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MORSE–NOVIKOV NUMBER FOR KNOTS AND LINKS

C. WEBER, A. PAJITNOV, L. RUDOLPH,

ABSTRACT. The Morse–Novikov number $\mathcal{MN}(L)$ of a link $L \subset S^3$ is defined as the minimal possible number of critical points of a Morse mapping $S^3 \setminus L \to S^1$ of a special type. Some properties of this invariant are studied: a lower estimate for it is obtained in terms of the Novikov numbers of L, which, in turn, are shown to be related to the classical invariants of links; the sharpness of the estimates obtained is discussed. It is proved that the Morse–Novikov number is subadditive with respect to the connected sum of knots. A conjecture is stated.

§1. Introduction

Let $K \subset S^3$ be a tame knot, and let $C_K = S^3 \setminus K$. We recall that K is *fibered* if there is a fibration $\phi\colon C_K \to S^1$ behaving "nicely" in a neighborhood of K.

The fibered knots form a large and natural class. For example, all algebraic knots are fibered [Mi1]. The fibration $\phi\colon C_K \to S^1$ provides supplementary invariants, which help to calculate the basic invariants of a fibered knot [Du]. A similar notion can be introduced for oriented links.

Consider an arbitrary link $L \subset S^3$. It is always possible to construct a Morse map $\phi\colon C_L = S^3 \setminus L \to S^1$ behaving "nicely" in a neighborhood of L. If L is not fibered, then necessarily ϕ has critical points. The study of such maps and related invariants of the link is our aim in the present paper.

The simplest invariant of the link L arising in this way is the *Morse–Novikov number* $\mathcal{MN}(L)$ of L defined as the minimal possible number of critical points of such a Morse map. This invariant can be studied with the help of the Morse–Novikov theory of maps to the circle. Originated with S. P. Novikov's paper [No], this theory has been developing in the past 20 years.

An important tool in the Morse–Novikov theory is the *Novikov inequalities*, which are an analog of the classical Morse inequalities and give lower estimates for the Morse–Novikov number in terms of the *Novikov numbers*. This allows us to prove that there are knots with an arbitrarily large Morse–Novikov number (see Proposition 6.1).

Since the Novikov numbers provide only a lower bound for the Morse–Novikov number, this bound may fail to be optimal. Indeed, we give examples of knots for which the Novikov homology groups (as well as the Novikov numbers) vanish, while the Morse–Novikov number does not (see §5). Thus, purely homological lower bounds are insufficient, and to obtain further information on the Morse–Novikov number we need other methods.

We prove that for each oriented link L there exist Morse maps $C_L \to S^1$ having only critical points of indices 1 and 2, and that $\mathcal{MN}(L)$ is equal to the minimal possible number of critical points of such a Morse map.

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The next step consists in using natural geometric operations on links, such as connected sum, and in investigating the behavior of the Morse–Novikov number under such operations. As a first result (see Proposition 6.2), we prove that

$$\mathcal{MN}(K_1 \# K_2) \leq \mathcal{MN}(K_1) + \mathcal{MN}(K_2),$$

where $\#$ is the operation of connected sum of oriented knots (see [Ro, p. 40]). Although our method is simple, it is new apparently. Given two Morse functions defined on the complements of two knots K_1 and K_2, respectively, we explicitly construct a circle-valued Morse function on the complement of $K_1 \# K_2$ (see §6). In the case of fibered knots, the converse is also true: if the knot $K_1 \# K_2$ is fibered, then both K_1 and K_2 are also fibered [Ga]. Thus, it is natural to pose the following question (M. Boileau, C. Weber):

Is it true that $\mathcal{MN}(K_1 \# K_2) = \mathcal{MN}(K_1) + \mathcal{MN}(K_2)$?

Our results bear witness to the positive answer.

Terminology and notation. Throughout the paper, we work in the C^∞-category. Thus, the functions, maps, curves, etc. are assumed to be of class C^∞ if the contrary is not stated explicitly.

