CLOSED ORBITS OF GRADIENT FLOWS AND LOGARITHMS OF NON-ABELIAN WITT VECTORS

A.V.PAJITNOV

Abstract. We consider the flows generated by generic gradients of Morse maps $f : M \to S^1$. To each such flow we associate an invariant counting the closed orbits of the flow. Each closed orbit is counted with the weight derived from its index and homotopy class. The resulting invariant is called the *eta function*, and lies in a suitable quotient of the Novikov completion of the group ring of the fundamental group of M. Its abelianization coincides with the logarithm of the twisted Lefschetz zeta function of the flow. For C^0-generic gradients we obtain a formula expressing the eta function in terms of the torsion of a special homotopy equivalence between the Novikov complex of the gradient flow and the completed simplicial chain complex of the universal cover.

Introduction

The study of the periodic orbits of a dynamical system via homotopy invariants is one of traditional subjects of the algebraic topology and dynamical system theory. Let us consider first a discrete dynamical system given by a continuous map $f : X \to X$ of a topological space X. The fundamental invariant of this dynamical system – the cardinality of the fixed point set Fix f – has a homotopy counterpart, namely the algebraic number of fixed points of f. The Lefschetz trace formula expresses this number (under some mild conditions on f and X) in terms of the traces of the homomorphisms induced by f in homology.

In order to investigate the sets Fix f^n for $n \to \infty$ Artin and Mazur [AM] introduced a zeta function which encodes the information about all the periodic points into a single power series in one variable t. This zeta function and its generalizations have proved very important in dynamical system theory. See the works [B] of V.Baladi and [Fel] of A.Fel'shtyn for a survey of the present state of the theory.

1991 *Mathematics Subject Classification.* Primary: 57R70; Secondary: 57R99.
Key words and phrases. Novikov Complex, gradient flow, Lefschetz zeta-function.

The Artin-Mazur zeta function has a homotopy counterpart – the *Lefschetz zeta function* introduced by S.Smale [Sm]. Here is the definition. Let Fix f denote the set of fixed points of f. Assume for simplicity that for any n the set Fix f^n is finite. Each $x \in \mathrm{Fix}\, f^k$ has an associated index $\nu(x) \in \mathbf{Z}$. Set

$$L_k(f) = \sum_{a \in \mathrm{Fix}\, f^k} \nu(a) \tag{1}$$

and define the Lefschetz zeta function by the following formula:

$$\zeta_L(t) = \exp\left(\sum_{k=0}^{\infty} \frac{L_k(f)}{k} t^k\right) \tag{2}$$

Here is the formula for ζ_L in terms of the homology invariants of f:

$$\zeta_L(t) = \prod_i \det(I - tf_{*,i})^{(-1)^{i+1}} \tag{3}$$

where $f_{*,i}$ stands for the homomorphism induced in $H_i(X)$ by f.

In order to encode more information on periodic points in one single power series, one counts the periodic points with weights belonging to some group ring (usually the group ring of some regular covering of X). An abelian zeta function of this kind was first introduced by D.Fried [Fr1], and the corresponding analog of the formula (3) was obtained. The first non-abelian generalization of zeta functions was introduced by R.Geoghegan and A.Nicas [GN] as one of the consequences of their parameterized Lefschetz-Nielsen fixed point theory. Their *Nielsen-Fuller invariant* counting periodic points of an orientation preserving diffeomorphism $f : M \to M$ belongs to the Hochschild homology of the group ring of the mapping torus of f. They obtain an analog of the formula (3) for this invariant. The analogs of Lefschetz zeta functions for continuous maps in the non-commutative setting were defined also in the papers [J], [L].

Proceeding to the dynamical systems generated by flows on manifolds, one would expect that the theory described above generalizes to this setting. Namely, there should exist an analog of Lefschetz zeta function (2) counting the closed orbits of the flow, and there should be a formula expressing this invariant in homotopy invariant terms. Such formulas exist in many important cases, but only for non-singular flows. See [Fr2] Th.10 for abelian invariants of homology proper Axiom A-No-Cycles semiflows and [GN] for non-abelian invariants of suspension flows.

The first results concerning zeta functions of flows with zeros appeared only very recently ([HL], [Pa6]). These papers deal with the flows generated by gradients of circle-valued Morse maps. The results of these papers show that the naturally arising zeta functions are no longer homotopy invariants of the underlying manifold itself (as it is the case for non-singular flows). There occurs a "correction term" which can be computed in terms of the Novikov complex (see [No]) associated with the flow. Both papers cited above deal with abelian invariants.

In the present paper we introduce *non-abelian* invariants counting closed orbits of gradient flows of circle-valued Morse maps. These invariants – we call them *eta functions* – generalize the logarithms of abelian zeta functions. (Note that the abelian zeta functions carry the same amount of information as their logarithms.) The definition of the non-abelian eta function for gradient flows is given in 0.1.6. The Main Theorem of the paper gives a formula expressing the eta function of the flow in terms of the torsion of a chain homotopy equivalence naturally associated with the flow.

Remarks on the terminology. Let R be a ring with a unit. Recall from [Mi2] the group $\overline{K}_1(R) = K_1(R)/\{0, [-1]\}$ where $[-1]$ denotes the element of order 2 corresponding to the unit $(-1) \in GL(R,1) \subset GL(R,\infty)$. A free based finitely generated chain complex C_* of right R-modules is called *R-complex.* Recall from [Mi2], §3 that for an acyclic R-complex C_* the torsion $\tau(C_*) \in \overline{K_1}(R)$ is defined. If $\phi : F_* \to D_*$ is a homotopy equivalence of R-complexes, then its *torsion* $\tau(\phi) \in \overline{K_1}(R)$ is defined to be the torsion of the algebraic mapping cone $C_*(\phi)$.

0.1. **Counting closed orbits of a gradient flow of a circle-valued Morse map.** In order to state the main theorem we shall need some preliminaries on circle-valued Morse theory. They are gathered in the three following subsections.

0.1.1. *Novikov rings.* Let G be a group, and $\xi : G \to \mathbf{Z}$ be a homomorphism. Set $\Lambda = \mathbf{Z}G$ and denote by $\widehat{\widehat{\Lambda}}$ the abelian group of all functions $G \to \mathbf{Z}$. Equivalently, $\widehat{\widehat{\Lambda}}$ is the set of all formal linear combinations $\lambda = \sum_{g\in G} n_g g$ (not necessarily finite) of the elements of G with integral coefficients n_g. For $\lambda \in \widehat{\widehat{\Lambda}}$ set $\operatorname{supp} \lambda = \{g \in G \mid n_g \neq 0\}$.

Set

$$\widehat{\Lambda}_\xi = \{\lambda \in \widehat{\widehat{\Lambda}} \mid \forall C \in \mathbf{Z} \quad \operatorname{supp} \lambda \cap \xi^{-1}([C, \infty[) \text{ is finite } \} \tag{4}$$

Then $\widehat{\Lambda}_\xi$ has a natural structure of a ring. We shall also need an analog of this ring with $\mathbf{Z}$ replaced by $\mathbf{Q}$. That is, let $\widehat{\widehat{\Lambda}}_{\mathbf{Q}}$ be the set of all the functions $G \to \mathbf{Q}$ and denote by $\widehat{\Lambda}_{\xi,\mathbf{Q}}$ the subgroup of $\widehat{\widehat{\Lambda}}$ formed by all the λ such that for every $C \in \mathbf{Z}$ the set $\xi^{-1}([C,\infty[) \cap \operatorname{supp} \lambda$ is finite. Then $\widehat{\Lambda}_{\xi,\mathbf{Q}}$ is a ring, and $\widehat{\Lambda}_\xi$ is a subring.

A basic example of a Novikov ring is provided by $G = \mathbf{Z}$ and the standard inclusion $\xi : \mathbf{Z} \hookrightarrow \mathbf{R}$. In this case $\widehat{\Lambda}_\xi$ is the ring of Laurent power series with integral coefficients and finite negative part.

One of the main algebraic instruments of the paper are the groups $K_1(\widehat{\Lambda}_\xi)$ and $K_1(\widehat{\Lambda}_{\xi,\mathbf{Q}})$. We shall need also the analogs of Whitehead groups. To introduce these groups define first a subset $\pm G$ of $\Lambda = \mathbf{Z}G$ by: $\pm G = \{\pm g | g \in G\}$. Then $\pm G \subset \widehat{\Lambda}_\xi^\bullet$ and this inclusion induces a homomorphism $\pm G \to K_1(\widehat{\Lambda}_\xi)$. The image of $\pm G$ via this homomorphism is denoted by $\pm \widehat{G}$ and the group $K_1(\widehat{\Lambda}_\xi)/\pm\widehat{G}$ is denoted by $\widehat{\mathrm{Wh}}(G,\xi)$. Similarly the quotient of $K_1(\widehat{\Lambda}_{\xi,\mathbf{Q}})$ by the image of $\pm G$ is denoted by $\widehat{\mathrm{Wh}}_{\mathbf{Q}}(G,\xi)$.

0.1.2. *The Novikov complex and its simple homotopy type.* Let M be a closed connected manifold, and let $f : M \to S^1$ be a Morse map. We assume that $f_* : H_1(M) \to H_1(S^1) = \mathbf{Z}$ is epimorphic. Let $\mathcal{P} : \widetilde{M} \to M$ be the universal covering; then $f \circ \mathcal{P}$ is homotopic to zero. The structure group of this covering (isomorphic to $\pi_1(M)$) will be denoted by G. There is a homomorphism $\xi : G \to \mathbf{Z}$, induced by f. Set $\Lambda = \mathbf{Z}G$, then the corresponding Novikov ring $\widehat{\Lambda}_\xi$ is defined. Let v be an f-gradient satisfying Transversality Condition (that is, for every pair of zeros p, q of v we have: $W^{st}(p) \pitchfork W^{un}(q)$ where $W^{st}(p)$ and $W^{un}(q)$ are respectively stable and unstable manifolds of p, resp. q). Choose for every critical point p of f an orientation of the stable manifold of p and a lifting of p to $\widetilde{M}$. To this data one associates a $\widehat{\Lambda}_\xi$-complex $C_*(v)$, called *the Novikov complex* such that the number of free generators of $C_k(v)$ equals the number of critical points of f of index k. This chain complex has the same homotopy type as the complex

$$\widehat{C}^\Delta_*(M,\xi) = C^\Delta_*(\widetilde{M}) \underset{\Lambda}{\otimes} \widehat{\Lambda}_\xi$$

where $C^\Delta_*(\widetilde{M})$ is the simplicial chain complex of $\widetilde{M}$ (see [Pa1]). Both $C_*(v)$ and $\widehat{C}^\Delta_*(M,\xi)$ are free based chain complexes so a chain homotopy equivalence between them has a naturally defined torsion in $\overline{K_1}(\widehat{\Lambda}_\xi)$. The bases in these complexes are determined by the choices of orientations and liftings of critical points (resp. simplices) to $\widetilde{M}$.

