Surgery on the Novikov complex

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Short summary.

Let M be a closed connected smooth manifold with $\dim M \geq 6$, and $\xi: \pi_1(M) \to \mathbf{Z}$ be an epimorphism. Denote by Λ the group ring of $\pi_1(M)$, and let Λ_ξ^- be its Novikov completion. Let D_* be a free based finitely generated chain complex over Λ_ξ^-. Assume that $D_i = 0$ for $i \leq 1$ and $i \geq n-1$ and that D_* has the same simple homotopy type as the Novikov-completed simplicial chain complex of the universal covering $\widetilde{M}$. Let N be an integer.

We prove that D_* can be realized, up to the terms of Λ_ξ^- of degree $\geq N$ as the Novikov complex of a Morse map $f: M \to S^1$ belonging to ξ.

Applications to Arnold's conjectures and to the theory of fibering of M over S^1 are given.

Introduction.

A. Classical Morse theory.

Classical Morse theory is based on a geometric construction of a chain complex from a Morse function on a manifold. This construction is due to Morse, Thom, Smale (see [18], [40], see also [44]). It is our starting point and we begin with a short expository account.

Let $f: M \to \mathbf{R}$ be a Morse function on a smooth closed manifold, and let v be a gradient-like vector field for f. We shall abbreviate "gradient-like vector field" to "gradient". A gradient v will be called perfect if the stable manifold $D(p, v)$ of any critical point p is transversal to the unstable manifold $D(q, -v)$ of any other critical point q. A version of the Kupka-Smale theorem [28] gives that any gradient can be approximated by a perfect one — see for example [27, Prop.2.0] — so we shall assume v perfect. In this case the number of $(-v)$-trajectories joining critical points p, q of f with $\operatorname{ind} p = \operatorname{ind} q + 1$ is finite. Here and elsewhere in the Introduction we identify the trajectories which differ by a parameter change $t \mapsto t + C$. If we choose orientations for all the stable manifolds each trajectory obtains a sign $+$ or $-$ and the sum of these signs is denoted by $n(p, q)$. Denote now by $C_s(f, v)$ the free abelian group generated by the critical points of f of index s, and define a map $\partial_s: C_s(f, v) \to C_{s-1}(f, v)$ by $\partial_s p = \sum n(p, q) q$, where q runs through the critical points of f of index $s - 1$.

Theorem 0.1. *$\partial_{s-1} \circ \partial_s = 0$ and the homology $H_*(C_*(f, v))$ is isomorphic to $H_*(M)$.*

A proof can be extracted from [18]. Elementary algebra then gives the classical Morse inequalities (in a version due to Pitcher [29]):

Corollary 0.2. *The number $m_k(f)$ of critical points of index k of a Morse function f is not less than $b_k(M) + q_k(M) + q_{k-1}(M)$. (Here $b_k(M)$, $q_k(M)$, are resp. rank and torsion numbers of the group $H_k(M)$.)*

Smale's proof of the h-cobordism theorem gives also the following optimality result for these inequalities:

Theorem 0.3 (Smale [40]). *If M is simply connected and $\dim M \geq 6$ then there exists a Morse function $f: M \to \mathbf{R}$ such that $m_k(f) = b_k(M) + q_k(M) + q_{k-1}(M)$ for any k.*

If $\pi_1(M) \neq 0$ these inequalities are not generally optimal and to improve the situation one considers a modified construction of the Morse complex, which takes into account the homotopy classes of v-trajectories joining the critical points.

We shall formulate the results for Morse functions $f: W \to [a, b]$ where W is a connected cobordism, $\partial W = V_0 \sqcup V_1$, $f^{-1}(a) = V_0$, $f^{-1}(b) = V_1$, and the critical points of f are in $W \setminus (V_0 \cup V_1)$. Let $\widetilde{W}$ be the universal covering of W, and $\widehat{V}_0$, $\widehat{V}_1$ be the preimages of V_0, resp. V_1 in $\widetilde{W}$.

Let Δ be any smooth triangulation of W such that V_0, V_1 are subcomplexes. Denote by $C_*^{\Delta}(\widetilde{W}, \widehat{V}_0)$ the corresponding simplicial chain complex.

Theorem 0.4. *For any perfect gradient v of f there is a free finitely generated based chain complex $C_*(f, v)$ of right $\mathbf{Z}\pi_1(W)$-modules such that*

1) *the free generators of* $C_k(f,v)$ *are the critical points of* f *of index* k,

2) $C_*(f,v)$ *is simple homotopy equivalent to* $C_*^{\Delta}(\widetilde{W},\widehat{V}_0)$.

This theorem is essentially contained in [19], for a detailed proof see for example [27, Thm.A.5].

Now the analog of the optimality Theorem 0.3 is the following theorem.

Theorem 0.5. *Under the assumption of* 0.4 *suppose further that* $\dim W \geq 6$, *and that the inclusions* $V_0 \subset W \supset V_1$ *induce isomorphisms in* π_0, π_1.

Then each free finitely generated based right $\mathbf{Z}\pi_1(W)$*-complex* D_*, *simple homotopy equivalent to* $C_*^{\Delta}(\widetilde{W},\widetilde{V}_0)$, *and such that* $D_* = 0$ *for* $* \leq 1$ *or* $* \geq n-1$, *is the Morse complex of a Morse function* $f : W \to [a,b]$.

This theorem can be proved, following the Barden–Mazur–Stallings proof of the s-cobordism theorem, see [37]. It was used by V. V. Sharko [37 - 39] to obtain the analogues of Morse–Pitcher inequalities for the non-simply connected case.

B. Morse–Novikov inequalities for Morse maps $f : M \to S^1$.

We proceed now to Morse theory of maps $M \to S^1 = \mathbf{R}/\mathbf{Z}$, which is the subject of the paper. Let $f : M \to S^1$ be a Morse map, not homotopic to zero. Denote the homotopy class of f by $\xi \in [M, S^1] = H^1(M,\mathbf{Z})$. Denote by $\mathcal{P} : \overline{M}_\xi \to M$ the infinite cyclic covering such that $\overline{M}_\xi$ is connected and that $\mathcal{P}^*\xi = 0$. (Since $\xi \neq 0$ such a covering exists and is unique.) The map $f \circ \mathcal{P} : \overline{M}_\xi \to S^1$ admits a lifting $\overline{f} : \overline{M}_\xi \to \mathbf{R}$, which is equivariant in the sense that $\overline{f}(x \cdot t) = \overline{f}(x) - k$, where t is a generator of the structure group of the covering, and k a nonzero integer. We choose t in such a way that $k > 0$.

Now $\overline{f}$ is a Morse function, but the number of the critical points of $\overline{f}$ is infinite, and the set of the trajectories of a gradient which are to be taken into account can well be infinite. To construct an analog of the Morse complex in this situation one needs a procedure to count the trajectories. The construction is due to S.P.Novikov [21]. Choose a perfect gradient v for f. Its lifting to $\overline{M}_\xi$, which is a perfect $\overline{f}$-gradient, will be denoted by the same letter v. For each critical point x of f one chooses a lifting $\overline{x}$ to $\overline{M}_\xi$. Let x, y be two critical points of f, $\operatorname{ind} x = \operatorname{ind} y + 1$. The number of v-trajectories joining $\overline{x}$ and $\overline{y}t^i$ is finite for each fixed i. We sum them up with the signs due to orientations and get an integer $n_i(x,y)$. Set by definition $n(x,y) = \sum n_i(x,y)t^i$; that is an element of the ring $\widehat{L} = \mathbf{Z}[[t]][t^{-1}]$ of the Laurent power series in t, which is a sort of completion of the group ring $L = \mathbf{Z}[\mathbf{Z}] = \mathbf{Z}[t^{\pm 1}]$ of the structure group of the covering. Consider the free $\widehat{L}$-module $C_p(f,v)$, generated by the critical points x of f of index p. Set $\partial_p(x) = \sum y \cdot n(x,y)$, where y runs through the critical points of y of index $(p-1)$.

Theorem 0.6. $\partial_{p-1} \cdot \partial_p = 0$ *and the homology* $H_p(C_*(f,v))$ *of the resulting complex is isomorphic to* $H_p(\overline{M}_\xi) \underset{L}{\otimes} \widehat{L}$ *as a right* $\widehat{L}$*-module.*

This theorem was announced by S.P.Novikov in [21], see also [22].For a complete proof of a more general result see [27,Main Theorem].

Using the fact that $\widehat{L}$ is a principal ideal domain, we obtain:

Corollary 0.7 (Novikov [21]). *The number* $m_p(f)$ *of critical points of index* p *of a not homotopic to zero Morse map* $f : M \to S^1$ *is not less than* $b_p(M,\xi) + q_p(M,\xi) +$

$q_{p-1}(M,\xi)$, *where* $\xi \in H^1(M,\mathbf{Z}) = [M,S^1]$ *is the homotopy class of* f, *and* $b_p(M,\xi)$, $q_p(M,\xi)$ *are, respectively, rank and torsion numbers of the* $\widehat{L}$*-module* $H_p(\overline{M}_\xi) \underset{L}{\otimes} \widehat{L}$.

These inequalities are optimal, as it was proved by M. Farber.

Theorem 0.8 (Farber [9]). *Let* M *be a closed manifold* $\dim M \geq 6$, $\pi_1(M) \cong \mathbf{Z}$ *and let* $\xi \in H^1(M,\mathbf{Z})$ *be any nonzero cohomology class.*

Then there is a Morse map $f : M \to S^1$, *belonging to* ξ *such that for every* p *we have* $m_p(f) = b_p(M,\xi) + q_p(M,\xi) + q_{p-1}(M,\xi)$.

The interesting feature of Novikov theory is that these inequalities are optimal even in the case of free abelian fundamental group for "generic" cohomology classes. This almost never happens to Morse functions (see [37], [38]). The following theorem was proved in [23] and [24] under some restrictions on the homotopy type of M; these restrictions can be removed by the results of the present paper.

A subgroup L' of a free abelian finitely generated group L will be called a *hyperplane* if L' is a direct summand of L and $\mathrm{rk}\ L' = \mathrm{rk}\ L - 1$.

Theorem 0.9 (Thm. 7.25 of the present paper). *Let* M *be a closed connected manifold,* $\dim M \geq 6$, $\pi_1(M) \cong \mathbf{Z}^m$.

Then there is a finite set of hyperplanes $\{\Delta_i\}_{i\in I}$ *in* $H^1(M,\mathbf{Z})$ *such that for a nonzero* $\xi \in H^1(M,\mathbf{Z}) \setminus \bigcup_i \Delta_i$ *there is a Morse map* $f : M \to S^1$, *such that*

$$m_p(f) = b_p(M,\xi) + q_p(M,\xi) + q_{p-1}(M,\xi) \quad \textit{for every } 0 \leq p \leq n.$$

The numbers $b_p(M,\xi), q_p(M,\xi)$ *depend only on the homotopy type of* M *and the connected component of* ξ *in* $H^1(M,\mathbf{R}) \setminus \bigcup_i (\Delta_i \otimes \mathbf{R})$.

The following corollary of this theorem is a partial generalization for the higher dimensions of Thurston's theorem on nonsingular 1-forms in 1-dimensional cohomology classes on 3-manifolds [42].

Corollary 0.10. *Let* M *be a connected closed manifold,* $\dim M \geq 6$, $\pi_1(M) = \mathbf{Z}^m$.

Then there exists a finite number of hyperplanes $\Delta_i \subset H^1(M,\mathbf{Z}) = \mathbf{Z}^m$ *such that for each nonzero class* $\gamma \in H^1(M,\mathbf{Z}) \setminus \cup_i \Delta_i$ *its representability by a fibration over a circle depends only on its connected component in the complement* $H^1(M,\mathbf{R}) \setminus \cup_i (\Delta_i \otimes \mathbf{R})$.

C. Statement of the main result.

We recall first the construction of the Novikov ring (see [21],[27, p.7-9],[35]). Let G be a group. Denote $\mathbf{Z}G$ by Λ. Denote by $\widehat{\widehat{\Lambda}}$ the abelian group of all the linear combinations of the form $\lambda = \sum_{g\in G} n_g g$ where $n_g \in \mathbf{Z}$, and the sum may be infinite. For $\lambda \in \widehat{\widehat{\Lambda}}, \lambda = \sum_{g\in G} n_g g$ denote by $\mathrm{supp}\,\lambda$ the subset of G consisting of all g such that $n_g \neq 0$. Now if $\xi : G \to \mathbf{Z}$ is a nonzero homomorphism, we denote by Λ_ξ^- the subset of $\widehat{\widehat{\Lambda}}$, consisting of such $\lambda \in \widehat{\widehat{\Lambda}}$ that for every $c \in \mathbf{Z}$ the set $\mathrm{supp}\,\lambda \cap \xi^{-1}(]-\infty, c])$ is finite. It is not difficult to verify that Λ_ξ^- has a natural ring structure, and the inclusion $\Lambda \hookrightarrow \Lambda_\xi^-$ is a ring homomorphism. If, for example, $G = \mathbf{Z}$ then Λ_ξ^- is easily identified with $\widehat{L} = \mathbf{Z}[[t]][t^{-1}]$, where t is the generator of $\mathbf{Z}$, satisfying $\xi(t) < 0$.

In the group of units $(\Lambda_\xi^-)^*$ there is a subgroup

$$U_\xi^- = \{\lambda \in \Lambda_\xi^- | \lambda = \varepsilon g(1+P), \text{ where } \varepsilon = \pm 1, g \in G, \ P \in \Lambda_\xi^-, \operatorname{supp} P \subset \xi^{-1}(]-\infty, -1])\}$$

We denote by $\mathrm{Wh}(G,\xi)$ the quotient of $K_1(\Lambda_\xi^-)$ by the image in $K_1(\Lambda_\xi^-)$ of U_ξ^-.

We introduce now a convention, which will be valid throughout all the paper: the term "*complex*" will mean "finite free nonnegative chain complex of finitely generated free right modules with a fixed basis". The base ring will be normally clear from the context, if not we give it explicitly (usually it is the Novikov ring).

We say that two complexes over Λ_ξ^- are simple homotopy equivalent if there exists a homotopy equivalence between them with torsion vanishing in $\mathrm{Wh}\,(G,\xi)$.

