduction to the corresponding part of Morse theory.

Subsections 2.1.5, 2.1.6, 2.3.2, 2.3.3, 2.4.1, 2.4.2 contain the standard Morse-theoretical techniques, generalized to the case of f-gradients. In Subsections 2.1.4 and Sections 2.2, 2.5 we generalize the techniques of [**38**] to the case of f-gradients.

The chapter 3 contains a construction which is a Morse-theoretic analog of triangulation of a manifold with simplices of arbitrary small diameter. This construction makes no use of triangulations of the manifold; we do not leave the framework of Morse theory, Morse functions and their gradients. The contents of this chapter does not differ much from the corresponding chapter of [**38**], we have only revised the exposition on order to make it easier to read.

Chapter 4 is technically the central part of the work. It deals with gradients of Morse functions on cobordisms. We construct a small perturbation of a given gradient, so that the gradient descent map, associated to the perturbed gradient, respects a given handle decomposition of the components of the boundary of the cobordism.

Chapter 5 contains the proof of the main results of the work. Let $\mathcal{GT}(f)$ be the set of all f-gradients, satisfying Transversality Condition. We introduce a subset $\mathcal{G}\mathfrak{C}\mathcal{CT}(f) \subset \mathcal{GT}(f)$ which is open and dense in C^0 topology (see Proposition 5.4). The next theorem asserts that C^0 generically the Novikov incidence coefficients are rational functions. Denote by $\widetilde{L}$ the subring of the ring $\mathbf{Q}(t)$ formed by all the rational functions of the form $\frac{P(t)}{t^m Q(t)}$ where $m \in \mathbf{N}$ and P, Q are polynomials with integer coefficients and $Q(0) = 1$.

Theorem (Sect. 5, Th. 5.5)

Let $v \in \mathcal{G}\mathfrak{C}\mathcal{CT}(f)$. Let $x, y \in S(f)$, $\operatorname{ind} x = \operatorname{ind} y + 1$. Then $n(x, y; v) \in \widetilde{L}$,

The gradients belonging to $\mathcal{G}\mathfrak{C}\mathcal{CT}(f)$ have remarkable properties. In particular, the incidence coefficients $n(x, y; v)$ are invariant under C^0 small perturbations of $\mathcal{GCT}(f)$, if this perturbation has support outside $S(f)$. Here is the precise statement.

Theorem (Sect. 5, Th. 5.7)

Let $v \in \mathcal{G}\mathfrak{CCT}(f)$. Let U be an open neighborhood of $S(f)$. Then there is $\delta > 0$ such that for every $w \in \mathcal{G}\mathfrak{CCT}(f)$ with $\|w - v\| < \delta$ and $v \mid U = w \mid U$, and for every $x, y \in S(f)$ with $\operatorname{ind} x = \operatorname{ind} y + 1$ *we have: $n(x, y; v) = n(x, y; w)$.*

There are some naturally arising versions of Novikov Complex defined over completions of group rings, for example the completion of the group ring $\mathbf{Z}[\pi_1(M)]$. (One may consider these versions as "equivariant analogs" of the usual Novikov Complex). There is the corresponding generalization of the Theorem 5.5 to this situation. To cite here this theorem, let $f : M \to S^1$ be a Morse map, which we assume to be primitive, that is the homomorphism $f_* : \pi_1(M) \to \pi_1(S^1) = \mathbf{Z}$ is epimorphic. To the homomorphism $\xi = f_*$ one associates a certain completion $\widehat{\Lambda}_\xi$ of the group ring $\Lambda = \mathbf{Z}[\pi_1(M)]$ (see Subsections 5.1.1 for definition). There is a non-commutative localization procedure, which allows to define a certain subring of $\widehat{\Lambda}_\xi$ which is a natural analog of the ring $\widetilde{L}$. This subring is denoted by $\operatorname{Im} \ell$ (see Subsection 5.3.2 for definition). The following statement is a part of the theorem 5.14 .

Theorem

Let $v \in \mathcal{G}\mathfrak{CCT}(f)$. Then for every $x, y \in S(f)$ with $\operatorname{ind} x = \operatorname{ind} y + 1$ *the Novikov incidence coefficient $\widehat{n}(\widehat{x}, \widehat{y}; v)$ is in* $\operatorname{Im} \ell$.

1.2.2. Preliminaries on Morse maps $M \to S^1$. In this Subsection we introduce some terminology which will be used also in Chapter 5.

Let M be a closed connected manifold and $f : M \to S^1$ be a Morse map, non homotopic to zero. We shall assume that the homotopy class $[f] \in [M, S^1] = H^1(M, \mathbf{Z})$ is indivisible (the general case is easily reduced to this one by considering the map $\widetilde{f} = f/n$ with some $n \in \mathbf{N}$).

The map f lifts to a Morse function $F : \bar{M} \to \mathbf{R}$. Let t be the generator of the structure group ($\approx \mathbf{Z}$) of $\mathcal{C}$, such that for every x we have: $F(xt) < F(x)$. (We invite the reader to look at the figure on the page 100 where one can visualize these constructions.) We shall assume that f is primitive, that is $f_* : H_1(M) \to H_1(S^1)$ is epimorphic (one can always reduce the situation to this case by considering the map F/n with some $n \in \mathbf{N}$.) In this case we have: $F(xt) = F(x) - 1$. To simplify the notation we shall assume that $1 \in S^1$ is a regular value of f, and therefore every $n \in \mathbf{N}$ is a regular value of F. Denote $f^{-1}(1)$ by V. Set $V_\alpha = F^{-1}(\alpha)$, $W = F^{-1}([0, 1])$, $V^- = F^{-1}(]-\infty, 1])$. Note that $V_\alpha t = V_{\alpha - 1}$.

The cobordism W can be thought of as the result of cutting M along V. We shall sometimes call W a *fundamental cobordism* of f. Denote Wt^s by W_s; then $\bar{M}$ is the union $\cup_{s \in \mathbf{Z}} W_s$, the neighbor copies W_s and W_{s+1} intersecting by V_{-s}. For any $n \in \mathbf{Z}$ the restriction of $\mathcal{C}$ to V_n is a diffeomorphism $V_n \to V$.

The $\mathcal{C}$-preimage of a subset $A \subset M$ will be denoted by $\bar{A}$. On the other hand, if $x \in M$ we shall reserve the symbol $\bar{x}$ for liftings of x to $\bar{M}$ (that is $\bar{x} \in \bar{M}$ and $\mathcal{C}(\bar{x}) = x$). Let v be an f-gradient. Lift v to $\bar{M}$; we obtain a complete vector field on $\bar{M}$, which is obviously an F-gradient. We shall denote this F-gradient by the same symbol v, since there is no possibility of confusion. To restriction of this F-gradient to any cobordism $W_{[\lambda,\mu]} = F^{-1}([\lambda, \mu])$ (where λ, μ are regular values of F) is an $(F \mid W_{[\lambda,\mu]})$-gradient. Note that if v satisfies Transversality Condition, then the F-gradient $v \mid W_{[\lambda,\mu]}$ satisfies Transversality Condition.

The set of all f-gradients will be denoted by $\mathcal{G}(f)$, and the set of all f-gradients satisfying Transversality Conditionwill be denoted by $\mathcal{GT}(f)$.

The Novikov incidence coefficient $n_i(p, q; v)$ equals by definition to the incidence coefficient $n(p, qt^i; v)$ of the critical points p and qt^i of the Morse function F, restricted to the cobordism $F^{-1}([a, b])$, where the interval $[a, b]$ is chosen to be sufficiently large as to contain $F(p)$ and $F(qt^i)$.

1.2.3. First step of the proof. We adopt here the terminology of the preceding subsection.

Let $p, q \in W$ be critical points of F, ind$p$ = ind$q + 1$. Set $n = \dim M$ and $l + 1 = $ ind$p$. For simplicity, assume that $F(p)$ is the lowest critical level of $F|W$, so that the interval $]0, F(p)[$ is regular. Similarly, we assume that the interval $]F(q), 1[$ is regular and that $\bar{p} \in W_0$, $\bar{q} \in W_1$. Let $S(p)$ denote the intersection of the stable manifold of p with V_0; this is an oriented l-dimensional embedded sphere in V_0. The intersection $S(q)$ of the unstable manifold of q with V_1 is a cooriented $(n - l - 1)$-dimensional sphere in V_1.

Let $v \in \mathcal{GT}(f)$. Let K_1 be the union of all unstable discs of critical points of $F|W$ and K_0 be the union of all stable discs of critical points of $F|W$. Let $(-v)^{\leadsto}$ be the diffeomorphism of $V_1 \setminus K_1$ to $V_0 \setminus K_0$ which sends each point x to the point of intersection of the $(-v)$- trajectory starting at x with V_0. Using the presentation of $\bar{M}$ described above, we obtain the identity

$$n_k(p, q; v) = ((-v)^{\leadsto})^k (S(p)) \# S(q) \tag{1}$$

Here are some comments on this formula. 1) The symbol $\#$ stands for the algebraic intersection index of two manifolds; it can be checked that this number is well defined in our situation. 2) $(-v)^{\leadsto}$ is not defined on all of V_1; nevertheless, we denote by $(-v)^{\leadsto}(A)$ the set $(-v)^{\leadsto}(A \setminus K_1)$, where $A \subset V_1$. 3) We identify V_1 and V_0 with V, so that the iterations of $(-v)^{\leadsto}$ make sense and are applicable to $S(p) \subset V_0$.

Should $(-v)^{\leadsto}$ be an everywhere defined diffeomorphism, the intersection index in (1) would coincide with the intersection index of some homology classes, and our result would be a consequence of linear algebra. Indeed, let $[p]$ be a homology class of the sphere $S(p)$ in $H_l(V)$, let $]q[\in H^l(V)$ be the cohomology class dual to $S(q)$, and let $L : H_*(V) \to H_*(V)$ be the homomorphism induced by $(-v)^{\leadsto}$; then $((-v)^{\leadsto})^k (S(p)) \# S(q) = \langle]q[, L^k([p]) \rangle t^k$ (here and in what follows we denote by $\langle \cdot, \cdot \rangle$ the canonical pairing between homology and cohomology). A simple computation shows that the power series $\sum_{k=0}^{\infty} \langle]q[, L^k([p]) t^k \rangle$ is a rational function with denominator $1 - \det L \cdot t$. (see Lemma 5.6 for details).

These considerations were well known to S.P.Novikov and V.I.Arnold in th end of 80s. They form a motivation for Novikov exponential growth conjecture. I learned the algebraic lemma 5.6 from V.I.Arnold lecture in Moscow in 1990. The paper [**2**] gave rise to anew domain of investigation: the theory of topological complexity of intersection. The basic object of study in this theory is the intersection of the form $A^n X \cap Y$, where X and Y are submanifolds of a manifold M and A is a diffeomorphism $M \to M$. One tries to establish the asymptotic properties of this intersection when $n \to \infty$. See the paper [**3**] and the bibliography there for the actual state of this theory.

1.2.4. Outline of the proof of the main theorem. The exposition in this section is rather informal, nevertheless we think that it will be useful to the reader to go through it, since it explains the main ideas of what follows.

We continue with the arguments of the previous subsection. If f has critical points, then the map $(-v)^{\leadsto}$ is no longer an everywhere defined diffeomorphism, and the indices $n_i(p,q;v)$ do not admit any simple homological interpretation. We shall show that for a C^0-generic f-gradient v we can still obtain a formula of type $n(p,q;v) = \sum_{k=0}^{\infty} \langle]q[, L^k([p])\rangle t^k$, where, however, the meaning of the objects $[p],]q[, L$ will be more complicated than in the model situation discussed in Subsection 1.2.3.

Namely, we shall show that for any f and any $p, q \in S(f)$ one can find an C^0-open-and-dense subset $G \subset \mathcal{GT}(f)$, disjoint compacts $A, B \subset V$, and a homomorphism $L : H_l(V \setminus B, A) \to H_l(V \setminus B, A)$ such that the following condition is fulfilled for any $v \in G$:

$S(p)$ does not intersect B, $S(q)$ does not intersect A, the group $H_*(V \setminus B, A)$ is finitely generated, and

$$n(p,q;v) = \sum_{k=0}^{\infty} \langle]q[, L^k([p])\rangle t^k, \tag{2}$$

where $[p]$ is the image in $H_l(V \setminus B, A)$ of the fundamental class $[S(p)]$, and $]q[$ is an element of $H^l(V \setminus B, A)$, dual to $S(q)$.

Remark 1.4.

The homomorphism L is *not* induced by any continuous map of the pair $(V \setminus B, A)$ to itself. $\triangle$

Now we describe what are the sets A, B, the homomorphism L, and the C^0-small perturbations of v satisfying (2). We use the terminology of Subsections 2.1.4, 2.1.5. We start with an auxiliary Morse function $\phi V \to \mathbf{R}$ and a ϕ-gradient u satisfying the almost transversality condition. For $i \leqslant n-1$ consider the set $D(\mathrm{ind}{\leqslant i}\ ; u)$ (recall from Definition 2.15 that it is the union of all the stable manifolds of v of dimension $\leqslant i$.) These sets form a sort of cellular decomposition of V, and we shall call $D(\mathrm{ind}{\leqslant i}\ ; u)$ *i-skeleton* of V. (This is a certain abuse of notation, but in this subsection we shall not use the term *skeleton* in its usual meaning). The sets $D(\mathrm{ind}{\leqslant j}\ ; -u)$ form the dual decomposition of V. We have also "thickenings" of this skeleton: the sets $B_\delta(\mathrm{ind}{\leqslant i}\ ; u), D_\delta(\mathrm{ind}{\leqslant i}\ ; u), C_\delta(\mathrm{ind}{\leqslant i}\ ; u)$. Here $B_\delta(\mathrm{ind}{\leqslant i}\ ; u)$ is an open set, and $D_\delta(\mathrm{ind}{\leqslant i}\ ; u), C_\delta(\mathrm{ind}{\leqslant i}\ ; u)$ are compacts. One may think of $D_\delta(\mathrm{ind}{\leqslant i}\ ; u)$ as of union of all handles of V of thickness δ and indices $\leqslant i$.

For δ sufficiently small the sets $D_\delta(\mathrm{ind}{\leqslant i}\ ; u)$ and $D_\delta(\mathrm{ind}{\leqslant n-i-2}\ ; -u)$ are disjoint.

Put $A = D_\delta(\mathrm{ind}{\leqslant l-1}\ ; u), \quad B = D_\delta(\mathrm{ind}{\leqslant n-l-2}\ ; -u)$.

Returning to the cobordism W with boundary $\partial W = V_0 \sqcup V_1$, we note that for every geometric object associated with V there are two copies of it that correspond to V_0 and to V_1, because both V_0 and V_1 are diffeomorphic to V. These objects will be denoted by the same letter but with indices 0 or 1, respectively. In this way, we obtain Morse functions ϕ_0, ϕ_1, compacts A_0, A_1, etc.

The homomorphism L will be constructed for vector fields v, satisfying additional restriction, namely condition $(\mathfrak{C}\mathcal{Y})$ (see Definition 4.7). The principal part of this condition is given by the following two formulas:

$$(-v)^{\rightsquigarrow}\Big(C_\delta(\mathrm{ind}{\leqslant}j\ ; u_1)\Big) \cup \Big(D_\delta(\mathrm{ind}{\leqslant}j{+}1\ ; v) \cap \partial_0 W\Big) \subset B_\delta(\mathrm{ind}{\leqslant}j\ , u_0)$$

(B1) for every j

$$\overset{\rightsquigarrow}{v}\Big(C_\delta(\mathrm{ind}{\leqslant}j\ ; -u_0)\Big) \cup \Big(D_\delta(\mathrm{ind}{\leqslant}j{+}1\ ; -v) \cap \partial_1 W\Big) \subset B_\delta(\mathrm{ind}{\leqslant}j\ ; -u_1)$$

(B0) for every j

(see Definitions 4.7 , 4.2 for the details).

The conditions (B0), (B1) may be regarded as analogs of the cellular approximation conditions; however,an additional restriction is imposed: the soles of the δ-thickened handles of v of dimension less than or equal to $j+1$ must belong to the δ-thickened j-skeleton of V.

If condition $(\mathfrak{C}\mathcal{Y})$ is satisfied, then we construct the homomorphism $L = \mathcal{H}_l(-v)$: $H_l(V \setminus B, A) \to H_l(V \setminus B, A)$. To clarify the idea of this construction, we make a simplifying assumption: we assume that the homology class $x \in H_l(V \setminus B, A)$ is represented by an oriented closed submanifold N of $V \setminus B$ of dimension l. Set $N' = (-v)^{\rightsquigarrow}(N_1)$; then N' is a submanifold of V_0 (non compact in general). The condition (B1) implies $N' \subset V_0 \setminus B_0$. The condition (B0) implies that of the discs $D(a, -v)$ the manifold N_1 can intersect only those for which $\mathrm{ind}a \leqslant l$. Therefore the set $N' \cup (D(\mathrm{ind}{\leqslant}l\ ; v) \cap V_0)$ is compact, and the set $N' \setminus B_\delta(\mathrm{ind}{\leqslant}l{-}1\ ; u_0)$ is also compact. Thus the fundamental class of N' modulo A_0 is well defined as an element of $H_l(V_0 \setminus B_0, A_0)$. Identifying V_0 and V we obtain an element of $H_*(V \setminus B, A)$, which is by definition $\mathcal{H}_l(-v)(x)$.

The detailed construction of $\mathcal{H}_l(-v)$ is the subject of the section 4.2.

Now we explain how to gain the properties (B0), (B1) by C^0-small perturbation of the initial f-gradient v (that is, how to prove that the condition $(\mathfrak{C}\mathcal{Y})$ is C^0-dense.)

As already noted, for a compact submanifold R in V_1, the submanifold $(-v)^{\rightsquigarrow}(R)$ in V_0 is not necessarily compact; to make it compact, we must add the soles of the descending disks of v (thus losing the manifold property). Therefore, it should be expected that the correspondence $\cdot \mapsto (-v)^{\rightsquigarrow}(\cdot)$ is well defined on an appropriate category of stratified manifolds. As shown in 2.2, this is indeed so. The results of 2.2 imply that (under some mild transversality conditions) the set

$$\mathcal{R}_j = (-v)^{\rightsquigarrow}\big(D(\mathrm{ind}{\leqslant}j\ ; u_1)\big) \cup \big(D(\mathrm{ind}{\leqslant}j{+}1\ ; v) \cap V_0\big) \tag{3}$$

is compact for every j. Since this set is a union of submanifolds of V_0 of dimension not exceeding j, we can push $\mathcal{R}_j$ away from $D(\mathrm{ind}{\leqslant}n{-}j{-}2\ ; -u_0)$; then, applying a diffeomorphism $\Phi(T, -u_0)$ for T sufficiently large, we obtain an isotopy of V_0 that pushes $\mathcal{R}_j$ into $B_\delta(\mathrm{ind}{\leqslant}j\ ; u_0)$. After some work with thickenings we obtain an isotopy of V_0 such that the set $(-v)^{\rightsquigarrow}(D_\delta(\mathrm{ind}{\leqslant}j\ ; u_1)) \cup (D_\delta(\mathrm{ind}{\leqslant}j{+}1\ ; v) \cap V_0)$ is also pushed inside $B_\delta(\mathrm{ind}{\leqslant}j\ ; u_0)$. This isotopy may be realized by modifying v; the new gradient will satisfy (B0) and (B1). This procedure is described in detail in Section 4.1. The resulting gradient can be chosen so as to be C^0-close to the initial one. In other words, the pair (ϕ_0, u_0) can be chosen in such a way that $\Phi(\tau, -u_0)$ pushes $V_0 \setminus B_\delta(\mathrm{ind}\leqslant n-j-2\,; -u_0)$ inside $B_\delta(\mathrm{ind}\leqslant j\,; u_0)$ already for τ sufficiently small. Such flows will be called *quick flows*. The construction of quick flows is the subject of the Chapter 3.

As we have already mentioned, our construction of quick flows differs from the usual constructions of "small handle decomposition" in that it uses no triangulations of the manifold. A description of the basic idea of this construction can be found at the beginning of Chapter 3.

1.2.5. Further remarks.

1. Note that the C^0 topology is very important for us during all the work. One of the reasons why the C^0 topology is relevant, lies in the following property of the integral curves of differential equations: the integral curves (on a given finite interval) of a C^0-*small* perturbations v' of a C^1-vector field v are C^0-small perturbations of the integral curves of v (see, e.g., [**4**], Chapter 5, §4]).

   See the section 2.6 for the detailed exposition of the C^0 continuity properties. We think in general that C^0 topology is much more natural and efficient in Morse theory than it seems at present.
2. The section 5.3 of the paper is devoted to the generalization of Novikov exponential growth conjecture for the case of incidence coefficients with values in group rings. These results (in a somewhat weaker form) were announced in [**30**]. Unfortunately I have no time to include in the present paper the proofs of the other results announced in [**30**]; that will be the subject of the second part of the present work.

1.3. Terminology

1.3.1. Terminological conventions. Here I have gathered some definitions and notational conventions which I use throughout the paper.

The closed disc of radius r and center a in the Euclidean space $\mathbf{R}^n$ is denoted by $D^n(a,r)$.

The open disc of radius r and center a in the Euclidean space $\mathbf{R}^n$ is denoted by $B^n(a,r)$.

Let v be a vector field on a manifold W (possibly, with boundary). We denote by $\gamma(x,t;v)$ the value at t of the integral curve of v starting at x. If $A \subset \mathbf{R}$ and $\gamma(x,\cdot;v)$ is defined on A, then $\gamma(x,A;v)$ denotes the set $\{\,\gamma(x,t;v) \mid t \in A\,\}$.

A subset $X \subset W$ is said to be *v-invariant* if $\gamma(x,t;v)$ is defined for every $t \geqslant 0$ and every $x \in X$ and $\gamma(x,[0,\infty[;v) \subset X$. A subset $X \subset W$ is *$(\pm v)$-invariant* if it is v-invariant and $(-v)$-invariant.

A subset $X \subset W$ is said to be *weakly v-invariant* if $\gamma(x,t;v) \in X$ whenever $x \in X$, $t \geqslant 0$, and $\gamma(x,t;v)$ is defined. Defining the notion of weak $(\pm v)$-invariance is left to the reader.

For any (time dependent) vector field w on a closed manifold M and any $t \in \mathbf{R}$, we denote by $\Phi(w,t)$ the diffeomorphism $x \mapsto \gamma(x,t;w)$ of M.

The closure of a set U is denoted by $\overline{U}$. The short bar is reserved for the objects related to the infinite cyclic covering (e.g., $\mathcal{P}\,\bar{M} \to M$).

Let $R : A \to B$ be a diffeomorphism of riemannian manifolds. For $x \in A$ set $E(R, x) = ||R'(x)||$. The number

$$\max\big(\sup_{x\in A} E(R,x), \sup_{y\in B} E(R^{-1}, y)\big)$$

is called *expansion constant* of R.

A cobordism is a compact manifold W together with a presentation $\partial W = \partial_0 W \sqcup \partial_1 W$ where $\partial_1 W$ and $\partial_0 W$ are compact manifolds of dimension $\dim W - 1$ without boundary (one or both of them can be empty). The manifold $W \setminus \partial W$ will be denoted by W°.

A riemannian cobordism is a cobordism endowed with a riemannian metric.

Let W be a riemannian cobordism of dimension n, $q \in W^\circ$. Let $\delta_0 > 0$ be so small, that the restriction of the exponential map $\exp_q : T_q W \to W$ to the disc $B^n(0, \delta_0)$ is a diffeomorphism onto its image, and this image belongs to W°. Let $0 < \delta < \delta_0$. Denote by $B_\delta(q)$, resp. $D_\delta(q)$ the $\exp_q$-images of the open euclidean ball $B^n(0,\delta)$, resp. closed euclidean ball $D^n(0,\delta)$ with the center in 0. Then $B_\delta(q)$, resp. $D_\delta(q)$ are riemannian δ-discs (open, and resp. closed). In what follows we shall use the notation $B_\delta(q), D_\delta(q)$ only when $0 < \delta_0 < \delta$.

The end of the proof of a theorem, lemma etc. is marked by $\square$; the end of a definition or a remark is marked by $\triangle$.

1.3.2. Some differences between the notation of this paper and that of [29], [34]. The gradients satisfying the almost transversality condition were called "almost good" in [**34**], [**35**], [**29**].

The diffeomorphism denoted by $(-v)^{\leadsto}$ in the present paper was denoted by $\overset{\leftsquigarrow}{v}$ in [**34**], [**35**], [**29**].Our present notation is the same as that of [**38**]; it seems more natural because this diffeomorphism is a shift in the *positive* direction along the integral curves of the field $-v$. In [**29**] the sets $B_\delta(\mathrm{ind}_{\leqslant s}\,; v)$, $D_\delta(\mathrm{ind}_{\leqslant s}\,; v)$, $D(\mathrm{ind}_{\leqslant s}\,; v)$ were denoted by $B_\delta(\leqslant s; v)$, $D_\delta(\leqslant s; v)$, $D(\leqslant s; v)$. In [**29**] the set $D(\mathrm{ind}_{\leqslant s}\,, v)$ was denoted by $K(\mathrm{ind}_{\leqslant s}\,; v)$.

CHAPTER 2

Morse functions and their gradients

2.1. Basic definitions and constructions

2.1.1. Basic definitions. Recall that *cobordism* for us is a compact manifold W together with a presentation $\partial W = \partial_0 W \sqcup \partial_1 W$ where $\partial_1 W$ and $\partial_0 W$ are compact manifolds of dimension $\dim W - 1$ without boundary (one or both of them can be empty). The manifold $W \setminus \partial W$ is denoted by W°.

Definition 2.1.
A *Morse function* $f : W \to [a,b]$ is a C^∞ map $f : W \to \mathbf{R}$, such that $f(W) \subset [a,b], f^{-1}(b) = \partial_1 W, f^{-1}(a) = \partial_0 W$, all the critical points of f are non-degenerate and belong to W°. △

We shall denote $f^{-1}(\lambda)$ by V_λ and $f^{-1}([\alpha,\beta])$ by $W_{[\alpha,\beta]}$ if there is no possibility of confusion.

Recall that for a critical point a of f there is a well defined bilinear form on $T_a M$, called *Hessian* of f, and denoted by $d^2 f(a)$. For a non-degenerate critical point a the dimension of the negative eigenspace of $d^2 f(a)$ is called *index* of a. The set of critical point of f is denoted by $S(f)$, the set of critical points of f of index k is denoted by $S_k(f)$.

To introduce the notion of gradient vector field for f recall first some notions and results from the general theory of vector fields. Our reference for that theory is the book by R.Abraham and J.Marsden [**1**], but we modify a bit their terminology. Let M be a manifold without boundary. We say that a zero p of a vector field v on M is *elementary* if the matrix of the derivative $v'(p)$ has no eigenvalues with zero real part. Recall the following Hadamard-Perron local stable manifold theorem ([**1**], p.85)

Theorem 2.2.
Let p be an elementary zero of a vector field v. Then there is a neighborhood $U(p)$ and two submanifolds $W^{un}_{loc}(p,v) \subset U(p), W^{st}_{loc}(p,v) \subset U(p)$, having the following properties:

$$W^{un}_{loc}(p,v) = \{x \in U(p) \mid \gamma(t,x;v) \text{ is defined and belongs to } U(p) \text{ for all } t \leqslant 0, \\ \text{and } \gamma(t,x;v) \xrightarrow[t\to-\infty]{} p$$

$$W^{st}_{loc}(p,v) = \{x \in U(p) \mid \gamma(t,x;v) \text{ is defined and belongs to} \\ U(p) \text{ for all } t \geqslant 0, \text{ and } \gamma(t,x;v) \xrightarrow[t\to\infty]{} p$$

$$W^{un}_{loc}(p,v) \pitchfork W^{st}_{loc}(p,v),$$

$W^{un}_{loc}(p,v) \cap W^{st}_{loc}(p,v) = \{p\}$.

The tangent spaces to $W^{st}_{loc}(p,v)$, resp. $W^{un}_{loc}(p,v)$ will be denoted by $T_-(p;v)$, resp. $T_+(p;v)$.

Let W be a manifold with boundary. Let v be a vector field on W, such that $\forall x \in \partial W : v(x) \neq 0$. We can apply the theorem above to every elementary zero p of $v \mid W^\circ$. Thus we obtain: a neighborhood $U(p)$, local unstable manifold $W^{un}_{loc}(p,v)$, local stable manifold $W^{st}_{loc}(p,v)$ and the corresponding subspaces $T_+(p;v), T_-(p;v)$ of T_pW.

Definition 2.3.

Let $f : W \to [a,b]$ be a Morse function on a cobordism W. A vector field v on W is called *gradient for* f, or simply *f-gradient*, if

1. For every $x \in W \setminus S(f)$ we have: $df(x)\big((v(x)\big) > 0$.
2. The set of zeros of v equals to $S(f)$ and each zero of v is elementary.
3. For every $p \in S(f)$ the vector subspaces $T_-(p,v)$ and $T_+(p,v)$ are orthogonal with respect to the bilinear form $d^2f(a)$. $\triangle$

$\triangle$

Remark 2.4.

The same definition is valid of course for Morse functions on manifolds without boundary (the case of closed manifolds is already contained in the definition). $\triangle$

The basic example of an f-gradient is riemannian gradient. Namely, let G be a riemannian metric on W; recall that *riemannian gradient* $\operatorname{grad} f$ is the vector field defined by the formula

$$\langle \operatorname{grad}\ f(x), h\rangle = df(x)(h) \text{ for } x \in M$$

Lemma 2.5.

Riemannian gradient of a Morse function f is an f-gradient in the sense of the preceding definition.

Proof. The property 1) and the first half of 2) are obvious. Proceeding to the local properties in a neighborhood of a critical point $p \in S(f)$, consider the geodesic coordinate system centered in p. We can assume therefore, that f and $v = \operatorname{grad} f$ are defined in a neighborhood U of $0 \in \mathbf{R}^n$, and that for every $x \in U$ the matrix $G(x)$ of our riemannian metric in U satisfies $G(x) = I + O(|x|^2)$. Write the Taylor decomposition of f as $f(x) = f(0) + 1/2\langle Ax, x\rangle + O(|x|^3)$ where $\langle\cdot,\cdot\rangle$ stands for the Euclidean scalar product, $|\cdot|$ stands for the Euclidean norm, and A is a non degenerate symmetric linear operator. Then we have: $f'(x)h = \langle Ax, h\rangle + O(|x|^2)(h)$, and $v(x) = Ax + O(|x|^2)$. Thus $v'(0) = A$ and all the eigenvalues of $v'(0)$ are real and nonzero. $\square$

Remark on the terminology

Our present terminology is different from that of [**34**] and from that of [**29**]. For the convenience of the reader we cite here the definitions from [**19**], [**29**] and [**34**], and establish relations between them.

Here is the definition of gradient-like vector field ([**19**], Def. 3.1):

Let f be a Morse function for the triad $(W^n; V, V')$. A vector field ξ on W^n is a gradient-like vector field for f if

1. $\xi(f) > 0$ throughout the complement of the set of critical points of f, and
2. given any critical point p of f there are coordinates $(\vec{x}, \vec{y}) = (x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_n)$ in a neighborhood U of p so that $f = f(p) - |\vec{x}|^2 + |\vec{y}|^2$ and ξ has coordinates $(-x_1, ..., -x_\lambda, x_{\lambda+1}, ..., x_n)$ throughout U.

Here is the definition of f-gradient and related objects from [**29**],§2.2:

Let $f : W \to [a, b]$ be a Morse function on W (for the definition see [**19**], def 2.3). Denote by $S(f)$ the set of critical points of f and by $S_k(f)$ the set of critical points of f of index k. For each

$p \in S(f)$ choose a chart $\Phi(p) : U(p) \to B^n(0, r)$ (where $U(p)$ is a neighborhood of p in W, and $n = \dim W$), such that $f \circ \Phi(p)^{-1}(x_1, ..., x_n) = f(p) - (x_1^2 + ... + x_k^2) + (x_{k+1}^2 + ... + x_n^2)$ where $k = \text{ind } p$. We call this chart *standard chart* and we consider it as a part of the data of the Morse function f. The neighborhood $U(p)$ is called *standard coordinate neighborhood.* We say (cf. [**19**], §3), that a vector field v on W is a *gradient* for f, or an *f-gradient* if 1) $df(v) > 0$, except for the critical points of f, and 2) for any $p \in S(f)$ the vector fields $\Phi(p)_*(v)$ and $(-x_1, ..., -x_k, x_{k+1}, ..., x_n)$ are equal in some neighborhood of 0 (where $k = \text{ind} p$).

Here is the definition of f-gradient and related objects from [**34**], def 1.1

Let $f : M \to \mathbf{R}$ be a Morse function on a closed manifold M. Denote $\dim M$ by n. The set of critical points of f will be denoted by $S(f)$. A chart $\Phi_p : U_p \to B^n(0, r_p)$ (where $p \in S(f), U_p$ is a neighborhood of p, $r_p > 0$) is called *standard chart for f around p of radius r_p* (or simply *f-chart*) if there is an extension of Φ_p to a chart $\widetilde{\Phi}_p : V_p \to B^n(0, r'_p)$, (where $\overline{U_p} \subset V_p$ and $r'_p > r_p$), such that $(f \circ \widetilde{\Phi}_p^{-1})(x_1, ..., x_n) = f(p) + \sum_{i=1}^n \alpha_i x_i^2$, where $\alpha_i < 0$ for $i \leqslant \text{ind}_f p$ and $\alpha_i > 0$ for $i > \text{ind}_f p$. The domain U_p is called *standard coordinate neighborhood.* Any such extension $\widetilde{\Phi}_p$ of Φ_p will be called *standard extension of* Φ_p

...

A family $\mathcal{U} = \{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}$ of f-charts is called *f-chart-system*, if the family $\{\overline{U_p}\}$ is disjoint.

...

Given an f-chart system $\mathcal{U} = \{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}$ we say, that a vector field v on M is an *f-gradient with respect to $\mathcal{U}$,* if

1) $\forall x \in M \setminus S(f)$ we have $df(v)(x) > 0$;

2) $\forall p \in S(f)$ we have $(\widetilde{\Phi}_p)_*(v) = (-x_1, ..., -x_k, x_{k+1}, ..., x_n)$, where $k = \text{ind}_f p$, and $\widetilde{\Phi}_p$ is some standard extension of Φ_p.

We say that a vector field v is an *f-gradient* if there is an f-chart system $\mathcal{U}$, such that v is an f-gradient with respect to $\mathcal{U}$.

The notion of gradient-like vector field from [**19**] coincides essentially with that of [**29**], except that in [**29**] we considered the standard coordinate system around the critical point as a part of the data of a Morse function.

The following simple lemma shows that the notion of f-gradient in the sense of [**34**] and the notion of gradient-like vector field are the same.

Lemma 2.6.

Let $f : W \to \mathbf{R}$ be a Morse functionn a cobordism W, v be a vector field on W. Then v is a gradient-like vector field, if and only if v is an f-gradient in the sense of [**34**]

Proof. To prove that every f-gradient in the sense of [**34**] is a gradient-like vector field for f (the other sense is obvious) consider the standard coordinate system $\Phi : U \to V$ for f and v around a critical point $p \in S_k(f)$. We have: $(f \circ \widetilde{\Phi}_p^{-1})(x_1, ..., x_n) = f(p) + \sum_{i=1}^n \alpha_i x_i^2$, where $\alpha_i < 0$ for $i \leqslant k$ and $\alpha_i > 0$ for $i > k$ and $(\widetilde{\Phi}_p)_*(v) = (-x_1, ..., -x_k, x_{k+1}, ..., x_n)$. Define a diffeomorphism $\Psi : \mathbf{R}^n \to \mathbf{R}^n$ by $\Psi(x_1, ..., x_n) = (\sqrt{|\alpha_1|}x_1, ..., \sqrt{|\alpha_n|}x_n)$. Then the chart $\Psi \circ \Phi$ is a standard chart for f around p, and it is obvious that $(\Psi \circ \Phi)_*(v)(x) = (-x_1, ..., -x_k, x_{k+1}, ..., x_n)$. □

Of course the notion of f-gradient suggested in the present paper is strictly wider than that of gradient-like vector field. In the present paper we shall sometimes use gradient-like vector fields. For brevity we shall sometimes use the term "f-Gradient" instead of "gradient-like vector field". The reason for using in our paper the term "f-gradient" already used in the previous paper in the different sense (we apologize to the reader for every inconvenience caused by this) is that we try to find the widest possible class of "gradients " for which the usual Morse-theoretic constructions like "cellular-like decomposition, formed by the descending separatrix discs" are still valid. The extension which we propose in this paper (Definition 2.3) is motivated by the fact that riemannian gradient of a Morse function is now an f-gradient (this was not true in general with the definitions of [**29**], [**34**]).

2.1.2. Local structure of Morse function and its gradient in a neighborhood of a critical point. In this subsection W is a cobordism, $f : W \to [a, b]$ is a Morse function on W, p is a critical point of f of index k, v is an f-gradient. Recall the famous Morse Lemma, asserting that f is equivalent to its Hessian $d^2f(p)$ locally in a neighborhood of p. The next lemma is a bit sharper version of the Morse Lemma.

Lemma 2.7.

Let L_-, L_+ be the subspaces of T_pW, such that $d^2f(p) \mid L_- < 0, d^2f(p) \mid L_+ > 0$, and L_- is orthogonal to L_+ with respect to the bilinear form $d^2f(a)$.

Then there is a chart $\varphi : U \to V$ at the point p, such that $\varphi(p) = 0$, and

$$f \circ \varphi^{-1}(x_1, ..., x_n) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2 \tag{4}$$

and $\varphi'(p)(L_-) = \mathbf{R}^k, \varphi'(p)(L_+) = \mathbf{R}^{n-k}$, where $\mathbf{R}^{n-k}$ stands for the subspace $\{x_j = 0 \mid 1 \leqslant j \leqslant k\}$, and $\mathbf{R}^k$ stands for the subspace $\{x_j = 0 \mid k+1 \leqslant j \leqslant n\}$.

Proof.

The Morse lemma provides a chart Φ which satisfies (4) and $\Phi(p) = 0$. Thus $(f \circ \Phi^{-1})(x) = f(p) + A(x,x)$ where A is the bilinear form on $\mathbf{R}^n$ with the matrix

$$A_{ij} = \begin{cases} 0, & if\ i \neq j \\ 1, & if\ i = j \leqslant k \\ -1, & if\ i = j > k \end{cases} \tag{5}$$

Let $K_+ = \Phi'(p)(L_+), K_- = \Phi'(p)(L_-)$. It suffices to find a linear isomorphism $\xi : \mathbf{R}^n \to \mathbf{R}^n$ such that $\xi(K_+) = \mathbf{R}^{n-k}, \xi(K_-) = \mathbf{R}^k$. Note that $A|K_+ > 0, A|K_- < 0, A(K_+, K_-) = 0$. From these properties one deduces easily the existence of the isomorphism ξ satisfying the properties above. □

A chart $\varphi : U \to V$, satisfying the conclusions of 2.7 is called *Morse chart for f at p (with respect to L_+, L_-)* . Morse charts provide a natural tool for the study of the local behavior of f and v nearby the critical point p.

Note that the linear subspaces $T_+(p,v), T_-(p,v) \subset T_pW$ satisfy:

$$d^2f(p)|T_+(p,v) > 0, \quad d^2f(p)|T_+(p,v) < 0,$$
$$d^2f(p)(T_+(p,v), T_-(p,v)) = 0$$

(Indeed, the third property is already in the definition of f-gradient. Since $f|W^{st}_{loc}(p,v)$ has a local maximum in p, we have $d^2f(p)|T_+(p,v) \geqslant 0$, and using the third property together with non-degeneracy of d^2f it is not difficult to obtain the strict positivity of $d^2f|T_+$.)

Set $L_- = T_-(p;v); L_+ = T_+(p;v)$, and let $\varphi : U \to V \subset \mathbf{R}^n$ be a Morse chart for f at p with respect to L_+, L_-. In the following definition we develop the notation for subsets, maps, etc. in $\mathbf{R}^n$, related to $\varphi_*(v)$ and to $f \circ \varphi^{-1}$ (see the figure on the page 96).

Definition 2.8.

Denote $\varphi_* v$ by $\widetilde{v}$, $f \circ \varphi^{-1}$ by $\widetilde{f}$.

We write a point $x \in \mathbf{R}^n$ as (x_+, x_-), where $x_+ \in \mathbf{R}^{n-k}$, and $x_- \in \mathbf{R}^k$. The orthogonal projection of $\mathbf{R}^n$ onto $\mathbf{R}^k$ will be denoted by p_-. The orthogonal projection of $\mathbf{R}^n$ onto $\mathbf{R}^{n-k}$ will be denoted by p_+.

Set

$$Q_r = D^k(0,r) \times D^{n-k}(0,r)$$
$$Q_r^\circ = B^k(0,r) \times B^{n-k}(0,r)$$
$$Q_r^- = Q_r \cap \{(x_+, x_-) \mid |x_-| \geqslant |x_+|\}$$
$$Q_r^+ = Q_r \cap \{(x_+, x_-) \mid |x_-| \leqslant |x_+|\}$$
$$C = \{(x_+, x_-) \mid |x_+| = |x_-|\}$$

The boundary ∂Q_r is the union of two subsets:

$$\partial_- Q_r = \{x \in \partial Q_r \mid |x_-| = r\}$$
$$\partial_+ Q_r = \{x \in \partial Q_r \mid |x_+| = r\}$$

Set

$$\partial_-^\circ Q_r = \partial_- Q_r \setminus \partial_+ Q_r$$
$$\partial_+^\circ Q_r = \partial_+ Q_r \setminus \partial_- Q_r$$

Let W^- be the local stable manifold, and W^+ be the local unstable manifold with respect to a neighborhood $U(p)$ of p. We can assume $U(p) \subset U$. Denote $\varphi(W^\pm(p,v))$ by $N_\pm$. Since $\widetilde{v}$ is an $\widetilde{f}$-gradient, we have $Q_r \cap N_\pm \subset Q_r^\pm$. Let $\widetilde{v}'(0) = A$. By definition, A preserves $\mathbf{R}^{n-k}$ and $\mathbf{R}^k$, and there is $\nu > 0$ such that $\langle Ax_+, x_+\rangle \geqslant \nu\langle x_+, x_+\rangle, \langle Ax_-, x_-\rangle \leqslant -\nu\langle x_-, x_-\rangle$. We have $\widetilde{v}(x) = Ax + \delta(x)$, where $\delta(x) = o(|x|)$. △

In what follows we shall assume that the size of the set Q_r is sufficiently small. The following condition (R) is satisfied for every r sufficiently small.

(R)
- $\widetilde{v}$ points inward Q_r in the points of $\partial_-^\circ Q_r$ and outward Q_r in $\partial_+^\circ Q_r$
- $p_- \mid (N_- \cap Q_r)$ is a diffeomorphism onto $D^k(0,r)$
- $p_- \mid (N_+ \cap Q_r)$ is a diffeomorphism onto $D^{n-k}(0,r)$
- for $x \in Q_r$ we have $|\delta(x)| \leqslant \dfrac{\nu}{10}x$
- $Q_r \subset \varphi(U(p))$

Up to the end of this subsection we assume that r satisfies the condition (R). Let $N_+(r), N_-(r)$ be the local stable, resp. local unstable manifolds of $\widetilde{v}|Q_r^\circ$. We have obviously $N_\pm(r) \subset N_\pm \cap Q_r^\circ$. Actually the inverse inclusion is also true. Indeed, let $x \in N_- \cap Q_r^\circ$. If the trajectory $\gamma(x, \cdot; \widetilde{v})$ quits Q_r°, it must intersect $\partial_+^\circ Q_r^\circ$ and then at the moment of the intersection the value $\widetilde{f}(\gamma(x,t;\widetilde{v}))$ will be strictly greater than $\widetilde{f}(0)$ which contradicts $x \in N_-$. Thus $\gamma(x,t;\widetilde{v}) \in Q_r^\circ$ for all $t \geqslant 0$, and $x \in N_\pm(r)$.

The next lemma will be useful in the sequel.

Lemma 2.9.
Assume that a $\widetilde{v}$-trajectory $\gamma(x,t;\widetilde{v})$ stays in Q_r for all $t \geqslant 0$. Then $\gamma(x,t;\widetilde{v}) \xrightarrow[t\to\infty]{} p$.

Proof. Using the fact that $\widetilde{f}(\gamma(x,t;\widetilde{v}))$ is an increasing function of t, it is easy to prove that $\gamma(x,t;\widetilde{v}) \in Q_r^-$ for every $t \geqslant 0$. Write

$$\gamma'(t) = (\gamma_+'(t), \gamma_-'(t)) = (A\gamma_+(t), A\gamma_-(t)) + \delta(\gamma(t))$$

So $\langle \gamma_-', \gamma_-\rangle = \langle A\gamma_-, \gamma_-\rangle + \langle \delta(\gamma), \gamma_-\rangle$. Since $|\gamma| \leqslant 2|\gamma_-|$ in Q_r^- we have $\langle \gamma_-', \gamma_-\rangle \leqslant -\frac{\nu}{2}\langle \gamma_-, \gamma_-\rangle$. Therefore $\gamma_-(t) \to 0$ as $t \to \infty$ and the same is true for $\gamma_+(t)$. □

2.1.3. Global structure of v-trajectories. In this subsection $f : W \to [a,b]$ is a Morse function on W, and v is an f-gradient.