A *Seifert surface* is an oriented compact 2-submanifold of S^3 with no closed components. The boundary $L = \partial S$ of a Seifert surface S is an oriented link; S is called a *Seifert surface for L.*

The 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is endowed with the orientation induced from the standard orientation of $\mathbb{R}^3$. If L is an oriented link, then the link with the opposite orientation is denoted by $-L$. Similarly, if S is a Seifert surface, then the Seifert surface with the opposite orientation is denoted by $-S$.

Let $L \subset S^3$ be a link. A Morse map $f: C_L \to S^1$ is said to be *regular* if each component L_i of L has a neighborhood framed as $S^1 \times D^2$ and such that $L_i \approx S^1 \times 0$ and the restriction $f|: S^1 \times (D^2 \setminus \{0\}) \to S^1$ is given by $(x, y) \mapsto y/|y|$.

If f is a Morse map of a manifold to $\mathbb{R}$ or to S^1, then we denote by $m_p(f)$ the number of critical points of f of index p.

Below, only (co)homology groups with integral coefficients are used.

The layout of the paper. In §2, we discuss the Novikov numbers of knots and links. The Novikov inequalities give a lower estimate for the number of critical points of a regular Morse map $f: C_L \to S^1$ in terms of the Novikov numbers of L (see Proposition 2.1 and Corollary 2.2). We relate the Novikov homology of the complement to the classical invariants of links.

In §3, we prove that there exists a minimal Morse function on the complement of any link L (see Theorem 3.3). Such a Morse function has only critical points of indices 1 and 2. In §4, we discuss the relationship between Morse functions on the complement of a link and free Seifert surfaces. §5 contains examples. In §6, we prove that the Morse–Novikov number is subadditive with respect to the connected sum of knots.

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The Novikov homology for the complements of knots in S^3 was first studied by A. Lazarev [La]. In particular, he obtained an analog of our Theorem 2.4 for the case of knots.

§2. Novikov homology and lower bounds for the Morse–Novikov number

Notation. Let L be an oriented link. Since the ambient space S^3 is oriented, so are the normal fibration and the normal circle bundle of L. For each component L_i of L, there is a unique element $\mu_i \in H_1(C_L)$ represented by any oriented fiber of the normal circle bundle of L_i. There is a unique cohomology class $\xi_L \in H^1(C_L)$ such that for each i we have $\xi_L(\mu_i) = 1$. We let $\overline{C_L} \to C_L$ be an infinite cyclic covering associated with this cohomology class.

Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ and $\widehat{\Lambda} = \mathbb{Z}[[t]][t^{-1}]$. We recall that $\widehat{\Lambda}$ is a principal ideal domain. The homology $H_*(\overline{C_L})$ is a Λ-module. We denote $H_*(\overline{C_L}) \otimes_\Lambda \widehat{\Lambda}$ by $\widehat{H}_*(L)$ and put $\widehat{b}_i(L) = \operatorname{rk}_{\widehat{\Lambda}} \widehat{H}_i(L)$. Let $\widehat{q}_i(L)$ be the *torsion number* of $\widehat{H}_i(L)$, i.e., the minimal number of $\widehat{\Lambda}$-generators of the torsion submodule of $\widehat{H}_i(L)$.

Proposition 2.1. *If $f\colon C_L \to S^1$ is a regular Morse function representing the cohomology class ξ_L, then*

$$m_i(f) \geq \widehat{b}_i(L) + \widehat{q}_i(L) + \widehat{q}_{i-1}(L). \tag{1}$$

The proof is a straightforward generalization of the proof of the Novikov inequalities [No] for closed manifolds (see [F] or [P1]). $\square$

Relations among Novikov numbers. The numbers $\widehat{b}_i(L)$ and $\widehat{q}_i(L)$ satisfy some algebraic relations.

1) $\widehat{H}_0(L) = 0$.

Indeed, $H_0(\overline{C_L}) \approx \mathbb{Z}$, and the element $t \in \Lambda$ acts as the identity; in other words, $1 - t = 0$, and we have $H_0(\overline{C_L}) \otimes_\Lambda \widehat{\Lambda} = 0$ because $1 - t$ is invertible in $\widehat{\Lambda}$. Therefore,

$$\widehat{b}_0(L) = \widehat{q}_0(L) = 0. \tag{2}$$

2) Here are consequences of the Poincaré duality:

$$\widehat{b}_i(L) = \widehat{b}_{3-i}(-L), \tag{3}$$

$$\widehat{q}_i(L) = \widehat{q}_{3-i-1}(-L). \tag{4}$$

For the proof, see [P, §3]. The arguments there pertain to the case of closed manifolds, but can easily be generalized to the present case because the Novikov homology of the boundary of the tubular neighborhood of L vanishes.