Theorem 0.11 ([25],[27, Thm.2.2]). *Let M be a connected closed manifold, $\pi_1(M) = G, \xi : G \to \mathbf{Z}$ be a nonzero homomorphism. Let $f : M \to S^1$ be a Morse map such that $f_* : \pi_1(M) \to \mathbf{Z}$ equals to ξ. Let v be a perfect gradient for f.*

Then there exists a complex $C_*(f,v)$ over Λ_ξ^- such that 1) *$C_p(f,v)$ is generated by the critical points of f of index p.*

2) *$C_*(f,v)$ is simple homotopy equivalent to $C_*^\Delta(\widetilde{M}) \underset{\Lambda}{\otimes} \Lambda_\xi^-$.*

(Here Δ stands for a smooth triangulation of M, and $C_*^\Delta(\widetilde{M})$ stands for the simplicial chain complex of $\widetilde{M}$, associated with Δ.)

This complex is called Novikov complex.

For an integer N and elements $a, b \in \Lambda_\xi^-$ we say that a is N-equivalent to b, if $\operatorname{supp}(a-b) \subset \xi^{-1}(]-\infty, -N])$. We say that two complexes D_*, D'_* are N-equivalent, if there is a one-to-one correspondence between their bases and the corresponding matrix entries $\partial_{ij}^{(p)}$, $\partial_{ij}'^{(p)}$ of differentials $\partial^{(p)} : D_p \to D_{p-1}$ and $\partial'^{(p)} : D'_{(p)} \to D'_{(p-1)}$ are N-equivalent. In this case we write $C_* \underset{N}{\sim} D_*$.

Here is the main result of the present paper.

Theorem 0.12. *Let M be a closed connected manifold, $\pi_1(M) = G, \xi : G \to \mathbf{Z}$ be an epimorphism. Assume that $\operatorname{Ker}\xi$ is a finitely presented group. Let $\dim M \geq 6$. Let D_* be a complex over Λ_ξ^-, simple homotopy equivalent to $C_*^\Delta(\widetilde{M}) \underset{\Lambda}{\otimes} \Lambda_\xi^-$, and such that $D_i = 0$ for $i \leq 1$ and $i \geq \dim M - 1$. Let N be an integer.*

Then there exists a Morse map $f : M \to S^1$ belonging to ξ, and a perfect gradient v for f, such that $C_*(f,v)$ is N-equivalent to D_*.

This theorem follows from Theorem 3.7 of the present paper. Before we present the contents of the paper section by section, we explain briefly the strategy of the proof.

There are at present several results concerning the number of critical points of Morse maps $f : M \to S^1$. One usually seeks a function having the minimal possible number of critical points in given homotopy class (which number is equal to zero in the cases considered by Stallings [41], Browder–Levine [3], Farrell [10] and Siebenmann [34], and in the cases, considered by Farber [9] and myself [23] is given by the Novikov numbers). The method proposed in [41], and developed for the high dimensions in [3] is as follows. One picks up a submanifold $N \subset M$ of codimension 1, dual to $\xi = [f]$ (we assume that ξ is an epimorphism). The cyclic covering $\overline{M}_\xi$ is now the union of translates of a fundamental domain W, each of which has two boundary components diffeomorphic to N.

First step: one makes surgery over N inside M, so as to obtain in the end a cobordism W with the desired homotopy properties (h-cobordism in [3], [10], [34], a cobordism with given homology in [9], a cobordism of a given homotopy type in [23]). To do this there is a obstruction in $\mathrm{Nil}\, \mathbf{Z}H$ in the case of [10]. Due to the methods, which I used in proofs in [23], I could find such an N only under some restrictions on the homotopy type of M.

Second step: one seeks a Morse function on the cobordism W, constant on the components of boundary, with the desired number of critical points. In the case of [10] there is an obstruction in $\mathrm{Wh}\, H$ to do this. In the cases of [3], [9] and [23] it is always possible, mainly because of vanishing of all the K-theoretic groups.

The method of the present paper is, in a sense, more direct. We start with an arbitrary Morse function $f_0 : M \to S^1$, $f_0 \in \xi$, which defines the Novikov complex $C_*(f_0, v_0)$. The complex D_*, which we want to realize as $C_*(f, v)$ (up to degree N) is simple homotopy equivalent to $C_*(f_0, v_0)$. By the algebraic simple homotopy type theory the complex D_* can be obtained from $C_*(f_0, v_0)$ by a finite number of the basic operations such as: adding and canceling of a trivial direct summand $\{0 \leftarrow \Lambda_\xi^- \xleftarrow{\mathrm{id}} \Lambda_\xi^- \leftarrow 0\}$, changing the basis by the elementary transformations. We realize each step (up to degree N) on the level of Morse maps f and gradients v, and that proves the theorem.

D. The contents of the paper.

The first section is purely algebraic. We prove there that if C_* and D_* are two complexes of the same simple homotopy type then one can be obtained from the other by a sequence of elementary transformations, which do not increase the length of the complexes. This is implicit in [5], see also [37 - 39]. We believe that our proof is shorter.

In §2 we collected all the auxiliary propositions about Morse functions on cobordisms. We prove that if a gradient v of a function on a cobordism is almost good (weakening of "perfect", see the definition in §2.B), the union of the stable manifolds of all the critical points of indices $\leq s$ is compact and in a sense stable under small perturbations of v (Lemma 2.11). For an almost good gradient we give several notions of incidence coefficients (with values in the group ring of the fundamental group) for two critical points p, q of adjacent indices. The second main result of §2 is that all these coefficients are equal (see Corollary 2.30).

In §3 we formulate (after some necessary definitions) Theorem 3.7, which implies Theorem 0.12. Then we deduce this theorem from the results of Sections 4,5,6.

As usual, elimination of critical points demands first the realization of the algebraic incidence coefficient geometrically. That is the contents of §4. The main result of this section is Theorem 4.1. One of technical difficulties to overcome is that we must perform the Whitney trick in such a way that the other incidence coefficients do not change. We have placed the exposition of the version of Whitney trick, which we need, into Appendix 2 (§9). It is close to [18,§6].

In §5 we do the cancellation and addition of a trivial direct summand. In §6 we deal with an elementary change of basis.

In §7 we present some applications of the main theorem. In 7.D we deal with the fibering obstructions. Theorem 0.12 implies that in order that ξ contains a fibering over a circle it is necessary and sufficient that $C_*^{\triangle}(\widetilde{M}) \underset{\Lambda}{\otimes} \Lambda_\xi^-$ be simple homotopy equivalent to zero (under the assumptions that $\dim M \geq 6$ and $\operatorname{Ker} \xi$ is finitely presented). That is: α) $C_*^{\triangle}(\widetilde{M}) \underset{\Lambda}{\otimes} \Lambda_\xi^-$ is acyclic and β) the torsion $\tau(C_*^{\triangle}(\widetilde{M}) \underset{\Lambda}{\otimes} \Lambda_\xi^-) \in \mathrm{Wh}\,(G, \xi)$ vanishes.

Now the Farrell–Siebenmann fibering obstruction theory gives another fibering obstruction, which belongs to $\mathrm{Wh}\,(G)$ and is defined if $\overline{M}_\xi$ is finitely dominated.

In the paper [26] the problem of comparing two obstructions was posed. This problem was solved by A. Ranicki [31]. We present a part of his results in §7. In 7.G we present an application to a version of Arnold's conjecture, namely to Lagrangian intersections in the cotangent bundle. We recall here the necessary definitions.

Let M be a closed manifold. Denote by λ the standard 1-form on T^*M then $\omega = d\lambda$ determines the standard symplectic structure on T^*M. Let φ_t be a symplectic isotopy of M then the 1-form $\alpha_t = (\varphi_t \mid M)^*(\lambda)$ is closed. A version of Arnold's conjecture (see conjecture $1'$ of [36]) says:

(C) If $\varphi_1(M)$ is transversal to M then the number of points of intersection of $\varphi_1(M)$ with M is not less then the minimal possible number of zeros of a Morse form, cohomological to α_1.

In the paper [36] the conjecture (C) is proved, using the method of [17], in the case $\dim M \geq 6$, $\pi_1(M) = \mathbf{Z}$, $[\alpha_1] \neq 0$.

The main theorem of the present paper allows us to prove the conjecture (C) under the following assumption (A): $n = \dim M \geq 6$, $\pi_1(M) = H \times \mathbf{Z}$, where H is a finitely presented group and $[\alpha_1]$ is up to a nonzero constant, the projection onto the second factor.

In §8 we gathered some auxiliary propositions on vector fields and their trajectories.

E. Some terminology and conventions.

For $a > 0$ the manifold $\mathbf{R}/a\mathbf{Z}$ will be denoted by $S^1_{(a)}$; $S^1_{(1)}$ is denoted by S^1.

Manifolds, functions, vector fields etc. are supposed to be of class C^∞, unless the contrary is said explicitly. When a notion concerning Morse maps $M \to S^1$ (or gradients of such maps) is similar to the corresponding notion for Morse functions $M \to \mathbf{R}$ we often leave it without special definition. In particular, it holds for definitions of regular Morse function, and for the definitions of §2.B.

We recall that "complex" means "free finite finitely generated nonnegative chain complex C_* of right R-modules along with a fixed basis in each C_i".

For $r > 0$ and $a \in \mathbf{R}^k$ we denote by $B^k(a, r)$ and by $D^k(a, r)$ the open and, respectively, closed disc of radius $r > 0$ and of center a in $\mathbf{R}^k$. The image of a smooth embedding of a closed disc (resp. open disc, sphere) is called *embedded disc* , resp. *embedded open disc*, *embedded sphere.*

If X is a topological space then $H_*(X)$ and $H^*(X)$ stand for homology, resp. cohomology with integer coefficients.

The letter t is used in this paper in two ways (both are standard): in §§2,8,9 it stands for a real parameter and in the Introduction and in §§3-7 it stands for a generator of the deck transformation group of a cyclic covering.

The end of a proof or the absence of the proof is marked by $\square$. The end of a remark, definition, or notation is marked by $\triangle$.

F.Remarks and acknowledgments.

This paper grew from the paper [23], where I proved the optimality result (Theorem 0.9) for the manifolds M with $\pi_2(M) = \pi_3(M) = \pi_4(M) = 0$. That suggested that there should be a version of 0.5 for maps to a circle.

There exists a project of F.Latour, concerning the problem of representing an arbitrary cohomology class $\xi \in H^1(M, \mathbf{R})$ (not necessarily rational) by a non-singular 1-form. (The results would intersect with these of the present paper on the Farrell's case.) It was announced in F.Latour's talk in Orsay in the summer of 1990. In this talk the Whitehead groups $Wh(G, \xi)$ were first introduced. His proofs seem to use special completion-type homotopy groups $\pi_*(M, \xi)$.

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§1. Simple homotopy type of complexes of a fixed length.

In this section our terminology and conventions are basically those of [6,Chapter 3] (except that our modules are right). For our further applications we must work with modules with a fixed basis, rather than with modules with preferred class of bases, as in [6]. So we give (and recall) some definitions.

Definition 1.1. Let R be a ring with unity such that $R^m \cong R^n$ implies $m = n$. Let G be a subgroup of the group R^* of units of R. A matrix in $GL(R)$ is *elementary*, if it is elementary or diagonal with only one entry, different from 1, which belongs to G. The subgroup E_G of $GL(R)$, generated by all the G-elementary matrices, is normal, and the quotient is denoted by $K_G(R)$. A free finitely generated right R-module E along with a fixed basis in it will be called *bf-module.* An *identification* is an isomorphism between two bf-modules, which preserves the bases. A *G-equivalence* is an isomorphism between two bf-modules such that $\tau(A)\epsilon K_G(R)$ vanishes, where A is the matrix of the isomorphism. A change of basis in a bf-module E is called G-elementary (resp. G-simple), if its matrix is G-elementary (resp. belongs to E_G). $\triangle$

We shall always assume that $(-1) \in G$. In this case it is easy to see that $K_G(R) \cong K_1(R)/Im(G \subset R^* = GL(1,R) \to K_1(R))$. The following lemma is obvious.

Lemma 1.2. *Let $f: E \to F$ be a G-equivalence of two bf-modules. Then there exists a bf-module H and a G-simple change of basis in $F \oplus H$ such that $f \oplus \mathrm{id}: E \oplus H \to F \oplus H$ is an identification (with respect to the new basis in $F \oplus H$).* $\square$

We are going to prove an analogue of this lemma for chain complexes.

Definitions 1.3. A chain complex C_* is called *n-bounded*, if $C_i = 0$ for $i < 0$ and for $i > n$. A *chain f-complex* is a free finitely generated chain complex of right R-modules. A *complex* is a chain f-complex C_* along with a fixed basis in each C_i. The r-th component of a chain map $f: C_* \to D_*$ will be noted $f_r: C_r \to D_r$. For $n \in \mathbf{Z}$ and an R-module H we denote by $Tv_*(n,H)$ the chain complex $\{0 \leftarrow H \overset{\mathrm{id}}{\leftarrow} H \leftarrow 0\}$, concentrated in dimensions n, $n+1$. If H is a bf-module we put a bf-structure on $Tv_*(n,H)$, introducing the same basis in both copies of H. The result is denoted by $Triv_*(n,H)$. A finite direct sum of bf-complexes of the type $Triv_*(n,H)$ is called *trivial complex.* The f-complex, obtained from a trivial complex, by forgetting the bases, will be also called trivial; in each case the meaning of the term "trivial" will be clear from the context.

Two complexes are *identical*, if there exists an isomorphism between them (called *identification*), which preserves the bases. An isomorphism f of two complexes is called *GI-simple*, if for each i the torsion of f_i vanishes in $K_G(R)$.A homotopy equivalence of two complexes is called *G-simple* if its torsion vanishes in $K_G(R)$. A *G-simple change of basis* in a complex C_* is a set of G-simple changes of basis in the components C_i of C_*.

Let $f: C_* \to D_*$ be a map of chain complexes. A map $\Phi: C_* \oplus \Gamma_* \to D_* \oplus \Delta_*$ is called *thickening* of f if 1) Γ_*, Δ_* are trivial, 2) the composition $p \circ \Phi \circ i$ is equal to f, where $i: C_* \to C_* \oplus \Gamma_*$, $p: D_* \oplus \Delta_* \to D_*$ are the obvious embedding and projection. A thickening is called *n-bounded* if all the considered chain complexes are n-bounded. $\triangle$

The following lemma contains two properties of torsion, which are easy consequences of Milnor's definition of torsion (see [6, § 16]).

Lemma 1.4. 1) *If $f : C_* \to D_*$ is a chain map of complexes, and Φ is a thickening of f then $\tau(\Phi) = \tau(f)$.*

2) *If $f : C_* \to D_*$ is a chain isomorphism, then $\tau(f) = \sum_i (-1)^i \tau(f_i)$.* □

The following proposition is the main aim of the section.