We consider here the behavior of v-trajectories on the whole of W, not only in a neighborhood of a critical point. Since $v \mid \partial W$ is transversal to ∂W and points inward W on $\partial_0 W$ and outward on $\partial_1 W$, there are 4 possible types of behavior for an integral curve of v (a, b in the formulas below are finite real numbers):

1. γ is defined on $[a, \infty[$, $\gamma(a) \in \partial_0 W$, $\gamma(]a,\infty[) \subset W^\circ$
2. γ is defined on $]-\infty, a]$, $\gamma(a) \in \partial_1 W$, $\gamma(]-\infty, a[) \subset W^\circ$
3. γ is defined on $[a,b]$, $\gamma(a) \in \partial_0 W$, $\gamma(b) \in \partial_1 W$, $\gamma(]a,b[) \subset W^\circ$
4. γ is defined on $]-\infty, \infty[$, $\gamma(]-\infty,\infty[) \subset W^\circ$

(see, e.g., [**29**], Prop. 8.9.) Since v is a gradient of a Morse function we can say more about the behavior of v-trajectories. First we study an integral curve in a neighborhood of a critical point. In the next lemma p is a critical point of f, and r is sufficiently small so that (R) holds. Set $I_r = \{t \in \mathbf{R} \mid \gamma(t) \in \varphi^{-1}(Q_r)\}$.

Lemma 2.10.
One of the following possibilities holds:

1. $I_r = [a, \infty[, \quad \gamma(t) \xrightarrow[t\to\infty]{} p,$
2. $I_r =]-\infty, a], \quad \gamma(t) \xrightarrow[t\to-\infty]{} p,$
3. $I_r = [a, b]$ *(where $a = b$ not excluded)*
4. $I_r =]-\infty, \infty[, \quad \gamma(t) = p$ *for every t.*
5. $I_r = \emptyset$

Proof. Assume that $I_r \neq \emptyset$, and let $\tau_0 \in I_r$. Consider the connected component J of τ_0 in I_r; then J is a closed interval, finite or infinite from one or both sides. The proof goes by considering subsequently these cases. We do three of them and leave the rest to the reader.

1. $J = \mathbf{R}$. Then $\mathrm{Im}\gamma \subset U(p) \cap (W^{st}_{loc} \cap W^{un}_{loc}) = \{p\}$ (by Lemma 2.9) and we obtain the case (4).
2. $J =]-\infty, \alpha]$. Denote $\varphi(\gamma(t))$ by $\widetilde{\gamma}(t)$. Then $\widetilde{\gamma}(t) \in Q_r$ for every $t \in]-\infty, \alpha]$, and by Lemma 2.9 we have $\gamma(t) \in W^{un}_{loc}$. Note that $\widetilde{\gamma}(\alpha) \in \partial Q_r$, and since $\widetilde{v}$ points inward Q_r on $\partial^\circ_- Q_r$, we have $\widetilde{\gamma}(\alpha) \in \partial_+ Q_r$. Therefore $f(\gamma(\alpha)) \geqslant f(p)$, and for every $t > \alpha$ we have $f(\gamma(t)) > f(p)$. Moreover for $t > \alpha$ sufficiently close to α, we have $\widetilde{\gamma}(t) \notin Q_r$. Assume now that $I_r \neq J$, let t_0 be the least number $> \alpha$, such that $\widetilde{\gamma}(t) \in Q_r$. Then $\widetilde{\gamma}(t_0) \notin \partial^\circ_+ Q_r$, therefore $\widetilde{\gamma}(t_0) \in \partial_- Q_r$, which is impossible since $\widetilde{f} \mid \partial_- Q_r \leqslant f(p)$. Thus $J = I_r$.
3. $J = [\alpha, \beta]$. The same reasoning as in the preceding point, proves that $\widetilde{\gamma}(\beta) \in \partial^+ Q_r$, and that $\widetilde{\gamma}(\tau) \notin Q_r$ for $\tau > \beta$. By symmetry we have: $\widetilde{\gamma}(\alpha) \in \partial_- Q_r$ and $\gamma(\tau) \notin Q_r$ for $\tau < \alpha$.

$\square$

From this lemma it follows easily that for a non-constant v-trajectory γ the four possibilities can occur:

1. γ is defined on $[a, \infty[, \gamma(a) \in \partial_0 W, \gamma(t) \xrightarrow[t\to\infty]{} p$, where $p \in S(f)$.
2. γ is defined on $]-\infty, a[, \ \gamma(a) \in \partial_1 W, \ \gamma(t) \xrightarrow[t\to-\infty]{} q$, where $q \in S(f)$.
3. γ is defined on $[a, b], \ \gamma(a) \in \partial_0 W, \ \gamma(b) \in \partial_1 W, \ \gamma(]a, b[) \subset W^\circ$
4. γ is defined on $]-\infty, \infty[, \ \gamma(t) \xrightarrow[t\to\infty]{} p$, where $p \in S(f)$, and $\gamma(t) \xrightarrow[t\to-\infty]{} q$, where $q \in S(f)$.

Definition 2.11.
Let p be a critical point of f. We denote by $D(p, v)$ the set of all $x \in W$, such that $\lim_{t\to\infty} \gamma(x, t; v) = p$, and we call it *descending disc.*

We denote by $D(p, -v)$ the set of all $x \in W$, such that $\lim_{t\to\infty} \gamma(x, t; -v) = p$, and we call it *ascending disc.*

$\triangle$

Remarks.

1. $D(p, v)$ is an analog of the global stable manifold in the dynamical system theory. Note, however, that the flow generated by v is not defined on the whole of $\mathbf{R}$ in the case $\partial W \neq \emptyset$ (this is one of the reasons to prefer this notation to the notation $W^{st}(p)$, used in the literature on dynamical systems).
2. By the Grobman-Hartman theorem the behavior of the v-trajectories belonging to $D(p, v)$ nearby p is *topologically* equivalent to that of the solutions of the standard system $x' = -x$ in $\mathbf{R}^n$. This is not the case in the differentiable category, and the behavior of trajectories nearby p can be complicated. The following lemmas show nevertheless that globally $D(p, v)$ is a good geometrical object. △

In the next three lemmas p is a critical point of f. In these lemmas we use the terminology from Definition 2.8 and we assume that r satisfies (R).

Lemma 2.12.
$D(p, v) \cap \varphi^{-1}(Q_r) = W^{st}_{loc}(p, v) \cap \varphi^{-1}(Q_r)$

Proof. Let $\widetilde{x} = \varphi(x) \in Q_r$. If $\gamma(\widetilde{x}, \cdot; \widetilde{v})$ stays forever in Q_r, then $x \in W^{st}_{loc}(p, v)$ by Lemma 2.9 . If $\gamma(\widetilde{x}, \cdot; \widetilde{v})$ leaves Q_r, then it must intersect $\partial_+ Q_r$ (since in $\partial^\circ_- Q_r \setminus C$ the vector field $\varphi_*(v)$ points inward Q_r) say at a moment t_0. Then for $t > t_0$ we have: $f(\gamma(x, t; v)) > f(p)$ and $x \notin D(p, v)$. □

Denote $f^{-1}(\lambda)$ by V_λ and $f^{-1}([\lambda, \mu])$ by $W_{[\lambda,\mu]}$.

Lemma 2.13.
We have:

1. $D(p, v) \cap W^\circ$ *is a submanifold of* W° *of dimension* $\operatorname{ind} p$.
2. $D(p, v) \cap V_\lambda$ *is a submanifold of* V_λ *of dimension* $\operatorname{ind} p - 1$.

Proof. The second part follows from the first, since $D(p, v)$ is transversal to V_λ. To prove 1) note that it suffices to prove that $D(p, v) \cap W^\circ$ is locally a submanifold of W° in a neighborhood of a point $x \in \varphi^{-1}(Q_\epsilon)$, where φ is a Morse chart. But this follows immediately from the preceding Lemma. □

The following Lemma justifies the using the term *descending disc* for $D(p, v)$.

Lemma 2.14.
Let $\lambda \in [a, f(p)]$ *be a regular value of* f. *Assume that for every* $x \in D(p, v) \cap W_{[\lambda,b]}$ *the* $(-v)$*-trajectory reaches* V_λ. *Then the pair* $(D(p, v), D(p, v) \cap V_\lambda)$ *is diffeomorphic to* $(D^k, \partial D^k)$, *where* $k = \operatorname{ind} p$.

Proof. For $0 \leqslant \rho \leqslant r$ set

$$\Delta_\rho = N_- \cap \{(x_+, x_-) \mid |x_-| \leqslant \rho\},$$
$$\Sigma_\rho = \partial \Delta_\rho = N_- \cap \{(x_+, x_-) \mid |x_-| = \rho\}$$
$$A(\rho, r) = N_- \cap \{(x_+, x_-) \mid \rho \leqslant |x_-| \leqslant r\}$$

For $0 < \rho < r$ we have: Δ_ρ is diffeomorphic to a k-disc, $A(\rho, r)$ is diffeomorphic to the annulus $S^{k-1} \times [0, 1]$. Each $(-v)$-trajectory, starting from a point $x \in A(\rho, r)$ reaches Σ_r at a moment $t(x)$ and reaches V_λ at a moment $T(x) > t(x)$, both $t(x)$ and $T(x)$ being C^∞ functions of x. It is easy now to construct a diffeomorphism $\Phi : \Delta_r \to D(p, v) \cap V_{[\lambda, f(p)]}$. Namely, set $\Phi \mid \Delta_\rho = \mathrm{id}$ and to define Φ outside Δ_ρ accelerate the movement along each $(-v)$-trajectory $\gamma(x, t; -v)$ for $x \in \Sigma_\rho$. We leave the details to the reader. □

2.1.4. Thickenings of descending discs and compactness properties. Let $f : W \to [a,b]$ be a Morse function on a riemannian cobordism W of dimension n, and v be an f-gradient.

Definition 2.15.
For $p \in S(f)$ set

$$B_\delta(p,v) = \{x \in M \mid \exists t \geqslant 0 : \gamma(x,t;v) \subset B_\delta(p)\}$$
$$D_\delta(p,v) = \{x \in M \mid \exists t \geqslant 0 : \gamma(x,t;v) \subset D_\delta(p)\},$$

Recall from Subsection 1.3.1 that in the notation $B_\delta(p), D_\delta(p)$ it is implicitly understood that δ is sufficiently small.

We denote by $D_\delta(v)$, resp. by $B_\delta(v)$ the union of all the $D_\delta(p,v)$, resp. $B_\delta(p,v)$. We denote by $D(v)$ the union of all the descending discs.

For $s \in \mathbf{N}, 0 \leqslant s \leqslant n$ we denote by $D(\text{ind}_{\leqslant s}\,; v)$ the union of $D(p,v)$ where p ranges over critical points of f of index $\leqslant s$. We denote by $B_\delta(\text{ind}_{\leqslant s}\,; v)$, resp. by $D_\delta(\text{ind}_{\leqslant s}\,; v)$ the union of $B_\delta(p,v)$, resp. of $D_\delta(p,v)$, where p ranges over critical points of f of index $\leqslant s$. △

Remark 2.16.
We shall use similar notation like $D_\delta(\text{ind}_{=s}\,; v)$ or $B_\delta(\text{ind}_{\geqslant s}\,; v)$, which are now clear without special definition. △

Lemma 2.17.
$B_\delta(p,v)$ is open.

Proof. Follows immediately from the continuous dependence of the integral trajectories of vector fields on their initial values. □

The things are not that simple for $D_\delta(p,v)$ or for $D(p,v)$. These sets are not necessarily closed. The general reason for that is the existence of v-trajectories, joining critical points: if there is a $(-v)$-trajectory joining two critical points p and q, then clearly $q \in \overline{D(p,v)} \setminus D(p,v)$. But for the union of all the $D(p,v)$ the compactness property is valid:

Lemma 2.18.

1. *$D(v)$ is compact.*
2. *$D_\delta(v)$ is compact*
3. *$D_\delta(v) = \overline{B_\delta(v)}$.*

Proof. 1) The complement $W \setminus D(v)$ consists of all points $x \in W$, such that $\gamma(x,\cdot;v)$ reaches $\partial_1 W$. This set is open by Corollary 2.68 [1].

2) similar to 1).

3) It follows from the inclusions $B_\delta(v) \subset D_\delta(v) \subset \overline{B_\delta(v)}$ □

[1] We shall use in this chapter the results of Section 2.6. The results of the section 2.6 makes no use of the results of the preceding sections; we placed it in the end in order not to interrupt the exposition.

As δ goes to 0, these δ-thickenings, for example, $B_\delta(p,v)$, should collapse to their "skeleton" $D(p,v)$, that is, $B_\delta(p,v)$ should form a fundamental system of neighborhoods for $D(p,v)$.

Again in general it is not true, but the analog of this property is true for the union of all $D(p,v)$; that is the subject of the following proposition.

Proposition 2.19.

1. *For every $\theta > 0$ we have: $D_\theta(v) = \cap_{\delta>\theta} B_\delta(v)$.*
2. *$D(v) = \cap_{\delta>0} B_\delta(v)$.*
3. *$\{B_\delta(v)\}_{\delta>\theta}$ form a fundamental system of neighborhoods for $D_\theta(v)$.*
4. *$\{B_\delta(v)\}_{\delta>0}$ form a fundamental system of neighborhoods for $D(v)$.*

Proof. 1) A point $x \in W$ *does not* belong to $D_\theta(v)$ if and only if the trajectory $\gamma(x,\cdot;v)$ reaches $\partial_1 W$ without intersecting the discs $D_\theta(p), p \in S(f)$. Then it does not intersect the discs $D_\delta(p)$ for some $\delta > \theta$. Similar argument proves 2).

3) Denote $B_\delta(v)$ by R_δ, and $D_\theta(v)$ by K (here δ is in some interval $I_=]\theta,\delta_0[$). Then:

1. $\overline{R_\delta} \subset R_{\delta'}$, if $\delta < \delta'$.
2. $\cap_{\delta>\theta} R_\delta = K$.
3. K and W are compacts.

These three properties imply in general, that $\{R_\delta\}_{\delta>\theta}$ form a fundamental system of neighborhoods of K. This is an exercise from an elementary general topology, which we leave to the reader (see also Lemma 2.35 , where a similar argument is exposed). □

2.1.5. Transversality conditions and Rearrangement Lemma. In this section $f : W \to [a,b]$ is a Morse function on a cobordism W and v is an f-gradient.

Definition 2.20.

We say that v satisfies *Transversality Condition* , if

$$\big(x,y \in S(f)\big) \Rightarrow \big(D(x,v) \cap W^\circ \pitchfork D(y,-v) \cap W^\circ\big)$$

We say that v satisfies *Almost Transversality Condition*, if

$$\big(x,y \in S(f)\ \&\ \mathrm{ind}x \leqslant \mathrm{ind}y\big) \Rightarrow \big(D(x,v) \cap W^\circ \pitchfork D(y,-v) \cap W^\circ\big)$$

△

One of the basic instruments in the Smale's theory of handlebodies is the construction from any Morse function on a cobordism W a so-called self-indexing Morse function, that is a function, such that $f(p) = \mathrm{ind}p$ for every $p \in S(f)$. The details of this construction can be found in [**19**], §4. The construction actually gives a bit more: if we start with an arbitrary Morse function g and a gradient-like vector field v for g, satisfying Transversality Condition, then it is possible to find a self-indexing Morse function f on the cobordism, so that v is also a f-gradient. The same is true for f-gradients; the corresponding Rearrangement Lemma is the main subject of the present subsection.

Definition 2.21.

A Morse function $\phi : W \to [\alpha,\beta]$ is called *adjusted to* (f,v), if:

1) $S(\phi) = S(f)$,

2) the function $f - \phi$ is constant in a neighborhood of V_0, in a neighborhood of V_1, and in a neighborhood of each point of $S(f)$,

3) v is also a ϕ-gradient. $\triangle$

Proposition 2.22.

Assume that $S(f) = \{p, q\}$, and that $D(p, \pm v) \cap D(q, \mp v) = \emptyset$. Let $\alpha, \beta \in]a, b[$. Then there is a Morse function $g : W \to [a, b]$, adjusted to (f, v), such that $g(p) = \alpha, g(q) = \beta$.

Proof. Note first, that we can assume that $f(p) \neq f(q)$ (this can be always achieved by a small perturbation of f with support in a neighborhood of $S(f)$). Assume, for example, that $f(p) < f(q)$. Denote $D_\delta(p, v) \cup D_\delta(p, -v)$ by $\mathcal{D}_\delta(p)$ and $B_\delta(p, v) \cup B_\delta(p, -v)$ by $\mathcal{B}_\delta(p)$. Similarly we define $\mathcal{D}_\delta(q)$ and $\mathcal{B}_\delta(q)$. Note, that $\mathcal{B}_\delta(q)$ and $\mathcal{B}_\delta(p)$ are open.

Lemma 2.23.

For $\delta > 0$ sufficiently small we have: $\mathcal{B}_\delta(p) \cap \mathcal{B}_\delta(q) = \emptyset$.

Proof. Choose any λ such that $f(p) < \lambda < f(q)$. Denote the cobordism $f^{-1}([a, \lambda])$ by W_0, and the cobordism $f^{-1}([\lambda, b])$ by W_1. Applying to W_0 and W_1 Proposition 2.19 we deduce from $D(p, -v) \cap D(q, v) = \emptyset$ the following:

For $\delta > 0$ sufficiently small the sets $D_\delta(p, -v) \cap f^{-1}(\lambda)$ and $D_\delta(q, v) \cap f^{-1}(\lambda)$ are disjoint. This implies the Lemma. $\square$

It follows from the Lemma that for $\delta > 0$ sufficiently small the sets $\mathcal{D}_\delta(p), \mathcal{D}_\delta(q)$ are disjoint and compact. Denote their intersections with $\partial_0 W$ by K_p, resp. K_q. Choose a C^∞ function $\alpha : \partial_0 W \to [0, 1]$, such that $\alpha \mid K_p = 1, \alpha \mid K_q = 0$. Note, that $W' = W \setminus (D(v) \cup D(-v))$ is the union of all the v-trajectories, starting at a point of $\partial_0 W \setminus D(v)$, and reaching $\partial_1 W$. Therefore there is a unique extension A of α to W', such that A is constant on each such trajectory. This function is smooth on W'. Remark that $W = W' \cup \mathcal{B}_\delta(p) \cup \mathcal{B}_\delta(q)$ and it follows from the definition that $A \mid W' \cap \mathcal{B}_\delta(p) = 1, A \mid W' \cap \mathcal{B}_\delta(q) = 0$. Therefore A extends to C^∞ function on W, which is constant on every v-trajectory. The rest of the proof is standard (see [**19**], §4). We reproduce it briefly for the sake of completeness.

One constructs first an auxiliary real C^∞ function of two real arguments $G : [a, b] \times [0, 1] \to [a, b]$, which will be considered as a real function $[a, b] \to [a, b]$ depending on the real parameter $s \in [0, 1]$. This function must satisfy the following conditions:

1. $\frac{\partial G_s(x)}{\partial x} > 0$ for every s, x.
2. $G_s(x) = x$ for x close to a or b and for every s.
3. $G_0(f(q)) = \beta$ and $\frac{\partial G_0(x)}{\partial x} = 1$ in a neighborhood of $x = f(q)$.
4. $G_1(f(p)) = \alpha$ and $\frac{\partial G_1(x)}{\partial x} = 1$ in a neighborhood of $x = f(p)$.

Then the function $x \mapsto G_{A(x)}(f(x))$ satisfy the conclusion of our Proposition. $\square$

Corollary 2.24.

Assume that v satisfies the almost transversality condition. Let $\prec$ be any partial order on the set $S(f)$, such that for every $p, q \in S(f)$ with $\mathrm{ind} p < \mathrm{ind} q$ we have: $p \prec q$. Then there is a Morse function $q : W \to [a, b]$, adjusted to (f, v), such that $f(p) < f(q) \Leftrightarrow p \prec q$. $\square$

Corollary 2.25.

Assume that v satisfies the almost transversality condition. Then for every s the set $D(\mathrm{ind}_{\leqslant s}\ , v)$ is compact.

Proof. By the preceding Corollary there is a function $\phi : W \to [a,b]$ and a regular value λ of ϕ, such that $\mathrm{ind} p \leqslant s \Leftrightarrow \phi(p) < \lambda$. Now apply Lemma 2.18 to the cobordism $\phi^{-1}([a,\lambda])$. □

The preceding Corollary has an analog for δ-thickenings of descending discs. To formulate it we need two more definitions.

Definition 2.26.

We say that f is *ordered* with an *ordering sequence* $a_0, ..., a_{n+1}$, if $a = a_0 < a_1 < ... < a_{n+1} = b$ are regular values of f such that $S_i(f) \subset f^{-1}(]a_i, a_{i+1}[)$. △

Definition 2.27.

We say that v is δ-*ordered* (where $\delta > 0$), if there is an ordered Morse function $\phi : W \to [a,b]$, adjusted to (f,v), with an ordering sequence $a_0, ..., a_{n+1}$, such that

$$\forall p \in S_k(f) : \quad D_\delta(p) \subset \phi^{-1}(]a_k, a_{k+1}[) \tag{6}$$

△

Remark 2.28.

If f-gradient v is δ-ordered, then it is δ'-ordered for some $\delta' > \delta$. Every f-gradient, satisfying almost transversality assumption, is δ-ordered for some δ (it follows from Corollary 2.24). △

Warning: Being δ-ordered does not imply almost transversality assumption.

Proposition 2.29.

If v is ϵ-ordered then $\forall \delta : 0 < \delta \leqslant \epsilon$ and $\forall s : 0 \leqslant s \leqslant n$ we have:

1. $D_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$ *is compact* ;
2. $D(\mathrm{ind}_{\leqslant s}\ ; v)$ *is compact* ;
3. $\bigcap_{\theta>0} B_\theta(\mathrm{ind}_{\leqslant s}\ ; v) = D(\mathrm{ind}_{\leqslant s}\ ; v)$;
4. $\bigcap_{\theta>\delta} B_\theta(\mathrm{ind}_{\leqslant s}\ ; v) = D_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$;
5. $\overline{B_\delta(\mathrm{ind}_{\leqslant s}\ ; v)} = D_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$.

Proof. Let $\phi : W \to [a,b]$ be the ordered Morse function , satisfying (6). Apply Lemmas 2.19 and 2.18 to the cobordism $\phi^{-1}([a_0, a_{s+1}])$. □

2.1.6. Adding a horizontal component to a vector field nearby the boundary. In this subsection we describe a (fairly standard) construction: modifying a vector field on a cobordism by adding a horizontal component near the boundary. See for example [**19**], page 62. The aim of this subsection is to give explicit estimates and to derive some properties for the resulting vector fields.

Let W be a cobordism, v be a vector field on W, transversal to $\partial_0 W$ (that is, for $x \in \partial_0 W$ we have $v(x) \notin T_x(W)$). We shall expose here the procedure of changing v nearby $\partial_0 W$. (If v is transversal to $\partial_1 W$, the similar procedure can be performed nearby $\partial_1 W$; we leave the statement of results to the reader.)

Let $a > 0$; assume that for every $x \in \partial_0 W$ the v-trajectory $\gamma(x, \cdot; v)$ is defined on $[0,a]$ and $\gamma(x, [0,a]; v) \subset W \setminus \partial_1 W$. The set $\gamma(\partial_0 W, [0,a]; v)$ will be called *closed* *collar* of $\partial_0 W$ of height a with respect to v, and denoted by C_0. Consider a map $\Psi : \partial_0 W \times [0,a] \to W$ defined by $\Psi(x,t) = \gamma(x,t;v)$. Then Ψ is a diffeomorphism on its image C_0. The tangent space $T_{(x,t)}\big(\partial_0 W \times [0,a]\big)$ being naturally isomorphic to $T_x(\partial_0 W) \oplus \mathbf{R}$, any vector field η on $\partial_0 W \times [0,a]$ can be written as follows: $\eta = \big(\eta_1(x,t), \eta_2(x,t)\big)$ where η_1 is a vector field on $\partial_0 W$ (it will be referred to as *the first component* of η) and η_2 is a function $\partial_0 W \to \mathbf{R}$ (it will be referred to as *the second component* of η). In this notation we have: $\big(\Psi^{-1}\big)_*(v) = (0,1)$.

Let ξ be a vector field on $\partial_0 W$, and let $h : [0,a] \to \mathbf{R}$ be a positive C^∞ function such that $h(t) = 0$ for t in a neighborhood of 0 and in a neighborhood of a. Define a vector field v' on $\partial_0 W \times [0,a]$ by $v'(x,t) = \big(h(t)\xi(x), 1\big)$ and define a vector field u on W by the following formula:

$$u = v \text{ in } W \setminus C_0;$$
$$u = \Psi_*(v') \text{ in } C_0$$

We shall call this vector field u *the result of adding to v a horizontal component proportional to ξ nearby $\partial_0 W$* (or simply the result of AHC-construction). The triple (a, ξ, h) will be called *input data* for AHC-construction. We shall also write $u = AHC(a, \xi, h; v)$. We shall establish three basic properties of this construction. They are all easily checked, and we leave the proofs to the reader .

AHC1. Estimate of $u - v$

Let E be the expansion constant of Ψ. We have:

$$||u - v|| \leqslant E||\xi||||h||$$

(where $||\cdot||$ is the C^0 norm). Moreover, the map $(\xi, h) \mapsto u$ is continuous with respect to C^∞ topology.

AHC2. The behavior of $(-u)$-trajectories

Let γ be a $(-v)$-trajectory starting at $x_0 \in W \setminus C_0$, and such that $y = \gamma(x, t_0; -u) \in \partial_0 W$. Then the $(-u)$-trajectory starting at x_0 reaches $\partial_0 W$ at the moment t_0 and $\gamma(x_0, t_0; -u) = \Phi(-\xi, T_0)(y)$, where $T_0 = \int_0^a h(\tau)d\tau$. Thus

$$(-u)^{\leadsto}_{[b,a]} = \Phi(-\xi, T_0) \circ (-v)^{\leadsto}_{[b,a]}$$

AHC3. Gradient property.

Assume that v is an f-gradient, where $f : W \to [a,b]$ is a Morse function on W. Set $m = \min_{x \in C_0} df(v)(x)$, $\quad M = \max_{x \in C_0} ||df(x)||$. If $||\xi||||h|| < \frac{m}{EM}$ then u is also an f-gradient.

2.2. s-submanifolds and gfn-systems

This section is an extended version of §2.1, 2.2 of [**38**].

2.2.1. s-submanifolds. Here we define and study s-submanifolds. The basic example of an s-submanifold is the set of all the descending discs of an f-gradient v, satisfying the almost transversality condition (where $f : M \to \mathbf{R}$ is a Morse function on a closed manifold.) The notion of s-submanifold of a manifold M is close to the notion of stratified manifold in the sense of Whitney-Thom. We have chosen to introduce the notion of s-submanifold and to deal with these objects, rather than to stick to the classical definition, since s-submanifolds carry less structure, and are more flexible than the classical stratified manifolds. It seems that if we impose additional restriction on v, these s-manifolds are actually Whitney-stratified manifolds. We hope to return to these questions in further publications.

Proceeding to the precise definitions, let $\mathbb{A} = \{A_0, ..., A_k\}$ be a finite sequence of subsets of a topological space X. We denote A_s also by $\mathbb{A}_{(s)}$, and we denote $A_0 \cup ... \cup A_s$ by $A_{\leqslant s}$ and also by $\mathbb{A}_{(\leqslant s)}$. We say that $\mathbb{A}$ is a *compact family* if $\mathbb{A}_{(\leqslant s)}$ is compact for every s.

Definition 2.30.

Let M be a manifold without boundary. A finite sequence $\mathbb{X} = \{X_0, ..., X_k\}$ of subsets of M is called *s-submanifold of M* (s for stratified) if

1. $\mathbb{X}$ is disjoint and each X_i is a submanifold of M of dimension i with the trivial normal bundle. [2]
2. $\mathbb{X}$ is a compact family. $\triangle$

$\triangle$

For an s-submanifold $\mathbb{X} = \{X_1, ..., X_s\}$ the largest k such that $X_k \neq \emptyset$ is called *dimension* of $\mathbb{X}$, and denoted by $\dim \mathbb{X}$. For a diffeomorphism $\Phi : M \to N$ and an s-submanifold $\mathbb{X}$ of M we denote by $\Phi(\mathbb{X})$ the s-submanifold of N defined by $\Phi(\mathbb{X})_{(i)} = \Phi(\mathbb{X}_{(i)})$.

If V is a submanifold of M and $\mathbb{X}$ is an s-submanifold of M, then we say that *V is transversal to $\mathbb{X}$* (notation: $V \pitchfork \mathbb{X}$) if $V \pitchfork \mathbb{X}_{(i)}$ for each i. If V is a compact submanifold of M, transversal to an s-submanifold $\mathbb{X}$, then the family $\{\mathbb{X}_{(i)} \cap V\}$ is an s-submanifold of V which will be denoted by $\mathbb{X} \cap V$.

If $\mathbb{X}, \mathbb{Y}$ are two s-submanifolds of M, we say, that $\mathbb{X}$ is *transversal to $\mathbb{Y}$* (notation: $\mathbb{X} \pitchfork \mathbb{Y}$) if $\mathbb{X}_{(i)} \pitchfork \mathbb{Y}_{(j)}$ for every i, j; we say that $\mathbb{X}$ is *almost transversal to $\mathbb{Y}$* (notation: $\mathbb{X} \nmid \mathbb{Y}$) if $\mathbb{X}_{(i)} \pitchfork \mathbb{Y}_{(j)}$ for $i + j < \dim M$.

Remark 2.31.

Note, that $\mathbb{X} \nmid \mathbb{Y}$ if and only if $\mathbb{X}_{(\leqslant i)} \cap \mathbb{Y}_{(\leqslant j)} = \emptyset$ whenever $i + j < \dim M$. $\triangle$

Let $f : W \to [a, b]$ be a Morse function on a cobordism W, v be an f-gradient. We denote by $\mathbb{D}(v)$ the family $\{D(\mathrm{ind}{=}i\ ; v)\}_{0 \leqslant i \leqslant \dim W}$ of subsets of W. For $\lambda \in \mathbf{R}$ denote by $\mathbb{D}_\lambda(v)$ the family $\{D(\mathrm{ind}{=}i{+}1\ ; v) \cap f^{-1}(\lambda)\}_{0 \leqslant i \leqslant \dim W - 1}$ of subsets of $V_\lambda = f^{-1}(\lambda)$.

Lemma 2.32.

Assume that v satisfies almost transversality assumption. Then:

1. *$\mathbb{D}(v)$ is a compact family.*
2. *If M is a closed manifold, then $\mathbb{D}(v)$ is an s-submanifold, transversal to $f^{-1}(\lambda)$ for every regular value λ of f.*
3. *If M is any cobordism, then for every regular value λ of f the family $\mathbb{D}_\lambda(v)$ is an s-submanifold of $f^{-1}(\lambda)$.*

Proof. 1) follows immediately from Proposition 2.18 . Note that the normal bundle to $D(p, v)$ is always trivial, which implies 2) and 3). $\square$

[2] We impose the triviality of the normal bundle since it makes easier some of the proofs, see for example the proof of the Proposition 2.33 .

We proceed to an analog for s-submanifolds of the Thom theorem on density of transversality property. In the proposition 2.33 we shall prove that if $\mathbb{X}, \mathbb{Y}$ are two s-submanifolds, it is always possible to perturb a little one of them say, $\mathbb{X}$, so that the result is almost transversal to $\mathbb{Y}$. First we need a definition.

Let M be a manifold without boundary. Denote by $\mathrm{Vectt}(M)$ the subset of $\mathrm{Vect}(M \times [0,1])$, formed by the vector fields $v \in \mathrm{Vect}(M \times [0,1])$, satisfying

1. v vanishes together with all partial derivatives in $M \times \{0\} \cup M \times \{1\}$.
2. The coordinate of v, corresponding to the factor $[0,1]$ is equal to zero.

One can consider the elements of $\mathrm{Vectt}(M)$ as the C^∞ vector fields on M, depending smoothly on the parameter $t \in [0,1]$. There is a natural topology on $\mathrm{Vect}(M \times [0,1]$, and $\mathrm{Vectt}(M)$ is a closed subspace of $\mathrm{Vect}(M \times [0,1])$ (see, e.g., [**29**],§8). From now on and to the end of the subsection we assume that M is compact. In this case the space $\mathrm{Vect}(M \times [0,1])$ as well as the space $\mathrm{Vectt}(M)$ has a *sup*-norm (with respect to a riemannian metric), which induces the C^0 *topology* on this space.

For every $v \in \mathrm{Vectt}(M)$ the integral curve $\gamma(x,t;v)$ is defined for all $t \in \mathbf{R}$. Recall that $\gamma(x,t;v)$ is also denoted by $\Phi(v,t)(x)$, and that for a fixed t the map $x \mapsto \Phi(v,t)(x)$ is a diffeomorphism of M.

Proposition 2.33.
Let $\mathbb{X}, \mathbb{Y}$ be s-submanifolds of a closed manifold M. Then the set of $v \in$ Vectt(M), such that $\Phi(v,1)(\mathbb{X}) \nmid \mathbb{Y}$ is open and dense in Vectt(M).

Proof. We shall give a sketch of the proof. A reader familiar with the differential topology will easily reconstruct the details. Denote by $\mathcal{T}(\mathbb{X},\mathbb{Y})$ the subset of all $v \in \mathrm{Vectt}(M)$, such that $\Phi(v,1)(\mathbb{X}) \nmid \mathbb{Y}$. Note first that $\mathcal{T}(\mathbb{X},\mathbb{Y})$ is open in C^0-topology, which follows easily from the results of §5 and the Remark 2.31 . To prove its C^∞-density we proceed by induction. Consider two auxiliary assertions:

$A(n)$: $\mathcal{T}(\mathbb{X},\mathbb{Y})$ is dense for every $\mathbb{X}$ and $\mathbb{Y}$ with $\dim \mathbb{X} + \dim \mathbb{Y} \leqslant n$.

$A_0(n)$: For every $\mathbb{X}, \mathbb{Y}$ with $\dim \mathbb{X} + \dim \mathbb{Y} \leqslant n$ the closure of $\mathcal{T}(\mathbb{X},\mathbb{Y})$ contains 0.

It is easy to show that $A_0(n) \Rightarrow A(n)$. Therefore it is sufficient to show that $A(n-1) \Rightarrow A_0(n)$. For this, let $\mathbb{X} = \{X_0, ..., X_k\}, \mathbb{Y} = \{Y_0, ..., Y_l\}$ be s-submanifolds of M, $\dim \mathbb{X} = k, \dim \mathbb{Y} = l, k + l = n$. Denote by $\mathbb{X}', \mathbb{Y}'$ the s-submanifolds $\{X_0, ..., X_{k-1}\}$, resp. $\{Y_0, ..., Y_{l-1}\}$. By the induction assumption there is a small vector field $w \in \mathrm{Vectt}(M)$, such that $\Phi(w,1)(\mathbb{X}') \nmid \mathbb{Y}$ and $\Phi(w,1)(\mathbb{X}) \nmid \mathbb{Y}'$. The manifolds which are not yet in general position are $X'_k = \Phi(w,1)(X_k)$ and Y_l. It suffices to consider the case $k + l < \dim M$. Consider three disjoint compacts: $\mathbb{X}'_{(\leqslant k-1)}, X'_k \cap Y_l, \mathbb{Y}_{(\leqslant l-1)}$.The compact $X'_k \cap Y_l$ is in X_k. Recall that the normal bundle to X'_k is trivial. Let U_0 be a neighborhood of $X'_k \cap Y_l$ in X'_k. Choose a tubular neighborhood U of U_0, fiberwise diffeomorphic to the product $U_0 \times \mathbf{R}^{n-k}$, such that $\overline{U} \cap (\mathbb{X}'_{(k-1)} \cup \mathbb{Y}_{(l-1)}) = \emptyset$. A standard argument using Sard Lemma (see, e.g., [**19**], p.46) implies that there is a small isotopy Ψ_t with support in U, such that $\Psi_1(X'_k) \pitchfork Y_l$. □

2.2.2. Good fundamental system of neighborhoods. The aim of this subsection is to introduce the notions of gfn-systems and ts-submanifolds. The ts-submanifolds are "thickenings of s-submanifolds" in the same sense as $B_\delta(v)$ is a thickening of $D(v)$. The notion of gfn-system is a generalization of the notion of ts-submanifold to the situation when we replace s-submanifolds by compact families of subsets of topological spaces. They appear in our setting when we deal with Morse functions on cobordisms with non-empty boundary, so that the descending discs are manifolds with boundary, non compact in general.

Proceeding to the definitions we start with good fundamental systems. Recall that a *fundamental system of neighborhoods* of a subset of A in a topological space X is a family $\{U_i\}_{i\in I}$ of open subsets of X, such that for every open set $U \supset A$ there is $i \in I$ with $U_i \subset U$.

Definition 2.34.

Let X be a topological space, $A \subset X$ a closed subset, I an open interval $]\delta_0, \delta_1[$. A *good fundamental system of neighborhoods of* A (abbreviation: gfn-system for A) is a family $\{A(\delta)\}_{\delta\in I}$ of open subsets of X, satisfying the following conditions:

1. for each $\delta \in I$ we have $A \subset A(\delta)$. Moreover, $\delta_1 < \delta_2 \Rightarrow A(\delta_1) \subset A(\delta_2)$,
2. for each $\delta \in I$ we have $\overline{A(\delta)} = \bigcap\limits_{\theta>\delta} A(\theta)$,
3. $A = \bigcap\limits_{\theta>\delta_0} A(\theta)$.

$\triangle$

Lemma 2.35.

Assume that X is compact. Let $I =]\delta_0, \delta_1[$. Let $\{A(\delta)\}_{\delta\in I}$ be a gfn-system for A. Then:

1. *The family $\{A(\theta)\}_{\theta>\delta_0}$ is a fundamental system of neighborhoods of A.*
2. *For every $\delta \in I$ the family $\{A(\theta)\}_{\theta>\delta}$ is a fundamental system of neighborhoods of $\overline{A(\delta)}$.*

Proof. (1) Let U be an open neighborhood of A. The sets $\{X \setminus \overline{A(\theta)}\}_{\theta>\delta_0}$ form an open covering of the compact $X \setminus U$. There is a finite subcovering, therefore there is $\theta > \delta_0$, such that $\overline{A(\theta)} \subset U$. (2) is proved similarly. $\square$

Next we define good fundamental systems of families of subsets.

Definition 2.36.

Let X be a topological space, $\mathbb{A} = \{A_0, ..., A_k\}$ be a compact family of subsets of X , I an open interval $]\epsilon_0, \epsilon_1[$. A *good fundamental system of neighborhoods of* $\mathbb{A}$ (abbreviation: gfn-system for $\mathbb{A}$) is a family $\mathbf{A} = \{A_s(\delta)\}_{\delta\in I, 0\leqslant s\leqslant k}$ of open subsets of X, satisfying the following conditions:

(FS1) For every $s : 0 \leqslant s \leqslant k$ and every $\delta \in I$ we have $A_s \subset A_s(\delta)$

(FS2) For every $s : 0 \leqslant s \leqslant k$ we have $\delta_1 < \delta_2 \Rightarrow A_s(\delta_1) \subset A_s(\delta_2)$

(FS3) For every $\delta \in I$ and every j with $0 \leqslant j \leqslant k$ we have $\overline{A_{\leqslant j}(\delta)} = \bigcap_{\theta>\delta}(A_{\leqslant j}(\theta))$

(FS4) For every j with $0 \leqslant j \leqslant k$ we have $A_{\leqslant j} = \bigcap_{\theta>0}(A_{\leqslant j}(\theta))$.

I is called *interval of definition* of the system. $\mathbb{A}$ is called the *core* of $\mathbf{A}$.

We shall denote $A_s(\delta)$ also by $\mathbf{A}_{(s)}(\delta)$ and $A_{\leqslant i}(\delta)$ also by $\mathbf{A}_{(\leqslant i)}(\delta)$. We shall denote A_s also by $A_s(\epsilon_0)$. $\triangle$ $\triangle$

Remark 2.37.

Assume that X is compact. Let $I =]\epsilon_0, \epsilon_1[$. Let $\mathbf{A} = \{A_s(\delta)\}_{\delta \in I, 0 \leqslant s \leqslant k}$ be a gfn-system for a compact family $\mathbb{A} = \{A_0, ..., A_k\}$ of subsets of X. Then for every $\delta \in [\epsilon_0, \epsilon_1[$ and every $0 \leqslant s \leqslant k$ the family $\{A_{\leqslant s}(\theta)\}_{\theta \in]\delta, \epsilon_1[}$ is a gfn-system for $\overline{A_{\leqslant s}(\delta)}$. △

If M is a manifold without boundary, $\mathbb{X}$ is an s-submanifold of M, and $\mathbf{X}$ is a gfn-system for $\mathbb{X}$, then we say that $\mathbf{X}$ is a *ts-submanifold with the core $\mathbb{X}$.* For a ts-submanifold $\mathbf{X} = \{X_s(\delta)\}_{\delta \in I, 0 \leqslant s \leqslant k}$ we shall denote $X_i(\delta)$ by $\mathbf{X}_{(i)}(\delta)$, and $X_{\leqslant i}(\delta)$ by $\mathbf{X}_{(\leqslant k)}(\delta)$.

The next lemma follows immediately from Proposition 2.29 . It provides the basic example of gfn-system and of ts-manifold.

Lemma 2.38.

Let W be a riemannian cobordism, $n = \dim M$. Let $f : W \to [a, b]$ be a Morse function and v be an ϵ-ordered f-gradient. Then the family $\{B_\delta(\mathrm{ind}{=}s\ ; v)\}_{\delta \in]0, \epsilon[, 0 \leqslant s \leqslant n}$ is a gfn-system for $\mathbb{D}(v)$. If W is a closed manifold, this family is a ts-submanifold with the core $\mathbb{D}(v)$. □

The gfn-system, introduced in the lemma, will be denoted by $\mathbf{D}(v)$. Note that there is no canonical choice of the interval of definition for this system.

2.3. Handle-like filtrations

It is well known since Smale (see e.g. [**46**], [**45**]) that to any Morse function on a closed manifold M one can associate a so-called *handle decomposition* of M. This section contains a version of this construction, which suits better to our purposes, than the original one. The object in our setting, which replaces the classical "handle corresponding to a critical point p" is the set $D_\delta(p, v)$ (see Definition 2.15). With our definition a systematic treatment of "δ-thin handles with $\delta \to 0$" is possible. On the other hand these objects are not very well suited for studying diffeomorphism type of manifolds (which was the original aim of the seminal papers of S.Smale, cited above). Indeed, we do not even introduce a smooth structure on $D_\delta(\mathrm{ind}{\leqslant}s\ ; v)$. We only identify the homotopy type of $D_\delta(\mathrm{ind}{\leqslant}s\ ; v)$ (under some mild restrictions on v).

2.3.1. δ-separated gradients: definition. Recall from [**8**], Ch. 5, that a *filtration* of a topological space X is a family $\{X_i\}_{i \in \mathbf{Z}}$ of subsets of X such that $i \leqslant j \Rightarrow X_i \subset X_j$. We shall say that a filtration $\{X_i\}$ is *complete* if $\cup_{i \in \mathbf{Z}} X_i = X$.

Let $f : W \to [a, b]$ be a Morse function on a riemannian cobordism W of dimension n, and v be an f-gradient. The very first filtration of W to introduce is of course the following: $W_s = D_\delta(\mathrm{ind}{\leqslant}s\ ; v)$ (where $\delta > 0$ is some fixed number). In general these W_s do not form a good filtration; W_s can well be non-compact. To obtain a filtration with "handle-like" properties we shall impose more restrictions on f and on v.

Let $\phi : W \to [a, b]$ be an ordered Morse function with an ordering sequence $(a = a_0 < a_1 ... < a_{n+1} = b)$. Let $\delta > 0$ and let v be a ϕ-gradient. Denote $\phi^{-1}([a_i, a_{i+1}])$ by W_i.

Definition 2.39.

We say that v is *δ-separated with respect to the ordering sequence* $(a_0, ..., a_{n+1})$, if

i) for every i and every $p \in S_i(f)$ we have $D_\delta(p) \subset W_i^\circ$.

ii) for every i and every $p \in S_i(f)$ there is a Morse function $\psi : W_i \to [a_i, a_{i+1}]$, adjusted to $(\phi \mid W_i, v)$ and a regular value λ of ψ such that

$$D_\delta(p) \subset \psi^{-1}(]a_i, \lambda[)$$

and for every $q \neq p, q \in S_i(f)$ we have

$$D_\delta(q) \subset \psi^{-1}(]\lambda, a_{i+1}[)$$

If there is no possibility of confusion we shall say that v is a δ-separated ϕ-gradient.

△

Definition 2.40.

