THE INCIDENCE COEFFICIENTS IN THE NOVIKOV COMPLEX ARE GENERICALLY RATIONAL FUNCTIONS

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Abstract. Let $f: M \to S^1$ be a Morse map. In [11], Novikov constructed an analog of the Morse complex for f. As its classical predecessor, this complex is constructed with the help of some additional data; in particular, a Riemannian metric or a gradient-like vector field must be fixed. The classical Morse complex is defined over the ring $\mathbb{Z}$, the Novikov complex is defined over the ring $\mathbb{Z}((t))$ of Laurent power series with integral coefficients and finite negative part, so that the matrix coefficients of the boundary operators are Laurent power series in one variable t. Novikov conjectured that (possibly, under some additional assumptions like analyticity or general position) the coefficients of each of this power series grow at most exponentially.

In this paper it is proved that for every Morse map f and every gradient-like vector field that is in general position with respect to the C^0-topology, all these power series are rational functions of t (obviously, this implies the exponential growth estimate).

Introduction

A. The Morse–Novikov theory. We briefly recall the classical construction of the Morse complex (see [8, 18, 27, 28]). Let $g: M \to \mathbb{R}$ be a Morse function and v a gradient-like vector field for g (obtained, e.g., from a Riemannian metric on M). Suppose that v satisfies the transversality condition. Then for every pair p, q of critical points of g satisfying $\operatorname{ind} p = \operatorname{ind} q + 1$, the set $L(p,q)$ of the orbits of g that join p and q is finite. For every critical point of g we fix an orientation of its stable manifold. Then to each orbit γ of v that joins p and q we can attribute a sign $\epsilon(\gamma) \in \{+1, -1\}$. Set $n(p,q) = \sum_{\gamma \in L(p,q)} \epsilon(\gamma)$. The integer $n(p,q)$ is called the *incidence coefficient* corresponding to p, q. We introduce the chain complex of free Abelian groups generated freely in dimension k by the critical points of g of index k. The boundary operator in this complex is defined by the formula $\partial_k p = \sum_q n(p,q) q$ (here p is a critical point of g of index k, and the summation index ranges over all critical points of g of index $k-1$). It can be proved that $\partial_k \circ \partial_{k+1} = 0$, and that the homology of the chain complex obtained is isomorphic to $H_*(M)$ (in more detail, this construction was described in [18, Appendix]). In the early 80s, S. P. Novikov generalized this construction to the case of maps $f: M \to S^1$ (in his work, this generalization have arisen in connection with the study of a problem on the motion of a solid in a fluid; see [9, 10]). The corresponding analog of the Morse complex

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is a free chain complex $C_*(f)$ defined over the ring $\mathbb{Z}((t))$. By definition, the ring $\mathbb{Z}((t))$ consists of all the power series $\lambda = \sum_{n=-\infty}^{\infty} a_i t^i$ such that $a_i \in \mathbb{Z}$ and the negative part of λ (i.e., the set $\{ k < 0 \mid a_k \neq 0 \}$) is finite. We shall assume that f is not homotopic to zero. The number of free generators of C_* is equal to the number of critical points of f, and the homology of C_* is isomorphic to the completed homology of the corresponding cyclic covering. The analogs of the Morse inequalities obtained in [11], as well as their applications, were then studied in a series of papers. M. Farber [3] proved that these inequalities are sharp in the case where $\pi_1 M = \mathbb{Z}$, $\dim M \geqslant 6$. J.-C. Sikorav [25] applied these inequalities to the theory of Lagrangian intersections. In [14, 15] the author obtained some results concerning the sharpness of these inequalities for $\pi_1 M = \mathbb{Z}^m$, $\dim M \geqslant 6$.

The study of the Novikov complex itself advanced slower. In [24], yet another version of the Novikov complex, defined over a completion of the group ring of $\pi_1(M)$, was given. In [16, 18] I proved that the simple homotopy type of this chain complex equals the simple homotopy type of the completed chain simplicial complex of the universal covering of M. In [17, 19] I proved a theorem on realization of the abstract chain complexes as the Novikov complexes of Morse maps $M \to S^1$. This theorem implies the results of [3] and [14]. The algebraic result by Ranicki [23] shows that the main theorem of [17, 19] implies also the Farrell fibration theorem [4].

In certain respects, the Morse–Novikov theory is similar to the classical one, but some objects and results of it differ drastically from their classical analogs. One of such objects is the boundary operator in the Novikov complex. For k fixed, the boundary operator $\partial\colon C_k(f) \to C_{k-1}(f)$ is represented by a matrix whose entries are in the ring of Laurent power series, i.e., $\partial_{(ij)} = \sum_{n=-N}^{\infty} a_n t^n$, where $a_n \in \mathbb{Z}$.

Since the outset S. P. Novikov conjectured that the power series $\partial_{(ij)}$ have some nice analytic properties. In particular, he conjectured that,

generically, the coefficients a_n of the series $\partial_{(ij)} = \sum_{n=-N}^{\infty} a_n t^n$ grow at most exponentially with n.

During several years, this conjecture has been discussed and adjusted. At present, there are different versions of it. As far as I know, the first published version appeared in the V. I. Arnol′d paper "Dynamics of Intersections", 1989. Arnol′d wrote in [1, p. 83]:

> "The author is indebted to S. P. Novikov who has communicated the following conjecture, which was the starting point of the present paper. Let $p\colon \widetilde{M} \to M$ be a covering of a compact manifold M with fiber $\mathbb{Z}^n$, and let α be a closed 1-form on M such that $p^*\alpha = df$, where $f\colon M \to \mathbb{R}$ is a Morse function. The Novikov conjecture states that, "generically" the number of the trajectories of the vector field $-\operatorname{grad} f$ on M starting at a critical point x of the function f of index k and connecting it with the critical points y having index $k-1$ and satisfying $f(y) \geqslant f(x) - n$, grows in n more slowly than some exponential, e^{an}."

Here is another version of the conjecture, published in 1993 by S. P. Novikov [13, p. 229]:

> "CONJECTURE. For any closed quantized analytic 1-form ω, the boundary operator ∂ in the Morse complex has all coefficients $a_{pq}^{(i)} \in K$ with positive

part convergent in some region $|t| \leqslant r_1^*(\omega) \leqslant r_1$. (If we replace S by $(-S)$ and t by t^{-1}, we have to replace r_1 by r_1^{-1}.) In particular, $r_1^* \neq 0$ and there will be coefficients $a_{pq}^{(i)}$ with radius of convergence not greater than $r_1(M,[\omega])$ if at least one jumping point really exists."[1]

In the present paper we prove that, C^0-generically, the incidence coefficients are rational functions of t of the form $\frac{P(t)}{t^k Q(t)}$, where $P(t)$ and $Q(t)$ are polynomials, $k \in \mathbb{N}$, and $Q(0) = 1$.

It should be noted that the main theorem in [19] implies that every polynomial of the form $1 + a_1 t + \cdots + a_n t^n$ can be realized as an *initial segment* of some incident coefficient in the Novikov complex of a Morse map $f\colon M \to S^1$ (where M is some manifold). In particular, this shows that for fixed n there are no restrictions on the initial segments of length n of the incidence coefficients.

For the first time, the results of the present paper appeared in the e-print dg-ga 9603006 (14 March 1996). Also, they were announced [22]. The present text is an extended version of that preprint. Some terminology and notation is changed, and the exposition in § 4 is revised.

Concluding this subsection, I would like to note that, when studying the algebraic properties of the Novikov complex and the Novikov completion, one often passes from localization to completion and *vice versa*. Now we see that these passages have geometric background.

B. The statement of the main theorem. Let M be a closed connected manifold and $f\colon M \to S^1$ a Morse map not homotopic to zero. We denote by $S(f)$ the set of critical points of f. The set of f-gradients of class C^∞ satisfying the transversality condition (see §1 for the terminology) will be denoted by $\mathcal{G}t(f)$. By the Kupka–Smale theorem, this set is residual in the set of all C^∞-gradients. Let $v \in \mathcal{G}t(f)$, and let $\bar{M} \xrightarrow{\mathcal{P}} M$ be the (uniqely determined) connected infinite cyclic covering for which $f \circ \mathcal{P}$ is homotopic to zero. We choose a lifting $\bar{f}\colon \bar{M} \to \mathbb{R}$ of $f \circ \mathcal{P}$ and denote by t the generator of the structure group ($\approx \mathbb{Z}$) of $\mathcal{P}$ such that $\bar{f}(xt) < \bar{f}(x)$. The t-invariant lifting of v to $\bar{M}$ will be denoted by the same letter v. For every critical point x of f, we fix a lifiting $\bar{x}$ of x to $\bar{M}$. If the orientations of the stable manifolds of critical points are fixed, then for every $x, y \in S(f)$ with $\operatorname{ind} x = \operatorname{ind} y + 1$ and every $k \in \mathbb{Z}$ the incidence coefficient $n_k(x, y; v)$ is defined (this is the algebraic number of $(-v)$-trajectories joining $\bar{x}$ and $\bar{y}t^k$). Consequently, the Novikov incidence coefficient $n(x, y; v) = \sum_{k\in\mathbb{Z}} n_k(x, y; v)t^k$, belonging to $\mathbb{Z}((t))$, is also well defined.

Main Theorem. *In the set $\mathcal{G}t(f)$ there is a subset $\mathcal{G}t_0(f)$ with the following properties.*

(1) *$\mathcal{G}t_0(f)$ is open and dense in $\mathcal{G}t(f)$ with respect to the C^0-topology.*
(2) *If $v \in \mathcal{G}t_0(f)$, $x, y \in S(f)$, and $\operatorname{ind} x = \operatorname{ind} y + 1$, then $n(x, y; v)$ is a rational function of t of the form $\frac{P(t)}{t^m Q(t)}$, where $P(t)$ and $Q(t)$ are polynomials with integral coefficients, $m \in \mathbb{N}$, and $Q(0) = 1$.*

[1]For convenience of the reader, we explain the terminology in this citation. S. P. Novikov denotes by K the ring denoted by $\mathbb{Z}((t))$ in the present paper; p, q correspond to critical points with $\operatorname{ind} p = \operatorname{ind} q + 1$, and $a_{pq}^{(i)}$ is $\partial_{(pq)}$ in our notation. A closed 1-form w is called *quantized* if its cohomology class belongs to $H^1(M, \mathbb{Z})$.

(3) *Let $v \in \mathcal{G}t_0(f)$, and let U be a neighborhood of $S(f)$. For every $w \in \mathcal{G}t_0(f)$ such that $w = v$ in U and w is sufficiently close to v in the C^0-topology, we have $n_k(x, y; v) = n_k(x, y; w)$ for any $x, y \in S(f)$ with $\operatorname{ind} x = \operatorname{ind} y + 1$.*

Remark. This theorem implies the exponential estimate mentioned in the Novikov conjecture above: every rational function of the form $\frac{P(t)}{Q(t)}$ with $Q(0) \neq 0$ has a nonzero radius of convergence at 0.

C. An outline of the proof, and the contents of the paper. Here we keep the terminology of the preceding subsection. Clearly, it suffices to consider the case of indivisible $[f] \in H^1(M, \mathbb{Z})$. Assume that $1 \in S^1$ is a regular value for $f\colon M \to S^1$. Cutting M along $V = f^{-1}(1)$, we obtain a cobordism W with boundary $\partial W = V_0 \sqcup V_1$, where both V_0 and V_1 are diffeomorphic to V. Also, we obtain a Morse function $F\colon W \to [0, 1]$ with $F^{-1}(1) = V_1, F^{-1}(0) = V_0$. Any f-gradient v gives rise to an F-gradient, which will be denoted again by v. The shift along the trajectories of the vector field $(-v)$ determines a diffeomorphism $\widetilde{v} : V_1 \setminus K_1 \to V_0 \setminus K_0$, where K_1 (respectively, K_2) is the union of the instable (respectively, stable) manifolds of all critical points of F. The infinite cyclic covering $\bar{M} \xrightarrow{\mathcal{P}} M$ is obtained by glueing countably many copies $W^{[i]}$, $i \in \mathbb{Z}$, of W one after another under the identification $V_0^{[i]} \approx V_1^{[i-1]}$. We identify $W^{[0]}$ with W. Let $p, q \in W$ be critical points of F, $\operatorname{ind} p = \operatorname{ind} q + 1$. We put $n = \dim M$ and $l + 1 = \operatorname{ind} p$. For simplicity, assume that $F(p)$ is the lowest critical level of F, so that the interval $]0, F(p)[$ is regular. Similarly, we assume that the interval $]F(q), 1[$ is regular and that $\bar{p} \in W^{[0]}$, $\bar{q} \in W^{[-1]}$. Let $S(p)$ denote the intersection of the stable manifold of p with V_0; this is an oriented l-dimensional embedded sphere in V_0. The intersection $S(q)$ of the unstable manifold of q with V_1 is a cooriented $(n - l - 1)$-dimensional sphere in V_1.

Let $v \in \mathcal{G}t(f)$. Using the presentation of $\bar{M}$ described above, we obtain the identity

$$n_k(p, q; v) = ((-v)^{\leadsto})^k (S(p)) \# S(q) \tag{1}$$

Here are some comments on this formula. 1) $(-v)^{\leadsto}$ is not defined on all of V_1; nevertheless, we denote by $(-v)^{\leadsto}(A)$ the set $(-v)^{\leadsto}(A \setminus K_1)$, where $A \subset V_1$. 2) We identify V_1 and V_0 with V, so that the iterations of $(-v)^{\leadsto}$ make sense and are applicable to $S(p) \subset V_0$. 3) The symbol $\#$ stands for the algebraic intersection index of two manifolds; it can be checked that this number is well defined in our situation.

Should $(-v)^{\leadsto}$ be an everywhere defined diffeomorphism, the intersection index in (1) would coincide with the intersection index of some homology classes, and our result would be a consequence of linear algebra. Indeed, let $[p]$ be a homology class of the sphere $S(p)$ in $H_l(V)$, let $]q[\in H^l(V)$ be the cohomology class dual to $S(q)$, and let $L\colon H_*(V) \to H_*(V)$ be the homomorphism induced by $(-v)^{\leadsto}$; then $((-v)^{\leadsto})^k(S(p)) \# S(q) = \langle]q[, L^k([p]) \rangle t^k$ (here and in what follows we denote by $\langle \cdot, \cdot \rangle$ the canonical pairing between homologies and cohomologies). A simple computation shows that the power series $\sum_{k=0}^{\infty} \langle]q[, L^k([p]) t^k \rangle$ is a rational function with denominator $1 - \det L \cdot t$.

If f has critical points, then the map $(-v)^{\leadsto}$ is no longer an everywhere defined diffeomorphism, and the indices $n_i(p,q;v)$ do not admit any simple homological interpretation. Our main goal in the present paper is to show that for a C^0-generic f-gradient v we can still obtain a formula of type $n(p,q;v) = \sum_{k=0}^{\infty} \langle]q[, L^k([p])t^k \rangle$, where, however, the meaning of the objects $[p],]q[, L$ will be more complicated than in the case where there are no critical points.

Namely, we shall show that for any f and any $p, q \in S(f)$ one can find an C^0-open-and-dense subset $G \subset \mathcal{G}t(f)$, disjoint compacta $A, B \subset V$, and a homomorphism $L\colon H_l(V \setminus B, A) \to H_l(V \setminus B, A)$ such that the following condition is fulfilled for any $v \in G$:

$S(p)$ does not intersect B, $S(q)$ does not intersect A, the group $H_*(V \setminus B, A)$ is finitely generated, and

$$n(p,q;v) = \sum_{k=0}^{\infty} \langle]q[, L^k([p]) \rangle t^k, \tag{2}$$

where $[p]$ is the image in $H_l(V \setminus B, A)$ of the fundamental class $[S(p)]$, and $]q[$ is an element of $H^l(V \setminus B, A)$, dual to $S(q)$.

Remark. The homomorphism L is *not* induced by any continuous map of the pair $(V \setminus B, A)$ to itself.

Now we describe what are the sets A, B, the homomorphism L, and the C^0-small perturbations of v satisfying (2). We use the terminology of Subsections 1.1–1.3 (see below). We start with an auxiliary Morse function $\phi\colon V \to \mathbb{R}$ and a ϕ-gradient u satisfying the almost transversality condition. For $\delta > 0$ sufficiently small and for any i the sets $D_\delta(\text{ind}{\leqslant}i; u)$ and $D_\delta(\text{ind}{\leqslant}n{-}2{-}i; -u)$ are disjoint compact subsets of V. We may think of $D_\delta(\text{ind}{\leqslant}i; u)$ as of the union of all handles of thickness δ and dimension at most i of the handle decomposition of V constructed via ϕ and u. This union will be referred to as the *δ-thickened i-skeleton* of V. We put $A = D_\delta(\text{ind}{\leqslant}l{-}1; u)$ and $B = D_\delta(\text{ind}{\leqslant}n{-}l{-}2; -u)$.

Returning to the cobordism W with boundary $\partial W = V_0 \sqcup V_1$, we note that for every geometric object associated with V there are two copies of it that correspond to V_0 and to V_1, because both V_0 and V_1 are diffeomorphic to V. These objects will be denoted by the same letter but with indices 0 or 1, respectively. In this way, we obtain Morse functions ϕ_0, ϕ_1, compacta A_0, A_1, etc.

In order to construct L, on v we must impose condition (RF) (see Definition 4.15 in Subsection 4.5); in the notation slightly different from that of §4, the principal part of this condition is given by the relations

$$\text{(RP3)} \quad \overline{(-v)^{\leadsto}(B_\delta(\text{ind}{\leqslant}j; u_1)) \cup (B_\delta(\text{ind}{\leqslant}j{+}1; v) \cap V_0)} \subset B_\delta(\text{ind}{\leqslant}j; u_0) \quad \forall j;$$

$$\text{(RP4)} \quad \overline{\overset{\leadsto}{v}(B_\delta(\text{ind}{\leqslant}j; -u_0)) \cup (B_\delta(\text{ind}{\leqslant}j{+}1; -v) \cap V_1)} \subset B_\delta(\text{ind}{\leqslant}j; -u_1) \ \forall j.$$

(The remaining parts of condition (RF) state, roughly, that $\delta > 0$ is sufficiently small.) Conditions (RP3) and (RP4) may be regarded as analogs of the cellular approximation conditions; however, what we demand is stronger: the soles of the

δ-thickened handles of v of dimension not exceeding $j+1$ must belong to the δ-thickened j-skeleton of V. The gradient u will be referred to as the *ranging flow* for (f, v).

If condition (RF) is satisfied, then we construct the homomorphism $L = H(v)\colon H_*(V \setminus B, A) \to H_*(V \setminus B, A)$. To clarify the idea of this construction, we consider the simplest possible situation. Namely, we assume that the function f has only one critical level; moreover, we shall define the homomorphism $H(v)$ only on the homology classes represented by oriented closed submanifolds of dimension l. Let $N \subset V$ be such a submanifold. Perturbing N (if necessary), we can arrange that the corresponding submanifold N_1 of V_1 intersect only the disks $D(a, -v)$, $a \in S(F)$, with $\operatorname{ind} a \leqslant l$. Therefore, although the manifold $(-v)^{\leadsto}(N_1) \subset V_0$ is not compact, the set $(-v)^{\leadsto}(N) \cup (D(\mathrm{ind}{\leqslant}l; v) \cap V_0)$ is compact. Condition (RP3) shows that the manifold $(-v)^{\leadsto}(N_1)$ does not intersect $D_\delta(\mathrm{ind}{\leqslant}n-1-2; u_0)$, which allows us to define the fundamental class of $(-v)^{\leadsto}(N)$ in the homology of the couple $(V \setminus D_\delta(\mathrm{ind}{\leqslant}n-l-2; -u_0),\ B_\delta(\mathrm{ind}{\leqslant}l-1; u_0))$. Indentifying V_0 and V, we arrive at an element of $H_*(V \setminus B, A)$, which will be denoted by $H(v)([N])$.

In full generality, the construction of $H(v)$ is done in Subsection 4.3. It is based on the formalism of *ranging systems* (actually, this technique applies to any cobordism W, not necessarily satisfying $V_0 \approx V_1$). As described in §4, this formalism also allows us to construct an *open* and dense set of gradients v satisfying (2).

Now we explain how to gain properties (RP3) and (RP4) by small perturbation of the initial f-gradient v.

As already noted, for a compact submanifold R in V_1, the submanifold $(-v)^{\leadsto}(R)$ in V_0 is not necessarily compact; to make it compact, we must add the soles of the descending disks of v (thus losing the manifold property). Therefore, it should be expected that the correspondence $\cdot \mapsto (-v)^{\leadsto}(\cdot)$ is well defined on an appropriate category of stratified objects. As shown in §2, this is indeed so. The results of §2 imply that (under some mild transversality conditions) the set

$$\mathcal{R}_j = (-v)^{\leadsto}\big(D(\mathrm{ind}{\leqslant}j; u_1)\big) \cup \big(D(\mathrm{ind}{\leqslant}j+1; v) \cap V_0\big) \tag{3}$$

is compact for every j. Since this set is a union of submanifolds of V_0 of dimension not exceeding j, we can push $\mathcal{R}_j$ away from $D(\mathrm{ind}{\leqslant}n-j-2; -u_0)$; then, applying a diffeomorphism $\Phi(T, -u_0)$ for T sufficiently large, we obtain an isotopy of V_0 that pushes $\mathcal{R}_j$ into $B_\delta(\mathrm{ind}{\leqslant}j; u_0)$. After some work with thickenings (see Subsections 2.2, 2.3) we obtain an isotopy of V_0 such that the set $(-v)^{\leadsto}(D_\delta(\mathrm{ind}{\leqslant}j; u_1)) \cup (D_\delta(\mathrm{ind}{\leqslant}j+1; v) \cap V_0)$ is also taken inside $B_\delta(\mathrm{ind}{\leqslant}j; u_0)$. This isotopy may be realized by modifying v; the new gradient will satisfy (RP3) and (RP4). This procedure is described in detail in Subsection 4.1. The resulting gradient can be chosen so as to be C^0-close to the initial one. In other words, the pair (ϕ_0, u_0) can be chosen in such a way that $\Phi(\tau, -u_0)$ pushes $V_0 \setminus B_\delta(\mathrm{ind}{\leqslant}n-j-2; -u_0)$ inside $B_\delta(\mathrm{ind}{\leqslant}j; u_0)$ already for τ sufficiently small. (Such flows will be called *quick flows.*) The precise formulation of the corresponding statement is contained in Subsection 1.6, and its proof in §3. Our construction of quick flows differs from the usual constructions of "small handle decomposition" in that it uses no triangulations of the manifold. A description of the basic idea of this construction can be found at the beginning of §3.

Finally, we note that the C^0-topology plays an important role in the statement and the proof of the main theorem. To make this more apparent, we recall that the integral curves (on a given finite interval) of a C^0-*small* perturbations v' of a C^1-vector field v are C^0-small perturbations of the integral curves of v (see, e.g., [2, Chapter 5, §4]). The properties of C^0-continuity that we use in this paper are collected in Subsection 1.4. They are based on the general results of the theory of differential equations. For completeness, to this paper we have included an Appendix (§5), where the detailed proof of these properties is given.

D. Further generalizations. The methods developed in the present paper make it possible to compute the Novikov complex explicitely. These computations and further generalizations are the subject of the second part of the paper (see [20]). We recall that for a Morse map $M \to S^1$, the Novikov complex can be defined over a special completion of the ring $\mathbb{Z}[\pi_1(M)]$. In [20], we proved that for a C^0-generic f-gradient the incidence coefficients in this complex belong in fact to the image in the Novikov ring of the corresponding localization of the group ring of the fundamental group. In the case of irrational Morse forms, we prove the analog of this result, but only for free Abelian coverings. We state and prove the corresponding exponential growth estimates for the incidence coefficients (coming from the universal covering) in the case of the Morse forms with degree of arbitrary irrationality.

Also, we obtain exponential growth estimates for the total number (as opposed to the algebraic number) of trajectories joining $\bar{x}$ and $\bar{y}t^k$. Note that the above-cited version of the conjecture in [1] involves precisely this total number of trajectories. At present, this estimate is obtained only for a C^0-dense set of gradients.

Also, we give an example of an explicit computation of the Novikov complex.