3) Obviously, from (4) and (2) it follows that

$$\widehat{q}_2(L) = 0. \tag{5}$$

4) Finally, we observe that $\chi(L) = \chi(C_L) = 0$.

Consequently, for every cellular decomposition of the pair $(S^3 \setminus \operatorname{Int} T(L), \partial T(L))$ we have $\sum (-1)^i n_i = 0$, where n_i is the number of the i-dimensional cells. Therefore, the Euler characteristic of the chain complex $C_*(\overline{C_L})$ of free Λ-modules vanishes, which implies in turn that

$$\widehat{b}_1(L) = \widehat{b}_2(L). \tag{6}$$

Thus, all the numbers $\widehat{b}_i, \widehat{q}_i$ are determined by two of them: $\widehat{b}_1 = \widehat{b}_2$ and $\widehat{q}_1$. By the preceding remarks, the Novikov inequalities for links in S^3 have the following final form.

Corollary 2.2. $m_1(f) \geq \widehat{b}_1(L) + \widehat{q}_1(L)$; $m_2(f) \geq \widehat{b}_1(L) + \widehat{q}_1(L)$. □

As a $\widehat{\Lambda}$-module, the homology $\widehat{H}_*(L)$ admits a decomposition into a direct sum of cyclic modules. The module $\widehat{H}_1(L)$ is related to the classical polynomials of knots and links. To establish this relationship, first we need a bit of algebra.

Λ-modules. Let M be a finitely generated Λ-module with a free finitely-generated resolvent. We consider the following exact sequence:

$$F_1 \xrightarrow{D} F_0 \longrightarrow M \longrightarrow 0, \tag{7}$$

where $F_0 \approx \Lambda^m$ and $F_1 \approx \Lambda^n$. It is assumed that $m \leq n$. The ideals $E_s(D)$ generated by the $(m-s) \times (m-s)$-subdeterminants of D are invariants of M. We denote by $\alpha_s(M)$ the GCD of all elements of the ideal $E_s(D)$. (In general, $\alpha_s(M) \notin E_s(D)$.)

We consider the completion $\widehat{M} = M \otimes_\Lambda \widehat{\Lambda}$, the corresponding exact sequence

$$\widehat{F}_1 \xrightarrow{\widehat{D}} \widehat{F}_0 \longrightarrow \widehat{M} \longrightarrow 0 \tag{8}$$

(which is exact by the right exactness of tensor product), and the invariants $E_s(\widehat{D})$ and $\alpha_s(\widehat{M})$. The key observation here is that

$$\alpha_s(\widehat{M}) = \alpha_s(M). \tag{9}$$

This is an immediate consequence of the following lemma.

Lemma 2.3. *For $a, b \in \Lambda$, we have $\mathrm{GCD}_\Lambda(a,b) = \mathrm{GCD}_{\widehat{\Lambda}}(a,b)$.*

Proof. Let $\alpha = \mathrm{GCD}(a,b)$, and let $\lambda \in \Lambda$ be any element dividing both a and b in $\widehat{\Lambda}$. We show that α is divisible by λ in $\widehat{\Lambda}$.

It suffices to consider the case where $a, b \in \mathbb{Z}[t]$, and a and b are coprime in $\mathbb{Z}[t]$, so that $aP + bQ = n \in \mathbb{Z}$. If $\lambda \in \mathbb{Z}[[t]]$ divides both a and b, then $\lambda | n$. We write $\lambda\mu = k$, where $k \in \mathbb{Z}$ and $\mu \in \mathbb{Z}[[t]]$. The element λ is not divisible by an integer, and we may assume that the same is true for μ. Then $\lambda\mu$ must be equal to 1, because otherwise reduction modulo any prime divisor of k leads to a contradiction. Therefore, λ is invertible in $\widehat{\Lambda}$ and divides every element in $\widehat{\Lambda}$. □

Since $\widehat{\Lambda}$ is a principal ideal domain, the module $\widehat{M}$ is isomorphic to the following sum of cyclic modules:(1)

$$\widehat{M} \approx \bigoplus_{s=0}^{m-1} \widehat{\Lambda}/\gamma_s\widehat{\Lambda}, \quad \text{where } \gamma_s = \alpha_s/\alpha_{s+1}. \tag{10}$$

In particular, $\gamma_{s+1}|\gamma_s$ for every s.