Let $f : W \to [a, b]$ be an arbitrary Morse function , v be an f-gradient. We say that v is *δ-separated* if it is δ-separated with respect to some ordered Morse function $\phi : W \to [a, b]$, adjusted to (f, v). △

It is obvious that each f-gradient satisfying Almost Transversality Condition is δ-separated for some $\delta > 0$.

2.3.2. Definition of handle-like filtrations. Now let $\phi : W \to [a, b]$ be an ordered Morse function with an ordering sequence $(a_0, ..., a_{n+1})$, where $n = \dim W$, let v be a δ-separated ϕ-gradient; let $\nu \in]0, \delta]$. Let s be an integer between 0 and $n + 1$. Set

$$W^{\{\leqslant s\}} = \phi^{-1}([a_0, a_{s+1}]); \quad W^{\{\geqslant s\}} = \phi^{-1}([a_s, a_{n+1}]) \tag{7}$$

$$W^{[\leqslant s]}(\nu) = D_\nu(\mathrm{ind}_{\leqslant s} \,; v) \cup \partial_0 W; \tag{8}$$

$$W^{\lceil s \rceil}(\nu) = \phi^{-1}\big([a_0, a_s]\big) \cup D_\nu(\mathrm{ind}_{=s} \,; v) \tag{9}$$

Note that $W^{\{\leqslant 0\}} = \phi^{-1}\big([a_0, a_1]\big), W^{\{\geqslant 0\}} = W$, $W^{[\leqslant 0]}(\nu) = \big(\cup_{p \in S_0(f)} D_\delta(p)\big) \cup \partial_0 W = W^{\lceil 0 \rceil}$. Extend the definition to the negative values of s as follows. For $s \leqslant -2$ define all these sets (except $W^{\{\geqslant -s\}}$) to be empty. For $s = -1$ the definitions of $W^{\{\leqslant s\}}$,$W^{[\leqslant s]}(\nu)$ are good as they are; we have $W^{\{\leqslant -1\}} = \partial_0 W = W^{[\leqslant -1]}(\nu)$. For $s \leqslant 0$ set $W^{\{\geqslant 0\}} = W$. Set $W^{\lceil -1 \rceil}(\nu) = W^{[\leqslant -1]}(\nu) = \partial_0 W$.

There is an ambiguity in this notation, since $W^{\{\leqslant s\}}$ depend on ϕ, and $W^{[\leqslant s]}(\nu), W^{\lceil s \rceil}(\nu)$ depend on v. Each time we use this notation it will be clear what are ϕ and v. For the rest of the section 2.3 we fix both ϕ and v, so there is no possibility of confusion. We have obviously: $W^{\{\leqslant s\}} \supset W^{\lceil s \rceil}(\nu) \supset W^{[\leqslant s]}(\nu)$. All the three filtrations will be referred to as *handle-like filtrations*.

We introduce one more family of subsets setting

$$U_s = \phi^{-1}\big([a_0, a_s]\big) \cup \Big(\bigcup_{\mathrm{ind} p = s} D(p, v) \Big)$$

for $s \geqslant 0$ and setting $U_{-1} = \partial_0 W$, $U_s = \emptyset$ for $s \leqslant -2$. The space U_s is obtained from $\phi^{-1}\big([a_0, a_s]\big)$ by attaching the following family of s-dimensional cells: $\{D(p, v) \cap \phi^{-1}\big([a_s, a_{s+1}]\big)\}_{p \in S_s(\phi)}$

2.3.3. Homotopical properties of the filtrations.
Proposition 2.41.
Let $s \in \mathbf{Z}$ and $\nu \in]0, \delta]$.

1. *The inclusions $W^{[\leqslant s]}(\nu) \subset W^{\lceil s \rceil}(\nu) \subset W^{\{\leqslant s\}}$ are homotopy equivalences.*
2. *The inclusion $U_s \subset W^{\lceil s \rceil}(\nu)$ is a homotopy equivalence.*
3. *The inclusions of pairs*
$$\left(W^{[\leqslant s]}(\nu), W^{[\leqslant s-1]}(\nu)\right) \subset \left(W^{\lceil s \rceil}, W^{\{\leqslant s-1\}}\right) \subset \left(W^{\{\leqslant s\}}, W^{\{\leqslant s-1\}}\right)$$
are homotopy equivalences.
4. *The inclusion*
$$\left(U_s, W^{\{\leqslant s-1\}}\right) \subset \left(W^{\lceil s \rceil}(\nu), W^{\{\leqslant s-1\}}\right)$$
is a homotopy equivalence.

Proof. We consider only the case $s \geqslant 0$, the rest is trivial. The assertions 1 and 2 follow immediately from 3 and 4. The proofs of 3 and 4 are based on the following lemma.

Lemma 2.42.
Let $(Y, B) \subset (X, A)$ be pairs of topological spaces. Assume that there is a homotopy $\Phi_t : X \to X, t \in [0, a]$, verifying the following three properties:

(i) *$\Phi_0 = id$,*
(ii) *the subsets Y, A and B are invariant under Φ_t for all t,*
(iii) *$\Phi_a(X) \subset Y, \Phi_a(A) \subset B$.*

Then the inclusion $i : (Y, B) \hookrightarrow (X, A)$ is a homotopy equivalence.

Proof. The map $\Phi_a : (X, A) \to (Y, B)$ is the homotopy inverse for i. Indeed, $\Phi_a \circ i : (Y, B) \to (Y, B)$ equals to $\Phi_a \mid (Y, B)$ which is homotopic to the identity map $(Y, B) \to (Y, B)$ via the homotopy $\Phi_t \mid (Y, B)$. Further, the map $i \circ \Phi_a$: $(X, A) \to (X, A)$ is homotopic to the identity via the homotopy Φ_t. □

To deduce the assertion (3) from Lemma 2.42 define a homotopy $\Phi_t : W \to W$ by $\Phi_t(x) = \Gamma(x, t; -v)$ (see Definition 2.69 and Corollary 2.70 for the definition and properties of $\Gamma(\cdot, \cdot; \cdot)$). Here $t \geqslant 0$; note that for $T > 0$ sufficiently large we have for every s: $\Phi_T(W^{\{\leqslant s\}}) \subset W^{[\leqslant s]}(\nu)$. (Indeed, there is $\lambda > 0$ such that $df(v)(x) \geqslant \lambda$ for $x \in W^{\{\leqslant s\}} \setminus \cup_{p \in S(f)} B_\nu(p)$. Therefore if T is sufficiently large, there exists $t_0 \leqslant T$ such that $\Gamma(x, t_0; -v) \in \partial_0 W \cup B_\nu(\mathrm{ind}_{\leqslant s}\, ; v)$.)

The part 4 of our assertion is a bit more delicate: we can not do only with deformations of the type "shift along the $(-v)$-trajectories". Note that the compacts $D_\delta(p, v) \cap W_{[a_s, a_{s+1}]}$ where $\mathrm{ind} p = s$ are disjoint, therefore it suffices to prove that for each $p \in S_s(f)$ the space $D(p, v) \cup W_{[a, a_s]}$ is a strong deformation retract of $D_\delta(p, v) \cup W_{[a, a_s]}$. [3] We shall proceed as if there were only one critical point p of index s; the general case differs from this one only by complications in the notation.

Let φ be a Morse chart for f at p. We shall use the terminology from Definition 2.8 . We assume that the condition (R) from the definition 2.8 is true, and also

[3] Recall that $A \subset X$ is *a strong deformation retract of X* if there is a homotopy $f_t : X \to X$, such that $f_0 = \mathrm{id}, f_t \mid A = \mathrm{id}, f_1(X) \subset A$.

that $\varphi^{-1}(Q_r) \subset D_\nu(p)$. Recall that the vector field $(-\widetilde{v})$ points outward Q_r in the points of $\partial_-^\circ Q_r$ and inward Q_r in the points of $\partial_+^\circ Q_r$. Set

$$R = \{z \in W_{[a_s, a_{s+1}]} \mid z = \gamma(x, t; -v) \text{ where } x \in \varphi^{-1}(Q_r), t \geqslant 0\};$$

$$R_- = \{z \in W_{[a_s, a_{s+1}]} \mid z = \gamma(x, t; -v) \text{ where } x \in \varphi^{-1}(\partial_- Q_r), t \geqslant 0\}$$

The sets R and R_- are weakly $(-v)$-invariant. Set $W_0 = W_{[a, a_s]}$. We have inclusions:

$$(U_s, W_0) \subset (R_- \cup U_s, W_0) \subset (R \cup W_0, W_0) \subset \big(W^{\lceil s \rceil}(\nu) \cup W_0, W_0\big)$$

The argument similar to the one used above shows that the first and the third inclusions are homotopy equivalences. To prove that the second inclusion is a homotopy equivalence, it suffices to prove that $R_- \cup U_s$ is a strong deformation retract of $R \cup W_0$. For this it suffices in turn to prove that $\partial_- Q_r \cup (D(p, v) \cap Q_r)$ is a strong deformation retract of Q_r. The last assertion is obvious since the pair $\big(Q_r, \partial_- Q_r \cup (D(p, v) \cap Q_r)\big)$ is homeomorphic to $(Q_r, \partial_- Q_r \cap \mathbf{R}^k)$. □

2.4. Morse complex

2.4.1. Definition of the Morse complex. In this section W is a riemannian cobordism of dimension n, $f : W \to [a, b]$ is a Morse function on W, v is an f-gradient. Let s be an integer between 0 and n. Consider the free abelian group C_s generated by all critical points of f of index s. Assuming that v satisfies Transversality Condition we shall introduce for every s a homomorphism $\partial_s : C_s \to C_{s-1}$ satisfying $\partial_{s-1} \circ \partial_s = 0$, so that we shall obtain a chain complex over $\mathbf{Z}$, called *Morse complex.* The construction of ∂_s makes use mainly of the orbits of v joining critical points of f, but it depends also on an additional arbitrary choice (namely on the choice of orientations of descending discs). The homology of the complex is isomorphic to $H_*(W, \partial_0 W)$.

Proceeding to precise definitions, choose an ordered Morse function $\phi : W \to [a, b]$ with an ordering sequence $(a_0 = a, a_1, ..., a_{n+1} = b)$, adjusted to (f, v). In the preceding subsection we have constructed a filtration $W^{\{\leqslant s\}}$. The filtration $W^{\{\leqslant s\}}$ is complete; we obtain thus a complete filtration in the singular chain complex of $(W, \partial_0 W)$. From the proposition 2.41 we know that the pair $\big(W^{\{\leqslant s\}}, W^{\{\leqslant s-1\}}\big)$ is homotopy equivalent to $\big(U_s, W^{\{\leqslant s-1\}}\big)$, and U_s is obtained from $W^{\{\leqslant s-1\}}$ by attaching the following family of s-dimensional cells: $\{D(p, v) \cap \phi^{-1}\big([a_s, a_{s+1}]\big)\}_{p \in S_s(\phi)}$. Therefore we have:

$$H_*\big(W^{\{\leqslant s\}}, W^{\{\leqslant s-1\}}\big) = \begin{cases} 0 \text{ if } * \neq s \\ (\mathbf{Z})^{\# S_s(\phi)} \text{ if } * = s \end{cases} \tag{10}$$

We shall denote $H_s\big(W^{\{\leqslant s\}}, W^{\{\leqslant s-1\}}\big)$ by C_s. The property (10) implies that the filtration $\{W^{\{\leqslant s\}}\}$ is a *cellular filtration* in the sense of [**8**], Ch 5, §1. (Note that in the subsection 2.3.2 we have defined $W^{\{\leqslant s\}}$ for all $s \in \mathbf{Z}$; in particular $W^{\{-1\}} = \partial_0 W$). Therefore, by [**8**], Ch. 5, Prop. 1.3 there is an isomorphism

$$H_*(W, \partial_0 W) \approx H_*(C_*, \partial_*).$$

Here $\partial_s : C_s = H_s\big(W^{\{\leqslant s\}}, W^{\{\leqslant s-1\}}\big) \to C_{s-1} = H_{s-1}\big(W^{\{\leqslant s-1\}}, W^{\{\leqslant s-2\}}\big)$ is the composition of the boundary homomorphism of the exact sequence of the pair

$\big(W^{\{\leqslant s-1\}}, W^{\{\leqslant s-2\}}\big)$ and the projection

$$H_{s-1}\big(W^{\{\leqslant s-1\}}\big) \to H_{s-1}\big(W^{\{\leqslant s-1\}}, W^{\{\leqslant s-2\}}\big)$$

The complex (C_*, ∂_*) is called *Morse complex*, and denoted by $C_*(\phi, v)$.

By Theorem 2.44 and Remark 2.43 below this complex depends only on v, but not on the choice of ordered Morse function ϕ, of which v is a gradient. So we shall often denote it by $C_*(v)$.

It occurs that the boundary operator ∂_* is computable from the geometric data (see Theorem 2.44 below).

Historical remark. The essential in this construction was discovered by S.Smale and used in his proof of the Poincare conjecture in dimensions $\geqslant 5$ (see [**46**]). In the book [**19**], §7 one finds almost the statement of the theorem 2.44 . Much later the interest to the subject was stimulated by the paper [**50**], where one finds the theorem 2.44 in its present form (for the case of riemannian gradients). The proof suggested in [**50**] is, however, based on different ideas, than those of Smale. A proof of Theorem 2.44 is contained in the Appendix to my paper [**28**] (only the case of pseudo-gradients is treated there, but we shall see a bit later that this does not diminish the generality). $\triangle$

Choose for each $p \in S_s(f)$ an orientation of the tangent space $T_-(p, v)$ to the local stable manifold. Then the manifold $D(p, v) \cap W^\circ$ obtains an orientation. Further, the manifold $D(p, -v)$ obtains a co-orientation (recall that a *co-orientation* of a submanifold N of a manifold M is an orientation of the normal bundle to N in M). The fundamental class of $D(p, v)$ modulo its boundary defines an element $d(p) \in C_s$. These elements form a free base in C_s. Write

$$\partial_s d(p) = \sum_{q \in S_{s-1}(f)} n(p, q) \cdot q$$

where $n(p, q) \in \mathbf{Z}$. We call $n(p, q)$ *homological incidence coefficient* corresponding to p and q. We shall also write $n(p, q; v)$ to stress that a priori this index depends on v (we shall see later that often it does not).

Assume now that v satisfies Transversality Condition. Denote $D(p, v) \cap \phi^{-1}(a_s)$ by $S(p, v)$ and $D(q, -v) \cap \phi^{-1}(a_s)$ by $S(q, -v)$. For each $p \in S_s(f), q \in S_{s-1}(f)$ the sets $S(p, v), S(q, -v)$ are compact submanifolds of complementary dimension in V_{a_s}, and they are transversal to each other. Therefore their intersection $N(p, q; v)$ is a finite set. Its points are in 1-1 correspondence with the orbits of v joining q to p. Due to our choice of orientations each point $x \in N(p, q; v)$ obtains a sign $+$ or $-$, denoted by $\varepsilon(x)$. (Namely, the manifold $S(p, v)$ is oriented as the boundary of $D(p, v)$. Further, $S(q, -v)$ is a cooriented submanifold of V_{a_s}, since $D(q, -v)$ is cooriented. Then in each point of $x \in S(p, v) \cap S(q, -v)$ we obtain an orientation of $T_x(S(p, v))$ and a coorientation of $T_x(S(q, -v))$. Since these two subspaces of $T_x V_{a_s}$ are complementary, we obtain a sign).

Set

$$\nu(p, q; v) = \sum_{x \in N(p, q; v)} \varepsilon(x)$$

Remark 2.43.

The number $\nu(p, q; v)$ depends only on v, but not on the choice of ordered Morse function ϕ, adjusted to (f, v). Indeed, the above definition is easily reformulated in terms of orbits of v: the number $\nu(p,q;v)$ is the algebraic number of v-orbits joining p with q.) △

2.4.2. Properties of incidence coefficients.

Theorem 2.44.

For all p,q with $\mathrm{ind}p = \mathrm{ind}q + 1$ we have: $n(p,q;v) = \nu(p,q;v)$.

Proof.

By the remark 2.43 we can assume that all the critical points of ϕ of index s lie on the same critical level of ϕ, say $c_s \in]a_s, a_{s+1}[$.

We shall say that two ϕ-gradients v and w are *tangent* if for every critical point $p \in S(\phi)$ we have: $T_-(p,v) = T_-(p,w), T_+(p,v) = T_+(p,v)$. Using partitions of unity, it is not difficult to show that every f-gradient is tangent to an f-Gradient.

Proposition 2.45.

Let w be another ϕ-gradient, satisfying Transversality Condition. Assume that v and w are tangent. Then for every $p,q \in S(f)$ with $\mathrm{ind}p = \mathrm{ind}q + 1$ we have

1. $\nu(p,q;v) = \nu(p,q;w)$
2. $n(p,q;v) = n(p,q;w)$.

Proof. 1) The number $\nu(p,q;v)$ is the intersection index of submanifolds $S(p,v), S(q,-v)$ in V_{a_s} (the first being oriented, the second cooriented). To prove $\nu(p,q;v) = \nu(p,q;w)$ it suffices to show that $S(p,v)$ is homotopic to $S(p,w)$ and $S(q,-v)$ is homotopic to $S(q,-w)$.

The proof of $S(p,v) \sim S(p,w)$ makes use of Morse chart φ for ϕ around p. Since v and w are tangent, we can use the same Morse chart φ for both v and w. We assume here therefore the terminology of Definition 2.8 . Let r be so small that the condition (R) from Definition 2.8 is valid for v and for w. Let $t \in [0,1]$ and consider the vector field $v_t = tw + (1-t)v$. In general it is not a ϕ-gradient, but it satisfies $d\phi(v_t)(x) > 0$ for every $x \notin S(\phi)$. This implies that every $(-v_t)$-trajectory, starting in $\partial_- Q_r$, reaches V_{a_s}.

Consider a continuous map $\psi_t : \partial_- Q_r \to \partial_0 W$ which associates to each point y of $\partial_- Q_r$ the point of intersection of $\gamma(y,\cdot;-v_t)$ with V_{a_s}. Set $\sigma(p,v) = \partial_- Q_r \cap D(p,v)$, $\sigma(p,w) = \partial_- Q_r \cap D(p,w)$. These are two oriented embedded spheres in $\partial_- Q_r$, which are homotopic to each other; denote the corresponding homotopy by H_t. Then $\psi_t \circ H_t$ is the homotopy which we were seeking for. The proof of $S(q,-v) \sim S(q,-w)$ is similar.

2) The proof is similar to 1): deforming w to v

one shows that the discs $D(p,v)$ and $D(p,w)$ define the same homology classes in $H_*(W^{\{\leqslant s\}}, W^{\{\leqslant s-1\}})$. We leave the details to the reader. □

In view of this proposition we can assume that the image of v in the Morse chart corresponding to p is the standard vector field $\boldsymbol{v}_s$ (where $s = \mathrm{ind}p$). Denote $\phi^{-1}([a, a_{s-2}])$ by W_0, $\phi^{-1}([a, a_{s-1}])$ by W_1, $\phi^{-1}([a, c_{s-1}[)$ by W'.

Fix some $p \in S_s(f)$ and denote $S(p,v)$ by M. Consider its fundamental class $[M]$ as an element of $H_{s-1}(W_1, W_0)$. Our theorem describes the decomposition of $[M]$ with respect to to the base $\{d(q)\}_{q \in S_{s-1}(f)}$ of $H_{s-1}(W_1, W_0)$ in terms of the algebraic intersection index of M with the spheres $S(q,-v)$. In order to find this decomposition we shall perform a homotopy of M in W_1. The inclusion $(W_1, W_0) \hookrightarrow (W, W')$ is a homotopy equivalence; therefore it suffices to calculate the image of $[M]$ in the group $H_{s-1}(W_1, W')$. Let $q \in S_{s-1}(\phi)$ and let $x_1, ..., x_l \in M$ be the points of intersection of M with $D(q, -v)$. Denote the closed $(s-1)$-disc $D^{s-1}(0,1)$ by Δ and $D^{s-1}(0,\lambda)$ by Δ_λ. The interiors of these discs will be denoted by Δ°, resp. Δ°_λ. For each $i : 1 \leqslant i \leqslant l$ choose an embedding $E_i : \Delta \to M$ such that $E_i(0) = x_i$ and E_i conserves the orientation. We assume also that the images of E_i are disjoint. The set $M' = M \setminus \bigcup_i E_i(\Delta^\circ)$ is a compact subset of the domain of definition of $v^{\leadsto}_{[a_s, a_{s-1}]}$ therefore the gradient descent along $(-v)$ will push M' below c_{s-1}. (In precise terms, denote $\Gamma(x, t; -v)$ by $H_t(x)$. Then for $T > 0$ sufficiently large we shall have: $H_t(M') \subset \phi^{-1}([a, c_{s-1}[)$.) Therefore the image of $[M]$ in $H_*(W_1, W')$ equals to the sum (over all i) of the fundamental classes modulo boundary of singular discs $H_t \circ E_i : \Delta \to (W_1, W')$. The same is of course true if instead of the embeddings E_i we consider their restrictions to Δ_λ, $0 < \lambda < 1$. Consider for example the point $x_1 \in M$, assume that $x_1 \in D(q, -v) \cap D(p, v)$, where $q \in S_{s-1}(\phi)$. Let ε be the intersection sign of $S(p, v)$ and $S(q, -v)$ in x_1. Let $\varphi : U \to V$ be a Morse chart for ϕ around q. We refer again to Definition 2.8 for the notation and definitions. Let T be so large that $\gamma(x_1, T; -v)$ belong to Int $\varphi^{-1}(Q_r)$. The map $\rho : y \mapsto \varphi(\gamma(E_1(y), T; -v))$ restricted to $y \in \Delta_\lambda$, where r is sufficiently small, is a diffeomorphism onto its image, which is in Int Q_r.

Consider now the local situation in Q_r The map $\rho'(0) : \mathbf{R}^{s-1} \to \mathbf{R}^n = \mathbf{R}^{s-1} \oplus \mathbf{R}^{n-s-1}$ is an injection, moreover, its first coordinate is an isomorphism $H : \mathbf{R}^{s-1} \to \mathbf{R}^{s-1}$. Note that the sign of det H equals to ε. Consider the linear map $\alpha : \mathbf{R}^{s-1} \to \mathbf{R}^{s-1} \oplus \mathbf{R}^{n-s+1}$ defined by $x \mapsto \left(\frac{r}{2\|H\|} Hx, 0\right)$. The restriction of α to the unit disc $D^{s-1}(0,1)$ is a diffeomorphism onto its image K which is contained in Int $Q_r \cap L_-$.

Apply now to ρ the diffeomorphism $\Phi(T', -\boldsymbol{v}_{s-1})$, where T' is a large enough real number. It is clear that if $\lambda > 0$ is small, the embedded disc $\Phi(T', -\boldsymbol{v}_{s-1})(\rho(\Delta_\lambda))$ will become larger and more and more flat. More precisely, for every $\epsilon > 0$ there is $\lambda > 0$ and $T' > 0$ such that the embedded disc $K' = \Phi(T', \boldsymbol{v}_{s-1})(\rho(\Delta_\lambda))$ is ϵ-close to the embedded disc K. (This is a matter of an easy computation which will be left to the reader.) Therefore the homology class of $(K', \partial K')$ in (Q_r, Q_r^-) is the same as that of the negative disc modulo its boundary, multiplied by ε. $\square$

2.5. Gradient descent map

In this section we deal with the following situation. Let W be a cobordism, $f : W \to [a, b]$ be a Morse function on W, and v be an f-gradient. Let $U_1 \subset \partial_1 W$ be the set of all x, such that the $(-v)$-trajectory $\gamma = \gamma(x, t; -v)$ starting at x reaches $\partial_0 W$. For $x \in U_1$ the point of intersection of γ with $\partial_0 W$ will be denoted by $(-v)^{\leadsto}(x)$. We obtain a C^∞ diffeomorphism $(-v)^{\leadsto} : U_1 \to U_0$, where U_0 is an open subset of $\partial_0 W$. In general of course $U_i \neq \partial_i W$. For $A \subset \partial_1 W$ denote $(-v)^{\leadsto}(A \cap U_1)$ by $(-v)^{\leadsto}(A)$. Note that if $L \subset \partial_1 W$ is a submanifold of $\partial_1 W$, then $(-v)^{\leadsto}(L)$ is a submanifold of $\partial_0 W$, but compactness of L does not imply compactness of $(-v)^{\leadsto}(L)$. To make $(-v)^{\leadsto}(L)$ compact, we should add the soles of descending discs, and the resulting object will not be a manifold. So one can think that the gradient descent map is well defined on the category of stratified manifolds. It is indeed so, and the corresponding construction is given in subsection 2.5.2 (subsection 2.5.1 contains some preliminaries). To explain it briefly, let $\mathbb{A} = \{A_0, ..., A_k\}$ be an s-submanifold of V_1 (see the definition in Subsection 2.2.1). Let A'_i denote the union of all $(-v)$-trajectories, starting in A_i. Set $B_i = A'_i \cup D(\text{ind}{=}i\ , v)$. The main observation is that $\{B_i\}_{0\leqslant i\leqslant k}$ is a compact family, provided that $\mathbb{A} \nmid \mathbb{D}(-v)$ (see Lemma 2.50). It is natural therefore to define an s-submanifold $(-v)^{\rightsquigarrow}(\mathbb{A})$ of $\partial_0 W$ setting $(-v)^{\rightsquigarrow}(\mathbb{A})_{(i)} = B_{i+1} \cap \partial_0 W$. The most technical part of the section is Subsection 2.5.3. Here we deal with thickenings. We construct a gfn-system for $\{B_i\}$ starting with a gfn-system for $\mathbb{A}$ and the gfn-system $\mathbf{D}(v)$.

2.5.1. General properties of tracks. In this subsection W is a cobordism, f is a Morse function on W, v is an f-gradient.

Definition 2.46.

Let $X \subset W$. The set $\{\gamma(x,t;-v) \mid x \in X, t \geqslant 0\}$ is denoted by $T(X,-v)$ and called *track of* X *with respect to* $(-v)$. △

The aim of this subsection is to establish the basic properties of $T(X,-v)$. Next lemma is obvious.

Lemma 2.47.

1. *For every* $A \subset W$ *we have* $T(\overline{A},-v) \subset \overline{T(A,-v)}$.
2. *If* $\{X_i\}_{i\in I}$ *is a family of subsets of* $\partial_1 W$, *then* $\cap_i T(X_i,-v) = T(\cap_i X_i,-v)$. □

In the following lemma we list the compactness properties of sets $T(X,-v)$. It can be considered as an analog of 2.18 .

Lemma 2.48.

1. *If* X *is compact, then* $T(X,-v) \cup D(v)$ *is compact.*
2. *If* X *is compact, and every* $(-v)$*-trajectory starting from a point of* X *reaches* $\partial_0 W$ *then* $T(X,-v)$ *is compact.*
3. *For any* X *we have* $\overline{T(X,-v) \cup D(v)} = T(\overline{X},-v) \cup D(v)$.
4. *For any* X *and* $\delta > 0$ *we have* $\overline{T(X,-v) \cup B_\delta(v)} = T(\overline{X},-v) \cup D_\delta(v)$.

Proof.

1. A point $y \in W$ *does not* belong to $T(X,-v) \cup D(v)$, if and only ifthe trajectory $\gamma(x,\cdot;-v)$ reaches $\partial_1 W$ without intersecting X. Therefore the complement of $T(X,-v) \cup D(v)$ in W is open.
2. For $x \in X$ let $\tau(x) \geqslant 0$ be the moment when $\gamma(x,t;-v)$ intersects $\partial_0 W$. Then $\tau(x)$ is continuous in x, so it is bounded by a constant, say, $C > 0$. Now let $y_i = \gamma(x_i,t_i;-v) \in T(X,-v)$ where $x_i \in X, 0 \leqslant t_i \leqslant C$. Choose a subsequence i_k, such that $\{x_{i_k}\}$ and $\{t_{i_k}\}$ converge, and we are over.
3. Since $\overline{X}$ is compact, $T(\overline{X},-v) \cup D(v)$ is also compact, and the inclusion $\overline{T(X,-v) \cup D(v)} \subset T(\overline{X},-v) \cup D(v)$ holds. The inverse inclusion follows from the preceding lemma.
4. Similar to (3).

□

2.5.2. Tracks of s-submanifolds. In this and in the following subsection $f : W \to [a,b]$ is a Morse function on a cobordism W, v is an f-gradient, satisfying the Almost Transversality Condition, $\mathbb{A} = \{A_0, ..., A_k\}$ is an s-submanifold of $\partial_1 W$, satisfying $\mathbb{A} \nmid \mathbb{D}_b(-v)$.

Definition 2.49.
For $0 \leqslant i \leqslant k+1$ set $TA_i(-v) = T(A_{i-1}, -v) \cup D(\mathrm{ind}{=}i\ ; v)$ (we set $A_{-1} = \emptyset$ by definition). Introduce a family $\mathbb{T}(\mathbb{A}, -v)$ of subsets of W by $\mathbb{T}(\mathbb{A}, -v) = \{TA_i(-v)\}_{0 \leqslant i \leqslant k+1}$ For $\lambda \in [a, b]$ introduce a family $(-v)^{\rightsquigarrow}_{[b,\lambda]}(\mathbb{A})$ of subsets of V_λ as follows: $(-v)^{\rightsquigarrow}_{[b,\lambda]}(\mathbb{A}) = \{TA_{i+1}(-v) \cap f^{-1}(\lambda)\}_{0 \leqslant i \leqslant k}$ of subsets of $f^{-1}(\lambda)$. If the values b, λ are clear from the context we shall abbreviate $(-v)^{\rightsquigarrow}_{[b,\lambda]}(\mathbb{A})$ to $(\overset{\rightsquigarrow}{-v})(\mathbb{A})$. The family $\mathbb{T}(\mathbb{A}, -v)$ will be called *track* of $\mathbb{A}$, and the family $v^{\rightsquigarrow}_{[b,\lambda]}(\mathbb{A})$ will be called $\overset{\rightsquigarrow}{v}$*-image* of $\mathbb{A}$. △ △

Lemma 2.50.

1. $\mathbb{T}(\mathbb{A}, -v)$ *and* $(-v)^{\rightsquigarrow}_{[b,\lambda]}(\mathbb{A})$ *are compact families .*
2. *If* λ *is a regular value of* f, *then* $(-v)^{\rightsquigarrow}_{[b,\lambda]}(\mathbb{A})$ *is an* s*-submanifold of* $f^{-1}(\lambda)$.

Proof. (1) Let $0 \leqslant s \leqslant k+1$. Denote $T(A_{\leqslant s-1}, -v) \cup D(\mathrm{ind}{\leqslant}s\ ; v)$ by $Q(s)$. Let $\phi : W \to [a, b]$ be an ordered Morse function, adjusted to (f, v), and $(a_0, ..., a_{n+1})$ be the ordering sequence for ϕ. The all the critical points of ϕ of indices $> s$ are above a_s and those of indices $\leqslant s$ are below a_s. Since $\mathbb{A} \nmid (\mathbb{D}(-v) \cap V_1)$, every $(-v)$-trajectory, starting at a point of $A_{\leqslant s-1}$ reaches $\phi^{-1}(a_{s+1})$. Then $Q(s) \cap \phi^{-1}([a_{s+1}, a_{n+1}])$ is compact by 2.48 (2) and $Q(s) \cap \phi^{-1}([a_0, a_{s+1}])$ is compact by 2.48 (1), therefore $Q(s)$ is compact. (2) is obvious. □

The assumption $\mathbb{A} \nmid \mathbb{D}_b(-v)$, adopted in the beginning of the present subsection is not very restrictive: given $\mathbb{A}$, we can always perturb v so that this assumption will hold. A more general statement is the subject of the following Proposition.

Proposition 2.51.
Let $\mathbb{B}$ *be an* s*-submanifold of* $\partial_1 W$, $\mathbb{C}$ *be an* s*-submanifold of* $\partial_0 W$. *Let* U *be a neighborhood of* ∂W. *Then there is an* f*-gradient* w, *satisfying Almost Transversality Condition, such that*

1. $\mathbb{D}_b(-w) \nmid \mathbb{B}$;
2. $(-w)^{\rightsquigarrow}(\mathbb{B}) \nmid \mathbb{C}$; $\quad \overset{\rightsquigarrow}{w}(\mathbb{C}) \nmid \mathbb{B}$

Moreover, w *can be chosen arbitrarily* C^∞ *close to* v, *and such that* supp $(u - w) \subset U$, *and* supp $(u - v) \cap \partial W = \emptyset$.

Proof. Choose $a > 0$ such that the collars $C_0 = \gamma(\partial_0 W, [0, a]; v)$ and $C_1 = \gamma(\partial_1 W, [0, a]; -v)$ are disjoint. Our new f-gradient w will satisfy supp $(w - v) \subset$ Int $(C_0 \cup C_1)$, which imply immediately Almost Transversality Condition. Apply to v the AHC-construction nearby the upper boundary $\partial_1 W$ of W. We choose the input data (a, ξ, h) as follows. Fix first a function h as to satisfy $\int_0^a h(\tau) d\tau = 1$. Choose then a vector field ξ on $\partial_1 W$ so that $\Phi(\xi, 1)\big(\mathbb{D}_b(-v)\big) \nmid \mathbb{B}$. (This is possible by 2.33), and, moreover, we can choose ξ so C^∞ small that $v_0 = AHC(a, \xi, h; v)$ is C^∞ close to v, and that v_0 is still an f-gradient. Now AHC2 from Subsection 2.1.6 implies that $\Phi(\xi, 1)\big(\mathbb{D}_b(-v)\big) = \mathbb{D}_b(-v_0)$ and the point 1) of our conclusions is true with v_0 instead of w. Therefore $\mathbb{T}(\mathbb{B}, -v_0)$ is defined and the s-submanifold $\mathbb{L} = (-v_0)^{\rightsquigarrow}(\mathbb{B})$ of $\partial_0 W$ is defined. Applying to $\mathbb{L}$ and to $\mathbb{C}$ a similar construction we obtain an f-gradient w, satisfying the 1) and the first half of 2) of our conclusion. Note that the first half of 2) implies the second, and the proof is over. $\square$

2.5.3. Tracks of ts-submanifolds. In this subsection we assume that our f-gradient v is δ_1-ordered.

Definition 2.52.
Let $\mathbf{A} = \{A_s(\delta)\}_{\delta\in]0,\delta_0[,0\leqslant s\leqslant k}$ be a ts-submanifold of V_1 with the core $\mathbb{A}$. For $0<\delta<\min(\delta_0,\delta_1)$ and $0\leqslant s\leqslant k+1$ set $TA_s(\delta,-v) = T(A_{s-1}(\delta),-v)\cup B_\delta(\text{ind}{=}s\ ;v)$ (where we set by definition $A_{-1}(\delta)=\emptyset$). $\triangle$ $\triangle$

We shall prove that for sufficiently small $\epsilon>0$ the system $\{TA_s(\delta,-v)\}_{\delta\in]0,\epsilon[,0\leqslant s\leqslant k}$ is a gfn-system for $\mathbb{T}(\mathbb{A},-v)$. Up to the end of this section $Q(s,\delta)$ stands for $T(\overline{A_{\leqslant s-1}(\delta)},-v)\cup D_\delta(\text{ind}{\leqslant}s\ ;v)$.

Proposition 2.53.
There is $\epsilon\in]0,\min(\delta_0,\delta_1)[$ *such that* $\{TA_s(\delta,-v)\}_{\delta\in]0,\epsilon[,\ 0\leqslant s\leqslant k}$ *is a gfn-system for* $\mathbb{T}(\mathbb{A},-v)$.

Proof. We shall first prove two lemmas, establishing the properties of the introduced family $\{TA_s\}$.

Lemma 2.54.
For every δ with $0<\delta\leqslant\min(\delta_0,\delta_1)$ and for every s we have

$$TA_{\leqslant s}(-v) = \cap_{\theta>0}TA_{\leqslant s}(\theta,-v) \tag{11}$$

$$Q(s,\delta) = \cap_{\theta>\delta}TA_{\leqslant s}(\theta,-v), \tag{12}$$

$$TA_{\leqslant s}(\delta,-v)\subset Q(s,\delta)\subset\overline{TA_{\leqslant s}(\delta,-v)} \tag{13}$$

Proof. To prove (1) note first that $TA_{\leqslant s}(-v)=\cap_{\theta>0}TA_{\leqslant s}(\theta,-v)$ (by 2.47) $\bigcap_{\theta>0}B_\theta(\text{ind}{\leqslant}s\ ;v)=D(\text{ind}{\leqslant}s\ ;v)$; (by 2.29) Since both $TA_{\leqslant s}(\theta,-v)$ and $B_\theta(\text{ind}{\leqslant}s\ ;v)$ are decreasing in θ families of subsets, the (1) holds. (2) is proved similarly. To prove (3), note that the first inclusion is straightforward, the second follows from 2.47 $\square$

Lemma 2.55.
Let $0<\epsilon\leqslant\min(\delta_0,\delta_1)$. Then (i)$\Leftrightarrow$(ii).

(i) *For every $0<\delta<\epsilon$ and every $s: 0\leqslant s\leqslant k+1$ the set $Q(s,\delta)$ is compact*
(ii) *$TA_s(\delta,-v)_{0<\delta<\epsilon,0\leqslant s\leqslant k+1}$ is a gfn-system for $\mathbb{T}(\mathbb{A},-v)$*

Proof. The properties (FS1), (FS2), (FS4) hold for the system $TA_s(\delta,-v)_{0<\delta<\epsilon,0\leqslant s\leqslant k+1}$ for every ϵ. So we have to check that (i) is equivalent to (FS3). And this follows immediately from the preceding lemma. $\square$

Now the question is reduced to the compactness of the subsets $Q(s,\delta)$. Since v is δ_1-ordered, there is a Morse function $\phi: W\to[a,b]$, adjusted to (f,v) with an ordering sequence $(a_0,a_1,\ldots,a_{n+1})$, such that for every $p\in S_s(f)$ we have: $D_{\delta_1}(p)\subset\phi^{-1}(]a_s,a_{s+1}[)$.

Lemma 2.56.
Let $0<\delta\leqslant\min(\delta_0,\delta_1)$. Then (i) $\Leftrightarrow$ (ii).
(i) $Q(s,\delta)$ is compact.

(ii) every $(-v)$-trajectory starting at a point of $\overline{A_{\leqslant s-1}(\delta)}$ reaches $\phi^{-1}(a_{s+1})$.

Proof. Assume that (ii) holds. Then $Q(s,\delta)\cap\phi^{-1}([a_{s+1},a_{n+1}])$ is compact by 2.48 (2) and $Q(s,\delta)\cap\phi^{-1}([a_0,a_{s+1}])$ is compact by 2.48 (1).

Assume now that (ii) does not hold. Then there is $x\in\overline{A_{\leqslant s-1}(\delta)}$ such that $\gamma(x,t;-v)$ converges to $p\in S(f)$ as $t\to\infty$, and $p\in\phi^{-1}(]a_{s+1},a_{n+1}[)$. But in this case $p\in\overline{Q(s,\delta)}\setminus Q(s,\delta)$. □

This lemma implies in particular that if $0<\delta'\leqslant\delta\leqslant\min(\delta_0,\delta_1)$ and $Q(s,\delta)$ is compact, then $Q(s,\delta')$ is also compact.

Definition 2.57.

Let $\lambda>0$. Let $\mathbf{A}$ be an ts-submanifold of $\partial_1 W$ with interval of definition $I=]0,\delta_0[$. We say that the pair $(\mathbf{A},v)$ is λ-ordered, if

1. v is λ-ordered
2. λ is in the interval of definition of $\mathbf{A}$
3. $Q(s,\lambda)$ is compact for every s.

△

Three preceding lemmas imply that if $(\mathbf{A},v)$ is λ-ordered, then $\{TA_s(\delta,-v)\}_{\delta\in]0,\lambda[,0\leqslant s\leqslant k}$ is a gfn-system. Therefore to prove our proposition it suffices to show that there is λ such that $(\mathbf{A},v)$ is λ-ordered. But this is obvious. Indeed, every $(-v)$-trajectory starting in $A_{\leqslant s-1}$ reaches $\phi^{-1}(a_{s+1})$. Recall that $\{A_{\leqslant s-1}(\theta)\}_{\theta>0}$ is a fundamental system of neighborhoods of the compact $A_{\leqslant s-1}$. An easy compactness argument shows that for some $\theta>0$ every $(-v)$-trajectory, starting in $A_{\leqslant s-1}(\theta)$ reaches $\phi^{-1}(a_{s+1})$ as well. □

We shall denote the gfn-system, introduced in 2.53 by $\mathbf{T}(\mathbf{A},-v)$ and call it *track* of $\mathbf{A}$ (with respect to $-v$). Note that there is no canonical choice of the interval of definition for this system.

2.6. Properties of continuity of integral curves of vector fields with respect to C^0 small perturbations of of vector fields

Let v be a C^1 vector field on a manifold M, γ be an integral curve of v. It is well known that given $t\geqslant 0$ the value $\gamma(t)$ depends continuously on the initial value $\gamma(0)$ and on the vector field itself (where the topology on the space of vector fields is the C^1- *topology.* Actually this continuity property is still true if we replace C^1 topology by C^0*-topology.* [4] It is due to this fact that the C^0 topology plays an important role in our work. The present section contains the statements of the continuity theorems we shall need in the sequel. We start with citing two propositions from [**38**]. Afterwards we deduce from them some properties (see Corollary 2.63 , Proposition 2.64 , Proposition 2.61) in order to obtain the results in the form ready for the applications in the further sections. The results which are used in the sequel are gathered in Subsection 2.6.2. Essentially all the contents of of the present section is already present in [**38**]; we have only rearranged the material.

[4] By the time when the preprint [**34**] was written I had no references for this property at hand. I am grateful to O.Viro for the indication that this "C^0 continuity property" is known and must be exposed somewhere in the literature.

2.6.1. Manifolds without boundary. For a manifold M (without boundary) we denote by $\mathrm{Vect}^1(M)$ (resp. $\mathrm{Vect}^1_0(M)$) the vector space of C^1 vector fields on M (resp. the vector space of C^1 vector fields on M with compact support). In this subsection M is a riemannian manifold without boundary, $v \in \mathrm{Vect}^1_0(M), n = \dim M$.

Proposition 2.58. ([38], Prop. 5.1)
Let $a, b \in M, t_0 \geqslant 0$ and $\gamma(a, t_0; v) = b$. Then for every open neighborhood U of $\gamma(a, [0, t_0]; v)$ and every open neighborhood R of b there exist $\delta > 0$ and an open neighborhood $S \subset U$ of a such that $\forall x \in S$ and $\forall w \in \mathrm{Vect}^1_0(M)$ with $\|w - v\| < \delta$ we have: $\gamma(x, t_0; w) \in R$ and $\gamma(x, [0, t_0]; w) \subset U$. □

Proposition 2.59. ([38], Prop. 5.1)
Let $a, b \in M, t_0 \geqslant 0$ and $\gamma(a, t_0; v) = b$. Let E be a submanifold without boundary of M of codimension 1, such that $b \in E$ and $v(b) \notin T_bE$.

Let U be an open neighborhood of $\gamma(a, [0, t_0]; v)$ and R be an open neighborhood of b. Then there is $\theta_0 > 0$ such that for every $\theta \in]0, \theta_0[$ there exist $\delta > 0$ and an open neighborhood $S \subset U$ of a, such that $\forall x \in S$ and $\forall w \in \mathrm{Vect}^1_0(M)$ with $\|w - v\| < \delta$ we have:

1. *$\gamma(x, [-\theta, t_0 + \theta]; w) \subset U$, and $\gamma(x, [t_0 - \theta, t_0 + \theta]; w) \subset R$.*
2. *There is a unique $\tau_0 = \tau_0(w, x) \in [t_0 - \theta, t_0 + \theta]$, such that $\gamma(x, \tau_0; w) \in E$.*
3. *If E is compact and t_0 is the unique t from $[0, t_0]$ such that $\gamma(x, t_0; v) \in E$, then $\forall y \in S$ the number $\tau_0(w, y)$ is the unique τ from $[-\theta, t_0 + \theta]$ such that $\gamma(y, \tau; w) \in E$.* □

For the further use we shall need a version of Proposition 2.58 ; this version (Proposition 2.61 below) is proved by the same method, so we shall expose only briefly the proof.

Definition 2.60.
We denote by $\|\cdot\|$ the *sup*-norm on $\mathrm{Vect}^1_0(M)$. Let $\delta > 0, v \in \mathrm{Vect}^1_0(M)$. We denote by $B(v, \delta)$ the set of all $w \in \mathrm{Vect}^1_0(M)$, such that $\|w - v\| < \delta$. The topology, induced in $\mathrm{Vect}^1_0(M)$ by the *sup*-norm will be called C^0-topology. △

Proposition 2.61.
Denote by $\mathcal{E} : M \times \mathbf{R} \times \mathrm{Vect}^1_0(M) \to M$ the map defined by

$$(x, t, v) \mapsto \gamma(x, t; v)$$

Then $\mathcal{E}$ is continuous with respect to the C^0 topology in $\mathrm{Vect}^1_0(M)$.