E. Remarks on the terminology. By a *cobordism* we mean a compact manifold W together with a fixed decomposition $\partial W = V_0 \sqcup V_1$, where V_0 and V_1 are disjoint closed manifolds; the set $W \setminus (V_0 \cup V_1)$ will be denoted by W°. A *Morse function* $f\colon W \to [a, b]$ on a cobordism W is a C^∞-function with $f^{-1}(b) = V_1$, $f^{-1}(a) = V_0$ for which all the critical points are contained in W° (this terminology is almost identical to that in [6, §§1, 2]; note that V_0 and V_1 may be empty).

Let v be a vector field on a manifold W (possibly, with boundary). We denote by $\gamma(x, t; v)$ the value at t of the integral curve of v starting at x. If $A \subset \mathbb{R}$ and $\gamma(x, \cdot; v)$ is defined on A, then $\gamma(x, A; v)$ denotes the set $\{\, \gamma(x, t; v) \mid t \in A \,\}$.

A subset $X \subset W$ is said to be v-*invariant* if $\gamma(x, t; v)$ is defined for every $t \geqslant 0$ and every $x \in X$ and $\gamma(x, [0, \infty[; v) \subset X$. A subset $X \subset W$ is $(\pm v)$-*invariant* if it is v-invariant and $(-v)$-invariant.

A subset $X \subset W$ is said to be *weakly* v-*invariant* if $\gamma(x, t; v) \in X$ whenever $x \in X$, $t \geqslant 0$, and $\gamma(x, t; v)$ is defined. Defining the notion of weak $(\pm v)$-invariance is left to the reader.

For any (time dependent) vector field w on a closed manifold M and any $t \in \mathbb{R}$, we denote by $\Phi(w, t)$ the diffeomorphism $x \mapsto \gamma(x, t; w)$ of M. The closure of a set U is denoted by $\overline{U}$. The short bar is reserved for the objects related to the infinite cyclic covering (e.g., $\mathcal{P}\colon \bar{M} \to M$).

Let ρ be a Riemannian metric on a manifold M. The corresponding norm on T_xM is denoted by $|\cdot|_\rho$; $\|\cdot\|$ stands for the C^0-norm on the space of vector fields on M and e for the Euclidean metric in $\mathbb{R}^n$. Whenever no confusion is possible, we drop the indices ρ and e in the notation.

The closed (respectively, open) disk of radius r centered at a in $\mathbb{R}^n$ is denoted by $D^n(a,r)$ (respectively, $B^n(a,r)$). The closed (respectively, open) disk of radius r centered at a in a Riemannian manifold M is denoted by $\mathcal{D}_r(a)$ (respectively, $\mathcal{B}_r(a)$). (We wish to avoid confusion with the sets $B_r(v)$ introduced in §1.)

Let $\rho, \varkappa$ be two Riemannian metrics on a manifold M, and let $x \in M$; the symbol $C(x;\rho,\varkappa)$ denotes the minimal number C such that $|h|_\varkappa \leqslant C|h|_\rho$ and $|h|_\rho \leqslant C|h|_\varkappa$ for any $h \in T_xM$. We say that ρ and $\varkappa$ are C-equivalent if $C(x;\rho,\varkappa) \leqslant C$ for every $x \in M$. Any chart $\Phi\colon U \to V \subset \mathbb{R}^n$ of M induces a metric ε on U from $\mathbb{R}^n$ (this metric will also be called *Euclidean*). We put $C(x;\rho|U,\varepsilon) = \mathcal{G}(x,\Phi)$, and for $K \subset U$ we denote by $\mathcal{G}(K,\Phi)$ the number $\sup_x \mathcal{G}(x,\Phi)$, where x ranges over the points of K.

The following differences between the notation adopted in this paper and that in [14–21] are worth mentioning. The gradients satisfying the almost transversality condition (see Subsection 1.3 below) were called "almost good" in [21] and [19]. The diffeomorphism associated with a vector field v on a cobordism W was denoted by $\overset{\leadsto}{v}$ in [19–21]; here we denote it by $(-v)^{\leadsto}$, which seems more natural because this diffeomorphism is a shift in the *positive* direction along the integral curves of the field $-v$. In [21], the sets $B_\delta(\text{ind}{\leqslant}s; v)$, $D_\delta(\text{ind}{\leqslant}s; v)$, $D(\text{ind}{\leqslant}s; v)$ were denoted by $B_\delta(\leq s; v)$, $D_\delta(\leq s; v)$, $D(\leq s; v)$. In [19, 20], the set $D(\text{ind}{\leqslant}s, v)$ was denoted by $K(\text{ind}{\leqslant}s; v)$.

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The conjecture that the incidence coefficients must be rational was formulated by P. Vogel about 1991; the motivation was that these coefficients are results of successive iterations of one and the same procedure applied to the cobordism W. M. Farber informed me that he formulated the rationality conjecture in seminars and in private discussions. The conjecture stated by him pertained to the case of an *arbitrary* gradient satisfying the transversality condition (in the sense of Subsection 1.3). This conjecture remains unproved. It should be noted that the first version of the Novikov conjecture on the exponential growth also pertained to the same case of an arbitrary gradient satisfying the transversality condition. The author thanks S. P. Novikov and M. Farber for clarifying this question to him.

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§1. Preliminaries on Morse functions and their gradients

In Subsections 1.1–1.3 we introduce (and recall) terminology related to Morse functions and their gradients. Also, we present the compactness properties of (the unions of) stable manifolds of critical points. In Subsection 1.4 we discuss some continuity properties of the unions of stable manifolds of the zeros of an f-gradient v under C^0-small perturbations of v. Subsection 1.5 contains explicit estimates (to be used throughout the paper) for the time spent by the trajectory of some standard vector fields (in $\mathbb{R}^n$) in annuli and other domains. In Subsection 1.6 we formulate a theorem on the existence of quick flows (Theorem 1.12). The only result to be used in the proof of the main theorem is Corollary 1.14, which is our version of the "small handle decomposition". Theorem 1.12 is reduced to Theorem 1.13 with the help of Proposition 1.15.

Remark. Unless otherwise stated, all objects are assumed to be of class C^∞.

1.1. Functions and gradients. In Subsections 1.1 and 1.2, M stands for a closed manifold.

Definition 1.1. Let $f\colon M \to \mathbb{R}$ be a Morse function on M. We put $n = \dim M$. The set of critical points of f will be denoted by $S(f)$. The set of critical points of f of index i will be denoted by $S_i(f)$. Let $p \in S_s(f)$. A chart $\Phi_p\colon U_p \to B^n(0, r_p)$, where U_p is a neighborhood of p and $r_p > 0$, is called a *standard Morse chart for f near p of radius r_p* (or, simply, an *f-chart*) if there is an extension of Φ_p to a chart $\widetilde{\Phi}_p\colon V_p \to B^n(0, r'_p)$, where $\overline{U_p} \subset V_p$, $r'_p > r_p$, and $(f \circ \widetilde{\Phi}_p^{-1})(x_1, \dots, x_n) = f(p) + \sum_{i=1}^n \alpha_i x_i^2$, where $\alpha_i < 0$ for $i \leqslant s$ and $\alpha_i > 0$ for $i > s$. The domain U_p is called a *standard coordinate neighborhood.* Any such extension $\widetilde{\Phi}_p$ of Φ_p will be called a *standard extension of Φ_p.*

The set $\Phi_p^{-1}(\mathbb{R}^s \times \{0\})$ (respectively, $\Phi_p^{-1}(\{0\} \times \mathbb{R}^{n-s})$ is called a *negative disk* (respectively, a *positive disk*). If for every i we have $\alpha_i = \pm 1$, we say that Φ_p is *strongly standard.*

A family $\mathcal{U} = \{\Phi_p\colon U_p \to B^n(0, r_p)\}_{p\in S(f)}$ of f-charts is called an *f-chart system* if the sets $\{\overline{U_p}\}$ are disjoint. We denote $\min_p r_p$ by $d(\mathcal{U})$, and $\max_p r_p$ by $D(\mathcal{U})$. If all the r_p are equal to r, we say that $\mathcal{U}$ is *of radius r.* The set $\Phi_p^{-1}(B^n(0, \lambda))$, where $\lambda \leqslant r_p$, will be denoted by $U_p(\lambda)$. For $\lambda \leqslant d(\mathcal{U})$ we put $\mathcal{U}(\lambda) = \bigcup_{p\in S(f)} U_p(\lambda)$.

Let $\mathcal{U} = \{\Phi_p\colon U_p \to B^n(0, r_p)\}_{p\in S(f)}$, $\mathcal{U}' = \{\Phi'_p\colon U'_p \to B^n(0, r'_p)\}_{p\in S(f)}$ be two f-chart systems. We say that $\mathcal{U}'$ is a *restriction* of $\mathcal{U}$ if for every $p \in S(f)$ we have $r'_p \leqslant r_p$, $U'_p \subset U_p$, $\Phi'_p = \Phi_p \mid U'_p$.

Given an f-chart system $\mathcal{U} = \{\Phi_p\}$, we say that a vector field v on M is an *f-gradient with respect to $\mathcal{U}$* if

1) $df(v)(x) > 0$ for any $x \in M \setminus S(f)$;

2) for every $p \in S(f)$ we have $(\widetilde{\Phi}_p)_*(v) = (-x_1, \dots, -x_k, x_{k+1}, \dots, x_n)$, where $k = \operatorname{ind}_f p$, and $\widetilde{\Phi}_p$ is some standard extension of Φ_p.

We say that a vector field v is an *f-gradient* if there is an f-chart system $\mathcal{U}$ such that v is an f-gradient with respect to $\mathcal{U}$.

Definition 1.2. Assume that M is a Riemannian manifold. Let $\delta_0 > 0$ be the radius of injectivity of M (i.e., for every $q \in M$ the restriction of the exponential map $\exp_q\colon T_qM \to M$ to the ball $B(0, \delta_0)$ is a diffeomorphism onto its image), and let $0 < \delta < \delta_0$. Suppose that $f\colon M \to \mathbb{R}$ is a Morse function, v is an f-gradient, $p \in S(f)$. We set

$$B_\delta(p, v) = \{\, x \in M \mid \exists t \geqslant 0 : \gamma(x, t; v) \subset \mathcal{B}_\delta(p) \,\}, \tag{4}$$

$$D_\delta(p, v) = \{\, x \in M \mid \exists t \geqslant 0 : \gamma(x, t; v) \subset \mathcal{D}_\delta(p) \,\}, \tag{5}$$

$$D(p, v) = \{\, x \in M \mid \lim_{t\to\infty} \gamma(x, t; v) = p \,\}. \tag{6}$$

$D(p,v)$ is nothing but the stable manifold of the point p relative to v; we also call $D(p,v)$ the *descending disk* of p.

For $s \in \mathbb{N}$ with $0 \leqslant s \leqslant n$, we denote by $D(\mathrm{ind}{\leqslant}s; v)$ the union of the sets $D(p,v)$, where p ranges over the critical points of f of index at most s. We denote by $B_\delta(\mathrm{ind}{\leqslant}s; v)$ (respectively, by $D_\delta(\mathrm{ind}{\leqslant}s; v)$) the union of the sets $B_\delta(p,v)$ (respectively, of the $D_\delta(p,v)$), where p ranges over the critical points of f of index at most s. We shall use similar notation like $D_\delta(\mathrm{ind}{=}s; v)$ or $B_\delta(\mathrm{ind}{\geqslant}s; v)$, the meaning of which is now clear. If $s = \dim M$, we abbreviate $D_\delta(\mathrm{ind}{\leqslant}s; v)$ to $D_\delta(v)$, etc.

Remarks. 1) Our definition of the standard Morse charts is wider than that used in [6, Definition 3.1] and in [21], since we allow the α_i in Definition 1.1 to be different from ± 1. It is not difficult to show that all results in [21, §2] can be carried over to the situation of our present definition, to which the terminology of [21, §2] concerning Morse functions and their gradients applies without changes. For the reader's convenience, in Subsection 1.3 we discuss the results and terminology of [21, §2] that we use here.

2) Obviously, the sets $B_\delta(\mathrm{ind}{\leqslant}s; v)$ and $D_\delta(\mathrm{ind}{\leqslant}s; v)$ can be defined for any $\delta > 0$; see [21, §2].

3) In the e-print [19] we used the notation $K(\mathrm{ind}{\leqslant}s; v)$ for the set denoted here by $D(\mathrm{ind}{\leqslant}s; v)$.

4) The notion of an f-gradient for a Morse map $f\colon M \to S^1$ can be defined in a similar way.

1.2. $\mathcal{M}$-flows.

Definition 1.3. Assume that M is Riemannian. We say that a triple $\mathcal{V} = (f, v, \mathcal{U})$ is an *$\mathcal{M}$-flow* on M ($\mathcal{M}$ for Morse) if (1) $f\colon M \to \mathbb{R}$ is a Morse function; (2) $\mathcal{U}$ is an f-chart system; (3) v is an f-gradient with respect to $\mathcal{U}$; (4) for each f-chart $\Phi_p\colon U_p \to B^n(0, r_p)$ in $\mathcal{U}$, the coordinate frame at p is orthonormal in the Riemannian metric of M.

Let $\mathcal{V} = (f, v, \mathcal{U})$ be an $\mathcal{M}$-flow on M, where $\mathcal{U} = \{\, \Phi_p\colon U_p \to B^n(0, r_p) \,\}_{p \in S(f)}$.

We put $\mathcal{G}(\mathcal{V}) = \max_p \mathcal{G}(U_p, \Phi_p)$. We say that $\mathcal{V}$ is *of radius* r if $\mathcal{U}$ is of radius r.

Let γ be a v-trajectory.

A) The number of the sets $U_p = U_p(r_p)$ intersected by γ is denoted by $N(\gamma)$. We put $N(\mathcal{V}) = \max_\gamma N(\gamma)$. The set $\{\, t \in \mathbb{R} \mid \gamma(t) \notin \bigcup_{p \in S(f)} U_p \,\}$ is a finite union of disjoint closed intervals; the sum of their lengths is denoted by $T(\gamma)$. The number $\max_\gamma T(\gamma)$ is denoted by $T(\mathcal{V})$. Let $\beta > 0$, $C > 0$. We say that $\mathcal{V}$ is *(C, β)-quick* if $\|v\| \leqslant C$ and $T(\mathcal{V}) \leqslant \beta$.

B) Let $\delta \leqslant d(\mathcal{U})$. The number of the sets $U_p(\delta)$ intersected by γ is denoted by $N(\gamma, \delta)$. We put $N(\mathcal{V}, \delta) = \max_\gamma N(\gamma, \delta)$. The set $\{\, t \in \mathbb{R} \mid \gamma(t) \notin \mathcal{U}(\delta) \,\}$ is a finite union of disjoint closed intervals; the sum of their lengths is denoted by $T(\gamma, \delta)$. Let $T(\mathcal{V}, \delta) = \max_\gamma T(\gamma, \delta)$.

For $\beta > 0$, $C > 0$, we say that $\mathcal{V}$ is *(C, β, δ)-quick* if $\|v\| \leqslant C$ and $T(\mathcal{V}, \delta) \leqslant \beta$.

Let $\mathcal{V}^{(i)} = (f^{(i)}, v^{(i)}, \mathcal{U}^{(i)})$, $i = 1, 2$, be two $\mathcal{M}$-flows on M. Setting $\mathcal{U}^{(i)} = \{\Phi_p^{(i)}\colon U_p^{(i)} \to B^n(0, r_p^{(i)})\}_{p \in S(f^{(i)})}$, we say that $\mathcal{V}^{(2)}$ *is subordinate to* $\mathcal{V}^{(1)}$ if (1) $\mathcal{U}^{(2)}$ is a restriction of $\mathcal{U}^{(1)}$; (2) $v^{(2)} = \phi \cdot v^{(1)}$, where $\phi\colon M \to \mathbf{R}^+$ is a C^∞-function such that $\phi(x) = 1$ for x in a neighborhood of the closure of $\bigcup_p U_p^{(2)}$.

Remark. There is an obvious analog of the terminology of Subsections 1.1, 1.2 for a more general situation of the Morse functions on compact cobordisms. We shall use this generalization freely.

1.3. The simplest properties of f-gradients. In this subsection, $f\colon W \to [a,b]$ is a Morse function on a compact cobordism, $V_0 = f^{-1}(a)$, $V_1 = f^{-1}(b)$, and v is an f-gradient. We denote $n = \dim W$. As in [21, §2], v is said to satisfy

the transversality condition if $\big(x, y \in S(f)\big) \Rightarrow \big(D(x,v) \pitchfork D(y,-v)\big)$;

the almost transversality condition if $\big(x, y \in S(f)$ and $\operatorname{ind} x \leqslant \operatorname{ind} y\big) \Rightarrow \big(D(x,v) \pitchfork D(y,-v)\big)$.

An $\mathcal{M}$-flow $(f, v, \mathcal{U})$ is said to satisfy the almost transversality condition (and is called an $\mathcal{AM}$-flow) if v satisfies the almost transversality condition.

In the terminology of [21, §2], a Morse function $\phi\colon W \to [\alpha,\beta]$ is said to be *adjusted to* (f,v) if: 1) $S(\phi) = S(f)$; 2) the function $f - \phi$ is constant in a neighborhood of V_0, in a neighborhood of V_1, and in a neighborhood of each point in $S(f)$; 3) v is also a ϕ-gradient

A Morse function $f\colon W \to [a,b]$ is said to be *ordered* with *ordering sequence* $a_0, \dots, a_{n+1}$ if the a_i are regular values of f such that $S_i(f) \subset f^{-1}(]a_i, a_{i+1}[)$.

We say that an f-gradient is *δ-separated* (here $\delta > 0$) if there is an ordered Morse function $\phi\colon W \to [a,b]$ adjusted to (f,v) and having an ordering sequence $a_0, \dots, a_{n+1}$ such that

(1) for every $p \in S(f)$ we have $\mathcal{D}_\delta(p) \subset \phi^{-1}(]a_k, a_{k+1}[)$, where $k = \operatorname{ind} p$;

(2) for every $0 \leqslant l \leqslant n$ and every $q \in S_l(f)$ there is a Morse function $\psi\colon W_l \to [a_l, a_{l+1}]$ adjusted to $(\phi|W_l, v|W_l)$, where $W_l = \phi^{-1}([a_l, a_{l+1}])$, and there is a regular value $\mu \in]a_l, a_{l+1}[$ of ψ such that $\mathcal{D}_\delta(q) \subset \phi^{-1}(]a_l, \mu[)$ and for every $r \in S_l(f)$ with $r \neq q$ we have $\mathcal{D}_\delta(r) \subset \phi^{-1}(]\mu, a_{l+1}[)$.

We note that if v is δ-separated, then $D_\delta(p,v) \cap D_\delta(q,-v) = \varnothing$ whenever $\operatorname{ind} p \leqslant \operatorname{ind} q$. With the help of the usual rearrangement procedure (see [6, Theorem 4.1]) it is easy to check that each gradient satisfying the almost transversality condition is δ-separated for some $\delta > 0$.

Proposition 1.4. *If v is δ_0-separated, then the following is true for any $\delta \in]0, \delta_0[$ and any s with $0 \leqslant s \leqslant n$:*

(1) $D_\delta(\mathrm{ind}{\leqslant}s; v)$ *is compact*;

(2) $D(\mathrm{ind}{\leqslant}s; v)$ *is compact*;

(3) $\bigcap_{\theta>0} B_\theta(\mathrm{ind}{\leqslant}s; v) = D(\mathrm{ind}{\leqslant}s; v)$;

(4) $\bigcap_{\theta>\delta} B_\theta(\mathrm{ind}{\leqslant}s; v) = D_\delta(\mathrm{ind}{\leqslant}s; v)$;

(5) $\overline{B_\delta(\mathrm{ind}{\leqslant}s; v)} = D_\delta(\mathrm{ind}{\leqslant}s; v)$.

Proof. Statements (1)–(4) were proved in [21, Lemma 2.11]. To obtain (5), we observe that, since v is δ-separated, it suffices to prove (5) for $s = n$, in which case this follows immediately from the inclusions $B_\delta(v) \subset D_\delta(v) \subset \overline{B_\delta(v)}$. $\square$

For two regular values λ, μ of f with $\lambda < \mu$ we denote by K_μ^+ the set $f^{-1}(\mu) \cap \bigcup_{p \in S(f) \cap f^{-1}([\lambda,\mu])} D(p,-v)$, by K_λ^- the set $f^{-1}(\lambda) \cap \bigcup_{p \in S(f) \cap f^{-1}([\lambda,\mu])} D(p,v)$, and by $(-v)^{\leadsto}_{[\mu,\lambda]}$ the C^∞-diffeomorphism $f^{-1}(\mu) \setminus K_\mu^+ \to f^{-1}(\lambda) \setminus K_\lambda^-$ that takes each point x to the point of intersection of $\gamma(x, \cdot; -v)$ with $f^{-1}(\lambda)$. For $X \subset f^{-1}(\mu)$ we denote (somewhat loosely) $(-v)^{\leadsto}_{[\mu,\lambda]}(X \setminus K_\mu^+)$ by $(-v)^{\leadsto}_{[\mu,\lambda]}(X)$. If the values of μ and λ are clear from the context, we suppress them in the notation.

1.4. The C^0-continuity properties. In this subsection, $f\colon W \to [a,b]$ is a Morse function on a compact Riemannian cobordism W, $f^{-1}(a) = V_0$, $f^{-1}(b) = V_1$, and v is an f-gradient.

Proposition 1.5. *Let $\delta > 0$. Let K be a compact set in $V_0 \cap B_\delta(v)$, let R_1 be an open neighborhood of $D_\delta(v) \cap V_0$, and let R_2 be an open neighborhood of $D(v) \cap V_0$. Then there is a number $\epsilon > 0$ such that for every f-gradient w satisfying $\|w - v\| < \epsilon$ we have*

(1) $K \subset B_\delta(w)$; (2) $D_\delta(w) \cap V_0 \subset R_1$; (3) $K(w) \cap V_0 \subset R_2$.

Proof. (1) can be deduced from Corollary 5.5 (see the Appendix) by an easy compactness argument. To prove (2), we observe that the set $V_0 \setminus R_1$ is compact, that each $(-v)$-trajectory starting at a point of $V_0 \setminus R_1$ reaches the boundary, and that $\tau(V_0 \setminus R_1, -v) \subset W \setminus \bigcup_{p \in S(f)} D_\delta(p)$. By Corollary 5.6, the same is true for every w sufficiently close to v in the C^0-topology. The proof of (3) is similar. $\square$

The next lemma is obvious.

Lemma 1.6. *Let $g\colon W \to [a,b]$ be a Morse function adjusted to (f,v). Then there is a number $\epsilon > 0$ such that every f-gradient w satisfying $\|w - v\| < \epsilon$ is a g-gradient.* $\square$

Corollary 1.7. *If v is an f-gradient satisfying the almost transversality condition (respectively, a δ-separated f-gradient), then there is a number $\epsilon > 0$ such that every f-gradient w satisfying $\|w - v\| < \epsilon$ satisfies the almost transversality condition (respectively, is δ-separated).* $\square$

Proposition 1.8. *Let $p \in S(f)$. Let U be an open set in V_0. Assume that for every $x \in D(p,v)$ the trajectory $\gamma(x, \cdot; -v)$ reaches the boundary and that $D(p,v) \cap V_0 \subset U$. Then there is a number $\epsilon > 0$ such that for every f-gradient w satisfying $\|w - v\| < \epsilon$ and every $x \in D(p,w)$ the trajectory $\gamma(x, \cdot; -w)$ reaches the boundary, and $D(p,w) \cap V_0 \subset U$.*

Proof. By [21, Theorem 2.6], there is a Morse function $\phi\colon W \to [a,b]$ adjusted to (f,v) and a regular value μ of ϕ such that $\phi(p) < \mu < \phi(q)$ for every $q \in S(f)$ with $q \neq p$. It remains to apply Lemma 1.6 and Proposition 1.5(3) to $\phi | \phi^{-1}([a,\mu])$. $\square$

1.5. Two lemmas on standard gradients in $\mathbb{R}^n$. In this subsection we view $\mathbb{R}^n$ as the product $\mathbb{R}^k \times \mathbb{R}^{n-k}$; a point $z \in \mathbb{R}^k$ is denoted by (x,y), where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k}$. The vector field with components $(x, -y)$ is denoted by v_0; the standard Euclidean norm in $\mathbb{R}^n$ is denoted by $|\cdot|$. Let f_0 be the function $(x,y) \mapsto -|x|^2 + |y|^2$. (Then $-2v_0$ is the Riemannian gradient of f_0 with respect to the Euclidean metric.) All v_0-trajectories are of the form $(x_0 e^t, y_0 e^{-t})$. With the help of this observation, the following two lemmas can be proved by simple computations, which are left to the reader.