Novikov numbers. From now on, $\widehat{M} = H_1(\overline{C_L})$. The corresponding element α_s will be called the *sth link polynomial*. The next theorem, which directly follows from the preceding arguments, provides computation of the Novikov numbers $\widehat{b}_1(L)$ and $\widehat{q}_1(L)$ in terms of these link polynomials.

Theorem 2.4. 1. *The Novikov number $\widehat{b}_1(L)$ is equal to the number of the polynomials α_s that are equal to zero.*

2. *The Novikov number $\widehat{q}_1(L)$ is equal to the number of the γ_s that are nonzero and nonmonic.* □

The case of knots. It turns out that for a knot K we always have $\widehat{b}_i(K) = 0$. For the first time, this was observed by A. Lazarev in [La]; for the proof, it suffices to note that the module $H_1(\overline{C_K})$ has a square matrix presentation; since the determinant of the matrix is the Alexander polynomial of the knot, it does not vanish.

(1)With the conventions $0/0 = 0$ and $\alpha_m = 1$.

§3. Minimal Morse functions

Let M be a closed manifold. A Morse map $f\colon M \to S^1$ is said to be *minimal* if for each k the number $m_k(f)$ is minimal on the class of all Morse functions homotopic to f. (Cf. the parallel definition [Sh, Definition 1.11] for real-valued Morse functions.)

Even in the case of real-valued Morse functions, minimal Morse functions do not always exist [Sh]. The problem is that, in general, the Morse numbers $m_k(f)$ cannot be minimized for all indices k simultaneously. In this section we show that in the case where $M = C_L$ minimal Morse functions do exist.

(Since the manifold C_L is not compact, the definition of a minimal Morse function should be modified slightly. We say that a Morse function $f\colon C_L \to S^1$ is *minimal* if it is regular and its Morse numbers $m_i(f)$ are minimal possible among all regular functions homotopic to f.)

Furthermore, we show that a minimal Morse function has critical points only of indices 1 and 2. (The latter fact is already suggested by the relations $\widehat{b}_0(L) = \widehat{b}_3(L) = \widehat{q}_3(L) = \widehat{q}_2(L) = \widehat{q}_0(L) = 0$.)

Notation. Let L be an oriented link with μ components and $f\colon C_L \to S^1$ a regular Morse function. We assume that $1 \in S^1$ is a regular value of f. (The general case differs from this one only by more complicated notation.)

We denote by $F\colon \overline{C_L} \to \mathbb{R}$ a lifting of f, and by T an open tubular neighborhood of L such that the restriction of f to the closure of T is standard.

Setting $X = S^3 \backslash T$, we let $\overline{X} \subset C_L$ be the inverse image of X. Then $\overline{X}$ is a noncompact manifold, and the boundary of X is an infinite cyclic covering of the union of μ tori, i.e., it is the disjoint union of μ copies of $S^1 \times \mathbb{R}$.

Consider the set $W = F^{-1}([0,1])$. This is a cobordism between two manifolds with boundary: $F^{-1}(0)$ and $F^{-1}(1)$. We choose a generator t of the (infinite cyclic) structure group of the covering so that $F(xt) < F(x)$. The total space $\overline{C_L}$ of the covering is the union of the cobordisms $W_n = t^n W$, $n \in \mathbb{Z}$.

Lemma 3.1. *There exists a regular Morse function $g\colon C_L \to S^1$ such that $m_p(g) \leq m_p(f)$ for each p and one of the regular level surfaces of g is connected.*

Proof. In the above notation, assume that $S = F^{-1}(0)$ is not connected. Let $m(f)$ denote the total number of critical points of f. We construct a new Morse function f_0 such that $m_p(f_0) \leq m_p(f)$ for each p and either

i) $m(f_0) < m(f)$, or

ii) $m(f_0) = m(f)$, and f_0 has a regular level surface S_0 such that the number of connected components of S_0 is strictly less than that of S.