Proof. Let $(a, t_0, v_0) \in M \times \mathbf{R} \times \mathrm{Vect}^1_0(M)$; let $b = \gamma(a, t_0; v_0)$. The continuity of $\mathcal{E}$ at the point (a, t_0, v_0) is equivalent to the following assertion:

for every neighborhood U of b there are: a neighborhood S of a, a number $\delta > 0$, a number $\theta > 0$, such that

$$\big(x \in S, w \in B(v, \delta), |t - t_0| < \theta\big) \Rightarrow \big(\gamma(x, t; w) \in U\big)$$

Subdividing the interval $[0, t_0]$ into small intervals it is easy to see that we can restrict our assertion to the case when the curve $\gamma(a, [0, t_0]; v_0)$ belongs to the domain of a single chart $\Phi : V \to V' \subset \mathbf{R}^n$. This case is reduced to the case of vector fields with compact support in $\mathbf{R}^n$. This last case is settled via the next lemma, which in turn is a direct consequence of [**4**], §4, Th. 3.

For $v \in \mathrm{Vect}_0^1(\mathbf{R}^n)$ we denote by $\|v\|_1$ the *sup*-norm of the derivative $dv : \mathbf{R}^n \to L(\mathbf{R}^n, \mathbf{R}^n)$.

Lemma 2.62.
Let $u, w \in \mathrm{Vect}_0^1(\mathbf{R}^n)$, $\|u - w\| < \alpha$, $\|u\|_1 \leqslant D$, where $D > 0$. Let γ, η be trajectories of, respectively, u, w, and assume that $|\gamma(0) - \eta(0)| \leqslant \epsilon$. Then for every $t \geqslant 0$ we have: $|\gamma(t) - \eta(t)| \leqslant \epsilon e^{Dt} + \frac{\alpha}{D}(e^{Dt} - 1)$. □

Corollary 2.63.
Let $a, b \in M, t_0 \geqslant 0, \gamma(a, t_0; v) = b$. Let E be a submanifold without boundary of M of codimension 1, such that $b \in E$ and $v(b) \notin T_b E$. Let U be an open neighborhood of b. There is $\theta_0 > 0$, such that for every $\theta \in]0, \theta_0[$ there is $\delta > 0$ and a neighborhood S of a and a function $\tau_0 : B(v, \delta) \times S \to \mathbf{R}$, such that

1. *$\forall (w, x) \in B(v, \delta) \times S$ we have $y \in \gamma(x, \tau_0(w, x); w) \in E \cap U$ and $w(y) \notin T_y E$.*
2. *$\tau_0(w, x) \in]t_0 - \theta, t_0 + \theta[$ and $\tau_0(w, x)$ is the unique number τ of this interval such that $\gamma(x, \tau; w) \in E$.*

Proof. Choose a neighborhood $R \subset U$ of b and $\delta_0 > 0$, such that for every $w \in B(v, \delta_0)$ and every $y \in R \cap E$ we have $w(y) \notin T_y E$. Then apply Proposition 2.59 with this R. There is $\theta_0 > 0$, such that for every $\theta \in]0, \theta_0[$ there is $\delta > 0$ and a neighborhood S of a and a function $\tau_0 : B(v, \delta) \times S \to \mathbf{R}$, satisfying the conclusion of 2.59 . For $(w, x) \in B(v, \delta) \times S$ set $y(w, x) = \gamma(x, \tau_0(w, x); w)$. It follows from our choice of R, that $w(y(w, x)) \notin T_{y(w,x)} E$. It remains only to diminish δ and S in order to force $\tau_0(w, x)$ to belong to $]t_0 - \theta, t_0 + \theta[$ instead of the interval $[t_0 - \theta, t_0 + \theta]$ from (2). For this just take δ' and S' corresponding to $\theta/2$. □

Proposition 2.64.
Let $a, b \in M, t_0 \geqslant 0, \gamma(a, t_0; v) = b$. Let E be a submanifold without boundary of M of codimension 1, such that $b \in E$ and $v(b) \notin T_b E$. Let $\delta > 0, \theta > 0$, let S be a neighborhood of a, and assume that $\tau_0 : B(v, \delta) \times S \to \mathbf{R}$ is a function satisfying (1) and (2) from the Corollary 2.63 . Then τ_0 is continuous on its domain.

Proof. Let $x_0 \in S, w_0 \in B(v, \delta)$, and consider the w_0-trajectory $\gamma(x_0, \cdot; w_0)$. Set $t_0' = \tau_0(w_0, x_0)$ and $y_0 = \gamma(x_0, t_0'; w_0)$. Then $w(y_0) \notin T_{y_0} E$. Thus we can apply the preceding Corollary to the trajectory $\gamma(x_0, \cdot; w_0)$. Let $\epsilon > 0$. We apply Corollary 2.63 with θ_0', satisfying $\theta_0' < \epsilon$, $[t_0' - \theta', t_0' + \theta'] \subset]t_0 - \theta, t_0 + \theta[$. We obtain a number $\delta' > 0$ a neighborhood S' of x_0 and a unction $\tau' : B(w, \delta') \times S' \to \mathbf{R}$, such that

1. For every $(w, x) \in B(w_0, \delta') \times S'$ we have: $y = \gamma(x, \tau'(w, x); w) \in E$ and $w(y) \notin T_y E$
2. $\tau'(w, x) \in]t_0' - \theta', t_0' + \theta'[$ and $\tau'(w, x)$ is the unique number τ of this interval such that $\gamma(x, t; w) \in E$.

We can diminish δ' and S' so that $B(w_0, \delta') \subset B(v, \delta)$ and $S' \subset S$. Then the domain of definition of τ' is in the domain of definition of τ_0 and our choice of θ' implies that $\tau'(w, x) = \tau_0(w, x)$. The condition (2) implies then that $\tau_0(w, x) \in]t_0 - \epsilon, t_0 + \epsilon[$ for $(w, x) \in B(w_0, \delta') \times S'$. Since $\epsilon > 0$ was arbitrary this proves continuity. □

In the proposition 2.65 we assume that the submanifold E and the vector field v satisfy the following assumptions:

($\mathcal{E}1$) There is $\alpha > 0$ such that for every $w \in B(v, \alpha)$ every w-trajectory intersects E at most once.

($\mathcal{E}2$) for every $a \in E$ we have: $v(a) \notin T_a E$.

Proposition 2.65.

Let $K \subset M$ be a compact. Let U be an open subset of E. Assume that for every $x \in K$ the trajectory $\gamma(x, \cdot; v)$ reaches E at the moment $\tau(x)$ and that $\gamma(x, \tau(x); v) \in U$. Then there is $\delta > 0$ and a continuous function $\tau : B(v, \delta) \times K \to \mathbf{R}_+$, such that

1. *For every $(w, x) \in B(v, \delta) \times K$ we have: $\gamma(x, \tau(w, x); w) \in U$*
2. *Denote by $A : B(v, \delta) \times K \to U$ the map defined by $A(w, x) = \gamma(w, \tau(x, w); w)$. For $w \in B(v, \delta)$ denote by A_w the map $K \to U$ defined by $A_w(x) = A(w, x)$. Then A is continuous, and A_w is continuous for every w and A_w is homotopic to A_v for every W.*

Proof. Apply Corollary 2.63 and choose for every $a \in K$ a neighborhood $S(a)$ of a, a number $\delta(a)$ and a function $\tau_{(a)} : B(v, \delta(a)) \times S(a) \to \mathbf{R}_+$ such that for every $x \in S(a)$ and $w \in B(v, \delta(a))$ we have: $\gamma(w, \tau_a(x, w); w) \in U$. Moreover the function $\tau_{(a)}$ is continuous at (v, a). We assume that $\delta(a) < \alpha$ for every a. Note that if a pair (w, x) is in the intersection of $B(v, \delta(a)) \times S(a)$ and $B(v, \delta(b)) \times S(b)$ (where $a, b \in K$) then $\tau_a(w, x) = \tau_b(w, x)$ (it follows from ($\mathcal{E}1$)). Choose a finite covering of K by sets $S(a_i), 1 \leqslant i \leqslant N$, and set $\delta = \min_i \delta(a_i), S = \cup_i S(a_i)$. We obtain a continuous function $\tau : B(v, \delta) \times S \to \mathbf{R}_+$. This function is continuous. Now the continuity of the map A, defined in the point 2) of our Proposition is obvious: just apply Proposition 2.61 and get that A is a composition of continuous maps. To obtain a homotopy between A_w and A_v set $H_t = A_{v+t(w-v)}$. □

2.6.2. Manifolds with boundary. Let W be a compact manifold with boundary ∂W. Let $\phi : \partial W \times [0, \infty[\to W$ be a collar of ∂W (see [**12**], p. 113 for the definition of collar and the theorem of existence). Consider the smooth manifold W' without boundary obtained by gluing W to $\partial W \times]-1, 1[$ via the diffeomorphism $\phi | \partial W \times [0, 1[$. The manifolds $W, \partial W \times]-1, 1[$ are embedded in the obvious way to W', so we shall consider $W, \partial W \times]-1, 1[$ as submanifolds of W'; Let $\xi : \partial W \times]-1, 1[\to]-1, 1[$ be the projection onto the second factor. Denote by $\mathrm{Vect}^1(W, \perp)$ the set of all C^1 vector fields v on W such that that for every $x \in \partial W$ we have: $v(x) \notin T_x(\partial W)$. Let $v \in \mathrm{Vect}^1(W, \perp)$. Denote by $\partial_v^+ W$, resp. by $\partial_v^- W$ the set of $x \in \partial W$, such that $v(x)$ points inward W, resp. outward W. (If the value of v is clear from the context we shall write simply $\partial^- W, \partial^+ W$.) Since W is compact, the set $\mathrm{Vect}^1(W, \perp)$ is open in $\mathrm{Vect}^1(W)$ with respect to C^0 topology. Moreover for given $v \in \mathrm{Vect}^1(W, \perp)$ there is $\delta > 0$ such that for $\|v - w\| < \delta$ we have: $\partial_v^+ W = \partial_w^+ W, \quad \partial_v^- W = \partial_w^- W$.

A vector field u on W' will be called *good extension* of v, if

(E1) $u | W = v$;

(E2) supp u is compact;

(E3) the restriction of u to $\partial_0 W \times [0, 1[$ satisfies $\xi'(x, t)(u(x, t)) \geqslant 0$ and $\xi'(x, t)(u(x, t)) > 0$ for every x and t in some interval $[0, \beta]$.

(E4) the restriction of u to $\partial_1 W \times [0, 1[$ satisfies $\xi'(x, t)(u(x, t)) \leqslant 0$ and $\xi'(x, t)(u(x, t)) < 0$ for every x and t in some interval $[0, \beta]$.

Remark 2.66.

It is not difficult to show using the partitions of unity that for any $v \in \mathrm{Vect}^1(W, \perp)$ good extensions of v exist. Moreover, one can show that if $v, w \in \mathrm{Vect}^1(W, \perp)$ and $\partial_v^{\pm} W = \partial_w^{\pm} W$, then one can choose good extensions $\widetilde{v}, \widetilde{w}$ of v, resp. of w, such that $\|\widetilde{v} - \widetilde{w}\|_{W'} \leqslant 2\|v - w\|_W$. △

Lemma 2.67.

Let $v \in \mathrm{Vect}^1(W, \perp)$ and let $\widetilde{v}$ be a good extension of v. Then the vector field $\widetilde{v}$ and the submanifold $\partial^+ W \subset W'$ satisfy $(\mathcal{E}1)$ and $(\mathcal{E}2)$ from Subsection 2.6.1.

Proof. The condition $(\mathcal{E}2)$ goes by definition. To prove $(\mathcal{E}1)$ note first that every $\widetilde{v}$- trajectory γ, which intersects $\partial^+ W$ at a moment say t_0, leaves necessarily the domain $\partial^+ W \times [0, \beta]$, passing by a point of $\partial^+ W \times \{\beta\}$. Afterwards it stays forever in $\partial^+ W \times [\beta, 1[$. So $(\mathcal{E}1)$ holds for v. If w is sufficiently C^0 close to $\widetilde{v}$ the property $(E3)$ above is still true for w and thus $(\mathcal{E}1)$ holds for w.

Corollary 2.68.

Let $v \in \mathrm{Vect}^1(W, \perp)$. let $x \in W$ and assume that the trajectory $\gamma(x, \cdot; v)$ reaches $\partial^+ W$. Then there is an open neighborhood S of x such that $\forall y \in S$ the trajectory $\gamma(y, \cdot; v)$ reaches $\partial^+ W$. □

Definition 2.69.

For $x \in W, t \geqslant 0$ define the element $\Gamma(x, t; v) \in W$ as follows:

1. if $\gamma(x, \cdot; v)$ is defined on $[0, t]$ then $\Gamma(x, t; v) = \gamma(x, t; v)$
2. if $\gamma(x, t_0; v) \in \partial^+ W$ for some $t_0 \leqslant t$, then $\Gamma(x, t; v) = \gamma(x, t_0; v)$

△

Corollary 2.70.

The map $(x, t) \mapsto \Gamma(x, t; v)$ is continuous.

Proof. Let $x \in W$. If $\gamma(x, \cdot; v)$ does not reach $\partial^+ W$ then the continuity of Γ in (x, t) for every t is obvious. If $\gamma(x, \cdot; v)$ reaches $\partial^+ W$, then the same is true for any y in a neighborhood of x; denote the moment of intersection of $\gamma(x, \cdot; v)$ with $\partial^+ W$ by $\tau_0(x)$. Then $\Gamma(x, \cdot; v) = \gamma(x, \min(t, \tau_0(x)); \widetilde{v})$ and the continuity is proved. □

Corollary 2.71.

Let $v \in \mathrm{Vect}^1(W, \perp)$. Let U be an open subset of $\partial^+ W$, and $K \subset W$ be a compact. Assume that for every $x \in K$ the v- trajectory $\gamma(x, \cdot; v)$ starting at K reaches $\partial^+ W$ and intersects it at $\alpha_v(x) \in U$.

Then the map $x \mapsto \alpha_v(x)$ is a continuous map $K \to U$. There is is $\lambda > 0$ such that for every w with $\|w - v\| < \lambda$ every w-trajectory starting at K reaches $\partial^+ W$, intersects it at a point $\alpha_w(x) \in U$, and the map $x \mapsto \alpha_w(x)$ is a continuous map $K \to U$ homotopic to α_v.

Proof. Let $\widetilde{v}$ be a good extension of v. The vector fields $\widetilde{v}$ and the submanifold $\partial^+ W$ of $W4$ satisfy $(\mathcal{E}1)$ and $(\mathcal{E}2)$, as it follows from Lemma 2.67 . Therefore Proposition 2.65 applies, and we obtain a number $\delta > 0$ and a continuous function $A : B(\widetilde{v}, \delta) \times K \to U$, such that $A(u, x) = \gamma(x, \tau(x, u); u)$. Set $u = \widetilde{v}$; we obtain a continuous function $A_{\widetilde{v}} : K \to U$ and a continuous function $x \mapsto \tau(x, \widetilde{v})$, which we shall denote by $\tau_{\widetilde{v}}$. Since $K \subset W$ for every $x \in K$ we have $\tau_{\widetilde{v}}(x) \geqslant 0$ and $\gamma(x, [0, \tau_{\widetilde{v}}(x)]; \widetilde{v}) \subset W$. Therefore $\alpha_v(x) = A_{\widetilde{v}}(x)$ and the map α_v is continuous.

Assume now that δ is so small that for every $\xi \in \mathrm{Vect}^1(W)$ with $\|\xi - v\|_W < \delta$ we have: $\xi \in \mathrm{Vect}^1(W, \perp)$ and $\partial_\xi^\pm W = \partial_v^\pm W$. Let $\lambda < \delta/2$; let $w \in \mathrm{Vect}^1(W)$ and $\|v - w\|_W < \lambda$. Let $\widetilde{w}$ be a good extension of v, satisfying $\|\widetilde{v} - \widetilde{w}\|_{W'} < \delta$. We have: $\gamma(x, \tau(\widetilde{w}, x); w) \in U \subset E$ for $x \in K$. Note that $\tau(x, \widetilde{w}) \geqslant 0$ (indeed, if $\tau(x, \widetilde{w}) < 0$ then for some $\theta \in]\tau(x, \widetilde{w}), 0[$ we have: $\gamma(x, \theta; \widetilde{w}) \in \partial^+ W \times]0, 1[$, and the relation $\gamma(x, 0; \widetilde{w}) \in W$ is no more possible). Further, $\gamma(x, [0, \tau(x, \widetilde{w})]; \widetilde{w}) \subset W$, therefore $\gamma(x, \tau(\widetilde{w}, x); w) = A(\widetilde{w}, x) = \alpha_w(x) \in U$ and α_w is continuous in x.

To prove that α_w is homotopic to α_v for $\|v - w\|_W < \lambda$, consider a vector field $\xi(t) = \widetilde{v} + t(\widetilde{w} - \widetilde{v})$ where $t \in [0, 1]$. Note that $\|\xi(t) - \widetilde{v}\|_{W'} < \delta$, therefore for every $t \in [0, 1]$ the vector field $\xi(t)$ is in $B(\widetilde{v}, \delta)$. The map $[0, 1] \times K \to U$ given by the following formula: $(t, x) \mapsto A(\xi(t), x)$ is then a homotopy between α_w and α_v. $\square$

For some further applications we shall need a generalization of Corollary 2.71 to the case of pairs of compacts (K_1, K_2). For the further references we shall formulate it as a separate statement. The proof is immediate generalization of the proof of 2.71 .

Corollary 2.72.

Assume the hypotheses and terminology of Corollary 2.71 . Let K' be a compact subset in K, and U' be an open subset of U. Assume that $\alpha_v(K') \subset U'$.

There is is $\lambda > 0$ such that for every w with $\|w - v\| < \lambda$ every w-trajectory starting at K', resp. at K reaches $\partial^+ W$, intersects it at a point $\alpha_w(x) \in U'$, resp $\alpha_w(x) \in U$, and the map $x \mapsto \alpha_w(x)$ defines a continuous map of pairs $(K, K') \to (U, U')$, which is homotopic to α_v, considered as a map of pairs $(K, K') \to (U, U')$. $\square$

2.6.3. Gradients of Morse functions. In this subsection $f : W \to [a, b]$ is a Morse function on a riemannian cobordism W, v is an f-gradient. We start with an elementary lemma.

Lemma 2.73.

Let $\phi : W \to [\alpha, \beta]$ be a Morse function adjusted to (f, v). Then there is $\delta > 0$, such that every f-gradient w, satisfying $\|w - v\| < \delta$ is also a ϕ-gradient.

Proof. To check the condition 1 from Definition 2.3 for the pair (ϕ, w) (the other two conditions hold obviously), choose for each $x \in S(f)$ a neighborhood $U(x)$ such that $(f - \phi)|U = const$. Set $U = \cup_x U(x)$ and $K = W \setminus U$. We have: $d\phi(x)(v(x)) > 0$ for every $x \in K$. Since K is compact, this implies that for w sufficiently close to v in C^0-topology we have: $d\phi(x)(w(x)) > 0$ for every $x \in K$. For every $x \in U \setminus S(f)$ we have: $d\phi(x)(wx)) = df(x)(wx)) > 0$, and the condition 1 is proved. $\square$

Corollary 2.74.

1. *Assume that v is a δ-ordered f-gradient. then every f-gradient w, sufficiently close to v in C^0-topology is δ-ordered.*
2. *Assume that v is a δ-separated f-gradient. Then every f-gradient w sufficiently close to v in C^0-topology is δ-separated.* $\square$

In the following two propositions $\lambda \in [a, b]$ is a regular value of f, p is a critical point of f, satisfying $f(p) \geqslant \lambda$, U is an open subset of $f^{-1}(\lambda)$. We denote $f(p)$ by c, and $f^{-1}(\lambda)$ by V'.

Proposition 2.75.

Assume that for every $x \in D(p, v) \cap f^{-1}([\lambda, c])$ the $(-v)$-trajectory $\gamma(x, \cdot; -v)$ reaches V'. Assume that $D(p, v) \cap V' \subset U$.

Then there is $\delta > 0$, such that for every f-gradient w, satisfying $\|v - w\| < \delta$ we have:

For every $x \in D(p, v) \cap f^{-1}([\lambda, c])$ the $(-w)$-trajectory $\gamma(x, \cdot; -w)$ reaches V', and $D(p, v) \cap V' \subset U$.

Proof. Let $W' = f^{-1}([\lambda, b])$. For every $q \in S(f) \cap W'$ we have $D(q, -v) \cap D(p, v) = \emptyset$, therefore applying Rearrangement Lemma 2.22 several times we can find a Morse function $\phi : W' \to [\lambda, b]$, adjusted to (f, v) and a regular value μ of ϕ, such that p is the unique critical point of ϕ in the domain $\phi^{-1}([\lambda, \mu])$. Consider the cobordism $W'' = \phi^{-1}([\lambda, \mu])$. Recall from Lemma 2.73 that any f-gradient w, sufficiently close to v in C^0-topology is also a ϕ-gradient. Therefore for w sufficiently close to v in C^0-topology a $(-w)$- trajectory starting at p necessarily reaches V', since there is only one critical point of ϕ in W''. Further, let $K = V' - U$. Every v- trajectory starting at K reaches $\phi^{-1}(\mu) = \partial_1 W''$. Therefore the same is true for every vector field w, sufficiently close to v in C^0-topology(by Corollary 2.71). Thus $D(p, w) \cap V' \subset V' \setminus K = U$ □

Assume the hypotheses of the preceding proposition. Choose an orientation of $D(p, v)$. Set $\Sigma(v) = D(p, v) \cap V'$. Then $\Sigma(v)$ is an embedded s-dimensional oriented sphere in V', where $s + 1 = \operatorname{ind} p$ (see Lemma 2.14). Therefore its fundamental class defines an element $\sigma(v) \in H_s(U)$. By Proposition 2.75 the same is true for any f-gradient w if only w is sufficiently close to v in C^0-topology.

Proposition 2.76.

Assume the hypotheses of Proposition 2.75 . Let U be a neighborhood of p.

Then there is $\delta > 0$, such that for every f-gradient w, satisfying $v|U = w|U$ and $\|v - w\| < \delta$ we have: $\sigma(v) = \sigma(w)$.

Proof. We assume the terminology from the proof of 2.75 . Set $\nu = \phi(p)$ and let $\epsilon > 0$ be so small that $D(p, v) \cap \phi^{-1}([\nu - \epsilon, \nu]) \subset U$. Set $\Sigma_0(v) = D(p, v) \cap \phi^{-1}(\nu - \epsilon)$, $\Sigma_0(v) = D(p, v) \cap \phi^{-1}(\nu - \epsilon)$, . Then

$$\Sigma(v) = (-v)^{\rightsquigarrow}_{[\nu-\epsilon,\lambda]}(\Sigma_0(v)), \quad \Sigma(w) = (-w)^{\rightsquigarrow}_{[\nu-\epsilon,\lambda]}(\Sigma_0(w))$$

Note that the condition $v|U = w|U$ implies $\Sigma_0(v) = \Sigma_0(w)$, therefore to obtain the conclusion of our Proposition just apply the Corollary 2.71 to the cobordism $\phi^{-1}([\lambda, \nu - \epsilon]) = W''$ and the vector field $(-v)|W''$. □

CHAPTER 3

Quick Flows

Let M be a closed manifold of dimension n, f be a Morse function on M, v be an f-gradient satisfying the Transversality Condition Consider the union of all the descending discs of indices $\leqslant k$, denote it by M_k. These descending discs form a filtration of M, which is a close analog of cellular decomposition. (In many cases the descending discs *do* give a real cell decomposition, but we neither study this subject in our paper, nor use the corresponding results.) The ascending discs form a "dual decomposition", denoted by $\{\widehat{M_k}\}_{0\leqslant k\leqslant n}$. The transversality condition implies that $M_k \cap \widehat{M}_{n-k-1} = \emptyset$. Let U be an open neighborhood of M_k, and V be an open neighborhood of $\widehat{M}_{n-k-1}$. It is clear that the gradient descent map $\Phi(T, -v)$ pushes $M \setminus V$ inside U, if only T is large enough. The main aim of this section is to find for any M such a function f and its gradient v, and the neighborhoods U, V, that the number T from above and the C^0 norm of v are small. We demand also that V and U be sufficiently small neighborhoods of M_k, respectively of $\widehat{M}_{n-k-1}$. The flows generated by such gradients are called *rapid flows*, since they push $M \setminus V$ to U rapidly.

The precise statement of the result is Theorem 3.4 which is the main result of the present chapter. This theorem will be used in the proof of C^0-density of condition ($\mathfrak{C}$) (Theorem 4.3); this last theorem is essential for the proof of Theorem 5.5 . Later on we shall use only the statement of Theorem 3.4 , but not the proof, so the reader interested only in the applications of the theorem can read the present introduction and Subsection 3.1.1 and skip the rest of the chapter.

One may try to give a construction of rapid flows using triangulations of M. Indeed, choose a smooth triangulation of M. Pick a Morse function f on M, such that the critical points of f of index p are the barycenters of the simplices of the triangulation. (The existence of such a function seems to be known in folklore since sixties; there is a recent work [**40**] of M.Pozniak on this subject.) It seems that if the triangulation of M is sufficiently fine (that is, the simplices are of small diameter) the function f and some f-gradient would give a solution of our problem.

I have chosen another approach, in which it is easier to obtain explicit estimates for times and norms. I do not make any use of triangulations of the manifold. Instead this approach suggests an alternative construction of "small handle decomposition" of a manifold, which may prove useful in other domains of topology.

Now I shall give a brief informal description of the construction. Start with an arbitrary Morse function $f : M \to \mathbf{R}$ such that the values of f at all the critical points are different. Then choose a finite sequence of regular values $\alpha_1, ..., \alpha_N$ of f, such that between α_i and α_{i+1} there is at most one critical value of f. We then think of M as of the union of slices $W_i = f^{-1}([\alpha_i, \alpha_{i+1}])$, which we choose to be very thin. Consider a slice W_i which has no critical points inside it. It looks like a cylinder $V_i \times [0, 1]$, where $V_i = f^{-1}(\alpha_i)$ (see the figure A on the page 98).

Now modify f inside W_i so as to make a fold (see the figure B on the page 98)[1]. Glue together the functions $\widetilde{f_i}$ and obtain a smooth function on W. Note that the trajectories of its gradient have become much shorter: they can intersect no more than three successive slices W_i. However each function $\widetilde{f_i}$ is not a Morse function: there are two level surfaces, consisting only of critical points. This must be repaired by perturbing $\widetilde{f_i}$ nearby these level surfaces with the help of an auxiliary Morse function on V_i (see the figure C on the page 98). Gluing together the perturbed functions we obtain the Morse function sought.

What is going on in one slice is the subject of section 4.1 of this chapter, "gluing back" is done in section 3.3, which contains also the end of the proof of the Theorems 3.12 and 3.13 . Section 3.1 contains all the new terminology we need. Unfortunately we do need a lot, since the precise estimates of times which a trajectory of a gradient of a Morse function can spend in different domains is not a standard item. In particular we shall distinguish between *rapid flows* (Definition 3.1) and *quick flows* (Definition 3.8). In both definitions the main item is the time which a trajectory of a flow can spend outside a neighborhood U of $S(f)$; but in the definition of rapid flows U is the union of some riemannian balls around the critical points of f, and in the definition of quick flows U is the union of coordinate neighborhoods corresponding to Morse charts. In the section 3.1 we also state the main result of this chapter (Theorem 3.4) and do first steps in its proof.

3.1. Introduction and statement of results

3.1.1. Theorem on the existence of rapid gradient flows. In this subsection $f : M \to \mathbf{R}$ is a Morse function on a closed manifold of dimension m.

Definition 3.1.

Let v be an f-gradient, satisfying Almost Transversality Condition. Let $\epsilon > 0, t \geqslant 0$. We say that the flow, generated by v is (t,ϵ)-*rapid*, if for every $s : 0 \leqslant s \leqslant m$ we have:

$$\Phi(v,t)\big(M \setminus B_\epsilon(\mathrm{ind}_{\leqslant s}\ ; v)\big) \subset B_\epsilon\big(\mathrm{ind}_{\leqslant m-1-s}\ ; -v\big) \tag{14}$$

$$\Phi(-v,t)\big(M \setminus B_\epsilon(\mathrm{ind}_{\leqslant s}\ ; -v)\big) \subset B_\epsilon\big(\mathrm{ind}_{\leqslant m-1-s}\ ; v\big) \tag{15}$$

△

Of course the definition would make sense for arbitrary f-gradient, not necessarily satisfying Almost Transversality Condition. The reason to restrict ourselves to the gradients, satisfying Almost Transversality Condition, is the following lemma.

Lemma 3.2.
For every $\epsilon > 0$ there is $t \geqslant 0$ such that the flow generated by v is (t,ϵ)-rapid.

Proof. If the lemma is true for some ν, then it is true for every $\nu' \geqslant \nu$, so it is sufficient to prove it for such ν, that v is ν-ordered. For such ν consider the compact subset

$$K = M \setminus \left(B_\epsilon(\mathrm{ind}_{\leqslant m-s-1}\ , -v) \cup \bigcup_{\mathrm{ind} p \leqslant s} B_\epsilon(p) \right)$$

[1] "fold" is "skladka" in Russian, which explains the term "S-construction"

There are no critical points in K, therefore $df(v)(x) > C > 0$ for $x \in K$. This implies the existence of $D > 0$, such that for every $x \in K$ there is $t_0 \leqslant D$, such that we have: $\gamma(x, t_0; -v) \in M \setminus K$. Since $M \setminus B_\epsilon(\mathrm{ind} \leqslant m-s-1\ , -v)$ is $(-v)$-invariant we have: $\gamma(x, t_0, -v) \in \cup_{\mathrm{ind} p \leqslant s} B_\epsilon(p)$, and since $B_\epsilon(\mathrm{ind} \leqslant s\ , v)$ is also $(-v)$-invariant, we have $\gamma(x, D; -v) \subset B_\epsilon(\mathrm{ind} \leqslant s\ , v)$. □

Similarly, given t and ϵ one easily constructs an f-gradient which is (t, ϵ)-rapid, but has maybe a very large C^0 norm. (Indeed, multiply a given f-gradient satisfying Almost Transversality Condition by a large constant outside $B = \cup_{\mathrm{ind} p \leqslant s} B_\epsilon(p)$ and smooth the result nearby the boundary ∂B.) The aim of the construction of rapid flows is to produce flows v which have small C^0 norm and are (t, ϵ)-rapid for a small t and ϵ. This is achieved by the theorem 3.4 which is the main result of the present chapter. We need a definition.

Definition 3.3.

Let $f : W \to [a, b]$ be a Morse function on a compact cobordism, and u, v be f-gradients. We say that u and v are *equivalent*, if $u(x) = \phi(x)v(x)$, where $\phi : W \to \mathbf{R}$ is a C^∞ function, such that $\phi(x) > 0$ for every x and $\phi(x) = 1$ for x in a neighborhood of $S(f)$. △

Theorem 3.4.

Let M be a closed riemannian manifold. Let $C > 0, t > 0$.

There is a Morse function $f : M \to \mathbf{R}$ and an f-gradient v, satisfying Almost Transversality Condition and for every $\epsilon > 0$ there is a (t, ϵ)-rapid f-gradient u with $\|u\| \leqslant C$, equivalent to v.

We shall deduce Theorem 3.4 from Theorem 3.12 , see page 57.

3.1.2. Terminology. The rapid f-gradients which we shall construct will be gradient-like vector fields. (The time which a v- trajectory spends in some domain nearby a critical point of f is much easier computed or estimated for a gradient-like vector field than for a general f-gradient.)

Let n, s be positive integers and $0 \leqslant s \leqslant n$; set

$$\mathcal{Q}_s(x) = -\sum_{i=1}^{s} x_i^2 - \sum_{i=s+1}^{n} x_i^2 \tag{16}$$

$$\mathfrak{n}_s(x) = (-x_1, ..., -x_s, x_{s+1}, ..., x_n) \tag{17}$$

Definition 3.5.

1. Quasi-Morse charts (qM-charts) for Morse functions.

Let p be a critical point of f of index s.

A chart $\Phi : U \to B^n(0, r)$, where U is a neighborhood of p, and $r > 0$ is called *quasi-Morse chart for f around p of radius r* (or simply *qM-chart for f*) if $\Phi(p) = 0$ and there is an extension of Φ to a chart $\widetilde{\Phi} : V \to B^n(0, r')$, where $\overline{U} \subset V$ and $r' > r$, such that

$$(f \circ \widetilde{\Phi}^{-1})(x_1, ..., x_n) = f(p) + \sum_{i=1}^{n} \alpha_i x_i^2$$

with $\alpha_i < 0$ for $i \leqslant s$ and $\alpha_i > 0$ for $i > s$. The domain U is called *quasi-Morse coordinate neighborhood.* Any such extension $\widetilde{\Phi}$ of Φ will be called *quasi-Morse extension of* Φ. We shall often abbreviate "quasi-Morse" to qM. The set $\Phi^{-1}(\mathbf{R}^s \times \{0\})$, resp. $\Phi^{-1}(\{0\} \times \mathbf{R}^{n-s})$, is called *negative disc,* resp. *positive disc.*

2. Quasi-Morse chart systems.

Assume that for every critical point $p \in S(f)$ there is given a quasi-Morse chart $\{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}$. The family $\mathcal{U} = \{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}$ is called *quasi-Morse chart system* , if the family $\{\overline{U_p}\}$ is disjoint and no one of $\overline{U_p}$ intersects ∂W. We denote $\min_p r_p$ by $d(\mathcal{U})$, and $\max_p r_p$ by $D(\mathcal{U})$. If all the r_p are equal to r, we shall say that $\mathcal{U}$ *is of radius* r. The set $\Phi_p^{-1}(B^n(0,\lambda))$, where $\lambda \leqslant r_p$ will be denoted by $U_p(\lambda)$. For $\lambda \leqslant d(\mathcal{U})$ we denote $\cup_{p \in S(f)} U_p(\lambda)$ by $\mathcal{U}(\lambda)$. In the case when W is riemannian, we denote $\max_p \mathcal{G}(U_p, \Phi_p)$ by $\mathcal{G}(\mathcal{U})$. Let $\mathcal{U} = \{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}$, $\mathcal{U}' = \{\Phi'_p : U'_p \to B^n(0, r'_p)\}_{p \in S(f)}$ be two qM-chart-systems for f. We say, that $\mathcal{U}'$ is *a restriction of* $\mathcal{U}$, if for every $p \in S(f)$ we have: $r'_p \leqslant r_p$, $U'_p \subset U_p$, $\Phi'_p = \Phi_p \mid U'_p$.

3. f-Gradients

Given an qM-chart system $\mathcal{U} = \{\Phi_p\}$ we say, that a vector field v on W is an *f-Gradient with respect to* $\mathcal{U}$, if

1) $\forall x \in W \setminus S(f)$ we have $df(v)(x) > 0$;

2) $\forall p \in S(f)$ we have $(\widetilde{\Phi}_p)_*(v)(x) = (-x_1, ..., -x_k, x_{k+1}, ..., x_n) = \mathfrak{n}_k(x)$, where k is the index of p, and $\widetilde{\Phi}_p$ is some qM extension of Φ_p.

Note that a vector field v is a gradient-like vector field for f if and only if it is an f-Gradient with respect to some qM-chart system.

4. $\mathcal{M}_0$-flows, $\mathcal{M}$-flows, $\mathcal{AM}_0$-flows, $\mathcal{AM}$-flows

If $\mathcal{U} = \{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}$ is a qM-chart system for f and v is an f-Gradient of f with respect to to $\mathcal{U}$, we shall say that the triple $\mathcal{V} = (f, v, \mathcal{U})$ is an *$\mathcal{M}_0$-flow.* We say that $\mathcal{V}$ *is of radius* r, if $\mathcal{U}$ is of radius r. We set $d(\mathcal{V}) = d(\mathcal{U})$, $D(\mathcal{V}) = D(\mathcal{U})$.

If W is riemannian, we shall say that the triple $\mathcal{V} = (f, v, \mathcal{U})$ is an *$\mathcal{M}$-flow* if moreover for each chart $\Phi_p : U_p \to B^n(0, r_p)$ from $\mathcal{U}$ the coordinate frame in p is orthonormal with respect to the riemannian metric.

An $\mathcal{M}_0$-flow $(f, v, \mathcal{U})$ (resp. an $\mathcal{M}$-flow) will be called *$\mathcal{AM}_0$-flow* (respectively, $\mathcal{AM}$-flow), if v satisfies the almost transversality condition.

Let $\mathcal{V} = (f, v, \mathcal{U})$ be an $\mathcal{M}$-flow on W, where $\mathcal{U} = \{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}$. We denote $\max_p \mathcal{G}(U_p, \Phi_p)$ by $\mathcal{G}(\mathcal{V})$.

5. Subordinate flows

Let $\mathcal{V}_1 = (f, v_1, \mathcal{U}_1)$ and $\mathcal{V}_2 = (f, v_2, \mathcal{U}_2)$ be two $\mathcal{M}_0$-flows on W (the Morse function is the same for both), where $\mathcal{U}_1 = \{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}$, and $\mathcal{U}_2 = \{\Psi_p : U_p \to B^n(0, r'_p)\}_{p \in S(f)}$ are the corresponding qM-chart systems; We say that $\mathcal{V}_2$ is *subordinate to* $\mathcal{V}_1$, if

1. $\mathcal{U}_2$ is a restriction of $\mathcal{U}_1$
2. $v_2 = \phi \cdot v_1$, where $\phi : W \to \mathbf{R}$ is a strictly positive C^∞ function equal to 1 in a neighborhood of the closure of $\cup_p U_q$.

$\triangle$

(Note that if $\mathcal{V}_2$ is subordinate to $\mathcal{V}_1$ then v_1 and v_2 are equivalent in the sense of Definition 3.3 .)

Remark 3.6.

Here are two observations, which may serve as some justification for having introduced so many definitions.

1. As we have already noted (Remark 2.1.1) each f-gradient is actually a gradient-like vector field. Still we keep the qM-charts in our terminology, since we shall often consider different Morse functions with the same f-gradient (for example, we shall multiply Morse functions by a constant).
2. The condition or orthonormality in the definition of $\mathcal{M}$-flow seems artificial, but it makes easier some constructions. Namely, it implies that the riemannian δ-balls around critical points and the δ-balls with respect to standard charts are very close to each other. The basic invariants of such flows , like $N(\mathcal{V}), T(\mathcal{V})$ (see Definition 3.8) can be defined of course for every $\mathcal{M}_0$-flow. $\triangle$

$\triangle$

Lemma 3.7.

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a quadratic form $f(x) = \sum_{i=1}^n \alpha_i x_i^2$, where $\alpha_i < 0$ for $i \leqslant s$ and $\alpha_i > 0$ for $i > s$. (Note that $\mathfrak{n}_s$ is an f-gradient.)

Let $R > R' > r' > r > 0$. Then there is a Morse function $g : \mathbf{R}^n \to \mathbf{R}$ such that

1. *$g(x) = f(x)$ outside $B^n(0, R')$*
2. *$g(x) = C\mathcal{Q}_s(x)$ in $B^n(0, r)$ where $C > 0$.*
3. *$S(g) = \{0\}$, and $g'(x)\big(v(x)\big) > 0$ for every $x \neq 0$.*

Proof. Choose a C^∞ function $h : \mathbf{R}^n \to [0, 1]$ such that supp $h \subset B^n(0, R')$ and $h(x) = 1$ for $x \in B^n(0, r')$. It is obvious that the function $x \mapsto Ch(x)\mathcal{Q}_s(x) + (1 - h(x))f(x)$ satisfies the conclusion of the lemma if only C is sufficiently small . $\square$

It follows from this lemma that if v is an f-Gradient with respect to a qM-chart system $\mathcal{U} = \{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}$ and γ is a v-trajectory, then for every $p \in S(f)$ and every $\lambda \leqslant r_p$ the set $I = \{t \in \mathbf{R} \mid \gamma(t) \in U_p(\lambda)\}$ is an open interval, empty, or finite, or infinite from one side.

Definition 3.8.

(Quick flows)

Let $\mathcal{V} = (f, v, \mathcal{U})$ be an $\mathcal{M}_0$-flow on W. Let γ be a v-trajectory.

A) The number of sets $U_p = U_p(r_p)$, intersected by γ, will be denoted by $N(\gamma)$. The number $\max_\gamma N(\gamma)$ will be denoted by $N(\mathcal{V})$. The set $\{t \in \mathbf{R} | \gamma(t) \notin \cup_{p \in S(f)} U_p\}$ is a finite union of disjoint closed intervals; the sum of their lengths will be denoted by $T(\gamma)$. The number $\max_\gamma T(\gamma)$ is denoted by $T(\mathcal{V})$.

B) Let $\beta > 0,\ C > 0$. We say, that $\mathcal{V}$ is (C, β)*-quick,* if $\|v\| \leqslant C$ and $T(\mathcal{V}) \leqslant \beta$.

C) Let $\delta \leqslant d(\mathcal{U})$. The number of sets $U_p(\delta)$, intersected by γ, will be denoted by $N(\gamma, \delta)$. The number $\max_\gamma N(\gamma, \delta)$ will be denoted by $N(\mathcal{V}, \delta)$. The set $\{t \in \mathbf{R} | \gamma(t) \notin \mathcal{U}(\delta)\}$ is a finite union of disjoint closed intervals and the sum of their lengths will be denoted by $T(\gamma, \delta)$. The number $\max_\gamma T(\gamma, \delta)$ will be denoted by $T(\mathcal{V}, \delta)$.

D) Let $\beta > 0,\ C > 0, \delta \leqslant d(\mathcal{U})$. We say, that $\mathcal{V}$ is (C, β, δ)*-quick,* if $\|v\| \leqslant C$ and $T(\mathcal{V}, \delta) \leqslant \beta$.

$\triangle$

3.1.3. Two lemmas on standard gradients in $\mathbf{R}^n$. In this subsection n, k are positive integers, and $0 \leqslant k \leqslant n$. We refer to $\mathbf{R}^n$ as to the product $\mathbf{R}^k \times \mathbf{R}^{n-k}$; a point $z \in \mathbf{R}^k$ is therefore written down as (x, y), where $x \in \mathbf{R}^k, y \in \mathbf{R}^{n-k}$. The Euclidean norm in $\mathbf{R}^n$ is denoted by $|\cdot|$.

We have: $\mathfrak{n}_k(x,y) = (-x, y)$; $\mathcal{Q}_k(x,y) = -|x|^2 + |y|^2$. The $\mathfrak{n}_s$-trajectories are of the form (x_0e^{-t}, y_0e^t). using this fact it is not difficult to prove the following two lemmas .

Lemma 3.9.

Let $R > r > 0$ and γ be a $\mathfrak{n}_s$-trajectory. Then the time, which γ spends in $B^n(0,R) \setminus B^n(0,r)$ is not more than $\ln\big((\frac{R}{r})^2 + \sqrt{(\frac{R}{r})^4 - 1}\,\big)$ *and the length of the corresponding part of γ is not more than $2R$.*

Proof. Let $\gamma(t) = (x(t), y(t))$ be a $\mathfrak{n}_s$-trajectory. We can assume that both x and y are never equal to 0. (Indeed, if $x(t_0) = 0$, then $x(t) = 0$ for all t and the time, respectively length, cited in the statement of the lemma, are equal to $\ln\frac{R}{r}$, respectively, to $R - r$.) Set $\mathcal{L}(x) = \ln(x + \sqrt{x^2 - 1})$. There is a unique t_0, such that $|x(t_0)| = |y(t_0)|$. This point is the minimum of $|\gamma(t)|$. Reparametrize γ so as to have $t_0 = 0$ and denote $|x(0)| = |y(0)|$ by α. Assuming $\sqrt{2}\alpha \leqslant R$, we calculate easily that γ quits $B(0,R)$ at the moment $\frac{1}{2}\mathcal{L}(\frac{R^2}{2\alpha^2})$. If $r < \sqrt{2}\alpha$, we are over. If not, the time spent between two spheres equals to $\mathcal{L}\big(\frac{R^2}{2\alpha^2}\big) - \mathcal{L}\big(\frac{r^2}{2\alpha^2}\big)$ and it suffices to prove that $\mathcal{L}(x) - \mathcal{L}(y) \leqslant \mathcal{L}(\frac{x}{y})$ for $x > y > 1$, but this is obvious (take the derivative in x). To obtain the estimate of the length, just note that the length can not be more, than the length of the intersection of the 1-dimensional hyperbole $H = \{(x,y) \in \mathbf{R}^2 \mid xy = \alpha\}$ with $B^2(0,R)$. $\square$

Lemma 3.10.

Let $r > 0$ and γ be a $\mathfrak{n}_k$-trajectory. Then the time, which γ spends in the set $\mathcal{Q}_k^{-1}([-r^2, r^2]) \setminus B^n(0,r)$ is not more than 2.

Proof. The function $\mathcal{Q}_k'(x)(v(x))$ is bounded below in the domain by r^2 and the variation of $\mathcal{Q}_k$ in this domain is not more than 2. $\square$

Corollary 3.11.

Let $\mathcal{V} = (f, v, \mathcal{U})$ be an $\mathcal{M}$-flow on a cobordism. Assume, that $\mathcal{V}$ is (C, β, α)-quick. Then $\mathcal{V}$ is $(C, \beta + 8N(\mathcal{V}), \alpha/2)$-quick.