Changing these choices leads to the changing of the torsion by adding an element of Im $(G \to \overline{K_1}(\widehat{\Lambda}_\xi))$. Thus the torsion is really well defined in the group $\widehat{\mathrm{Wh}}(G,\xi)$ (introduced above in the section 0.1.1).

0.1.3. *Vector fields and closed orbits: terminology.* Let w be a C^∞ vector field on a closed manifold M. The set of all closed orbits of w is denoted by $Cl(w)$. A vector field w will be called *Kupka-Smale* if every zero and every closed orbit is hyperbolic, and the stable manifold of every zero of v is transversal to the unstable manifold of every zero of v. For a Kupka-Smale vector field w and a closed orbit $\gamma \in Cl(w)$ let $\varepsilon(\gamma) \in \{-1, 1\}$ denote the index of the corresponding Poincare map; let $m(\gamma)$ denote the multiplicity of γ. Let $f : M \to S^1$ be a Morse map. Gradient-like vector fields for f (in the sense of the definition in [Mi1], §3.1) will be also called *f-gradients* . The set of all Kupka-Smale f-gradients will be denoted by $\mathcal{G}(f)$.

0.1.4. *Witt vectors and their logarithms.* Let A be a ring with unit (non-commutative in general). We assume that $\mathbf{Q} \subset A$. Let $\rho : A \to A$ be an automorphism. Consider the *ρ-twisted polynomial ring* $A_\rho[t]$ (where the multiplication satisfies $at = t\rho(a)$ for $a \in A$) and embed it into the corresponding formal power series ring $P = A_\rho[[t]]$, which is in turn a subring of the ring $R = A_\rho((t)) = A_\rho[[t]][t^{-1}]$ of Laurent powers series with finite negative part.

The aim of the Section 1 is to construct a homomorphism $\mathcal{L}$ defined on $K_1(P)$ and with values in an abelian group which we shall now describe. For $n \in \mathbf{N}$ set $P_n = At^n \subset P$. Let P'_n be the abelian subgroup of P_n generated by all the commutators $[x, y] = xy - yx$ where $x \in P_k, y \in P_s, k + s = n$. Set

$$\bar{P}_n = P_n/P'_n, \quad P' = \prod_{n\geqslant 0} P'_n, \quad \bar{P} = \prod_{n\geqslant 0} \bar{P}_n = P/P' \tag{5}$$

(In general P' is not an ideal of P, so $\bar{P}$ has only the structure of a $\mathbf{Q}$-vector space.)

The homomorphism $\mathcal{L}$ takes its values in $\bar{P}$ and has in particular the following logarithm-like property:

For $x \in P$ the image of tx in $\bar{P}$ is equal to $\mathcal{L}([\exp tx])$.

The main technical tool in the construction of $\mathcal{L}$ is the group of Witt vectors in P. A *Witt vector* is an element of $P = A_\rho[[t]]$ of the form $1 + a_1t + a_2t^2 + \ldots$, with $a_i \in A$. The Witt vectors form a multiplicative subgroup W of the full group of units $P^\bullet$. The image of W in $K_1(P)$ will be denoted by $\widetilde{W}$. We first construct $\mathcal{L}$ on the group $\widetilde{W}$ and then extend it to the whole of $K_1(P)$ (section 1). The resulting extension $\mathcal{L} : K_1(P) \to \bar{P}$ satisfies $\mathcal{L}(K_1(A)) = 0$.

For our topological applications we need rather a homomorphism defined on the group $K_1(R)$. The construction of such homomorphism follows immediately from the main theorem of the paper [PR] of A.Ranicki and the author. The corollary which we need of the main theorem of [PR] says:

Proposition 0.1 ([PR], Corollary 0.1). *The homomorphism $\widetilde{W} \to K_1\big(A_\rho((t))\big)$ induced by the inclusion $W \hookrightarrow (A_\rho[[t]])^\bullet$ admits a left inverse $\widehat{B}_2 : K_1\big(A_\rho((t))\big) \to \widetilde{W}$, such that*

1. *$\widehat{B}_2$ vanishes on the image of $K_1(A)$ in $K_1\big(A_\rho((t))\big)$*
2. *$\widehat{B}_2$ vanishes on the image in $K_1\big(A_\rho((t))\big)$ of the 1×1-matrix (t).*

$\square$

Set $\mathfrak{L} = \mathcal{L} \circ \widehat{B}_2$; then the homomorphism $\mathfrak{L} : K_1\big(A_\rho((t))\big) \to \bar{P}$ vanishes on the image of $K_1(A)$ in $K_1\big(A_\rho((t))\big)$ and the composition $K_1(A_\rho[[t]]) \to K_1\big(A_\rho((t))\big) \xrightarrow{\mathfrak{L}} \bar{P}$ equals to $\mathcal{L}$.

0.1.5. *Logarithms in the context of group rings.* The results about Witt vectors in power series rings or Laurent series rings will be applied to the Novikov completions of group rings.

Let G be a group and $\xi : G \to \mathbf{Z}$ be an epimorphism. Let $H = \operatorname{Ker} \xi$. Choose any element t with $\xi(t) = -1$. The ring $\mathbf{Q}G$ is then identified naturally with ρ-twisted Laurent polynomial ring $A_\rho[t, t^{-1}]$, where $A = \mathbf{Q}H$, and $\rho(x) = t^{-1}xt$ (so that the multiplication in this ring satisfies $at = t\rho(a)$ for $a \in A$). The Novikov ring $\widehat{\Lambda}_{\xi,\mathbf{Q}}$ is identified with the twisted Laurent series ring $R = A_\rho((t))$.

Let Γ be the set of conjugacy classes of G, and let Γ_n denote the set of all the conjugacy classes contained in $G_{(n)} = \xi^{-1}(n)$. Consider the $\mathbf{Q}$-vector space $\mathbf{Q}\Gamma_n$ spanned by Γ_n, and set $\mathfrak{G} = \prod_{n \leqslant 0} \mathbf{Q}\Gamma_n$.

Lemma 2.1 shows that $\mathfrak{G} \approx \bar{P}$ where $P = (\mathbf{Q}H)_\rho[[t]]$. Thus we obtain a homomorphism $K_1(\widehat{\Lambda}_{\xi,\mathbf{Q}}) \to \mathfrak{G}$. Up to natural identifications it is still the same homomorphism $\mathfrak{L} : K_1(R) \to \bar{P}$, so we keep the same symbol $\mathfrak{L}$ to denote it. (The advantage of the notation $\mathfrak{L} : K_1(\widehat{\Lambda}_{\xi,\mathbf{Q}}) \to \mathfrak{G}$ is that the objects $K_1(\widehat{\Lambda}_{\xi,\mathbf{Q}}), \mathfrak{G}$ do not depend on the particular choice of the element t.)

In subsection 2.1 we show that $\mathfrak{L}$ factors through $\widehat{\mathrm{Wh}}_{\mathbf{Q}}(G, \xi) = K_1(\widehat{\Lambda}_{\xi,\mathbf{Q}})/ \pm \widehat{G}$. Composing with the embedding $\widehat{\Lambda}_\xi \subset \widehat{\Lambda}_{\xi,\mathbf{Q}}$ we obtain now a homomorphism from $\widehat{\mathrm{Wh}}(G, \xi)$ to $\mathfrak{G}$. We shall denote it by the same symbol $\mathfrak{L}$ since there is no possibility of confusion.

0.1.6. *Non abelian eta function for gradient flows.* Let $f : M \to S^1$ be a Morse map and let v be an f-gradient. Let γ be a closed orbit of $(-v)$. There is no natural choice of a base point in γ, and the class of γ in $\pi_1(M)$ is ill-defined, but the class of γ is well-defined as the element of the set Γ of conjugacy classes of $\pi_1(M)$. This class will be denoted by $\{\gamma\}$. We are about to introduce the eta-function counting the classes $\{\gamma\}$ corresponding to the closed orbits of $-v$.

Let $\xi = f_* : \pi_1(M) \to \mathbf{Z}$. We assume that ξ is epimorphic. Applying the terminology of the subsection (0.1.5) we obtain the subsets $\Gamma_n \subset \Gamma$ and the rational vector space $\mathfrak{G}$. Assume now that v is a Kupka-Smale vector field. Here is the definition of the non-abelian eta function of $-v$:

$$\eta_L(-v) = \sum_{\gamma \in Cl(-v)} \frac{\varepsilon(\gamma)}{m(\gamma)}\{\gamma\} \in \mathfrak{G} \tag{6}$$

(Using the Kupka-Smale condition, it is not difficult to check that for a given n there is only a finite number of closed orbits γ with $\{\gamma\} \in \Gamma_n$, and, therefore, the right hand side of (6) is indeed a well-defined element of $\mathfrak{G}$.)

Recall the notation $\widehat{C}^\Delta_*(M,\xi) = C^\Delta_*(\widetilde{M}) \underset{\Lambda}{\otimes} \widehat{\Lambda}_\xi$ from subsection 0.1.2.

Main Theorem

There is a subset $\mathcal{G}_0(f) \subset \mathcal{G}(f)$ with the following properties:

1. *$\mathcal{G}_0(f)$ is open and dense in $\mathcal{G}(f)$ with respect to C^0 topology*
2. *For every $v \in \mathcal{G}_0(f)$ there is a chain homotopy equivalence*

$$\phi : C_*(v) \xrightarrow{\sim} \widehat{C}^\Delta_*(M,\xi) \tag{7}$$

such that

$$\mathfrak{L}(\tau(\phi)) = -\eta_L(-v). \tag{8}$$

0.1.7. *Main lines of the proof.* In the paper [Pa4], we constructed for a given Morse map $f : M \to S^1$ a special class of f-gradients which form a subset, C^0-open-and-dense in the subset of all f-gradients satisfying Transversality Condition. For any f-gradient v in this class the boundary operators in the Novikov complex, associated to v, are not merely power series, but rational functions. The main construction in the proof is a group homomorphism $h(v)$ between homology groups of certain spaces, (see §4 of [Pa4]) which we call *homological gradient descent.*

To explain the roots of this notion, let $\lambda \in S^1$ be a regular value of f, and consider the integral curve γ of $(-v)$ starting at $x \in V = f^{-1}(\lambda)$. If this curve does not converge to a critical point of f, then it will intersect again the submanifold V at some point, say, $H(x)$. The map H is thus a (not everywhere defined) smooth map of V to V, and the *homological gradient descent operator* is a substitute for the homomorphism induced by H in homology. This operator is defined for a C^0-generic v-gradient satisfying Transversality Condition.