Proposition 1.5. *Let $f: C_* \to D_*$ be a G-simple homotopy equivalence between two n-bounded complexes, where $n \geq 1$. Then there exists an n-bounded isomorphic thickening $I: C_* \oplus \Gamma_* \to D_* \oplus \Delta_*$, which becomes an identification after a G-simple change of basis in $D_* \oplus \Delta_*$.*

Remark 1.6. The proposition says that if two complexes C_*, D_* are n-bounded and G-simple homotopy equivalent then (up to identification) one can be obtained from the other by adding and collapsing trivial direct summands together with G-elementary changes of basis. Further, we can assume that all the intermediate complexes are n-bounded. △

We shall give the proof in two steps.

A. Homotopy type.

In this part we deal with chain complexes without fixed basis.

Proposition 1.7. *Let $g: C_* \to D_*$ be a homotopy equivalence of two n-bounded chain f-complexes.*

Then there exists an n-bounded isomorphic thickening $J: C_* \oplus F_* \to D_* \oplus F'_*$.

Note. The case $n \leq 0$ is trivial, so we assume $n \geq 1$. △

Lemma 1.8. *Under the hypotheses of 1.7 there is an n-bounded trivial complex Q_*, and a monomorphism $h: C_* \to D_* \oplus Q_*$ such that the quotient chain complex F_* is trivial, and the composition of the projection $D_* \oplus Q_* \to D_*$ with h equals g.*

Proof. Denote by $\overline{Q}_*$ the direct sum of the chain f-complexes $Tv_*(i, C_i)$ where $0 \leq i \leq n-1$. It is an n-bounded trivial chain f-complex. Now we define a chain map $\overline{h}: C_* \to D_* \oplus \overline{Q}_*$ by the following formulas (note that $\overline{Q}_0 = C_0$, $\overline{Q}_i = C_i \oplus C_{i-1}$ for $0 < i < n$, and $\overline{Q}_n = C_{n-1}$):

If $c \in C_0$ then $\overline{h}(c) = (g(c), c)$;
if $c \in C_i$ for $0 < i < n$ then $\overline{h}(c) = (g(c), c, \partial c)$;
if $c \in C_n$ then $\overline{h}(c) = (g(c), \partial c)$.

One sees easily that $\overline{h}$ is a chain homotopy equivalence which is a split monomorphism in the dimensions $\leq n-1$. This implies also that $\overline{h}_n: C_n \to D_n \oplus \overline{Q}_n$ is a monomorphism. (Indeed if $\overline{h}_n(c) = 0$ then c is a cycle with the zero g_n-image, therefore a boundary, therefore zero). Denote the quotient $D_* \oplus \overline{Q}_* / \operatorname{Im} \overline{h}$ by P_*. The modules P_i are free for $0 \leq i \leq n-1$; this permits to split P_* as a direct sum of n-bounded chain complexes of the form $\{0 \leftarrow H \xleftarrow{r} H \leftarrow 0\}$, where H is stably free and r is an isomorphism. Therefore there is a trivial n-bounded Γ_* such that $P_* \oplus \Gamma_*$ is trivial. We set $Q_* = \overline{Q}_* \oplus \Gamma_*$, $h = (\overline{h}, 0)$ and the lemma is proved. □

Proof of 1.7. By 1.8 we get an exact sequence $0 \to C_* \xrightarrow{h} D_* \oplus Q_* \to F_* \to 0$ which splits, since F_* is free and acyclic ([6, 13.2]). Therefore we get an isomorphism $J\colon C_* \oplus F_* \to D_* \oplus Q_*$; the property $p \circ J \circ i = g$ is clear from the properties of h. $\triangle$

B. Simple homotopy type.

Remark 1.9. If Proposition 1.5 holds for some n-bounded thickening of $f\colon C_* \to D_*$ then it holds for f itself. $\triangle$

Lemma 1.10. *Proposition* 1.5 *is true if* f *is a* GI*-simple isomorphism.*

Proof. For each C_i we choose a bf-module H_i and a G-simple change of basis in $D_i \oplus H_i$ such that $f_i \oplus \mathrm{id}: C_i \oplus H_i \to D_i \oplus H_i$ is an identification with respect to the new basis. Denote by Δ_* the direct sum of the complexes $\mathrm{Triv}_*(i, H_i)$ for $0 \leq i \leq n-1$ with $\mathrm{Triv}_*(n-1, H_n)$. It is now clear that the thickening $f \oplus \mathrm{id}: C_* \oplus \Delta_* \to D_* \oplus \Delta_*$ satisfies the conclusions of 1.5 for a suitable G-simple change of basis in $D_* \oplus \Delta_*$. $\square$

Lemma 1.11. *Proposition* 1.5 *is true if* f *is an isomorphism.*

Proof. By 1.9, 1.10 it suffices to construct an n-bounded thickening of f, which is a GI-simple isomorphism. Choose a bf-module H and an isomorphism $a\colon H \to H$ such that $\tau(a) = -\tau(f_0)$ in $K_G(R)$. Denote $\mathrm{Triv}_*(0, H)$ by L_*. Consider the automorphism α of $\mathrm{Triv}_*(0, H)$, given by $\alpha_0 = \alpha_1 = a$. Consider the map $f \oplus \alpha\colon C_* \oplus L_* \to D_* \oplus L_*$. We have: $\tau(f_0 \oplus \alpha_0) = 0$, $\tau(f_1 \oplus \alpha_1) = \tau(f_1) - \tau(f_0)$. Next we add a trivial summand to kill $\tau(f_1 \oplus \alpha_1)$; we continue in the same way, and after the n-th step we get an isomorphism of n-bounded complexes $A\colon C_* \oplus \Gamma_* \to D_* \oplus \Delta_*$, where $\tau(A_i) = 0$ for $0 \leq i \leq n-1$ and $\tau(A_n) = \tau(f_n) - \tau(f_{n-1}) + \ldots + (-1)^n \tau(f_0)$. Now 1.4.2) implies $\tau(A_n) = 0$. $\square$

Proof of 1.5. By 1.7 there is an n-bounded thickening J of f which is an isomorphism. Now $\tau(J) = 0$ by 1.4.1) and we apply 1.11. $\square$

§2. Stable manifolds of gradients of Morse functions on cobordisms.

A. Compactness properties for vector fields of gradient type.

Let W be a compact cobordism, $\partial W = V_0 \sqcup V_1$. Denote $W \setminus \partial W$ by $\overset{\circ}{W}$. For a vector field v on W we denote by $S(v)$ the set of zeros of v. We assume here the terminology of §8, in particular, of §8.C.

Definition 2.1. A C^∞ vector field v on W is *of gradient type*, if

1) v is transversal to the boundary and points inward W on V_0 and outward W on V_1;

2) For each $x \in W$ the v-trajectory, starting at x, satisfies one of two conditions: α) it reaches V_1; β) for every open neighbourhood U of $S(v)$ there exists t_0 such that for $t > t_0$ we have $\gamma(x,t;v) \in U$. $\triangle$

In the case β) we say that the trajectory hangs on $S(v)$. Note that we do not demand in the definition that the trajectory, hanging on $S(v)$, converges to a point of $S(v)$.

Let X be a topological space, and $C \subset X$. Recall ([7,p.34]) that a family of neighbourhoods $\{V_i\}_{i\in I}$ of C is called *fundamental system of neighbourhoods of C* if for every open set $U \supset C$ there is $i \in I$ such that $V_i \subset U$.

Assume,that X and C are compact, and that $I =]\delta, \infty[$, where $\delta \geq 0$, and that $\delta < \theta < \theta'$ implies $\overline{V_\theta} \subset V_{\theta'}$, and that $\bigcap_{\theta>\delta} V_\theta = C$. Then it is easy to verify that $\{V_\theta\}_{\theta\in I}$ is a fundamental system of neighbourhoods of C. (Indeed, let U be a neighbourhood of C. Then $X \setminus U \subset X \setminus C = \bigcup_{\theta>\delta} (X \setminus V_\theta) = \bigcup_{\theta>\delta} (X \setminus \overline{V_\theta})$, which implies $U \supset \overline{V_\theta}$ for some $\theta > \delta$.)

For example, let X be a compact metric space, A be any subset of X, and $\delta > 0$. Denote by $B_\delta(A)$, resp. $D_\delta(A)$, the set of such $x \in X$ that $d(x,A) < \delta$, resp. $d(x,A) \leq \delta$. Since the function $x \mapsto d(x,A)$ for a fixed A is continuous ([7,(3.11.8)]), $B_\delta(A)$ is open, and $D_\delta(A)$ is closed. If $0 < \delta < \theta$ we have $\overline{B_\delta(A)} \subset D_\delta(A) \subset B_\theta(A)$. We have also $D_\delta(A) = \bigcap_{\theta>\delta} B_\theta(A)$. Therefore, the sets $B_\theta(A)$ for $\theta > \delta$ form a fundamental system of neighbourhoods of $D_\delta(A)$.

Definition 2.2. Choose a riemannian metric on W. For a vector field v of gradient type, and $\delta > 0$ we denote by $S_\delta(v)$, resp. $\overline{S}_\delta(v)$ the sets $B_\delta(S(v))$, resp. $D_\delta(S(v))$.

For any $A \subset W$ we denote $B_\delta(A,v)$ and $D_\delta(A,v)$ the union of all $(-v)$-trajectories, starting at a point of $B_\delta(A)$, resp. $D_\delta(A)$. That is

$$B_\delta(A,v) = \{\gamma(x,t;-v) | x \in B_\delta(A),\ t \geq 0\};$$

$$D_\delta(A,v) = \{\gamma(x,t;-v) | x \in D_\delta(A),\ t \geq 0\}.$$

The sets $B_\delta(S(v),v)$ and $D_\delta(S(v),v)$ will be denoted $B_\delta(v)$, resp. $D_\delta(v)$. We denote by $K(v)$ the set of all $x \in W$ such that the v-trajectory γ, starting at x, hangs on $S(v)$. $\triangle$

Lemma 2.3. 1) *$B_\delta(A,v)$ is open for any A.*

2) *$D_\delta(v)$ is compact and the sets $B_\theta(v)$ for $\theta > \delta$ form a fundamental system of neighbourhoods of $D_\delta(v)$.*

3) *$K(v)$ is compact and the sets $B_\theta(v)$ for $\theta > 0$ form a fundamental system of neighbourhoods of $K(v)$.*

Proof. 1) follows from Propositions 8.10 and 8.11. To prove 2) note that (by 2.1.2)) the set $W \setminus D_\delta(v)$ consists of such points $x \in W$ that the v-trajectory, starting at x, reaches V_1 without intersecting the compact $\overline{S}_\delta(v)$. Therefore, $D_\delta(v)$ is closed and $D_\delta(v) = \bigcap_{\theta>\delta} B_\theta(v)$. Furthermore, $\overline{B_\theta(v)} \subset D_\theta(v) \subset B_{\theta'}(v)$ for $\theta' > \theta$. Therefore we obtain 2). 3) is proved similarly. $\square$

Lemma 2.4. *Let v be a gradient-type vector field on W, $\delta > 0$, A be any subset of W, Q be a compact in $B_\delta(A, v)$, U be an open neighbourhood of $D_\delta(v)$, V be an open neighbourhood of $K(v)$.*

Then for any gradient-type vector field w sufficiently close to v we have: 1) $Q \subset B_\delta(A, w)$, 2) $U \supset D_\delta(w)$, 3) $V \supset K(w)$.

Proof. 1) Let $x \in Q$. There is $T \geq 0$ such that $\gamma(x, T; v) \in B_\delta(A)$. Applying 8.10 (in the case when γ reaches the boundary at T) or 8.11 (in the case $\gamma(x, T; v) \in \overset{\circ}{W}$), we see that for every y from a neighbourhood $A(x)$ of x in W and every w from a neighbourhood $\mathcal{V}$ of v in $Vect^\infty(W)$ there is $T(y, w) \in \mathbf{R}$ such that $\gamma(y, T(y, w); w) \in B_\delta(A)$. Choose from $\{A(x)\}$ a finite covering of Q; then the intersection of the corresponding neighbourhoods $\mathcal{V}$ is the neighbourhood sought.

2) By 2.3.2) there exists $\theta > \delta$ such that $D_\theta(v) \subset U$. If w is sufficiently close to v, we have $\overline{S}_\delta(w) \subset S_\theta(v)$. Denote by $X(w)$ the union of all $(-w)$-trajectories, starting at $\overline{S}_\theta(v)$. It suffices to prove that $X(w) \subset U$ for all w, close to v. For every $y \in W \setminus U$ the v-trajectory, starting at x is contained in $W \setminus \overline{S}_\theta(v)$ and reaches V_1 at some moment $T(y) \geq 0$. By 8.10 the same holds for every z from a neighbourhood $A(y)$ of y and for every w from some neighbourhood $\mathcal{V}(v)$ of v. Choosing from $\{A(y)\}$ a finite covering of $W \setminus U$ we obtain that for every w from a neighbourhood $\mathcal{U}$ of v and every $y \in W \setminus U$ the w-trajectory, starting at y, reaches V_1 without intersecting $\overline{S}_\theta(v)$, therefore $X(w) \subset U$.

3) is proved similarly. $\square$

B. Morse functions on cobordisms: recollections and definitions.

Let $f : W \to [a, b]$ be a Morse function on W (for the definition see [18, def 2.3]). Denote by $S(f)$ the set of critical points of f and by $S_k(f)$ the set of critical points of f of index k. For each $p \in S(f)$ choose a chart $\Phi(p) : U(p) \to B^n(0, r)$ (where $U(p)$ is a neighbourhood of p in W, and $n = \dim W$) such that $f \circ \Phi(p)^{-1}(x_1, \ldots, x_n) = -(x_1^2 + \ldots + x_k^2) + (x_{k+1}^2 + \ldots + x_n^2)$ where $k = \text{ind } p$. We call this chart *standard chart* and we consider it as a part of the data of the Morse function f. The neighbourhood $U(p)$ is called *standard coordinate neighbourhood.* We say (cf. [18, §3]) that a vector field v on W is a *gradient* for f, or an *f-gradient* if 1) $df(v) > 0$, except for the critical points of f, and 2) for any $p \in S(f)$ in a sufficiently small neighbourhood of p the field v equals the gradient of f with respect to the euclidean metric in the standard chart.

If v is an f-gradient, the set $S_k(f)$ will also be denoted by $S_k(v)$, and $S(f)$ by $S(v)$.