Successively applying this construction, we prove the lemma.

Proceeding to the construction, we observe that at least one of the two inclusions

$$W \hookleftarrow S \hookrightarrow tW$$

induces a noninjective map in H_0. (Indeed, we recall that $\overline{C_L}$ is connected. Among the paths in $\overline{C_L}$ joining different connected components of $S = F^{-1}(0)$ we can choose one intersecting the minimal number of cobordisms W_n. Obviously, this minimal number must be equal to 1.)

Suppose, for instance, that the homomorphism $H_0(S) \to H_0(W)$ has nontrivial kernel. By the standard rearrangement argument (see [Mi2, §4]), we may assume that the function $F\colon W \to [0,1]$ is *ordered*, i.e., there are regular values

$$0 = \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 = 1$$

such that for each i all the points of index i of F are in $F^{-1}([\alpha_i, \alpha_{i+1}])$.

We choose a gradient-like vector field v for f. We require that v behave "nicely" in a tubular neighborhood T of L. Namely, we recall that we have a framing $\Phi\colon L \times D^2 \xrightarrow{\approx} T$ of T such that for each $x \in L$ the restriction of $f \circ \Phi$ to the punctured disk $x \times (D^2 \setminus \{0\})$ is given by the formula $(x, y) \mapsto y/|y|$. We require that the vector field $w = \Phi_*^{-1}(v)$ on $L \times D^2$ be tangent to $x \times D^2$ for every $x \in L$ and be given by the formula $w(y_1, y_2) = (y_1, -y_2)$ in the disk $x \times D^2 \approx D^2$. Thus, in particular, the integral curves of v in T are meridians. We also require that v satisfy the following transversality condition: the stable manifold of every critical point is transversal to the unstable manifold of any other critical point.

Consider the following inclusions

$$F^{-1}(0) \subset F^{-1}([0, \alpha_1]) \subset F^{-1}([0, \alpha_2]) \subset W.$$

The second (fourth) set in this sequence is obtained from the first (third) by attaching handles of index 0 (respectively, 2 and 3). Therefore, the first and third inclusions induce monomorphisms in H_0. Thus, the second inclusion homomorphism is not injective, whence it follows that there is a critical point p of index 1 such that the stable manifold $D(p, v)$ of p with respect to the flow v intersects $F^{-1}(\alpha_1)$ at two points lying in two different connected components of $F^{-1}(\alpha_1)$. There are two possibilities:

1. These two connected components are descended diffeomorphically by the gradient shift down to $F^{-1}(0)$.

2. One (or both) of them is the intersection of $F^{-1}(\alpha_1)$ with ascending disks of critical points of index 0.

In the first case, applying the rearrangement procedure, we push the critical point p downwards to obtain a new Morse function $\widetilde{F}\colon W \to [0, 1]$ with the following properties:

1) the vector field v is still a gradient-like vector field for $\widetilde{F}$;

2) the critical point p belongs to the smallest critical level of $\widetilde{F}$;

3) if ϵ is small, then the regular level surface corresponding to $\widetilde{F}(p) + \epsilon$ has $k - 1$ connected components (where k is the number of connected components of S).

In the second case, we observe that there is a v-trajectory joining a critical point of index 1 with a critical point of index 0, and the standard cancellation argument (see [Mi2, Theorem 8.1]) gives a Morse function $\widetilde{F}\colon W \to [0, 1]$ having two critical points less than F. In the case where the homomorphism $H_0(S) \to H_0(tW)$ is not injective, we apply the same argument to the function $-F$. Lemma 3.1 is proved. □

Lemma 3.2. *If f has a connected regular level surface S, then there is a Morse function g having no critical points of index 0 or 3 and such that S is one of the level surfaces of g. Furthermore, $m_1(g) \leq m_1(f)$ and $m_2(g) \leq m_2(f)$.*

Proof. The cobordism W is necessarily connected (by the Poincaré–Lefschetz duality). Thus, the standard cancellation procedure provides a Morse function $\widetilde{F}$ with $m_i(F) = m_i(\widetilde{F})$ for $i \geq 2$, $m_1(\widetilde{F}) \leq m_1(F)$, and $m_0(\widetilde{F}) = 0$.