Note that the closed orbits of v correspond to the periodic points of the map H. This makes possible to give a formula for η_L in terms of the homological gradient descent operator (see the formula (24)). On the other hand it turns out that there is a homotopy equivalence (7) such that its torsion is computable in terms of homological gradient descent operator (see (22)). Comparing these two expressions we obtain the theorem.

0.2. **Acknowledgements.** Many thanks are due to the referee for careful reading of the manuscript, and for pointing out two errors in the first version of the paper.

I am happy to thank A.Ranicki for the attention he paid to this work and for our collaboration on the subject of the Whitehead groups of the Novikov rings. Our joint work has lead to the paper [PR], the results of which allowed a strengthening of the results of the first version of the present paper.

I am grateful to V.Turaev for valuable discussions and to A.Fel'shtyn for useful information about zeta functions.

After this paper has been accepted for publication I received a very interesting preprint of D.Schütz [Sch]. Schütz's work contains in particular a generalization of the results of the present paper to the case of closed 1-forms within an irrational cohomology class. Many thanks to D.Schütz for having sent me his paper.

A part of the work was done during my stay at IHES in December 1998 and May 1999. I am grateful to IHES for hospitality and financial support.

1. Witt vectors and homomorphism $\mathcal{L}$

We shall be working here with the notation of 0.1.4. Thus A is a ring containing $\mathbf{Q}$, $\rho : A \to A$ is an automorphism, $P = A_\rho[[t]]$ is the corresponding twisted power series ring, $R = A_\rho((t))$ is the twisted Laurent series ring, etc. For $x \in R, x = \sum_i x_i t^i$ set $\nu(x) = \min\{k \in \mathbf{Z} \mid x_k \neq 0\}$. Set $P_+ = \{x \in P | \nu(x) > 0\}$. There is a natural topology in P, namely $a_n \to a$ if $\nu(a_n - a) \to \infty$. Note that the closure $\overline{[P, P]}$ is in P'. Our aim in this section is to construct the homomorphism

$\mathcal{L} : K_1(P) \to \bar{P}$. We do it in three steps: first we define a logarithm-like homomorphism on the group of Witt vectors W, then we factor it through $\widetilde{W} = \mathrm{Im}\,(W \to K_1(P))$; finally we extend it to the whole of $K_1(P)$ using the computation of $K_1(P)$ from [Pa1]. (Note by the way, that we do not use here the results of [PR].)

1.1. **The Baker-Campbell-Hausdorff-Dynkin formula in $A[[t]]$.** Define the exponent and logarithm as usual by the following formulas

(9)
$$\log(1+\mu) = \mu - \frac{\mu^2}{2} + \frac{\mu^3}{3} - \dots + (-1)^{n-1}\frac{\mu^n}{n} + \dots, \text{ where } \mu \in P_+,$$

$$\exp y = e^y = 1 + y + \frac{y^2}{2} + \dots \frac{y^n}{n!} + \dots \text{ where } y \in P_+ \tag{10}$$

It is clear that both series converge in P. A standard computation shows that $\log\exp y = y$ for $y \in P_+$, and $\exp\log x = x$ for $x \in W$. For $X, Y \in P_+$ set

$$\zeta(X,Y) = \log(e^X e^Y) \in A_\rho[[t]] \tag{11}$$

The well-known Baker-Campbell-Hausdorff-Dynkin computation (see for example [Po], Lecture 6, or [Bou] Ch. 6, §4) implies:

$$\zeta(X,Y) = \sum_{n=1}^{\infty} \zeta_n(X,Y) \tag{12}$$

where $\zeta_n(X,Y)$ is a Lie polynomial in X, Y of degree n. The power series in the right hand side of (12) converges since $\nu(\zeta_n(X,Y)) \geqslant n$. Moreover, for $n > 1$ we have: $\zeta_n(X,Y) \in [P,P] \subset P'$. Therefore if μ, ν are the units in W of the form e^X, e^Y with $X, Y \in P_+$, we have $\log(\mu\nu) - \log\mu - \log\nu \in P'$. Since every element of W is of the form e^X with $X \in P_+$, we obtain that the composition

$$W \xrightarrow{\log} A_\rho[[t]] = P \xrightarrow{pr} \bar{P} \tag{13}$$

is a group homomorphism, which will be denoted by Log . In the next section we shall see that this homomorphism can be factored through the group $\widetilde{W} = \mathrm{Im}\,(W \to K_1(P))$. The first step in the proof is the next lemma. (Note, that the symbol $[P^\bullet, P^\bullet]$ denotes the commutator of the group $P^\bullet$ in the group-theoretic sense, that is $[P^\bullet, P^\bullet]$ is the subgroup of $P^\bullet$ generated by the set $\{ghg^{-1}h^{-1} | g, h \in P^\bullet\}$.)

Lemma 1.1. *The homomorphism* Log *vanishes on the subgroup* $W \cap [P^\bullet, P^\bullet]$ *of* W.

Proof. Let $m \in W$. The elements of the form $\alpha m \alpha^{-1}$, where $\alpha \in A^\bullet$ will be called *A-conjugate* to m. It is obvious that the value of Log on A-conjugate elements is the same.

Let $X \in W \cap [P^\bullet, P^\bullet]$. Assume first that $X = aba^{-1}b^{-1}$ where $a, b \in P^\bullet$. Write $a = \alpha m, b = \beta n$ where $\alpha, \beta \in A^\bullet$ and $m, n \in W$. A direct computation gives: $X = m'n'(m'')^{-1}(n'')^{-1} \cdot \alpha\beta\alpha^{-1}\beta^{-1}$, where m', m'' are A-conjugate to m, and n', n'' are A-conjugate to n. Since $X \in W$ we have $\alpha\beta\alpha^{-1}\beta^{-1} = 1$, and Log $(X) = 0$. The general case (X is a product of several commutators) is done by an easy induction argument. □

1.2. **Homomorphism $\mathcal{L}$.** Apply the results of the previous section to the ring $\mathcal{A}_n = \mathcal{M}_n(A)$ of $n \times n$-matrices over A. We have the power series ring $\mathcal{P}_n = \mathcal{A}_n[[t]]$, the multiplicative subgroup $\mathcal{U}_n = 1 + t\mathcal{P}_n \subset \mathcal{P}_n^\bullet$ of the group of units, the Laurent power series ring $\mathcal{R}_n$ etc. By the results of the preceding subsection the composition

$$\mathcal{U}_n \xrightarrow{\log} \mathcal{A}_n[[t]] = \mathcal{P}_n \xrightarrow{pr} \bar{\mathcal{P}}_n \tag{14}$$

is a group homomorphism. We have the trace map $\mathrm{Tr} : \mathcal{P}_n \to A[[t]]$ defined by $\mathrm{Tr}(\sum_i \alpha_i t^i) = \sum_i \mathrm{Tr}(\alpha_i) t^i$. It is easy to check that Tr factors through the quotients to define correctly the group homomorphism $\bar{\mathcal{P}}_n \to \bar{P}$, which we shall denote by the same symbol Tr. The composition $\mathrm{Tr} \circ pr \circ \log$ will be denoted by $\mathcal{L}_n : \mathcal{U}_n \to \bar{P}$; it is a group homomorphism. It is easy to check that the map $\mathcal{L}_n$ behaves well with respect to the natural stabilization map $\mathcal{U}_n \hookrightarrow \mathcal{U}_{n+1}$. Set $\mathcal{U} = \varinjlim \mathcal{U}_n$ and obtain a homomorphism $\mathcal{U} \to \bar{P}$, which will be denoted by $\mathcal{L}$. Let $\widetilde{\mathcal{U}}$ be the image of $\mathcal{U}$ via the homomorphism $\mathcal{U} \hookrightarrow GL(P) \to K_1(P)$. It follows from the Lemma 1.1 that $\mathcal{L}$ factors through $\widetilde{\mathcal{U}}$; we keep the same notation $\mathcal{L}$ for the resulting homomorphism $\widetilde{\mathcal{U}} \to \bar{P}$. It is obvious that $\widetilde{\mathcal{U}} \supset \widetilde{W} = \mathrm{Im}\,(W \to K_1(P))$ and that the restriction of $\mathcal{L}$ to $\widetilde{W}$ coincides with Log .

Proposition 1.2. 1. $\widetilde{\mathcal{U}} = \widetilde{W}$.

2. *The inclusions $W \xhookrightarrow{i} P$, $A \xhookrightarrow{j} P$ and the projection $P \xrightarrow{p} A$ induce the following split exact sequence*

$$0 \longrightarrow \widetilde{W} \xrightarrow{i_*} K_1(P) \underset{j_*}{\overset{p_*}{\rightleftarrows}} K_1(A) \longrightarrow 0 \tag{15}$$

Proof. A slightly weaker version of this proposition is contained in [Pa1], Lemma 1.1; it is based on the argument due essentially to A.Suslin. Let us consider an element $Z \in K_1(P)$ with $p_*(Z) = 0$. Then Z can be reduced to a square matrix S of the form $1 + \mu$ where μ

has the coefficients in P_+. Each diagonal element of S is then invertible in P and using the standard Gauss elimination method we can reduce S to a diagonal matrix Δ without changing the class of S in K_1. The class of Δ in K_1 belongs obviously to $\widetilde{W}$. $\square$

In particular, $\mathcal{L}$ can be extended in a functorial way to a homomorphism of the whole of $K_1(P)$ to $\bar{P}$: just define $\mathcal{L}$ to be zero on the image of j_*.

Note that $\operatorname{Im} \mathcal{L} = \bar{P}_+$.

2. Laurent series rings and homomorphism $\mathfrak{L}$

As we have already mentioned (see 0.1.4) the homomorphism $\mathcal{L}$ gives rise to the homomorphism $\mathfrak{L} : K_1(R) \to \bar{P}$. We shall work mainly with the case of group rings. This case has some particularities which we shall now explain.