Lemma 1.9. *Let $R > r > 0$, and let γ be a v_0-trajectory. Then the time that γ spends in $B^n(0,R) \setminus B^n(0,r)$ is at most $\log\left((\frac{R}{r})^2 + \sqrt{(\frac{R}{r})^4 - 1}\, \right)$, and the length of the corresponding part of γ is at most $2R$.* $\square$

Lemma 1.10. *Let $r > 0$, and let γ be a v_0-trajectory. Then the time that γ spends in the set $f_0^{-1}([-r^2, r^2]) \setminus B^n(0, r)$ is at most 2.* $\square$

Corollary 1.11. *Let $\mathcal{V} = (f, v, \mathcal{U})$ be an $\mathcal{M}$-flow. If $\mathcal{V}$ is (C, β, α)-quick, then $\mathcal{V}$ is $(C, \beta + 8N(\mathcal{V}), \alpha/2)$-quick.*

Proof. By Lemma 1.9, the time spent by a v-trajectory in $\mathcal{U}(\alpha) \setminus \mathcal{U}(\alpha/2)$ does not exceed $8N(\mathcal{V})$. $\square$

1.6. Theorems on the existence of quick flows.

Theorem 1.12. *Let M be a closed Riemannian manifold of dimension n. Let $C > 0$, $\beta > 0$, $A > 1$. There is an $\mathcal{AM}$-flow $\mathcal{V} = (f, v, \mathcal{U})$ of radius r, and for every $\mu \in]0, r[$ there is an $\mathcal{AM}$-flow $\mathcal{W} = (f, w, \mathcal{U}')$ subordinate to $\mathcal{V}$ and having the following properties*:

(1) *$\mathcal{W}$ is of radius μ and is (C, β)-quick*; (2) $N(\mathcal{W}) \leqslant n2^n$; (3) $\mathcal{G}(\mathcal{W}) \leqslant A$.

Theorem 1.13. *Let M be a closed Riemannian manifold of dimension n. Let $B > 0$, $\mu_0 > 0$. There exists an $\mathcal{AM}$-flow $\mathcal{V} = (f, v, \mathcal{U})$ on M that has the following properties*:

(1) $\mathcal{G}(\mathcal{V}) \leqslant 2$ and $D(\mathcal{V}) < \mu_0$;
(2) $N(\mathcal{V}) \leqslant n2^n$;
(3) *$\mathcal{V}$ is $(B, R(n))$-quick, where* $R(n) = 100 + n2^{n+7}$.

The brief outline of the proof of Theorems 1.12 and 1.13 is as follows. The implication 1.13 $\Rightarrow$ 1.12 is proved below. The proof of the fact that $1.12(n) \Rightarrow 1.13(n+1)$ is given in §3. The same argument proves also 1.13(1).

In the proof of the main theorem we shall use only the following consequence of Theorem 1.12.

Corollary 1.14. *Let M be a closed Riemannian manifold of dimension n. Let $C > 0$, $\beta > 0$. There exists an $\mathcal{AM}$-flow $\mathcal{V} = (f, v, \mathcal{U})$ on M with the property that for every sufficiently small $\delta > 0$ there is an $\mathcal{M}$-flow $\mathcal{W} = (f, w, \mathcal{U}')$ subordinate to $\mathcal{V}$ and such that $\|w\| \leqslant C$ and for every $t \geqslant \beta$ and every s with $0 \leqslant s \leqslant n$ we have*

$$\Phi(-w, t)\big(M \setminus B_\delta(\mathrm{ind}_{\leqslant n-s-1}; -v)\big) \subset B_\delta(\mathrm{ind}_{\leqslant s}; v), \tag{7}$$

$$\Phi(w, t)\big(M \setminus B_\delta(\mathrm{ind}_{\leqslant n-s-1}; v)\big) \subset B_\delta(\mathrm{ind}_{\leqslant s}; -v). \tag{8}$$

Proof. Let $\mathcal{V} = (f, v, \mathcal{U})$ be an $\mathcal{M}$-flow of radius r that satisfies the conclusions of Theorem 1.12 with respect to C, β, and $A = 2$. Let $\delta > 0$ be any number so small that $\delta < r$ and 1) v is δ-separated; 2) for every $p \in S(f)$ the disk $\mathcal{D}_\delta(p)$ is included in U_p.

Let $\mathcal{W} = (f, w, \mathcal{U}')$ be an $\mathcal{M}$-flow satisfying conditions (1)–(3) of Theorem 1.12 with respect to $\delta/2$. We claim that $\mathcal{W}$ possesses the required properties. Indeed, we have $\|w\| \leqslant C$. In order to prove (7), we observe that $B_\delta(\mathrm{ind}_{\leqslant s}; v) \subset M \setminus B_\delta(\mathrm{ind}_{\leqslant n-s-1}; -v)$ and that both sets are $(-v)$-invariant; therefore, for every $x \in M \setminus B_\delta(\mathrm{ind}_{\leqslant n-s-1}; -v)$ it suffices to find $p \in S(f)$ and $t \leqslant \beta$ such that $\gamma(x, t; w) \in \mathcal{B}_\delta(p)$. This is possible because $\mathcal{W}$ is $(C, \beta, \delta/2)$-quick. The proof of (8) is similar. $\square$

FIGURE 1

To deduce Theorem 1.12 from Theorem 1.13 we need a special construction, the properties of which are the subject of the next proposition. For a given $\mathcal{M}$-flow $\mathcal{V}$, this construction produces another one $\mathcal{V}'$ for which $D(\mathcal{V}')$ is as small as we like, but $T(\mathcal{V}')$ can be controlled in terms of the parameters of $\mathcal{V}$.

Proposition 1.15. *Let $\mathcal{V} = (f, v, \mathcal{U})$, where $\mathcal{U} = \{\, \Phi_p \colon U_p \to B^n(0, r_p)\,\}_{p\in S(f)}$, be a $\mathcal{M}$-flow such that $\mathcal{G}(\mathcal{V}) \leqslant 2$, $\|v\| \leqslant B$. Let $C \geqslant B$. Assume that to each $p \in S(f)$ a number $\mu_p \in \,]0, r_p[$ is assigned. Then there is an $\mathcal{M}$-flow $\mathcal{V}' = (f, w, \mathcal{U}')$ subordinate to $\mathcal{U}$ (here $\mathcal{U}' = \{\Phi'_p \colon U'_p \to B^n(0, \mu_p)\}_{p\in S(f)}$ and $\Phi'_p = \Phi_p | U'_p$) for which $\|w\| \leqslant C$ and*

$$C \cdot T(\mathcal{V}') \leqslant B \cdot T(\mathcal{V}) + 5N(\mathcal{V}) \cdot D(\mathcal{V}). \tag{9}$$

Proof. Off the set $\bigcup_p U_p$ we put $w = \frac{C}{B} v$. In each U_p we construct w separately with the help of a standard chart. Namely, $(\Phi_p)_*(v)$ is a standard vector field v_0 on $\mathbb{R}^k \times \mathbb{R}^{n-k}$, where $k = \operatorname{ind} p$. Let $\lambda \colon \mathbb{R}_+ \to \mathbb{R}$ be a C^∞-function such that

$$\begin{cases} \lambda(t) = 1 & \text{if } 0 \leqslant t \leqslant \mu_p; \\ \lambda(t) \text{ is between 1 and } \frac{C}{2t} & \text{if } t \in [\mu_p, \mu_p + \delta]; \\ \lambda(t) = \frac{C}{2t} & \text{if } \mu_p + \delta \leqslant t \leqslant r_p - \delta; \\ \lambda(t) \text{ is between } \frac{C}{2t} \text{ and } \frac{C}{B} & \text{if } t \in [r_p - \delta, r_p]; \\ \lambda(t) = \frac{C}{B} & \text{if } r_p \leqslant t \end{cases}$$

(see Figure 1). The number $\delta \in \,]0, \frac{r_p - \mu_p}{2}[$ will be chosen later.

We define a vector field w_0 on $\mathbb{R}^n$ by the formula $w_0(x) = \lambda(|x|) v_0(x)$. Observe that $(\Phi_p^{-1})_*(w_0)$ can be glued to the field w already defined on $M \setminus \bigcup_p U_p$ (we may use standard extensions); performing this construction for each critical point p, we obtain an $\mathcal{M}$-flow $\mathcal{V}' = (f, w, \mathcal{U}')$ subordinate to $\mathcal{V}$.

The Euclidean norm of $w_0(x)$ equals $\lambda(|x|)|x|$, and using the inequality $\mathcal{G}(\mathcal{V}) \leqslant 2$ it is easy to check that $\|w\| \leqslant C$. To obtain an estimate for $T(\mathcal{V})$, we note that the time spent by a w-trajectory off the set $\bigcup_p U_p$ does not exceed $T(\mathcal{V})\frac{B}{C}$. Therefore, in order to obtain (9), it suffices to prove that for each $p \in S(f)$ the time spent by a w-trajectory in $U_p \setminus U_p(\mu_p)$ is at most $\frac{5}{C} r_p$. This is a consequence of the following two observations (γ stands for a w_0-trajectory):

(1) the time spent by γ in each of the annuli $\mathcal{A}_1 = \{\mu_p \leqslant |x| \leqslant \mu_p + \delta\}$, $\mathcal{A}_2 = \{r_p - \delta \leqslant |x| \leqslant r_p\}$ tends to zero as $\delta \to 0$ (this follows from Lemma 1.9);

(2) the Euclidean length of the part of γ in the annulus $\mathcal{A}_3 = \{\mu_p + \delta \leqslant |x| \leqslant r_p - \delta\}$ does not exceed $2r_p$. The Euclidean norm of γ' in $\mathcal{A}_3$ equals $\frac{C}{2}$. □

Proof of the implication 1.12 ⇒ 1.13. Given (C, β, A), we choose $B > 0$, $\mu_0 > 0$ in such a way that

$$B \leqslant C, \quad B \leqslant C\beta \frac{1}{3R(n)}, \quad \mu_0 \leqslant C\beta \frac{1}{15n2^n}.$$

Let $\mathcal{V}_0 = (f, v, \mathcal{U}_0)$ be an $\mathcal{M}$-flow satisfying the conclusions of Theorem 1.13 with respect to B, μ_0.

Fixing $r < d(\mathcal{V}_0)$ so small that $\mathcal{G}(U_p, \Phi_p) < A$ for every $p \in S(f)$, for each chart $\{\Psi_p : U_p \to B^n(0, r_p)\}_{p \in S(f)}$ of $\mathcal{V}_0$ we consider the restriction $\Phi_p = \Psi_p | U_p(r)$, form the corresponding f-chart system $\mathcal{U} = \{\Phi_p\}$, and introduce the $\mathcal{M}$-flow $\mathcal{V} = (f, v, \mathcal{U})$.

We claim that $\mathcal{V}$ satisfies the conclusions of Theorem 1.13. Indeed, let $0 < \mu < r$. Applying Proposition 1.15 to the flow $\mathcal{V}_0$ and to the numbers $\mu_p = \mu$, we obtain an $\mathcal{M}$-flow $\mathcal{W} = (f, w, \mathcal{U}')$ subordinate to $\mathcal{V}_0$, where $\mathcal{U} = \{\Phi'_p\}_{p \in S(f)}$ and $\Phi'_p = \Phi_p | U_p(\mu)$. This $\mathcal{M}$-flow is (C, β)-quick, because $\|w\| \leqslant C$ and $T(\mathcal{V}') \leqslant (\frac{\beta T(\mathcal{V}_0)}{3R(n)} + \frac{5}{C} n 2^n \mu_0) \leqslant \frac{2}{3}\beta$ (by (9)). Properties (2) and (3) follow by construction. □

§2. Stratifications and transversality

In Subsection 2.1 we introduce the notion of an s-submanifold and develop some relevant technique. The s-submanifolds are analogs of stratified manifolds more convenient for our purposes than the classical stratified manifolds. The main example of s-manifolds is provided by the descending disks of the gradient of a Morse function. In Subsection 2.2 we introduce *ts-submanifolds*, which are "thickened" s-submanifolds. These are analogs of the tubular neighborhoods of stratified manifolds. In Subsection 2.3 we consider the behavior of s- and ts-submanifolds under the gradient descent. The main result of §2 is Proposition 2.12, which will be used in §4.

2.1. s-Submanifolds. Let $\mathbb{A} = \{A_0, \dots, A_k\}$ be a finite sequence of subsets of a topological space X. For $0 \leqslant s \leqslant k$, we denote A_s also by $\mathbb{A}_{(s)}$; the set $A_0 \cup \dots \cup A_s$ is denoted by $A_{\leqslant s}$ and also by $\mathbb{A}_{(\leqslant s)}$. We say that $\mathbb{A}$ is a *compact family* if $\mathbb{A}_{(\leqslant s)}$ is compact for every s with $0 \leqslant s \leqslant k$.

Definition 2.1. Let M be a manifold without boundary. A finite sequence $\mathbb{X} = \{X_0, \dots, X_k\}$ of subsets of M is called an *s-submanifold of M* (s for "stratified") if

(1) $X_i \cap X_j = \varnothing$ for $i \neq j$, and each X_i is a submanifold of M of dimension i with trivial normal bundle;
(2) $\mathbb{X}$ is a compact family.

For an s-submanifold $\mathbb{X} = \{X_1, \dots, X_s\}$, the largest k such that $X_k \neq \varnothing$ is called the *dimension* of $\mathbb{X}$ and is denoted by $\dim \mathbb{X}$. For a diffeomorphism $\Phi\colon M \to N$ and an s-submanifold $\mathbb{X}$ of M, we denote by $\Phi(\mathbb{X})$ the s-submanifold of N defined by $\Phi(\mathbb{X})_{(i)} = \Phi(\mathbb{X}_{(i)})$.

If V is a submanifold of M and $\mathbb{X}$ is an s-submanifold of M, then we say that V is *transversal* to $\mathbb{X}$ (in symbols: $V \pitchfork \mathbb{X}$) if $V \pitchfork \mathbb{X}_{(i)}$ for each i. If V is a compact submanifold of M transversal to an s-submanifold $\mathbb{X}$, then the family $\{\mathbb{X}_{(i)} \cap V\}$ is an s-submanifold of V to be denoted by $\mathbb{X} \cap V$.

Let $\mathbb{X}$, $\mathbb{Y}$ be two s-submanifolds of M. We say that $\mathbb{X}$ is *transversal* to $\mathbb{Y}$ (in symbols: $\mathbb{X} \pitchfork \mathbb{Y}$) if $\mathbb{X}_{(i)} \pitchfork \mathbb{Y}_{(j)}$ for every i, j; we say that $\mathbb{X}$ is *almost transversal* to $\mathbb{Y}$ (in symbols: $\mathbb{X} \nmid \mathbb{Y}$) if $\mathbb{X}_{(i)} \pitchfork \mathbb{Y}_{(j)}$ for any i, j with $i + j < \dim M$. We note that $\mathbb{X} \nmid \mathbb{Y}$ if and only if $\mathbb{X}_{(\leqslant i)} \cap \mathbb{Y}_{(\leqslant j)} = \phi$ whenever $i + j < \dim M$.

Let $f\colon M \to \mathbb{R}$ be a Morse function, where M is a closed manifold or a compact cobordism, and let v be an f-gradient We denote by $\mathbb{D}(v)$ the family $\{D(\mathrm{ind} {\leqslant} i; v)\}_{0 \leqslant i \leqslant \dim M}$, and by $\mathbb{D}_\lambda(v)$ the family $\{D(\mathrm{ind} {\leqslant} i; v) \cap f^{-1}(\lambda)\}_{0 \leqslant i \leqslant \dim M}$.

Lemma 2.2. *Assume that v satisfies the almost transversality condition. Then*:

(1) *$\mathbb{D}(v)$ is a compact family*;
(2) *if M is a closed manifold, then $\mathbb{D}(v)$ is an s-submanifold transversal to $f^{-1}(\lambda)$ for every regular value λ of f*;
(3) *if M is a cobordism, then for every regular value λ of f the family $\mathbb{D}_\lambda(v)$ is an s-submanifold of $f^{-1}(\lambda)$.*

Proof. Proposition 1.4 implies statement (1) immediately. Since the normal bundle to $D(p, v)$ is always trivial, statements (2) and (3) follow. $\square$

If W is a compact manifold (possibly, with boundary), we denote by $\mathrm{Vect}^\infty(W)$ the space of all vector fields of class C^∞ equipped with the natural topology (see [19, §8]).

Let M be a closed manifold. We denote by $\mathrm{Vectt}(M)$ the subspace of $\mathrm{Vect}^\infty(M \times [0,1])$ consisting of all vector fields of class C^∞ that have the second coordinate 0 and vanish with all their derivatives on $M \times \{0,1\}$. The subspace $\mathrm{Vectt}(M)$ is closed.

The following transversality result is proved by induction on $\dim \mathbb{X} + \dim \mathbb{Y}$ with the help of a standard general position argument like that used in [6, Lemma 5.3].

Theorem 2.3. *Let $\mathbb{X}, \mathbb{Y}$ be s-submanifolds of a closed manifold M. Then the set of $v \in \mathrm{Vectt}(M)$ such that $\Phi(v, 1)(\mathbb{X}) \nmid \mathbb{Y}$ is open and dense in* $\mathrm{Vectt}(M)$. $\square$

2.2. Good fundamental systems of neighborhoods, and ts-submanifolds.

Definition 2.4. Let X be a topological space, $\mathbb{A} = \{A_0, \dots, A_k\}$ a compact family of subsets of X, I an open interval $]0, \delta_0[$. A *good fundamental system of neighborhoods of* $\mathbb{A}$ (briefly: gfn-system for $\mathbb{A}$) is a family $\mathbf{A} = \{A_s(\delta)\}_{\delta \in I, 0 \leqslant s \leqslant k}$ of open subsets of X satisfying the following conditions:

(FS1) for every s with $0 \leqslant s \leqslant k$ and every $\delta \in I$ we have $A_s \subset A_s(\delta)$;

(FS2) for every s with $0 \leqslant s \leqslant k$ we have $\delta_1 < \delta_2 \Rightarrow A_s(\delta_1) \subset A_s(\delta_2)$;

(FS3) for every $\delta \in I$ and every j with $0 \leqslant j \leqslant k$ we have $\overline{A_{\leqslant j}(\delta)} = \bigcap_{\theta > \delta} (A_{\leqslant j}(\theta))$;

(FS4) for every j with $0 \leqslant j \leqslant k$ we have $A_{\leqslant j} = \bigcap_{\theta > 0} (A_{\leqslant i}(\theta))$.

I is called an *interval of definition* of the gfn-system. $\mathbf{A}$; $\mathbb{A}$ is called the *core* of $\mathbf{A}$.

We shall denote $A_s(\delta)$ also by $\mathbf{A}_{(s)}(\delta)$ and $A_{\leqslant i}(\delta)$ also by $\mathbf{A}_{(\leqslant i)}(\delta)$.

The following lemma is proved by an easy argument from general topology.

Lemma 2.5. *If X is compact and $\{A_s(\delta)\}_{\delta \in I, 0 \leqslant s \leqslant k}$ is a* gfn-*system for a compact family $\mathbb{A}$ in X, then for every s with $0 \leqslant s \leqslant k$ we have the following statements*:

(1) *for any $\delta \in I$ the family $\{A_{\leqslant s}(\theta)\}_{\theta > \delta}$ is a fundamental system of neighborhoods of $\overline{A_{\leqslant s}(\delta)}$*;

(2) *the family $\{A_{\leqslant s}(\theta)\}_{\theta > 0}$ is a fundamental system of neighborhoods of $A_{\leqslant s}$.* $\square$

Let M be a manifold without boundary, $\mathbb{X}$ an s-submanifold of M, and $\mathbf{X}$ a gfn-system for $\mathbb{X}$. We say that $\mathbf{X}$ is a *ts-submanifold of M* with *core* $\mathbb{X}$. For a ts-submanifold $\mathbf{X} = \{X_s(\delta)\}_{\delta \in I, 0 \leqslant s \leqslant k}$, we shall denote $X_i(\delta)$ by $\mathbf{X}_{(i)}(\delta)$, and $X_{\leqslant i}(\delta)$ by $\mathbf{X}_{(\leqslant k)}(\delta)$.

Let $I =]0, \delta_0[$, and let $Z = \{Z(\delta)\}_{\delta \in I}$ be a family of subsets of some space X. Let $0 < \epsilon_0 < \delta_0$. The family $\{Z(\delta)\}_{\delta \in]0, \epsilon_0[}$ will be called a *restriction* of Z.

The basic example of a gfn-system and a ts-manifold is given by the next lemma, which follows immediately from Proposition 1.4.

Lemma 2.6. *Let M be a closed Riemannian manifold or a Riemanian cobordism. Let $f\colon M \to \mathbb{R}$ be a Morse function and v a δ-separated f-gradient. Then the family $\{B_\delta(\mathrm{ind}{=}s; v)\}_{\delta \in]0, \epsilon[, 0 \leqslant s \leqslant n}$ is a* gfn-*system for $\mathbb{D}(v)$. If M is a closed manifold, this family is a ts-submanifold with core $\mathbb{D}(v)$.* $\square$

The gfn-system introduced in the lemma will be denoted by $\mathbf{D}(v)$. We note that there is no canonical way for choosing an interval of definition for this system. We shall say that $\lambda > 0$ *is in an interval of definition of* $\mathbf{D}(v)$ if there is $\epsilon > \lambda$ such that $\{B_\delta(\mathrm{ind}{=}s; v)\}_{\delta \in]0, \epsilon[, 0 \leqslant s \leqslant n}$ is a gfn-system. Proposition 1.4 implies that if v is λ-separated, then λ is in an interval of definition of $\mathbf{D}(v)$.

2.3. Tracks of subsets, of s-submanifolds, and of ts-submanifolds. Let $f\colon W \to [a, b]$ be a Morse function on a compact Riemannian cobordism W with $f^{-1}(b) = V_1, f^{-1}(a) = V_0$, and let v be an f-gradient.

Let $X \subset V_1$. The set $\{\, \gamma(x, t; -v) \mid t \geqslant 0,\ x \in X \,\}$ is called the *track* of X (with respect to v) and is denoted by $T(X, v)$.

Lemma 2.7. (1) *If X is compact, then $T(X,v)\cup D(v)$ is compact.*

(2) *If X is compact, and every $(-v)$-trajectory starting at a point of X reaches V_0, then $T(X,v)$ is compact.*

(3) *For any X we have $\overline{T(X,v)\cup D(v)} = T(\overline{X},v)\cup D(v)$.*

(4) *For any X and any $\delta>0$ we have $\overline{T(X,v)\cup B_\delta(v)} = T(\overline{X},v)\cup D_\delta(v)$.*

Proof. Statements (1) and (2) are obvious. To prove (3), we note that $T(X,v)\cup D(v)\subset T(\overline{X},v)\cup D(v)\subset \overline{T(X,v)}\cup D(v)$, which implies (3) in view of (1). Statement (4) is proved similarly. $\square$

If X is a submanifold of V_1 of dimension k, then $T(X,v)\cap W^\circ$ is a submanifold of W° of dimension $k+1$. If λ is a regular value of f, then $T(X,v)\cap f^{-1}(\lambda)$ is a submanifold of dimension k of $f^{-1}(\lambda)$.

¿From this point till the end of the section it is assumed that v is an f-gradient satisfying the almost transversality condition. Let $\mathbb{A}$ be an s-submanifold $\{A_0,\dots,A_k\}$ of V_1 such that $\mathbb{A}\nmid\mathbb{D}_b(-v)$.

Definition 2.8. For $0\leqslant i\leqslant k+1$, let $TA_i(v)$ denote the set $T(A_{i-1},v)\cup D(\mathrm{ind}{=}i;v)$ (here we put $A_{-1}=\phi$). We denote by $\mathbb{T}(\mathbb{A},v)$ the family $\{TA_i(v)\}_{0\leqslant i\leqslant k+1}$ of subsets of W and by $(-v)^{\leadsto}_{[b,\lambda]}(\mathbb{A})$ the family $\{TA_{i+1}(v)\cap f^{-1}(\lambda)\}_{0\leqslant i\leqslant k}$ of subsets of $f^{-1}(\lambda)$. If the values b,λ are clear from the context, we shall abbreviate $(-v)^{\leadsto}_{[b,\lambda]}(\mathbb{A})$ to $(-v)^{\leadsto}(\mathbb{A})$. The family $\mathbb{T}(\mathbb{A},v)$ will be called the *track* of $\mathbb{A}$, and the family $(-v)^{\leadsto}_{[b,\lambda]}(\mathbb{A})$ will be called the $(-v)^{\leadsto}$-*image* of $\mathbb{A}$.