Proof. By Lemma 3.9 the time, which a v-trajectory can spend in $\mathcal{U}(\alpha) \setminus \mathcal{U}(\alpha/2)$ is not more than $8N(\mathcal{V})$. $\square$

3.1.4. Theorems on the existence of quick flows. In this subsection W is a riemannian cobordism of dimension n ($\partial_0 W$ and $\partial_1 W$ can both be empty).

Theorem 3.12.

Let $C > 0, \beta > 0, A > 1$. There is an $\mathcal{AM}$-flow $\mathcal{V} = (f, v, \mathcal{U})$ on W of radius r, and for every $\mu \in]0, r[$ there is an $\mathcal{AM}$-flow $\mathcal{W} = (f, w, \mathcal{U}')$, subordinate to $\mathcal{V}$, and such that

(1) *$\mathcal{W}$ is of radius μ and is (C, β)-quick;*
(2) *$N(\mathcal{W}) \leqslant n2^n$;*

(3) $\mathcal{G}(\mathcal{W}) \leqslant A$.

Theorem 3.13.

Let $B > 0, \mu_0 > 0$. Then there is an $\mathcal{AM}$-flow $\mathcal{V} = (f, v, \mathcal{U})$ on W with the following properties:

(1) $\mathcal{G}(\mathcal{V}) \leqslant 2$ *and* $D(\mathcal{V}) < \mu_0$;

(2) $N(\mathcal{V}) \leqslant n2^n$;

(3) $\mathcal{V}$ *is $(B, S(n))$-quick, where* $S(n) = 30 + n2^{n+5}$.

The scheme of the proof of 3.12 and 3.13 is as follows. We shall use the notation 3.12 (n) which means "Theorem 3.12 for manifolds of dimension n. The proof of 3.13 $\Rightarrow$ 3.12 is given below. The proof of 3.12 (n) $\Rightarrow$ 3.13 (n+1) is done in §2 and §3 with the help of (S)-construction. The same argument proves also 3.13 for 1-dimensional manifolds.

Proof of Theorem 3.4 from 3.12

Let $(f, v, \mathcal{U})$ be the $\mathcal{AM}$-flow of radius r, satisfying the conclusion of 3.12 with respect to the manifold M, the numbers C, t and $A = 2$. Let $\varkappa > 0$ be any number, so small that

1. $\varkappa < r$ and $\varkappa < \epsilon$
2. v is $\varkappa$-ordered
3. for every $p \in S(f)$ the disc $D_\epsilon(p)$ is in U_p.

Let $\mathcal{W} = (f, w, \mathcal{U}')$ be the $\mathcal{M}$-flow, subordinate to $\mathcal{V}$ and satisfying 3.12 with respect to the number $\varkappa/2$. I claim that it satisfies our conclusions. Indeed, $\|w\| \leqslant C$ by definition. To show that the flow generated by u is (t, ϵ)-rapid, it suffices to check (14, 15) with $\varkappa$ instead of ϵ. The set $M \setminus B_\varkappa(\mathrm{ind}{\leqslant}n-s-1\,; -v)$ is $(-v)$-invariant and $(-w)$-invariant, so we must only check that for every $M \setminus B_\varkappa(\mathrm{ind}{\leqslant}n-s-1\,; -v)$ there is $p \in S(f)$ and $t_0 \leqslant t$ such that $\gamma(x, t_0; -w) \in B_\varkappa(p)$. This is possible, since $U_p(\varkappa/2) \subset B_\varkappa(p)$ and $\mathcal{W}$ is $(C, t, \varkappa/2)$-quick. The proof of 15 is similar. □

Before proceeding to the proof of 3.13 $\Rightarrow$3.12 we explain informally the main idea of the proof. The principal difference between the statements of these two theorems is that 3.13 asserts the existence of *one* flow with prescribed quickness properties, and 3.12 asserts the existence of a whole family of such flows, subordinate to a given one. All these flows must have the prescribed quickness properties, and in this family there are flows of arbitrary small radius. Assume for example that we have a flow $\mathcal{V} = (f, v, \mathcal{U})$ of radius r, satisfying the conclusions of 3.13 . The problem is of course that any v-trajectory which passes very close to a critical point p spends a lot of time nearby p, in particular in the spherical slice $U_p(r) \setminus U_p(\mu)$, if μ is small. But if we seek the flow of radius μ with good quickness properties, we can multiply v in $U_p(r) \setminus U_p(\mu)$ by a large positive real function, so that the (Euclidean) norm of the resulting field becomes constant in this domain, which will diminish dramatically the time spent in $U_p(r) \setminus U_p(\mu)$.

These considerations are formalized in the following proposition. In this proposition we present a construction, producing from a given $\mathcal{M}$-flow $\mathcal{V}$ another one $\mathcal{V}'$, for which $D(\mathcal{V}')$ is as small as we like, but $T(\mathcal{V}')$ is still under control.

Proposition 3.14.

Let $\mathcal{V} = (f, v, \mathcal{U})$ be an $\mathcal{M}$-flow on W with $\mathcal{G}(\mathcal{V}) \leqslant 2, \|v\| \leqslant B$. Write its atlas as $\mathcal{U} = \{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}$. Let $C \geqslant B$. Assume that to each $p \in S(f)$ a number $\mu_p \in]0, r_p[$ is assigned. Then there is an $\mathcal{M}$-flow $\mathcal{V}' = (f, w, \mathcal{U}')$, subordinate *to $\mathcal{U}$, with $\mathcal{U}' = \{\Phi_p' : U_p' \to B^n(0,\mu_p)\}_{p\in S(f)}$ and $\Phi_p' = \Phi_p \mid U_p'$ and satisfying: $\|w\| \leqslant C$ and*

$$C \cdot T(\mathcal{V}') \leqslant B \cdot T(\mathcal{V}) + 5N(\mathcal{V}) \cdot D(\mathcal{V}) \tag{18}$$

Proof. Outside $\cup_p U_p$ we set $w = \frac{C}{B}v$. We shall construct w in each U_p separately with the help of the standard chart. Namely, $(\Phi_p)_*(v)$ is the standard vector field $\mathfrak{n}_k$ on $\mathbf{R}^k \times \mathbf{R}^{n-k}$, where $k = \operatorname{ind} p$. Let $\lambda : \mathbf{R}_+ \to \mathbf{R}$ be a C^∞ function such that

$$\begin{cases} \lambda(t) = 1 \quad \text{for} \quad 0 \leqslant t \leqslant \mu_p \\ \min(1, \dfrac{C}{2t}) \leqslant \lambda(t) \leqslant \max(1, \dfrac{C}{2t}) \text{ for } t \in [\mu_p, \mu_p + \delta] \\ \lambda(t) = \dfrac{C}{2t} \quad \text{for} \quad \mu_p + \delta \leqslant t \leqslant r_p - \delta \\ \min(\dfrac{C}{B}, \dfrac{C}{2t}) \leqslant \lambda(t) \leqslant \max(\dfrac{C}{B}, \dfrac{C}{2t}) \text{ for } t \in [r_p - \delta, r_p] \\ \lambda(t) = \dfrac{C}{B} \quad \text{for} \quad r_p \leqslant t \end{cases}$$

(see the figure on the page 97). The number $\delta \in]0, \frac{r_p - \mu_p}{2}[$ will be chosen later.

Define a vector field w_0 in $\mathbf{R}^n$ by $w_0(x) = \lambda(|x|)\mathfrak{n}_k(x)$. Note that $(\Phi_p^{-1})_*(w_0)$ can be glued to w previously defined on $M \setminus \cup_p U_p$ (with the help of a standard extension); performing this construction for each p we obtain an $\mathcal{M}$-flow $\mathcal{V}' = (f, w, \mathcal{U}')$, subordinate to $\mathcal{V}$.

The Euclidean norm of $w_0(x)$ equals $\lambda(|x|)|x|$ and using $\mathcal{G}(\mathcal{V}) \leqslant 2$ it is easy to check that $\|w\| \leqslant C$. To obtain the estimate for $T(\mathcal{V})$ note that the time which a w-trajectory spends outside $\cup_p U_p$ is not more than $T(\mathcal{V})\frac{B}{C}$. Therefore to obtain (18) it suffices to prove that for each $p \in S(f)$ the time which a w-trajectory can spend in $U_p \setminus U_p(\mu_p)$ is not more than $\frac{5}{C}r_p$. This is a consequence of two following observations (where γ stands for a w_0-trajectory).

1. The time spent by γ in each of the annuli $\mathcal{A}_1 = \{\mu_p \leqslant |x| \leqslant \mu_p + \delta\}$, $\mathcal{A}_2 = \{r_p - \delta \leqslant |x| \leqslant r_p\}$ goes to zero when $\delta \to 0$ (it follows from 3.9)
2. The Euclidean length of the part of γ in the annulus $\mathcal{A}_3 = \{\mu_p + \delta \leqslant |x| \leqslant r_p - \delta\}$ is not more than $2r_p$. The Euclidean norm of γ' in $\mathcal{A}_3$ equals $\frac{C}{2}$. $\square$

Proof of Theorem 3.12 from Theorem 3.13 . We are given C, β, A. Choose $B > 0, \mu_0 > 0$ as to satisfy

$$B \leqslant C, \quad B \leqslant C\beta\frac{1}{3S(n)}, \quad \mu_0 \leqslant C\beta\frac{1}{15n2^n}$$

and let $\mathcal{V}_0 = (f, v, \mathcal{U}_0)$ be an $\mathcal{M}$-flow satisfying the conclusions of 3.13 with respect to B, μ_0. Write $\mathcal{U}_0 = \{\Psi_p : U_p \to B^n(0, r_p)\}_{p\in S(f)}$.

Choose $r < d(\mathcal{V}_0)$ so small that $\mathcal{G}(U_p(r), \Psi_p) < A$ for every $p \in S(f)$. (This is possible since the coordinate frame of Ψ_p at p is orthonormal with respect to the riemannian metric of W). For each chart $\{\Psi_p : U_p \to B^n(0, r_p)\}_{p\in S(f)}$ of $\mathcal{U}_0$ consider its restriction $\Phi_p = \Psi_p|U_p(r)$, form the corresponding qM-chart system $\mathcal{U} = \{\Phi_p\}$ and denote by $\mathcal{V}$ the $\mathcal{M}$-flow $\mathcal{V} = (f, v, \mathcal{U})$.

I claim that $\mathcal{V}$ satisfy the conclusions of 3.13 . Indeed, let $0 < \mu < r$. Apply the proposition 18 to the flow $\mathcal{V}_0$ and to the set of numbers $\mu_p = \mu$ for each $p \in S(f)$. We obtain an $\mathcal{M}$-flow $\mathcal{W} = (f, w, \mathcal{U}')$, subordinate to $\mathcal{V}_0$, where $\mathcal{U} = \{\Phi_p'\}_{p\in S(f)}$ and

$\Phi'_p = \Phi_p | U_p(\mu)$. This $\mathcal{M}$-flow is (C,β)-quick, since $\|w\| \leqslant C$ and (by (9)) $T(\mathcal{V}') \leqslant \left(\frac{\beta T(\mathcal{V}_0)}{3S(n)} + \frac{5}{C} n 2^n \mu_0\right) \leqslant \frac{2}{3}\beta$. The properties (2) and (3) go by the construction. $\square$

3.2. S-Construction

The aim of this section is to present the S-construction. Given a Morse function without critical points on a cobordism W, the S-construction produces another Morse function on W. This new function behaves like "fold" (see the figure on the page 98 and the informal introduction in the beginning of the present chapter).

The corresponding theorem is stated below (theorem 3.15), and the proof occupy the rest of the section. In Subsection 3.2.1 we make some auxiliary constructions and give some definitions. The output of the construction, namely, Morse function F and its gradient u are introduced in Subsection 3.2.2. The proof of the properties of F and u, listed in the theorem 3.15 , occupies the end of Subsection 3.2.2 and Subsection 3.2.3.

Proceeding to the statement of the theorem 3.15 , let W be a riemannian cobordism of dimension $n+1$, where $n \geqslant 0$. In the present section we denote by $|\cdot|$ the norm induced by the metric on the tangent spaces, and by $\|\cdot\|$ the corresponding C^0 norm in the space of vector fields.

Theorem 3.15.

Let $g : W \to [a,b]$ be a Morse function without critical points, $g^{-1}(a) = \partial_0 W, g^{-1}(b) = \partial_1 W$. Let $C > 0$ and let w be a g-gradient, such that $\|w\| \leqslant C$. Denote $g^{-1}(\frac{2a+b}{3})$ by $V_{1/3}$, $g^{-1}(\frac{2b+a}{3})$ by $V_{2/3}$, $g^{-1}(\frac{a+b}{2})$ by $V_{1/2}$. Denote $g^{-1}([a, \frac{2a+b}{3}])$ by W_0, $g^{-1}([\frac{2b+a}{3}, b])$ by W_1, $g^{-1}([\frac{2a+b}{3}, \frac{2b+a}{3}])$ by $W_{\frac{1}{2}}$. Denote by $\mathrm{grad}(g)$ the riemannian gradient of g and by $\mathrm{grd}(g)$ the vector field $\mathrm{grad}(g)/|\mathrm{grad}(g)|$. Denote by T the maximal length of the interval of definition of a $\mathrm{grd}(g)$-trajectory. Then there is $\nu_0 > 0$ such that:

For every s-submanifold $\mathbb{X}$ of $\partial_0 W$, s-submanifold $\mathbb{Y}$ of $\partial_1 W$, every $\mathcal{AM}$-flow $\mathcal{V}_1 = (F_1, u_1, \mathcal{U}_1)$ on $V_{1/3}$ and $\mathcal{AM}$-flow $\mathcal{V}_2 = (F_2, u_2, \mathcal{U}_2)$ on $V_{2/3}$, and every $\mu \leqslant \min(\nu_0, d(\mathcal{V}_1), d(\mathcal{V}_2))$ there is an $\mathcal{AM}$-flow $\mathcal{V} = (F, u, \mathcal{U})$ on W, having the following properties:

1. *$F : W \to [a,b], S(F) = S(F_1) \cup S(F_2)$; in a neighborhood of ∂W we have: $F = g, u = w$.*
2. *$\mathcal{V}$ is of radius μ; $\mathcal{G}(\mathcal{V}) \leqslant \frac{3}{2}\max(\mathcal{G}(\mathcal{V}_1), \mathcal{G}(\mathcal{V}_2))$.*
3. *$V_{\frac{1}{3}}, V_{\frac{2}{3}}, W_{\frac{1}{2}}$ are $(\pm u)$-invariant, W_0 is u-invariant and weakly $(-u)$-invariant, W_1 is $(-u)$-invariant and weakly u-invariant.*
4. *Assume that $\mathcal{V}_1$ is $(C_1, \beta_1, \mu/2)$-quick and $\mathcal{V}_2$ is $(C_2, \beta_2, \mu/2)$-quick. Then $\mathcal{V}$ is*
   $(3/2(C + C_1 + C_2),\ \beta_1 + \beta_2 + 5 + 4\frac{T}{C}\ , \mu)$-quick.
5. *Let γ be an u-trajectory. If $\mathrm{Im}\,\gamma \subset W_{\frac{1}{2}}$, then $N(\gamma,\mu) \leqslant N(\mathcal{V}_1,\mu) + N(\mathcal{V}_2,\mu)$. If $\mathrm{Im}\,\gamma \subset W_0$, then $N(\gamma,\mu) \leqslant N(\mathcal{V}_1,\mu)$. If $\mathrm{Im}\,\gamma \subset W_1$, then $N(\gamma,\mu) \leqslant N(\mathcal{V}_2,\mu)$.*
6. *$\mathbb{D}_a(u) \pitchfork \mathbb{X}$; $\mathbb{D}_b(-u) \pitchfork \mathbb{Y}$.*

Comments. To visualize the constructions we suggest three figures (see the page 98) which we have already mentioned. What is drawn exactly is the case $\partial_0 W = \partial_1 W = S^1$. On the figure A we see the cobordism $W \approx S^1 \times [0,1]$, embedded in $\mathbf{R}^3$, the function g being the vertical projection onto $[0,1]$. The figure

B presents the manifold W', diffeomorphic to W but embedded differently into $\mathbf{R}^3$. The projection π onto the vertical, restricted to W', has the same behavior as the functions $\chi \circ g, \chi \circ f$ (see Subsection 3.2.1). The function $\pi \mid W'$ has two critical levels: $\frac{2}{3}$ and $\frac{1}{3}$, and the level surfaces correspond to $V_{\frac{1}{3}}$ and to $V_{\frac{2}{3}}$. Finally, on the figure C we find a further perturbed embedding W'' of $S^1 \times [0,1]$ to $\mathbf{R}^3$, and our resulting function F is the vertical projection, restricted to W''.

The submanifolds $V_{\frac{1}{3}}, V_{\frac{2}{3}}$ correspond to the critical level surfaces of $\widetilde{f}_i$ on the figure B. The flows $\mathcal{V}_1$ and $\mathcal{V}_2$ correspond to the perturbations of $\widetilde{f}_i$ which are necessary to turn $\widetilde{f}_i$ to a Morse function. $\triangle$

In the proof we shall assume that $a = 0, b = 1$, since the general case is easily reduced to this one by an affine transformation of $\mathbf{R}$.

3.2.1. Auxiliary constructions and the choice of ν_0.

1. Function f and its gradient v

Lemma 3.16.
There is a Morse function $f : W \to [0,1]$ without critical points and an f-gradient v, such that:

1. $\|v\| \leqslant C$;
2. *in a neighborhood of ∂W we have: $f = g$ and $v = w$;*
3. $f^{-1}(1/3) = V_{1/3},\ f^{-1}(2/3) = V_{2/3},\ f^{-1}(1/2) = V_{1/2}$;
4. *for x in a neighborhood of $V_{1/3} \cup V_{1/2} \cup V_{2/3}$ we have: $|v(x)| = C$ and $df(v)(x) = C$;*
5. *for $\lambda = 1/3,\ 1/2,\ 2/3$ and $x \in g^{-1}(\lambda)$ we have: $v(x) \perp g^{-1}(\lambda)$;*
6. *The maximal length of the domain of a v-trajectory is not more than $2T/C$.*

Proof. Let U be an open neighborhood of ∂W, such that $U \cap (V_{1/3} \cup V_{2/3} \cup V_{1/2}) = \emptyset$ and let $h : W \to [0,1]$ be a C^∞ function such that supp $h \subset U$ and for x in a neighborhood of ∂W we have $h(x) = 1$. Set $v(x) = h(x)w(x) + (1-h(x))C\mathrm{grd}(g)$. It is obvious that v is a g-gradient, satisfying (1) and (5). We have also $v(x) = w(x)$ nearby ∂W, as well as $|v(x)| = C$ nearby $V_{1/3} \cup V_{2/3} \cup V_{1/2}$. (6) holds also, if only U was chosen sufficiently small. To construct f define it first of all in a neighborhood of $V_{1/3} \cup V_{2/3} \cup V_{1/2}$ directly by (4), and in a neighborhood of ∂W as equal to g. Now extend f to each of the cobordisms $W_0, W_1, g^{-1}([\frac{1}{3}, \frac{1}{2}]), g^{-1}([\frac{1}{2}, \frac{2}{3}])$ using the usual procedure of gluing two functions having the same gradient (this procedure is described in details in [21, Corollary 8.14]). $\square$

For $\lambda \in [0,1]$ we denote $f^{-1}(\lambda)$ by V_λ. Fix some $\epsilon \in]0, \frac{1}{12}[$. For $\nu > 0$ sufficiently small the map $(x,\tau) \mapsto \gamma(x,\tau; v/C)$ is defined on $V_i \times [-\nu, \nu]$, where $i = 1/3, 1/2, 2/3$, on $V_0 \times [0,\nu]$ and on $V_1 \times [-\nu, 0]$. The corresponding embeddings will be denoted by

$$\Psi_0(\nu) : V_0 \times [0,\nu] \to W, \tag{19}$$

$$\Psi_1(\nu) : V_1 \times [-\nu, 0] \to W, \tag{20}$$

$$\Psi_i(\nu) : V_i \times [-\nu, \nu] \to W, \qquad \text{for} \quad i = \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \tag{21}$$

The image of $\Psi_i(\nu)$ will be denoted by $\mathrm{Tb}_i(\nu)$ (for $i = 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$).

2. Choice of ν_0.

Let ν_0 satisfy the following condition $(\mathfrak{R})$:

$$(\mathfrak{R})\begin{cases}(1) & 2\nu_0 < C\\ (2) & \text{For } i = 0, 1/3, 1/2, 2/3, 1 \text{ we have:}\\ & (2A)\quad \mathrm{Tb}_i(2\nu_0) \subset]i-\epsilon, i+\epsilon[\\ & (2B)\quad \text{The riemannian metric induced by } \Psi_i(2\nu_0) \text{ from } W\\ & \text{and the product metric on the domain of } \Psi_i(2\nu_0) \text{ are}\\ & \dfrac{3}{2}- \text{ equivalent.}\\ (3) & \text{For } i = \dfrac{1}{3}, \dfrac{1}{2}, \dfrac{2}{3} \text{ we have: } |v(x)| = df(v(x)) = C \text{ for } x \text{ in a}\\ & \text{neighborhood of } \mathrm{Tb}_i(\nu_0)\end{cases}$$

We shall prove that ν_0 satisfy the conclusions of our theorem. So let $\mathbb{X}, \mathbb{Y}, \mathcal{V}_1 = (F_1, u_1, \mathcal{U}_1), \mathcal{V}_2 = (F_2, u_2, \mathcal{U}_2)$ be as in the statement of the theorem, where $\mathcal{U}_1 = \{\Phi_{1p} : U_{1p} \to B^n(0, r_{1p})\}_{p\in S(F_1)}$ and $\mathcal{U}_2 = \{\Phi_{2q} : U_{2q} \to B^n(0, r_{2q})\}_{q\in S(F_2)}$. Let $\mu \leqslant \min(\nu_0, d(\mathcal{V}_1), d(\mathcal{V}_2))$. We shall denote $\Psi_i(\mu)$ by Ψ_i. Using $(\mathfrak{R})$ it is not difficult to check the following property:

$$\text{(22)}\qquad \text{for} \quad |\lambda| \leqslant 2\mu \quad \text{and} \quad i = \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \text{ we have}$$
$$f^{-1}(i+\lambda) = \Psi_i(V_i \times \{\lambda\})$$

3. Auxiliary functions, vector fields and s-submanifolds

For $\lambda \in [0, 1/3]$ we denote by $\mathbb{L}_\lambda(u_1)$ the s-submanifold $v^{\leadsto}_{[1/3,\lambda]}(\mathbb{D}(u_1))$ of V_λ. For $\lambda \in [1/3, 1]$ we denote by $\mathbb{L}_\lambda(u_1)$ the s-submanifold $(-v)^{\leadsto}_{[1/3,\lambda]}\mathbb{D}(u_1)$ of V_λ. For $\lambda \in [2/3, 1]$ we denote by $\mathbb{L}_\lambda(-u_2)$ the s-submanifold $(-v)^{\leadsto}_{[2/3,\lambda]}(\mathbb{D}(-u_2))$ and for $\lambda \in [0, 2/3]$ we denote by $\mathbb{L}_\lambda(-u_2)$ the s-submanifold $v^{\leadsto}_{[2/3,\lambda]}(\mathbb{D}(-u_2))$. Note that since v have no zeros, $v^{\leadsto}_{[\alpha,\beta]}$ is a diffeomorphism of V_α onto V_β for any $\beta < \alpha$. Note also that $\mathbb{L}_{2/3}(-u_2)$ equals to $\mathbb{D}(-u_2)$ and $\mathbb{L}_{1/3}(u_1)$ equals to $\mathbb{D}(u_1)$.

Let $\chi : [0, 1] \to [0, 1]$ be a C^∞ function, with the following properties:

$$\begin{gathered}\chi(x) = x \quad \text{for} \quad x \in [0, \epsilon] \cup [1-\epsilon, 1];\\ \chi(x) = 1 - x \quad \text{for} \quad x \in [1/2-\epsilon, 1/2+\epsilon];\\ \chi(x) = 2/3 - (x-1/3)^2 \text{ for } x \in [1/3-\epsilon, 1/3+\epsilon];\\ \chi'(x) > 0 \text{ for } x \in [0, 1/3[\, \cup\,]2/3, 1];\\ \chi(x) = 1/3 + (x-2/3)^2 \text{ for } x \in [2/3-\epsilon, 2/3+\epsilon];\\ \chi'(x) < 0 \text{ for } x \in]1/3, 2/3[.\end{gathered}$$

(see the graph of χ on the page 99).

Denote $\chi \circ f$ by ψ. The following property follows immediately from $(\mathfrak{R})$:

$$(23)\quad (\psi \circ \Psi_{\frac{1}{3}})(x,\tau) = 2/3 - \tau^2,$$
$$(\psi \circ \Psi_{\frac{2}{3}})(x,\tau) = 1/3 + \tau^2,$$
$$(\psi \circ \Psi_{\frac{1}{2}})(x,\tau) = 1 - \tau$$

Let $B : \mathbf{R} \to [0,1]$ be a C^∞ function such that $B(t) = 0 \quad \text{for} \quad |t| \geqslant 5\mu/3 \quad \text{and} \quad B(t) = 1 \quad \text{for} \quad |t| \leqslant 4\mu/3$. Let $B_1 : \mathbf{R} \to \mathbf{R}^+$ be a C^∞ function such that supp $B_1 \subset]\mu, 2\mu[$ and that $\int_0^\infty B_1(t)dt = C$.

Let z_0 be a C^∞ vector field on V_0 such that $\Phi(z_0,1)(\mathbb{L}_0(u_1)) \pitchfork \mathbb{X}$. Let z_1 be a C^∞ vector field on V_1 such that $\Phi(z_1,1)(\mathbb{L}_1(-u_2)) \pitchfork \mathbb{Y}$. Let $z_{1/2}$ be a C^∞ vector field on $V_{1/2}$ such that $\Phi(z_{1/2},1)(\mathbb{L}_{1/2}(u_1)) \pitchfork \mathbb{L}_{1/2}(-u_2)$. We shall assume that z_i (where $i = 0, 1/2, 1$) are chosen so small, that $\sup_\tau |B_1(\tau)| \cdot \|z_i\| < C/9$.

3.2.2. Morse function F and its gradient u.

1. Morse function F.

Let $a_1, a_2 > 0$. Set: $F(y) = \psi(y) \quad \text{for} \quad y \in W \setminus (\mathrm{Tb}_{\frac{1}{3}}(2\mu) \cup \mathrm{Tb}_{\frac{2}{3}}(2\mu))$;

$(F \circ \Psi_{\frac{1}{3}})(x,\tau) = a_1 B(\tau) F_1(x) + 2/3 - \tau^2 \quad \text{for} \quad (x,\tau) \in V_{1/3} \times [-2\mu, 2\mu]$;

$(F \circ \Psi_{\frac{2}{3}})(x,\tau) = a_2 B(\tau) F_2(x) + 1/3 + \tau^2 \quad \text{for} \quad (x,\tau) \in V_{2/3} \times [-2\mu, 2\mu]$

It follows from (3.2) that these formulas define correctly a smooth function $F : W \to \mathbf{R}$, which equals to f nearby ∂W. To find critical points of F note that $S(F)$ is contained obviously in $\mathrm{Tb}_{\frac{1}{3}}(2\mu) \cup \mathrm{Tb}_{\frac{2}{3}}(2\mu)$. For $(x,\tau) \in V_{1/3} \times [-2\mu, 2\mu]$ we have

$$d(F \circ \Psi_1)(x,\tau) = (a_1 B'(\tau) F_1(x) - 2\tau)d\tau + a_1 B(\tau) dF_1(x).$$

For a_1 small enough this can vanish only for $\tau = 0$. We conclude therefore (applying the same reasoning to $\mathrm{Tb}_{\frac{2}{3}}(2\mu)$), that $S(F) = S(F_1) \cup S(F_2)$ if only a_i are small enough, and we make this assumption from now on. To prove that F is a Morse function we shall explicit the standard charts for F. Let $p \in S(F_1)$ and write $(F_1 \circ \Phi_{1p}^{-1})(x) = F_1(p) + \sum_i \alpha_i x_i^2$. Consider the chart $\Phi_{1p} \times \mathrm{id} : U_{1p} \times] - \mu, \mu[\to B^n(0, r_{1p}) \times] - \mu, \mu[$ of the manifold $V_{1/3} \times] - \mu, \mu[$ around the point $(p, 0)$. We have

$$F \circ \Psi_{\frac{1}{3}} \circ (\Phi_{1p} \times \mathrm{id})^{-1}(x,\tau) = a_1 F_1(p) + 2/3 + a_1 \sum_i \alpha_i x_i^2 - \tau^2$$

therefore the chart

$$\big((\Phi_{1p} \times \mathrm{id}) \mid (\Phi_{1p} \times \mathrm{id})^{-1}(B^{n+1}(0,\mu)\big) \circ (\Psi_{\frac{1}{3}} \mid \ \mathrm{Tb}_{\frac{1}{3}}(\mu))^{-1}$$

is a standard chart of radius μ for F at $p \in S(F_1)$. These charts together with the similar ones for $q \in S(F_2)$ give an F-chart-system of radius μ. We shall denote this system by $\mathcal{U}$. Note that $\mathrm{ind}_F p = \mathrm{ind}_{F_1} p + 1$ for $p \in S(F_1)$ and that $\mathrm{ind}_F q = \mathrm{ind}_{F_2} q$ for $q \in S(F_2)$. Note also that if a_1 and a_2 are chosen sufficiently small, then $F(\mathrm{Tb}_{\frac{1}{3}}(2\mu)) \subset [2/3 - \epsilon, 2/3 + \epsilon]$, and $F(\mathrm{Tb}_{\frac{2}{3}}(2\mu)) \subset [1/3 - \epsilon, 1/3 + \epsilon]$. In this case also $F^{-1}(1/2) = V_{1/2} \cup V_\theta \cup V_{\theta'}$, where $\epsilon < \theta < 1/3 - \epsilon, \quad 2/3 + \epsilon < \theta' < 1 - \epsilon$.

2. F-gradient u

Set $\mathrm{Tub} = \mathrm{Tb}_0(2\mu) \cup \mathrm{Tb}_{\frac{1}{3}}(2\mu) \cup \mathrm{Tb}_{\frac{1}{2}}(2\mu) \cup \mathrm{Tb}_{\frac{2}{3}}(2\mu) \cup \mathrm{Tb}_1(2\mu)$

Define a vector field u on W as follows:

$$u(y) = v(y) \text{ for } y \in (W_0 \setminus \mathrm{Tub}) \cup (W_2 \setminus \mathrm{Tub})$$

$$u(y) = -v(y) \text{ for } y \in (W_{\frac{1}{2}} \setminus \mathrm{Tub})$$

$$\big((\Psi_0^{-1})_*(u)\big)(x,\tau) = (-B_1(\tau)z_0(x),\ C);$$

$$\big((\Psi_1^{-1})_*(u)\big)(x,\tau) = (B_1(-\tau)z_1(x),\ C);$$

$$\big((\Psi_{\frac{1}{3}}^{-1})_*(u)\big)(x,\tau) = \big(B(\tau)u_1(x),\ -B(\tau)\tau - (1-B(\tau))\cdot C\cdot \mathrm{sgn}\tau\big);$$

$$\big((\Psi_{\frac{1}{2}}^{-1})_*(u)\big)(x,\tau) = (-B_1(-\tau)z_{1/2}(x),\ -C);$$

$$\big((\Psi_{\frac{2}{3}}^{-1})_*(u)\big)(x,\tau) = \big(B(\tau)u_2(x),\ B(\tau)\tau + (1-B(\tau))\cdot C\cdot \mathrm{sgn}\tau\big).$$

An easy computation using the definition of F and u shows that these formulas define correctly a C^∞ vector field on W, and that $(F,u,\mathcal{U})$ is an $\mathcal{M}$-flow on W of radius μ, if only $z_0, z_1, z_{1/2}$ are small enough (which assumption we make from now on). It is also easy to see that u is an f-gradient in $W_0 \setminus V_{1/3}$ and $W_1 \setminus V_{2/3}$; $(-u)$ is an f-gradient in $W_{\frac{1}{2}} \setminus (V_{1/3} \cup V_{2/3})$.

We claim that $\mathcal{V} = (F,u,\mathcal{U})$ satisfies all the conclusions of 3.1. (1) and (2) follow immediately from the construction. To prove (3) note that $V_{1/3}$ is a closed submanifold of W° and $u \mid V_{1/3}$ is tangent to $V_{1/3}$; therefore $V_{1/3}$ is $(\pm u)$-invariant (same for $V_{2/3}$). To prove that $W_{\frac{1}{2}}$ is u-invariant, let $x \in W^\circ_{\frac{1}{2}}$. The trajectory $\gamma(x,\cdot;u)$ cannot intersect $V_{\frac{1}{3}}$ or $V_{\frac{2}{3}}$, since they are $(\pm u)$-invariant. Therefore $\gamma(x,\cdot;u)$ has to stay in $W^\circ_{\frac{1}{2}}$ forever.

3.2.3. Properties of F and u.

1. Estimate of the quickness of $\mathcal{V}$

To obtain the estimate of $\|u\|$ note that the inequality $|u(x)| \leqslant 3/2(C + C_1 + C_2)$ is to be checked only for $x \in \cup_i \mathrm{Tb}_i(2\mu)$, where it is a matter of a simple computation; we leave it to the reader. To estimate the time, which an u-trajectory spends outside $\mathcal{U}(\mu)$, note first that for a trajectory, starting at a point of $V_{1/3}$ (resp. $V_{2/3}$), this time is not more than β_1 (resp. β_2), since it is actually a u_1- (resp. u_2)-trajectory. Now let γ be an u-trajectory, passing by a point of $W^\circ_{\frac{1}{2}}$. In the end of the preceding subsection we proved, that γ stays in $W^\circ_{\frac{1}{2}}$ forever. Since u is a $(-f)$-gradient in $W^\circ_{\frac{1}{2}}$, the function $t \mapsto f(\gamma(t))$ is strictly decreasing and $\lim_{t\to-\infty} f(\gamma(t)) = 2/3$, $\lim_{t\to\infty} f(\gamma(t)) = 1/3$. This implies $\lim_{t\to-\infty}\gamma(t) = q \in S(F_2)$ and $\lim_{t\to\infty}\gamma(t) = p \in S(F_1)$. We shall estimate the time which γ spends outside $\mathcal{U}(\mu)$ between the various level surfaces of f.

1) $f(\gamma(t)) \in [2/3-\mu/2, 2/3]$. By (22) and (23) this condition is equivalent to: $\gamma(t) \in \Psi_{\frac{2}{3}}(V_{2/3} \times [-\mu/2, 0[)$. The curve $\Psi_{\frac{2}{3}}^{-1}(\gamma(t))$ is a product of an u_2-trajectory and the curve $\tau \mapsto \alpha e^\tau$ with $\alpha < 0$. Since $\Psi_{\frac{2}{3}}^{-1}(\mathcal{U}(\mu))$ contains $U_p(\mu/2)\times]-\mu/2,\mu/2[$ for every $p \in S(F_2)$ the time which $\gamma(t)$ spends in $\Psi_{\frac{2}{3}}(V_{2/3} \times [-\mu/2, 0[)$ outside $\Psi_{\frac{2}{3}}^{-1}(\mathcal{U}(\mu))$ is not more than β_2.

2) $f(\gamma(t)) \in [2/3-2\mu, 2/3-\mu/2]$, in other words, $\gamma(t) \in \Psi_{\frac{2}{3}}(V_{2/3}\times[-2\mu,-\mu/2])$. The vector field $(\Psi_{\frac{2}{3}}^{-1})_*(u)$ equals to $(B(\tau)u_2(x), \varkappa(\tau))$, where $\varkappa(\tau) \leqslant \tau$. Therefore the total time which γ can spend in this domain is not more than $\ln\big(2\mu/(\mu/2)\big) < 2$.

3) $f(\gamma(t)) \in [1/2, 2/3-2\mu]$. In the domain $f^{-1}([1/2, 2/3-2\mu])$ we have $u = v$. Therefore the time is $\leqslant 2T/C$.

4) $f(\gamma(t)) \in [1/2-2\mu, 1/2]$. Here $\gamma(t) \in \Psi_{\frac{1}{2}}(V_{1/2}\times[-2\mu, 0])$. The second coordinate of $(\Psi_{\frac{1}{2}}^{-1})_*(u)$ is equal to $(-C)$, therefore the total time spent here is not more than $2\mu/C \leqslant 1$.

Similarly to the cases 1) - 3) above one shows that the time which γ spends in $f^{-1}(]1/3, 1/2-2\mu])\setminus\mathcal{U}(\mu)$ is not more than $2T/C+2+\beta_1$. Summing up, we obtain that the time which γ can spend in $W_{\frac{1}{2}}^{\circ}\setminus\mathcal{U}(\mu)$ is not more than $\beta_1+\beta_2+4T/C+5$.

Similar analysis of behavior of u-trajectories in W_0 and W_1 shows that the same estimate holds in these cases also.

2. Estimate of $N(\gamma,\mu)$

Let $\gamma(\cdot)$ be an u-trajectory in $W_{\frac{1}{2}}$. If γ is in $V_{1/3}$, (resp. in $V_{2/3}$), then obviously $N(\gamma,\mu) \leqslant N(\mathcal{V}_1,\mu)$ (resp. $N(\gamma,\mu) \leqslant N(\mathcal{V}_2,\mu)$). Assume that $\operatorname{Im}\gamma \subset W_{\frac{1}{2}}^{\circ}$. Let $\alpha \in \mathbf{R}$ (resp. $\beta \in \mathbf{R}$) be the unique number, such that $f(\gamma(\alpha)) = 2/3-\mu$ (resp. $f(\gamma(\beta)) = 1/3+\mu$). We have $\cup_{p\in S(F_i)}U_p(\mu) \subset \mathrm{Tb}_i(\mu)$ (for $i = 1,2$). Therefore $\gamma(t) \in \mathcal{U}(\mu)$ can occur only if $t \geqslant \beta$ or $t \leqslant \alpha$. For $t \leqslant \alpha$ (resp. $t \geqslant \beta$) the curve $\Psi_{\frac{2}{3}}^{-1}(\gamma(t))$ (resp. $\Psi_{\frac{1}{3}}^{-1}(\gamma(t))$) is an integral curve of the vector field $(u_2(x),\tau)$ (resp. $(u_1(x),-\tau)$). Since an integral curve of u_2 (resp. u_1) can intersect no more than $N(\mathcal{V}_2,\mu)$ (resp. $N(\mathcal{V}_1,\mu)$) standard coordinate neighborhoods of radius μ, the first part of (5) follows. The case of curves in W_0 and W_1 is considered similarly.

3. Transversality properties

The next lemma implies that $\mathcal{V}$ satisfies Almost Transversality Condition. It implies also the point (6) of our conclusions.

Lemma 3.17.

1. *The family* $\{\cup_{p\in S_i(F_1)}D_p(-u)\}_{0\leqslant i\leqslant n}$ *equals to* $\mathbb{D}(-u_1)$.
   The family $\{\cup_{q\in S_i(F_2)}D_q(u)\}_{0\leqslant i\leqslant n}$ *equals to* $\mathbb{D}(u_2)$.
2. *For each* $\lambda \in [0,1]$ *the family* $\{(\cup_{q\in S_i(F_2)}D_q(-u)) \cap V_\lambda\}_{0\leqslant i\leqslant n}$ *is an s-submanifold of* V_λ, *which is equal to:*
   (a) $\Phi(z_1,1)(\mathbb{L}_0(-u_2))$ *if* $\lambda = 1$,
   (b) $\mathbb{L}_\lambda(-u_2)$ *if* $\lambda \in [1/2, 1-\epsilon]$,
   (c) $\emptyset$ *if* $\lambda \leqslant 1/3$.
3. *For each* $\lambda \in [0,1]$ *the family* $\{(\cup_{p\in S_i(F_1)}D_p(u))\cap V_\lambda\}_{0\leqslant i\leqslant n}$ *is an s-submanifold of* V_λ, *which is equal to:*
   (a) $\Phi(z_0,1)(\mathbb{L}_0(u_1))$ *for* $\lambda = 0$;
   (b) $\mathbb{L}_\lambda(u_1)$ *for* $\lambda \in [\epsilon, 1/2-2\mu]$;
   (c) $\Phi(z_{1/2},1)(\mathbb{L}_{1/2}(u_1))$ *for* $\lambda = 1/2$;
   (d) $\emptyset$ *for* $\lambda \geqslant 2/3$;
4. *u satisfies Almost Transversality Condition, and we have:* $\mathbb{D}(u)\cap V_0 \pitchfork \mathbb{X}$, $\mathbb{D}(-u)\cap V_1 \pitchfork \mathbb{Y}$

Proof. (1) For every $p \in S_i(F_1)$ the positive disc of F in p belongs to $V_{1/3}$, which implies immediately the first assertion; the second is proved similarly.

(2) It is not difficult to see that it suffices to prove the assertion for $\lambda \in [-2\mu + 2/3, 2\mu + 2/3]$. For these values of λ it follows from the analysis of the behavior of u-trajectories in $\mathrm{Tb}_{\frac{2}{3}}(2\mu)$, carried out in Subsection 3.2.3 1) above. (3) is proved similarly. (4): To prove that u satisfies Almost Transversality Condition let $p, q \in S(F)$, $\mathrm{ind}p \leqslant \mathrm{ind}q$. Let γ be an $(-u)$-trajectory joining p with q. The case $p, q \in S(F_1)$ or $p, q \in S(F_2)$ follows easily from (1). Let $p \in S(F_1), q \in S(F_2)$. Denote by z the (unique) point of intersection of γ with $V_{1/2}$, $\mathrm{ind}_F p$ by k, $\mathrm{ind}_F q$ by r. Then $z \in \big(\Phi(z_{1/2}, 1)(\mathbb{L}_{1/2}(u_1))\big)_{k-1}$ and $z \in \mathbb{L}_{\frac{1}{2}}(-u_2)_{n-r}$, which is impossible by the choice of $z_{1/2}$. The last point is already proved. $\square$

3.3. Proof of the theorem on the existence of quick flows

3.3.1. Introduction. Let W be a riemannian cobordism of dimension $n+1$. We shall deduce from 3.12 the theorem 3.18 below, which is somehow stronger than 3.13 . Indeed, we have included in the conclusions of the theorem 3.18 not only the restrictions (1) – (3) from 3.13 on the resulting $\mathcal{M}$-flow $\mathcal{V}$, but also the restrictions $(\partial 1) - (\partial 3)$ which concern the behavior of $\mathcal{V}$ nearby ∂W (the conditions $(\partial 1) - (\partial 3)$ are empty when $\partial W = \emptyset$).

Theorem 3.18.

Let $g : W \to [a, b]$ be a Morse function on W. Let $B > 0, \mu_0 > 0$. Let u be a g-gradient with $\|u\| \leqslant B/2$. Let $\mathbb{X}$ be an s-submanifold of $\partial_0 W$, $\mathbb{Y}$ be an s-submanifold of $\partial_1 W$. There is an $\mathcal{AM}$-flow $\mathcal{V} = (f, v, \mathcal{U})$ on W, such that

1. *$\mathcal{G}(\mathcal{V}) \leqslant 2$, $D(\mathcal{V}) \leqslant \mu_0$;*
2. *$N(\mathcal{V}) \leqslant (n+1)2^{n+1}$;*
3. *$\mathcal{V}$ is $(B, S(n))$-quick, where $S(n) = 30 + n2^{n+5}$*

Moreover:

- $(\partial 1)$ *$f(W) = [a, b]$ and in a neighborhood of ∂W we have: $f = g, u = v$.*
- $(\partial 2)$ *$\mathbb{D}_a(v) \pitchfork \mathbb{X}$; $\mathbb{D}_b(-v) \pitchfork \mathbb{Y}$*
- $(\partial 3)$ *If a v-trajectory γ starts at a point of $\partial_0 W$ then $\gamma(t)$ converges to a critical point of f as $t \to \infty$. If a $(-v)$-trajectory θ starts at a point of $\partial_1 W$ then $\theta(t)$ converges to a critical point of f as $t \to \infty$. Moreover for such trajectories γ we have: $N(\gamma) \leqslant n2^n$, $T(\gamma) \leqslant 8 + n2^{n+4}$.*

We shall prove the theorem by induction on the number of critical points of g. The lemma 3.20 below shows how to do it: if 3.18 is true for two "halves" of a cobordism W, then it is true also for the whole W. To make our notation shorter we shall introduce some terminological conventions. In the next definition and in the lemma 3.20 $g : W \to [a, b]$ is a Morse function.

Definition 3.19.

1. An *input data* is a quintuple $(B, \mu_0, u, \mathbb{X}, \mathbb{Y})$ where $B > 0, \mu_0 > 0$ are real numbers, u is a g-gradient with $\|u\| \leqslant B/2$, $\mathbb{X}, \mathbb{Y}$ are s-submanifolds of $\partial_0 W$, resp. of $\partial_1 W$.
2. The pair (g, W) will be called *good*, if the theorem 3.18 is true for (g, W) and any input data $(B, \mu_0, u, \mathbb{X}, \mathbb{Y})$.