2.1. **The case of group rings.** Here we consider a group G and an epimorphism $\xi : G \to \mathbf{Z}$. We shall work here with the terminology of 0.1.4. Thus $H = \operatorname{Ker} \xi$, Γ stands for the set of conjugacy classes of G, Γ_n is the set of all the conjugacy classes contained in $G_{(n)} = \xi^{-1}(n)$. Recall also the abelian group $\mathfrak{G} = \prod_{n\leqslant 0} \mathbf{Q}\Gamma_n$. Set $\mathfrak{G}_+ = \prod_{n<0} \mathbf{Q}\Gamma_n$. We choose and fix any element $t \in G_{(-1)}$.

Lemma 2.1. *There is an isomorphism* $\bar{P}_+ \xrightarrow{i} \mathfrak{G}_+$, *such that the following diagram is commutative*

$$\begin{tikzcd} \bar{P}_+ \arrow[rr, "i"] & & \mathfrak{G}_+ \\ & P \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Here the oblique arrows are natural projections (they are only abelian groups homomorphisms, not ring homomorphisms)

Proof. It is convenient to use another description of $\mathfrak{G}$, made by analogy with $\bar{P}$. Namely, let $n \geqslant 0$. Let $\bar{R}'_n$ be the abelian subgroup of R_n, generated by all the commutators $[x, y]$ where $x \in R_k, y \in R_s, k+s = n$ (we do not assume that k, s are positive). Set $\bar{R}_n = R_n/R'_n$ and consider an abelian group $\bar{R} = \prod_{i=0}^{\infty} \bar{R}_i$, so that $\bar{R}$ is a quotient of R. Then $\bar{R}$ is identified in an obvious way with $\mathfrak{G}$. There is an inclusion $P'_n \subset R'_n$, and the natural projection $\pi_n : \bar{P}_n \to \bar{R}_n$, and to prove the lemma it suffices to prove that π_n is an isomorphism for $n > 0$. The automorphism $\rho : A \to A$ extends to an automorphism $\rho : P \to P$ of the ring P by the following formula: $\rho(\sum_i a_i t^i) = \sum \rho(a_i)t^i$. The set P' is preserved by ρ, so ρ determines an isomorphism of abelian groups $\bar\rho : \bar P \to \bar P$. Note that for $n > 0$ the homomorphism $\bar\rho_n : \bar P_n \to \bar P_n$ equals id. (Indeed, let $a \in P, a = bt$ with $b \in P_{n-1}$. Write $a - \rho(a) = bt - \rho(bt) = t\rho(b) - \rho(bt) = \rho(tb - bt)$. The last element is obviously in $\bar P$.) Now we can prove the proposition. Let $xy - yx \in R'_n$, where $x = at^p, y = bt^{-q}$ with $a, b \in A_0, p, q \geqslant 0$ and $p - q = n > 0$. Consider the element $yx = t^{-p}(t^p bt^{-q}a)t^p \in P_n$. Its class in $\bar P_n$ equals the class of $t^p bt^{-q}a$, which equals the class of $at^p bt^{-q} = xy$ and we are over. $\square$

Thus the homomorphism $\mathfrak{L} : K_1(R) \to \bar P$ can be identified with a homomorphism $K_1(\widehat\Lambda_{\xi,\mathbf{Q}}) \to \mathfrak{G}$, which will be denoted by the same symbol $\mathfrak{L}$. It is easy to see that $\mathfrak{L}$ does not depend on a particular choice of $t \in G_{(-1)}$ so it depends only on G and ξ.

It follows now from Proposition 0.1 that $\mathfrak{L}$ vanishes on the subgroup $\pm\widehat G$ of $K_1(\widehat\Lambda_{\xi,\mathbf{Q}})$. Composing with the embedding $\widehat\Lambda_\xi \subset \widehat\Lambda_{\xi,\mathbf{Q}}$ we obtain thus a homomorphism $\mathfrak{L} : K_1(\widehat\Lambda_\xi) \to \mathfrak{G}$, which factors through $\widehat{\mathrm{Wh}}(G, \xi)$ and the image of which is contained in $\mathfrak{G}_+$.

3. Application of the logarithms to the study of the groups $K_1(P), K_1(R)$

We have seen that the group $K_1(\widehat\Lambda_\xi)$ admits a natural splitting. The last two direct summands of this splitting, namely the Whitehead group $K_1(\mathbf{Z}H)$ and the *Nil*-group of H are well known in K-theory. This is not the case with the group $\widetilde W$, the structure of which is not yet well understood. The situation is not clear even if we consider K_1 of the rational Novikov ring $\widehat\Lambda_{\xi,\mathbf{Q}}$. The logarithm maps, defined above, can shed some light on the structure of these groups. In the rest of this section we shall be working in the terminology of 0.1.4. Thus $\mathbf{Q} \subset A, P = A_\rho[[t]], R = A_\rho((t))$. We shall describe explicitly the kernel of the map $\mathrm{Log} : W \to \bar P$. This provides a family of non-vanishing elements in the group $\widetilde W = \mathrm{Im}\,(I : W \to K_1(P))$, since $\mathrm{Log} = \mathcal{L} \circ I$ and the elements of W, which are not in $\mathrm{Ker\,Log}$ are not in the $\mathrm{Ker}\, I$. (Note that since the map $\widetilde W \to K_1(R)$ is a split monomorphism, we get automatically the information about the group $\mathrm{Im}\,(W \to K_1(R))$.)

3.1. **Power series.** We have already mentioned that $K_1(P) \approx \widetilde W \oplus K_1(A)$. Thus the computation of $K_1(P)$ is reduced to the computation of $K_1(A)$ and the image of W in $K_1(P)$. Recall from Section 1 a commutative diagram:

$$
\begin{tikzcd}
W \arrow[r, "I"] \arrow[dr, "\mathrm{Log}"'] & K_1(P) \arrow[d, "\mathcal{L}"] \\
 & \bar{P}_+
\end{tikzcd}
$$

Thus Ker $I \subset$ Ker Log . In particular $W_1 = W \cap [P^\bullet, P^\bullet]$ is in Ker Log . [1] Since Log is continuous in the natural topology of P, the closure $\overline{W_1}$ is also in Ker Log . It turns out that for the rings A satisfying the property (*) below the kernel of Log coincides with $\overline{W_1}$.

(*) Every element of A is a sum of invertible elements

(This condition is satisfied, for example, for the rational group rings.) Let us denote by $\overline{\mathrm{Log}}$ the homomorphism $W/\overline{W_1} \to \bar{P}_+$ derived from Log .

Theorem 3.1. *If A satisfies (*), then $\overline{\mathrm{Log}} : W/\overline{W_1} \to \bar{P}_+$ is an isomorphism.*

Proof. Our task is to check the injectivity. Let $\alpha \in W$ belong to Ker $\overline{\mathrm{Log}}$. We shall find a sequence of elements $\beta_n \in W_1$ such that $\nu(1 - \alpha \cdot \beta_1 \cdot \dots \cdot \beta_n) \to \infty$ and $\nu(1 - \beta_n) \to \infty$ as $n \to \infty$. This will prove our result since the element $\beta = \lim_{n\to\infty} \beta_1 \cdot \dots \cdot \beta_n$ is in $\overline{W_1}$ and $\alpha = \beta^{-1}$. The existence of such a sequence of elements follows from the next lemma.

Lemma 3.2. *Let $\alpha \in W, \nu(1-\alpha) = n$. Assume that $\nu(\mathrm{Log}\,(\alpha)) \geqslant n+1$. Then there is an element $X \in W_1, \nu(1-X) = n$, such that $\nu(1-\alpha X) \geqslant n+1$.*

Proof. Let $\alpha = 1 + x + O(t^{n+1})$ (where $x \in P_n$ and $O(t^{n+1})$ stands for "terms of degree $\geqslant n+1$"). Then $\log \alpha = x + O(t^{n+1})$, and thus $x \in P'_n$. Let us denote by $P^\bullet_n$ the subgroup of $P^\bullet$ consisting of all elements $\xi = \xi_0 + \xi_1 + \dots + \xi_n + \dots$, such that $\xi_0 = 1$ and $\xi_i = 0$ for $0 < i < n$. Our lemma will be proved if we show the following:

Lemma 3.3. *Let $z \in P'_n, n > 0$. There is $\xi \in P^\bullet_n \cap W_1$, such that $\xi_n = z$.*

[1] Recall, that the symbol $[P^\bullet, P^\bullet]$ denotes the commutator of the group $P^\bullet$ in the group-theoretic sense, that is $[P^\bullet, P^\bullet]$ is the subgroup in $P^\bullet$ generated by the set $\{ghg^{-1}h^{-1} | g, h \in P^\bullet\}$.

Proof. Let us call an element $z \in P_n$ *regular* if $z = \xi_n$ for some $\xi \in P_n^\bullet \cap W_1$. Note that a sum of regular elements is again regular. Thus it suffices to prove that for any elements $x \in P_s, y \in P_r$ with $s+r = n$ and $s, r \geqslant 0$ the commutator $xy - yx$ is regular. We consider two cases:

1. One of $\nu(x), \nu(y)$ is zero. Assume for example that $\nu(x) = 0$. In view of the condition $(*)$ we can assume that x is invertible. Then $xy - yx = \xi_n$ where $\xi = x(1+yx)x^{-1}(1+yx)^{-1}$.
2. Both $\nu(x), \nu(y)$ are strictly positive. Let $\xi = (1-x)(1-y)(1-x)^{-1}(1-y)^{-1}$. Expanding this expression, one shows easily that the first non-zero term of $\xi - 1$ equals $xy - yx$.

□

Corollary 3.4. *If A satisfies $(*)$ then* $\overline{\text{Ker }(W \to K_1(R))} = \overline{\text{Ker } I} = \overline{W_1} = \text{Ker Log}$. □

4. Counting closed orbits: preliminaries

The proof of Main Theorem uses the techniques from the author's papers [Pa3], [Pa4], [Pa6]. In this section we recall the necessary material following essentially [Pa6]. The main objects in this section are C^0-*generic gradients* of a Morse map $f : M \to S^1$. Subsections 4.1 and 4.2 contain recollections of some more or less standard material in Morse theory, centered around Morse complexes and handle decompositions. The notion of *gradient descent homomorphism* associated to a given f-gradient is the contents of the section 4.4. This homomorphism is defined only for some special class of gradients. The C^0-generic condition $(\mathfrak{C})$ on vector fields which suffices to define such a homomorphism is the contents of the section 4.3. Recall from [Pa1] that for every f-gradient v satisfying Transversality Condition there is a chain homotopy equivalence $\widetilde{C}_*(v) \to \widehat{C}_*^\Delta(M, \xi)$. In the paper [Pa6] we proved that for a C^0-generic gradient satisfying Transversality Condition the chain homotopy equivalence above can be chosen so that its torsion is computable in terms of the gradient descent homomorphism associated to v. The resulting formula (22) for the torsion is given in the subsection 4.5. The proof of Main Theorem will be obtained in the Section 5 by identifying the right hand side of (22) with the non-abelian eta function of the gradient flow.