Applying the same procedure to the Morse function $-F$, we get rid of the critical points of index 3. □

As an easy consequence, we get the existence of minimal Morse functions.

Theorem 3.3. *There exists a minimal regular Morse function $f\colon C_L \to S^1$. This Morse function has critical points of indices 1 and 2 only.*

Proof. We take a regular Morse function $f\colon C_L \to S^1$ such that $\min\{m_1(f), m_2(f)\}$ is minimal possible on the class of all regular Morse functions. Applying Lemmas 3.1

and 3.2, we obtain a regular Morse function g with

$$m_0(g) = m_3(g) = 0 \quad \text{and} \quad m_1(g) = m_2(g) \le \min\{m_1(f),\, m_2(f)\},$$

which is obviously minimal. $\square$

§4. Morse maps and free Seifert surfaces

For a topological space X, we put $h_1(X) = \operatorname{rk} H_1(X)$. A connected Seifert surface $F \subset S^3$ is said to be *free* if $\pi_1(S^3 \setminus F)$ is a free group (necessarily, with $h_1(F)$ generators).

Let $N(F) \approx F \times [0,1]$ be a tubular neighborhood of F. We observe that if F is free, then $S^3 \setminus N(F)$ is homeomorphic to a handlebody with $h_1(F)$ handles. Indeed, applying the Dehn lemma, we see that the space $S^3 \setminus N(F)$ is homeomorphic to some handlebody [H, pp. 56–58]. The rank of the first homology group of this handlebody is easily calculated by the Alexander duality and is equal to $h_1(F)$.

We see that F is a free Seifert surface if and only if $\partial N(F)$ induces a Hegaard splitting of S^3.

The minimal genus of a free Seifert surface S with $\partial S = L$ is called the *free genus* of the link L. In this section, we indicate some relationships between free Seifert surfaces and Morse functions on the complement of a link.

Here, we consider a special class of Morse functions. A regular Morse function $f\colon C_L \to S^1$ is said to be *moderate* if

1) $m_0(f) = m_3(f) = 0$,
2) all critical values corresponding to critical points of the same index coincide,
3) every regular level surface is connected.

An easy application of the results of the preceding section shows that moderate functions always exist.

Let $f\colon C_L \to S^1$ be a moderate function, and let $\theta_1, \theta_2 \in S^1$ regular values of f such that the critical values of index 1 (respectively, 2) are in the interval $]\theta_1, \theta_2[$ (respectively, $]\theta_2, \theta_1[$). To simplify notation, we assume that $\theta_1 = 0$ and $\theta_2 = \pi$. We set $m = m_1(f) = m_2(f)$. Let g_1 be the genus of $f^{-1}(0)$ and g_2 the genus of $f^{-1}(\pi)$.

Lemma 4.1. *$g_2 - g_1 = m$.*

Proof. The surface $f^{-1}(\pi)$ is obtained from $f^{-1}(0)$ by m surgeries of index 1. Therefore, $\chi(f^{-1}(\pi)) = \chi(f^{-1}(0)) - 2m$, i.e., $2 - 2g_2 = 2 - 2g_1 - 2m$. $\square$

Lemma 4.2. *The surface $f^{-1}(\pi)$ is a free Seifert surface for L.*

Proof. The complement of the tubular neighborhood of $f^{-1}(\pi)$ is $f^{-1}(S^1 \setminus [\pi - \epsilon, \pi + \epsilon])$. The complement is obtained from $f^{-1}(0)$ by attaching m handles of index 1 corresponding to the critical points of f of index 1 and m handles of index 1 corresponding to the critical points of $(-f)$ of index 1. $\square$

§5. Examples

5.1. A trivial link L_0 with μ components. Obviously, $\widehat{b}_1(L) = \widehat{b}_2(L) = \mu - 1$, and it is also easy to construct a Morse function $C_L \to S^1$ with $m_1(f) = m_2(f) = \mu - 1$. Thus, in this case estimates (1) are sharp.