Lemma 2.9. (1) *$\mathbb{T}(\mathbb{A},v)$ and $(-v)^{\leadsto}_{[b,\lambda]}(\mathbb{A})$ are compact families.*

(2) *If λ is a regular value of f, then $(-v)^{\leadsto}_{[b,\lambda]}(\mathbb{A})$ is an s-submanifold of $f^{-1}(\lambda)$.*

Proof. (1) Let $0\leqslant s\leqslant k+1$, and let $Q(s)=T(A_{\leqslant s-1},v)\cup D(\mathrm{ind}{\leqslant}s;v)$. Let $\phi\colon W\to[a,b]$ be an ordered Morse function adjusted to (f,v), and let $(a_0,\dots,a_{n+1})$ be the ordering sequence for ϕ. Since $\mathbb{A}\nmid(\mathbb{D}(-v)\cap V_1)$, every $(-v)$-trajectory starting at a point of $A_{\leqslant s-1}$ reaches $\phi^{-1}(a_{s+1})$. Then $Q(s)\cap\phi^{-1}([a_{s+1},a_{n+1}])$ is compact by 2.7(2), and $Q(s)\cap\phi^{-1}([a_0,a_{s+1}])$ is compact by 2.7(1), whence $Q(s)$ is compact. Statement (2) is obvious. $\square$

Now we introduce the notion of the track of a ts-submanifold.

Definition 2.10. Let $\mathbf{A}=\{A_s(\delta)\}_{\delta\in]0,\delta_0[,0\leqslant s\leqslant k}$ be a ts-submanifold of V_1 with core $\mathbb{A}$. Assume that v is δ_1-separated. For $0<\delta<\min(\delta_0,\delta_1)$ and $0\leqslant s\leqslant k+1$, we put $TA_s(\delta,v)=T(A_{s-1}(\delta),v)\cup B_\delta(\mathrm{ind}{=}s;v)$ (by definition, $A_{-1}(\delta)=\varnothing$).

We prove that some restriction of $\{TA_s(\delta,v)\}$ is a gfn-system for $\mathbb{T}(\mathbb{A},v)$. Up to the end of this section, $Q(s,\delta)$ stands for $T(\overline{A_{\leqslant s-1}(\delta)},v)\cup D_\delta(\mathrm{ind}{\leqslant}s;v)$.

Lemma 2.11. *Let $0<\epsilon\leqslant\min(\delta_0,\delta_1)$. The following statements are equivalent.*

(i) *For every $0<\delta<\epsilon$ and every $0\leqslant s\leqslant k+1$ the set $Q(s,\delta)$ is compact.*

(ii) *$TA_s(\delta,v)_{0<\delta<\epsilon,0\leqslant s\leqslant k+1}$ is a* gfn-*system for $\mathbb{T}(\mathbb{A},v)$.*

Proof. It is easily seen that for every $\delta\in]0,\min(\delta_0,\delta_1)[$ we have

$$Q(s,\delta)=\bigcap\nolimits_{\theta>\delta}TA_{\leqslant s}(\theta,v),\qquad TA_{\leqslant s}(v)=\bigcap\nolimits_{\theta>0}TA_{\leqslant s}(\theta,v),\tag{2.1}$$

$$TA_{\leqslant s}(\delta,v)\subset Q(s,\delta)\subset\overline{TA_{\leqslant s}(\delta,v)}.\tag{2.2}$$

Now, to prove the implication (ii) $\Rightarrow$ (i) it suffices to observe that (FS3) and (2.1) imply that $Q(s,\delta) = \overline{TA_{\leqslant s}(\delta, v)}$. To prove the reverse implication, we note that (FS1), (FS2), and (FS4) are true for every $\delta \in \,]0, \min(\delta_0, \delta_1)[$; now (2.2) shows that (i) implies (FS3). $\square$

Proposition 2.12. *There is* $\epsilon \in \,]0, \min(\delta_0, \delta_1)[$ *such that* $\{TA_s(\delta, v)\}_{\delta \in]0,\epsilon[, 0 \leqslant s \leqslant k}$ *is a* gfn*-system for* $\mathbb{T}(\mathbb{A}, v)$.

Proof. Let $\phi\colon W \to [a_0, a_{n+1}]$ be an ordered Morse function adjusted to (f, v), and let $(a_0, \ldots, a_{n+1})$ be the ordering sequence for ϕ. Since $\mathbb{A} \nmid \mathbb{D}_b(-v)$, every $(-v)$-trajectory starting at a point in $A_{\leqslant s-1}$ reaches $\phi^{-1}(a_{s+1})$. The set $A_{\leqslant s-1}$ is compact, and the collection $\{A_{\leqslant s-1}(\theta)\}_{\theta > 0}$ is a gfn-system for $A_{\leqslant s-1}$; therefore, an easy compactness argument based on Corollary 5.4 shows that there is $\theta > 0$ such that each $(-v)$-trajectory starting at a point of $A_{\leqslant s-1}(\theta)$ reaches $\phi^{-1}(a_{s+1})$.

We claim that the number $\epsilon = \min(\delta_0, \delta_1, \theta)$ is the required one. Indeed, let $\delta < \epsilon$. Then each $(-v)$-trajectory starting at a point of $\overline{A_{\leqslant s-1}(\delta)}$ reaches $\phi^{-1}(a_{s+1})$, and the set $T(\overline{A_{\leqslant s-1}(\delta)}, v)$ is compact by 2.7(2). We put $C(\delta) = (-v)^{\rightsquigarrow}_{[b, a_{s+1}]}(\overline{A_{\leqslant s-1}(\delta)})$; this is a compact set. The intersection of $Q(s,\delta)$ and $\phi^{-1}([a_{s+1}, a_{n+1}])$ is compact by the above, and its intersection with $\phi^{-1}([a_0, a_{s+1}])$ is compact by Lemma 2.7(4) applied to the cobordism $\phi^{-1}([a_0, a_{s+1}])$ and the compact $C(\delta) \subset \phi^{-1}(a_{s+1})$. $\square$

The gfn-system introduced in Proposition 2.12 will be denoted by $\mathbf{T}(\mathbf{A}, v)$; we call it the *track* of $\mathbf{A}$. Note that there is no canonical choice of an interval of definition for this system. We shall say that $\lambda > 0$ *belongs to an interval of definition of* $\mathbf{T}(\mathbf{A}, v)$ if there is $\epsilon > \lambda$ such that $\{TA_s(\delta, v)\}_{\delta \in]0,\epsilon[, 0 \leqslant s \leqslant k}$ is a gfn-system. The next two lemmas contain some properties of $\mathbf{T}(\mathbf{A}, v)$ to be used in §4.

Lemma 2.13. *Assume that* $\delta > 0$ *is in an interval of definition of* $\mathbf{T}(\mathbf{A}, v)$. *Then for* $\lambda = a, b$ *we have* $\overline{\mathbf{T}(\mathbf{A}, v)_{(\leqslant s)}(\delta)} \cap f^{-1}(\lambda) = \overline{\mathbf{T}(\mathbf{A}, v)_{(\leqslant s)}(\delta) \cap f^{-1}(\lambda)}$.

Proof. Obvious. $\square$

Consider regular values α, β of f such that $a < \alpha < \beta < b$ and there are no critical points of f in $f^{-1}([a, \alpha] \cup [\beta, b])$; next, let $\widetilde{f}$, $\widetilde{v}$ denote the restrictions of f,v to $\widetilde{W} = f^{-1}([\alpha, \beta])$. We put $\widetilde{V}_0 = f^{-1}(\alpha)$, $\widetilde{V}_1 = f^{-1}(\beta)$. Let $\widetilde{\mathbf{C}} = \{\widetilde{C}_s(\delta)\}_{\delta \in]0,\delta_0[, 0 \leqslant s \leqslant k}$ be a *ts*-submanifold of $\widetilde{V}_1$ with core $\widetilde{\mathbb{A}} = (-v)^{\rightsquigarrow}_{[b,\beta]}(\mathbb{A})$. We assume that for every s with $0 \leqslant s \leqslant k$ and every $\delta \in \,]0, \delta_0[$ we have $(-v)^{\rightsquigarrow}_{[b,\beta]}(A_s(\delta)) \subset C_s(\delta)$ and that $\widetilde{v}$ is δ_1-separated.

Lemma 2.14. *If* $0 < \nu < \min(\delta_0, \delta_1)$ *and* ν *is in an interval of definition of* $\mathbf{T}(\widetilde{\mathbf{C}}, \widetilde{v})$, *then* ν *is in an interval of definition of* $\mathbf{T}(\mathbf{A}, v)$.

Proof. Let $\theta \in \,]0, \min(\delta_0, \delta_1)[$ be such that $\{TC_s(\delta, \widetilde{v})\}_{\delta \in]0,\theta[, 0 \leq s \leq k+1}$ is a gfn-system. It suffices to prove that for $\delta \in \,]0, \theta[$ and $0 \leqslant s \leqslant k+1$ the set $Q(s,\delta)$ is compact, which reduces easily to the compactness of the set $Q(s,\delta) \cap f^{-1}([\alpha, \beta])$. Observe that if $Z' \subset Z$ are compact sets in $\widetilde{V}_1$, then $T(Z', \widetilde{v}_1)$ is closed in $T(Z, v)$; therefore, $Q(s,\delta) \cap f^{-1}([\alpha, \beta])$ is closed in $T(\overline{C_{\leqslant s-1}(\delta)}, \widetilde{v}) \cup D_\delta(\mathrm{ind}_{\leqslant s}; \widetilde{v})$. Thus, this set is compact. $\square$

FIGURE 2

§3. THE S-CONSTRUCTION

In this section we prove the implication $1.12(n) \Rightarrow 1.13(n+1)$, thus finishing the proof of Theorem 1.12. Recall that Theorem 1.13 claims the existence of $\mathcal{M}$-flows $\mathcal{V}$ having arbitrarily small norm and radius, but with bounded $T(\mathcal{V})$.

We give a construction of such flows. First, we present an informal description of this construction. Starting with an arbitrary Morse function $f\colon M \to \mathbb{R}$ with the property that the values of f at all critical points are different, we choose a finite sequence of regular values $\alpha_1, \dots, \alpha_n$ of f such that between α_i and α_{i+1} there is at most one critical value of f. Then, we view M as the union of the cobordisms $W_i = f^{-1}([\alpha_i, \alpha_{i+1}])$. Making α_i and α_{i+1} suficiently close to each other, we can arrange that the "slices" W_i be as thin as we wish. Consider a cobordism W_i having no critical points. Then W_i is diffeomorphic to the cylinder $V_i \times [0,1]$, where $V_i = f^{-1}(\alpha_i)$ (see Figure 2a; here $V_i = S^1$, the manifold $V_1 \times [0,1]$ is embedded in $\mathbb{R}^3$, and the function f is the projection to the z-axis). Now we change f inside W_i so as to make a fold. We denote by $\widetilde{f}_i$ the resulting function (see Figure 2b; here another embedding of $V_i \times [0,1]$ in $\mathbb{R}^3$ is depicted, and the function $\widetilde{f}_i$ is the composition of the new embedding and the projection to the z-axis).

Glueing the functions $\widetilde{f}_i$ together, we obtain a function $\widetilde{f}$ on the manifold M. Observe that the trajectories of the gradient have become much shorter: none of them can intersect more than three subsequent cobordisms W_i. Surely, this function $\widetilde{f}$ is not a Morse function, because there are two level surfaces formed by critical points. This must be remedied by perturbing $\widetilde{f}_i$ near these level surfaces with the help of some auxiliary Morse functions on V_i (see Figure 2c).

Having constructed new Morse functions $\bar{f}_i$ on the cobordisms W_i in this way, we glue them together to get a Morse function on W; the latter satisfies the conclusions of Theorem 1.13 with $n+1$ in place of n. The process of constructing $\bar{f}_i$ starting with f_i will be called the *S-construction* (S is for *skladka*, the Russian for fold).

The functions without critical points on cobordisms and on their fold-like modifications are studied in Subsections 3.1 and 3.2.1. The process of glueing the modified functions together is described in Subsection 3.2, where we prove the implication $1.12(n) \Rightarrow 1.13(n+1)$.

3.1. The S-construction. Let W be a compact Riemannian cobordism. Throughout §3 we denote by $|\cdot|$ the norm on tangent spaces induced by the Riemannian metric and by $||\cdot||$ the corresponding C^0-norm in the space of vector fields.

Theorem 3.1. *Let $g\colon W \to [a,b]$ be a Morse function without critical points, and let $g^{-1}(a) = V_0, g^{-1}(b) = V_1$. For $C > 0$, let w be a g-gradient such that $||w|| \leqslant C$. We denote $V_{1/3} = g^{-1}(\frac{2a+b}{3})$, $V_{2/3} = g^{-1}(\frac{2b+a}{3})$, $V_{1/2} = g^{-1}(\frac{a+b}{2})$, $W_0 = g^{-1}([a, \frac{2a+b}{3}])$, $W_1 = g^{-1}([\frac{2b+a}{3}, b])$, $W_{1/2} = g^{-1}([\frac{2a+b}{3}, \frac{2b+a}{3}])$. We denote by $\operatorname{grad}(g)$ the Riemannian gradient of g and by $\operatorname{grd}(g)$ the vector field $\operatorname{grad}(g)/|\operatorname{grad}(g)|$. Let T be the maximal length of the interval of definition of a $\operatorname{grd}(g)$-trajectory. Then there is a number $\nu_0 > 0$ such that for every s-submanifold $\mathbb{X}$ of V_0, every s-submanifold $\mathbb{Y}$ of V_1, every $\mathcal{AM}$-flow $\mathcal{V}_1 = (F_1, u_1, \mathcal{U}_1)$ on $V_{1/3}$, every $\mathcal{AM}$-flow $\mathcal{V}_2 = (F_2, u_2, \mathcal{U}_2)$ on $V_{2/3}$, and every $\mu \leqslant \min(\nu_0, d(\mathcal{V}_1), d(\mathcal{V}_2))$ there is an $\mathcal{AM}$-flow $\mathcal{V} = (F, u, \mathcal{U})$ on W having the following properties:*

1. *$F\colon W \to [a,b]$, $V_0 = F^{-1}(a)$, $V_1 = F^{-1}(b)$, $S(F) = S(F_1) \cup S(F_2)$, and in a neighborhood of ∂W we have $F = g$, $u = w$;*
2. *$\mathcal{V}$ is of radius μ, and $\mathcal{G}(\mathcal{V}) \leqslant \frac{3}{2}\max(\mathcal{G}(\mathcal{V}_1), \mathcal{G}(\mathcal{V}_2))$,*
3. *$V_{1/3}$, $V_{2/3}$, $W_{1/2}$ are $(\pm u)$-invariant, W_0 is u-invariant and weakly $(-u)$-invariant, and W_1 is $(-u)$-invariant and weakly u-invariant;*
4. *if $\mathcal{V}_1$ is a $(C_1, \beta_1, \mu/2)$-quick flow and $\mathcal{V}_2$ is a $(C_2, \beta_2, \mu/2)$-quick flow, then $\mathcal{V}$ is a $(3/2(C + C_1 + C_2), \beta_1 + \beta_2 + 5 + 4\frac{T}{C}\ \mu)$-quick flow;*
5. *if γ is a u-trajectory such that $\operatorname{Im}\gamma \subset W_{1/2}$, then $N(\gamma,\mu) \leqslant N(\mathcal{V}_1,\mu) + N(\mathcal{V}_2,\mu)$; if $\operatorname{Im}\gamma \subset W_0$, then $N(\gamma,\mu) \leqslant N(\mathcal{V}_1,\mu)$, and if $\operatorname{Im}\gamma \subset W_1$, then $N(\gamma,\mu) \leqslant N(\mathcal{V}_2,\mu)$;*
6. *$\mathbb{D}_a(u) \nmid \mathbb{X}$; $\mathbb{D}_b(-u) \nmid \mathbb{Y}$.*

Remark. The submanifolds $V_{\frac{1}{3}}, V_{\frac{2}{3}}$ correspond to the critical level surfaces of $\widetilde{f_i}$ in Figure 2b. The flows $\mathcal{V}_1$ and $\mathcal{V}_2$ correspond to perturbations of $\widetilde{f_i}$ needed for turning $\widetilde{f_i}$ into a Morse function.

Proof. The proof occupies the rest of Subsection 3.1. We assume that $a = 0$, $b = 1$, since the general case is easily reduced to this one by an affine transformation of $\mathbb{R}$.

3.1.1. The function f, its gradient v, and the choice of ν_0.

Lemma 3.2. *There is a Morse function $f\colon W \to [0,1]$ having no critical points and an f-gradient v with the following properties:*

1. *$||v|| \leqslant C$;*
2. *for x in a neighborhood of $V_0 \cup V_1$ we have $f = g$ and $v = w$;*
3. *$f^{-1}(1/3) = V_{1/3}$, $f^{-1}(2/3) = V_{2/3}$, $f^{-1}(1/2) = V_{1/2}$;*
4. *for any x in a neighborhood of $V_{1/3} \cup V_{1/2} \cup V_{2/3}$ we have $|v(x)| = C$ and $df(v)(x) = C$;*
5. *for $\lambda = 1/3, 1/2, 2/3$ and $x \in g^{-1}(\lambda)$ we have $v(x) \perp g^{-1}(\lambda)$;*

(6) *the maximal length of the interval of definition of a v-trajectory does not exceed $2T/C$.*

Proof. Let U be an open neighborhood of $V_0 \cup V_1$ satisfying $U \cap (V_{1/3} \cup V_{2/3} \cup V_{1/2}) = \varnothing$ and let $h: W \to [0,1]$ be a C^∞-function such that $\operatorname{supp} h \subset U$ and for any x in a neighborhood of ∂W we have $h(x) = 1$. We set $v(x) = h(x)w(x) + (1 - h(x))C \operatorname{grd}(g)$. Obviously, that v is a g-gradient satisfying (1) and (5). Also, we have $v(x) = w(x)$ near ∂W, and $|v(x)| = C$ near $V_{1/3} \cup V_{2/3} \cup V_{1/2}$. Property (6) will be fulfilled if U is chosen sufficiently small. To construct f, first we define this function in a neighborhood of $V_{1/3} \cup V_{2/3} \cup V_{1/2}$ directly by (4), and in a neighborhood of ∂W we put $f = g$. Next, we extend f to each of the cobordisms W_0, W_1, $g^{-1}([\frac{1}{3}, \frac{1}{2}])$, $g^{-1}([\frac{1}{2}, \frac{2}{3}])$ by the usual procedure of glueing two functions that have one and the same gradient (this procedure is described in detail in [21, Corollary 8.14]). $\square$

For $\lambda \in [0,1]$ we denote $V_\lambda = f^{-1}(\lambda)$. Let $\epsilon \in]0, \frac{1}{2}[$. If $\nu > 0$ is sufficiently small, then the map $(x,\tau) \mapsto \gamma(x,\tau; v/C)$ is defined on $V_i \times [-\nu, \nu]$ (where $i = 1/3, 1/2, 2/3$), on $V_0 \times [0,\nu]$, and on $V_1 \times [-\nu, 0]$. The corresponding embeddings will be denoted as follows:

$$\Psi_0(\nu): V_0 \times [0,\nu] \to W, \quad \Psi_1(\nu): V_1 \times [-\nu, 0] \to W, \tag{10}$$

$$\Psi_i(\nu): V_i \times [-\nu,\nu] \to W \quad \text{if } i = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}. \tag{11}$$

The image of $\Psi_i(\nu)$ will be denoted by $\mathrm{Tb}_i(\nu)$ (here $i = 0, 1/3, 1/2, 2/3, 1$).

We require that ν_0 satisfy the following condition $(\mathfrak{R})$:

$$(\mathfrak{R}) \begin{cases} (1) & 2\nu_0 < C; \\ (2) & \text{for } i = 0, 1/3, 1/2, 2/3, 1 \text{ we have} \\ & \quad \text{(2A)} \ f(\mathrm{Tb}_i(2\nu_0)) \subset]i - \epsilon, i + \epsilon[, \\ & \quad \text{(2B)} \ \text{the Riemannian metric induced by } \Psi_i(2\nu_0) \\ & \qquad\quad \text{from } W \text{ and the product metric on the domain} \\ & \qquad\quad \text{of } \Psi_i(2\nu_0) \text{ are } \tfrac{3}{2}\text{-equivalent;} \\ (3) & \text{For } i = 1/3, 1/2, 2/3 \text{ we have } |v(x)| = df(v(x)) = C \\ & \text{for any } x \text{ in a neighborhood of the set } \mathrm{Tb}_i(\nu_0). \end{cases}$$

We prove that such a number ν_0 satisfies the conclusions of our theorem. So, let $\mathbb{X}, \mathbb{Y}, \mathcal{V}_1 = (F_1, u_1, \mathcal{U}_1), \mathcal{V}_2 = (F_2, u_2, \mathcal{U}_2)$ be as in the statement of the theorem, where $\mathcal{U}_1 = \{\Phi_{1p}: U_{1p} \to B^n(0, r_{1p})\}_{p \in S(F_1)}$, $\mathcal{U}_2 = \{\Phi_{2q}: U_{2q} \to B^n(0, r_{2q})\}_{q \in S(F_2)}$, $n = \dim W - 1$. Let $\mu \leqslant \min(\nu_0, d(\mathcal{V}_1), d(\mathcal{V}_2))$. We put $\Psi_i = \Psi_i(\mu)$. Using $(\mathfrak{R})$, it is easy to check the following property:

$$\text{if } |\lambda| \leqslant 2\mu \text{ and } i = 1/3, 1/2, 2/3, \text{ then } f^{-1}(i + \lambda) = \Psi_i(V_i \times \{\lambda\}). \tag{12}$$

For $\lambda \in [0, 1/3]$, we denote by $\mathbb{L}_\lambda(u_1)$ the s-submanifold $(-v)^{\leadsto}_{[1/3,\lambda]}(\mathbb{D}(u_1))$ of the manifold V_λ. For $\lambda \in [1/3, 1]$, let $\mathbb{L}_\lambda(u_1)$ be the s-submanifold $(-v)^{\leadsto}_{[1/3,\lambda]}\mathbb{D}(u_1)$ of V_λ. For $\lambda \in [2/3, 1]$, we denote by $\mathbb{L}_\lambda(-u_2)$ the s-submanifold $(-v)^{\leadsto}_{[2/3,\lambda]}(\mathbb{D}(-u_2))$,

FIGURE 3

and for $\lambda \in [0, 2/3]$, $\mathbb{L}_\lambda(-u_2)$ will stand for the s-submanifold $(-v)^{\rightsquigarrow}_{[2/3,\lambda]}(\mathbb{D}(-u_2))$. Since v has no zeros, $(-v)^{\rightsquigarrow}_{[\alpha,\beta]}$ is a diffeomorphism of V_α onto V_β for any $\beta < \alpha$. Also, we note that $\mathbb{L}_{2/3}(-u_2)$ coincides with $\mathbb{D}(-u_2)$, and $\mathbb{L}_{1/3}(u_1)$ coincides with $\mathbb{D}(u_1)$.

3.1.2. Auxiliary functions and vector fields. Let $\chi\colon [0,1] \to [0,1]$ be a C^∞-function with the following properties:

$$
\begin{aligned}
\chi(x) &= x && \text{if } x \in [0,\epsilon] \cup [1-\epsilon, 1];\\
\chi(x) &= 1 - x && \text{if } x \in [1/2-\epsilon, 1/2+\epsilon];\\
\chi(x) &= 2/3 - (x-1/3)^2 && \text{if } x \in [1/3-\epsilon, 1/3+\epsilon];\\
\chi'(x) &> 0 && \text{if } x \in [0, 1/3[\cup]2/3, 1];\\
\chi(x) &= 1/3 + (x-2/3)^2 && \text{if } x \in [2/3-\epsilon, 2/3+\epsilon];\\
\chi'(x) &< 0 && \text{if } x \in]1/3, 2/3[
\end{aligned}
$$

(the graph of χ is shown in Figure 3).