4.1. **Morse functions and their gradients.** Let v be a C^1 vector field on a manifold M. The value of the integral curve of v passing by x at $t = 0$ will be denoted by $\gamma(x, t; v)$. The set of all critical points of a Morse function f will be denoted by $S(f)$; the set of all critical points of f of index k will be denoted by $S_k(f)$.

Let $f : W \to [a,b]$ be a Morse function on a cobordism W, so that $f^{-1}(b) = \partial_1 W, f^{-1}(a) = \partial_0 W, \partial W = \partial_1 W \sqcup \partial_0 W$. We shall call gradient-like vector fields of f (in the sense of the definition in [Mi1], §3.1) also *f-gradients*. The set of all f-gradients will be denoted by $G(f)$. Let $v \in G(f)$. Set $U_1 = \{x \in \partial_1 W | \gamma(x, \cdot; v) \text{ reaches } \partial_0 W\}$. Then U_1 is an open subset of $\partial_1 W$ and the gradient descent along the trajectories of v determines a diffeomorphism of the open subset U_1 of $\partial_1 W$ onto an open subset U_0 of $\partial_0 W$. This diffeomorphism will be denoted by $(-v)^{\leadsto}$ and we abbreviate $(-v)^{\leadsto}(X \cap U_1)$ to $(-v)^{\leadsto}(X)$. For $x \in S(f)$ we denote by $D(x,v)$ the *descending disc of* x, that is the set of all $y \in W$, such that $\gamma(y,t;v) \xrightarrow[t\to\infty]{} x$.

We say that v satisfies *Almost Transversality Condition*, if

$$\big(x, y \in S(f) \ \& \ \mathrm{ind} x \leqslant \mathrm{ind} y\big) \Rightarrow \big(D(x,v) \pitchfork D(y,-v)\big)$$

We say that v satisfies *Transversality Condition*, if

$$\big(x, y \in S(f)\big) \Rightarrow \big(D(x,v) \pitchfork D(y,-v)\big)$$

The set of all f-gradients satisfying Transversality Condition, resp. Almost Transversality Condition will be denoted by $GT(f)$, resp. by $GA(f)$.

A very useful class of Morse functions is that of *ordered functions*, (which is a slightly wider class than the class of self-indexing functions of S.Smale). We say that $f : W \to [a,b]$ is *ordered* Morse function with an *ordering sequence* $(a_0, \dots, a_{n+1})$, if $a = a_0 < a_1 < \dots < a_{n+1} = b$ are regular values of f such that $S_i(f) \subset f^{-1}(]a_i, a_{i+1}[)$. (Here $n = \dim W$.)

In many cases the considerations involving gradients of arbitrary Morse functions can be reduced to gradients of ordered Morse functions. (This is for example the case with the definition of the Morse-Thom-Smale-Witten complex of a Morse function.)

A Morse function $\phi : W \to [\alpha, \beta]$ is called *adjusted to* (f,v), if:

1) $S(\phi) = S(f)$, and v is also a ϕ-gradient.

2) The function $f - \phi$ is constant in a neighborhood of $\partial_0 W$, in a neighborhood of $\partial_1 W$, and in a neighborhood of each point of $S(f)$.

One can show that for an arbitrary Morse function f and an f-gradient satisfying the Almost Transversality Condition, there is an ordered Morse function g, adjusted to (f,v).

4.2. **δ-thin handle decompositions.** In this subsection W is a riemannian cobordism of dimension n, $f : W \to [a,b]$ is a Morse function on W, and v is an f-gradient. We denote $W \setminus \partial W$ by W°.

Let $p \in W^\circ$. Let $\delta > 0$. Assume that for some $\delta_0 > \delta$ the restriction of the exponential map $\exp_p : T_pW \to W$ to the disc $B^n(0,\delta_0)$ is a diffeomorphism on its image. Denote by $B_\delta(p)$ (resp. $D_\delta(p)$) the riemannian open ball (resp. closed ball) of radius δ centered in p. We shall use the notation $B_\delta(p), D_\delta(p)$ only when the assumption above on δ holds. Set

$$B_\delta(p,v) = \{x \in W \mid \exists t \geqslant 0 : \gamma(x,t;v) \in B_\delta(p)\}$$
$$D_\delta(p,v) = \{x \in W \mid \exists t \geqslant 0 : \gamma(x,t;v) \in D_\delta(p)\}$$

We denote by $D(\mathrm{ind}\leqslant s\,; v)$ the union of $D(p,v)$ where p ranges over critical points of f of index $\leqslant s$. We denote by $B_\delta(\mathrm{ind}\leqslant s\,; v)$, resp. by $D_\delta(\mathrm{ind}\leqslant s\,; v)$ the union of $B_\delta(p,v)$, resp. of $D_\delta(p,v)$, where p ranges over critical points of f of index $\leqslant s$. We shall use similar notation like $D_\delta(\mathrm{ind}{=}s\,; v)$ or $B_\delta(\mathrm{ind}\geqslant s\,; v)$, which is now clear without special definition. Set

$$C_\delta(\mathrm{ind}\leqslant s\,; v) = W \setminus B_\delta(\mathrm{ind}\leqslant n-s-1\,; -v)$$

Let $\phi : W \to [a,b]$ be an ordered Morse function with an ordering sequence $(a_0 < a_1 \cdots < a_{n+1})$. Let w be a ϕ-gradient. Denote $\phi^{-1}([a_i,a_{i+1}])$ by W_i.

Definition 4.1. We say that w is *δ-separated with respect to ϕ* (and the ordering sequence $(a_0,\dots,a_{n+1})$), if

i) for every i and every $p \in S_i(f)$ we have $D_\delta(p) \subset W_i^\circ$;

ii) for every i and every $p \in S_i(f)$ there is a Morse function $\psi : W_i \to [a_i,a_{i+1}]$, adjusted to $(\phi \mid W_i, w)$ and a regular value λ of ψ such that

$$D_\delta(p) \subset \psi^{-1}(]a_i,\lambda[)$$

and for every $q \in S_i(f), q \neq p$ we have

$$D_\delta(q) \subset \psi^{-1}(]\lambda, a_{i+1}[)$$

We say that w is *δ-separated* if it is δ-separated with respect to some ordered Morse function $\phi : W \to [a,b]$, adjusted to (f,v). Each f-gradient satisfying Almost Transversality Condition is δ-separated for some $\delta > 0$.

Proposition 4.2 ([Pa6], Prop. 3.2, 4.1). *If v is δ_0-separated, then $\forall \delta \in [0,\delta_0]$ and $\forall s : 0 \leqslant s \leqslant n$*

1. *$D_\delta(\mathrm{ind}\leqslant s\,; v)$ is compact.*
2. $\bigcap_{\theta>0} B_\theta(\mathrm{ind}\leqslant s\,; v) = D(\mathrm{ind}\leqslant s\,; v)$

3. *$B_\delta(\mathrm{ind}_{\leqslant s}\ ; v) = \mathrm{Int}\ D_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$ and $D_\delta(\mathrm{ind}_{\leqslant s}\ ; v) \subset C_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$.*
4. *$H_*(D_\delta(\mathrm{ind}_{\leqslant s}\ ; v) \cup \partial_0 W, D_\delta(\mathrm{ind}_{\leqslant s-1}\ ; v) \cup \partial_0 W)$ equals 0 if $* \neq s$ and is a free module generated by the classes of the descending discs $D(p, v)$ with $p \in S_s(f)$.*

Thus the collection of descending discs $D(p, v)$ form a sort of stratified manifold, and the open sets $B_\delta(v)$ form a family of δ-thin neighborhoods of this manifold.

We shall often denote $D_\delta(\mathrm{ind}_{\leqslant s}\ ; v)$ by $W^{\{\leqslant s\}}$ if the values of v, f, δ are clear from the context.

4.3. **Condition ($\mathfrak{C}$).** In this subsection we recall the condition ($\mathfrak{C}$) on the gradient v. If this condition holds, the gradient descent map, corresponding to v can be endowed with a structure, resembling closely the cellular maps between CW-complexes. We shall explain this cellular-like structure in the section 4.4. We begin by stating the condition ($\mathfrak{C}$). Let $f : W \to [a, b]$ be a Morse function on a riemannian cobordism W, v be an f-gradient.

Definition 4.3. ([Pa6], Def. 4.5)

We say that v *satisfies condition* ($\mathfrak{C}$) if there are objects 1) - 4), listed below, with the properties (1 - 3) below.

Objects:

1) An ordered Morse function ϕ_1 on $\partial_1 W$ and a ϕ_1-gradient u_1.
2) An ordered Morse function ϕ_0 on $\partial_0 W$ and a ϕ_0-gradient u_0.
3) An ordered Morse function ϕ on W adjusted to (f, v).
4) A number $\delta > 0$.

Properties:

(1) u_0 is δ-separated with respect to ϕ_0, u_1 is δ-separated with respect to ϕ_1, v is δ-separated with respect to ϕ.

$$(2) \qquad (-v)^{\leadsto}\Big(C_\delta(\mathrm{ind}_{\leqslant j}\ ; u_1)\Big) \cup \Big(D_\delta(\mathrm{ind}_{\leqslant j+1}\ ; v) \cap \partial_0 W\Big) \subset B_\delta(\mathrm{ind}_{\leqslant j}\ , u_0) \text{ for every } j$$

$$(3) \qquad \overset{\leadsto}{v}\Big(C_\delta(\mathrm{ind}_{\leqslant j}\ ; -u_0)\Big) \cup \Big(D_\delta(\mathrm{ind}_{\leqslant j+1}\ ; -v) \cap \partial_1 W\Big) \subset B_\delta(\mathrm{ind}_{\leqslant j}\ ; -u_1) \text{ for every } j$$

$\triangle$

The set of all f gradients satisfying $(\mathfrak{C})$ will be denoted by $GC(f)$.