Let $\epsilon > 0$. We say that two f-gradients v and w are *ϵ-fitted*, if for each $p \in S(f)$ the disc $D^n(0, \epsilon)$ belongs to the image of the standard chart $\Phi(p)$ and $v(x) = w(x)$ in $\Phi(p)^{-1}(D(0, \epsilon))$.

If $\phi : W \to [a', b']$ is another Morse function, we say that *ϕ is adjusted to the pair (f, v)*, if 1) $S(\phi) = S(f)$, 2) v is also a ϕ-gradient, 3) the function $\phi - f$ is constant in a neighbourhood of any critical point p of f, as well as in a neighbourhood of V_0 and in a neighbourhood of V_1. The following lemma is proved similarly to [18, lemma2.8].

Lemma 2.5. *Let $f : W \to [a, b]$ be a Morse function, v be an f-gradient, $S(f) = \{p_1, \ldots, p_k\}$.*

Then there is $\delta > 0$ such that $\forall \epsilon_1, \ldots, \epsilon_k \in]-\delta, \delta[$ there is a Morse function $g : W \to [a, b]$ such that g is adjusted to (f, v), and $g(p_i) = f(p_i) + \epsilon_i$. $\square$

It is easy to see that an f-gradient is always a vector field of gradient type (i.e. it satisfies 2.1). For $p \in S(f)$ we denote by $D(p, v)$, resp. $D(p, -v)$, the union of all v-trajectories γ, converging to p as the parameter tends to ∞, resp. $-\infty$. We call $D(p, v)$ *descending disc* of p (although in general it is not homeomorphic to a disc). Recall ([27, Lemma A.0]) that $D(p, v) \cap \overset{\circ}{W}$ is a smooth submanifold of $\overset{\circ}{W}$ and $D(p, v) \cap f^{-1}(\lambda)$ is a smooth submanifold of $f^{-1}(\lambda)$ for any regular value λ. Denote $D(p, v) \cup D(p, -v)$ by $\Upsilon(p)$. Recall the following Rearrangement Theorem from [18, Th.4.1].

Theorem 2.6. *Let $f : W \to [a, b]$ be a Morse function, v an f-gradient, $c, d \in]a, b[$. Assume that $S(f) = \{p, q\}$ and that $\Upsilon(p) \cap \Upsilon(q) = \emptyset$.*

Then there is a Morse function $g : W \to [a, b]$, adjusted to (f, v), such that $g(p) = c, g(q) = d$. $\square$

A *curve* in a space X is a continuous map of an interval of $\mathbf{R}$ (bounded or not, closed or half-closed) to X. If $a, b \in X$, we shall say that a curve γ in X, defined on an interval I *joins* a with b, if there are points α, β in the closure of I in $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$, such that $\alpha < \beta$ and $\lim\limits_{t \to \alpha} \gamma(t) = a, \quad \lim\limits_{t \to \beta} \gamma(t) = b.$

For example, if v is a gradient of a Morse function f, and $p, q \in S(f)$, and γ is a v-trajectory then γ joins p with q if and only if the domain of γ equals $\mathbf{R}$, and $\lim\limits_{t \to -\infty} \gamma(t) = p, \quad \lim\limits_{t \to \infty} \gamma(t) = q$. The union of the images of all $(-v)$-trajectories, joining p with q, equals $D(p, v) \cap D(q, -v)$.

Denote by K^+ (resp. K^-) the union of $D(p, -v)$ (resp. $D(p, v)$) over $p \in S(f)$. The sets K^+, K^- are compact by 2.3.3). For each $x \in V_1 \setminus K^+$ the trajectory $\gamma(x, t; -v)$ reaches V_0 at a moment $T = T(x)$. The correspondence $x \mapsto \gamma(x, T(x); -v)$ is a bijective map of $V_1 \setminus K^+$ to $V_0 \setminus K^-$, which will be denoted by $\widehat{v}_{[b,a]}$, or, if there is no possibility of confusion, by $\widehat{v}$. By 8.12 it is a diffeomorphism.

If $p, q \in S(f), x \in \overset{\circ}{W}$, we say that *$D(p, v)$ is transversal to $D(q, -v)$ in x*, if $(D(p, v) \cap \overset{\circ}{W}) \pitchfork (D(q, -v) \cap \overset{\circ}{W})$ in x. We say that *$D(p, v)$ is transversal to $D(q, -v)$*, if $D(p, v) \cap \overset{\circ}{W}$ is transversal to $D(q, -v) \cap \overset{\circ}{W}$, notation: $D(p, v) \pitchfork D(q, -v)$. Note that for every regular value $\lambda \in [f(p), f(q)]$ of f we have

$$\big(D(p, v) \pitchfork D(q, -v)\big) \Leftrightarrow \big(D(p, v) \cap f^{-1}(\lambda) \pitchfork D(q, -v) \cap f^{-1}(\lambda) \text{ as submanifolds of } f^{-1}(\lambda)\big).$$

We say that $D(p, v)$ is *oriented*, resp. *cooriented*, if $D(p, v) \cap \overset{\circ}{W}$ is oriented, resp. cooriented.†

† A submanifold N of a manifold N' is called *cooriented*, if the normal bundle to N in N'

Let $f : W \to [a, b]$ be a Morse function. An f-gradient v will be called:

1) *perfect* , if $\Big(x, y \in S(f)\Big) \Rightarrow \Big(D(x, v) \pitchfork D(y, -v)\Big)$

2) *good*, if $\Big(x, y \in S(f)$ and $\operatorname{ind} x \le \operatorname{ind} y + 1\Big) \Rightarrow \Big(D(x, v) \pitchfork D(y, -v)\Big)$

3) *almost good*, if $\Big(x, y \in S(f)$ and $\operatorname{ind} x \le \operatorname{ind} y\Big) \Rightarrow \Big(D(x, v) \pitchfork D(y, -v)\Big)$.

Note that for $\operatorname{ind} x \le \operatorname{ind} y, x \ne y$ the condition $D(x, v) \pitchfork D(y, -v)$ is equivalent to the condition $D(x, v) \cap D(y, -v) = \emptyset$. By [27, Lemma A.1] there is a perfect f-gradient for any Morse function f. It is obvious that if v is a perfect (resp. good, almost good) f-gradient then $(-v)$ is a perfect (resp. good, almost good) $(-f)$-gradient.

Recall from [4, p.95] that a Morse function $f : W \to [a, b]$ is called *ordered* if

$$\Big(x, y \in S(f) \ \text{ and } \ \operatorname{ind} x < \operatorname{ind} y\Big) \Rightarrow \Big(f(x) < f(y)\Big).$$

An ordered function induces an order (in the sense of [16, ch.2 §9]) on the set $S(f)$, in which all the elements of $S_k(f)$ precede those in $S_l(f)$, whenever $k < l$. Namely, for $x, y \in S(f)$ set $x \prec y \underset{\text{def}}{=} \big(f(x) < f(y)$ or $x = y\big)$. Denote $\dim W$ by n. For an ordered Morse function $f : W \to [a, b]$ a sequence $\vec{a} = (a_0, \dots, a_{n+1})$ of regular values of f is called *ordering sequence*, if $a_0 = a, a_{n+1} = b$, for every l between 0 and n we have $a_l \le a_{l+1}$, and all the critical points of f of index l are contained in $f^{-1}([a_l, a_{l+1}])$.

C. Ordered functions and compactness properties for almost good gradients.

Lemma 2.7. *Let v be a gradient for a Morse function $f : W \to [a, b]$. Then* i) *and* ii) *are equivalent:*

i) *v is almost good;*

ii) *For any order on $S(f)$ such that any element of $S_k(f)$ precedes any element of $S_l(f)$ whenever $k < l$, there exists an ordered Morse function φ adjusted to (f, v) such that φ induces this order on $S(\varphi) = S(f)$.*

Proof. ii) $\Rightarrow$ i). Let $p, q \in S(f)$, $\operatorname{ind} p \le \operatorname{ind} q$. Then by ii) we can choose a Morse function φ with a regular value λ such that v is a φ-gradient and $\varphi(p) < \lambda < \varphi(q)$, which implies that $D(p, v) \cap D(q, -v) = \emptyset$.

i) $\Rightarrow$ ii). Apply first 2.5 to get a Morse function with different critical values and then apply (several times) 2.6. □

Lemma 2.8. *Let $f : W \to [a, b]$ be a Morse function on a cobordism, and v be an almost good gradient for f. Let $p, q \in S(f)$,* $\operatorname{ind} p = \operatorname{ind} q + 1$, *and let γ be a $(-v)$-trajectory, joining p with q. Denote the closure of the image of γ by Γ. Then there exists a neighbourhood U of Γ such that for any critical points p', q' of indices respectively* $\operatorname{ind} p$, $\operatorname{ind} q$, *we have*:

1) *If there exists a $(-v)$-trajectory, joining p' with some point of U then there exists also a $(-v)$-trajectory, joining p' with q.*

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is oriented. A choice of orientation of the normal bundle to N in N' is called *coorientation* of N.

2) *If there exists a $(-v)$-trajectory, joining some point of U with q' then there exists also a $(-v)$-trajectory, joining p with q'.*

Proof. Denote $\operatorname{ind} p$ by k and denote by Σ_q the set of all critical points p' of f of index k such that there are no $(-v)$-trajectories, joining p' with q. By the preceding lemma there is a Morse function $\phi : W \to [a,b]$, adjusted to (f,v), and having two regular values $\lambda < \mu$ such that the only critical points in $W_0 = \phi^{-1}([\lambda,\mu])$ are p, p' and q, and $\phi(q) < \phi(p') < \phi(p)$. By 2.6 there is a Morse function $\psi : W_0 \to [\lambda,\mu]$, adjusted to (f,v), and such that for some $\nu \in]\lambda,\mu[$ we have $\psi(p') < \nu < \psi(q) < \psi(p)$. Denote $\psi^{-1}(]\nu,\mu[)$ by $U(p')$; it is an open neighbourhood of Γ such that there are no $(-v)$-trajectories from p' to a point of $U(p')$. Denote by U_1 the intersection of the family $\{U(p')\}_{p'\in\Sigma_q}$. It satisfies 1). Similarly, one constructs U_2, satisfying 2). Set $U = U_1 \cap U_2$. $\square$

Lemma 2.9. *Let $f: W \to [a,b]$ be a Morse function and let v be an almost good f-gradient. Then any f-gradient w, sufficiently close to v, is almost good.*

Proof. Let $p, q \in S(f)$, $\operatorname{ind} p \le \operatorname{ind} q$. By 2.7 v is a φ-gradient for a Morse function φ on W such that $\varphi(p) < \lambda < \varphi(q)$ for some regular value λ of φ. Any w-trajectory, joining q with p, must intersect $\varphi^{-1}(\lambda)$, which is impossible for w sufficiently close to v, since in this case $w \mid \varphi^{-1}(\lambda)$ points inward $\varphi^{-1}([\lambda,\infty))$. Therefore for w close to v we have $D(p,w) \cap D(q,-w) = \emptyset$ which implies the corollary. $\square$

For almost good gradients there is a refined version of Lemma 2.3. To formulate it we need some more terminology.

Definition 2.10. Let $f: W \to [a,b]$ be a Morse function, v be an f-gradient, s be a natural number. Denote by $D_\delta(\le s; v)$, $B_\delta(\le s; v)$, $K(\le s; v)$ the unions of, respectively, $D_\delta(p,v)$, $B_\delta(p,v)$, $K(p,v)$, where p runs through the critical points of f of indices $\le s$. If $s = \dim W$,we sometimes drop s from the notation. We have not kept f in the notation, since these sets depend only on v. Note that $D_\delta(\le r; -v)$ now stands for the union of $D_\delta(p,-v)$ where p runs through the set of critical points of f of index $\ge n-r$. Denote by $D_\delta(\ge s; v)$, $B_\delta(\ge s; v)$, $K(\ge s; v)$ the unions of, respectively, $D_\delta(p,v)$, $B_\delta(p,v)$, $K(p,v)$, where p runs through the critical points of f of indices $\ge s$. $\triangle$

Lemma 2.11. *Let $f: W \to [a,b]$ be a Morse function, v be an almost good gradient for f, s be a natural number. Then for all $\delta > 0$ sufficiently small we have:*

1) *$B_\delta(\le s; v)$ and $B_\delta(\ge s; v)$ are open.*

2) *$D_\delta(\le s; v)$ and $K(\le s; v)$ are compact. The sets $B_\theta(\le s; v)$ for $\theta > \delta$, resp. for $\theta > 0$, form a fundamental system of neighbourhoods of $D_\delta(\le s; v)$, resp. $K(\le s; v)$.*

3) *Let U be an open neighbourhood of $D_\delta(\le s; v)$, resp. $K(\le s; v)$. Then if w is an f-gradient, sufficiently close to v, we have $D_\delta(\le s; w) \subset U$, resp. $K(\le s; w) \subset U$.*

4) *Let Q be a compact in $B_\delta(\le s; v)$, resp. $B_\delta(\ge s; v)$. Then if w is an f-gradient, sufficiently close to v, we have $Q \subset B_\delta(\le s; w)$, resp. $Q \subset B_\delta(\ge s; w)$.*

5) *If p is a critical point of f of index $s+1$ then $D_\delta(\le s; v) \cup D_\delta(p; v)$ and $K(\le s; v) \cup D_\delta(p; v)$ are compact, and if U is an open neighbourhood of $D_\delta(\le s; v) \cup D_\delta(p; v)$, resp. of $K(\le s; v) \cup D_\delta(p; v)$ then for every f-gradient w, sufficiently close to v, we have $D_\delta(\le s; w) \cup D_\delta(p; w) \subset U$, resp. $K(\le s; w) \cup D_\delta(p; w) \subset U$.*

Proof. 1) follows from 2.3.1). 4) follows from 2.4.1). By Lemma 2.7 we can choose an ordered Morse function $\varphi: W \to [\alpha,\beta]$, such that for some regular values $\lambda < \mu$, all the critical points of φ of indices $\leq s$ are below the level λ and p is the only critical point between λ and μ. Then for $\delta > 0$ and $\theta > 0$ sufficiently small the set $D_\delta(\leq s; v)$, resp. $B_\theta(\leq s; v)$, is just the set $D_\delta(v)$, resp. $B_\theta(v)$ with respect to the restriction of the Morse function φ to the cobordism $\varphi^{-1}([\alpha, \lambda])$ and the set $D_\delta(\leq s; v) \cup D_\delta(p, v)$ is the set $D_\delta(v)$ with respect to $\varphi|\varphi^{-1}([\alpha, \mu])$. Apply now 2.3 and get 2) and the first part of 5) of our lemma. Note, further that an f-gradient w, sufficiently close to v, is a vector field of gradient type on the cobordism $\varphi^{-1}([\alpha, \lambda])$. Apply now 2.4 and get 4) and the rest of 5). □

Definition 2.12. For a Morse function $f : W \to [a, b]$ and two regular values α, β of f, where $\alpha < \beta$ the sets $D_\delta(\leq s; v)$, resp $B_\delta(\leq s; v)$ etc., defined with respect to the cobordism $f^{-1}([\alpha, \beta])$, will be denoted by $D_\delta(\leq s; [\alpha, \beta]; v)$, resp. $B_\delta(\leq s; [\alpha, \beta]; v)$ etc. As in 2.10, if $s = \dim W$, we drop it from the notation, e.g. $D_\delta([\alpha, \beta]; v)$,resp. $B_\delta([\alpha, \beta]; v)$. △

D. Regular functions.

We say that a cobordism W is *regular*, if V_0, V_1 and W are connected and the inclusions $V_0 \hookrightarrow W \hookleftarrow V_1$ induce isomorphisms in π_1. We say that a Morse function $f: W \to [a, b]$ on a regular cobordism is *regular*, if it has no critical points of indices 0, 1, $n-1$, n and for each regular value λ the inclusion $f^{-1}(\lambda) \subset W$ induces an isomorphism in π_0 and π_1. It is obvious that for any two regular values α, β of a regular function $f : W \to [a, b]$ the inclusions $f^{-1}(\alpha) \hookrightarrow f^{-1}([\alpha, \beta]) \hookleftarrow f^{-1}(\beta)$ induce isomorphisms in π_0 and π_1.