3. A v-trajectory starting at a point of $\partial_0 W$ or finishing at a point of $\partial_1 W$ will be called *boundary trajectory.*

$\triangle$

Let λ be a regular value of g. Set $W_0 = g^{-1}([a,\lambda]), \quad W_1 = g^{-1}([\lambda, b])$. Set $g_0 = g \mid W_0, g_1 = g \mid W_1$.

Lemma 3.20.
If (g_0, W_0) and (g_1, W_1) are good, then (g, W) is good.

Proof. Let $(B, \mu_0, u, \mathbb{X}, \mathbb{Y})$ be an input data. Set $u_0 = u \mid W_0, u_1 = u \mid W_1$. Apply 3.18 to (g_0, W_0) and to the following input data: $(B, \mu_0, u_0, \mathbb{X}, \emptyset)$. We obtain an $\mathcal{AM}$-flow $\mathcal{V}_0 = (f_0, v_0, \mathcal{U}_0)$ on W_0. Next apply Theorem 3.18 to the function $g_1 : W_1 \to [\lambda, b]$ and to the following input data: $(B, \mu_0, u_1, \mathbb{D}_\lambda(-v_0), \mathbb{Y})$. We obtain an $\mathcal{AM}$-flow $\mathcal{V}_1 = (f_1, v_1, \mathcal{U}_1)$ on W_1. Note that the condition $(\partial 1)$ from 3.18 allows to glue together the $\mathcal{M}$-flows $\mathcal{V}_0$ on W_0 and $\mathcal{V}_1$ on W_1 to an $\mathcal{M}$-flow $\mathcal{V} = (f, v, \mathcal{U})$ on W. I claim that $\mathcal{V}$ satisfies the conclusions of 3.18 . We must check first that $\mathcal{V}$ is an $\mathcal{M}$-flow. This follows, since $\mathcal{V}_0$ and $\mathcal{V}_1$ are $\mathcal{AM}$-flows and $\mathbb{D}_\lambda(v_1) \nmid \mathbb{D}_\lambda(-v_0)$. Next we check that $\mathcal{V}$ verifies the conclusions (1) – (3) of 3.13 . The (1) follows from the properties of $\mathcal{V}_0, \mathcal{V}_1$. To prove (2) let γ be a v-trajectory. If it stays forever in one of W_0, W_1, then we are over since the corresponding condition is true for $\mathcal{V}_0, \mathcal{V}_1$. If γ intersects $V_\lambda = f^{-1}(\lambda)$, say, $\gamma(t_0) \in V_\lambda$, then $\gamma(t) \in W_1$ for $t \geqslant t_0$ and $\gamma(t) \in W_0$ for $t \leqslant t_0$. Now the inequality $N(\gamma) \leqslant (n+1)2^{n+1}$ follows from the condition $(\partial 3)$ which holds for $\mathcal{V}_0$ and for $\mathcal{V}_1$. The proof of (3) is similar.

Now we proceed to check $(\partial 1) - (\partial 3)$ for $\mathcal{V}$. The properties $(\partial 1)$ and $(\partial 3)$ follow immediately from the corresponding properties of $\mathcal{V}_0, \mathcal{V}_1$. To check $(\partial 3)$ note that no v-trajectory starting at a point of W_0 reaches $\partial_1 W$ (this follows since the property $(\partial 3)$ holds for $\mathcal{V}_1$). Therefore the condition $\mathbb{D}(-v) \nmid \mathbb{Y}$ is equivalent to $\mathbb{D}(-v \mid W_1) \nmid \mathbb{Y}$, which holds by the assumption. $\square$

Therefore it suffices to deduce 3.18 (n+1) from 3.12 (n) for Morse functions $g : W \to [a, b]$ without critical points and for those with one critical point. This is the aim of the subsections 3.3.2, 3.3.3. We shall not use the full force of 3.12 (n), but only the following corollary of 3.12 (n).

Lemma 3.21.
Let N be a closed riemannian manifold of dimension n. Let $D > 0, \beta > 0, A > 1$. There is $\nu_0 > 0$ such that for every $\mu < \nu_0$ there is an $\mathcal{AM}$-flow $\mathcal{W}$, such that

1. *$\mathcal{W}$ is of radius μ and is (D, β)-quick;*
2. *$N(\mathcal{W}) \leqslant n2^n$;*
3. *$\mathcal{G}(\mathcal{W}) \leqslant A$.* $\square$

3.3.2. Functions without critical points. Here we prove 3.12 (n+1) for Morse functions without critical points. The theorem 3.15 applies in this situation and we shall first of all state a corollary of (3.15 + 3.21).

Corollary 3.22.
Let $g : W \to [a, b]$ be a Morse function without critical points. Let $(B, \mu_0, w, \mathbb{L}, \mathbb{K})$ be an input data for g. Denote by T the maximal length of the interval of definition of $\mathrm{grd}(g)$*-trajectory.*

Then there is an $\mathcal{M}$-flow $\mathcal{V} = (F, u, \mathcal{U})$ of radius μ on W, such that

1. *In a neighborhood of ∂W we have: $F = g, u = w$.*
2. $\mathcal{G}(\mathcal{V}) \leqslant 2, \mu \leqslant \mu_0$
3. $N(\mathcal{V}) \leqslant n2^{n+1}$,
4. *$\mathcal{V}$ is $(B, 7 + n2^{n+4} + 8\frac{T}{B})$-quick*
5. *Every boundary trajectory γ of v converges to a critical point of F. Moreover we have for such trajectories: $N(\gamma) \leqslant n2^n$.*
6. $\mathbb{D}_a(u) \nmid \mathbb{L}, \quad \mathbb{D}_b(-u) \nmid \mathbb{K}$.

Proof. As in the proof of 3.15 set $g^{-1}(\frac{2a+b}{3}) = V_{\frac{1}{3}}, g^{-1}(\frac{2b+a}{3}) = V_{\frac{2}{3}}$. Apply the lemma 3.21 to $V_{1/3}$ and to $V_{2/3}$ and obtain two $\mathcal{AM}$-flows $\mathcal{V}_1, \mathcal{V}_2$ of some radius $\mu < \mu_0$ on $V_{1/3}, V_{2/3}$, such that $\mathcal{G}(\mathcal{V}_i) \leqslant 2, N(\mathcal{V}_i) \leqslant n2^n$ and that $\mathcal{V}_i$ are $(\frac{B}{12}, 1)$-quick. Note that $\mathcal{V}_i$ are $(\frac{B}{12}, 1 + n2^{n+3}, \mu/2)$-quick (by 3.11). Now just apply 3.15 and produce an $\mathcal{AM}$-flow $\mathcal{V} = (F, u, \mathcal{U})$ on W of radius μ, which is $(B, 7 + n2^{n+4} + 8\frac{T}{B})$-quick. We have also $N(\mathcal{V}) \leqslant n2^{n+1}$. The other points of our conclusion also follow from the properties of $\mathcal{V}$ listed in the 3.15 . □

Of course this corollary does not yet give an $\mathcal{AM}$-flow satisfying the conclusions of 3.18 since T can be arbitrarily large. So what we shall do is to cut W to thin slices so that for each slice W_s the corresponding time T_s is small, then apply the 3.22 to each of W_s and then glue the resulting flows together.

Proceeding to the precise constructions, recall that we are given the input data $(B, \mu_0, u, \mathbb{X}, \mathbb{Y})$. Choose a sequence $a = a_0 < a_1 < a_2 < ... < a_N = b$ of regular values of g, so close to each other, that for each s the time which a $\mathrm{grd}(g)$-trajectory spends in $W_s = g^{-1}([a_s, a_{s+1}])$ is not more than $B/8$. We shall define by induction in s a sequence of $\mathcal{AM}$-flows $\mathcal{V}_i = (F_i, u_i, \mathcal{U}_i)$ on W_s. Assume that $\mathcal{V}_i$ is already constructed for $i \leqslant s - 1$. Apply 3.22 to W_s, choosing the following input data: $(B, \mu_0, u, \mathbb{L}_s, \mathbb{K}_s)$ where $\mathbb{L}_s$ and $\mathbb{K}_s$ are defined as follows:

$$\begin{array}{ll} s = 0 : & \mathbb{L}_s = \mathbb{X}, \quad \mathbb{K}_s = \emptyset \\ 0 < s < N - 1 : & \mathbb{L}_s = \mathbb{D}_{a_s}(-u_{s-1}), \quad \mathbb{K}_s = \emptyset \\ s = N - 1 : & \mathbb{L}_s = \mathbb{D}_{a_s}(-u_{s-1}), \quad \mathbb{K}_s = \mathbb{Y} \end{array}$$

We obtain thus a sequence of $\mathcal{AM}$-flows $\mathcal{V}_i$ of radius $\mu_i \leqslant \mu_0$, each of $\mathcal{V}_i$ is $(B, 8 + n2^{n+4})$-quick, and satisfies $N(\mathcal{V}_i) \leqslant n2^{n+1}$. Now glue all the $\mathcal{V}_i$ together (this is possible, since nearby every $g^{-1}(a_i)$ the functions f_i, f_{i+1} coincide with g and the vector fields u_i, u_{i+1} coincide with u). We obtain an $\mathcal{M}$-flow $\mathcal{V} = (F, u, \mathcal{U})$ on W, and I claim that it satisfies the conclusions of 3.18 . Indeed, note first that $\mathcal{V}$ is an $\mathcal{AM}$-flow (this follows from the restrictions $\mathbb{D}_{a_s}(u_s) \nmid \mathbb{D}_{a_s}(-u_{s-1})$, imposed on $\mathcal{V}_s$ in the process of construction). Further, (1) and (2) follow immediately (actually, we obtain here a bit stronger estimate $N(\mathcal{V}) \leqslant n2^{n+1}$). As to the estimate for the quickness, note that no v-trajectory can intersect more than two of W_s. Thus $\mathcal{V}$ is $(B, 2(8 + n2^{n+4}))$-quick. The other conditions follow immediately. □

3.3.3. Functions with one critical point. Here we prove 3.18 (n+1) for functions $g : W \to [a, b]$ with one critical point. Let p be this critical point, let k be its index, and let $(B, \mu_0, u, \mathbb{X}, \mathbb{Y})$ be our input data for construction of the flow $\mathcal{V}$. Changing if necessary the g-gradient u in a neighborhood of p, we can assume that there is a Morse chart $\Phi : U \to B^n(0, r) \subset \mathbf{R}^{n+1}$ for g such that $\Phi(p) = 0$ and $\Phi_*(v) = \mathfrak{n}_k$.

Lemma 3.23.

There is $\delta_0, 0 < \delta_0 < r$, such that for every $\delta < \delta_0$ the time which a v-trajectory can spend in $g^{-1}([g(p) - \delta^2, g(p) + \delta^2]) \setminus U_p(\delta/2)$ is not more than 6.

Proof. Denote $g^{-1}([g(p) - \delta^2, g(p) + \delta^2])$ by S_δ. Let γ be a w-trajectory. The function $dg(w)$ is bounded from below in $S_\delta \setminus U_p$, therefore for $\delta > 0$ sufficiently small the time which γ spends in $S_\delta \setminus U_p$ is less than 1. The time, which γ spends in $U_p \cap \big(S_\delta \setminus U_p(\delta)\big)$ is not more than 2 (by 1.10), and the time which γ spends in $U_p(\delta) \setminus U_p(\delta/2)$ is not more than $\ln 8 \leqslant 3$ (by 1.9). $\square$

Now let $\delta > 0$ be so small that

1. $\delta < \mu_0, \delta < \delta_0$
2. $\mathcal{G}(U(\delta), \Phi \mid U(\delta)) \leqslant 2$

Set

$$\begin{aligned} W_+ &= g^{-1}\big([g(p) + \delta^2, b]\big), & g_+ &= g \mid W_+ \\ W_- &= g^{-1}\big([a, g(p) - \delta^2]\big), & g_- &= g \mid W_- \\ W_- &= g^{-1}\big([g(p) - \delta^2, g(p) + \delta^2]\big), & g_0 &= g \mid W_0 \\ V_- &= g^{-1}(g(p) - \delta^2) \\ V_+ &= g^{-1}(g(p) + \delta^2) \\ S_+ &= D(p, -v) \cap g^{-1}(g(p) + \delta^2) & & . \end{aligned}$$

By abuse of notation we keep the symbol u to denote all the restrictions of u to: $W_+, W_-, ...$. Apply to g_+ Theorem 3.18 (proved in 3.3.2). Choose as the input data the following string: $(B, \mu_0, u, S_+, \mathbb{Y})$.

Denote the resulting $\mathcal{AM}$-flow by $\mathcal{V}_+ = (f_+, v_+, \mathcal{U}_+)$. Glue it to the $\mathcal{AM}$-flow $\mathcal{V}_0 = (g \mid W_0, u, \{\Phi \mid U(\delta/2)\})$. We obtain an $\mathcal{AM}$-flow on $W_+ \cup W_0$, which will be denoted by $\mathcal{V}_1 = (F_1, v_1, \mathcal{U}_1)$. Apply now Theorem 3.18 to g_-, choosing as the input data the following string: $(B, \mu_0, u, \mathbb{X}, \mathbb{D}_{f(p)-\delta^2}(v_1))$. We obtain an $\mathcal{AM}$-flow. Denote it by $\mathcal{V}_- = (f_-, v_-, \mathcal{U}_-)$ and glue it to $\mathcal{V}_1$.

Denote the resulting $\mathcal{M}$-flow by $\mathcal{V} = (f, v, \mathcal{U})$. I claim that it satisfies the conclusions of 3.18 . Indeed, the condition (1) goes by construction, as well as the condition $\|v\| \leqslant B$. To check (2) note that it is already proved for trajectories of v, which stay forever in W_- or W_+. Assume that a v-trajectory γ intersects V_+. If $\lim_{t \to -\infty} \gamma(t) = p$, then $N(\gamma) \leqslant 1 + n2^n$. If γ intersects V_- then $N(\gamma) \leqslant 1 + n2^{n+1}$. The time which a v-trajectory γ can spend outside $\cup_p U(p)$ is estimated from above using the lemma 3.23 and the estimates of Corollary 3.22 .

The properties $(\partial 1) - (\partial 3)$ of Theorem 3.18 are verified without difficulties. $\square$

3.4. Final remark

The results of this chapter are essentially contained in [**38**]. In particular, the theorem 3.12 is a generalization of the Theorem 1.12 from [**38**] to the case of cobordisms (in [**38**] it was stated only for closed manifolds). The theorem 3.13 is a generalization of the theorem 1.13 from [**38**] to the case of cobordisms (also we diminished the corresponding constant from $100{+}n2^{n+7}$ in [**38**] to $30{+}n2^{n+5}$; we think that one can diminish it still more). The theorems 1.12 and 1.13 of [**38**] are in turn exactly the theorems 1.17, resp. 1.18 of the preprint [**34**].

CHAPTER 4

Regularizing the gradient descent map

Let W be a riemannian cobordism of dimension n, $f: W \to [a,b]$ be a Morse function and v be an f-gradient. Consider the open set $U_1 \subset \partial_1 W$ of all the points x, such that the trajectory $\gamma(x,\cdot;-v)$ reaches $\partial_0 W$ and intersects it it at some point y. The correspondence $x \mapsto y$ is a diffeomorphism of U_1 onto an open subset $U_0 \subset \partial_0 W$, we denote it by $(-v)^{\rightsquigarrow}$. The aim of this chapter is to prove an analog of cellular approximation theorem for the map $(-v)^{\rightsquigarrow}$. The problem is that $(-v)^{\rightsquigarrow}$ in general can not be extended to a continuous map on the whole of $\partial_1 W$.

Recall however from 2.5 that $(-v)^{\rightsquigarrow}$ is well defined on the category of s-submanifolds (under some mild transversality condition, see Subsection 2.5.2). The collection of descending discs of a gradient of a Morse function form an s-submanifold (if the gradient satisfies Almost Transversality Condition); we pick a Morse function ϕ_1 on $\partial_1 W$ and a ϕ_1-gradient u_1, satisfying Almost Transversality Condition. Consider the corresponding s-submanifold $\mathbb{D}(u_1)$, formed by the descending discs of u_1. Denote $\mathbb{D}(u_1)$ by $\mathbb{A}$ and $\cup_{i\leqslant j} D(\mathrm{ind}{=}i\ ; u_1)$ by $A_{\leqslant j}$. Proceed similarly with $\partial_0 W$ and obtain the s-submanifold $\mathbb{B} = \mathbb{D}(u_0)$, where u_0 is a gradient of a Morse functionon $\partial_0 W$, and u_0 satisfies Almost Transversality Condition. The decomposition of a manifold to the union of the descending discs of a gradient resembles much of a cellular decomposition, so one can try to reproduce the cellular approximation theorem in this situation. We can not, of course, expect, that, perturbing v, we obtain for the new gradient u the direct analog of the cellular approximation condition: $(-u)^{\rightsquigarrow}(A_{\leqslant j}) \subset B_{\leqslant j}$. We can, though, deform v so that for the new gradient u the following will hold:

$$(-u)^{\rightsquigarrow}(A_{\leqslant j}) \subset D_\delta(\mathrm{ind}{\leqslant}j\ ; u_0) \tag{24}$$

Indeed, first perturb v so as to make the submanifold $(-u)^{\rightsquigarrow}(\mathbb{A})$ transversal to the s-submanifold $\widehat{\mathbb{B}} = \mathbb{D}(-u_0)$. Then $(-u)^{\rightsquigarrow}(A_{\leqslant j})$ does not intersect $\mathbb{D}_{\leqslant n-2-j}(-u_0)$ and for T sufficiently large the diffeomorphism $\Phi(T,-u_0)$ pushes $(-u)^{\rightsquigarrow}(A_{\leqslant j})$ into a small neighborhood of $\mathbb{D}_{\leqslant j}(-u_0)$. We can modify v in a small neighborhood of $\partial_0 W$, so that for the new vector field $\widetilde{v}$ the following holds: $(-\widetilde{v})^{\rightsquigarrow} = \Phi(T,-u_0)\circ(-v)^{\rightsquigarrow}$. This is done by adding to v a horizontal component proportional to u_0 (see Subsection 2.1.6 for definition of the construction of adding a horizontal component). Thus for $\widetilde{v}$ the (24) holds.

It is essential for us that the property (24) (and actually a bit stronger property) can be achieved by a C^0-small perturbation $\widetilde{v}$ of v. The proof of this result (Theorem 4.3) is based on the main theorem of the previous chapter.

4.1. Condition ($\mathfrak{C}$)

4.1.1. Condition ($\mathfrak{C}$): the definition and the statement of the density theorem.

Terminology

Let $g : W \to [a,b]$ be a Morse function on a cobordism W of dimension n, v be a g-gradient.

Let $\lambda \in [a,b]$ be a regular value of f. Let $A \subset \partial_1 W$. We say, that v *descends* A *to* $f^{-1}(\lambda)$, if every $(-v)$-trajectory starting in A reaches $f^{-1}(\lambda)$. Let $X \subset W$. We say that v *descends* A *to* $f^{-1}(\lambda)$ *without intersecting* X, if v descends A to $f^{-1}(\lambda)$ and $T(A,-v) \cap X = \emptyset$.

We leave to the reader to define the meaning of "v *lifts* B *to* $f^{-1}(\lambda)$" and "v *lifts* B *to* $f^{-1}(\lambda)$ *without intersecting* Y", where $B \subset \partial_0 W, Y \subset f^{-1}([a,\lambda])$. $\triangle$

Recall the sets $B_\delta(\mathrm{ind}_{\leqslant s}\ ; v), D_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$ from Definition 2.15 and introduce one more set with similar properties: set $C_\delta(\mathrm{ind}_{\leqslant s}\ ; v) = W \setminus B_\delta(\mathrm{ind}_{\geqslant n-s-1}\ ; -v)$. The following lemma is proved by standard Morse-theoretical arguments.

Lemma 4.1.

Assume that v is δ-separated with respect to ϕ and the ordering sequence $(a_0, ..., a_{n+1})$. Then for every s we have:

$$\partial_0 W \cup B_\delta(\mathrm{ind}_{\leqslant s}\ ; v) \subset \phi^{-1}([a_0, a_{s+1}]) \subset C_\delta(\mathrm{ind}_{\leqslant s}\ ; v) \cup \partial_0 W$$

and these inclusions are homotopy equivalences. $\square$

Now we can formulate the condition ($\mathfrak{C}$).

Definition 4.2.

We say, that v *satisfies condition* ($\mathfrak{C}$) if there are objects 1) - 4), listed below, with the properties (A), (B0), (B1) below.

Objects:

1) An ordered Morse function ϕ_1 on $\partial_1 W$ with ordering sequence $(\alpha_0, ..., \alpha_n)$, and a ϕ_1-gradient u_1.
2) An ordered Morse function ϕ_0 on $\partial_0 W$ with ordering sequence $(\beta_0, ..., \beta_n)$, and a ϕ_0-gradient u_0.
3) An ordered Morse function ϕ on W with ordering sequence $(a_0, ..., a_{n+1})$, adjusted to (f, v).
4) A number $\delta > 0$.

Properties:

(A) u_0 is δ-separated with respect to ϕ_0, u_1 is δ-separated with respect to ϕ_1, v is δ-separated with respect to ϕ.

(B1)

$$(-v)^{\leadsto}\Big(C_\delta(\mathrm{ind}_{\leqslant j}\ ; u_1)\Big) \cup \Big(D_\delta(\mathrm{ind}_{\leqslant j+1}\ ; v) \cap \partial_0 W\Big) \subset B_\delta(\mathrm{ind}_{\leqslant j}\ , u_0) \text{ for every } j$$

(B0)

$$\overset{\leadsto}{v}\Big(C_\delta(\mathrm{ind}_{\leqslant j}\ ; -u_0)\Big) \cup \Big(D_\delta(\mathrm{ind}_{\leqslant j+1}\ ; -v) \cap \partial_1 W\Big) \subset B_\delta(\mathrm{ind}_{\leqslant j}\ ; -u_1) \text{ for every } j$$

$\triangle$

The set of all f gradients satisfying ($\mathfrak{C}$) will be denoted by $\mathcal{G}\mathfrak{C}(f)$.

Comments:

1. As we have already mentioned in the introduction, the conditions (B0), (B1) together form an analog of cellular approximation condition. Indeed, the sets $C_\delta(\text{ind}\leqslant j\ ; u_1)$ and $B_\delta(\text{ind}\leqslant j\ ; u_0)$ are "thickenings" of $D(\text{ind}\leqslant j\ ; u_1)$, resp. of $D(\text{ind}\leqslant j\ ; u_0)$ (the first one being "thicker" than the second). Thus the condition (B1) requires that $(-v)^{\leadsto}$ sends a certain thickening of a j-skeleton of $\partial_1 W$ to a certain thickening of a j-skeleton of $\partial_0 W$. Warning: (B1) actually requires more than that: for every j the soles of δ-handles of W of indices $\leqslant j+1$ must belong to the set $B_\delta(\text{ind}\leqslant j\ ; u_0)$.
2. The condition ($\mathfrak{C}$) is stronger than the condition ($\mathcal{C}$) from [**31**], §1.2. This is obvious, since ($\mathfrak{C}$) is formulated similarly to ($\mathcal{C}$), except that in ($\mathcal{C}$) we have demanded $(-v)^{\leadsto}\big(\phi^{-1}([a_0, a_{j+1}])\big) \subset B_\delta(\text{ind}\leqslant j\ ; u_0)$. The condition ($\mathcal{C}$) is in turn stronger, than the condition (RP) from [**38**], §4, this is a bit less obvious, and we shall not prove it here.
3. If v satisfies condition ($\mathfrak{C}$), then v obviously satisfies almost transversality assumption. Note that if we are given a ϕ_0-gradient u_0, a ϕ_1-gradient u_1, a ϕ-gradient v, which satisfy Almost Transversality Condition, then the condition (A) is always true if δ is sufficiently small.

Theorem 4.3.
$\mathcal{G}\mathfrak{C}(f)$ is open and dense in $\mathcal{G}(f)$ with respect to C^0 topology. Moreover, if v_0 is any f-gradient then one can choose a C^0-small perturbation v of v_0 such that $v \in \mathcal{G}\mathfrak{C}(f)$, $v = v_0$ in a neighborhood of ∂W, and $v = v_0$ in a neighborhood of every critical point of f.

The proof occupies the subsections 4.1.2 – 4.1.5.

Convention: We shall say that ($\mathfrak{C}$) is *open*, resp. *dense* if the set of all f-gradients satisfying ($\mathfrak{C}$) is open, resp. dense in $\mathcal{G}(f)$. $\triangle$

4.1.2. C^0-openness of $\mathfrak{C}$.

Let v be an f-gradient satisfying ($\mathfrak{C}$). Choose the corresponding functions ϕ_0, ϕ_1, ϕ, their gradients u_0, u_1, and a number $\delta > 0$ satisfying (A) and (B0), (B1). Fix now the string $\mathcal{S} = (\delta, u_0, u_1, \phi_0, \phi_1, \phi)$. For a ϕ-gradient w we shall denote the condition (A) with respect to w and $\mathcal{S}$ by (A)(w). Similarly, we shall denote the conditions (B1), (B0) with respect to w and $\mathcal{S}$ by (B1)(w), resp. (B0)(w). We know that (A)(v)& (B1)(v) & (B0)(v) holds; we shall prove that (A)(w)& (B1)(w) & (B0)(w) holds for every w sufficiently C^0 close to v.

Let w be another f-gradient. Corollary 2.74 implies that if $||w-v||$ is sufficiently small, then w is a ϕ-gradient and is also δ-separated, so the condition A(w) is satisfied also for w.

It is convenient to reformulate the conditions (B1)(w)&(B0)(w). Introduce three new conditions:

$\beta_1(w)$: For every j we have: w descends $C_\delta(\text{ind}\leqslant j\ ; u_1)$ to $\phi^{-1}(a_{j+1})$ without intersecting $\cup_{\text{ind} p \geqslant j+1} D_\delta(p)$.

$\beta_2(w)$: For every j we have: w lifts $C_\delta(\text{ind}\leqslant n-j-2\ ; -u_0)$ to $\phi^{-1}(a_{j+2})$ without intersecting $\cup_{\text{ind}p\leqslant j+1} D_\delta(p)$.

$\beta_3(w)$: For every j the sets:

$$R_j^+(w) = (-v)^{\leadsto}_{[b,a_{j+2}]}(C_\delta(\text{ind}\leqslant j\ ; u_1)),$$

$$R_j^-(w) = v^{\leadsto}_{[a,a_{j+2}]}(C_\delta(\text{ind}\leqslant n-j-2\ ; -u_0))$$

are disjoint.

Note that $\beta_1(w)$ and $\beta_2(w)$ are in a sense dual to each other.

Lemma 4.4.
Assume that $A(w)$ holds. Then (B1)(w)&(B0)$(w)$$\Leftrightarrow$ $\beta_1(w)$&$\beta_1(w)$&$\beta_3(w)$

Proof. $\Rightarrow$

The condition β_1 follows from (B1), β_2 follows from (B0). To prove β_3, note that if $R_j^+(w)\cap R_j^-(w)\neq\emptyset$, then there is a $(-v)$-trajectory joining $x\in C_\delta(\text{ind}\leqslant j\ ; u_1)$ with $y\in C_\delta(\text{ind}\leqslant n-j-2\ ; -u_0) = \partial_0 W\setminus B_\delta(\text{ind}\leqslant j\ ; u_0)$. And this contradicts (B1).

$\Leftarrow$

β_1 says that for every j we have:

$$\left(\bigcup_{\text{ind}p\geqslant j+1} D_\delta(p,-v)\right)\cap\partial_1 W\subset\partial_1 W\setminus C_\delta(\text{ind}\leqslant j\ ; u_1) = B_\delta(\text{ind}\leqslant n-2-j\ ; -u_1).$$

That is, $D_\delta(\text{ind}\leqslant n-1-j\ ; -v)\cap\partial_1 W\subset B_\delta(\text{ind}\leqslant n-2-j\ ; -u_1)$. Further, β_3 implies that for every j every v-trajectory starting in $C_\delta(\text{ind}\leqslant n-j-2\ ; -u_0)$ and reaching $\partial_1 W$, cannot intersect $\partial_1 W$ at a point of $C_\delta(\text{ind}\leqslant j\ ; u_1)$. Therefore it intersects $\partial_1 W$ at a point of $B_\delta(\text{ind}\leqslant n-2-j\ ; -u_1)$. Therefore (B0)$(w)$ holds. The proof of (B1)(w) is similar. □

Returning to the proof of C^0-openness of $(\mathfrak{C})$, note that the conditions $\beta_1(w)$ and $\beta_2(w)$ are obviously open. We know that $\beta_1(v)$ and $\beta_2(v)$ hold, therefore for every j the sets $R_j^+(v), R_j^-(v)$ are compact. Since they are disjoint (by $\beta_3(v)$), we can choose two disjoint open subsets U_j, V_j of $\phi^{-1}(a_{j+2})$ such that $R_j^+(v)\subset U_j, R_j^-(v)\subset V_j$. Then for every w sufficiently close to v and for every j we have (by Corollary 2.71):

$$(-w)^{\leadsto}_{[b,a_{j+2}]}\big(C_\delta(\text{ind}\leqslant j\ ; u_1)\big)\subset U_j,\quad w^{\leadsto}_{[a,a_{j+2}]}\big(C_\delta(\text{ind}\leqslant n-2-j\ ; -u_0)\big)\subset V_j$$

and $\beta_3(w)$ holds. Therefore $\beta_1(w)$&$\beta_2(w)$&$\beta_3(w)$ is C^0 open and we have proved C^0 openness of $(\mathfrak{C})$.

4.1.3. C^0-density of $(\mathfrak{C})$: The idea of the proof. In this subsection I present the idea of the proof on the intuitive level.

Choose some pair (ϕ_0, u_0), where ϕ_0 is an ordered Morse function on $\partial_0 W$, and u_0 is an ϕ_0-gradient. Let us try to perturb a given f-gradient v_0 in order to obtain (B1). The set

$$\mathfrak{N}_j(v,\delta) = (-v)^{\leadsto}\Big(C_\delta(\text{ind}\leqslant j\ ; u_1)\Big)\cup\Big(D_\delta\big(\text{ind}\leqslant j+1\ ; v\big)\cap\partial_0 W\Big)$$

contains in particular the set

$$\mathfrak{N}_j(v) = (-v)^{\leadsto}\big(D(\text{ind}\leqslant j\ ; u_1)\big)\cup\big(D(\text{ind}\leqslant j+1\ ; v)\cap\partial_0 W\big)$$

We know from Proposition 2.51 and Lemma 2.50 that perturbing v if necessary we can assume that $\{\mathfrak{N}_j(v)\}$ form an s-submanifold, transversal to to $\mathbb{D}(-u_0)$. In particular $\mathfrak{N}_j(v)$ is disjoint from $D(\mathrm{ind}\leqslant i\ , -u_0)$ for $i+j < n-1$. Using Proposition 2.53 we conclude that for $\epsilon > 0$ small enough the sets $\mathfrak{N}_j(v,\epsilon)$ and $D_\epsilon(\mathrm{ind}\leqslant i\ ; -u_0)$ are disjoint if $i+j < n-1$.

Now we shall modify v by adding a horizontal component nearby $\partial_0 W$. Let ξ be a vector field equivalent to u_0 (see Definition 3.3). Then for some $T > 0$ we have

$$\Phi(T, -\xi)(\partial_0 W \setminus B_\epsilon(\mathrm{ind}\leqslant i\ , -u_0)) \subset B_\epsilon(\mathrm{ind}\leqslant n-2-i\ ; u_0$$

Thus if we apply to the AHC-construction with respect to ξ and to the function h satisfying $\int_0^l h(\tau)d\tau = T$, we shall have for the resulting vector field w:

$$\mathfrak{N}_j(w) \subset B_\epsilon(\mathrm{ind}\leqslant n-2-i\ , u_0)$$

(see AHC2, page 29; actually we should also take some care in order that w be an f-gradient).

Of course if the vector field u_0 and ξ were chosen arbitrarily then T may be large and therefore $||v-w||$ may also be large.

Now I shall explain why it is possible to choose T and $||v-w||$ small. This step uses in essential way the Theorem 3.4 . We shall chose the pair (ϕ_0, u_0) in a special way. Let $C > 0, t > 0$ be real numbers and $(\phi_0, u8)$ be the pair, satisfying the conclusions of 3.4 with respect to C, t. Choose $\widetilde{u}_0$, equivalent u_0 with $||\widetilde{u}_0|| \leqslant C$, and having the property that the flow generated by $(-\widetilde{u}_0)$ pushes $\partial_0 W \setminus B_\epsilon(\mathrm{ind}\leqslant i\ ; -u_0)$ inside $B_\epsilon(\mathrm{ind}\leqslant n-2-i\ ; u_0)$, and the time necessary for this procedure is $\leqslant t$. Now set $\xi = \widetilde{u}_0$ and to v a horizontal component proportional to ξ nearby $\partial_0 W$. The input data (a, ξ, h) for AHC-construction will be chosen so that $\int_0^a h(\tau)d\tau = t$. The estimates from AHC1 (page 29) shows that $||v-w||$ will be small enough if only C and t are chosen sufficiently small.

4.1.4. C^0-density of ($\mathfrak{C}$): terminology. Let v_0 be an f-gradient. Choose regular values of $\alpha, \beta \in]a, b[$ of f, such that $a < \alpha < \beta < b$ and that the intervals $[a,\alpha], [\beta, b]$ be regular. Denote $f^{-1}(\lambda)$ by V_λ, so that $\partial_1 W = V_b, \partial_0 W = V_a$. Denote $f^{-1}([\alpha,\beta])$ by W'. The map $(-v_0)^{\leadsto}_{[b,\beta]} : V_b \to V_\beta$ and the map $(v_0)^{\leadsto}_{[a,\alpha]} : V_a \to V_\alpha$ are diffeomorphisms. We shall denote these maps by Ψ_1, resp. Ψ_0. Set

$$m = \inf_{x\in W\setminus W'} ||df(v)(x)||, \quad M = \sup_{x\in W\setminus W'} ||df(x)||$$

Let E_0, E_1 be the expansion constants for Ψ_0, resp. Ψ_1. Set $E = \max(E_0, E_1)$.

Now we shall define the horizontal components which will be added to v. Let $\epsilon > 0$. Our aim is to construct a vector field v, such that (B1) and (B0) hold and $||v - v_0|| < \epsilon$. Choose some closed collars $C_0 = \gamma(\partial_0 W, [0,l]; v)$ and $C_1 = \gamma(\partial_1 W, [0,l]; -v)$ so that $C_0 \subset \phi^{-1}([a,\alpha[), C_1 \subset \phi^{-1}(]\beta, b])$. Choose any positive C^∞ function h on $[0,l]$ such that $h(x) = 0$ for x in a neighborhood of 0 and in a neighborhood of l. Denote $\sup_{\tau\in[0,l]} h(\tau)$ by H. Set $t_0 = \int_0^l h(\tau)d\tau$. Choose a real number $A > 0$ such that

$$A < \frac{m}{EMH}, \quad A < \frac{\epsilon}{2EH} \tag{25}$$

Remark 4.5.

It is easy to deduce from AHC1, AHC3, that with this choice of A, t_0, h, H we have: Let ξ be a vector field $\partial_0 W$ with $||\xi|| \leqslant A$. Let v be an f-gradient. Let w be the result of AHC-construction with the input data (l, ξ, h), applied to v. Then w is also an f-gradient annd $||w - v|| \leqslant \epsilon/2$. (Similar assertion holds for adding a horizontal component nearby $\partial_1 W$.) $\triangle$

Let $\phi_0 : \partial_0 W \to \mathbf{R}$ be a Morse function and u_0 be a ϕ_0-gradient, satisfying the conclusions of Theorem 3.4 with respect to $C = A, t = t_0$. We can obviously assume that ϕ_0 is ordered; let $(\alpha_0, ...\alpha_n)$ be the corresponding ordering sequence.

Similarly, let $\phi_1 : \partial_1 W \to \mathbf{R}$ be a Morse function and u_1 be a ϕ_1-gradient, satisfying the conclusions of Theorem 3.4 with respect to $C = A, t = t_0$. We assume that ϕ_1 is ordered; let $(\beta_0, ...\beta_n)$ be the corresponding ordering sequence.

Assume that u_0, u_1 are δ-separated. We obtain the corresponding gfn-systems $\mathbf{D}(u_1), \mathbf{D}(u_0), \mathbf{D}(-u_1), \mathbf{D}(-u_0)$ with the cores $\mathbb{D}(u_0), \mathbb{D}(u_1), ...$ We fix $]0, \delta[$ as the interval of definition for these systems.

The diffeomorphism Ψ_1 identifies V_b with V_β, therefore we can transfer to V_β the function ϕ_1 and its gradient u_1. The resulting function and gradient will be denoted by ϕ_1', resp. u_1'. We obtain also gfn-systems: $\mathbf{A}^- = \Psi_1(\mathbf{D}(u_1))$ and $\mathbf{A}^+ = \Psi_1(\mathbf{D}(-u_1))$ in V_β with the cores $\mathbb{D}(u_1')$, resp. $\mathbb{D}(-u_1')$. (*Warning:* Ψ_1 is not in general an isometry, so $\mathbf{A}^-$ need not be the same as $\mathbf{D}(u_1')$.) A similar construction gives gfn-systems $\mathbf{B}^-, \mathbf{B}^+$ on V_α with the cores $\mathbb{D}(u_0)$, resp. $\mathbb{D}(-u_0)$.

4.1.5. C^0-density of $\mathcal{GC}(f)$: perturbation of a given gradient. In this subsection we finish the proof of Theorem 4.3 . Namely, we show how to perturb v_0 in order to obtain a gradient v, satisfying the condition $(\mathfrak{C})$.

A. First modification of the gradient

Applying Proposition 2.51 we obtain an $(f \mid W')$-gradient v_1, having the following properties:

(i) v_1 satisfy Almost Transversality Condition;
(ii) $v_1 = v$ in a neighborhood of $V_\alpha \cup V_\beta$;
(iii) the s-submanifolds $\mathbb{T}(\mathbb{D}(u_1'), -v_1)$ and $\mathbb{T}(\mathbb{D}(-u_0'), v_1)$ are defined
(iv) $(-v_1)^{\rightsquigarrow}_{[\beta,\alpha]}(\mathbb{D}(u_1')) \pitchfork \mathbb{D}(-u_0')$ and $(v_1)^{\rightsquigarrow}_{[\alpha,\beta]}(\mathbb{D}(-u_0')) \pitchfork \mathbb{D}(u_1')$.

Moreover, v_1 can be chosen arbitrarily C^∞ close to v_0, thus we can assume $||v_1 - v_0|| < \epsilon/2$. Recall from Subsection 2.5.3 that there is $\lambda > 0$ such that the pairs $(\mathbf{A}, v_1)$ and $(\mathbf{B}, -v_1)$ are λ-ordered. Therefore we can consider $]0, \lambda[$ as the interval of definition of the gfn-systems $\mathbf{T}(\mathbf{A}^-, -v_1), \mathbf{T}(\mathbf{B}^+, v_1)$.

Applying Proposition 2.53 we deduce from (iv) above that there $\nu < \lambda$, such that for every s:

$$(26) \quad (-v_1)^{\rightsquigarrow}_{[\beta,\alpha]}\Big(\mathbf{A}^-_{(\leqslant s)}(\nu)\Big) \cup \Big(D_\nu(\mathrm{ind}{\leqslant} s+1\ ; v_1) \cap V_\alpha\Big)$$
$$\text{does not intersect } \mathbf{B}^+_{(\leqslant n-2-s)}(\nu)$$

and

$$(27) \quad (v_1)^{\rightsquigarrow}_{[\alpha,\beta]}\Big(\mathbf{B}^+_{(\leqslant s)}(\nu)\Big) \cup \Big(D_\nu(\mathrm{ind}{\leqslant} s+1\ ; -v_1) \cap V_\beta\Big)$$
$$\text{does not intersect } \mathbf{A}^-_{(\leqslant n-2-s)}(\nu)$$

We shall choose ν so small that v_1 is ν-separated. Extend the vector field v_1 to the whole of W setting $v_1(x) = v_0(x)$ for $x \in W \setminus W'$. Then v_1 is a ν-separated f-gradient. Choose an ordered Morse function ϕ on W', adjusted to (f, v_1) and extend ϕ to the whole of W, setting $\phi(x) = f(x)$ for $x \in W \setminus W'$.

B. Second modification of the gradient

Here we start with the gradient v_1 constructed in previous subsection. The second modification consists in adding to v_1 horizontal components nearby $\partial_1 W$ and $\partial_0 W$. We have chosen a number $\nu > 0$. Choose now (by Theorem 3.4) a ϕ_0-gradient $\widetilde{u}_0$ and a ϕ_1-gradient $\widetilde{u}_1$, equivalent to u_0, resp. to u_1 and such that $\widetilde{u}_0, \widetilde{u}_1$ are (t_0, ν)-rapid and satisfy $\|\widetilde{u}_0\| \leqslant A, \|\widetilde{u}_1\| \leqslant A$. That is, for $i = 0, 1$ and for every s we have

$$\Phi(\widetilde{u}_i, t_0)(C_\nu(\text{ind}_{\leqslant s}\ ; \widetilde{u}_i)) \subset B_\nu(\text{ind}_{\leqslant s}\ ; \widetilde{u}_i)$$

$$\Phi(-\widetilde{u}_i, t_0)(C_\nu(\text{ind}_{\leqslant s}\ ; -\widetilde{u}_i) \subset B_\nu(\text{ind}_{\leqslant s}\ ; -\widetilde{u}_1)$$

Apply to v_1 the AHC-construction nearby $\partial_0 W$ with the input data $(l, \widetilde{u}_0, h)$ and nearby $\partial_1 W$ with the input data $(l, \widetilde{u}_1, h)$ (recall that l and h were chosen in the beginning of the subsection 4.1.4).

Denote the resulting vector field by v. Then $\|v - v_0\| \leqslant \epsilon$, and that v is an f-gradient (see Remark 4.5). Since supp $(v_1 - v) \subset W \setminus W'$ and $f \mid W \setminus W' = \phi \mid W \setminus W'$ the vector field v is also a ϕ-gradient. To compute $\overset{\leadsto}{v}$ and $(-v)^{\leadsto}$, denote the diffeomorphism $(-v)^{\leadsto}_{[b,\beta]}$ by $\widetilde{\mathbf{\Psi}}_1$ and $v^{\leadsto}_{[a,\alpha]}$ by $\widetilde{\mathbf{\Psi}}_0$.

From the construction of v it follows (see AHC2), page 29) that for every s we have:

$$\widetilde{\mathbf{\Psi}}_1\Big(C_\nu(\text{ind}_{\leqslant s}\ ; u_1)\Big) \subset \mathbf{A}^-_{(\leqslant s)}(\nu) \tag{28}$$

$$\widetilde{\mathbf{\Psi}}_1^{-1}\Big(V_\beta \setminus \mathbf{A}^+_{(\leqslant s)}(\nu)\Big) \subset B_\nu(\text{ind}_{\leqslant n-2-s}\ ; -u_1) \tag{29}$$

Similarly,

$$\widetilde{\mathbf{\Psi}}_0\Big(C_\nu(\text{ind}_{\leqslant s}\ ; -u_0\Big) \subset \mathbf{B}^+_{(\leqslant s)}(\nu) \tag{30}$$

$$\widetilde{\mathbf{\Psi}}_0^{-1}\Big(V_\alpha \setminus \mathbf{B}^+_{(\leqslant n-s-2)}(\nu)\Big) \subset B_\nu(\text{ind}_{\leqslant s}\ ; u_0) \tag{31}$$

Write now

$$(-v)^{\leadsto}_{[b,a]} = \widetilde{\mathbf{\Psi}}_0^{-1} \circ (-v_1)^{\leadsto}_{[\beta,\alpha]} \circ \widetilde{\mathbf{\Psi}}_1$$

$$v^{\leadsto}_{[b,a]} = \widetilde{\mathbf{\Psi}}_1^{-1} \circ (v_1)^{\leadsto}_{[\alpha,\beta]} \circ \widetilde{\mathbf{\Psi}}_0$$

and combine (28 - 31) and (26, 27) to note that (B1) and (B0) hold for v with $\delta = \nu$ and for every s. The proof of Theorem 4.3 is finished. $\square$

4.1.6. Cyclic cobordisms and condition $(\mathfrak{C}\mathcal{Y})$. The condition $(\mathfrak{C})$ and the related techniques, developed in this chapter are the basic ingredients in the proof of the main results of our work (Theorems 5.5 , 5.14). In this theorems we apply the results of the present chapter in the special situation when the parts $\partial_1 W, \partial_0 W$ of the cobordism are diffeomorphic (this cobordism is obtained as the result of cutting a manifold M along a regular value of a Morse map $M \to S^1$). The present subsection contains a version of Theorem 4.3 , adapted to this situation.

Definition 4.6.
A cyclic cobordism W is a riemannian cobordism together with an isometry $\Phi : \partial_0 W \to \partial_1 W$. △

Let $f : W \to [a, b]$ be a Morse function on a cyclic cobordism W of dimension n, v be an f-gradient.