We put $\psi = \chi \circ f$. The following properties follow immediately from $(\mathfrak{R})$:

(13)

$$(\psi \circ \Psi_{1/3})(x,\tau) = 2/3 - \tau^2, \quad (\psi \circ \Psi_{2/3})(x,\tau) = 1/3 + \tau^2, \quad (\psi \circ \Psi_{1/2})(x,\tau) = 1 - \tau.$$

Let $B\colon \mathbb{R} \to [0,1]$ be a C^∞-function such that $B(t) = 0$ if $|t| \geqslant 5\mu/3$ and $B(t) = 1$ if $|t| \leqslant 4\mu/3$, and let $B_1 : \mathbb{R} \to \mathbb{R}^+$ be a C^∞-function such that $\operatorname{supp} B_1 \subset]\mu, 2\mu[$ and $\int_0^\infty B_1(t)dt = C$.

We choose C^∞-vector fields $z_0, z_1, z_{1/2}$ on $V_0, V_1, V_{1/2}$, respectively, in such a way that $\Phi(z_0,1)(\mathbb{L}_0(u_1)) \pitchfork \mathbb{X}$, $\Phi(z_1,1)(\mathbb{L}_1(-u_2)) \pitchfork \mathbb{Y}$, $\Phi(z_{1/2},1)(\mathbb{L}_{1/2}(u_1)) \pitchfork \mathbb{L}_{1/2}(-u_2)$. We assume that the z_i $(i = 0, 1/2, 1)$ are chosen so small that $\sup_\tau |B_1(\tau)| \cdot \|z_i\| < C/9$.

3.1.3. The Morse function F. Let $a_1, a_2 > 0$. We put

$$\begin{aligned}
F(y) &= \psi(y) && \text{if } y \in W \setminus (\mathrm{Tb}_{\frac{1}{3}}(2\mu) \cup \mathrm{Tb}_{\frac{2}{3}}(2\mu));\\
(F\circ\Psi_{1/3})(x,\tau) &= a_1 B(\tau) F_1(x) + 2/3 - \tau^2 && \text{if } (x,\tau) \in V_{1/3} \times [-2\mu, 2\mu];\\
(F\circ\Psi_{2/3})(x,\tau) &= a_2 B(\tau) F_2(x) + 1/3 + \tau^2 && \text{if } (x,\tau) \in V_{2/3} \times [-2\mu, 2\mu].
\end{aligned}$$

Relations (13) imply that the above formulas define a smooth function $F\colon W \to \mathbb{R}$, which is equal to f near ∂W. In order to find the critical points of F, we observe that, obviously, $S(F)$ is contained in $\mathrm{Tb}_{1/3}(2\mu) \cup \mathrm{Tb}_{2/3}(2\mu)$. For $(x,\tau) \in V_{1/3} \times [-2\mu, 2\mu]$ we have $d(F\circ\Psi_1)(x,\tau) = (a_1 B'(\tau) F_1(x) - 2\tau)d\tau + a_1 B(\tau) dF_1(x)$. If a_1 is sufficiently small, then this expression can vanish only for $\tau = 0$. Applying the same argument to $\mathrm{Tb}_{2/3}(2\mu)$, we conclude that $S(F) = S(F_1) \cup S(F_2)$ if the a_i are sufficiently small, which we assume from now on. In order to prove that F is a Morse function, we indicate standard charts for F. Let $p \in S(F_1)$; we write $(F_1 \circ \Phi_{1p}^{-1})(x) = F_1(p) + \sum_i \alpha_i x_i^2$. Consider the chart $\Phi_{1p} \times \mathrm{id}\colon U_{1p} \times\,]-\mu,\mu[\, \to B^n(0, r_{1p}) \times\,]-\mu,\mu[$ of the manifold $V_{1/3} \times\,]-\mu,\mu[$ around the point $(p,0)$. We have

$$F \circ \Psi_{1/3} \circ (\Phi_{1p} \times \mathrm{id})^{-1}(x,\tau) = F_1(p) + 2/3 + a_1 \sum_i \alpha_i x_i^2 - \tau^2.$$

Therefore, the chart

$$\big((\Phi_{1p} \times \mathrm{id}) \mid (\Phi_{1p} \times \mathrm{id})^{-1}(B^{n+1}(0,\mu)\big) \circ (\Psi_{1/3} \mid Tb_{1/3}(\mu))^{-1}$$

is a standard chart of radius μ for F at $p \in S(F_1)$. Together with the similar ones for $q \in S(F_2)$, these charts give an F-chart-system of radius μ; this system will be denoted by $\mathcal{U}$. Observe that $\mathrm{ind}_F\, p = \mathrm{ind}_{F_1} p + 1$ for $p \in S(F_1)$ and $\mathrm{ind}_F\, q = \mathrm{ind}_{F_2} q$ for $q \in S(F_2)$. Also, we note that if a_1 and a_2 are sufficiently small, then $F(\mathrm{Tb}_{1/3}(2\mu)) \subset [2/3-\epsilon, 2/3+\epsilon]$ and $F(\mathrm{Tb}_{2/3}(2\mu)) \subset [1/3-\epsilon, 1/3+\epsilon]$, and in this case $F^{-1}(1/2) = V_{1/2} \cup V_\theta \cup V_{\theta'}$, where $\epsilon < \theta < 1/3-\epsilon$, $2/3+\epsilon < \theta' < 1-\epsilon$.

3.1.4. The F-gradient u. We define a vector field u on W as follows:

$$\begin{aligned}
u(y) = v(y) \quad &\text{if } y \in \big(W_0 \setminus (\mathrm{Tb}_{1/3}(2\mu) \cup \mathrm{Tb}_0(2\mu))\big) \cup\\
&\cup \big(W_1 \setminus (\mathrm{Tb}_{2/3}(2\mu) \cup \mathrm{Tb}_{1/2}(2\mu))\big);\\
u(y) = -v(y) \ &\text{if } y \in W_{1/2} \setminus \big(\mathrm{Tb}_{1/3}(2\mu) \cup \mathrm{Tb}_{2/3}(2\mu) \cup \mathrm{Tb}_{1/2}(2\mu)\big);\\
\big((\Psi_0^{-1})_*(u)\big)(x,\tau) &= (-B_1(\tau) z_0(x),\ C);\\
\big((\Psi_1^{-1})_*(u)\big)(x,\tau) &= (B_1(-\tau) z_1(x), C);\\
\big((\Psi_{1/3}^{-1})_*(u)\big)(x,\tau) &= \big(B(\tau) u_1(x), -B(\tau)\tau - (1 - B(\tau)) \cdot C \cdot \mathrm{sgn}\,\tau\big);\\
\big((\Psi_{1/2}^{-1})_*(u)\big)(x,\tau) &= (-B_1(-\tau) z_{1/2}(x), -C);\\
\big((\Psi_{2/3}^{-1})_*(u)\big)(x,\tau) &= \big(B(\tau) u_2(x), B(\tau)\tau + (1 - B(\tau)) \cdot C \cdot \mathrm{sgn}\,\tau\big).
\end{aligned}$$

A simple computation using only the definitions of F and u shows that the above formulas consistently define a C^∞-vector field on W, and that $(F, u, \mathcal{U})$ is an $\mathcal{M}$-flow on W of radius μ if z_0, z_1, $z_{1/2}$ are sufficiently small (which we assume in what follows). It is equally easy to check that $\operatorname{supp}(u-v) \subset \big(\bigcup_{i=0,1,3/2,2,3} \mathrm{Tb}_i(2\mu)\big)$, and that u is an f-gradient in $W_0 \setminus V_{1/3}$ and $W_{2/3} \setminus V_{2/3}$; moreover, $(-u)$ is an f-gradient in $W_{1/3} \setminus (V_{1/3} \cup V_{2/3})$.

We claim that $\mathcal{V} = (F, u, \mathcal{U})$ satisfies all conclusions of Theorem 3.1. Statements (1) and (2) follow immediately from the above construction. To prove (3), we note that $V_{1/3}$ is a closed submanifold of W° and u is tangent to $V_{1/3}$; therefore, $V_{1/3}$ is $(\pm u)$-invariant (the same is true for $V_{2/3}$). We prove that $W_{1/2}$ is u-invariant. Let $x \in W^\circ_{1/2}$. The trajectory $\gamma(x,\cdot;u)$ cannot intersect $V_{1/3}$ or $V_{2/3}$ because these sets are $(\pm u)$-invariant. Therefore, $\gamma(x,\cdot;u)$ must stay in $W^\circ_{1/2}$ forever.

3.1.5. Estimating the quickness of $\mathcal{V}$. In order to estimate $\|u\|$, we note that the inequality $|u(x)| \leqslant 3/2(C + C_1 + C_2)$ must be checked only for $x \in \bigcup_i \mathrm{Tb}_i(2\mu)$; this is the matter of a simple computation, which we leave to the reader. In order to estimate the time spent by an u-trajectory off $\mathcal{U}(\mu)$, first we observe that for a trajectory starting at a point of $V_{1/3}$ (respectively, $V_{2/3}$), this time does not exceed β_1 (respectively, β_2), because actually this trajectory is a u_1- (respectively, u_2-) trajectory. Now, let γ be the u-trajectory passing through a point of $W^\circ_{1/3}$. Statement (3) shows that it stays in $W^\circ_{1/3}$ forever. Since u is a $(-f)$-gradient in $W^\circ_{1/3}$, the function $t \mapsto f(\gamma(t))$ is strictly monotone decreasing, and $\lim_{t\to-\infty} f(\gamma(t)) = 2/3$, $\lim_{t\to\infty} f(\gamma(t)) = 1/3$, whence $\lim_{t\to-\infty}\gamma(t) = q \in S(F_2)$ and $\lim_{t\to\infty}\gamma(t) = p \in S(F_1)$. We are going to estimate the time spent by γ off $\mathcal{U}(\mu)$ between various level surfaces of f.

1) Let $f(\gamma(t)) \in [2/3-\mu/2, 2/3]$. By (12) and (13), this is equivalent to $\gamma(t) \in \Psi_{2/3}(V_{2/3} \times [-\mu/2, 0[)$. The curve $\Psi^{-1}_{2/3}(\gamma(t))$ is the product of a u_2-trajectory and the curve $\tau \mapsto \alpha e^\tau$ with $\alpha < 0$. Since $\Psi^{-1}_{2/3}(\mathcal{U}(\mu))$ contains $U_p(\mu/2) \times]-\mu/2, \mu/2[$ for every $p \in S(F_2)$, the time spent by $\gamma(t)$ in $\Psi_{2/3}(V_{2/3} \times [-\mu/2, 0[)$ but off $\Psi^{-1}_{2/3}(\mathcal{U}(\mu))$ does not exceed β_2.

2) Let $f(\gamma(t)) \in [2/3 - 2\mu, 2/3 - \mu/2]$, or, in other words, $\gamma(t) \in \Psi_{2/3}(V_{2/3} \times [-2\mu, -\mu/2])$. The vector field $(\Psi^{-1}_{2/3})_*(u)$ is equal to $(B(\tau)u_2(x), \varkappa(\tau))$, where $\varkappa(\tau) \leqslant \tau$. Therefore, the total time that γ can spend in this domain does not exceed $\log\big(2\mu/(\mu/2)\big) < 2$.

3) Let $f(\gamma(t)) \in [1/2, 2/3 - 2\mu]$. In the domain $f^{-1}([1/2, 2/3-2\mu])$ we have $u = v$. Therefore, the time in question is at most $2T/C$.

4) Let $f(\gamma(t)) \in [1/2 - 2\mu, 1/2]$. Here $\gamma(t) \in \Psi_{1/2}(V_{1/2} \times [-2\mu, 0])$. Since the second coordinate of $(\Psi^{-1}_{1/2})_*(u)$ is equal to $(-C)$, the total time spent here does not exceed $2\mu/C \leqslant 1$.

In a similar way we can show that the time spent by γ in $f^{-1}(]1/3, 1/2 - 2\mu]) \setminus \mathcal{U}(\mu)$ is at most $2T/C + 2 + \beta_1$. Thus, the time that γ can spend in $W^\circ_{1/2} \setminus \mathcal{U}(\mu)$ does not exceed $\beta_1 + \beta_2 + 4T/C + 5$.

Similar analysis of the behavior of the u-trajectories in W_0 and W_1 shows that the same estimate remains true in these cases.

3.1.6. Estimating the number $N(\gamma,\mu)$. Let $\gamma(\cdot)$ be a u-trajectory in $W_{1/2}$. If γ is in $V_{1/3}$ (respectively, in $V_{2/3}$), then, obviously, $N(\gamma,\mu) \leqslant N(\mathcal{V}_1,\mu)$ (respectively, $N(\gamma,\mu) \leqslant N(\mathcal{V}_2,\mu)$). Assume that $\operatorname{Im}\gamma \subset W^\circ_{1/2}$. Let $\alpha \in \mathbb{R}$ (respectively, $\beta \in \mathbb{R}$) be the (only) number such that $f(\gamma(\alpha)) = 2/3 - \mu$ (respectively, $f(\gamma(\beta)) =$

$1/3+\mu)$. We have $\bigcup_{p\in S(F_i)} U_p(\mu) \subset \mathrm{Tb}_i(\mu)$, $i=1,2$. Therefore, the relation $\gamma(t)\in \mathcal{U}(\mu)$ may be true only if $t\geqslant \beta$ or $t\leqslant \alpha$. For $t\leqslant\alpha$ (respectively, $t\geqslant\beta$) the curve $\Psi^{-1}_{2/3}(\gamma(t))$ (respectively, $\Psi^{-1}_{1/3}(\gamma(t))$) is an integral curve of the vector field $(u_2(x),\tau)$ (respectively, $(u_1(x),-\tau)$). Since an integral curve of u_2 (respectively, u_1) can intersect at most $N(\mathcal{V}_2,\mu)$ (respectively, $N(\mathcal{V}_1,\mu)$) standard coordinate neighborhoods of radius μ, the first part of (5) follows. The case of curves in W_0 and W_1 can be treated similarly.

3.1.7. Transversality properties. The next lemma implies that $\mathcal{V}$ satisfies the almost transversality condition and that statement (6) in Theorem 3.1 is true.

Lemma 3.3.

(1) *The families* $\{\bigcup_{p\in S_i(F_1)} D_p(-u)\}_{0\leqslant i\leqslant n}$ *and* $\{\bigcup_{q\in S_i(F_2)} D_q(u)\}_{0\leqslant i\leqslant n}$ *coincide with* $\mathbb{D}(u_1)$ *and* $\mathbb{D}(u_2)$, *respectively.*

(2) *For every* $\lambda\in[0,1]$, *the family* $\{(\bigcup_{q\in S_i(F_2)} D_q(-u))\cap V_\lambda\}_{0\leqslant i\leqslant n}$ *is an s-submanifold of* V_λ *coinciding with*

(a) $\Phi(z_1,1)(\mathbb{L}_0(-u_2))$ *if* $\lambda=1$; (b) $\mathbb{L}_\lambda(-u_2)$ *if* $\lambda\in[1/2,1-\epsilon]$;
(c) $\varnothing$ *if* $\lambda\leqslant 1/3$.

(3) *For every* $\lambda\in[0,1]$, *the family* $\{(\bigcup_{p\in S_i(F_1)} D_p(u))\cap V_\lambda\}_{0\leqslant i\leqslant n}$ *is an s-submanifold of* V_λ *coinciding with*

(a) $\Phi(z_0,1)(\mathbb{L}_0(u_1))$ *if* $\lambda=0$; (b) $\mathbb{L}_\lambda(u_1)$ *if* $\lambda\in[\epsilon,1/2-2\mu]$;
(c) $\Phi(z_{1/2},1)(\mathbb{L}_{1/2}(u_1))$ *if* $\lambda=1/2$; (d) $\varnothing$ *if* $\lambda\geqslant 2/3$.

(4) u *is an* F*-gradient satisfying the almost transversality condition, and we have* $\mathbb{D}(u)\cap V_0 \nmid \mathbb{X}$, $\mathbb{D}(-u)\cap V_1\nmid \mathbb{Y}$.

Proof. (1) For every $p\in S_i(F_1)$, the positive disk of F at p lies in $V_{1/3}$, which implies the first assertion immediately; the second one is proved similarly.

(2) As is easily seen, it suffices to prove the assertion for $\lambda\in[-2\mu+2/3, 2\mu+2/3]$, and in this case it follows from the analysis of the behavior of the u-trajectories in $\mathrm{Tb}_{2/3}(2\mu)$ carried out in Subsection 3.1.5 above. Statement (3) is proved similarly.

(4) In order to prove that u satisfies the almost transversality condition, we let $p,q\in S(F)$, $\operatorname{ind} p\leqslant \operatorname{ind} q$. Let γ be a $(-u)$-trajectory joining p with q. The case where $p,q\in S(F_1)$ or $p,q\in S(F_2)$ follows easily from statement (1). Let $p\in S(F_1)$, $q\in S(F_2)$, and let z be the (only) point of intersection of γ with $V_{1/2}$. We put $k=\operatorname{ind}_F p$, $r=\operatorname{ind}_F q$. Then $z\in\big(\Phi(z_{1/2},1)(\mathbb{L}_{1/2}(u_1))\big)_{k-1}$ and $z\in\mathbb{L}(-u_2)_{n-r-1}$, which is impossible by the choice of $z_{1/2}$. The statement at the end of (4) has already been proved. $\square$

3.2. The proof of the implication $1.12(n)\Rightarrow 1.13(n+1)$. First, the existence of $\mathcal{M}$-flows with quickness satisfying the required estimates will be established for the functions without critical points on cobordisms, then for the functions with one critical point. The proof will be finished by the usual induction procedure. In the remaining part of §3, we fix a natural number n and assume that $1.12(n)$ is true. We shall use only the following obvious consequence of $1.12(n)$.

Lemma 3.4. *Let M be a closed Riemannian manifold of dimension n. Let $D > 0$, $\beta > 0$, $A > 1$. For every sufficiently small $\mu > 0$ there is an $\mathcal{AM}$-flow $\mathcal{W}$ such that (1) $\mathcal{W}$ is of radius μ and is (D, β)-quick; (2) $N(\mathcal{W}) \leqslant n2^n$; (3) $\mathcal{G}(\mathcal{W}) \leqslant A$.* $\square$

3.2.1. Functions without critical points.

Lemma 3.5. *Let W be an $(n+1)$-dimensional Riemannian cobordism. Suppose that $g \colon W \to [a, b]$ is a Morse function without critical points. Let $C > 0$, and let w be a g-gradient such that $\|w\| \leqslant C$. Let $\mathbb{X}$ be an s-submanifold of V_0, $\mathbb{Y}$ an s-submanifold of V_1. Let $\mu_0 > 0$. Then there is an $\mathcal{AM}$-flow $\mathcal{V} = (F, u, \mathcal{U})$ on W with the following properties*:

(1) *in a neighborhood of ∂W we have $u = w$ and $F = g$;*

(2) *$D(\mathcal{V}) \leqslant \mu_0$, $\mathcal{G}(\mathcal{V}) \leqslant 2$;*

(3) *$\mathcal{V}$ is $(2C, 16 + n2^{n+5})$-quick, and $N(\mathcal{V}) \leqslant n2^{n+1}$;*

(4) *$\mathbb{D}_a(u) \pitchfork \mathbb{X}$, $\mathbb{D}_b(-u) \pitchfork \mathbb{Y}$;*

(5) *there are $\alpha, \beta \in \mathbb{R}$ with $a < \alpha < \beta < b$ such that $g^{-1}(\alpha)$, $g^{-1}(\beta)$ and $g^{-1}([\alpha, \beta])$ are $(\pm u)$-invariant, $g^{-1}([a, \alpha[)$ is u-invariant, $g^{-1}(]\beta, b])$ is $(-u)$-invariant, and for any u-trajectory γ in $g^{-1}([a, \alpha[)$ or in $g^{-1}(]\beta, b])$ we have $N(\gamma) \leqslant n2^n$.*

Proof. Let $N > 0$ be a natural number. For $0 \leqslant s \leqslant N$ we put $a_s = a + \frac{b-a}{N}s$, $b_i = \frac{2a_i + a_{i+1}}{3}$, $c_i = \frac{a_i + 2a_{i+1}}{3}$. The cobordism $g^{-1}([a_s, a_{s+1}])$ will be denoted by W_s. We choose N so large that for each s the time spent by a grd(g)-trajectory in W_s be not greater than $C/4$. By induction on i, for every $0 \leqslant i \leqslant N-1$ we define a sequence of $\mathcal{M}$-flows $\mathcal{V}_i = (F_i, u_i, \mathcal{U}_i)$ of radius $\varkappa_i \leqslant \mu_0$ on W_i.

Assume that $\mathcal{V}_{i-1} = (F_{i-1}, u_{i-1}, \mathcal{U}_{i-1})$ has already been constructed. By Theorem 3.1, there is a number $\nu_0 > 0$ corresponding to W_i. Using Lemma 3.4, we find two $\mathcal{M}$-flows $\mathcal{W}_1$ and $\mathcal{W}_2$ on $g^{-1}(b_i)$ and on $g^{-1}(c_i)$, respectively, such that $\mathcal{W}_s$ is of radius $\varkappa < \nu_0$ and is (D, β)-quick, and $N(\mathcal{W}_s) \leqslant n2^n$, $\mathcal{G}(\mathcal{W}_s) \leqslant A$ $(s = 1, 2)$. The values of $D, \bar{\beta}, A$ will be chosen later.

By Corollary 1.11, the flow $\mathcal{W}_i$ is $(D, \beta + n2^{n+3}, \varkappa/2)$-quick, and Theorem 3.1 applies to produce a $(\frac{3}{2}(C+2D), 2\beta + 6 + n2^{n+4}, \varkappa)$-quick $\mathcal{AM}$-flow $\mathcal{V}_i = (F_i, u_i, \mathcal{U}_i)$ on W_i of radius $\varkappa$ and satisfying $\mathcal{G}(\mathcal{V}_i) < \frac{3}{2}A$.

Moreover, this flow can be chosen in such a way that $\mathbb{D}(u) \cap g^{-1}(a_i)$ be almost transversal to $\mathbb{D}(-u_{i-1}) \cap g^{-1}(a_i)$. For $i = N-1$ we also demand that $\mathbb{D}(-u) \cap g^{-1}(b) \pitchfork \mathbb{Y}$, and for $i = 0$ we demand that $\mathbb{D}(u) \cap g^{-1}(a) \pitchfork \mathbb{X}$.

Thus, for every $0 \leqslant i \leqslant N-1$ we have an $\mathcal{AM}$-flow $\mathcal{V}_i$ on W_i. We glue them together to obtain an $\mathcal{AM}$-flow $\mathcal{V} = (F, u, \mathcal{U})$ on W. It is easy to check that $\mathcal{V}$ satisfies the almost transversality condition and that $N(\mathcal{V}) \leqslant n2^{n+1}$, $T(\mathcal{V}) \leqslant 2(2\beta + 6 + n2^{n+4})$ (similar statements were proved in Subsections 3.1.6, 3.1.7). Statement (5) in the conclusion of the lemma will be true if we put $\alpha = b_1$, $\beta = c_{N-1}$.

To complete the proof of the lemma, it remains to set $\beta = 1$, $A = \frac{4}{3}$, $D = C/3$. $\square$

3.2.2. Functions with one critical point.

Lemma 3.6. *Let W be an $(n+1)$-dimensional Riemannian cobordism. Let $g \colon W \to [a, b]$ be a Morse function with one critical point p and a strongly standard chart $\Phi_p : U_p \to B^{n+1}(0, r_p)$ around p. Let w be a g-gradient with respect to*

$\{\Phi_p\}$. *If* $\delta > 0$ *is sufficiently small, then the time that a* w*-trajectory can spend in* $g^{-1}([g(p) - \delta^2, g(p) + \delta^2]) \setminus U_p(\delta/2)$ *does not exceed* 6.

Proof. Let $S_\delta = g^{-1}([g(p)-\delta^2, g(p)+\delta^2])$, and let γ be a w-trajectory. The function $dg(w)$ is lower bounded on $S_\delta \setminus U_p$; therefore, for $\delta > 0$ sufficiently small the time spent by γ in $S_\delta \setminus U_p$ is at most 1. The time spent by γ in $U_p \cap \big(S_\delta \setminus U_p(\delta)\big)$ is at most 2 (see Lemma 1.10), and the time spent by γ in $U_p(\delta) \setminus U_p(\delta/2)$ cannot exceed $\ln 8 \leqslant 3$ (see Lemma 1.9). $\square$

Lemma 3.7. *Let* W *be an* $(n+1)$*-dimensional Riemannian cobordism. Let* $(g, w, \mathcal{U})$ *be an* $\mathcal{M}$*-flow on* W*, where* $g\colon W \to [a, b]$ *is a Morse function with one critical point* p*,* $g(p) = c$*, and the* g*-chart around* p *is strongly standard. Assume that* $\|w\| \leqslant C$*. Let* $\mu_0 > 0$*. Let* $\mathbb{X}$ *be an* s*-submanifold of* V_0*, and* $\mathbb{Y}$ *an* s*-submanifold of* V_1*. Then there is an* $\mathcal{AM}$*-flow* $\mathcal{V} = (F, u, \mathcal{U})$ *on* W *with the following properties.*

(1) *In a neighborhood of* ∂W *we have* $u = w$ *and* $F = g$.