Lemma 2.13. *Let W be a regular cobordism,* $\dim W = n \geq 6$. *Let $f: W \to [a, b]$ be an ordered Morse function without critical points of indices* 0, 1, $n-1$, n.

Then f is regular.

Proof. Since f is ordered, there is a sequence $a = c_1 < \ldots < c_{r+1} = b$ of regular values of f such that for every $1 \leq i \leq r$ all the critical points in $f^{-1}([c_i, c_{i+1}])$ are of the same index n_i and $n_i < n_{i+1}$. We shall prove that $\forall i, 1 \leq i \leq r$ the inclusions $f^{-1}(c_i) \subset f^{-1}([c_i, c_{i+1}]) \supset f^{-1}(c_{i+1})$ induce isomorphisms in π_0 and π_1. It is obvious for π_0 and every i, as well as for π_1, if $2 < n_i < n-2$; we shall now prove it for π_1 and $n_i = 2$ (the case $n_i = n-2$ is similar). The inclusion $I : f^{-1}(a) = f^{-1}(c_1) \subset f^{-1}([c_1, c_2])$ induces an epimorphism in π_1, since all the critical points in $f^{-1}([c_1, c_2])$ are of index 2. But the composition $f^{-1}(a) \hookrightarrow f^{-1}([c_1, c_2]) \hookrightarrow W$ induces an isomorphism in π_1, therefore I induces an isomorphism in π_1. The inclusion $f^{-1}(c_2) \subset f^{-1}([c_1, c_2])$ induces an isomorphism in π_1, since all the critical points of $(-f)$ in $f^{-1}([c_1, c_2])$ are of index $n-2 \geq 4$. □

Remark 2.14. By an argument, similar to that of [15, p. 35-36], on any regular cobordism W, $\dim W = n \geq 6$ there is a Morse function without critical points of indices 0, 1, $n-1$, n; therefore by the preceding lemma, there exists a regular one. △

Lemma 2.15. *Let $f: W \to [a, b]$ be a Morse function on a regular cobordism,* $\dim W = n \geq 6$. *Assume that* $S(f) = \{p, q\}$, $n-2 \geq \operatorname{ind} p$, $\operatorname{ind} q \geq 2$, $\operatorname{ind} p = \operatorname{ind} q - 1$, $f(p) > f(q)$.

Then f is regular.

Proof. One of two numbers $\operatorname{ind} p$, $\operatorname{ind} q$ must be between 3 and $n-3$, which proves the lemma. □

Lemma 2.16. *Let $f: W \to [a,b]$ be a Morse function without critical points of indices $0, 1, n-1, n$, let v be a perfect f-gradient, let $p \in S(f)$, and λ be a regular value of f.*

Then 1) *$D(p,v) \cap f^{-1}(\lambda)$ is non empty and connected for $\lambda < f(p)$;*

2) *$D(p,-v) \cap f^{-1}(\lambda)$ is non empty and connected for $\lambda > f(p)$.*

Proof. 1) Choose $\mu \in]\lambda, f(p)[$ such that $[\mu, f(p)[$ is regular. Denote $D(p,v) \cap f^{-1}(\mu)$ by S_μ, it is an embedded sphere. Set

$$K^+ = K([\lambda,\mu]; -v) \cap f^{-1}(\mu), \quad K^- = K([\lambda,\mu]; v) \cap f^{-1}(\lambda).$$

From our hypotheses it follows that K^+ is a finite union of submanifolds of $f^{-1}(\mu)$, which have codimension ≥ 2 and are transversal to S_μ. therefore $S_\mu \setminus K^+$ is non empty and connected. Now it suffices to note that $D(p,v) \cap f^{-1}(\lambda)$ is the image of $S_\mu \setminus K^+$ with respect to the diffeomorphism $\widehat{v}_{[\mu,\lambda]} : f^{-1}(\mu) \setminus K^+ \to f^{-1}(\lambda) \setminus K^-$.

2) is proved similarly. $\square$

E. Incidence coefficient $\nu(p,q;v)$.

Definition 2.17. Let X be a topological space, $p, q \in X$. Denote by $\Gamma(p,q)$ the set of curves $\gamma : \mathbf{R} \to X$ such that $\lim_{t \to -\infty} \gamma(t) = p$, $\lim_{t \to \infty} \gamma(t) = q$, where we identify two curves, which differ by a parameter change $t \mapsto t + C$. Denote by $\overline{\mathbf{R}}$ the compactification of $\mathbf{R}$ by two points $-\infty$ and ∞. Denote by $\pi(p,q)$ the set of homotopy classes (with fixed ends) of maps $\overline{\mathbf{R}} \to X$ such that $-\infty \mapsto p, \infty \mapsto q$. Each $\gamma \in \Gamma(p,q)$ determines correctly an element of $\pi(p,q)$, which we denote by $[\gamma]$. An *s-curve* (s for sign) is a pair (g, ε), where $\varepsilon = \pm 1, g \in \Gamma(p,q)$. If $\mathcal{A}$ and $\mathcal{B}$ are two sets of s-curves then they are *homotopic*, if there is a bijection $\Phi : \mathcal{A} \to \mathcal{B}$, such that $\Phi(g,\varepsilon) = (g', \varepsilon')$ and $[g'] = [g]$, notation: $\mathcal{A} \sim \mathcal{B}$. $\triangle$

Definition 2.18. Let $f: W \to [a,b]$ be any Morse function, v be an f-gradient. Let $p, q \in S(f)$ such that $\operatorname{ind} p = \operatorname{ind} q + 1$, and $D(p,v) \pitchfork D(q,-v)$. Suppose that $D(p,v)$ and $D(q,v)$ are oriented. Then $D(q,-v)$ is cooriented, and each $(-v)$-trajectory γ, joining p with q, obtains a sign $\varepsilon(\gamma) = \pm$. The set of all the s-curves (γ, ε), where γ is the class in $\Gamma(p,q)$ of a $(-v)$-trajectory, joining p with q, and $\varepsilon = \varepsilon(\gamma)$ is the sign of this trajectory, is denoted by $N(p,q;v)$. (Actually this set, as well as the sign of each s-curve in it, depends also on the orientation of $D(p,v)$ and on the orientation of $D(q,v)$; $\varepsilon(\gamma)$ depends also on v, but we drop it from the notation in order not to overburden the text.) If there are some fixed liftings $\tilde{p}, \tilde{q}$ of p and q to $\widetilde{W}$, we denote by $\tilde{\gamma}$ the unique lifting to $\widetilde{W}$ of the curve γ, which joins $\tilde{p}$ with a point $\tilde{q} \cdot g, g \in \pi_1(W)$. We denote this g by $g(\gamma)$. In the case when $N(p,q)$ is finite, we set

$$\nu(p,q;v) = \sum \varepsilon(\gamma) \cdot g(\gamma) \in \mathbf{Z}\pi_1(W),$$

where γ ranges over the set of images in $\Gamma(p,q)$ of $(-v)$-trajectories, joining p with q. We call $\nu(p,q;v)$ *incidence coefficient of* p, q.

For example, if v is a good f-gradient then $D(p,v) \pitchfork D(q,-v)$ whenever $\operatorname{ind} p = \operatorname{ind} q + 1$, and therefore, the incidence coefficient is defined for all $p, q \in S(f)$ such that $\operatorname{ind} p = \operatorname{ind} q + 1$.$\triangle$

Lemma 2.19. *Let $f : W \to [a,b]$ be a Morse function, v be an almost good f-gradient. Let $p, q \in S(f), \operatorname{ind} p = \operatorname{ind} q + 1$. Assume that $D(p,v) \pitchfork D(q,-v)$. Let $\epsilon > 0$.*

Then: 1) $N(p,q;v)$ *is finite.*

2) *Let* w *be an* f*-gradient,* ϵ*-fitted to* v*. If* w *is sufficiently close to* v*, we have* $N(p,q;v) \sim N(p,q;w)$*.*

Proof. Find (by 2.7) a Morse function $\phi : W \to [a,b]$, adjusted to (f,v), and such that for some regular values $\lambda < \mu < \nu$ of ϕ the only critical point of ϕ in $\phi^{-1}([\lambda,\mu])$ (resp. in $\phi^{-1}([\mu,\nu])$ is q(resp.p). Now the intersections $D(p,v) \cap \phi^{-1}(\mu)$ and $D(q,-v) \cap \phi^{-1}(\mu)$ are embedded spheres of complementary dimensions, which are transversal, and our lemma follows. $\square$

The following lemma is obvious.

Lemma 2.20. *Let* $f : W \to [a,b]$*,* $g : W \to [a',b']$ *be two Morse functions,* v *and* u *be* f*-,resp.* g*-gradients. Let* $x, y \in S(f) \cap S(g)$*. Assume that in a neighbourhood of* $\{x,y\}$ f *equals* g *and* v *equals* u*. Assume that every* $(-v)$*-trajectory, joining* x *with* y *does not intersect* $\operatorname{supp}(v-u)$*, and that every* $(-u)$*-trajectory, joining* x *with* y *does not intersect* $\operatorname{supp}(v-u)$ *(this assumption is true, for example, if one of the sets* $D(x,v), D(y,-v)$ *does not intersect* $\operatorname{supp}(u-v)$*).*

Then: 1) *Each* $(-v)$*-trajectory, joining* x *with* y*, is a* $(-u)$*-trajectory, and vice versa.*

2) *Let* γ *be a* $(-v)$*-trajectory, joining* x *with* y*. Then*

$$\big(D(x,v) \pitchfork D(y,-v) \text{ in the points of } \gamma\big) \Leftrightarrow \big(D(x,u) \pitchfork D(y,-u) \text{ in the points of } \gamma\big).$$

3) *If* $\operatorname{ind} x = \operatorname{ind} y + 1$ *and* $D(x,v) \pitchfork D(y,-v)$ *then* $N(x,y;v) = N(x,y;u)$*.* $\square$

F. Incidence coefficient $\overline{\nu}(p,q;v)$**.**

Let $f : W \to [a,b]$ be a regular Morse function on a regular cobordism, v be an almost good gradient for f. Let $p,\ q \in S(f)$, $\operatorname{ind} p = \operatorname{ind} q + 1$. Fix some orientation of $D(p,v)$ and of $D(q,v)$. The latter defines a coorientation of $D(q,-v)$. Fix some liftings $\widetilde{p}$ and $\widetilde{q}$ of $p,\ q$ to $\widetilde{W}$. Fix an ordered function φ on W such that v is also a gradient for φ. With respect to all these data we define the incidence coefficient $\overline{\nu}(p,q;v)$ as follows. Let λ be a regular value of φ such that all critical points of f of indices $\leq$ ind p, are in the set $\varphi^{-1}(]-\infty,\lambda[)$, and all the critical points of f of indices $>$ ind p are in the set $\varphi^{-1}(]\lambda,\infty[)$. The intersections $S_p(v) = D(p,v) \cap \varphi^{-1}(\lambda)$ and $S_q(-v) = D(q,-v) \cap \varphi^{-1}(\lambda)$ are the smoothly embedded spheres in $\varphi^{-1}(\lambda)$ of complementary dimensions, $S_p(v)$ is oriented, $S_q(-v)$ is cooriented. Put them into general position. The resulting intersection index as element of $\mathbf{Z}\pi_1(\varphi^{-1}(\lambda)) = \mathbf{Z}\pi_1(W)$ is called *incidence coefficient of* p,q and is denoted $\overline{\nu}(p,q;v)$. (It does not depend, obviously, on the choice of λ and we shall see in Section. K that neither does it depend on the choice of φ.)

Lemma 2.21. *Let* $\epsilon > 0$*. Let* w *be an* f*-gradient,* ϵ*-fitted to* v*. Then if* w *is sufficiently close to* v*, we have* $\overline{\nu}(p,q;v) = \overline{\nu}(p,q;w)$*.*

Proof. If w is ϵ-fitted and sufficiently close to v, then w is also an almost good φ-gradient. The family $tv + (1-t)w$ of φ-gradients determine homotopies $S_p(v) \sim S_p(w)$ and $S_q(-v) \sim S_q(-w)$. $\square$

The following lemma is easily proved, using the standard transversality arguments.