Theorem 4.4. *([Pa6], Th. 4.6) $GC(f)$ is open and dense in $G(f)$ with respect to C^0 topology. Moreover, if v_0 is any f-gradient then one can choose a C^0 small perturbation v of v_0 such that $v \in GC(f)$ and $v = v_0$ in a neighborhood of ∂W.*

4.4. **Homological gradient descent and a cellular approximation for $(-v)^{\leadsto}$.** As we have already mentioned the application $(-v)^{\leadsto}$ is not everywhere defined. But if the gradient v satisfies the condition $(\mathfrak{C})$, we can associate to v some family of continuous maps which plays the role of "cellular approximation" of $(-v)^{\leadsto}$, and a homomorphism $\mathcal{H}(-v)$ (*homological gradient descent*) which is a substitute for the homomorphism induced by $(-v)^{\leadsto}$ in homology.

So let v be an f-gradient satisfying $(\mathfrak{C})$. It will be convenient to denote $\partial_0 W$ by V_a, and $\partial_1 W$ by V_b. We have the corresponding functions ϕ_0, ϕ_1 on V_a, resp. V_b; and their gradients u_0, u_1 give rise to the corresponding handle-like filtrations

$$V_a^{\{\leqslant s\}} = D_\delta(\mathrm{ind}_{\leqslant s}\,; u_0), \quad V_b^{\{\leqslant s\}} = D_\delta(\mathrm{ind}_{\leqslant s}\,; u_1) \tag{16}$$

Consider the set Q_s of all $x \in V_b^{\{\leqslant s\}}$ where $(-v)^{\leadsto}$ is *not* defined. Equivalently,

$$Q_s = \{x \in V_b^{\{\leqslant s\}} | \gamma(x, t; -v) \text{ converge to a point in } S(f)\}.$$

This is a compact set, and the condition $(\mathfrak{C})$ implies that this set is a subset of $D(\mathrm{ind}_{\geqslant n-s}\,; -v)$. Therefore there is a neighborhood U of Q_s in $V_b^{\{\leqslant s\}}$ such that $(-v)^{\leadsto}(U)$ is in $D_\delta(\mathrm{ind}_{\leqslant s}\,; v) \cap V_a$ and this last set is in Int $V_a^{\{\leqslant s-1\}}$ (again by $(\mathfrak{C})$). It follows that the map $(-v)^{\leadsto}$ gives rise to a well-defined continuous map

$$V_b^{\{\leqslant s\}} / V_b^{\{\leqslant s-1\}} \to V_a^{\{\leqslant s\}} / V_a^{\{\leqslant s-1\}} \tag{17}$$

The family of these maps (one map for each $s : 0 \leqslant s \leqslant n-1$) is the substitute for "cellular approximation" of $(-v)^{\leadsto}$. Technically it is more convenient for us to quotient out a part of $V_b^{\{\leqslant s\}}$ a bit larger than $V_b^{\{\leqslant s-1\}}$. Recall that ϕ_1 is an ordered Morse function; let $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ be the ordering sequence for ϕ_1 with respect to which v is δ-separated. Let

$$V_b^{(s)} = \phi_1^{-1}([\alpha_0, \alpha_{s+1}]), \quad V_b^{[s]} = D_\delta(\mathrm{ind}_{\leqslant s}\,; u_0) \cup V_b^{(s-1)}$$

Then $V_b^{\{\leqslant s\}} \subset V_b^{[s]}$ and it is easy to see that the map (17) is factored through the map

$$v\!\downarrow\colon V_b^{[s]}/V_b^{(s-1)} \to V_a^{[s]}/V_a^{(s-1)} \tag{18}$$

The pair $(V_b^{[s]}, V_b^{(s-1)})$ is homotopy equivalent to $(V_b^{\{\leqslant s\}}, V_b^{\{\leqslant s-1\}})$, but it has the advantage that $V_b^{[s]}/V_b^{(s-1)}$ splits as the wedge of spaces corresponding to the descending discs. Namely, for $p \in S_s(f)$ let $\Sigma(p) = D(p,v)/D(p,v)\cap V_b^{(s-1)}$ and $\Sigma_\delta(p) = D_\delta(p,v)/D_\delta(p,v)\cap V_b^{(s-1)}$. Then $\Sigma(p)$ is homeomorphic to a s-dimensional sphere, the inclusion $\Sigma(p) \hookrightarrow \Sigma_\delta(p)$ is a homotopy equivalence, and $V_b^{[s]}/V_b^{(s-1)}$ is the wedge of all $\Sigma_\delta(p)$ with $p \in S_s(f)$. The homomorphism induced by $v\downarrow$ in homology is called *homological gradient descent* and denoted $\mathcal{H}_s(-v)$. The key property of the homological gradient descent allowing in particular to compute the boundary operators in the Novikov complex in terms of $\mathcal{H}_s(-v)$ is the following:

Let N be an oriented submanifold of V_b, such that $N \subset V_b^{[s]}$ and $N \setminus \operatorname{Int} V_b^{(s-1)}$ is compact. Then the manifold $N' = (-v)^{\rightsquigarrow}(N)$ is in $V_a^{[s]}$ and $N' \setminus \operatorname{Int} V_a^{(s-1)}$ is compact and the fundamental class of N' modulo $V_a^{(s-1)}$ equals $\mathcal{H}_s(-v)\big([N]\big)$.

4.5. **Circle valued Morse maps.** Now we can proceed to the circle-valued Morse maps. First of all we introduce the basic terminology. Let $f: M \to S^1$ be such a map. Assume that f is primitive, that is $f_*: H_1(M) \to H_1(S^1) = \mathbf{Z}$ is epimorphic. To simplify the notation we shall assume that $1 \in S^1$ is a regular value for f. Denote $f^{-1}(1)$ by V. Let $\mathcal{C}: \bar{M} \to M$ be the infinite cyclic covering, associated to f, and $F: \bar{M} \to \mathbf{R}$ be a lifting of f. Set

$$V_\alpha = F^{-1}(\alpha), \quad W = F^{-1}\big([0,1]\big), \quad V^- = F^{-1}\big(]-\infty, 1]\big).$$

Thus the cobordism W is the result of cutting M along V. The structure group of $\mathcal{C}$ is isomorphic to $\mathbf{Z}$ and we choose the generator t of this group so as to satisfy $V_\alpha t = V_{\alpha-1}$. Denote Wt^s by W_s; then $\bar{M}$ is the union $\cup_{s\in\mathbf{Z}} W_s$, the neighbor copies W_{s+1} and W_s intersecting by V_{-s}. For any $k \in \mathbf{Z}$ the restriction of $\mathcal{C}$ to V_k is a diffeomorphism $V_k \to V$. Endow M with an arbitrary riemannian metric and lift it to a t-invariant riemannian metric on $\bar{M}$. Now W is a riemannian cobordism, and $t^{-1}: \partial_0 W = V_0 \to \partial_1 W = V_1$ is an isometry. We shall say that v satisfies condition $(\mathfrak{C}')$ if the $(F|W)$-gradient v satisfies the condition $(\mathfrak{C})$ from subsection 4.3, and, moreover, the Morse functions ϕ_0, ϕ_1 and their gradients u_0, u_1 can be chosen so as to satisfy $\phi_0(xt) = \phi_1(x), t_*(u_1) = u_0$. The set of Kupka-Smale f-gradients v satisfying

$(\mathfrak{C}')$ will be denoted by $\mathcal{G}_0(f)$. The set $\mathcal{G}_0(f)$ is C^0-open and dense in $\mathcal{G}(f)$ (this is a version of the theorem 4.4, see [Pa6], §8). Let $v \in \mathcal{G}_0(f)$. The condition $(\mathfrak{C}')$ provides a Morse function ϕ_1 on V_1 together with its gradient u_1, and a Morse function $\phi_0 = \phi_1 \circ t^{-1}$ on V_0 together with its gradient $u_0 = t_*(u_1)$. For every $k \in \mathbf{Z}$ we obtain also an ordered Morse function $\phi_k = \phi_0 \circ t^k : V_k \to \mathbf{R}$ and a ϕ_k-gradient $u_k = (t^{-k})_*(u_0)$.

The universal covering $\mathcal{P} : \widetilde{M} \to M$ factors through $\mathcal{C}$, that is, there is a covering $p : \widetilde{M} \to \bar{M}$ with structure group $H = \operatorname{Ker} \xi$ such that $\mathcal{C} \circ p = \mathcal{P}$. For a subset $X \subset \bar{M}$ we shall denote $p^{-1}(X)$ also by $\widetilde{X}$. Let $v \in \mathcal{G}_0(f)$. For $k \in \mathbf{Z}$ we have the H-invariant filtrations $\widetilde{V}_k^{\{\leqslant s\}}$ of $\widetilde{V}_k$ and the corresponding equivariant version of the homological gradient descent. That is for every $k \in \mathbf{Z}, s \in \mathbf{N}$ we have a continuous H-equivariant map

$$\widetilde{v}\!\downarrow : \widetilde{V}_k^{[s]}/\widetilde{V}_k^{(s-1)} \to \widetilde{V}_{k-1}^{[s]}/\widetilde{V}_{k-1}^{(s-1)} \tag{19}$$

and the homomorphism $\widetilde{\mathcal{H}}(-v)$, induced by $\widetilde{v}\downarrow$ in homology. Let us introduce the group

$$T_s = \bigoplus_{k\in\mathbf{Z}} H_*(\widetilde{V}_k^{[s]}/\widetilde{V}_k^{(s-1)}) \tag{20}$$

This abelian group has the obvious structure of free $\mathbf{Z}G$-module, and the homomorphism $(v\downarrow)_*$ is a $\mathbf{Z}G$-homomorphism of this module. Let τ_s be its matrix. Note that the matrix entries of τ_s are in $\mathbf{Z}G_{(-1)}$. Thus the matrix $1-\tau_s$ is invertible over the ring $\widehat{\Lambda}_\xi$ and the image of $1-\tau_s$ in $K_1(\widehat{\Lambda}_\xi)$ belongs to $\widehat{W}$. In the statement of the theorem below we shall keep the notation $1-\tau_s$ for the image of this element in $\widehat{W}$.