(2) $D(\mathcal{V}) \leqslant \mu_0$; $\mathcal{G}(\mathcal{V}) \leqslant 2$.

(3) $\mathcal{V}$ *is* $(2C, 40 + n2^{n+6})$*-quick, and* $N(\mathcal{V}) \leqslant 1 + n2^{n+1}$.

(4) $\mathbb{D}_a(u) \nmid \mathbb{X}$, $\mathbb{D}_b(-u) \nmid \mathbb{Y}$.

(5) *There are* a', $b' \in \mathbb{R}$ *with* $a < a' < c < b' < b$ *such that* $g^{-1}(a')$, $g^{-1}(b')$, $g^{-1}([a', b'])$ *are* $(\pm u)$*-invariant,* $g^{-1}([a, a'[)$ *is* u*-invariant,* $g^{-1}(]b', b])$ *is* $(-u)$*-invariant, and for any* u*-trajectory* γ *in* $g^{-1}([a, a'[)$ *or in* $g^{-1}(]b', b])$ *we have* $N(\gamma) \leqslant n2^n$.

Proof. Let $\delta \in\ ]0, \mu_0[$ be smaller than the radius of the standard chart Φ_p of g around p; also, we require that the conclusions of Lemma 3.6 be true and that $\mathcal{G}(U_p(\delta), \Phi_p) \leqslant 2$. Denoting $g^{-1}(c-\delta^2)$ by V_- , $g^{-1}([a, c-\delta^2])$ by W_- , $g^{-1}(c+\delta^2)$ by V_+ , $g^{-1}([c+\delta^2, b])$ by W_+ , and $g^{-1}([c-\delta^2, c+\delta^2])$ by W', we apply Lemma 3.5 to the function $g|W_+$, the $(g|W_+)$-gradient w, the number μ_0, the s-submanifold $\mathbb{Y}$ of V_1, and the s-submanifold $D(p, -w) \cap V_+$ of V_+ (the latter consists of one compact manifold). Let $\mathcal{V}_+ = (F_+, u_+, \mathcal{U}_+)$ be the resulting flow. The corresponding values α, β that satisfy (5) of Lemma 3.5 with respect to $\mathcal{V}_+$ will be denoted by b'', b'; we have $b'' < b'$. Let $\Phi'_p\colon U'_p \to B^{n+1}(0, \delta/2)$ be the restriction of the g-chart around p, and let $\mathcal{V}'$ denote the $\mathcal{M}$-flow $(g|W', w|W', \Phi'_p)$. We glue $\mathcal{V}_+$ and $\mathcal{V}'$ together, obtaining an $\mathcal{M}$-flow $\mathcal{V}_1 = (F_1, u_1, \mathcal{U}_1)$. This flow satisfies the almost transversality condition because $\mathbb{D}(u_+) \cap V_+ \nmid D(p, -w) \cap V_+$. Now we apply Lemma 3.5 to the function $g|W_-$, the $(g|W_-)$-gradient w, the number μ_0, the s-submanifold $\mathbb{D}(u_1) \cap V_-$ of V_-, and the s-submanifold $\mathbb{X}$ of V_0. This results in an $\mathcal{M}$-flow $\mathcal{V}_- = (F_-, u_-, \mathcal{U}_-)$. The corresponding values α, β satisfying (5) of Lemma 3.5 with respect to $\mathcal{V}_-$ will be denoted by a', a'', so that $a' < a''$. Glueing $\mathcal{V}_-$ and $\mathcal{V}_1$, together, we obtain an $\mathcal{M}$-flow $\mathcal{V} = (F, u, \mathcal{U})$ on W, which satisfies the almost transversality condition condition $\mathbb{D}(u_1) \cap V_- \nmid \mathbb{D}(-u_-) \cap V_-$.

We claim that the $\mathcal{M}$-flow $\mathcal{V}$ and the numbers a', b' satisfy the conclusions of our lemma. Indeed, the required properties follow immediately from the properties of $\mathcal{V}_+$ and $\mathcal{V}_-$, except for the estimates of $T(\gamma)$ and $N(\gamma)$ for the u-trajectories γ belonging to $W' = g^{-1}([a'', b''])$. Let γ be such a trajectory. Assume that γ passes through a point of $g^{-1}([a'', c-\delta^2])$ (the other possibilities can be analyzed similarly). It is easily seen that in this case $\lim_{t\to-\infty} \gamma(t) = q \in S(F) \cap g^{-1}(a'')$, γ intersects $g^{-1}(c-\delta^2)$, and either $\lim_{t\to\infty} \gamma(t) = p$, or $\lim_{t\to\infty} \gamma(t) = q' \in S(F) \cap g^{-1}(b'')$. In the first case we have $N(\gamma) \leqslant n2^n$ and $T(\gamma) \leqslant 6 + T(\mathcal{V}_-) \leqslant 22 + n2^{n+5}$. In the second case, $N(\gamma) \leqslant 1 + n2^{n+1}$ and $T(\gamma) \leqslant 6 + T(\mathcal{V}_-) + T(\mathcal{V}_+) \leqslant 40 + n2^{n+6}$. $\square$

3.2.3. End of the proof of the implication $1.12(n) \Rightarrow 1.13(n{+}1)$**.** We are given $B > 0$ and $\mu_0 > 0$. Let $\mathcal{V}_0 = (g, w, \mathcal{U})$ be an $\mathcal{M}$-flow on M such that $\|w\| \leqslant B/2$, all the charts of $\mathcal{U}$ are strongly standard, and if $p, q \in S(g)$ and $p \neq q$, then $g(p) \neq g(q)$ (the existence of such a flow is an easy exercise in Morse theory). Let $p_1, \dots, p_r$ be the critical points of g, $g(p_i) < g(p_{i+1})$. We put $a_i = g(p_i)$, $b_i = \frac{a_i + a_{i+1}}{2}$ $(1 \leqslant i \leqslant r-1)$, $b_0 = a_1 - 1$, $b_r = a_r + 1$, $W_i = g^{-1}([b_i, b_{i+1}])$ and $V_i = g^{-1}(b_i)$. Using induction on i, we construct a sequence of $\mathcal{M}$-flows $\mathcal{V}_i = (f_i, v_i, \mathcal{U}_i)$ on W_i.

Assume that $\mathcal{V}_{i-1} = (f_{i-1}, v_{i-1}, \mathcal{U}_{i-1})$ is already constructed. Applying Lemma 3.7 to the restriction of $\mathcal{W}$ to W_i, we obtain an $\mathcal{AM}$ flow $\mathcal{V}_i = (f_i, v_i, \mathcal{U}_i)$. Also, we require that $\mathbb{D}(v_i) \cap V_i \nmid \mathbb{D}(-v_{i-1}) \cap V_i$, and that $D(\mathcal{V}_i) \leq \mu_0'$, where μ_0' is a number strictly less than μ_0.

Glueing all the $\mathcal{V}_i$ together, we obtain an $\mathcal{M}$-flow $\mathcal{V} = (f, v, \mathcal{U})$ on M. An easy induction argument (using property (5) in Lemma 3.7 for the flows $\mathcal{V}_i$) shows that $\mathcal{V}$ satisfies the almost transversality condition. The estimates required for $N(\gamma)$ and $T(\gamma)$ can be obtained in the same way as the estimates in Lemma 3.7. $\square$

§4. Ranging systems and the proof of the main theorem

This section is central in the paper. Here we define the set $\mathcal{G}t_0(f)$ and prove the properties of it stated in the main theorem. We shall use the definitions and notation introduced in the Introduction.

First, we consider a cobordism W not necessarily satisfying $V_0 \approx V_1$, and a Morse function f on W. In Subsection 4.1 we introduce a condition (labeled (RP)) on f-gradients (this is an analog of the cellular approximation condition for the "maps" $\overset{\leadsto}{v}$ and $(-v)^{\leadsto}$) and prove that the set of f-gradents satisfying (RP) is C^0-dense. It is plausible that this set is also C^0-open, but presently I cannot prove this.

Instead of proving the openness of this set, we include (RP) into a wider condition. Namely, let p, q be critical points of f with $\operatorname{ind} p = \operatorname{ind} q + 1$. In Subsection 4.3 we introduce condition $(RS; p, q)$; the set of f-gradients satisfying $(RS; p, q)$ is C^0-open, and if an f-gradient v satisfies $(RS; p, q)$, then the incidence coefficient $nm(p, q; v)$ in the Morse complex admits interpretation in homological terms (see Proposition 4.12; we add the letter "m" in the notation to avoid confusion with the Novikov incidence coefficient $n(p, q; v)$). In Subsection 4.4 we show that (RP) implies $(RS; p, q)$ for every pair p, q.

In Subsection 4.5 we adapt this techniques to the case of the cobordisms W for which the two parts V_0 and V_1 of the boundary are diffeomorphic. Such cobordisms will be called *cyclic*. A Morse map $f \colon M \to S^1$ generates a cyclic cobordism, and if the corresponding "cyclic" analog of the condition $(RS; p, q)$ is fulfilled for an f-gradient v, then the Novikov incidence coefficients $n(p, q; v)$ can be computed by formula (2) from the Introduction (this is proved in Subsection 4.5).

4.1. The set of f-gradients satisfying condition (RP) is C_0-dense. In this subsection, W is a Riemannian cobordism, $f \colon W \to [a, b]$ a Morse function, v an f-gradient satisfying the almost transversality condition, $n = \dim W$.

Definition 4.1. Let $\mathcal{V}_0 = (\phi_0, u_0, \mathcal{U}_0)$, $\mathcal{V}_1 = (\phi_1, u_1, \mathcal{U}_1)$ be $\mathcal{AM}$-flows on V_0 and V_1. We say that $(\mathcal{V}_0, \mathcal{V}_1)$ is a *ranging pair* for (f, v) if $\mathbb{D}_a(v) \pitchfork \mathbb{D}(-u_0)$, $\mathbb{D}_b(-v) \pitchfork \mathbb{D}(u_1)$, and there is a number $\delta > 0$ such that for any $0 \leqslant s \leqslant n-1$ the following conditions are fulfilled:

(RP1) δ lies in intervals of definition of $\mathbf{T}(\mathbf{D}(u_1), v)$ and of $\mathbf{T}(\mathbf{D}(-u_0), -v)$;

(RP2) the gradients v, u_0, and u_1 are δ-separated;

(RP3) $\overline{\mathbf{T}(\mathbf{D}(u_1), v)_{(\leqslant s+1)}(\delta)} \cap V_0 \subset \mathbf{D}(u_0)_{(\leqslant s)}(\delta)$;

(RP4) $\overline{\mathbf{T}(\mathbf{D}(-u_0), -v)_{(\leqslant s+1)}(\delta)} \cap V_1 \subset \mathbf{D}(-u_1)_{(\leqslant s)}(\delta)$.

We say that v satisfies condition (RP) if (f, v) has a ranging pair.

Proceeding to the construction of ranging pairs, we let v be an f-gradient satisfying the almost transversality condition and choose $\alpha, \beta \in]a, b[$ such that $\alpha < \beta$ and the intervals $[a, \alpha]$, $[\beta, b]$ are regular. Let $\mu > 0$ be so small that the map $(x, \tau) \mapsto \gamma(x, \tau; v)$ is defined on $V_0 \times [0, 3\mu]$ and on $V_1 \times [-3\mu, 0]$; these maps will be denoted by $\Psi_0 \colon V_0 \times [0, 3\mu] \to W$ and $\Psi_1 \colon V_1 \times [-3\mu, 0] \to W$. Also, we assume that μ is so small that $\operatorname{Im}\Psi_0 \subset f^{-1}([a, \alpha[)$, $\operatorname{Im}\Psi_1 \subset f^{-1}(]\beta, b])$, and the Ψ_0- (respectively, Ψ_1-) induced Riemannian metric is 2-equivalent to the product metric. The Ψ_i-image of $V_i \times [\alpha, \beta]$ $(i = 0, 1)$ will be denoted by $U_i(\alpha, \beta)$.

Lemma 4.2. *Let $\mathbb{A}$, $\mathbb{B}$ be s-submanifolds of V_1 (respectively, V_0). There is an f-gradient v_1 satisfying the almost transversality condition that is C^∞-close to v and satisfies the following conditions*:

1. $\operatorname{supp}(v - v_1) \subset \big(U_0(2\mu, 3\mu) \cup U_1(-3\mu, -2\mu)\big)$;
2. $\mathbb{D}_b(-v_1) \pitchfork \mathbb{A}$;
3. $\overset{\rightsquigarrow}{(-v_1)}(\mathbb{A}) \pitchfork \mathbb{B}$, $(v_1)^{\rightsquigarrow}(\mathbb{B}) \pitchfork \mathbb{A}$.

Proof. First, we note that any f-gradient v_1 satisfying (1) satisfies the almost transversality condition. Also, we observe that the first part of condition (3) implies the second. Let $\xi \in \operatorname{Vectt}(V_0)$ be a small vector field such that $\Phi(\xi, \mu)(\mathbb{D}_b(-v_1)) \pitchfork \mathbb{A}$ (see Theorem 2.3). Then the vector field v_0 that equals v everywhere off $U_1(-3\mu, -2\mu)$ and equals $(\Psi_0)_*((\xi) \times \frac{d}{dt})$ in $U_1(-3\mu, -2\mu)$ satisfies (1) and (2). Therefore, the s-submanifold $\overset{\rightsquigarrow}{(-v_0)}(\mathbb{A})$ of V_0 is well defined. Applying the same procedure to v_0 (near V_0), we get an f-gradient v_1 satisfying (1), (2), and the first part of (3). $\square$

Theorem 4.3. *Let $\epsilon > 0$. There is an f-gradient w satisfying the almost transversality condition and a ranging pair $(\mathcal{V}_0, \mathcal{V}_1)$ for (f, w) such that $\|w - v\| \leqslant \epsilon$ and $\operatorname{supp}(w - v) \subset U_0(\mu, 3\mu) \cup U_1(-3\mu, -\mu)$.*

Proof. Let ρ_1 denote the maximum of $|\operatorname{grad} f(x)|$, where $x \in \operatorname{Im}\Psi_0 \cup \Psi_1$. Let ρ_0 denote the minimum of $df(v)(x)$, where $x \in \operatorname{Im}\Psi_0 \cup \Psi_1$. We fix $\epsilon' < \epsilon$ so small that $\epsilon' < \rho_0/\rho_1$.

Let $\mathcal{V}_1 = (\phi_1, u_1, \mathcal{U}_1)$ and $\mathcal{V}_0 = (\phi_0, u_0, \mathcal{U}_0)$ be $\mathcal{AM}$-flows on V_1 and V_0 satisfying the conclusions of Corollary 1.14 relative to $C = \epsilon'/4$, $\beta = \mu/3$.

Let $\delta_0 > 0$ be so small that u_1, u_0 are δ_0-separated; then δ_0 belongs to an interval of definition of $\mathbf{D}(-u_0)$, $\mathbf{D}(u_1)$. Let u be an f-gradient satisfying the almost transversality condition and the conclusions of the preceding lemma with $\mathbb{A} = \mathbb{D}(u_1)$, $\mathbb{B} = \mathbb{D}(-u_0)$, and such that $\|u - v\| < \epsilon'/2$. We denote by $\widetilde{W}$ the cobordism $f^{-1}([\alpha, \beta])$ and by $\widetilde{f}$, $\widetilde{u}$ the restrictions of f (respectively, u) to $\widetilde{W}$, and put $\widetilde{V}_0 = f^{-1}(\alpha)$, $\widetilde{V}_1 = f^{-1}(\beta)$. Let $\widetilde{\mathbf{A}}$ (respectively, $\widetilde{\mathbf{B}}$) be the image of $\mathbf{D}(u_1)$ (respectively, $\mathbf{D}(-u_0)$), under the diffeomorphism $(-u)^{\leadsto}_{[b,\beta]} \colon V_1 \to \widetilde{V}_1$ (respectively, $u^{\leadsto}_{[a,\alpha]} \colon V_0 \to \widetilde{V}_0$). We fix $]0, \delta_0[$ as an interval of defnition of $\widetilde{\mathbf{A}}$ (respectively, $\widetilde{\mathbf{B}}$). The cores of $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ will be denoted by $\widetilde{\mathbb{A}}$ and $\widetilde{\mathbb{B}}$. Note that $\widetilde{\mathbb{A}} \nmid (\widetilde{u})^{\leadsto}(\widetilde{\mathbb{B}})$, $\widetilde{\mathbb{B}} \nmid (-\widetilde{u})^{\leadsto}(\widetilde{\mathbb{A}})$. We choose numbers $\delta_1 > 0$ such that $\widetilde{u}$ is δ_1-separated and choose $\theta > 0$ such that $\theta < \delta_0$, $\theta < \delta_1$, θ belongs to intervals of definition of $\mathbf{T}(\widetilde{\mathbf{A}}, \widetilde{u})$ and of $\mathbf{T}(\widetilde{\mathbf{B}}, -\widetilde{u})$, and for every $0 \leqslant s \leqslant n-1$ we have

(14)
$$\mathbf{T}(\widetilde{\mathbf{A}}, \widetilde{u})_{(\leqslant s)}(\theta) \cap \widetilde{\mathbf{B}}_{(\leqslant n-s-2)}(\theta) = \varnothing; \qquad \mathbf{T}(\widetilde{\mathbf{B}}, -\widetilde{u})_{(\leqslant s)}(\theta) \cap \widetilde{\mathbf{A}}_{(\leqslant n-s-2)}(\theta) = \varnothing.$$

Let $\mathcal{W}_i = (\phi_i, u_i', \mathcal{U}_i')$ $(i = 0, 1)$ be an $\mathcal{M}$-flow on V_i subordinate to $\mathcal{V}_i$ and satisfying conditions (7) and (8) of Corollary 1.14 with some $\delta \in]0, \theta[$. Let $\varkappa \colon [0, 2\mu] \to [0, 1]$ be a C^∞-function such that $\operatorname{supp} \varkappa \subset]\mu, 2\mu[$ and $\varkappa(t) = 1$ for $t \in [\frac{4}{3}\mu, \frac{5}{3}\mu]$.

We define a vector field w on W by setting

$$w(x) = u(x) \qquad \text{if } x \notin U_0(\mu, 2\mu) \cup U_1(-2\mu, -\mu); \tag{15}$$

$$((\Psi_0^{-1})_* w)(y, t) = (\varkappa(t) u_0'(y), 1) \qquad \text{if } y \in V_0,\ \ t \in [\mu, 2\mu]; \tag{16}$$

$$((\Psi_1^{-1})_* w)(y, t) = (\varkappa(-t) u_1'(y), 1) \qquad \text{if } y \in V_1,\ \ t \in [-2\mu, -\mu]. \tag{17}$$

We claim that w and $(\mathcal{V}_0, \mathcal{V}_1)$ satisfy the conclusion of the theorem. Our choice of ϵ' implies that w is an f-gradient. By construction, w satisfies the almost transversality condition and $\|w - v\| < \epsilon$. The condition $\operatorname{supp}(w - v) \subset U_0(\mu, 3\mu) \cup U_1(-3\mu, -\mu)$ follows from statement (1) in Lemma 4.12 and from (15) above. In order to prove that $(\mathcal{V}_0, \mathcal{V}_1)$ is a ranging pair for (f, w), we recall that the gradients u_0, u_1, $\widetilde{u}$ are θ-separated, whence u_0, u_1, and w are δ-separated, and (RP2) follows. Relations (15)–(17) imply that the $(-w)$-trajectories starting at a point $x \in V_1$ behave as follows:

(A) $\gamma(x, \cdot; -w)$ reaches $\widetilde{V}_1$ and intersects it at $x_1 = (-u)^{\leadsto}_{[b,\beta]}(\gamma(x, \tau(x); -u_1'))$, where $\tau(x) \geqslant \mu/3$. Next, two cases may occur:

(B1) $\gamma(x_1, t; -\widetilde{u})$ converges to some point $p \in S(f)$ as $t \to \infty$. Then the same is true for $\gamma(x, \cdot; -w)$;

(B2) $\gamma(x_1, \cdot; -\widetilde{u})$ reaches $\widetilde{V}_0$ and intersects this set at a point x_2. Then the same is true for $\gamma(x, \cdot; -w)$ and, moreover,

(C) $\gamma(x, \cdot; -w)$ reaches V_0 and intersects it at $x_3 = \gamma((-u)^{\leadsto}_{[\alpha,a]}(x_2), \tau(x_2); -u_0')$, where $\tau(x_2) \geqslant \mu/3$.

(The w-trajectories starting at the points of V_0 behave similarly.) Thus $(-w)^{\leadsto}_{[b,\beta]}(\mathbb{D}(u_1)) = \widetilde{\mathbb{A}}$ and $w^{\leadsto}_{[a,\alpha]}(\mathbb{D}(-u_0)) = \widetilde{\mathbb{B}}$, whence we see that $\mathbb{D}_b(-w) \nmid \mathbb{D}(u_1)$, and $\mathbb{D}_a(w) \nmid \mathbb{D}(-u_0)$. Further, statement (A) above implies that

(18) for any $\lambda < \delta_0$ and any s with $0 \leqslant s \leqslant n-1$ we have $(-w)^{\leadsto}_{[b,\beta]}(\mathbf{D}(u_1)_{(s)}(\lambda)) \subset \widetilde{\mathbf{A}}_{(s)}(\lambda)$

Therefore, Lemma 2.14 applies to show that θ lies in an interval of definition of $\mathbf{T}(\mathbf{D}(u_1), w)$. Thus, the first part of (RP1) is checked. Similarly, θ lies in an interval of definition of $\mathbf{T}(\mathbf{D}(-u_0), -w)$. Moreover, (14) implies that for every $z \in \mathbf{T}(\mathbf{D}(u_1), u)_{(\leqslant s)}(\theta) \cap \widetilde{V}_0$ we have $(-u)^{\leadsto}_{[\alpha,a]}(z) \in V_0 \setminus B_\theta(\mathrm{ind}{\leqslant}n-s-2, -u_0)$. Therefore, relations (4) and (7) (see §1) imply that $\mathbf{T}(\mathbf{D}(u_1), w)_{(\leqslant s)}(\theta) \cap V_0 \subset B_\delta(\mathrm{ind}{\leqslant}s, u_0)$, which yields (RP3) because $\delta < \theta$. The proof of (RP4) is similar. $\square$

4.2. General remarks on intersection indices. Before passing to ranging systems, we recall some well-known facts about fundamental classes of manifolds.

Let M be a manifold without boundary, $\dim M = m$. Let $X \subset M$, and let N be an oriented submanifold of M such that $N \setminus \mathrm{Int}\, X$ is compact. Then the orientation class $\mu_{N \setminus \mathrm{Int}\, X} \in H_n(N, N \cap \mathrm{Int}\, X)$ is well defined, where $n = \dim N$ (see [7, Theorem A8]). The image of it in $H_n(M, X)$ will be denoted by $[N]_{M,X}$ (or simply by $[N]$ if this does not lead to confusion). The excision isomorphisms $H_*(X \setminus B, A \setminus B) \to H_*(X, A)$ will be denoted by Exc. The next lemma follows immediately from [7, Theorem A8].

Lemma 4.4. 1. *Let Y be a closed subset of* $\mathrm{Int}\, X$. *Then $[N \setminus Y]_{M\setminus Y, X\setminus Y}$ coincides with the image of $[N]_{M,X}$ under* $\mathrm{Exc}^{-1}\colon H_*(M, X) \xrightarrow{\approx} H_*(M \setminus Y, X \setminus Y)$.

2. *Let M' be a manifold, U an open subset of M', $X' \subset U$. Let $\phi\colon M \to U$ be a diffeomorphism such that $\phi(X) \subset X'$. We put $N' = \phi(N)$ and denote by $\Phi\colon (M, X) \to (M', X')$ the resulting map of pairs. Then $[N']_{M',X'} = \Phi_*([N]_{M,X})$.* $\square$

Let L be a compact cooriented submanifold of M without boundary. Then the canonical coorientation class $]L[\in H^{m-l}(M, M \setminus L)$ is well defined, where $l = \dim L$. Assuming that $X \cap L = \varnothing$, $N \pitchfork L$, $n + l = m$, we denote by j the embedding $(M, X) \hookrightarrow (M, M \setminus L)$. The following lemma is well known.