Lemma 2.22. *Assume the hypotheses of* 2.17 *and* 2.18. *Assume further that W is a regular cobordism, f is a regular function, and v is a good f-gradient.*

Then $\nu(p,q;v) = \overline{\nu}(p,q;v)$. $\square$

G. Incidence coefficient $\overline{\overline{\nu}}(p,q;v)$.

Let $f : W \to [a,b]$ be a regular Morse function on a regular cobordism, v be an f-gradient. Let $p \in S_k(f)$ and α, β be regular values of f such that $\alpha < f(p) < \beta$. Assume that every $(-v)$-trajectory γ such that $\lim_{t\to-\infty}\gamma(t) = p$, reaches $f^{-1}(\alpha)$. In this case the intersection $D(p,v) \cap f^{-1}([\alpha,\beta])$ is an embedded disc, which will be denoted by $(p)(v,\alpha)$, or simply $(p)(v)$, if there is no possibility of confusion. The boundary of this disc belongs to $f^{-1}(\alpha)$. If a lifting $\widetilde{p}$ of p to $\widetilde{W}$ is fixed then a lifting $I : (p)(v) \to \widetilde{W}$ of the inclusion map is defined. If an orientation of $D(p,v)$ is fixed then the I_*-image of the fundamental class of $(p)(v)$ modulo boundary determines an element

$$[p](v,\alpha) \in H_k(\widetilde{f^{-1}([\alpha,\beta])}, \widetilde{f^{-1}(\alpha)}).$$

If there is no possibility of confusion, we shall denote this element by $[p](v)$.

Lemma 2.23. *Let $\epsilon > 0$. Under the above assumptions let w be an f-gradient, ϵ-fitted to v.*

Then if w is sufficiently close to v, we have $[p](v) = [p](w)$.

Proof. Denote $f(p)$ by c, and for $d \in [\alpha, c[$ denote $D(p,v) \cap f^{-1}(d)$ by $S_d(v)$ (it is an embedded sphere). Let $\Phi(p) : U(p) \to B^n(0,r)$ be a standard chart ($n = \dim W$) and let d be so close to c that $S_d(v) \subset \Phi^{-1}(p)(B^n(0,\epsilon))$. Then $S_d(v) = S_d(w)$ and from 8.10 one deduces easily that there is a neighbourhood $\mathcal{V}$ of v in $Vect^\infty(W,\perp)$ such that for $w \in \mathcal{V}$ every $(-w)$-trajectory γ such that $\lim_{t\to-\infty}\gamma(t) = p$, reaches $f^{-1}(\alpha)$. We can assume that $\mathcal{V}$ is convex, and then the family $tv + (1-t)w$ determines a homotopy between the discs $(p)(v)$ and $(p)(w)$ such that the boundary stays in $f^{-1}(\alpha)$. $\square$

Assume now, furthermore that v is an almost good f-gradient. Let $\phi : W \to [a,b]$ be an ordered Morse function, adjusted to (f,v), and let $\vec{a} = (a_0, \dots, a_{n+1})$ be an ordering sequence for ϕ. Denote $\phi^{-1}([a_k, a_{k+1}])$ by W_k and $\phi^{-1}(a_k)$ by V_k; the inclusions $V_k \subset W_k \subset W$ induce isomorphisms in π_0, π_1 for $0 \le k \le n$. Denote by $C_k(\phi, \vec{a})$ the right $\mathbf{Z}\pi_1(W)$-module $H_k(\widetilde{W_{k+1}}, \widetilde{W_k})$. The differential ∂_k of the exact sequence of the triplet $(\widetilde{W_{k+1}}, \widetilde{W_k}, \widetilde{W_{k-1}})$ turns $C_*(\phi, \vec{a})$ into a chain $\mathbf{Z}\pi_1(W)$-complex. For $p \in S_k(\phi)$ we shall denote $(p)(v, a_k)$ by $(p)(v,\phi)$ because we want to stress the dependance of this manifold on the function ϕ, and the value of $\vec{a}$ is normally clear from the context. The manifold $(p)(v,\phi)$ is an embedded disc, and $\partial(p)(v,\phi) \subset \phi^{-1}(a_k)$. For each $p \in S(f)$ fix an orientation of $D(p,v)$ and a lifting $\widetilde{p}$ of p to $\widetilde{W}$; then a lifting of $(p)(v,\phi)$ to $\widetilde{W}$ is defined, and the image of the fundamental class of $(p)(v,\phi)$ modulo boundary defines an element $[p](v,\phi) \in C_k(\phi, \vec{a})$. An argument, similar to that of [18,§3] shows that $C_k(\phi, \vec{a})$ is a free $\mathbf{Z}\pi_1(W)$-module with the basis $\{[p](v,\phi)\}_{p\in S_k(f)}$; we shall denote this free based chain complex over $\mathbf{Z}\pi_1(W)$ by $C_*(v, \phi, \vec{a})$. For $p \in S_k(\phi), q \in S_{k-1}(\phi)$ define an element $\overline{\overline{\nu}}(p,q;v) \in \mathbf{Z}\pi_1(W)$ by the following formula:

$$\partial_k([p](v,\phi)) = \sum [q](v,\phi) \cdot \overline{\overline{\nu}}(p,q;v) \in \mathbf{Z}\pi_1(W),$$

where the sum ranges over all critical points q of f of index $k-1$.

The proof of the following lemma is similar to the proof of Lemma 7.2 in [18].

Lemma 2.24. *In the terminology of Subsections F and G we have:* $\nu(p,q;v) = \overline{\overline{\nu}}(p,q;v)$. $\square$

H. Incidence coefficient $\nu_\lambda(p,q;v)$.

We shall need a definition of intersection index of two submanifolds in a situation slightly more general, than usual. Let M be a connected manifold without boundary. Let $X, Y \subset M$ be compact subsets of M, L a compact manifold (with boundary), $\phi : L \to M$ a smooth map, N a compact submanifold (with boundary) of M, and assume that $\dim L + \dim N = \dim M$. Denote by $(\mathcal{I})$ the conjunction of the following conditions:

$(\mathcal{I}0) X \cap Y = \emptyset$;

$(\mathcal{L}1)$ $\phi(L) \cap Y = \emptyset$, $\phi(\partial L) \subset X$; $(\mathcal{L}2)$ L is oriented; $(\mathcal{L}3)$ a lifting $\tilde{\phi} : L \to \widetilde{M}$ of ϕ is fixed;

$(\mathcal{N}1)$ $\partial N \subset Y, N \cap X = \emptyset$;$(\mathcal{N}2)$ N is cooriented; $(\mathcal{N}3)$ a lifting $\tilde{i} : N \to \widetilde{M}$ of the inclusion map $i : N \hookrightarrow M$ is fixed.

Suppose that X, Y, ϕ, N satisfy $(\mathcal{I})$. Arbitrarily close to ϕ there exist smooth maps $\psi : L \to M$ such that $\psi(L) \cap Y = \emptyset, \psi(\partial L) \subset X$, and ψ is transversal to N. The lifting $\tilde{\phi}$ determines a lifting $\tilde{\psi}$ of ψ. For each $x \in \psi^{-1}(N)$ denote by $\varepsilon(x)$ the sign of intersection of L with N at x, and by $g(x)$ the element of $\pi_1(M)$ such that $\tilde{\psi}(x) = \tilde{i}(\psi(x)) \cdot g(x)$. The sum of $\varepsilon(x) \cdot g(x)$, where $x \in \psi^{-1}(N)$, is an element of $\mathbf{Z}\pi_1(M)$. The standard transversality argument shows that this element does not depend on the choice of ψ. We call it *intersection index* of ϕ and N and denote it by $I(\phi, N; X, Y)$. (Note that this index depends also on the choices of $\tilde{\phi}, \tilde{i}$, and on the orientation of L and on the coorientation of N.) Fix some X, Y, N, satisfying $(\mathcal{I}0)$, and $(\mathcal{N}1) - (\mathcal{N}3)$. Then if for $i = 1, 2, \phi_i : L_i \to M$ are smooth maps, satisfying $(\mathcal{L}1)-(\mathcal{L}3)$, and $\varepsilon_i = \pm 1$, and $g_i \in \pi_1(M)$, the usual operations of disjoint sum, orientation reversing, and changing of the lifting by a deck transformation, define in an obvious way a map $L_1 \sqcup L_2 \to M$, denoted by $\varepsilon_1 \cdot \phi_1 \cdot g_1 + \varepsilon_2 \cdot \phi_2 \cdot g_2$, which satisfies $(\mathcal{L}1) - (\mathcal{L}3)$. It is obvious that

$$I(\varepsilon_1 \cdot \phi_1 \cdot g_1 + \varepsilon_2 \cdot \phi_2 \cdot g_2, N; X, Y) = \varepsilon_1 I(\phi_1, N; X, Y) g_1 + \varepsilon_2 I(\phi_2, N; X, Y) g_2 \qquad (2.1)$$

To formulate homotopy invariance properties of $I(\phi, N; X, Y)$ we need a definition.

Fix some X, Y, satisfying $(\mathcal{I}0)$. Then if $\phi : L \to M$ satisfies $(\mathcal{L}1) - (\mathcal{L}3)$, the $\tilde{\phi}_*$-image of the fundamental class of L modulo ∂L in the group $H_l(\widetilde{M} \setminus \widehat{Y}, \widehat{X})$, (where $l = \dim L$, and $\widehat{Y}, \widehat{X}$ are the preimages of Y, resp. X in $\widetilde{M}$) will be denoted by $\phi\langle L \rangle)$.

Definition 2.25. For given X, Y we say that two cooriented submanifolds N_0, N_1 are (X, Y)-homotopic, if there is a continuous map $F : N_0 \times [0, 1] \to M$, satisfying the following properties:

1) $F | N_0 \times]0, 1[$ is smooth.

2) $(\operatorname{Im} F) \cap X = \emptyset, F(\partial N_0 \times [0, 1]) \subset Y$.

3) There is an $\epsilon > 0$ such that $F | N_0 \times \{t\}$ does not depend on t for $t < \epsilon$ and for $t > 1 - \epsilon$.

4) $F | N_0 \times \{0\}$ equals to the inclusion map; $\operatorname{Im} (F | N_0 \times \{1\}) = N_1$.

5) For each $t \in [0, 1]$ the map $F | N_0 \times \{t\}$ is a smooth embedding.

6) If we denote by $\Phi : N_0 \times [0,1] \to M \times [0,1]$ the map $(n,t) \mapsto (F(n,t),t)$ then $\mathrm{Im}\,(\Phi | N_0 \times]0,1[)$ is a cooriented submanifold of $M \times]0,1[$, and this coorientation induces the given coorientations of $N_0 \times \{t\}$ in $M \times \{t\}$ for $t < \epsilon$ and of $N_1 \times \{t\}$ in $M \times \{t\}$ for $t > 1 - \epsilon$.

7) there is a lifting $\widetilde{F} : N_0 \times [0,1] \to \widetilde{M}$ of F such that $\widetilde{F}|N_0 \times \{0\}, \widetilde{F}|N_1 \times \{1\}$ are equal to the fixed liftings of the inclusion maps $N_0 \hookrightarrow M, N_1 \hookrightarrow M$. $\triangle$

The standard transversality arguments prove the following lemma.

Lemma 2.26. 1) *Fix N, X, Y, satisfying $(\mathcal{I}0)$ and $(\mathcal{N}1) - (\mathcal{N}3)$.*

Then $I(\phi, N; X, Y)$ depend only on $\phi\langle L \rangle \in H_l(\widetilde{M} \setminus \widehat{Y}, \widehat{X})$.

2) *Assume that ϕ, N_i, X, Y satisfy $(\mathcal{I})$ for $i = 0, 1$, and that N_0 and N_1 are (X,Y)-homotopic. Then $I(\phi, N_0; X, Y) = I(\phi, N_1; X, Y)$.*

3) *If L' (resp. N') is a submanifold of the interior of L, (resp. N), and $\varphi(L \setminus L') \subset X$, (resp. $N \setminus N' \subset Y$) then $I(\varphi, N; X, Y) = I(\varphi | L', N'; X, Y)$.*

4) *If for given $\varphi : L \to M$ and $N \subset M$ there is another pair X', Y' of compacts in M, satisfying $(\mathcal{I})$ then $I(\varphi, N; X, Y) = I(\varphi, N'; X', Y')$* . $\square$

Now we can give a definition of the incidence coefficient for an almost good gradient in terms of the initial (not necessarily ordered) Morse function. Let $f : W \to [a,b]$ be a regular Morse function, v be an almost good f-gradient. Let $p, q \in S(f)$, ind $p =$ ind q + 1. Choose and fix some orientations of $D(p,v), D(q,v)$ and some liftings $\tilde{p}, \tilde{q}$ of p and q to $\widetilde{W}$. Choose a riemannian metric on W such that it is euclidean in standard charts. If $f(p) \le f(q)$ we define the incidence coefficient to be 0. If not, let $f(p) = d, f(q) = c$, ind $p = s$.

Fix a regular value $\lambda \in]c,d[$ of f. Let $\alpha, \beta \in]c,d[$ be regular values of f such that $\alpha < \lambda < \beta$ and that $\alpha - c, d - \beta$ are so small that the intervals $]c,\alpha[,]\beta,d[$ are regular and the intersections $D(p,v) \cap f^{-1}([\beta,d])$, and $D(q,-v) \cap f^{-1}([c,\alpha])$ belong to the standard coordinate neighbourhoods of p, resp. q. These intersections are then smoothly embedded discs of dimensions s and $n-s+1$. Their boundaries $D(p,v) \cap f^{-1}(\beta)$, $D(q,-v) \cap f^{-1}(\alpha)$ are smoothly embedded spheres in $f^{-1}(\beta)$, resp. $f^{-1}(\alpha)$, of dimension $s-1$, resp. $n-s$; they will be denoted by $S_\beta(v)$, resp. $S_\alpha(v)$. Since v and $(-v)$ are almost good, the sets $K(\le s; v)$ and $K(\le n-s; -v)$ are compact and do not intersect; the same is true for $K(\le s-1; v)$ and $K(\le n-s+1; -v)$. Therefore, by 2.11.2) for $\delta > 0$ sufficiently small we have:

$$D_\delta(\le s; v) \cap D_\delta(\le n-s; -v) = \emptyset = D_\delta(\le s-1; v) \cap D_\delta(\le n-s+1; -v). \tag{2.2}$$