5.2. There are knots for which estimates (1) are *not* sharp. Here is a family of such examples. Let K be any nontrivial knot, and let K' be the nontwisted Whitehead double of K, as defined, e.g., in [Ro, p. 39].

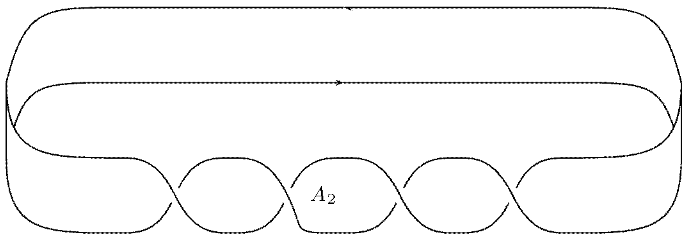

FIGURE 1

Since the Alexander polynomial of K is equal to 1, Theorem 2.4 shows that $\widehat{H}_*(K) = 0$. However, K is not fibered. Indeed, the genus of K is equal to 1. On the other hand, the degree of the Alexander polynomial of a fibered knot equals twice the genus of the knot.

The same argument works if we replace K' by any nontrivial knot with Alexander polynomial equal to 1.

It would be of interest to compute the Morse–Novikov number for these knots.

5.3. In this example, we show that changing the orientation of some of the components of a link can influence the Morse–Novikov number.

We orient the annulus $S^1 \times [-1, 1]$ arbitrarily and embed it in S^3 in such a way that the core of the annulus be unknotted and the linking coefficient of the boundary components (oriented as the boundary of the annulus) be equal to n. We obtain an oriented link in S^3. (In Figure 1, the reader can visualize the case where $n = 2$.) The standard Seifert matrix computation shows that $H_1(\overline{C_L}) \approx \Lambda/n(1-t)\Lambda$, whence $\widehat{q}_1(L) = 1$, so that $\mathcal{MN}(L) \geq 2$. (We believe that actually $\mathcal{MN}(L) = 2$.)

Reversing the orientation on one of the two components of the link, we obtain a torus link, which is fibered (see [Ro, p. 337]).

§6. CONNECTED SUM

One of the possible approaches to the computation of the invariant $\mathcal{MN}(L)$ is to study its behavior with respect to various natural operations on links or knots. The simplest operation of this type is the *connected sum*. We note that if $L = L_1 \# L_2$, then we have isomorphism

$$H_1(\overline{C_L}) \approx H_1(\overline{C_{L_1}}) \oplus H_1(\overline{C_{L_2}}) \tag{11}$$

(see [Ro, p. 179]). Therefore, the same is true for the Novikov homology.

Proposition 6.1. *There are knots with an arbitrarily large Morse–Novikov number.*

Proof. If K is a knot with nonmonic Alexander polynomial, then the decomposition of $\widehat{H}_1(K)$ into a sum of cyclic $\widehat{\Lambda}$-modules is nontrivial, whence $q_1(nK) \geq n$, where nK is a connected sum of n copies of K. □

Proposition 6.2. *If K_1 and K_2 are two oriented knots, then*

$$\mathcal{MN}(K_1 \# K_2) \leq \mathcal{MN}(K_1) + \mathcal{MN}(K_2).$$

Proof. Let $f_i \colon C_{K_i} \to S^1$ be minimal Morse functions. We may assume that both K_1 and K_2 contain the point $0 \in S^3$, that in a small disk $D(0, \epsilon)$ both K_1 and K_2 coincide