Lemma 4.5. *The set $N \cap L$ is finite, and the intersection index $N \sharp L$ is equal to $\langle j^*(]L[), [N]\rangle$.* $\square$

4.3. Ranging systems and the incidence coefficients in the Morse complex. In this subsection, $f\colon W \to [a, b]$ is a Morse function on a Riemannian cobordism W, v is an f-gradient, $f^{-1}(a) = V_0$, $f^{-1}(b) = V_1$.

Definition 4.6. Let $\Lambda = \{\lambda_0, \dots, \lambda_k\}$ be a finite set of regular values of f with the following properties: $\lambda_0 = a$, $\lambda_k = b$; for each $0 \leqslant i \leqslant k-1$ we have $\lambda_i < \lambda_{i+1}$; there is exactly one critical value of f in $[\lambda_i, \lambda_{i+1}]$. The values λ_i, λ_{i+1} will be called *adjacent*. The set of pairs $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ is called a *ranging system* for (f, v) if

(RS1) for any $\lambda \in \Lambda$, A_λ and B_λ are disjoint compacta in $f^{-1}(\lambda)$;

(RS2) if $\lambda, \mu \in \Lambda$ are adjacent, then for every $p \in S(f) \cap f^{-1}([\lambda, \mu])$ we have either

$$\text{i) } D(p, v) \cap f^{-1}(\lambda) \subset \mathrm{Int}\, A_\lambda, \quad \text{or} \quad \text{ii) } D(p, -v) \cap f^{-1}(\mu) \subset \mathrm{Int}\, B_\mu;$$

(RS3) if $\lambda, \mu \in \Lambda$ are adjacent, then $(-v)^{\leadsto}_{[\mu,\lambda]}(A_\mu) \subset \mathrm{Int}\, A_\lambda$ and $v^{\leadsto}_{[\lambda,\mu]}(B_\lambda) \subset \mathrm{Int}\, B_\mu$.

The next lemma is obvious.

Lemma 4.7. 1. *Let $S = \{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ be a ranging system for (f, v), and let $\mu, \nu \in \Lambda$. Then $S' = \{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda, \mu\leqslant\lambda\leqslant\nu}$ is a ranging system for $\big(f|f^{-1}([\mu,\nu]), v\big)$.*

2. *Let $c \in \,]a, b[$ be a regular value of f. Let $\{(A'_{\lambda'}, B'_{\lambda'})\}_{\lambda'\in\Lambda'}$, $\{(A''_{\lambda''}, B''_{\lambda''})\}_{\lambda''\in\Lambda''}$ be ranging systems for $(f|f^{-1}([a, c]), v)$ and for $(f|f^{-1}([c, b]), v)$, respectively. Assume that $A''_c \subset A'_c$, $B'_c \subset B''_c$, and $A'_c \cap B''_c = \varnothing$. Then the system $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ defined by relations* (C) *below is a ranging system for (f, v).*

$$\text{(C)} \qquad \begin{cases} \Lambda = \Lambda' \cup \Lambda''; \\ A_\lambda = A'_\lambda \quad \text{and} \quad B_\lambda = B'_\lambda \text{ if } \lambda \in \Lambda' \setminus \{c\}, \\ A_\lambda = A''_\lambda \quad \text{and} \quad B_\lambda = B''_\lambda \text{ if } \lambda \in \Lambda'' \setminus \{c\}; \\ A_c = A'_c, \qquad B_c = B''_c. \end{cases}$$

In statements 4.8–4.10, $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ denotes a ranging system for (f, v).

Lemma 4.8. *There is a number $\epsilon > 0$ such that for any f-gradient w with $\|v - w\| < \epsilon$ the ranging system $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ is also a ranging system for (f, w).*

Proposition 4.9. *If N is a submanifold of $V_1 \setminus B_b$, and $N \setminus \operatorname{Int} A_b$ is compact, then $N' = v^{\leadsto}_{[b,a]}(N)$ is a submanifold of $V_0 \setminus B_a$ such that $N' \setminus \operatorname{Int} A_a$ is compact.*

Proposition 4.10. *There is a homomorphism $H(v)\colon H_*(V_1 \setminus B_b, A_b) \longrightarrow H_*(V_0 \setminus B_a, A_a)$ such that*

(1) *if N is an oriented submanifold of V_1 satisfying the assumptions of Proposition 4.6, then $H(v)([N]) = [v^{\leadsto}_{[b,a]}(N)]$;*
(2) *there is a number $\epsilon > 0$ such that for every f-gradient w with $\|w - v\| < \epsilon$ we have $H(v) = H(w)$.*

An easy induction argument shows that it suffices to prove each of statements 4.8–4.10 in the case where $\operatorname{card}\Lambda = 1$. Let $S(f) = S1(f) \sqcup S2(f)$, where for every $p \in S1(f)$ (respectively, $p \in S2(f)$) condition i) (respectively, ii) (see (RS2)) is fulfilled. We choose Morse functions $\phi_1, \phi_2\colon W \to [a, b]$ adjusted to (f, v) such that there are regular values μ_1 of ϕ_1 and μ_2 of ϕ_2 with the following properties: (1) for every $p \in S1(f)$ we have $\phi_1(p) < \mu_1$ and $\phi_2(p) > \mu_2$; (2) for every $p \in S2(f)$ we have: $\phi_1(p) > \mu_1$ and $\phi_2(p) < \mu_2$.

Proof of Lemma 4.8. Proposition 1.15 implies that condition (RS2) is fulfilled for all f-gradients w sufficiently close to v. Passing to (RS3), we consider the cobordism $\phi_1^{-1}([\mu_1, b])$ and the ϕ_1-gradient v. Let $Z_+ = T(A_b, v) \cap \phi_1^{-1}(\mu_1)$. Since every $(-v)$-trajectory starting in A_b reaches $\phi_1^{-1}(\mu_1)$, the set Z_+ is compact. Consider the cobordism $\phi_1^{-1}([a, \mu_1])$ and the compact set $V_0 \setminus \operatorname{Int} A_a \subset V_0$. Every v-trajectory starting at a point in $V_0 \setminus \operatorname{Int} A_a$ reaches $\phi_1^{-1}(\mu_1)$, and the compact sets Z_+ and $Z_- = \phi_1^{-1}(\mu_1) \cap T(V_0 \setminus \operatorname{Int} A_a, -v)$ are disjoint. In $\phi_1^{-1}(\mu_1)$ we can choose disjoint open neighborhoods U_+ of Z_+ and U_- of Z_-. By Corollary 5.6, there exists $\epsilon'' > 0$ such that for every $w \in \operatorname{Vect}^1(W, \perp)$ with $\|w - v\| < \epsilon''$ we have

$$T(A_b, w) \cap \phi_1^{-1}(\mu_1) \subset U_+ \quad \text{and} \quad T(V_0 \setminus \operatorname{Int} A_a, -w) \cap \phi_1^{-1}(\mu_1) \subset U_-.$$

Therefore, for all f-gradients w sufficiently C^0-close to v we have $\overset{\leadsto}{w}(A_b) \subset \operatorname{Int} A_a$. The second part of (RS3) can be proved similarly. $\square$

For $\delta > 0$, we put $D1_\delta(v) = V_0 \cap \bigcup_{p\in S1(f)} D_\delta(p,v)$ and $D1_\delta(-v) = V_1 \cap \bigcup_{p\in S1(f)} D_\delta(p,-v)$. By abuse of notation, the intersection of $\bigcup_{p\in S1(f)} D(p,v)$ with V_0 will be denoted by $D1_0(v)$. We put $\Delta(\delta,-v) = D2_\delta(-v) \cup v^{\leadsto}(B_a)$ and $\nabla(\delta,v) = D1_\delta(v) \cup (-v)^{\leadsto}(A_b)$. Similar notation like $D2_\delta(-v)$, $\Delta(0,-v)$, $D2_0(v)$, etc. will be used without special explanations. If $\delta > 0$ is sufficiently small, then

(D1) $\qquad D_\delta(p) \in \phi_1^{-1}(]a,\mu_1[)$ and $D_\delta(p) \subset \phi_2^{-1}(]\mu_2,b[)$ for any $p \in S1(f)$;

(D2) $\qquad D_\delta(p) \in \phi_1^{-1}(]\mu_2,b[)$ and $D_\delta(p) \subset \phi_2^{-1}(]a,\mu_2[)$ for any $p \in S2(f)$.

Applying Lemmas 2.6 and 2.7 to the functions ϕ_1, ϕ_2 and to their restrictions, it is easy to prove that if $\delta > 0$ is sufficiently small, then

$$\nabla(\delta,v) \subset \operatorname{Int} A_a, \quad \Delta(\delta,-v) \subset \operatorname{Int} B_b, \quad \Delta(\delta,-v) \cap D1_\delta(-v) = \varnothing.$$

We fix some $\delta > 0$ satisfying (D1) and (D2).

Proof of Proposition 4.9. The set $N' \setminus \operatorname{Int} A_a$ is a closed subset of $(-v)^{\leadsto}\big(N \setminus (\operatorname{Int} A_b \cup B_\delta(-v))\big)$. Since the set $N \setminus \big(\operatorname{Int} A_b \cup B_\delta(-v)\big)$ is a compact subset of the domain of $(-v)^{\leadsto}$, we obtain the required statement. $\square$

We pass to the proof of proposition 4.10.

The homomorphism $H(v;\mu',\mu;U)\colon H_*(V_1 \setminus B_b, A_b) \to H_*(V_0 \setminus B_a, A_a)$**.** Let $0 \leqslant \mu' < \mu \leqslant \delta$. Let U be any subset of V_1 such that

(I) $$\Delta(0,-v) \subset U \subset B_b \quad \text{and} \quad U \cap D1_\delta(-v) = \varnothing$$

(for example, $U = \Delta(\delta,-v)$). We denote by $\mathcal{H}(v;\mu',\mu;U)$ the following sequence of homomorphisms:

(19)
$$H_*(V_1 \setminus B_b, A_b) \xrightarrow{I_*} H_*\big(V_1 \setminus U, A_b \cup D1_\mu(-v)\big)$$
(20)
$$\xrightarrow{\mathrm{Exc}^{-1}} H_*\big(V_1 \setminus (U \cup D1_{\mu'}(-v)), (A_b \cup D1_\mu(-v)) \setminus D1_{\mu'}(-v)\big)$$
$$\xrightarrow{\overset{\leadsto}{v}_*} H_*(V_0 \setminus B_a, A_a).$$

(Here I is the corresponding inclusion. We observe that the last arrow is well defined because $v^{\leadsto}(B_a) \subset U$ and $D_0(-v) \cap V_1 \subset D2_0(-v) \cup D1_{\mu'}(-v)$.) The composition $(-v)^{\leadsto}_* \circ \mathrm{Exc}^{-1} \circ I_*$ of the homomorphisms in this sequence will be denoted by $H(v;\mu',\mu;U)$.

The homomorphism $H(v)\colon H_*(V_1 \setminus B_b, A_b) \to H_*(V_0 \setminus B_a, A_a)$**.** Suppose that U' is another subset of B_b satisfying (I) and such that $U \subset U'$. Then the inclusion $V \setminus U' \subset V_1 \setminus U$ induces a map of the sequence $\mathcal{H}(v;\mu',\mu;U')$ to the sequence $\mathcal{H}(v;\mu',\mu;U)$, which is identical on the first term $H_*(V_1 \setminus B_b, A_b)$ and on the last term $H_*(V_0 \setminus B_a, A_a)$. It follows that $H(v;\mu',\mu;U)$ does not depend on the choice of U. In a similar way, we can check that $H(v;\mu',\mu;U)$ is independent of the choice of μ', μ, δ, and of a representation $S(f)=S1(f)\sqcup S2(f)$ (if there are several representations of this type). Therefore, this homomorphism $H(v)$ is entirely determined by v and the ranging system $\{(A_\lambda,B_\lambda)\}_{\lambda\in\Lambda}$.

In order to prove part (1) of Proposition 4.10, it suffices to consider the sequence $\mathcal{H}(v;0,\delta;U)$, where U is a compact set, and to apply Lemma 4.4.

Passing to the proof of part (2), we fix some μ', μ with $0<\mu'<\mu<\delta$ and an open subset U such that $\Delta(0,-v)\subset U\subset \operatorname{Int} B_b$ and $D1_\delta(-v)\subset V_1\setminus\overline{U}$. The proof of the next lemma is similar to the above proof of Lemma 4.8 and will be omitted.

Lemma 4.11. *There is a number $\epsilon>0$ such that for every f-gradient w with $\|w-v\|<\epsilon$ we have*

$$\Delta(0,-w)\subset U;\quad \nabla(\delta,w)\subset \operatorname{Int} A_a; \tag{21}$$

$$D1_0(-w)\subset B1_{\mu'}(-v);\quad D1_\mu(-v)\subset B1_\delta(-w);\quad D1_\delta(-w)\subset V_1\setminus\overline{U}.\quad\square \tag{22}$$

Now the proof of part (2) of Proposition 4.9 can be completed as follows. Let $R=V_1\setminus\big(U\cup B1_{\mu'}(-v)\big)$ and $Q=\big(A_b\cup D1_\mu(-v)\big)\setminus B1_{\mu'}(-v)$. Then R is a subset of the domain of $(-v)^{\leadsto}$, and $(-v)^{\leadsto}$ maps the pair of compact sets (R,Q) to the pair of open sets $(V_0\setminus B_a,\operatorname{Int} A_a)$. By Corollary 5.6, there exists $\kappa>0$ such that for every $u\in \operatorname{Vect}^1(W,\perp)$ with $\|u-v\|<\kappa$, each $(-u)$-trajectory starting at a point $x\in R$ reaches V_0; if it intersects V_0 at a point $L(u,x)$, then $x\in V_0\setminus B_a$, and for $x\in Q$ we have $L(u,x)\in\operatorname{Int} A_a$. By [21, 8.10], the map $(u,x)\mapsto L(u,x)$ is continuous with respect to the C^1-topology.

In particular, for every f-gradient w with $\|w-v\|<\kappa$, $(-w)^{\leadsto}$ determines a map of pairs $(R,Q)\to(V_0\setminus B_a,\operatorname{Int} A_a)$. This map is homotopic to $(-v)^{\leadsto}$ via the homotopy $\mathcal{H}_t(x)=L(tv+(1-t)w,x)$. Therefore, $H(v)$ coincides with the composition of the homomorphisms in the following sequence:

$$(\overline{\mathcal{H}})\qquad \begin{cases} H_*(V_1\setminus B_b,A_b)\xrightarrow{I_*} H_*(V_1\setminus U, A_b\cup D1_\mu(-v)) \\ \qquad\xrightarrow{\operatorname{Exc}^{-1}} H_*\big(V_1\setminus(U\cup D1_{\mu'}(-v)),(A_b\cup D1_\mu(-v))\setminus D1_{\mu'}(-v)\big) \\ \qquad\xrightarrow{\overset{\leadsto}{w}_*} H_*(V_0\setminus B_a,A_a). \end{cases}$$

Let $\epsilon>0$ satisfy the conclusions of Proposition 4.11; we set $\epsilon_0=\min(\epsilon,\kappa)$. Let w be an f-gradient with $\|w-v\|<\epsilon_0$. Then there is a morphism of the sequence $(\overline{\mathcal{H}})$ to the sequence $\mathcal{H}(w;0,\delta;U)$ such that the homomorphisms on the first term and on the last term are identical (the two homomorphisms in the middle are induced by inclusions). This shows that the composition of the homomorphisms in $(\overline{\mathcal{H}})$ coincides with $H(w)$, and Proposition 4.10 follows. $\square$

Remark. If $\{(A_\lambda,B_\lambda)\}_{\lambda\in\Lambda}$ is a ranging system and $\lambda,\mu\in\Lambda$, then, obviously, the system $\{(A_\lambda,B_\lambda)\}_{\sigma\in\Lambda,\lambda<\sigma<\mu}$ is a ranging system for $(f|f^{-1}([\lambda,\mu]),v)$. The corresponding homomorphism will be denoted by $H(v,[\mu,\lambda])$.

Ranging systems provide a convenient tool for computing the incidence coefficients in Morse complexes.

Let $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ be a ranging system for $f\colon W \to [a,b]$. Let $p \in S(f)$, and let $\mu(p)$ and $\sigma(p)$ be the (uniquely determined) adjacent elements of Λ such that $f(p) \in]\mu(p), \sigma(p)[$. We denote by $S_+(p,v)$ the sphere $D(p,-v) \cap f^{-1}(\sigma(p))$ and by $S_-(p,v)$ the sphere $D(p,v) \cap f^{-1}(\mu(p))$. If this does not lead to confusion, we denote these spheres simply by $S_-(p)$ and $S_+(p)$.

We say that a ranging system $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ is *adapted* to a pair p, q of critical points in $S(f)$, where $\operatorname{ind} p = \operatorname{ind} q + 1$, if $S_-(p) \cap B_{\mu(p)} = \varnothing$ and $S_+(q) \cap A_{\sigma(q)} = \varnothing$. Let $l+1 = \operatorname{ind} p$; then $\operatorname{ind} q = l$. We put $n = \dim W$. If $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ is adapted to p, q, we denote by $[S_-(p)]$ also the image of this fundamental class in $H_l(f^{-1}(\mu(p)) \setminus B_{\mu(p)}, A_{\mu(p)})$. Let $j(v)$ denote the inclusion $\big(f^{-1}(\sigma(q)) \setminus B_{\sigma(q)}, A_{\sigma(q)}\big) \hookrightarrow \big(f^{-1}(\sigma(q)), f^{-1}(\sigma(q)) \setminus S_+(q)\big)$, and let $]]S_+(q,v)[[$ be the $(j(v))^*$-image of the class dual to $S_+(q)$ in the group $H^l(f^{-1}(\sigma(q)) \setminus B_{\sigma(q)}, A_{\sigma(q)})$.

Proposition 4.12. *Let $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ be a ranging system adapted to p, q.*

1. *There is $\epsilon > 0$ such that for every f-gradient v with $\|w - v\| < \epsilon$ the system $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ is a ranging system for (f, w) adapted to p, q.*
2. *Let $U(p)$ be a neighborhood of p. There is $\epsilon > 0$ such that for every f-gradient w with $w|U(p) = v|U(p)$ and $\|w - v\| < \epsilon$ we have $[S_-(p,w)] = [S_-(p,v)]$.*
3. *Let $U(q)$ be a neighborhood of q. There is $\epsilon > 0$ such that for every f-gradient w with $w|U(q) = v|U(q)$ and $\|w - v\| < \epsilon$ we have $]]S_-(q,w)[[=]]S_-(q,v)[[$.*
4. *If v is an f-gradient satisfying the transversality condition, then*

$$n(p,q;v) = \big\langle]]S_+(q)[[, H_{[\mu(p),\sigma(q)]}(v)([S_-(p)])\big\rangle.$$

Proof. (1) This is a direct consequence of Proposition 1.8 and Lemma 4.8.

(2) Let $\delta > 0$ be so small that $\Sigma_\delta \subset U(p)$, where $\Sigma_\delta = D(p,v) \cap f^{-1}([f(p) - \delta, f(p)])$. Let W_0 denote the cobordism $f^{-1}([\mu(p), f(p) - \delta])$. The Morse function $f|W_0$ has no critical points. Let w be any f-gradient such that $w|U(p) = vU(p)$. Consider the homotopy $H_t = (-tv - (1-t)w)^{\leadsto}$ from $(-v)^{\leadsto}$ to $(-w)^{\leadsto}$. For $\|w - v\|$ sufficiently small, $H_t(\Sigma_\delta)$ is contained in $f^{-1}(\mu(p)) \setminus B_\mu(p)$, which proves our statement.

The proof of (3) is similar to that of (2).

(4) We recall that the incidence coefficient $n(p,q;v)$ is the algebraic number of the v-trajectories joining p and q, or, equivalently, the algebraic intersection index of $D(p,v) \cap f^{-1}(\lambda)$ and $D(q,-v) \cap f^{-1}(\lambda)$ in $f^{-1}(\lambda)$ for every regular value $\lambda \in]f(q), f(p)[$. It remains to put $\lambda = \sigma(q)$ and apply Lemma 4.5 and Proposition 4.10. $\square$

If (f, v) has a ranging system adapted to p, q, we say that v satisfies condition $(RP; p, q)$.

4.4. Construction of ranging systems from ranging pairs. Let $f\colon W \to [a,b]$ be a Morse function on a Riemannian cobordism, $\dim W = n$, and let v be an f-gradient satisfying the almost transversality condition. Assume that (f, v) has a ranging pair $(\mathcal{V}_0, \mathcal{V}_1)$. In this subsection we show that this ranging pair generates a family of ranging systems for (f, v). Moreover, for every $p, q \in S(f)$ with $\operatorname{ind} p = \operatorname{ind} q + 1$, in this family there is a ranging system adapted to p, q.

Construction 4.13. Let $\Lambda = \{\lambda_0, \dots, \lambda_k\}$ be a set of regular values of f such that $\lambda_i < \lambda_{i+1}, \lambda_0 = a, \lambda_k = b$, and in each $[\lambda_i, \lambda_{i+1}]$ there is only one critical value of f. We choose some $\delta' > \delta$ in the interval of definition of $\mathbf{T}(\mathbf{D}(u_1), v)$ and $\mathbf{T}(\mathbf{D}(-u_0), -v)$ such that for every $0 \leqslant s \leqslant n-1$ we have

$$\overline{\mathbf{T}(\mathbf{D}(u_1), v)_{(\leqslant s+1)}(\delta')} \cap V_0 \subset \mathbf{D}(u_0)_{(\leqslant s)}(\delta),$$
$$\overline{\mathbf{T}(\mathbf{D}(-u_0), -v)_{(\leqslant s+1)}(\delta')} \cap V_1 \subset \mathbf{D}(-u_1)_{(\leqslant s)}(\delta)$$

(such a number δ' exists because δ belongs to an interval of definition of $\mathbf{T}(\mathbf{D}(u_1), v)$ and of $\mathbf{T}(\mathbf{D}(-u_0), -v)$). Also, we require that the gradients v, u_0, u_1 be δ'-separated.

Now, for each integer s with $0 \leqslant s \leqslant n$ we define compact subsets $A_\lambda^{(s)}$ and $B_\lambda^{(s)}$ of $f^{-1}(\lambda)$. We set $A_\lambda^{(0)} = B_\lambda^{(0)} = \varnothing$. Let $\delta = \delta_k < \delta_{k-1} < \dots < \delta_1 < \delta_0 = \delta'$ be a sequence of real numbers, and let $1 \leqslant s \leqslant n$. We put

$$\begin{cases} A_{\lambda_l}^{(s)} = \overline{\mathbf{T}(\mathbf{D}(u_1), v)_{(\leqslant s)}(\delta_l)} \cap f^{-1}(\lambda_l) & \text{if } 0 < l \leqslant k, \\ A_a^{(s)} = \overline{\mathbf{D}(u_0)_{(\leqslant s-1)}(\delta)} = D_\delta(\operatorname{ind} \leqslant s-1; u_0); \end{cases}$$
$$\begin{cases} B_{\lambda_l}^{(s)} = \overline{\mathbf{T}(\mathbf{D}(-u_0), -v)_{(\leqslant s)}(\delta_{k-l})} \cap f^{-1}(\lambda_l) & \text{if } 0 \leqslant l < k, \\ B_b^{(s)} = \overline{\mathbf{D}(-u_1)_{(\leqslant s-1)}(\delta)} = D_\delta(\operatorname{ind} \leqslant s-1; -u_1). \end{cases}$$

Clearly, $A_b^{(s)} = D_\delta(\operatorname{ind} \leqslant s-1; u_1)$ and $B_a^{(s)} = D_\delta(\operatorname{ind} \leqslant s-1; -u_0)$.

Lemma 4.14.

(1) $A_\lambda^{(r)} \cap B_\lambda^{(s)} = \varnothing$ *if* $r + s \leqslant n$.

(2) *If* $0 \leqslant s \leqslant n-1$, *then* $\{(A_\lambda^{(s)}, B_\lambda^{(n-s-1)})\}_{\lambda \in \Lambda}$ *is a ranging system for* (f, v) *adapted to every pair* p, q *of critical points of* f *with* $\operatorname{ind} p = s+1$, $\operatorname{ind} q = s$.