Definition 4.7.
We say that v satisfies $(\mathfrak{C}\mathcal{Y})$ if v satisfies condition $(\mathfrak{C})$ (see Subsection 4.1.1), and, moreover, the Morse functions ϕ_1, ϕ_0 and their gradients u_1, u_0 from the definition 4.2 can be chosen so that $\phi_1 \circ \Phi = \phi_0, u_1 = \Phi_*(u_0)$. The set of all f-gradients satisfying $(\mathfrak{C}\mathcal{Y})$ will be denoted by $\mathcal{G}\mathfrak{C}\mathcal{Y}(f)$. △

Theorem 4.8.
$\mathcal{G}\mathfrak{C}\mathcal{Y}(f)$ is open and dense in $\mathcal{G}(f)$ with respect to C^0 topology. Moreover, if v_0 is any f-gradient then one can choose a C^0 small perturbation v of v_0 such that $v \in \mathcal{G}\mathfrak{C}\mathcal{Y}(f)$ and $v = v_0$ in a neighborhood of ∂W.

Proof. It goes exactly as the proof of Theorem 4.3 . To prove C^0-openness of the set $\mathcal{G}\mathfrak{C}\mathcal{Y}(f)$ we do not need to change whatever it be in the first part of the proof of Theorem 4.3 .

For the proof of C^0 density just note that in the proof of C^0 density of $(\mathfrak{C})$ (Subsection 4.1.4) we can choose $\phi_0 = \phi_1 \circ \Phi, u_1 = \Phi_*(u_0)$. □

4.2. Homological gradient descent

Now we shall deduce consequences from the condition $(\mathfrak{C})$. In this section $f : W \to [a, b]$ is a Morse function on a riemannian cobordism W of dimension n, and v is an f-gradient, satisfying condition $(\mathfrak{C})$. Recall that the condition $(\mathfrak{C})$ requires the existence of Morse functions $\phi_0 : \partial_0 W \to \mathbf{R}, \phi_1 : \partial_1 W \to \mathbf{R}$ and their gradients u_0, u_1, satisfying the conditions (B1) and (B0) from Subsection 4.1.1 There must be also an ordered Morse function $\phi : W \to [a, b]$, adjusted to (f, v) with an ordering sequence $a = a_0 < a_1 < ... < a_{n+1} = b$. We shall denote $\phi^{-1}(\lambda)$ by V_λ, so that $\partial_1 W = V_b, \partial_0 W = V_a$. From the gradients u_0, u_1 we obtain the filtrations of V_b and of V_a, namely $V_b^{\{\leqslant s\}}, V_a^{\{\leqslant s\}}$ (s ranges over integers between 0 and $n - 1$). We have proved that every f-gradient w, sufficiently C^0 close to v satisfies the condition $(\mathfrak{C})$ with respect to the same objects $\phi_0, \phi_1, u_0, u_1, \phi$ (see Subsection 4.1.2).

Theorem 4.9.
For every integer $s : 0 \leqslant s \leqslant n$ there is a homomorphism

$$\mathcal{H}_s(-v) : H_*(V_b^{\{\leqslant s\}}, V_b^{\{\leqslant s-1\}}) \to H_*(V_a^{\{\leqslant s\}}, V_a^{\{\leqslant s-1\}})$$

with the following properties:

1) *Let N be an oriented submanifold of V_b, contained in $V_b^{\{\leqslant s\}}$ such that $N \setminus \operatorname{Int} V_b^{\{\leqslant s-1\}}$ is compact. Then the manifold $N' = (-v)^{\rightsquigarrow}_{[b,a]}(N)$ is in $V_a^{\{\leqslant s\}}$ and $N' \setminus \operatorname{Int} V_a^{\{\leqslant s-1\}}$ is compact and the fundamental class of N' modulo $V_a^{\{\leqslant s-1\}}$ equals to $\mathcal{H}_s(-v)\big([N]\big)$.*
2) *There is an $\epsilon > 0$ such that for every f-gradient w with $\|w - v\| \leqslant \epsilon$ and every s we have: $\mathcal{H}_s(-v) = \mathcal{H}_s(-w)$.*

The proof of the theorem occupies Subsections 4.2.1, 4.2.2, 4.2.3.

4.2.1. Main idea of the proof and the definition of $\mathcal{H}_s(-v)$. We shall first give an informal description of $\mathcal{H}_s(-v)$. Let $\alpha \in H_m(V_b^{\{\leqslant s\}}, V_b^{\{\leqslant s-1\}})$. Assume for simplicity that α is represented by a closed oriented submanifold A of V_b, $\dim A = m$, contained in $V_b^{\{\leqslant s\}}$. Denote $V_{a_{s+1}}$ by V' for brevity. We descend A along the $(-v)$-trajectories from V_b to V'. The condition $(\mathfrak{C})$ implies that the whole of $V_b^{\{\leqslant s\}}$ descends to V' without hanging on critical points of f, that is, every $(-v)$-trajectory starting in $V_b^{\{\leqslant s\}}$ reaches $V_{a_{s+1}}$. Therefore $A' = (-v)^{\rightsquigarrow}_{[b,a_{s+1}]}(A)$ is a closed oriented submanifold of V'. Try now to apply the same procedure in order to descend A' to V_a. In general A' intersects the upper soles of the ascending discs $D(p,-v)$ with $\operatorname{ind} p \leqslant s$, and so $(-v)^{\rightsquigarrow}_{[a_s+1,a]}(A')$ is a submanifold of V_a, which is not closed. But note that the intersection of the upper soles of these ascending discs with A' is contained in the neighborhood $\cup_{\operatorname{ind} p \leqslant s} B_\delta(p,-v)$ of $\cup_{\operatorname{ind} p \leqslant s} D(p,-v)$, and this neighborhood is carried to $V_a^{\{\leqslant s-1\}}$ by $(-v)^{\rightsquigarrow}_{[a_{s+1},a]}$. So the fundamental class of $(-v)^{\rightsquigarrow}(A')$ can still be defined modulo $V_a^{\{\leqslant s-1\}}$, and this was our aim.

We proceed now to the precise definitions. The homomorphism $\mathcal{H}_s(-v)$ will be defined as the composition of two homomorphisms: $\mathcal{H}1_s(-v)$ and $\mathcal{H}0_s(-v)$. (Informally, $\mathcal{H}1$ corresponds to the descent from the level b to the level a_{s+1}, and $\mathcal{H}0$ corresponds to the descent from a_{s+1} to a.) We shall denote the maps $(-v)^{\rightsquigarrow}_{[b,a_{s+1}]}$ and $(-v)^{\rightsquigarrow}_{[a_{s+1},a]}$ by $(-v1)^{\rightsquigarrow}$, and respectively $(-v0)^{\rightsquigarrow}$. Every $(-v)$- trajectory starting in $V_b^{\{\leqslant s\}}$ reaches $V_{a_{s+1}}$. Therefore $V_b^{\{\leqslant s\}}$ is in the domain of $(-v1)^{\rightsquigarrow}$ and $(-v1)^{\rightsquigarrow}$ defines a homeomorphism of the pair $(V_b^{\{\leqslant s\}}, V_b^{\{\leqslant s-1\}})$ to its image, which is a pair of compact subsets of V'.

Now let A, B, C be subsets of V', satisfying the following conditions (R1) - (R4):

$$A \supset B \supset C; \quad \overline{C} \subset \operatorname{Int} B \tag{R1}$$

$$(-v1)^{\rightsquigarrow}(V_b^{\{\leqslant s\}}) \subset A, \quad (-v1)^{\rightsquigarrow}(V_b^{\{\leqslant s-1\}}) \subset B, \tag{R2}$$

$$\Big(\bigcup_{\operatorname{ind} p \leqslant s} D(p,-v)\Big) \cap V' \subset C \tag{R3}$$

$$(-v0)^{\rightsquigarrow}(A) \subset V_a^{\{\leqslant s\}}, \quad (-v0)^{\rightsquigarrow}(B) \subset V_a^{\{\leqslant s-1\}} \tag{R4}$$

A bit later we shall check that such subsets exist. Now we shall construct $\mathcal{H}_s(-v)$ assuming that such subsets exist, and we shall show that $\mathcal{H}_s(-v)$ does not depend on the choice of A, B, C satisfying (R1) - (R4).

The composition of $(-v1)^{\rightsquigarrow}$ with the inclusion defines a continuous map

$$\phi : (V_b^{\{\leqslant s\}}, V_b^{\{\leqslant s-1\}}) \to (A, B)$$

Set $\mathcal{H}1_s(-v) = \phi_*$. Consider the excision isomorphism

$$Exc : H_*(A \setminus C, B \setminus C) \xrightarrow{\approx} H_*(A, B) \tag{32}$$

The conditions (R3) and (R4) imply that the gradient descent from a_{s+1} to a defines a continuous map of pairs

$$\rho : (A \setminus C, B \setminus C) \to (V_a^{\{\leqslant s\}}, V_a^{\{\leqslant s-1\}}) \tag{33}$$

Set $\mathcal{H}0_s(-v) = \rho_* \circ (Exc)^{-1}$, and $\mathcal{H}_s(-v) = \mathcal{H}0_s(-v) \circ \mathcal{H}1_s(-v)$.

Now we check that the homomorphism $\mathcal{H}_s(-v)$ does not depend on the choice of A, B, C. Let $(A, B, C), (A', B', C')$ be two triplets satisfying (R1) - (R4); denote the corresponding homomorphisms by $\mathcal{H}, \mathcal{H}'$. It is easy to see that $A \cap A', B \cap B', C \cap C'$ satisfy again the conditions (R1) - (R4). Therefore it suffices to verify $\mathcal{H} = \mathcal{H}'$ assuming $A \supset A', B \supset B', C \supset C'$. This follows from an easy functoriality argument, which we leave to the reader.

In the following subsection we give two examples of triples (A, B, C), satisfying the conditions (R1) - (R4).

4.2.2. Sets $T_k(v), t_k(v)$.

Let k be an integer, $0 \leqslant k \leqslant n$.

Definition 4.10.
Denote by $T_k(v)$ the set of all $y \in \phi^{-1}([a, a_{k+1}])$ such that either
i) $\gamma(y, \cdot; -v)$ reaches V_a and intersects it at a point $z \in V_a^{\{\leqslant k-1\}}$ or
ii) $\lim_{t\to\infty} \gamma(y, t; -v) = p$, where $p \in S(f)$. △

One can reformulate this definition slightly. Consider the cobordism $W' = \phi^{-1}([a, a_{k+1}])$. We have the restriction $\phi \mid W' : W' \to [a, a_{k+1}]$ of the Morse function ϕ and the restriction $v \mid W'$ of its gradient v. Then $T_k(v) = T\Big(V_a^{\{\leqslant k-1\}}, v \mid W'\Big) \cup D(-v \mid W')$ (see the definition of $T(\cdot, \cdot)$ in Subsection (2.5.1)). This implies immediately that $T_k(v)$ is compact. Before proceeding to the following definition recall that we have fixed $\delta > 0$ satisfying ($\mathfrak{C}$).

Definition 4.11.
Denote by $t_k(v)$ the set of all $y \in \phi^{-1}([a, a_{k+1}])$ such that either
i) $\gamma(y, \cdot; -v)$ reaches V_a and intersects it at a point $z \in V_a^{[\leqslant k-1]}(\delta)$ or
ii) $\lim_{t\to\infty} \gamma(y, t; -v) = p$, where $p \in S(f)$. △

We have obviously $t_k(v) \subset T_k(v)$. The same argument as used for $T_k(v)$ shows that $t_k(v)$ is compact.

Lemma 4.12.
There is $\delta' > \delta$ such that v is δ'-separated and

$$\bigcup_{\mathrm{ind} p \leqslant k} D_{\delta'}(p, -v) \cap V_{a_{k+1}} \subset t_k(v) \tag{34}$$

Proof. The condition (B1) implies that

$$D_\delta(\text{ind}_{\leqslant k}\ ; v) \cap \partial_0 W \subset B_\delta(\text{ind}_{\leqslant k-1}\ ; u_0)$$

Since v is δ-separated, it is δ''-separated for some $\delta'' > \delta$ and the interval of definition of the gfn-system $\mathbf{D}(v)$ contains $]0, \delta''[$. Therefore $\{B_\theta(\text{ind}_{\leqslant k}\ ; v) \cap \partial_0 W\}_{\theta > \delta}$ is a fundamental system of neighborhoods for $D_\delta(\text{ind}_{\leqslant k}\ ; v) \cap \partial_0 W$. Thus for some $\delta' : \delta < \delta' < \delta''$ we have: $D_{\delta'}(\text{ind}_{\leqslant k}\ ; v) \cap \partial_0 W \subset B_\delta(\text{ind}_{\leqslant k-1}\ ; u_0)$. The lemma follows. $\square$

Remark 4.13.
The previous lemma implies that

$$t_k(v) \cap V_{a_{k+1}} \subset \text{Int}\ \big(T_k(v) \cap V_{a_{k+1}}\big)$$

Indeed, the inclusion $t_k(v) \subset T_k(v)$ is obvious. Further, let $z \in t_k(v) \cap V_{a_{k+1}}$. If $z \in \cup_{\text{ind} p \leqslant k} D(p, -v)$, then z is contained in

$$\Big(\bigcup_{\text{ind} p \leqslant k} B_\delta(p, -v)\Big) \cap V_{a_{k+1}}$$

which is in $t_k(v)$. If $\gamma(z, \cdot; -v)$ reaches V_a, then it intersects V_a at a point $z' \in V_a^{[\leqslant k-1]}(\delta)$, which is in the interior of $V_a^{\{\leqslant k-1\}}$, and this implies our assertion. A similar argument proves that $t_k(v) \cap \phi^{-1}\big([a, a_{k+1}[\big)$ is in Int $\big(T_k(v) \cap \phi^{-1}\big([a, a_{k+1}[\big)\big)$.

Returning to the definition of sets A, B, C from Subsection 4.2.1, set

$$\begin{cases} A = T_{s+1}(v) \cap V', \\ B = T_s(v) \cap V', \\ C = \Big(\bigcup_{\text{ind} p \leqslant s} D(p, -v)\Big) \cap V' \end{cases} \tag{35}$$

It is now obvious that this triple satisfies (R1) - (R4). Another triple satisfying the conditions is

$$\begin{cases} A = t_{s+1}(v) \cap V', \\ B = t_s(v) \cap V'; \\ C = \Big(\bigcup_{\text{ind} p \leqslant s} D(p, -v)\Big) \cap V' \end{cases} \tag{36}$$

$\triangle$

4.2.3. Properties of $\mathcal{H}_s$. *Proof of 1)*

Consider the submanifold $N'' = (-v1)^{\rightsquigarrow}(N)$ of $V' = V_{a_{s+1}}$. Since $(-v1)^{\rightsquigarrow}$ is a diffeomorphism of the open subset Int $V_b^{\{\leqslant s-1\}}$ of V_b onto its image in V', the manifold $N'' \setminus (-v1)^{\rightsquigarrow}\big(\text{Int}\ V_b^{\{\leqslant s-1\}}\big)$ is compact.

Thus the compactness of $N' \setminus \text{Int}\ V_a^{\{\leqslant s-1\}}$ follows from the next lemma.

Lemma 4.14.
Let L be a submanifold of V', such that $L \setminus (-v1)^{\rightsquigarrow}(\text{Int}\ V_b^{\{\leqslant s-1\}})$ is compact. Set $L' = (-v0)^{\rightsquigarrow}(L)$. Then $L' \setminus \text{Int}\ V_a^{\{\leqslant s-1\}}$ is compact.

Proof. The set $L \setminus \mathrm{Int}\ T_s$ is compact. Since $(-v0)^{\rightsquigarrow}$ is defined on $V' \setminus \mathrm{Int}\ T_s$, the set

$$(-v0)^{\rightsquigarrow}(L' \setminus \mathrm{Int}\ T_s) = (-v0)^{\rightsquigarrow}(L') \setminus (-v0)^{\rightsquigarrow}(\mathrm{Int}\ T_s)$$

is a compact subset in V_a. Note that $\mathrm{Int}\ V_a^{\{\leqslant s-1\}}$ is an open set containing $(-v0)^{\rightsquigarrow}(\mathrm{Int}\ T_s)$, and this implies finally that $L' \setminus \mathrm{Int}\ V_a^{\{\leqslant s-1\}}$ is compact. □

To verify that $[N'] = \mathcal{H}_s(-v)([N])$, use the definition of $\mathcal{H}_s(-v)$ with the help of A, B, C as in (35) and check through the definitions of N' and of $\mathcal{H}_s(-v)$. We leave the details to the reader.

Proof of 2)

Here it is convenient to use for the definition of $\mathcal{H}_s(-v)$ the triple (A, B, C), where

$$\begin{cases} A = T_{s+1}(v) \cap V', \\ B = T_s(v) \cap V'; \\ C = \Big(\bigcup\limits_{\mathrm{ind} p \leqslant s} B_\delta(p, -v) \Big) \cap V' \end{cases} \tag{37}$$

In the rest of this subsection we shall abbreviate $V_b^{\{\leqslant s\}}$ to R_b, $V_b^{\{\leqslant s-1\}}$ to Q_b, $V_a^{\{\leqslant s\}}$ to R_a, $V_a^{\{\leqslant s-1\}}$ to Q_a. First we shall show that the triple (A, B, C) defined in (37) satisfies the conditions (R1) - (R4) not only for v but also for every ϕ-gradient w sufficiently C^0 close to v. Of course it is true for (R1), since it does not depend on the gradient.

Proceeding to (R2) recall that (by Remark 4.13) $t_{s+1}(v) \cap V' \subset \mathrm{Int}\ A$, and $t_s(v) \cap V' \subset \mathrm{Int}\ B$. Therefore the pair of compacts

$$\Big((-v1)^{\rightsquigarrow}(R_b), (-v1)^{\rightsquigarrow}(Q_b) \Big)$$

is in $(\mathrm{Int}\ A, \mathrm{Int}\ B)$. Now it follows from Corollary 2.71 that there is $\epsilon > 0$ such that for every vector field w on $\phi^{-1}([a_{s+1}, b])$ with $\|w - v\| < \epsilon$ each $(-w)$-trajectory starting in $V_b^{\{\leqslant s\}}$ reaches V' and

$$\Big((-w)^{\rightsquigarrow}(R_b), (-w)^{\rightsquigarrow}(Q_b) \Big) \subset (\mathrm{Int}\ A, \mathrm{Int}\ B) \tag{38}$$

Therefore for every ϕ-gradient w with $\|w - v\| < \epsilon$ the condition (R2) is true with respect to w.

Since C is open, the property (R3) with respect to w is true if w is sufficiently close to v by Proposition 2.75 . Proceeding to (R4) note that

$$(-v0)^{\rightsquigarrow}(A) \subset V_a^{[\leqslant s]}(\delta) \subset \mathrm{Int}\ V_a^{\{\leqslant s\}}$$

So each v-trajectory starting in $V_a^{\{\geqslant s\}}$ reaches V' and intersects it at a point outside A. Then the same is true for every w-trajectory starting in $V_a^{\{\geqslant s\}}$ if only w is sufficiently C^0 close to v. This implies the first half of the property (R4) with respect to w. Similarly we verify the second half.

Thus we can use the triple (A, B, C) defined in (37) for the definition of $\mathcal{H}_s(-w)$. It is left to show that $\mathcal{H}_s(-w) = \mathcal{H}_s(-v)$ for all w sufficiently C^0 close to v.

Lemma 4.15.

For every ϕ-gradient w sufficiently C^0-close to v we have: $\mathcal{H}1_s(-w) = \mathcal{H}1_s(-v)$.

Proof. Apply the Corollary 2.72 to the cobordism $\phi^{-1}([a_{s+1}, b])$, and the vector field $(-v)$. □

Lemma 4.16.
For every ϕ-gradient w sufficiently C^0-close to v we have: $\mathcal{H}0_s(-w) = \mathcal{H}0_s(-v)$.

Proof. Recall that $A \setminus C$ is in the domain of definition of $(-v)^{\leadsto}_{[a_{s+1},a]}$. Moreover, the condition $(\mathfrak{C})$ implies that $(-v)^{\leadsto}_{[a_{s+1},a]}$ sends the pair of compacts $(A \setminus C, B \setminus C)$ to the pair $\big(V_a^{[\leqslant s]}(\delta), V_a^{[\leqslant s-1]}(\delta)\big)$ which is in the pair of open sets $(\mathrm{Int}\ R_a, \mathrm{Int}\ Q_a)$. Therefore (by Corollary 2.72) for every w sufficiently C^0 close to v the map $(-w)^{\leadsto}_{[a_{s+1},a]}$ is defined on $A \setminus C$, sends the pair $(A \setminus C, B \setminus C)$ to $(\mathrm{Int}\ R_a, \mathrm{Int}\ Q_a)$, and is homotopic to $(-v)^{\leadsto}_{[a_{s+1},a]}$ as a map of these pairs. □

4.2.4. A pairing between $C_s(u_1)$ and $C_s(v)$.

Here we present a construction related to the homological gradient descent homomorphism. This construction will be used in Chapter 4 for computation of Novikov incidence coefficients.

Let $0 \leqslant s \leqslant n-1$. As in the previous Subsection we denote $\phi^{-1}(a_{s+1})$ by V'. Every $(-v)$-trajectory starting at $C_\delta(\mathrm{ind}_{\leqslant s}\ ; u_1)$ reaches V'. Therefore the map $(-v)^{\leadsto}_{[b,a_{s+1}]}$ defines a homeomorphism of the pair

$$\big(\mathrm{Int}\ C_\delta(\mathrm{ind}_{\leqslant s}\ ; u_1), \quad \mathrm{Int}\ C_\delta(\mathrm{ind}_{\leqslant s-1}\ ; u_1)\big)$$

to the pair (N_s, N_{s-1}) of open subsets of V'. Let τ be the restriction of $(-v)^{\leadsto}_{[b,a_{s+1}]}$ to the pair $((\partial_1 W)^{\{\leqslant s\}}, (\partial_1 W)^{\{\leqslant s-1\}})$. Then $\mathrm{Im}\ \tau$ is a compact subpair $(n_s, n_{s-1}) \subset (N_s, N_{s-1})$. Let $q \in S_s(f)$. Consider the submanifold $S(q) = D(q, -v) \cap V'$ of V'; it is a $(n-s-1)$-dimensional cooriented embedded sphere. Denote by $]S_q[$ the cohomology class in $H^s(V', V' \setminus S(q))$, dual to $S(q)$. We have: $S(q) \cap n_{s-1} = \emptyset$ (this follows from $(\mathfrak{C})$). Therefore the pair (N_s, N_{s-1}) is a subpair of $(V', V' \setminus S(q))$ and we can induce the class $]S_q[$ to the cohomology of (N_s, N_{s-1}). The resulting cohomology class will be denoted again by $]S_q[$. For $x \in C_s(u_1) = H_s((\partial_1 W)^{\{\leqslant s\}}, (\partial_1 W)^{\{\leqslant s-1\}})$ set

$$\langle q, x \rangle_v = \langle]S_q[, \tau_*(x) \rangle$$

(The index v reminds that the pairing depends on the vector field v.) This pairing has some natural properties, gathered in the proposition 4.17 .

Proposition 4.17.

1. *Let L be an s-dimensional oriented submanifold of $\partial_1 W$, such that $L \subset (\partial_1 W)^{\{\leqslant s\}}$ and $L \setminus \mathrm{Int}\ (\partial_1 W)^{\{\leqslant s-1\}}$ is compact. Then $L' = (-v)^{\leadsto}_{[b,a_{s+1}]}(L)$ is an s-dimensional oriented submanifold of V', such that $L' \subset n_s$ and $L' \setminus \mathrm{Int}\ n_{s-1}$ is compact. If $L' \pitchfork S(q)$ then*

$$L' \# S(q) = \langle q, [L] \rangle_v$$

2. *Let U be a neighborhood of $S(f)$. There is $\delta > 0$, such that for every ϕ-gradient w with $v|U = w|U$ and $\|v - w\| < \delta$ we have: $\langle q, x \rangle_v = \langle q, x \rangle_w$*

Proof. 1) is obvious. Proceeding to (2), note first that the objects $N_s, N_{s-1}, n_s, n_{s-1}, \tau, S_q$ etc., introduced in the beginning of the present subsection depend on v, and we shall now show it in the notation, writing, respectively $N_s(v), N_{s-1}(v), n_s(v), n_{s-1}(v), \tau(v), S_q(v)$ etc. in order to make explicit the dependence on the gradient.

Recall that if w is sufficiently C^0 close to v then w still satisfies ($\mathfrak{C}$) with respect to the same string $\mathcal{S} = (\delta, u_0, u_1, \phi_0, \phi_1, \phi)$. In particular, every $(-w)$-trajectory starting in $(\partial_1 W)^{\{\leqslant s\}}$ reaches V'. It follows easily that if only w is sufficiently close to v, we have: $n_s(w) \subset N_s(v), n_{s-1}(w) \subset N_{s-1}(v)$.

The compacts $n_{s-1}(v), S(q,v)$ are disjoint subsets of V'. Therefore there are disjoint open subsets Λ, Γ of V' with $n_{s-1}(v) \subset \Lambda, S(q,v) \subset \Gamma$. We have the following inclusions:

$$(39) \qquad \big(n_s(v), n_{s-1}(v)\big) \xhookrightarrow{i(v)} (V', \Lambda) \xhookrightarrow{j} (V', V' \setminus \Gamma) \xhookrightarrow{k(v)} (V', V' \setminus S(q,v))$$

Note that, obviously

$$(40) \qquad \langle p, x\rangle_v = \Big\langle \big(k(v)\big)^* \big(]S_q(v)[\big), \big(j \circ i(v)\big)_* \big(\tau_*(v)(x)\big)\Big\rangle$$

If w is sufficiently C^0 close to v, then still $n_{s-1}(w) \subset \Lambda; , S(q,w) \subset \Gamma$ and the formula (40) still holds if we replace v by w. Therefore it suffices to show that for w sufficiently C^0 close to v we have

$$(41) \qquad i(v) \circ \tau(v) \sim i(w) \circ \tau(w), \quad \big(k(v)\big)^* \big(]S_q(v)[\big) = \big(k(w)\big)^* \big(]S_q(w)[\big)$$

The first follows from Corollary 2.72 . To prove the second, note that the class

$$\big(k(v)\big)^* \big(]S_q(v)[\big) \in H^s(V', V' \setminus \Gamma)$$

is the Lefschetz-dual to the fundamental class $[S_q(v)] \in H_{n-s-1}(\Gamma)$. Therefore it suffices to prove that if $v|U = w|U$ and w is sufficiently close to v in C^0-topologythen the fundamental classes of $S_q(v)$ and of $S_q(w)$ in $H_*(\Gamma)$ are equal. Now just apply Proposition 2.76 . ☐

4.2.5. Homological gradient descent: equivariant version. Let $p : \widehat{W} \to W$ be a regular covering of W with structure group H. For a subset $A \subset W$ we denote by $\widehat{A}$ the set $p^{-1}(A)$. We obtain thus an H-invariant filtration $\widehat{V}_b^{\{\leqslant s\}}$ of $\widehat{V}_b$ and an H-invariant filtration $\widehat{V}_a^{\{\leqslant s\}}$ of $\widehat{V}_a$. Recall from Section 2.1 the notation $K^+ = \cup_{x \in S(f)} D(x, -v), K^- = \cup_{x \in S(f)} D(x, v)$. For $x \in \widehat{\partial_1 W} \setminus \widehat{K}^+$ consider the $(-v)$-trajectory $\gamma(q(x), t; -v)$. It reaches $\partial_0 W$, and the lifting $\widehat{\gamma}$ of γ to $\widehat{W}$ is an integral curve of $-p^*v$. This integral curve reaches $\widehat{\partial_0 W}$, and the intersection point of $\widehat{\gamma}$ with $\widehat{\partial_0 W}$ will be denoted by $(-\widehat{v})^{\rightsquigarrow}_{[b,a]}(x)$. We obtain thus a diffeomorphism

$$(-\widehat{v})^{\rightsquigarrow}_{[b,a]}(x) : \widehat{\partial_1 W} \setminus \widehat{K}^+ \longrightarrow \widehat{\partial_0 W} \setminus \widehat{K}^-$$

which commutes with the action of H.

The homology $H_*\big((\widehat{\partial_1 W})^{\{\leqslant s\}}, (\widehat{\partial_1 W})^{\{\leqslant s-1\}}\big)$ is a right $\mathbf{Z}H$-module, and Proposition 2.41 implies that it is a free $\mathbf{Z}H$-module with the base formed by liftings of descending discs of critical points of ϕ_1 of index s. The following theorem is an "equivariant version" of theorem 4.9 . The proof is a direct generalization of the proof of Theorem 4.9 and we leave it to the reader.

Theorem 4.18.

For every integer $s : 0 \leqslant s \leqslant n$ there is a homomorphism

$$\widehat{\mathcal{H}}_s(-v) : H_*\big(\widehat{V}_b^{\{\leqslant s\}}, \widehat{V}_b^{\{\leqslant s-1\}}\big) \to H_*\big(\widehat{V}_a^{\{\leqslant s\}}, \widehat{V}_a^{\{\leqslant s-1\}}\big)$$

with the following properties:

1) *Let N be an oriented submanifold of $\widehat{V}_b$, contained in $\widehat{V}_b^{\{\leqslant s\}}$ and such that $N \setminus$ $\mathrm{Int}\ \widehat{V}_b^{\{\leqslant s-1\}}$ is compact. Then the manifold $N' = (-\widehat{v})^{\leadsto}_{[b,a]}(N)$ is in $\widehat{V}_a^{\{\leqslant s\}}$ and $N' \setminus \mathrm{Int}\ \widehat{V}_a^{\{\leqslant s-1\}}$ is compact and the fundamental class of N' modulo $\widehat{V}_a^{\{\leqslant s-1\}}$ equals to $\widehat{\mathcal{H}}_s(-v)([N])$.*
2) *There is an $\epsilon > 0$ such that for every f-gradient w with $\|w - v\| \leqslant \epsilon$ and every s we have: $\widehat{\mathcal{H}}_s(-v) = \widehat{\mathcal{H}}_s(-w)$.* $\square$

We proceed to the equivariant analogs of the results of Subsection 4.2.4. Let $0 \leqslant s \leqslant n-1$. We assume the terminology of Subsection 4.2.4. We denote $H_s\big((\widehat{\partial_1 W})^{\{\leqslant s\}}, (\widehat{\partial_1 W})^{\{\leqslant s-1\}}\big)$ by $\widehat{C}_s(u_1)$. It is a free right $\mathbf{Z}H$-module. There is an H-equivariant homeomorphism

$$\widehat{\tau}(v) : \big((\widehat{\partial_1 W})^{\{\leqslant s\}}, (\widehat{\partial_1 W})^{\{\leqslant s-1\}}\big) \to (\widehat{n}_s(v), \widehat{n}_{s-1}(v))$$

Let $q \in S_s(f)$ and choose a lifting $\widehat{q}$ of q to $\widehat{W}$. Then we obtain a lifting $S(\widehat{q}, v)$ of $S(q, v)$ to $\widehat{W}$, and a cohomology class $]S_{\widehat{q}}[\in H^s(\widehat{N}_s(v), \widehat{N}_{s-1}(v))$.

For $x \in H_s((\widehat{\partial_1 W})^{\{\leqslant s\}}, (\widehat{\partial_1 W})^{\{\leqslant s-1\}})$ set

$$\langle \widehat{q}, x\rangle_v = \langle]S_{\widehat{q}}[, (\widehat{\tau}(v))_*(x)\rangle$$

Thus $\langle \widehat{q}, x\rangle_v$ is an integer. If we fix $\widehat{q}$ and consider x as the variable, we obtain a homomorphism $\widehat{C}_s(u_1) \to \mathbf{Z}$.

Next we give an equivariant version of Proposition 4.17 . The first two points of this proposition are direct generalization of the corresponding points of Proposition 4.17 . The point (3) is obvious, and (4) follows from the fact that for a given singular chain $y \in H_*(\widehat{n}_s(v), \widehat{n}_{s-1}(v))$ there is only finite number of $h \in H$, such that $\mathrm{supp}\ y \cap S(\widehat{q}, v) \neq 0$.

Proposition 4.19.

1. *Let L be an s-dimensional oriented submanifold of $\widehat{\partial_1 W}$, such that $L \subset (\widehat{\partial_1 W})^{\{\leqslant s\}}$ and $L \setminus \mathrm{Int}\ (\widehat{\partial_1 W})^{\{\leqslant s\}}$ is compact. Then $L' = (-\widehat{v})^{\leadsto}_{[b,a_{s+1}]}(L)$ is an s-dimensional oriented submanifold of $\widehat{V}'$, such that $L' \subset \widehat{n}_s(v)$ and $L' \setminus \mathrm{Int}\ \widehat{n}_{s-1}(v)$ is compact. If $L' \pitchfork S(\widehat{q}, v)$ then*

$$L' \# S(\widehat{q}, v) = \langle \widehat{q}, [L]\rangle_v.$$

2. *Let U be a neighborhood of $S(f)$. There is $\delta > 0$, such that for every ϕ-gradient w with $v|U = w|U$ and $\|v - w\| < \delta$ we have: $\langle \widehat{q}, x\rangle_v = \langle \widehat{q}, x\rangle_w$.*
3. *Let $h \in H$. Then $\langle \widehat{q}h, xh\rangle_v = \langle \widehat{q}, x\rangle_v$.*
4. *For every x the set of $h \in H$ such that $\langle \widehat{q}h, x\rangle_v \neq 0$ is finite.*

$\square$

CHAPTER 5

C^0-generic rationality of the boundary operators in Novikov Complex

In this chapter we prove the theorems which are the aim of our work. We refer to the corresponding parts of the Introduction (1.2.3 and 1.2.4) for the exposition of the main ideas of the proof and we proceed directly to the precise definitions and statements of results.

5.1. Basic terminology

In the Subsections 5.1.1, 5.1.2 we recall the basic definitions concerning algebraic (Subsection 5.1.1) and geometric (Subsection 5.1.2) framework of Novikov theory, so that our exposition is self-contained at least in what concerns the terminology.

5.1.1. Novikov rings. We shall denote the integer Laurent polynomial ring $\mathbf{Z}[t, t^{-1}]$ by L. The very first example of Novikov ring is the ring $\mathbf{Z}((t))$, which consists by definition of all the Laurent series $\lambda = \sum_{-\infty}^{\infty} a_n t^n$ with integer coefficients and a finite negative part (so that $\lambda \in \mathbf{Z}((t))$ if $\forall n : a_n \in \mathbf{Z}$ and there is $N = N(\lambda)$ such that $a_n = 0$ for $n < N(\lambda)$). We shall often denote $\mathbf{Z}((t))$ by $\widehat{L}$. The localization of L with respect to to the set of all the polynomials of the form $1 + tP(t)$, where P is a polynomial, is denoted by $\widetilde{L}$, we have: $L \subset \widetilde{L} \subset \widehat{L}$.

To consider "equivariant versions" of the Novikov Complex one needs to introduce some more complicated rings. Let G be a group, $\xi : G \to \mathbf{R}$ be a group homomorphism. Set $\Lambda = \mathbf{Z}G$, and denote by $\widehat{\widehat{\Lambda}}$ the abelian group of all the formal linear combinations of the form $\sum_{g \in G} n_g g$ (infinite in general). For $\lambda \in \widehat{\widehat{\Lambda}}, \lambda = \sum_{g \in G} n_g g$ set as usual $\operatorname{supp} \lambda = \{g \in G \mid n_g \neq 0\}$. Consider the subset $\widehat{\Lambda}_\xi \subset \widehat{\widehat{\Lambda}}$ defined by

$$\widehat{\Lambda}_\xi = \{\lambda \in \widehat{\widehat{L}} \mid \forall c \in \mathbf{R} \text{ the set } \operatorname{supp} \lambda \cap \xi^{-1}([c, \infty[) \text{ is finite}\}$$

It is not difficult to see that $\widehat{\Lambda}_\xi$ has a natural structure of a ring so that the embedding $\Lambda \hookrightarrow \widehat{\Lambda}_\xi$ is a homomorphism.[1] The most important for us is the case of a group epimorphism $\xi : G \to \mathbf{Z}$. In this case set $H = \operatorname{Ker} \xi$. For $n \in \mathbf{Z}$ denote $\xi^{-1}(n)$ by $G_{(n)}$ and $\{x \in \mathbf{Z}G \mid \operatorname{supp} x \subset G_{(n)}\}$ by $\mathbf{Z}G_{(n)}$. Denote $\xi^{-1}(]-\infty, -1])$ by G_- and $\{x \in \mathbf{Z}G \mid \operatorname{supp} x \subset G_-\}$ by $\mathbf{Z}G_-$. Choose $\theta \in G_{(-1)}$. It is easy to see that

[1] In all my earlier papers on this subject I used the notation Λ_ξ^- for this ring in order to stress that the power series $\lambda \in \Lambda_\xi^-$ are allowed to have the terms with $\xi(g)$ an arbitrary large *negative* number. Now it seems to me more important to stress that the definition of the ring uses the completion procedure. The symbol $\widehat{}$ is included into the notation of the ring in the original paper [**23**], and in many other papers, see e.g. [**44**].

$\widehat{\mathbf{Z}G_\xi}$ is identified with the ring of power series of the form $\{a_{-n}\theta^n + ... + a_1\theta + ... \mid a_i \in \mathbf{Z}H\}$.

5.1.2. Morse maps $M \to S^1$. The circle S^1 is for us the quotient $\mathbf{R}/\mathbf{Z}$. Let M be a closed connected manifold. A *Morse map* $f : M \to S^1$ is a C^∞ map, such that that all critical points of f are non-degenerate. As in

Subsection 2.1.1 the set of all critical points of f is denoted by $S(f)$, and the set of all critical points of index k is denoted by $S_k(f)$.

We leave to the reader to formulate the definition of f-gradient of a Morse map (following the definition 2.3 and Remark 2.4). The set of all gradients of f is denoted by $\mathcal{G}(f)$. Since M is closed, the integral curves of vector fields on M are defined on the whole of $\mathbf{R}$. For $p \in S(f)$ the stable manifold $D(p, v)$ and the unstable manifold $D(p, -v)$ are defined. These are immersed smooth manifolds in M. We have the corresponding notions of gradients satisfying Transversality Condition and, respectively, Almost Transversality Condition. The set of all f-gradients satisfying Transversality Condition, resp. Almost Transversality Condition, will be denoted by $\mathcal{GT}(f)$, resp. by $\mathcal{GA}(f)$. (Warning: In our present setting there are no analogs of Lemma 2.18 Proposition 2.29 , mainly by the reason that the v-trajectories can pass infinitely many times more and more closely to the initial position.)

The next lemma implies that $\mathcal{GT}(f)$ is C^0 dense in $\mathcal{G}(f)$. This is a version of Kupka-Smale theorem. (It would be more precise to say: "a version of a result, weaker than Kupka-Smale theorem", since we make no assertions concerning the closed orbits of the considered vector field.) The proof of Kupka-Smale theorem given in [**39**] works in our case with only minor modifications. We leave the details to the reader.

Lemma 5.1.
Let $v \in \mathcal{G}(f)$. Let U be a neighborhood of $S(f)$, $\mathcal{V}$ be a neighborhood of v in $\mathcal{G}(f)$ with respect to C^∞ topology. Then there is an f-gradient v_0, such that $v_0 \in \mathcal{GT}(f), v_0 \in \mathcal{V}, \mathrm{supp}\ (v - v_0) \subset U$, and there is a neighborhood U_0 of $S(f)$, such that $v_0|U_0 = v|U_0$. □

We shall assume that f is not homotopic to 0 (otherwise f lifts to a Morse function $M \to \mathbf{R}$ and the situation is the same as considered previously in Chapter 1). There is a unique connected infinite cyclic covering $\mathcal{C} : \bar{M} \to M$ such that $f \circ \mathcal{C}$ is homotopy to zero. The map f lifts to a Morse function $F : \bar{M} \to \mathbf{R}$. Let t be the generator of the structure group ($\approx \mathbf{Z}$) of $\mathcal{C}$, such that for every x we have: $F(xt) < F(x)$. (We invite the reader to look at the figure on the page 100 where one can visualize the majority of the constructions of this subsection.) We shall assume that f is primitive, that is $f_* : H_1(M) \to H_1(S^1)$ is epimorphic (one can always reduce the situation to this case by considering the map F/n with some $n \in \mathbf{N}$.) In this case we have: $F(xt) = F(x) - 1$. To simplify the notation we shall assume that $1 \in S^1$ is a regular value of f, and therefore every $n \in \mathbf{N}$ is a regular value of F. Denote $f^{-1}(1)$ by V. Set $V_\alpha = F^{-1}(\alpha)$, $W = F^{-1}([0, 1])$, $V^- = F^{-1}(] - \infty, 1])$. The cobordism W can be thought of as the result of cutting M along V. Note that $V_\alpha t = V_{\alpha - 1}$. Denote Wt^s by W_s; then $\bar{M}$ is the union $\cup_{s \in \mathbf{Z}} W_s$, the neighbor copies W_s and W_{s+1} intersecting by V_{-s}. For any $n \in \mathbf{Z}$ the restriction of $\mathcal{C}$ to V_n is a diffeomorphism $V_n \to V$. Endow M with an arbitrary riemannian metric and lift it to a t-invariant riemannian metric on $\bar{M}$. Now W is a riemannian cobordism, and actually it is a cyclic cobordism with respect to the isometry $t^{-1}: \partial_0 W = V_0 \to \partial_1 W = V_1$. The $\mathcal{C}$-preimage of a subset $A \subset M$ will be denoted by $\bar{A}$. On the other hand, if $x \in M$ we shall reserve the symbol $\bar{x}$ for liftings of x to $\bar{M}$ (that is $\bar{x} \in \bar{M}$ and $\mathcal{C}(\bar{x}) = x$). Let $v \in \mathcal{G}(f)$. Lift v to $\bar{M}$; we obtain a complete vector field on $\bar{M}$, which is obviously an F-gradient. To restriction of this F-gradient to any cobordism $W_{[\lambda,\mu]} = F^{-1}([\lambda,\mu])$ (where λ, μ are regular values of F) is an $(F \mid W_{[\lambda,\mu]})$-gradient. We shall denote this F-gradient by the same symbol v, since there is no possibility of confusion. Note that if v satisfies Transversality Condition, then the F-gradient $v \mid W_{[\lambda,\mu]}$ satisfies Transversality Condition.

We shall consider "equivariant" versions of Novikov Complex; that requires considering one more covering (which can be, for example, universal covering). Let $\mathcal{P}: \widehat{M} \to M$ be a regular covering with structure group G. Assume that $f \circ \mathcal{P}$ is homotopic to zero; then the covering $\mathcal{P}$ is factored through $\mathcal{C}$ as follows:

$$\begin{tikzcd} \widehat{M} \arrow[r, "p"] \arrow[dr, "\mathcal{P}"'] & \bar{M} \arrow[d, "\mathcal{C}"] \\ & M \end{tikzcd} \tag{42}$$

Therefore there is a lifting of $f: M \to S^1$ to a Morse function $\widehat{F} = \bar{F} \circ p: \widehat{M} \to \mathbf{R}$. Further, the homomorphism $f_*: \pi_1 M \to \mathbf{Z}$ can be factored through a homomorphism $\xi: G \to \mathbf{Z}$. Recall that we assume that ξ is epimorphic. So, applying the terminology of Subsection 5.1.1, $H = \operatorname{Ker}\xi, G_- = \xi^{-1}(]-\infty, 0])$.

5.1.3. Novikov Complex: the definition. We assume the terminology of Subsection 5.1.2. Let $v \in \mathcal{GT}(f)$. For each point $x \in S(f)$ choose a lifting $\bar{x}$ of x to $\bar{M}$. Moreover, for each point $x \in S(f)$ choose an orientation of the local stable manifold $D(x, v)$, and lift it to an orientation of the corresponding local stable manifold of $\bar{x} \in S(F)$.

These choices made, we can define the Novikov Complex. The rth term of this complex is defined to be the free $\mathbf{Z}((t))$-module generated by $S_r(f)$. For $x, y \in S(f)$ satisfying $\operatorname{ind} x = \operatorname{ind} y + 1$ define am element $n(x, y) \in \mathbf{Z}((t))$ called *Novikov incidence coefficient* as follows: For $k \in \mathbf{Z}$ consider the set $\Gamma_k(\bar{x}, \bar{y})$ of all v-orbits joining $\bar{x}$ with $\bar{y}t^k$. This is a finite set and for every $\gamma \in \Gamma_k(\bar{x}, \bar{y})$ the orientation sign $\varepsilon(\gamma) = \pm 1$ is defined. (Indeed, choose $\lambda, \mu \in \mathbf{R}$ so that $\bar{x}, \bar{y}t^k \in F^{-1}([\lambda, \mu])$; then all the trajectories of $\Gamma_k(\bar{x}, \bar{y})$ are in the cobordism $F^{-1}([\lambda, \mu])$ and we can apply the argument from the page 37). Set

$$n_k(\bar{x}, \bar{y}) = \sum_{\gamma \in \Gamma_k(x,y)} \varepsilon(\gamma)$$

and

$$n(x, y) = \sum_k n_k(\bar{x}, \bar{y}) \cdot t^k$$

Now the boundary operator is defined on the generators of C_r as follows:

$$\partial_r x = \sum_{y \in S_{r-1}(f)} y \cdot n(x, y)$$

Since all $n_k(x,y)$ depend on v we shall sometimes include v in the notation, writing $n_k(\bar{x},\bar{y};v)$, $n(\bar{x},\bar{y};v)$, and $C_*(v)$. The dependence on the particular choice of liftings of critical points to $\bar{M}$ is less important, and we shall often write for example $n(x,y)$ instead of $n(\bar{x},\bar{y})$. Note also that the resulting complex does not depend on the particular choice of the function f for which v is gradient.