Theorem 4.5 ([Pa6], Corollary 7.16). *Let $v \in \mathcal{G}_0(f)$. There is a chain homotopy equivalence*

$$\phi : \widetilde{C}_*(v) \to \widehat{C}_*^{\Delta}(M, \xi) \tag{21}$$

such that

$$\tau(\phi) = \prod_{s=0}^{n-1} (1-\tau_s)^{(-1)^s} \tag{22}$$

□

In [Pa6] this theorem is stated for the case of the maximal abelian covering instead of universal covering, but the arguments are carried over to the present case without difficulty.

Remark 4.6. We take here the opportunity to note that the cited formula from [Pa6], and, consequently, the formula (3) of the main theorem of [Pa6] contain a sign error. With the sign convention of [Mi2], accepted in [Pa6], the formula (3) of the main theorem of [Pa6] should read: $\tau(\phi|G) = -\overline{\zeta_L(-v)}$ instead of $\tau(\phi|G) = \overline{\zeta_L(-v)}$ in the paper.

5. Proof of the main theorem

Recall from 4.5 the subset $\mathcal{G}_0(f) \subset \mathcal{G}(f)$. Let $v \in \mathcal{G}_0(f)$. In order to prove the main theorem it remains to show that the homomorphism $\mathfrak{L}$ evaluated on the right hand side of (22) equals to $-\eta_L(-v)$. The manifold V being identified with $V_0 \subset \bar{M}$ inherits from V_0 the handle-like filtrations $V^{[s]}, V^{(s)}$. Let $Cl^{[s]}(-v)$ be the subset of all the closed orbits of $(-v)$ passing through a point of $V^{\{\leqslant s\}} \setminus V^{\{\leqslant s-1\}}$. It follows from the property $(\mathfrak{C}')$ that the set $Cl(-v)$ is the disjoint union of its subsets $Cl^{[s]}(-v)$. Therefore if we define

$$\eta_s(-v) = \sum_{\gamma \in Cl^{[s]}(-v)} \varepsilon(\gamma) \frac{\{\gamma\}}{m(\gamma)}, \tag{23}$$

it suffices to prove that for every s we have:

$$(-1)^{s+1} \mathfrak{L}(1 - \tau_s) = \eta_s(-v) \tag{24}$$

So we fix s up to the end of the proof. We shall abbreviate $Cl^{[s]}(-v)$ to Cl. Recall from 4.5 that for every $k \in \mathbf{Z}$ we have a continuous map

$$\widetilde{v}\!\downarrow : \widetilde{V}_k^{[s]} / \widetilde{V}_k^{(s-1)} \to \widetilde{V}_{k-1}^{[s]} / \widetilde{V}_{k-1}^{(s-1)} \tag{25}$$

constructed from the gradient descent map.

Let $B_k = \widetilde{V}_k^{[s]} / \widetilde{V}_k^{(s-1)}$ and $B = \sqcup_k B_k$. We shall denote by ω_k the point $[\widetilde{V}_k^{(s-1)}]$ of B_k. The point ω_0 will be also denoted by $*$. There is a natural action of G on B, derived from the action of G on $\widetilde{M}$; every element $g \in G_{(-n)}$ sends B_k homeomorphically to B_{k-n} and ω_k to ω_{k-n}. In particular every B_k is H-invariant. The map $\widetilde{v}\!\downarrow : B_k \to B_{k-1}$ is H-equivariant and $\tau_s = (\widetilde{v}\!\downarrow)_* : \oplus_k H_s(B_k) \to \oplus_k H_s(B_{k-1})$.

Now we shall make an additional auxiliary choice: we choose and fix for every critical point $p \in S_s(f)$ its lifting to $\widetilde{V}_0$. The system of these liftings will be denoted by Υ. The reason for introducing Υ into the game is the following: the series $\eta_s(-v)$ is well defined only in $\mathfrak{G}$ which is a quotient of $P = \mathbf{Q}H_\rho[[t]]$. The choice of Υ enables us to construct a lifting of these power series to P. On the other hand the choice of Υ determines a base in the free $\mathbf{Z}G$-module $T_s = \oplus_k H_s(B_k)$ from (20) and so we obtain also a representative for $\mathfrak{L}(1-\tau_s)$ in P. Finally we shall compare the two resulting elements of P by a simple application of Lefschetz-Dold fixed point formula.

Note first of all that Υ determines liftings of the discs $D_\delta(p,v)$ to $\widetilde{V}_0 \subset \widetilde{M}$, and liftings of the thickened spheres $\Sigma_\delta(p)$ to B_0. The union of these lifted spheres is a subspace $\beta \subset B_0$, homeomorphic to the wedge of the thickened spheres themselves. Now

$$B_k = \vee_{g\in G_{(k)}} \beta\cdot g, \quad B = \cup_{g\in G}\beta\cdot g.$$

The homology classes in B_0 of the liftings of the spheres Σ_p form a $\mathbf{Z}$-base in $H_s(\beta)$ and a $\mathbf{Z}G$-base in $H_s(B)$. The element of this base corresponding to a critical point $p\in S_s(f)$ will be denoted by $[p]$. Now we can construct from Υ a lifting of $\mathfrak{L}(1-\tau_s)\in\bar{P}$ to $P=\mathbf{Q}H_\rho[[t]]$. Let $(R(k)_{qp})$ be the matrix of the $\mathbf{Z}G$-homomorphism τ_s^k with respect to our base. That is $\tau_s^k([p]) = \sum_{q\in S_s(f)}[q]\cdot R(k)_{qp}$ with $R(k)_{qp}\in\mathbf{Z}G_{(-k)}$. Set $\varkappa_k = \sum_p R(k)_{pp}\in\mathbf{Z}G_{(-k)}$ then the element

$$K(v) = -\sum_{k=1}^{\infty}\frac{\varkappa_k}{k}$$

is obviously a lifting to $P=\mathbf{Q}H_\rho[[t]]$ of $\mathfrak{L}(1-\tau_s)$.

To lift to P the right hand side of (23) we shall first translate our data to the language of fixed point theory. We shall say that a point $a\in\beta\setminus\{*\}$ is a *G-fixed-point* of $(\widetilde{v}\downarrow)^k$, if $(\widetilde{v}\downarrow)^k(a)=a\cdot g$. The element $g\in G_{(-k)}$ is uniquely determined by a and will be denoted by $g(a)$. The set of all G-fixed points of $(\widetilde{v}\downarrow)^k$ will be denoted by $GF(k)$. The set of all G-fixed points of $(\widetilde{v}\downarrow)^k$ with given $g(a)=g$ will be denoted by $GF(k,g)$. Thus $GF(k)=\sqcup_{g\in G_{(-k)}}GF(k,g)$. By analogy with standard fixed point theory we define the multiplicity $\mu(a)$ and the index ind$a$ for every $a\in GF(k)$.

Let $a\in GF(k)$. Let $a_i=(\widetilde{v}\downarrow)^i(a)$ and let $\widehat{a}_i$ be (the unique) point in β belonging to the G-orbit of a_i. The set of all $\widehat{a}_i, i\in\mathbf{N}$ will be called the *quasiorbit of* a, and denoted by $\mathcal{Q}(a)$; it is a finite subset of β of cardinality $\frac{k}{\mu(a)}$. Consider the integral curve γ of $(-v)$ in $\widetilde{M}$, such that $\gamma(0)=a$; then for some $T>0$ we have

$$\gamma(T) = (\widetilde{v}\downarrow)^k(a) = a\cdot g(a)\in\widetilde{V}_{-k}$$

The map $\mathcal{P}\circ\gamma:[0,T]\to M$ is then a closed orbit. Thus we obtain a map $\alpha: GF(k)\to Cl$ whose image is exactly the subset $Cl_k\subset Cl$, consisting of all $\gamma\in Cl$ with $f_*([\gamma])=-k$. For every $\gamma\in Cl_k$ the set $\alpha^{-1}(\gamma)$ is the quasiorbit of some $a\in GF_k$. (Note that the set $\mathcal{P}(\alpha^{-1}(\gamma))$ is the intersection of the orbit γ with V.) Further, for every $a\in\alpha^{-1}(\gamma)$

the projection of $g(a)$ to the set Γ of conjugacy classes of G equals to $\{\gamma\}$. Moreover, $\varepsilon(\gamma) = \mathrm{ind} a$ and $m(\gamma) = \mu(a)$. Thus the following power series

$$\nu(v) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{a \in GF_k} (\mathrm{ind} a) g(a) \tag{26}$$

is a lifting to $P = \mathbf{Q}H_\rho[[t]]$ of $\eta_s(-v)$.

To prove our theorem it remains to show that $\nu(v) = (-1)^{s+1} K(v)$. This follows obviously from the next lemma.

Lemma 5.1.

$$\varkappa_k = (-1)^s \sum_{a \in GF(k)} (\mathrm{ind} a) g(a) \tag{27}$$

Proof. The set $GF(k, g)$ is exactly the fixed point set of the following composition:

$$\beta = B_0 \xrightarrow{(\widetilde{v}\downarrow)^k} B_{(-k)} = \bigvee_{h \in G_{(-k)}} \beta \cdot h \xrightarrow{\pi_g} \beta \tag{28}$$

where π_g is the map which sends every component of the wedge to $*$, except the component βg, and this one is sent to β via the map $x \mapsto xg^{-1}$. The point $* \in \beta$ is a fixed point of this map, and its index equals to 1. Applying the Lefschetz-Dold fixed point formula, the proofs of the Lemma and the main theorem are now complete. □

6. Further remarks

In this section we shall discuss some particular cases of our results and open questions.

6.1. **Fibrations over a circle.** Consider the case of maps $f : M \to S^1$ without critical points. The main theorem then implies that the torsion of the acyclic $\widehat{\Lambda}_\xi$-complex $\widehat{C}_*^\Delta(M, \xi) = C_*^\Delta(\widetilde{M}) \underset{\Lambda}{\otimes} \widehat{\Lambda}_\xi$ satisfies

$$\mathfrak{L}(\tau(\widehat{C}_*^\Delta(M, \xi))) = -\eta_L(-v) \tag{29}$$

The gradient descent in this case gives rise to an everywhere defined diffeomorphism, say, g of $V = f^{-1}(\lambda)$ to itself (where λ is a regular value of f). The closed orbits of v are in one-to-one correspondence with periodic points of g, and the formula (29) thus gives an expression of a certain power series obtained via counting periodic points (eta function of g) in homotopy invariant terms. It would be interesting to compare this formula with other non-abelian invariants counting periodic points of diffeomorphisms like the invariants of Jiang (see [J]) and Geoghegan-Nicas (see [GN]).