(3) *For every* $0 \leqslant s \leqslant n-1$ *the pair* $(V_1 \setminus B_b^{(n-s-1)}, A_b^{(s)})$ *is homotopy equivalent to a finite CW-pair that has cells of dimension* s *only.*

Proof. (1) For $\lambda = a$, $\lambda = b$, this follows immediately from condition (RP2). Let $0 < l < k$, and let $\lambda = \lambda_l$. Then $\delta_l, \delta_{k-l} < \delta'$, $A_\lambda^{(r)} \subset T(\mathbf{D}(u_1)_{(\leqslant r-1)}(\delta'), v) \cup B_{\delta'}(\operatorname{ind} \leqslant r; v)$, and $B_\lambda^{(s)} \subset T(\mathbf{D}(-u_0)_{(\leqslant s-1)}(\delta'), -v) \cup B_{\delta'}(\operatorname{ind} \leqslant s, -v)$. Therefore, if $A_\lambda^{(r)} \cap B_\lambda^{(s)} \neq \varnothing$, there is a $(-v)$-trajectory that joins

1) a point of $\mathbf{D}(u_1)_{(\leqslant r-1)}(\delta')$ with a point of $\mathbf{D}(-u_0)_{(\leqslant s-1)}(\delta')$, or

2) a point of $\mathbf{D}(u_1)_{(\leqslant r-1)}(\delta')$ with a point of $B_{\delta'}(p)$, where $p \in S(f)$, $\operatorname{ind} p \geqslant n-s$, or

3) a point of $\mathbf{D}(-u_0)_{(\leqslant s-1)}(\delta')$ with a point of $B_{\delta'}(q)$, where $q \in S(f)$, $\operatorname{ind} q \leqslant r$, or

4) a point of $B_{\delta'}(p)$ with a point of $B_{\delta'}(q)$, where $p, q \in S(f)$, $\operatorname{ind} q \geqslant n-s$, $\operatorname{ind} p \leqslant r$.

Case 4) is impossible because v is δ'-separated, case 1) is impossible because condition (RP4) is fulfilled and u_0 is δ'-separated. Cases 2) and 3) are eliminated similarly.

(2) Condition (RS1) follows from (1). To prove (RS2), we note that if $\operatorname{ind} p = r$, then for every $\lambda \in \Lambda$ the set $D(p, v) \cap f^{-1}(\lambda)$ is contained in $A_\lambda^{(r)}$, and $D(p, -v) \cap f^{-1}(\lambda)$ is contained in $B_\lambda^{(n-r)}$. Therefore, for s given, either $\operatorname{ind} p \leqslant s$ and then $D(p, v) \cap f^{-1}(\lambda)$ is a subset of $A_\lambda^{(s)}$, or $\operatorname{ind} p > s$ and then $D(p, -v) \cap f^{-1}(\lambda)$ is a subset of $B_\lambda^{(n-s-1)}$.

A similar argument proves that the system $\{(A_\lambda^{(l)}, B_\lambda^{(n-l-1)})\}$ is adapted to any pair $p, q \in S(f)$ with $\operatorname{ind} p = \operatorname{ind} q + 1 = l + 1$.

Statement (3) can be proved by a standard argument of the Morse theory. $\square$

4.5. Cyclic ranging systems, the incidence coefficients in the Novikov complex, and the proof of the main theorem.

Definition 4.15. A Riemannian cobordism W with $\partial W = V_0 \sqcup V_1$ is said to be *cyclic* if there is an isometry $\Phi\colon V_0 \to V_1$ (such an isometry will be regarded as part of the structure of a cyclic cobordism).

Let $f\colon W \to [a, b]$ be a Morse function on a cyclic cobordism W, and let v be an f-gradient satisfying the almost transversality condition. An $\mathcal{M}$-flow $\mathcal{V}_0 = (f, v, \mathcal{U})$ on V_0 is called a *ranging flow* for (f, v) if $(\mathcal{V}_0, \Phi(\mathcal{V}_0))$ is a ranging pair for (f, v).

We say that v satisfies condition (RF) if there is a ranging flow for (f, v).

The proof of the following theorem is the same as the proof of Theorem 4.3, with the only difference that we must choose $\mathcal{V}_1 = \Phi(\mathcal{V}_0)$ and $\mathcal{W}_1 = \Phi(\mathcal{W}_0)$. We refer the reader to the beginning of Subsection 4.1 for the definition of the neighborhoods $U_0(\mu, 3\mu)$, and $U_1(-3\mu, -\mu)$.

Theorem 4.16. *Let $f\colon W \to [a, b]$ be a Morse function on a cyclic cobordism W. Let $\epsilon > 0$. There is an f-gradient w satisfying the almost transversality condition and a ranging flow $(\mathcal{V}_0, \mathcal{V}_1)$ for (f, w) such that $\|w - v\| \leqslant \epsilon$ and $\operatorname{supp}(w - v) \subset U_0(\mu, 3\mu) \cup U_1(-3\mu, -\mu)$.*

Definition 4.17. A ranging system $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ for (f, v) is said to be *cyclic* if $\Phi(A_a) = A_b$, $\Phi(B_a) = B_b$. A cyclic ranging system is said to be of *finite type* if $H_*(V_0 \setminus B_a, A_a)$ is finitely generated.

¿From the results of Subsections 4.1–4.4 it can easily be deduced that the "cyclic" analogs of these results are also true. We mention only the fact that for every p, q and every ranging flow there is a finite type cyclic ranging system adapted to p, q. Next, if $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ is a cyclic ranging system, then, composing the homomorphism $H(v)$ with Φ_*, we obtain a homomorphism $h(v)\colon H_*(V_0 \setminus B_a, A_a) \to H_*(V_0 \setminus B_a, A_a)$ satisfying $h(v)([N]) = [\overset{\leadsto}{(-v)}_{[b,a]}(N) \circ \Phi]$.

The language of cyclic ranging systems allows us to obtain formula (2) for the Novikov incidence coefficients (see the Introduction). Namely, let M be a closed Riemannian manifold, and let $f\colon M \to S^1$ be a Morse map belonging to an indivisible class $[f] \in H^1(M, \mathbb{Z})$. Let v be an f-gradient satisfying the transversality condition, and let p, q be critical points of f such that $\operatorname{ind} p = \operatorname{ind} q + 1$. We put $\operatorname{ind} q = l$ and choose a regular value c of f such that the maximal critical value of f not exceeding c is $f(q)$. To simplify the notation, we assume that $c = 0$. Cutting M along c, we obtain a Riemannian cyclic cobordism W, a Morse function $F\colon W \to [0, 1]$, and an f-gradient satisfying the transversality condition (the latter will still be denoted by v). Let $V_0 = f^{-1}(0)$, $V_1 = f^{-1}(1)$, and let $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ be a cyclic ranging system on W adapted to p, q.

We recall that the infinite cyclic covering $\bar M$ is composed of the copies $W^{[i]}$ of W $(i \in \mathbb{Z})$. We lift f to a Morse function $\bar f\colon \bar M \to \mathbb{R}$ such that $f(W^{[0]}) = [0,1]$ and lift p and q to $\bar M$ in such a way that $\bar q \in W^{[-1]}$, $\bar p \in W^{[0]}$. For $m < n$, we denote the cobordism $f^{-1}([m,n])$ by $W[m,n]$.

Let $[[p]] \in H_l(V_0 \setminus B_0, A_0)$ denote the class $H_{[\lambda_-(p),0]}(v)([S_-(p)])$. The cohomology class $]]S_+(q,v)[[\in H^l(V_0 \setminus B_0, A_0)$ will be denoted by $]]q[[$.

Proposition 4.18. *We have*

$$\begin{aligned} n_k(\bar p, \bar q; v) &= \big\langle]]q[[, (h(v))^k[[p]]\big\rangle, \quad && k \geqslant 0, \\ n_k(\bar p, \bar q; v) &= 0 && k < 0. \end{aligned}$$

Proof. Since $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda}$ is a cyclic ranging system, we can construct a ranging system on $W[-k,1]$ by glueing together the systems on the cobordisms $W^{[i]}$, where $-k \leqslant i \leqslant 0$. Proposition 4.12 implies that $n_k(\bar p, \bar q t^k) = \big\langle]]S_+(\bar q t^k)[[, H_{[\lambda_-(p),-k]}(v)([S_-(p)])\big\rangle$. Since the ranging system on $W[-k,1]$ is invariant with respect to the action of t, it is easy to check that the expression on the right is equal to $\big\langle]]q[[, (h(v))^k[[p]]\big\rangle$. $\square$

Proof of the main theorem. Let $p, q \in S(f)$, $\operatorname{ind} p = \operatorname{ind} q + 1$, and let $\mathcal{G}t_0(f,p,q)$ denote the set of all $v \in \mathcal{G}t(f)$ such that there is a cyclic ranging system of finite type on W adapted to p, q. Proposition 4.12 implies that $\mathcal{G}t_0(f;p,q)$ is a C^0-open set. The set of all f-gradients having a ranging flow is C^0-dense in the set of all f-gradients (Theorem 4.16). Every ranging flow generates a cyclic ranging system of finite type, adapted to (p,q). Therefore, for every $v \in \mathcal{G}t(f)$ and every $\varepsilon > 0$ there is an f-gradient w with $\|v - w\| < \varepsilon$ and such that w has a cyclic ranging system of finite type adapted to (p,q). Perturbing w if necessary, we may assume that w satisfies the transversality condition. Thus, $\mathcal{G}t_0(f;p,q)$ is C^0-dense in $\mathcal{G}t(f)$.

To deduce statement (2), it suffices to apply the following lemma (already cited informally in the Introduction).

Lemma 4.19. *Let G be a finitely generated Abelian group, A an endomorphism of G, and $\lambda\colon G \to \mathbb{Z}$ a homomorphism. Then for every $p \in G$ the series $\sum_{k\geqslant 0} \lambda(A^k p)t^k \in \mathbb{Z}[[t]]$ is a rational function of t of the form $\frac{P(t)}{Q(t)}$, where P, Q are polynomials and $Q(0) = 1$.*

Proof. It suffices to treat the case of a free Abelian group G. Consider a free $\mathbb{Z}[[t]]$-module $R = G[[t]]$ and the homomorphism $\phi\colon R \to R$ given by $\phi = 1 - At$. Then ϕ is invertible, and $\phi^{-1} = \sum_{k\geqslant 0} A^k t^k$. On the other hand, the inverse of ϕ is given by Cramér's formulas, which yield rational functions with denominator $Q(t) = \det(1 - At)$. $\square$

To prove statement (3) of the theorem, we note that $h(v)$ is not affected by the small C^0-perturbations of v (see Proposition 4.10), and the same in true for the homology and cohomology classes $[[p]]$ and $]]q[[$ (see Proposition 4.12). $\square$

§5. Appendix. C^0-perturbations of vector fields and of their integral curves

For a manifold M without boundary we denote by $\mathrm{Vect}^1(M)$ (respectively, $\mathrm{Vect}^1_0(M)$) the vector space of C^1 vector fields on M (respectively, the vector space of C^1 vector fields on M with compact support).

5.1. Manifolds without boundary. In this subsection M is a Riemannian manifold without boundary, $v \in \mathrm{Vect}^1_0(M)$, $n = \dim M$.

Proposition 5.1. *Suppose that $a, b \in M$, $t_0 \geqslant 0$, and $\gamma(a, t_0; v) = b$. For every open neighborhood U of $\gamma(x, [0, t_0]; v)$ and every open neighborhood R of b there exists a number $\delta > 0$ and an open neighborhood $S \subset U$ of a such that for any $x \in S$ and any $w \in \mathrm{Vect}^1_0(M)$ with $\|w - v\| < \delta$ we have $\gamma(x, t_0; w) \in R$ and $\gamma(x, [0, t_0]; w) \subset U$.*

Proposition 5.2. *Suppose that $a, b \in M$, $t_0 \geqslant 0$, and $\gamma(a, t_0; v) = b$. Let E be a submanifold without boundary of M of codimension 1 such that $b \in E$ and $v(b) \notin T_bE$. If U is an open neighborhood of $\gamma(a, [0, t_0]; v)$ and R an open neighborhood of b, then there exists $\theta_0 > 0$ such that for every $\theta \in]0, \theta_0[$ there is a number $\delta > 0$ and an open neighborhood $S \subset U$ of a such that for any $x \in S$ and any $w \in \mathrm{Vect}^1_0(M)$ with $\|w - v\| < \delta$ we have*

1. *$\gamma(x, [-\theta, t_0 + \theta]; w) \subset U$ and $\gamma(x, [t_0 - \theta, t_0 + \theta]; w) \subset R$;*
2. *there is a unique $\tau_0 = \tau_0(w, x) \in [t_0 - \theta, t_0 + \theta]$ such that $\gamma(x, \tau_0; w) \in E$;*
3. *if E is compact and t_0 is the only $t \in [0, t_0]$ such that $\gamma(x, t_0; v) \in E$, then for any $y \in S$ the number $\tau_0(w, y)$ is the only $\tau \in [-\theta, t_0 + \theta]$ such that $\gamma(y, \tau; w) \in E$.*

Proof of Proposition 5.1. The case where $M = \mathbb{R}^n$ (with the Euclidean metric) can be settled immediately with the help of the next lemma, which, in turn, is a direct consequence of [2, §4, Theorem 3]. For $v \in \mathrm{Vect}^1_0(\mathbb{R}^n)$, we denote by $\|v\|_1$ the norm of the derivative $dv\colon \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^n)$, i.e., $\|v\|_1 = \sup_{x \in \mathbb{R}^n} \|dv(x)\|$.

Lemma 5.3. *Suppose that $u, w \in \mathrm{Vect}^1_0(\mathbb{R}^n)$, $\|u - w\| < \alpha$, $\|u\|_1 \leqslant D$, where $D > 0$. Let γ, η be trajectories of the vector fields u, w (respectively) such that $|\gamma(0) - \eta(0)| \leqslant \epsilon$. Then for every $t \geqslant 0$ we have $|\gamma(t) - \eta(t)| \leqslant \epsilon e^{Dt} + \frac{\alpha}{D}(e^{Dt} - 1)$.* □

The general case is obtained by splitting $[a, b]$ into small intervals $I_k = [a_k, a_{k+1}]$ in such a way that each of the I_k belong to the domain of a chart. We leave the details to the reader. □

Proof of Proposition 5.2. 1) *The case where $t_0 = 0$.* We represent $\mathbb{R}^n$ as the product $\mathbb{R}^1 \times \mathbb{R}^{n-1}$; the elements $z \in \mathbb{R}^n$ will be referred to as pairs $z = (x, y)$ with $x \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$; x is called the *first coordinate* of z, and y is the *second coordinate* of z. We denote $\mathbb{R}^n_+ = \{(x, y) \mid x > 0\}$, $\mathbb{R}^n_- = \{(x, y) \mid x < 0\}$. For $r > 0$ we put $W_r =]-r, r[\times B^{n-1}(0, r)$. Let $h\colon \mathbb{R}^n \to [0, 1]$ be a C^∞-function such that $\operatorname{supp} h \subset W_2$ and $h(x) = 1$ for $x \in W_1$. We choose a chart $\phi\colon W \to \mathbb{R}^n$ of M such that $a \in W \subset U \cap R$, $\phi(a) = 0$, $\phi(W \cap E) = \{0\} \times \mathbb{R}^{n-1}$, and the first coordinate of $\phi_* v(a)$ is 1. For a vector field w on M, we denote by $\widetilde{w}$ the vector field $h \cdot \phi_* w$ on $\mathbb{R}^n$. Let $C > 0$ be a constant such that the metric on $\mathbb{R}^n$ induced by ϕ from M and the Euclidean metric are C-equivalent on $\overline{W_2}$. Then for every $w \in \mathrm{Vect}^1_0(M)$ we have $\|\widetilde{w}\|_e \leqslant C\|w\|_\rho$. Let $r \in]0,1[$ be so small that the first coordinate of $\phi_* v(x)$ be at least $1/2$ for $x \in W_r$, let $\theta > 0$ be so small that

$$\gamma(0,[-\theta,\theta];\widetilde{v}) \subset W_r, \quad \gamma(0,\theta;\widetilde{v}) \in \mathbb{R}^n_+, \quad \text{and} \quad \gamma(0,-\theta;\widetilde{v}) \in \mathbb{R}^n_-.$$

By Lemma 5.3, there is $\delta_0 > 0$ and a neighborhood $S_0 \subset W_r$ such that

$$\gamma(x,[-\theta,\theta];u) \subset W_r, \quad \gamma(x,\theta;u) \in \mathbb{R}^n_+, \quad \text{and} \quad \gamma(x,-\theta;u) \in \mathbb{R}^n_-$$

whenever $x \in S_0$, $u \in \mathrm{Vect}^1_0(\mathbb{R}^n), \|u - \widetilde{v}\|_e < \delta_0$. Therefore, if $x \in S_0$, $u \in \mathrm{Vect}^1_0(\mathbb{R}^n)$, and $\|u - \widetilde{v}\|_e < \delta_0$, then there is $\tau_0 = \tau_0(x,u) \in]-\theta,\theta[$ such that $\gamma(x,\tau_0) \in \{0\} \times \mathbb{R}^{n-1}$. Next, if $\|u - \widetilde{v}\|_e < \delta_1 = \min(\delta_0, 1/2)$, then the first coordinate of u is positive on W_r, whence we see that for every $x \in S_0$ there is precisely one $\tau_0 = \tau_0(x,u) \in [-\theta,\theta]$ with $\gamma(x,\tau_0;u) \in \{0\} \times \mathbb{R}^{n-1}$. Now it is easy to check that statements (1) and (2) in the proposition are true with $\delta = \delta_1/C$ and $S = \phi^{-1}(S_0)$. Statement (3) is an immediate consequence of (2).

2) *The general case.* Applied to the point b and $t_0 = 0$, the above part of the proof shows that there is $\theta_0 > 0$ such that for every $\theta \in]0,\theta_0[$ there is $\delta_0(\theta) > 0$ and a neighborhood $R_0(\theta) \subset U \cap R$ of b such that for every $x \in R_0(\theta)$ and every $w \in \mathrm{Vect}^1_0(M)$ with $\|w - v\| < \delta_0(\theta)$ we have

$$\gamma(x,[-\theta,\theta];w) \subset U; \tag{23}$$

(24) there exists a unique $\tau_0 = \tau_0(w,x) \in [-\theta,\theta]$ such that $\gamma(x,\tau_0;w) \in E$.

Let $\theta_1 > 0$ be a number such that $\gamma(x,[-\theta_1,t_0+\theta_1];v) \subset U$ and $\gamma(x,[t_0-\theta_1,t_0+\theta_1];w) \subset R$. Applying Proposition 5.1, we readily check that for any $\theta \in]0,\theta_0[$ there exists $\delta_1(\theta) > 0$ and an open neighborhood $S(\theta) \subset U$ of a such that for every $y \in S(\theta)$ and every $w \in \mathrm{Vect}^1_0(M)$ with $\|v - w\| < \delta_1(\theta)$ we have

$$\gamma(y,[-\theta_1,t_0+\theta_1];w) \subset U, \qquad \gamma(y,[t_0-\theta_1,t_0+\theta_1];w) \subset R, \tag{25}$$

$$\gamma(y,t_0;w) \in R_0(\theta). \tag{26}$$

We claim that for every $\theta \in]0,\min(\theta_0,\theta_1)[$ the number $\delta_2(\theta) = \min(\delta_1(\theta),\delta_0(\theta))$ and the neighborhood $S(\theta)$ of a satisfy conditions (1) and (2) of Proposition 5.2. Indeed, Condition (1) is an immediate consequence of (25). To prove (2), we note that the existence of a unique $\tau_0 \in [t_0-\theta,t_0+\theta]$ such that $\gamma(y,\tau_0;w) \in E$ is equivalent to the existence of a unique $\tau_1 \in [-\theta,\theta]$ such that $\gamma(\gamma(y,t_0;w),\tau_1;w) \in E$, and the latter condition is fulfilled if $y \in S(\theta)$ and $\|w - v\| \leqslant \delta_2(\theta)$ (this follows from (24) and (26)). Thus, if E is noncompact, the proof is complete. Let E be compact. We choose $\theta_3 > 0$ such that $\gamma(a,[-\theta_3,t_0[,v) \cap E = \varnothing$. For every $\theta \in]0,\theta_3[$, we find $\delta_3(\theta) > 0$ and a neighborhood $S_3(\theta)$ of a such that $\gamma(x,[-\theta_3,t_0-\theta];w) \subset M \setminus E$ if $x \in S_3(\theta)$ and $w \in \mathrm{Vect}^1_0(M)$, $\|w - v\| < \delta_3(\theta)$. Then for every $\theta \in]0,\min(\theta_1,\theta_2,\theta_3)[$ the number $\min(\delta_1(\theta),\delta_2(\theta),\delta_3(\theta))$, and the neighborhood $S_3(\theta) \cap S(\theta)$ of a satisfy conditions (1)–(3) of Proposition 5.2. $\square$

5.2. Manifolds with boundary. Let W be a compact Riemannian manifold with boundary. As in [21, Subsection 8C], we denote by $\mathrm{Vect}^1(W, \perp)$ the space of C^1 vector fields v on W such that $v(x) \notin T_x(\partial W)$ for $x \in \partial W$. We fix an embedding of W in a closed manifold M without boundary satisfying $\dim M = \dim W$ (e.g., as M we can take the double of W). Also, we fix a Riemannian metric on M extending that on W (the existence of such a metric is easily proved by using the standard partition of unity argument). The same argument proves that every C^1 vector field on W can be extended to a vector field $\widetilde{v} \in \mathrm{Vect}^1(M)$ such that $\|\widetilde{v}\| \leqslant 2\|v\|$. The corollaries below are "C^0-analogs" of Propositions 8.10 and 8.11 in [21], and can be proved in the same way if we use Propositions 5.1, 5.2 of the present paper instead of [21, Proposition 8.2].

Corollary 5.4. *Let $v \in \mathrm{Vect}^1(W, \perp)$, $x \in W$. Assume that the v-trajectory $\gamma(x, \cdot; v)$ reaches the boundary at a moment $T \geqslant 0$. Let U be an open neighborhood of $\gamma(x, [0, T]; v)$ and R an open neighborhood of $\gamma(x, T; v)$. There is a number $\delta > 0$ and a neighborhood S of x such that for every $y \in S$ and every $w \in \mathrm{Vect}^1(W, \perp)$ with $\|w - v\| < \delta$ the trajectory $\gamma(y, \cdot; w)$ reaches the boundary at a moment $T(y, w) \geqslant 0$, and $\gamma(y, [0, T(y, w)]; w) \subset U$, $\gamma(y, T(y, w); w) \subset R$.* $\square$

Corollary 5.5. *Suppose that $v \in \mathrm{Vect}^1(W, \perp)$, $x \in W$, $T \geqslant 0$, and that $\gamma(x, T; v) \in W^\circ$. Let U be an open neighborhood of $\gamma(x, [0, T]; v)$, and R an open neighborhood of $\gamma(x, T; v)$. There is a number $\delta > 0$ and a neighborhood S of x such that for every $y \in S$ and every $w \in \mathrm{Vect}^1(W, \perp)$ with $\|w - v\| < \delta$ the trajectory $\gamma(y, \cdot; w)$ is defined on $[0, T]$ and $\gamma(y, [0, T]; w) \subset U$, $\gamma(y, T; w) \in R$.* $\square$

Let $v \in \mathrm{Vect}^1(W, \perp)$, and let V_0 (respectively, V_1) denote the set of all points $x \in \partial W$ at which $v(x)$ is directed inwards (respectively, outwards) relative to W. For a subset $Z \subset W$, we denote by $\tau(Z, v)$ the set $\{\gamma(z, t; v) \mid z \in Z,\ t \geqslant 0\}$. The next corollary can be deduced from Corollary 5.4 with the help of an easy compactness argument.

Corollary 5.6. *Let $K \subset W$ be a compact set such that every v-trajectory starting at a point of K reaches the boundary. Let U be an open neighborhood of $\tau(K, v)$. If $R \subset V_1$ is an open neighborhood of $\tau(K, v) \cap V_1$, then there exists $\delta > 0$ such that for every $w \in \mathrm{Vect}^1(W, \perp)$ with $\|w - v\| < \delta$, each w-trajectory starting at a point of K reaches the boundary and we have $\tau(K, w) \subset U$, $\tau(K, w) \cap V_1 \subset R$.* $\square$

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