Denote by $A_+(v)$ the set $K([\lambda,\beta]; -v) \cap S_\beta(v)$ and by $B_+(v)$ the set $S_\beta(v) \setminus B_\delta([\lambda,\beta]; -v)$. $A_+(v)$ and $B_+(v)$ are disjoint compacts in $S_\beta(v)$. Let P be an $(s-1)$-dimensional compact submanifold of $S_\beta(v)$, satisfying the following conditions:

$$IntP \supset B_+(v), \quad P \cap A_+(v) = \emptyset \tag{2.3}$$

Denote by $A_-(v)$ the set $K([\alpha,\lambda]; v) \cap S_\alpha(v)$ and by $B_-(v)$ the set $S_\alpha(v) \setminus B_\delta([\alpha,\lambda]; v)$. $A_-(v)$ and $B_-(v)$ are disjoint compacts in $S_\alpha(v)$. Let Q be an $(n-s)$-dimensional compact submanifold of $S_\alpha(v)$, satisfying the following conditions:

$$IntQ \supset B_-(v), \quad Q \cap A_-(v) = \emptyset \tag{2.4}$$

A quintuple $(\alpha, \beta, \delta, P, Q)$ of elements, satisfying the above conditions, will be called *input data* with respect to v and λ. From an input data we shall now construct an element in $\mathbf{Z}\pi_1(W)$. Denote $D_\delta(\leq s-1; v) \cap f^{-1}(\lambda)$ by $X(v)$, and $D_\delta(\leq n-s; -v) \cap f^{-1}(\lambda)$ by $Y(v)$. These sets are compact and (2.2) implies, that $X(v) \cap Y(v) = \emptyset$, and, moreover, $(X(v) \cup D(p,v)) \cap Y(v) = \emptyset = X(v) \cap (Y(v) \cup D(q,-v))$. Conditions (2.3) imply that $\partial P \subset B_\delta([\lambda, \beta], -v)$ and from (2.2) it follows that $\partial P \subset B_\delta(\geq n-s+1, [\lambda,\beta], -v)$. From (2.3) it follows that P is contained in the domain of $\widehat{v}_{[\beta,\lambda]}$. The image $\widehat{v}(P)$ is contained in $D(p,v) \cap f^{-1}(\lambda)$, and by the above $\widehat{v}(\partial P) \subset B_\delta(\leq s-1, [\lambda,\beta], v) \subset X(v)$. Similarly, Q is contained in the domain of $\widehat{(-v)}_{[\alpha,\lambda]}$, and $\widehat{(-v)}(Q) \subset D(q,-v) \cap f^{-1}(\lambda), \widehat{(-v)}(\partial Q) \subset B_\delta(\leq n-s, [\alpha,\lambda]; -v) \subset Y(v)$. Denote $\widehat{(-v)}(Q)$ by $Q(v)$. The manifold P is oriented, $Q(v)$ is a cooriented submanifold of $f^{-1}(\lambda)$, the sum of their dimensions is equal to dim $f^{-1}(\lambda)$, and since $f^{-1}(\lambda) \hookrightarrow W$ induce an isomorphism in π_1, the liftings $\tilde{p}$, $\tilde{q}$ determine some liftings of $\widehat{v}|P$ and of $Q(v)$ to $\widetilde{f^{-1}(\lambda)}$. Therefore, the conditions $(\mathcal{I})$ from the beginning of this subsection are fulfilled for the manifold $f^{-1}(\lambda)$, the pair of non-intersecting compacts $X(v)$ and $Y(v)$, the map $\widehat{v}|P : P \to f^{-1}(\lambda)$, and the submanifold $Q(v) \subset f^{-1}(\lambda)$. Therefore for every input data $(\alpha, \beta, \delta, P, Q)$ the intersection index $I(\widehat{v}|P : P \hookrightarrow f^{-1}(\lambda), Q(v); X(v), Y(v)) \in \mathbf{Z}\pi_1(W)$ is defined. Using 2.26 3),4) it is easy to verify that this index does not depend on the choice of the input data. So we denote it by $\nu_\lambda(p,q;v)$.

K. Further properties of incidence coefficients.

First we shall prove that $\nu_\lambda(p,q;v)$ does not depend on λ.

Lemma 2.27. *Let $\epsilon > 0$. Let w be an f-gradient, ϵ-fitted to v. Then if w is close enough to v, we have: $\nu_\lambda(p,q;v) = \nu_\lambda(p,q;w)$.*

Proof. We shall choose the input data for defining $\nu_\lambda(p,q;v)$ in a special way. Namely, choose β so close to d that $D(p,v) \cap f^{-1}([\beta,d]) \subset D_\epsilon(p)$, and $D(q,-v) \cap f^{-1}([c,\alpha]) \subset D_\epsilon(q)$, where $D_\epsilon(p)$ and $D_\epsilon(q)$ stand for the preimages of ϵ-discs in the standard charts. In this case $S_\beta(v) = S_\beta(w)$ and $S_\alpha(v) = S_\alpha(w)$ for every w, ϵ-fitted to v, and we denote $S_\beta(v)$, resp. $S_\alpha(v)$ by S_β, resp. S_α. Choose $\delta > 0$ such that (2.2) is satisfied. Let $\theta \in]0, \delta[$. Choose P and Q in such a way that

$$IntP \supset S_\beta \setminus B_\theta([\lambda,\beta]; -v); \quad P \cap A_+(v) = \emptyset \tag{2.5}$$

$$IntQ \supset S_\alpha \setminus B_\theta([\alpha,\lambda]; v); \quad Q \cap A_-(v) = \emptyset \tag{2.6}$$

We shall verify that A) for this choice of the input data there is a neighbourhood $\mathcal{V}$ of v in $C^\infty(W)$ such that for $w \in \mathcal{V}$ this input data is also an input data for w, and B) that the resulting intersection index $\nu_\lambda(p,q;v)$ equals $\nu_\lambda(p,q;w)$.

From 2.11.2),3) it follows that there is a convex neighbourhood $\mathcal{U}$ of v such that for $w, w' \in \mathcal{U}$ we have

$$D_\delta(\leq s; w) \cap D_\delta(\leq n-s; -w') = \emptyset = D_\delta(\leq s-1; w) \cap D_\delta(\leq n-s+1; -w') \tag{2.7}$$

$$D_\theta(\leq s-1, [\lambda,\beta], w) \subset B_\delta(\leq s-1, [\lambda,\beta], w') \tag{2.8}$$

$$D_\theta(\leq n-s,[\alpha,\lambda],-w)\subset B_\delta(\leq n-s,[\alpha,\lambda];-w') \tag{2.9}$$

$$P\cap A_+(w)=\emptyset=Q\cap A_-(w) \tag{2.10}$$

$$IntP\supset S_\beta\setminus B_\theta([\lambda,\beta];-w);\quad IntQ\supset S_\alpha\setminus B_\theta([\alpha,\lambda];w) \tag{2.11}$$

Therefore $(\alpha,\beta,\delta,P,Q)$ is an input data for every $w\in\mathcal{V}$, and A) is proved. Further, $\partial P\subset B_\theta([\lambda,\beta],-w)$ and (2.7) implies that $\partial P\subset B_\theta(\geq n-s+1,[\lambda,\beta],-w)$. Therefore, $\widehat{w}(\partial P)\subset B_\theta(\leq s-1,[\lambda,\beta],w)\cap f^{-1}(\lambda)$,and the latter is a subset of $X(v)$ for $w\in\mathcal{V}$ by (2.8). Similarly, $(w\in\mathcal{V})\Rightarrow\big((\widehat{-w})(\partial Q)\subset Y(v)\big)$. (2.7) implies that the quadruple $(\widehat{w}|P,(\widehat{-w})(Q);X(v),Y(v))$ also satisfies the conditions $(\mathcal{I})$. Applying 2.26 we obtain $I(\widehat{w}|P,(\widehat{-w})(Q);X(v),Y(v))=\nu_\lambda(p,q;w)$. Now the family of gradients $tw+(1-t)v$ determines

1) a $(X(v),Y(v))$-homotopy between $(\widehat{-w})(Q)$ and $(\widehat{-v})(Q)$,

2) a homotopy H_t between the maps $\widehat{w}|P$ and $\widehat{v}|P$ such that $H_t(\partial P)\subset X(v)$ and $H_t(P)\cap Y(v)=\emptyset$.

Therefore by 2.26 1),2) we have: $\nu_\lambda(p,q;v)=I(\widehat{v}|P,(\widehat{-v})(Q);X(v),Y(v))=I(\widehat{w}|P,(\widehat{-w})(Q);X(v),Y(v))=\nu_\lambda(p,q;w)$. $\square$

Remark 2.28. If v is a good gradient then $\widehat{v}:P\to f^{-1}(\lambda)$ is already transversal to $Q(v)$, therefore their intersection index $\nu_\lambda(p,q;v)$ equals $\nu(p,q;v)$ (as defined in 2.18). $\triangle$

The proof of the following proposition is similar to that of [27, Lemma A.1].

Proposition 2.29. *Let v be an f-gradient, $\mathcal{V}$ be a neighbourhood of v in $Vect^\infty(W)$. Then if $\epsilon>0$ is sufficiently small, there is a perfect f-gradient $w\in\mathcal{V}$ such that w is ϵ-fitted to v.* $\square$

Corollary 2.30. *Let v be an almost good gradient for a regular Morse function $f:W\to[a,b]$, let $p,q\in S(f)$, $\operatorname{ind}p=\operatorname{ind}q+1$. Then the incidence coefficient $\bar\nu(p,q;v)$ from Subsection F equals $\nu_\lambda(p,q;v)$. In particular, the first does not depend on the choice of an auxiliary ordered Morse function, the second does not depend on the choice of a regular value λ.*

Proof. Fix an $\epsilon>0$. Then if w is a good gradient for f, ϵ-fitted and close enough to v, we have $\bar\nu(p,q;v)=\bar\nu(p,q;w)$ (by 2.21), and $\nu_\lambda(p,q;v)=\nu_\lambda(p,q;w)$ (by 2.27). But since w is good, we have $\bar\nu(p,q;w)=\nu(p,q;w)$ (by 2.22)) and $\nu(p,q;w)=\nu_\lambda(p,q;w)$ (by 2.28). $\square$

Definition 2.31. Let $f:W\to[a,b]$ be a Morse function, v be an f-gradient, $[\lambda,\mu]\subset\,]a,b[$ be a regular interval of f. Assume that $df(v)=1$ in $f^{-1}([\lambda,\mu])$. Let $h_\tau,\tau\in[0,1]$ be an isotopy of $V=f^{-1}(\lambda)$ such that $h_\tau(x)=x$ for all τ in some neighbourhood of $\{0\}$ and $h_\tau(x)=h_1(x)$ for all τ in some neighbourhood of $\{1\}$. Denote by Θ the diffeomorphism $V\times[0,\mu-\lambda]\to f^{-1}([\lambda,\mu])$, defined by $(x,\tau)\mapsto\gamma(x,\tau;v)$. Then $\Theta^*(v)=(0,1)$ and $(f\circ\Theta)(x,\tau)=\lambda+\tau$. The vector field $\tilde u(x,\tau)=\big(\frac{\partial}{\partial\tau}h_{\tau/(\mu-\lambda)}(x),1\big)$ is an $f\circ\Theta$-gradient. Denote by u the vector field on W, which equals v outside $Im(\Theta)$ and equals $\Theta_*(\tilde u)$ in $Im\Theta$. We shall say that *u is associated to the vector field v, interval $[\lambda,\mu]$ and the isotopy h_t of $f^{-1}(\lambda)$* . Note that supp $(u-v)\subset\Theta(\text{supp }h\times[0,\mu-\lambda])$.$\triangle$

Corollary 2.32. *Let $f : W \to [a,b]$ be a regular Morse function, v be an almost good f-gradient. In the above terminology let $h : V \times [0,1] \to V$ be an isotopy. Assume that for $\forall x, y \in S(f)$ with $\operatorname{ind} x \le \operatorname{ind} y$ we have: either $D(x,v) \cap \operatorname{supp} h$ or $D(y,-v) \cap \operatorname{supp} h$ is empty. Let $p \in S_s(f), q \in S_{s-1}(f)$, and assume that $K(\le s-1; v) \cap \operatorname{supp} h = K(\le n-s; -v) \cap \operatorname{supp} h = \emptyset$. Denote by u the f-gradient, associated to v and h.*

Then u is almost good and $\nu(p,q;v) = \nu(p,q;u)$.

Proof. 2.20 implies that u is almost good. To calculate $\nu(p,q;u)$ choose an input data $(\alpha, \beta, \delta, P, Q)$ with respect to v and λ in such a way that $\alpha < \lambda < \mu < \beta$, and δ so small that

$$D_\delta(\le s; u) \cap D_\delta(\le n-s; -u) = \emptyset = D_\delta(\le s-1; u) \cap D_\delta(\le n-s+1; -u), \tag{2.12}$$

$$D_\delta(\le s-1; v) \cap \operatorname{supp} h = D_\delta(\le n-s; -v) \cap \operatorname{supp} h = \emptyset \tag{2.13}$$

Since $[\lambda, \mu]$ is regular, we have $K([\lambda,\beta]; -v) = K([\mu,\beta]; -v) = K([\lambda,\beta]; -u) = K([\mu,\beta]; -u)$ and $B_\delta([\lambda,\beta]; -v) = B_\delta([\mu,\beta]; -v) = B_\delta([\lambda,\beta]; -u) = B_\delta([\mu,\beta]; -u)$. Similarly, $K([\alpha,\lambda]; v) = K([\alpha,\mu]; v) = K([\alpha,\lambda]; u) = K([\alpha,\mu]; u)$, and $B_\delta([\alpha,\lambda]; v) = B_\delta([\alpha,\lambda]; u) = B_\delta([\alpha,\mu]; v) = B_\delta([\alpha,\mu]; u)$. Therefore, $(\alpha, \beta, \delta, P, Q)$ is also an input data for u and λ and for u and μ.

Denote $\Theta^{-1}\big(D_\delta(\le s-1; v) \cap f^{-1}(\lambda)\big)$ by $\widehat{X}$, $\Theta^{-1}\big(D_\delta(\le n-s; -v) \cap f^{-1}(\lambda)\big)$ by $\widehat{Y}$. Condition (2.13) imply, that $\Theta^{-1}\big(D_\delta(\le s-1; u)\big) = \Theta^{-1}\big(D_\delta(\le s-1; v)\big) = \widehat{X} \times [0, \lambda - \mu]$ and $\Theta^{-1}\big(D_\delta(\le n-s; -u)\big) = \Theta^{-1}\big(D_\delta(\le n-s; -v)\big) = \widehat{Y} \times [0, \lambda - \mu]$. Denote $\Theta^{-1} \circ \widehat{v}_{[\beta,\lambda]}$ by θ, $\Theta^{-1}\big(\widehat{(-v)}_{[\alpha,\lambda]}(Q)\big)$ by $\widehat{Q}$. Then $\nu(p,q;v) = \nu_\lambda(p,q;v) = I(\theta : P \hookrightarrow V \times \{0\}, \widehat{Q}; \widehat{X}, \widehat{Y})$ and $\nu(p,q;u) = \nu_\mu(p,q;u) = I(h_1 \circ \theta, h_1(\widehat{Q}); \widehat{X}, \widehat{Y})$. These two intersection indices are equal by 2.26. $\square$

§3. Proof of the main theorem.