6.2. **Case of abelian fundamental group.** Let us assume now that $G = \pi_1(M)$ is abelian, so that $G = H \times \mathbf{Z}$ and $\xi : G \to \mathbf{Z}$ is the projection onto the second direct summand. In this case the map $W \to \widehat{W}$ is an isomorphism (since for any commutative ring R we have $K_1(R) \approx R^\bullet \oplus SK_1(R)$). The element $\tau(\phi)$ from the main theorem can therefore be identified with an element of W. Another attractive feature of the commutative case is that the exponent of the element $\eta(-v)$ is well defined and belongs to the Novikov ring. This exponent is called *homological zeta function* and denoted by $\zeta_L(-v)$. The main theorem can now be reformulated as follows:

$$\tau(\phi) = (\zeta_L(-v))^{-1} \tag{30}$$

where both sides are elements of the group W. This formula is proved in [Pa6]. Actually one can say more: both the sides of (30) are rational functions (and not merely power series). See [Pa6] for details.

6.3. **The simplest possible example.** Let $M = S^1$, and consider the identity map $f = \mathrm{id} : S^1 \to S^1$. Then $\bar{M} = \widetilde{M} = \mathbf{R}$, choose the identity map as the lift of f to $\mathbf{R}$. Let t be the generator of $\mathbf{Z}$ acting on $\mathbf{R}$ as follows: $x \mapsto x - 1$. For every $k \geqslant 1$ there is a unique orbit of $(-v)$ in the class t^k; this orbit has index 1. Thus $\eta_L(-v) = \sum_{k\geqslant 1} t^k/k$. The $\mathbf{Z}[t,t^{-1}]$-complex $C_*(\mathbf{R})$ is of the form

$$\{0 \longleftarrow \Lambda \xleftarrow{\partial} \Lambda \longleftarrow 0\} \tag{31}$$

where $\partial(1) = 1 - t$. Tensoring over $\widehat{\Lambda} = \mathbf{Z}((t))$ makes this complex acyclic with torsion $\tau = \tau(C_*(\mathbf{R}) \underset{\Lambda}{\otimes} \widehat{\Lambda}) = 1 - t$ (see [Mi2], p.387) Thus

$$\ln \tau = \ln(1-t) = -\sum_{k\geqslant 1} \frac{t^k}{k} = -\eta(-v)$$

6.4. **About Witt vectors.** The theorem 3.1 together with Corollary 3.4 implies that $\overline{\mathrm{Log}} : W/\overline{W_0} \to \bar{P}_+$ is an isomorphism, where $W_0 = \mathrm{Ker}\,(W \to K_1(R))$. We do not know whether $\overline{W_0} \subset \mathrm{Ker}\,(W \to K_1(P))$.

<u>Problem</u> Compute $\overline{W_0}/W_0$. This abelian group is an invariant of the ring A and its automorphism ρ, which will be denoted by $\mathfrak{W}(A,\rho)$. Is it true that $\mathfrak{W}(A,\rho)$ vanishes in simple cases? for example when $A = \mathbf{Q}H$ where $H = \mathrm{Ker}\,(\xi : G \to \mathbf{Z})$ is a finite group? (when G is abelian the group $\mathfrak{W}(\mathbf{Q}H, \mathrm{id})$ vanishes by the obvious reasons).

This context suggests that the natural power series topology in the Novikov rings may have some deeper geometrical sense. It would be interesting to investigate the "continuous Whitehead groups" of the Novikov rings, defined by

$$K_1^c(\widehat{\Lambda}_\xi) = \mathrm{GL}\ (\infty, \widehat{\Lambda}_\xi)\Big/\overline{\mathcal{E}}$$

(where $\mathcal{E} = [\mathrm{GL}\ (\infty, \widehat{\Lambda}_\xi), \mathrm{GL}\ (\infty, \widehat{\Lambda}_\xi)]$ is the commutator of the general linear group, and $\overline{\mathcal{E}}$ is its closure).

6.5. **Two questions about gradient flows. 1.** Note that both parts of the equality (8) are defined for every Kupka-Smale f-gradient v, not only for C^0-generic one.

Question: Does the equality (8) hold also for arbitrary Kupka-Smale gradients?

2. Assume now that $\eta_L(-v) = 0$. Then the element $\tau(\phi)$, which is in the subgroup $\widetilde{W} \subset K_1(R)$ by (22), belongs to Ker $\mathfrak{L} : K_1(R) \to \bar{P}_+$. Therefore, the element $\tau(\phi)$ belongs to the subgroup $\overline{W_0}/W_0 \subset K_1(R)$ where $W_0 =$ Ker J and $\overline{W_0}$ is the closure of the subgroup W_0 in the natural topology, see 3 and 6.4.

Question: What is the geometric meaning of the image of $\tau(\phi)$ in $\mathfrak{W}(\mathbf{Q}H, \rho) = \overline{W_0}/W_0$? Can this element be interpreted in terms of closed orbits of the flow?

6.6. **Constructions with non-commutative localization.** One can show (see [Pa5]) that for C^0-generic gradients the Novikov complex $C_*(v)$ can be defined over a non-commutative localization $\Lambda_{(\xi)}$ of the group ring $\Lambda = \mathbf{Z}\pi_1(M)$. In the paper [FR] M.Farber and A.Ranicki give a purely algebraic construction of a chain complex $C_*^{FR}(M, f)$ over $\Lambda_{(\xi)}$ built from Morse-theoretic data associated to a circle-valued Morse function $f : M \to S^1$ and counting the localized homology of the universal covering. In the paper [R] A.Ranicki constructs an explicit isomorphism between the Novikov complex and the complex C_*^{FR}. It would be interesting to compare this isomorphism with the chain equivalence ϕ constructed in the present paper.

References

[AM] M.Artin, B.Mazur, *On periodic points*, Annals of Math. **102** (1965), 82 – 99.

[B] V.Baladi *Periodic orbits and dynamical spectra*, Ergodic theory and dynamical spectra, **18** (1998), 255 - 292.

[Bou] N.Bourbaki, *Groupes de Lie, Algèbres de Lie* , Springer, 1985

[Do] A.Dold *Lectures on Algebraic Topology* , Springer, 1972.

[FR] M.Farber, A.Ranicki, *The Morse-Novikov theory of circle-valued functions and non-commutative localization*, Proc. 1998 Moscow Conference for S.P.Novikov's 60th Birthday, Proc. Steklov Inst. 225, 381–388 (1999); e-print dg-ga/9812122

[Fel] A.Fel'shtyn, *Dynamical zeta functions, Nielsen Theory and Reidemeister torsion,* Memoirs of AMS **147** (2000).

[Fr1] D.Fried, *Periodic points and twisted coefficients*, Lect. Notes in Math., **1007**, (1983) 261–293.

[Fr2] D.Fried, *Homological Identities for closed orbits*, Inv. Math., **71**, (1983) 419–442.

[G] D.Grayson, *Higher Algebraic K-theory 2,* Lecture Notes in Math., **551**, Springer, (1976) 217 – 240

[GN] R.Geoghegan, A.Nicas, *Trace and torsion in the theory of flows*, Topology, **33**, (1994) 683 – 719

[J] B.Jiang, *Estimation of the number of periodic orbits* Pacific J. Math. **172**, 151–185 (1996)

[HL] M.Hutchings, Y-J.Lee *Circle-valued Morse theory, Reidemeister torsion and Seiberg-Witten invariants of 3-manifolds*, Topology, **38**, (1999), 861 – 888

[Ki] Kiang Tsai-han *The theory of Fixed point classes*, Springer, 1989.

[L] W.Lück, *The Universal Functorial Lefschetz Invariant* Fund. Math. 161, 167–215 (1999)

[Mi1] J. Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, 1965.

[Mi2] J.Milnor, *Whitehead Torsion*, Bull. Amer. Math. Soc. **72** (1966), 358 - 426.

[No] S.P.Novikov, *Mnogoznachnye funktsii i funktsionaly. Analog teorii Morsa*, Doklady AN SSSR **260** (1981), 31-35. English translation: S.P.Novikov. *Many-valued functions and functionals. An analogue of Morse theory*, Sov.Math.Dokl. **24** (1981), 222-226.

[Pa1] A.V.Pazhitnov, *On the Novikov complex for rational Morse forms*, Annales de la Faculté de Sciences de Toulouse, **4** (1995), 297 – 338.

[Pa2] A.V.Pajitnov, *Surgery on the Novikov Complex*, K-theory **10** (1996), 323-412.

[Pa3] A.V.Pajitnov, *Rationality and exponential growth properties of the boundary operators in the Novikov Complex*, Mathematical Research Letters, **3** (1996), 541-548.

[Pa4] A.Pajitnov *Ratsional'nost' granichnyh operatorov v komplekse Novikova v sluchae obschego polozheniya* Algebra i Analiz, **9**, no.5 (1997), p. 92–139. English translation: *The incidence coefficients in the Novikov complex are generically rational functions* St. Petersburg Mathematical Journal, **9**, no.5 (1998), p. 969–1006

[Pa5] A.V.Pajitnov, *Incidence coefficients in the Novikov Complex for Morse forms: rationality and exponential growth properties*, e-print dg-ga/9604004 20 Apr 96.

[Pa6] A.V.Pajitnov, *Simple homotopy type of the Novikov Complex and Lefschetz ζ-function of the gradient flow*, Russian Math.Surveys, 1999, **54** No. 1 [2]

[PR] A.V.Pajitnov, A.Ranicki, *The Whitehead group of the Novikov ring*, to appear in "K-theory"

[2] available also as e-print dg-ga/9706014.

[Po] M.M.Postnikov, *Gruppy i algebry Li* Moskva, Nauka, 1982 (in Russian)

[R] A.Ranicki, *The algebraic construction of the Novikov complex of a circle-valued Morse function* e-print, math.AT/9903090, 15 mar 99

[Ra] A.Ranicki, *High-dimensional knot theory*, Springer Mathematical Monograph, Springer (1998)

[Sch] D.Schütz, *Gradient flows of closed 1-forms and their closed orbits*, math.DG/0009055

[S] L.Siebenmann, *A total Whitehead torsion obstruction to fibering over the circle*, Comment. Math. Helv. **45** (1970), 1–48.

[Sm] S. Smale, *Differential dynamical systems*, Bull. Amer. Math. Soc. **73** (1967) 747–817.

UMR 6629 CNRS, Université de Nantes, Département de Mathématiques, 2, rue de la Houssinière, 44072, Nantes Cedex France

E-mail address: pajitnov@math.univ-nantes.fr