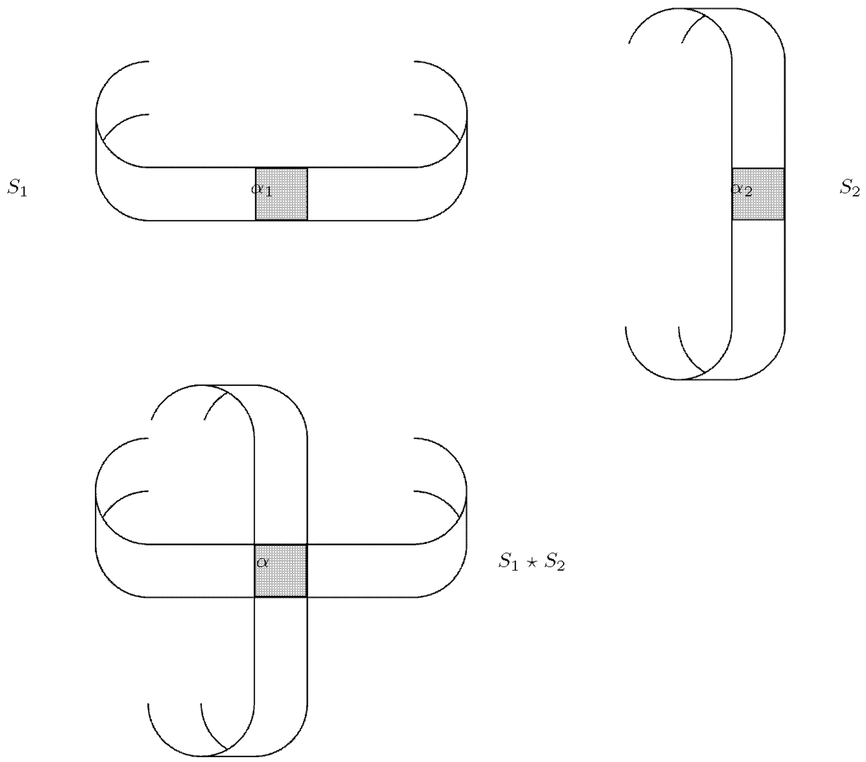

FIGURE 2

with one and the same segment of a straight line, and that the family of level surfaces of f_i in $D(0,\epsilon)$ consists of plane half-disks. Let $\Phi\colon S^3 \to S^3$ be the diffeomorphism defined by $\Phi(x) = I(-x)$, where I is the inversion with respect to the point 0 and the sphere $S(0,\epsilon/2)$.

The map Φ is isotopic to the identity; therefore, the knot $K_2' = \Phi(K_2)$ is isotopic to K_2. Now, attaching $K_2' \cap D(0,\epsilon)$ to $K_1 \cap \big(S^3 \setminus D(0,\epsilon)\big) + \pi$, we obtain a knot isotopic to $K_1 \# K_2$, and gluing together the functions $f_1|D(0,\epsilon)+\pi$ and $f_2 \circ \Phi|S^3 \setminus D(0,\epsilon)$, we obtain a map $f\colon C_K \to S^1$ with Morse number equal to the sum of those of f_1 and f_2.$\square$

We remark that a similar proposition is true for the case of links.

Another natural operation on links is the *Murasugi sum* [Ga]. The 4-*gon Murasugi sum* is defined for pairs (S_1,α_1) and (S_2,α_2), where the S_i are Seifert surfaces and the α_i are *patches* on them. Instead of reproducing the formal definition, we invite the reader to look at Figure 2. The surfaces S_1 and S_2 are glued together along the patches α_1 and α_2. The resulting surface is denoted by $S_1 \star S_2$; it contains a 4-gon $\alpha = \alpha_1 = \alpha_2$.

To formulate the corresponding conjecture, we let S be a Seifert surface and take $L = \partial S$. We define $\mathcal{MN}(S)$ as the minimal possible number of critical points of a Morse function $C_L \to S^1$ having S as a regular level surface.

Conjecture 6.3. *Let $S = S_1 \star S_2$. Then*

$$\mathcal{MN}(S_1 \star S_2) \leq \mathcal{MN}(S_1) + \mathcal{MN}(S_2). \tag{12}$$

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Section de Mathématiques, 2-4, rue du Lièvre, Case Postale 240, CH-1211, Genève 24, Switzerland

E-mail address: `Claude.Weber@math.unige.ch`

UMR 6629 CNRS, Université de Nantes, Département de Mathématiques, 2, rue de la Houssinière, 44072, Nantes Cedex France

E-mail address: `pajitnov@math.univ-nantes.fr`

Department of Mathematics, Clark University, Worcester, MA 01610, USA

E-mail address: `lrudolph@black.clarku.edu`

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Translated by THE AUTHORS