The next lemma says that $C_*(v)$ is indeed a chain complex.

Lemma 5.2.
For every r we have: $\partial_r \circ \partial_{r+1} = 0$

Proof. We shall assume that for every $x \in S(f)$ the lifting $\bar{x}$ is in W_0. Then for every $x, y \in S(f)$ the Novikov incidence coefficient $n(\bar{x},\bar{y};v)$ is actually an element of $\mathbf{Z}[[t]]$. Denote by $\partial_r^{(n)}$ the quotient homomorphism

$$C_r/t^n C_r \to C_{r-1}/t^n C_{r-1}$$

It suffices to prove that $\partial_r^{(n)} \circ \partial_{r+1}^{(n)} = 0$ for every n. But $\partial_r^{(n)}$ is easily identified with the boundary operator in the Morse complex corresponding to the cobordism $F^{-1}([-n,0])$. □

The main aim of the present paper is to investigate the properties of the incidence coefficients $n(x,y;v)$, but not the chain homotopy properties of $C_*(v)$. We hope to give the systematic treatment of these properties in the second part of this work. So we shall leave the next proposition without proof. (This proposition is proved in [**28**] for the case when the f-gradient v is an f-Gradient. The proof is carried over to the general case without difficulties.)

Proposition 5.3.

$$H_*(C_*(v)) \approx H_*(\bar{M}) \underset{\mathbf{Z}[t,t^{-1}]}{\otimes} \mathbf{Z}((t))$$

□

5.2. Condition $(\mathfrak{C}\mathcal{C})$, and C^0-generic rationality of Novikov incidence coefficients

We assume the terminology of 5.1.2.

We say that v satisfies condition $(\mathfrak{C}\mathcal{C})$ ($\mathfrak{C}$ for *cellular* and $\mathcal{C}$ for *circle*) if the $(F|W)$-gradient v satisfies the condition $(\mathfrak{C}\mathcal{Y})$ from Subsection 4.1.6. The set of f-gradients v satisfying $(\mathfrak{C}\mathcal{C})$ will be denoted by $\mathcal{G}\mathfrak{C}\mathcal{C}(f)$. First of all we establish that the properties $(\mathfrak{C}\mathcal{C})$ and $(\mathfrak{C}\mathcal{C}\mathcal{T})$ are C^0-generic.

Proposition 5.4.

1. *The set $\mathcal{G}\mathfrak{C}\mathcal{C}(f)$ is open and dense in $\mathcal{G}(f)$ with respect to C^0-topology. The set $\mathcal{G}\mathfrak{C}\mathcal{C}\mathcal{T}(f)$ is open and dense in $\mathcal{G}\mathcal{T}(f)$ with respect to C^0-topology.*
2. *Let $v \in \mathcal{G}(f)$, let $\mathcal{V}$ be a C^0-neighborhood of v in $\mathcal{G}(f)$. Then there is $v_0 \in \mathcal{G}\mathfrak{C}\mathcal{C}(f)$, such that $v_0 \in \mathcal{V}$ and $v = v_0$ in some neighborhood of $S(f)$.*
3. *Let $v \in \mathcal{G}\mathcal{T}(f)$, let $\mathcal{V}$ be a C^0-neighborhood of v in $\mathcal{G}(f)$. Ten there is $v_0 \in \mathcal{G}\mathfrak{C}\mathcal{C}(f)$, such that $v_0 \in \mathcal{V}$ and $v = v_0$ in some neighborhood of $S(f)$.*

Proof. It suffices to prove 2) and 3). 2) follows immediately from Theorem 4.8 . To prove 3) let v_0 be an f-gradient such that $v_0 \in \mathcal{G}\mathfrak{C}\mathcal{C}(f) \cap \mathcal{V}$ and $v_0 = v$ in a neighborhood of $S(f)$ (such v_0 exists by 2)). Now to obtain an f-gradient satisfying the assertion 3), apply Lemma 5.1 , and recall that $\mathcal{G}\mathfrak{C}\mathcal{C}(f) \cap \mathcal{V}$ is open. $\square$

Therefore $\mathcal{G}\mathfrak{C}\mathcal{C}\mathcal{T}(f) = \mathcal{G}\mathfrak{C}\mathcal{C}(f) \cap \mathcal{G}\mathcal{T}(f)$ is C^0-open-and-dense in $\mathcal{G}\mathcal{T}(f)$.

Theorem 5.5.

Let $v \in \mathcal{G}\mathfrak{C}\mathcal{C}\mathcal{T}(f)$. Let $x, y \in S(f)$, $\operatorname{ind} x = \operatorname{ind} y + 1$. Then $n(x, y; v) \in \widetilde{L}$, that is $n(x, y; v) = \frac{P(t)}{t^m Q(t)}$ where $m \in \mathbf{N}$ and P, Q are polynomials with integer coefficients and $Q(0) = 1$.

Proof. Let $v \in \mathcal{G}\mathfrak{C}\mathcal{C}(f)$. The condition $(\mathfrak{C}\mathcal{Y})$ provides a Morse function ϕ_1 on V_1 together with its gradient u_1, and a Morse function $\phi_0 = \phi_1 \circ t^{-1}$ on V_0 together with its gradient $u_0 = (t)_*(u_1)$. For every $k \in \mathbf{Z}$ we obtain also a Morse function $\phi_0 \circ t^{-k}$ on V_{-k} and a $\phi_0 \circ t^{-k}$-gradient $u_k = (t^k)_*(u_0)$. We have also an ordered Morse function $\phi : W \to [0, 1]$, adjusted to (f, v), with an ordered sequence $(a_0, ..., a_{n+1})$. Note that $v \mid W_k$ satisfies $(\mathfrak{C})$ with respect to the string $\mathcal{S}_k = \big(\delta, (t^k)_*(u_0), (t^k)_*(u_1), \phi_0 \circ t^{-k}, \phi_1 \circ t^{-k}\big)$.

Let $x, y \in S(f), \operatorname{ind} x = s + 1, \operatorname{ind} y = s$. Set $V' = \phi^{-1}(a_{s+1})$. We can assume that $\bar{x} \in W, \bar{y} \in W_1$. Then $n_k(x, y; v) = 0$ for $k \leqslant 0$. Consider the oriented submanifold $X = D(\bar{x}, v) \cap V_0$ of V_0.

Note that $X \subset V_0^{\{\leqslant s\}}$ and that $X \setminus \operatorname{Int} V_0^{\{\leqslant s-1\}}$ is compact. (Indeed, let $\Sigma = D(\bar{x}, v) \cap V'$. Then Σ is an embedded s-dimensional sphere in V'. Using the terminology of Subsection 4.2.1 write $X = (-v0)^{\rightsquigarrow}(\Sigma)$. Then the compactness of $X \setminus \operatorname{Int} V_0^{\{\leqslant s-1\}}$ follows from Lemma 4.14). Denote by $[X]$ the homology class of X modulo $V_0^{\{\leqslant s-1\}}$.

Applying Theorem 4.9 1) several times we deduce by induction that for every k the submanifold $X_k = (-v)^{\rightsquigarrow}_{[0,-k]}(X)$ of V_{-k} is in $V_{-k}^{\{\leqslant s\}}$ and $X_k \setminus \operatorname{Int} (V_{-k})^{\{\leqslant s-1\}}$ is compact. Set $V'_{-(k+1)} = V' \cdot t^{k+1}$, and $X'_k = (-v)^{\rightsquigarrow}_{[-k,-k-1+a_{s+1}]}(X_k)$.

Set $Y_k = D(\bar{y}t^k, -v) \cap V'_{-k}$; this is a cooriented embedded $(n - s - 1)$-sphere. Since v satisfies the Transversality Condition we have: $X'_k \pitchfork Y_k$. The proposition 4.17 implies that

$$n_k(x, y; v) = X'_k \# Y_k = \langle \bar{y}t^k, [X_k] \rangle_v \tag{43}$$

Recall from Section 4.2 the homomorphism

$$\mathcal{H}_s(-v) : H_s(V_1^{\{\leqslant s\}}, V_1^{\{\leqslant s-1\}}) \to H_s(V_0^{\{\leqslant s\}}, V_0^{\{\leqslant s-1\}})$$

associated with the cobordism W and and set $h = \mathcal{H}_s(-v) \circ (t_*^{-1})$. Then h is an endomorphism of $H_s(V_0^{\{\leqslant s\}}, V_0^{\{\leqslant s-1\}})$.

Apply Theorem 4.9 to obtain:

$$n_k(x, y; v) = \langle \bar{y}, h^k\left([X]\right) \rangle_v$$

Now the proof is finished by the application of the next lemma.

Lemma 5.6.

Let G be a finitely generated abelian group, A be an endomorphism of G and $\lambda : G \to \mathbf{Z}$ be a homomorphism. Then for every $p \in G$ the series $\sum_{k \geqslant 0} \lambda(A^k p) t^k \in \mathbf{Z}[[t]]$ is a rational function of t of the form $\frac{P(t)}{Q(t)}$, where P, Q are polynomials with integral coefficients and $Q(0) = 1$.

Proof. It is sufficient to consider the case of free finitely generated abelian group G. Consider a free finitely generated $\mathbf{Z}[[t]]$-module $R = G[[t]]$ and a homomorphism $\phi : R \to R$, given by $\phi = 1 - At$. Then ϕ is invertible, the inverse homomorphism is given by the formula $\phi^{-1} = \sum_{k \geqslant 0} A^k t^k$. On the other hand the inverse of ϕ is given by Cramer formulas, which imply that each matrix entry of the homomorphism, inverse to ϕ is the ratio of two integer polynomials. Moreover the denominator equals to det $(1 - At)$. □ □

The next result deals with continuity properties of $n(x, y; v)$ with respect to small perturbations of v.

Theorem 5.7.
Let $v \in \mathcal{G}\mathfrak{C}\mathcal{T}(f)$. Let U be an open neighborhood of $S(f)$. Then there is $\delta > 0$ such that for every $w \in \mathcal{G}\mathfrak{C}\mathcal{T}(f)$ with $\|w - v\| < \delta$ and $v \mid U = w \mid U$, and for every $x, y \in S(f)$ with $\mathrm{ind} x = \mathrm{ind} y + 1$ we have: $n(x, y; v) = n(x, y; w)$

Proof. We shall use the terminology of the proof of 5.5 , but we modify it a bit. Namely, we denote $D(\bar{x}, v) \cap V_0$ by $X(v)$ in order to stress the dependence on the vector field v. We assume the similar convention for the sets X_k, Y_k, the endomorphism h, the sphere Σ etc. Thus we have:

$$n_k(x, y; v) = \left\langle \bar{y}, (h(v))^k ([X(v)]) \right\rangle_v$$

Recall from Theorem 4.8 that if $w \in \mathcal{GT}(f)$ is sufficiently C^0 close to v, then w satisfies $(\mathfrak{C}\mathcal{C})$ with respect to to the same string $S = (\delta, u_0, u_1, \phi_0, \phi_1)$. Therefore the constructions done in the proof of Theorem 5.5 apply also to w; thus we obtain the corresponding objects $X(w), X_k(w), Y_k(v)$ etc. Thus the formula (5.2) holds with w substituted instead of v.

The endomorphism $h(v) = (t^{-1})_* \circ \mathcal{H}_s(-v)$ is invariant with respect to C^0-small perturbations of v, see Theorem 4.9 . Therefore we have only to show that if $v|U = w|U$ and $\|u - w\|$ is small enough, we have: $\langle \cdot, \cdot \rangle_v = \langle \cdot, \cdot \rangle_w$ and $[X(v)] = [X(w)]$. The first property follows from Proposition 4.17 .

Proceed to the second one. We assume the terminology of Subsection 4.2.3. In particular, we use for the definition of $\mathcal{H}, \mathcal{H}0$ the triple (A, B, C) from (37). Note that $\Sigma(v) \subset \mathrm{Int}\ A$. Denote by $s(v)$ the fundamental class of $\Sigma(v)$ reduced modulo B; then $[X(v)] = \mathcal{H}0_s(-v)(s(v))$. If w is sufficiently C^0 close to v, then $\Sigma(w) \subset \mathrm{Int}\ A$ and $s(v) = s(w)$ by Proposition 2.75 and 2.76 .

Lemma 4.16 applies to show that $\mathcal{H}0_s(-v) = \mathcal{H}0_s(-w)$ and that finishes the proof. □

5.3. Incidence coefficients with values in completions of group rings

5.3.1. Definitions. In this section we deal with "equivariant" analogs of Novikov complex. Let $f : M \to S^1$ be a Morse map and $\mathcal{P} : \widehat{M} \to M$ be a regular covering satisfying $f \circ \mathcal{P} = 0$. We assume here the terminology of Section 5.1.

Let $v \in \mathcal{GT}(f)$. For every $x \in S(f)$ we choose a lifting $\widehat{x}$ of x to $\widehat{M}$, and an orientation of the descending disc $D(x, v)$. Set $\bar{x} = p(\widehat{x})$. Let $x, y \in S(f), \mathrm{ind} x = \mathrm{ind} y + 1$. The set $\Gamma(x, y)$ of all $(-v)$-trajectories joining x to y splits naturally into disjoint parts defined as follows: Let $g \in G$. We say that $\gamma \in \Gamma_g(\widehat{x}, \widehat{y})$ if the lifting of γ to $\widehat{M}$ starting at $\widehat{x}$ finishes at $\widehat{y} \cdot g$. Then $\{\Gamma_g(\widehat{x}, \widehat{y})\}_{g \in G}$ is a family of disjoint subsets of $\Gamma(x, y)$ and their union equals to $\Gamma(x, y)$. Note also that the union of

$\Gamma_g(\widehat{x},\widehat{y})$ over all $g\in\xi^{-1}(k)$ is in a natural bijective correspondence with $\Gamma_k(\bar{x},\bar{y})$. In particular, the set $\cup_{\xi(g)>C}\Gamma_g(\widehat{x},\widehat{y})$ is finite for every $C\in\mathbf{R}$. Set

$$\widehat{n}_g(\widehat{x},\widehat{y}) = \sum_{\gamma\in\Gamma_g(\widehat{x},\widehat{y})} \varepsilon(\gamma)\cdot g \tag{44}$$

$$\widehat{n}(\widehat{x},\widehat{y}) = \sum_{g\in G} \widehat{n}_g(\widehat{x},\widehat{y}) \tag{45}$$

Then $\widehat{n}_g(\widehat{x},\widehat{y})$ is an element of $\mathbf{Z}G$ and $\widehat{n}(\widehat{x},\widehat{y})$ is a priori an element of $\widehat{\widehat{\Lambda}}$. Since for a given C there is only a finite number of $(-v)$-trajectories joining $\widehat{x}$ to $\widehat{y}g$ with $\xi(g)>C$, we obtain that $\widehat{n}(\widehat{x},\widehat{y})$ is actually an element of $\widehat{\Lambda}_\xi$. Note that the Novikov incidence coefficient depends on the particular choice of the liftings of the critical points to $\widehat{M}$: it is easy to obtain the following formula:

$$\widehat{n}(\widehat{x}g_1,\widehat{y}g_2) = g_2^{-1}\widehat{n}(\widehat{x},\widehat{y})g_1 \tag{46}$$

Remark 5.8.

Let $\widehat{C}_r(f)$ be the free right $\widehat{\Lambda}_\xi$-module generated by $S_r(f)$. Similarly to the Subsection 5.1.3 one can prove that setting for $x\in S_r(f)$

$$\widehat{\partial}_r(x) = \sum_{y\in S_{r-1}(f)} x\cdot\widehat{n}(\widehat{x},\widehat{y})$$

we obtain a homomorphism of $\widehat{C}_r(v)$ to $\widehat{C}_{r-1}(v)$ satisfying $\widehat{\partial}_{r+1}\circ\widehat{\partial}_r = 0$. The homology $H_*(\widehat{C}_*(v))$ of the resulting complex is isomorphic to the homology of the completed chain complex $C_*^\Delta(\widehat{M})\underset{\Lambda}{\otimes}\widehat{\Lambda}_\xi$. We shall not give the proofs here. See [**28**] where these theorems are proved in the case when v is an f-Gradient. $\triangle$

5.3.2. Some non-commutative algebra. In order to formulate an "equivariant" analog of theorem 5.5 we need first of all an analog of the localization ring $\widetilde{L}$. This analog will be introduced via the following non-commutative localization procedure.

Recollection ([**7**], p. 255). Let R be a (non-commutative) ring. Assume that for every n a subset S_n of the ring $Mat(n\times n,R)$ is given; set $S=\sqcup_n S_n$. For each matrix $\alpha\in S_n$ and each pair $1\leqslant i,j\leqslant n$ add to the ring R the generators $\bar{\alpha}_{i,j}$ subject to the relations $A\cdot\bar{A}=\bar{A}\cdot A=1$, where $\bar{A}=(\bar{\alpha}_{i,j})$. We obtain a ring, denoted by $S^{-1}R$, together with a homomorphism $\ell: R\longrightarrow S^{-1}R$. If R is a subring of a ring L, and every $\alpha\in S$ is invertible over L, then there is a commutative diagram:

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr] & L \\ & S^{-1}R \arrow[u, "\ell"'] \end{tikzcd} \tag{47}$$

Apply this procedure to our ring Λ, setting

$$S_n = \{1+A \mid A\in Mat(n\times n,\mathbf{Z}G_{(-1)})\}$$

We obtain a ring which will be denoted by $\widetilde{\Lambda}_\xi$. Since every matrix in S_n is invertible in $\widehat{\Lambda}_\xi$ (the inverse of $1+A$ being given by the formula $\sum_{i\geqslant 0}(-1)^i A^i$) we have a commutative diagram:

$$\begin{tikzcd} \Lambda \arrow[r, hook] \arrow[dr] & \widehat{\Lambda}_\xi \\ & \widetilde{\Lambda}_\xi \arrow[u, "\ell"'] \end{tikzcd} \tag{48}$$

We shall see that the Novikov incidence coefficients of C^0 generic gradients of Morse maps $M \to S^1$ are between the simplest possible elements of $\widetilde{\Lambda}_\xi$. They will be all of the form $g_1 Y(1+A)^{-1}Xg_2$, where $A \in Mat(m\times m, \mathbf{Z}G_{(-1)})$, $\quad X, Y \in (\mathbf{Z}H)^m$, $\quad g_1, g_2 \in G$. We shall give a special name for this class of elements of $S^{-1}R$.

Definition 5.9.

An element $a \in \widehat{\Lambda}_\xi$ *is of type* $(\mathcal{L})$ if there are vectors $X = (X_i)_{1\leqslant i\leqslant n}, Y = (Y_i)_{1\leqslant i\leqslant n}$ with $X_i, Y_j \in \mathbf{Z}H$, elements $g_1, g_2 \in G$ and an $(m\times m)$-matrix $A = (a_{ij})$ with entries in $\mathbf{Z}G_{(-1)}$ such that

$$a = g_1 \sum_{s\geqslant 0}\left(\sum_{i,j} Y_i a_{ij}^{(s)} X_i\right) g_2$$

where $a_{ij}^{(s)}$ are the matrix entries of A^s. $\triangle$

The elements of type $(\mathcal{L})$ are obviously in Im ℓ.

In the rest of this subsection we develop some basic non-commutative linear algebra over the ring $R = \mathbf{Z}H$. This non-commutative linear-algebraic constructions will be used in the following. Let F be a right R-module. Recall that we have chosen an element $\theta \in G_{(-1)}$. We shall say that an abelian group homomorphism $\xi : F \to F$ is θ-semilinear, if $\xi(x\lambda) = \xi(x)\cdot\theta\lambda\theta^{-1}$ for every $\lambda \in R$. Assume that F is a free right R-module with free base $(e_i)_{1\leqslant i\leqslant m}$. Associate to each θ-semilinear homomorphism $\xi : F \to F$ the matrix $\widehat{\xi} = (\xi_{ij})_{1\leqslant i,j\leqslant m}$ as follows: $\xi(e_i) = \sum_{j=1}^m e_j \xi_{ji}$.

Lemma 5.10.

Let $x = \sum_{i=1}^m e_i x_i$ be a vector in F, and $n \in \mathbf{N}$. Then

$$\xi^n(x) = \sum_{i,j} e_j \eta_{ji}^{(n)} x_i \theta^{-n} \tag{49}$$

where $\eta_{ji}^{(n)}$ stands for the matrix entries of the matrix $(\widehat{\xi}\theta)^n$.

Proof. Standard computation by induction in n $\square$

Corollary 5.11.

Let $\lambda : F \to R$ be a homomorphism of right R-modules. Then the element

$$\sum_{k\geqslant 0} \lambda(\xi^k(x))\cdot\theta^k$$

of $\widehat{\Lambda}_\xi$ *is of type* $(\mathcal{L})$.

Proof. Just apply λ to the right hand side of (49). $\square$

The homomorphisms $F \to R$ to which we shall apply our Corollary, will be constructed with the help of the following procedure. Let $l : F \to \mathbf{Z}$ be a homomorphism of abelian groups such that for every $x \in F$ the set $\{h \in H \mid l(xh) \neq 0\}$ is finite. Define a homomorphism $\underline{l} : F \to R$ as follows:

$$\underline{l}(x) = \sum_{h \in H} l(xh^{-1}) \cdot h$$

Then $\underline{l}$ is obviously a right $\mathbf{Z}H$-module homomorphism.

5.3.3. Elements of exponential growth in Novikov rings. Let $x \in \mathbf{Z}((t))$, $x = \sum_n x_n t^n$. We say, that x is *of exponential growth*, if there are $A, B > 0$ such that $|x_n| \leqslant Ae^{nB}$ for every n. The elements of exponential growth form obviously a subring of $\mathbf{Z}((t))$. In the present subsection we suggest an analog of this notion for elements of Novikov completion of group rings.

We assume the terminology of Subsection 5.1.1. In particular, $\Lambda = \mathbf{Z}G$. For an element $\lambda \in \Lambda, \lambda = \sum_{g \in G} n_g g$ set $|\lambda| = \sum |n_g|$. Thus $|\cdot|$ is the restriction to $\mathbf{Z}G$ of the L^1-norm on the Banach algebra $L^1(G)$, see [**5**], Ch 1, §2.2. In particular, we have: $|\lambda\mu| \leqslant |\lambda| \cdot |\mu|$. Recall that $\widehat{\Lambda}_\xi$ is the Novikov ring associated with a homomorphism $\xi : G \to \mathbf{R}$. For $\lambda \in \widehat{\Lambda}_\xi, \lambda = \sum_{g \in G} n_g g$ and $c \in \mathbf{R}$ set $\lambda_{\llcorner c \lrcorner} = \sum_{\xi(g) \geqslant c} n_g g$. It follows from the definition of $\widehat{\Lambda}_\xi$ that $\lambda_{\llcorner c \lrcorner} \in \mathbf{Z}G$ for every c.

Definition 5.12.

We say that $\lambda \in \widehat{\Lambda}_\xi$ is *of exponential growth* if there are $A, B > 0$, such that for every $c \in \mathbf{R}$ we have: $|\lambda_{\llcorner c \lrcorner}| \leqslant Ae^{-Bc}$. [2] The subset of all $\lambda \in \widehat{\Lambda}_\xi$ of exponential growth will be denoted by $\Lambda_\xi^{e.g.}$. $\triangle$

It is easy to see that $\Lambda_\xi^{e.g.}$ is a subring of $\widehat{\Lambda}_\xi$.

Lemma 5.13.

Elements of type $(\mathcal{L})$ *are of exponential growth.*

Proof. Let $\lambda \in \widehat{\Lambda}_\xi$ be a element of type $(\mathcal{L})$, i.e.

$$\lambda = g_1 \Big(\sum_{s \geqslant 0} X_i a_{ij}^{(s)} Y_j \Big) g_2, \quad A^s = (a_{ij}^s), \quad a_{ij} \in \mathbf{Z}G_{(-1)}, \quad g_i \in G$$

We can assume $g_1 = g_2 = 1, X_i = \delta_{ik}, Y_i = \delta_{js}$. Thus we must show that each matrix entry of $\sum_{s \geqslant 0} A^s$ is in $\Lambda_\xi^{e.g.}$. Set $||A|| = \max_{i,j} |a_{ij}|$. Then for every $A, B \in Mat(m \times m, \mathbf{Z}H)$ we have $||AB|| \leqslant ||A|| ||B|| m$, and for every $s \geqslant 0$ we have $||A^s|| \leqslant ||A||^s m^{s-1}$. This implies easily that if N is a natural number greater than $\max(2, ||A|| m)$, then $|| \sum_{s=0}^k A^s|| \leqslant Ne^{k \ln N}$ and our result is proved. $\square$

5.3.4. Rationality of Novikov incidence coefficients.
Theorem 5.14.

[2] One should probably say *not more than exponential growth.* We have chosen our present terminology for brevity.

Let $v \in \mathcal{G}\mathfrak{CCT}(f)$. *Then for every* $x, y \in S(f)$ *with* $\operatorname{ind} x = \operatorname{ind} y + 1$ *the Novikov incidence coefficient* $\widehat{n}(\widehat{x}, \widehat{y}; v)$ *is of type* $(\mathcal{L})$. *Therefore* $\widehat{n}(\widehat{x}, \widehat{y}; v)$ *is in* $\operatorname{Im} \ell$, *and is of exponential growth.*

Proof. The proof follows the lines of the proof of Theorem 5.5 , and we shall use the terminology from there. Choose furthermore an element $\theta \in G_{(-1)}$. In view of (46) we can choose liftings $\widehat{x}$ of x and $\widehat{y}$ of y to $\widehat{M}$ as we like. Choose them so that their projections $p(\widehat{x}), p(\widehat{y})$ to $\bar{M}$ satisfy $p(\widehat{x}) \subset W, p(\widehat{y}) \subset W_1$. Set $\bar{x} = p(\widehat{x}), \bar{y} = p(\widehat{y})$.

Thus the descending disc $D(\bar{x}, v)$ lifts to $\widehat{M}$ in a natural way, and the manifolds $X_k \subset V_{-k}$ are also lifted to some manifolds $\widehat{X}_k \subset \widehat{V}_{-k}$. Note that for every $\rho \in H$ the point $\widehat{y}\rho\theta^k$ is a lifting of $\bar{y}t^k$. The Novikov incidence coefficient $\widehat{n}(\widehat{x}, \widehat{y}; v)$ equals to $\sum_{k \geqslant 0, \rho \in H} \widehat{n}_{\rho\theta^k}(\widehat{x}, \widehat{y}; v)$.

To compute the generic term of this sum we apply Proposition 4.19 to obtain a formula analogous to (43):

$$\widehat{n}_{\rho\theta^k}(\widehat{x}, \widehat{y}) = \langle \widehat{y}\rho\theta^k, [\widehat{X}_k] \rangle_v \cdot \rho\theta^k \tag{50}$$

Define a θ-semilinear endomorphism $\widehat{h}$ of $\widehat{C}_s(u_0)$ setting $\widehat{h} = \theta_*^{-1} \circ \widehat{\mathcal{H}}_s(-v \mid W_1)$. Applying Theorem 4.18 it is not difficult to see that $[\widehat{X}_k] = \theta_*^k \circ \widehat{h}^k$. Therefore

$$\widehat{n}_{\rho\theta^k}(\widehat{x}, \widehat{y}) = \langle \widehat{y}\rho, \widehat{h}^k([\widehat{X}]) \rangle = \langle \widehat{y}, \widehat{h}^k\big([\widehat{X}]\big)\rho^{-1} \rangle \cdot \rho$$

Denote by Υ the homomorphism $\langle \widehat{y}, \cdot \rangle : \widehat{C}_l(u_0) \to \mathbf{Z}$ and set $\lambda = \underline{\Upsilon}$, see the Subsection 5.3.2 for the meaning of $\underline{\ \ }$). Then

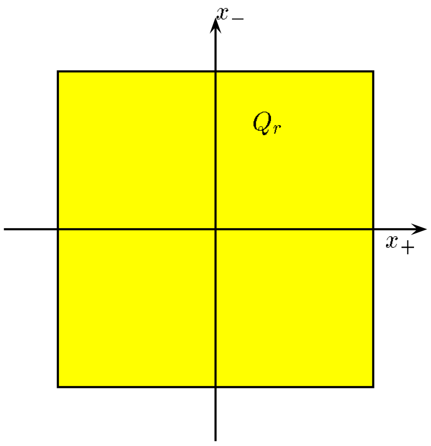
$$\sum_{\rho} \widehat{n}_{\rho\theta^k}(\widehat{x}, \widehat{y}) \cdot \rho\theta^k = \lambda\big(\widehat{h}^k([\widehat{X}])\big)\theta^k$$

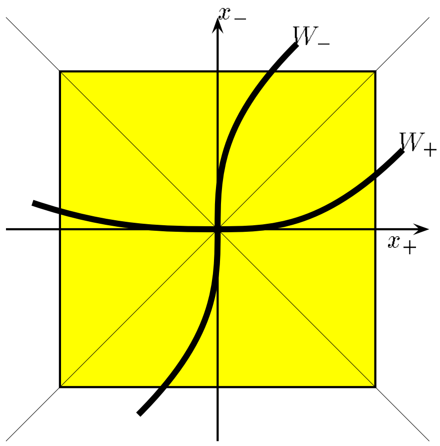
Thus

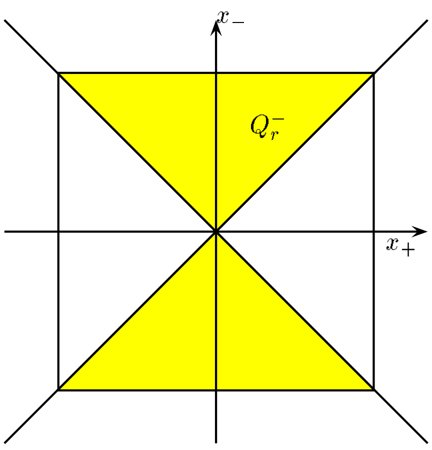
$$\widehat{n}(\widehat{x}, \widehat{y}; v) = \sum_{k \geqslant 0} \lambda\big(\widehat{h}^k([\widehat{X}])\big)\theta^k$$

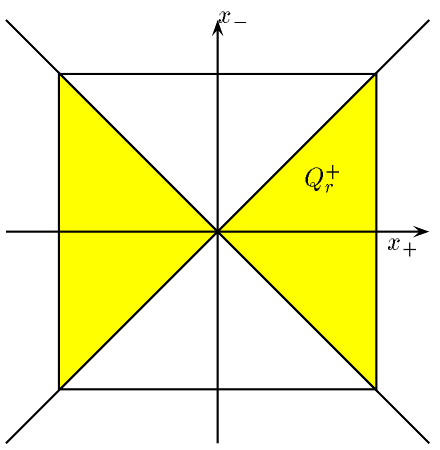
and it suffices to apply Corollary 5.11 to finish the proof. $\square$

Figures


x_-
Q_r
x_+


x_-
W_-
W_+
x_+


x_-
Q_r^-
x_+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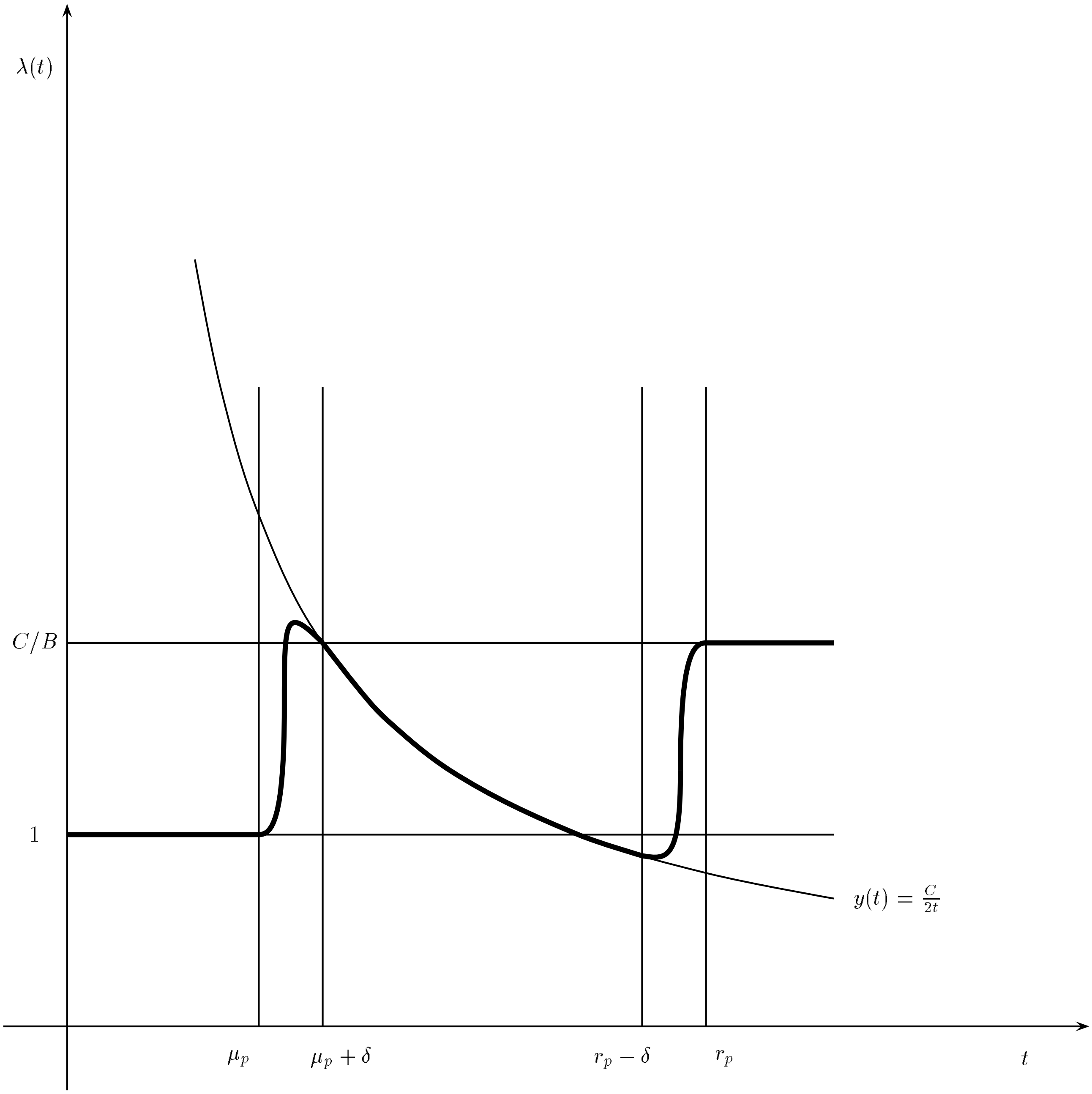
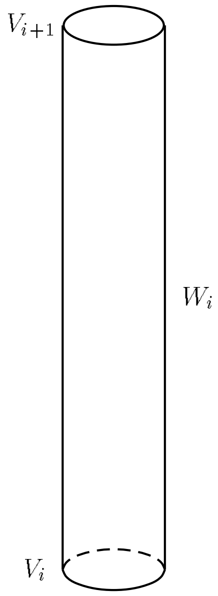
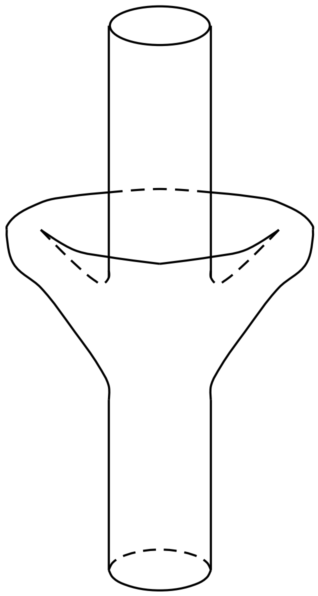
Q_r^+
x_+

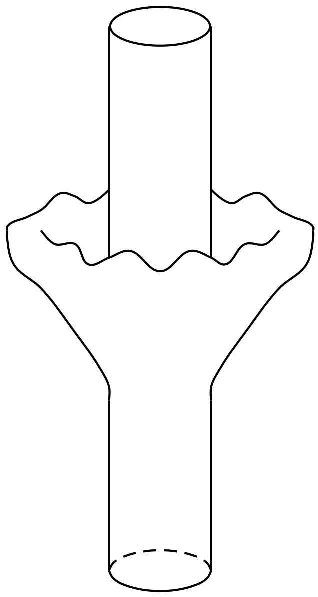

$\lambda(t)$
C/B
1
$y(t) = \frac{C}{2t}$
μ_p
$\mu_p + \delta$
$r_p - \delta$
r_p
t



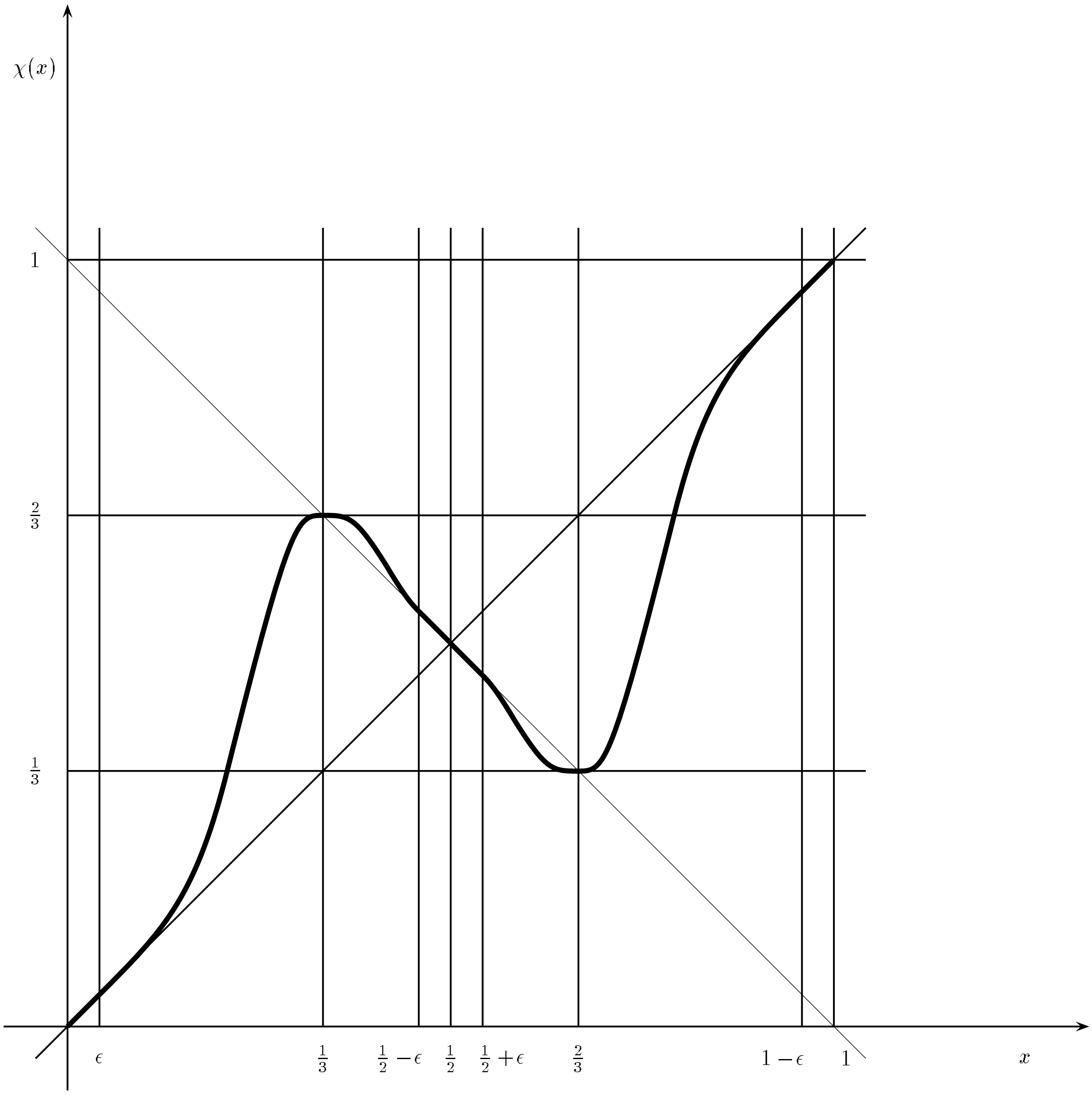
A



B



C



$\chi(x)$
1
$\frac{2}{3}$
$\frac{1}{3}$
ϵ
$\frac{1}{3}$
$\frac{1}{2}-\epsilon$
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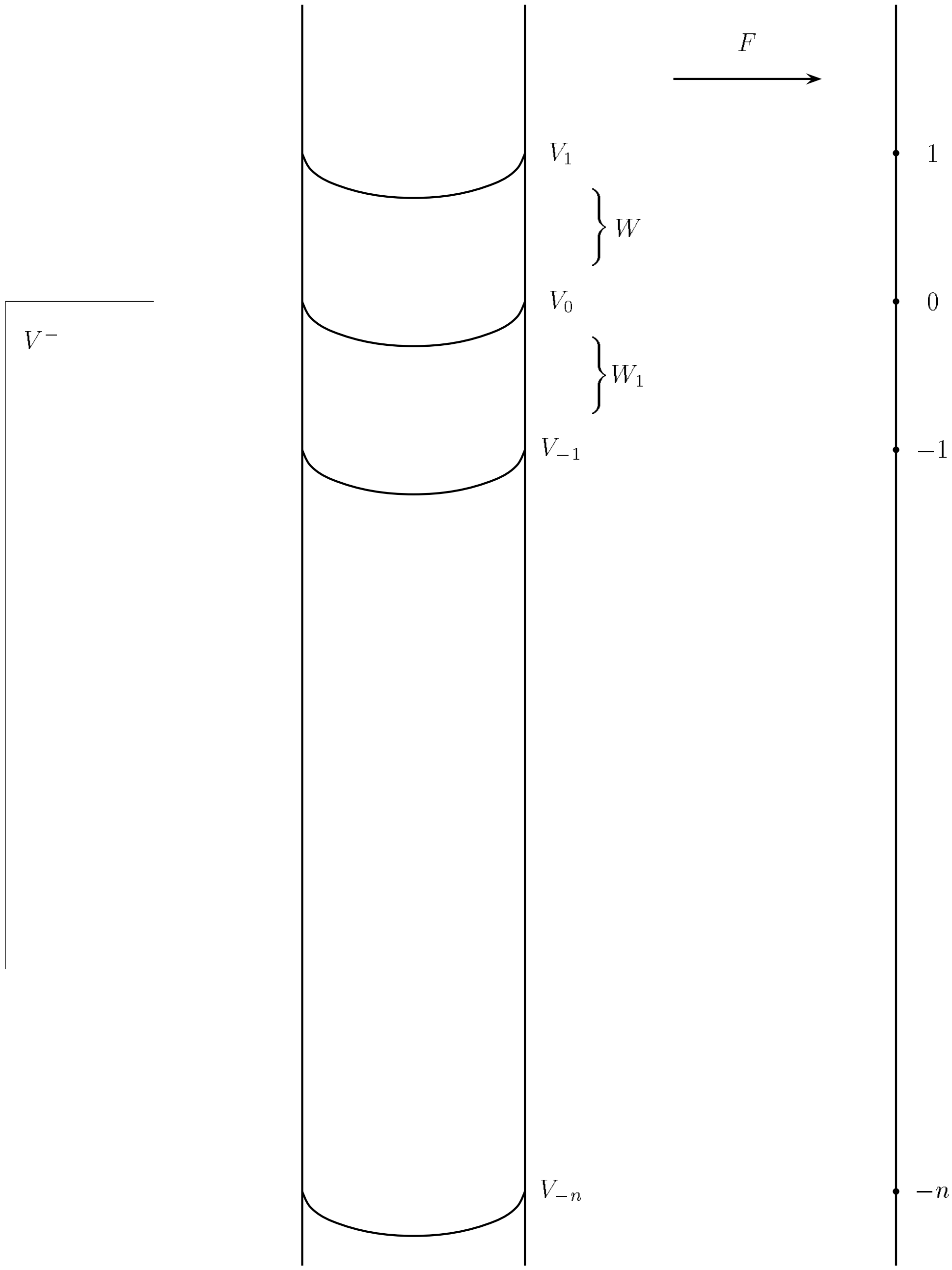
$1-\epsilon$
1
x


F
V_1
1
W
V_0
0
V^-
W_1
V_{-1}
-1
V_{-n}
$-n$

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