First we shall recall and reformulate in our present terminology the construction of Novikov complex from [27,§2, p.9-12]. We assume here the terminology of the first paragraph of subsection B of the Introduction. We shall often identify an element of $H^1(M)$ with the corresponding class in $[M, S^1]$.

Definition 3.1. Let M be a closed connected manifold, $\xi \in H^1(M)$ be a nonzero homology class. A triple of objects (f, v, E) will be called *M-triple belonging to* ξ (M for Morse), if:

1) $f: M \to S^1$ is a Morse map belonging to ξ,

3) v is a good f-gradient,

4) E is a map, which associates to each $x \in S(f)$ some lifting of x to $\widetilde{M}$ and an orientation of $D(x, v)$.$\triangle$

Usually the value of ξ is clear from the context, and we shall abbreviate "M-triple belonging to ξ" to "M-triple".

Recollections 3.2. Let $\kappa = (f, v, E)$ be an M-triple. Let $c \in S_i(f), d \in S_{i-1}(f)$, and let γ be a $(-v)$-trajectory, joining c with d. E determines an orientation of $D(p, v)$, a coorientation of $D(q, -v)$, therefore it determines a sign $\varepsilon(\gamma)$ of intersection of $D(p, v)$ with $D(q, -v)$ along γ. Denote by $\widetilde{c}, \widetilde{d}$ the liftings of c, resp. d, to $\widetilde{M}$, determined by E. Let $g(\gamma)$ be the element of G such that the lifting $\widetilde{\gamma}$ of γ to $\widetilde{M}$, starting at $\widetilde{c}$, finishes at $\widetilde{d} \cdot g(\gamma)$. Lemma 2.1 from [27] implies, that in each homotopy class of paths, joining c with d, there is (up to a parameter change $t \mapsto t + C$) only a finite number of $(-v)$-trajectories, and that the formal sum of $\varepsilon(\gamma)g(\gamma)$ over all the $(-v)$-trajectories, joining c with d, is an element of Λ_ξ^-. (In [27] we supposed v to be perfect, but the same proofs work for good v as well.) This element is called incidence coefficient of c and d and is denoted by $n(c, d; v)$ (when the value of v is clear, we drop it from the notation). Denote by $\widehat{C}_i$ the free right Λ_ξ^--module, generated by the critical points of f of index i. Define a homomorphism $\partial_i : \widehat{C}_i \to \widehat{C}_{i-1}$ by the formula $\partial_i(c) = \sum d \cdot n(c, d)$, where $c \in S_i(f)$ and d runs through all the critical points of f of index $(i-1)$. Theorem 2.2 of [27] implies that for every i we have $\partial_{i-1} \circ \partial_i = 0$ and that the resulting complex has the same simple homotopy type, as $C_*^\triangle(\widetilde{M}) \underset{\Lambda}{\otimes} \Lambda_\xi^-$. We shall denote this complex here by $C_*(v, E)$ (and not by $C_*(v)$ as in [27]) to stress its dependance on E. $\triangle$

Assume now that the cohomology class $\xi \in H^1(M)$ is indivisible, or, equivalently that the corresponding homomorphism $\pi_1(M) \to \mathbf{Z}$ is epimorphic. In this case the covering $\mathcal{P}$ is induced from $\mathbf{R} \to S^1$ by any map belonging to ξ. A function $\phi : \overline{M}_\xi \to \mathbf{R}$ is called equivariant, if $\phi(x \cdot t) = \phi(x) - 1$. An equivariant function $\phi : \overline{M}_\xi \to \mathbf{R}$ is called *lifting* of the corresponding quotient map $M \to S^1$.

For an arcwise connected topological space a cohomology class in $H^1(X)$ will be called *regular* if the corresponding homomorphism $\pi_1(X) \to \mathbf{Z}$ is epimorphic and its kernel is a finitely presented group. Let M be a closed connected manifold. A Morse map $f: M \to S^1$ belonging to a class $\xi \in H^1(M)$, is called *regular*, if it has no critical points of indices 0, 1, $n-1$, n, and for any regular value c the preimage $f^{-1}(c)$ is connected and the inclusion $f^{-1}(c) \hookrightarrow M$ induces an isomorphism $\pi_1(f^{-1}(c)) \to \mathrm{Ker}\,(\xi: \pi_1(M) \to \mathbf{Z})$.

Definition 3.3. Let M be a closed connected manifold, $\xi \in H^1(M)$ be a regular class.

A quadruple of objects $(f, \overline{f}, v, E)$ will be called *r-quad* (abbreviation for: "a quadruple of regular Morse data belonging to the regular class $\xi \in H^1(M)$"), if

1) $f: M \to S^1$ is a regular Morse map belonging to ξ,

2) $\overline{f}: \overline{M}_\xi \to \mathbf{R}$ is a lifting of f,

3) v is a good f-gradient,

4) E is a map, which associates to each $x \in S(f)$ some lifting of x to $\widetilde{M}$ and an orientation of $D(x, v)$.$\triangle$

Usually the value of ξ is clear from the context, and we shall abbreviate "r-quad belonging to ξ" to "r-quad".

The following lemma and remark show that r-quads exist.

Lemma 3.4. *Let M be a closed connected manifold, $\dim M \geq 6$, $\xi \in H^1(M)$ be a regular cohomology class.*

Then there exists a regular Morse map $f: M \to S^1$.

Proof. An argument, similar to that of [10, p. 325] shows, that there is a closed connected cooriented submanifold $N \subset M$ of codimension 1, such that this inclusion induces an isomorphism $\pi_1(N) \approx \operatorname{Ker} \xi$, and such that the Pontryagin – Thom construction, applied to N, gives a map, belonging to ξ. Cut M along N and get a regular cobordism W with two identical components $V_0 \approx N \approx V_1$ of the boundary. By Remark 2.14 there is a regular Morse function $h : W \to [0, 1]$. Let v be a C^∞ vector field on a neighbourhood U of N in M, such that $\forall x \in N$ we have $v(x) \notin T_x N$, and that v is positively oriented. It induces a vector field $\widetilde{v}$ in a neighbourhood of ∂W in W, which points inward W in V_0 and outward W in V_1. Therefore there is an f-gradient w, which equals $\widetilde{v}$ in a neighbourhood of ∂W, and (by 8.14) there is a Morse function $g : W \to [0, 1]$, such that $g(x) = T_0(x, w)$ in a neighbourhood of V_0, and $g(x) = 1 - T_1(x, w)$ in a neighbourhood of V_1. Now we glue back V_0 to V_1 and get a map $M \to S^1$ sought.$\square$

Remark 3.5. By [27,Prop. 2.0] for any Morse map $f : M \to S^1$, any f-gradient v, any $\epsilon > 0$ sufficiently small, and any neighbourhood $\mathcal{V}$ of v in $Vect^\infty(M)$, there is a perfect f-gradient $w \in \mathcal{V}$, ϵ-fitted to v. $\triangle$

Corollary 3.6. *Let $\xi \in H^1(M)$ be a regular class.*

Then in the simple homotopy type of $C_*^\triangle(\widetilde{M}) \underset{\Lambda}{\otimes} \Lambda_\xi^-$ there is a complex D_* such that $D_i = 0$ for $i \leq 1$ and for $i \geq n - 1$, where $n = \dim M$.$\square$

Theorem 0.12 follows obviously from the following one.

Theorem 3.7. *Let M be a closed connected manifold, $\dim M = n \geq 6$, $\xi \in H^1(M)$ be a regular class, $\eta = (f, \overline{f}, v, E)$ be an r-quad belonging to ξ. Let D_* be a complex over Λ_ξ^-, simple homotopy equivalent to $C_*(v, E)$. Assume that $D_i = 0$ for $i \leq 1$, $i \geq n - 1$. Let N be an integer.*

Then there exists an r-quad $\kappa = (g, \overline{g}, w, E')$ belonging to ξ such that $C_*(w, E')$ is N-equivalent to D_*.

Definition 3.8. Let C_* be a complex over a ring R. If e is an element of the fixed basis of C_* and $\lambda \in (R)^*$ we denote by $[e; \lambda]C_*$ the complex, obtained from C_* by changing the basis element e to $e \cdot \lambda$. If e, e' are two distinct elements of the same degree of the fixed basis of C_* and $a \in R$ we denote by $[e, e'; a]C_*$ the complex, obtained from C_* by changing the basis elements e to $e + e' \cdot a$.

If G is a group and $\xi : G \to \mathbf{Z}$ a homomorphism, we set $G^- = \{g \in G \mid \xi(g) \leq 0\}$; $G^= = \{g \in G \mid \xi(g) < 0\}$. Denote $\operatorname{Ker} \xi$ by H. If ξ is an epimorphism, we choose and fix an element $\theta \in G$ such that $\xi(\theta) = -1$. Then the Novikov ring from Subsection C of the Introduction is identified with the ring of power series of the form $a = a_{-n}\theta^{-n} + \ldots + a_0 + \ldots + a_n\theta^n + \ldots$. The group U_ξ^- is identified with the group of all $\lambda \in \Lambda_\xi^-$ of the form $\lambda = \varepsilon g(1 + a_1\theta + \ldots + a_N\theta^N + \ldots)$ where $\varepsilon = \pm 1, g \in G, a_i \in \mathbf{Z}H$. Two elements $a = a_{-n}\theta^{-n} + \ldots + a_0 + \ldots$ and $b = b_{-n}\theta^{-n} + \ldots + b_0 + \ldots$ are N-equivalent, if and only if $a_i = b_i$ for all $i \leq N - 1$. For $a \in \Lambda_\xi^-$ set $h(a) = \max(0, \max_{g \in \operatorname{supp}(a)} \xi(g))$.

For a complex C_* over Λ_ξ^-, the following operations will be called *basic*:

1°) $C_* \to [e; \lambda]C_*$, where $\lambda = \pm g$, $g \in G$.

2°) $C_* \to [e; \lambda]C_*$, where $\lambda = 1 + \mu$, $\operatorname{supp} \mu \subset G^=$.

3°) $C_* \to [e, e'; a]C_*$, where $a \in \Lambda_\xi^-$.

4°) $C_* \to C_* \oplus \operatorname{Triv}_*(k, \Lambda_\xi^-)$, where $2 \leq k \leq n - 3$.

5°) $C_* \oplus \operatorname{Triv}_*(k, \Lambda_\xi^-) \to C_*$, where $2 \leq k \leq n - 3$.

If A_* and B_* are complexes over Λ_ξ^- and A_* can be obtained from B_* by a finite sequence of basic operations, we denote by $\rho(A_*, B_*)$ the minimal possible length of such a sequence; if not we set $\rho(A_*, B_*) = \infty$.△

It is clear that $\rho(A_*, B_*) = \rho(B_*, A_*)$ and that if $\rho(A_*, B_*)$ and $\rho(B_*, C_*)$ are finite then $\rho(A_*, C_*)$ is also finite and less or equal to $\rho(A_*, B_*) + \rho(B_*, C_*)$.

Proof of Theorem 3.7 from the results of §§4 - 6. Proposition 1.5, applied to the complexes C'_* and D'_*, where $C'_i = C_{i+2}(v, E), D'_i = D_{i+2}$, to the ring $R = \Lambda_\xi^-$ and to the group of units $G = U_\xi^- \subset R^*$, implies that D_* can be obtained from $C_*(v, E)$ by a finite sequence of basic operations. Therefore $\rho(C_*(v, E), D_*) < \infty$. We prove 3.7 by induction in $\rho(C_*(v, E), D_*)$. If $\rho(C_*(v, E), D_*) = s$ then there is a complex F_* such that $\rho(C_*(v, E), F_*) \leq s - 1$ and $\rho(F_*, D_*) = 1$. Therefore D_* can be obtained from F_* by one basic operation. Suppose for example that it is an operation of the type 3°) with $h(a) = k$. By the induction assumption there is an r-quad $(\varphi, \overline{\varphi}, u, E'')$ such that $C_*(u, E'') \underset{N+k}{\sim} F_*$. Now apply Theorem 6.1.3) and we are over. The other cases are done similarly. □

The rest of this section contains some terminology and a lemma which will be used in §§4 - 6.

Definition 3.9. In Sections 4 - 6 M stands always for a closed connected manifold of dimension ≥ 6, n stands for $\dim M$, ξ for a regular class in $H^1(M)$, $\overline{M}$ for $\overline{M}_\xi$, $\mathcal{Q}$: $\widetilde{M} \to \overline{M}$ for the universal covering projection, $\mathcal{P} : \overline{M} \to M$ for the cyclic covering projection. Normally we shall denote equivariant functions $\overline{M} \to \mathbf{R}$ by overlined letters, e.g. $\overline{f} : \overline{M} \to \mathbf{R}$; the corresponding quotient map $M \to S^1$ (belonging to ξ) will be denoted by the corresponding simple letter, here f. The $\mathcal{Q}$-images of points (or subsets) in $\widetilde{M}$ will be noted by overlines, e.g. $\mathcal{Q}(x) = \overline{x}$. We shall say that $y \in \overline{M}$ is a t-shift of $x \in \overline{M}$, if $y = x \cdot t^s$ for some $s \in \mathbf{Z}$. For an r-quad $(f, \overline{f}, v, E)$ we shall often identify E with the set of corresponding liftings of the points of $S(f)$, thus "$e \in E$" means always "$(e \in \widetilde{M})$&(e is a lifting, determined by E, of a critical point of f)". In this case we shall denote the corresponding set of liftings of $S_k(f)$ by E_k.

For a vector field v on M we note its liftings to $\overline{M}$ and $\widetilde{M}$ by the same letter v. Therefore which is the ambient space of a "v-trajectory" will be clear from the context. If v is a perfect, resp. good, almost good f-gradient then its lifting to $\overline{M}$ is also perfect, resp. good, almost good $\overline{f}$-gradient.

Let $f_1, f_2 : M \to S^1$ be two Morse maps, v_i be f_i-gradients ($i = 1, 2$), $\epsilon > 0$. For $p \in S(f_1)$ denote by $\Phi_1 : U_1(p) \to B^n(0, r_1(p))$ the standard chart with respect to f_1, where $U_1(p)$ is the standard coordinate neighbourhood of p. Similarly, for $q \in S(f_2)$ denote by $\Phi_2 : U_2(q) \to B^n(0, r_2(q))$ the standard chart, where $U_2(q)$ is the standard coordinate neighbourhood of q with respect to f_2. We say that the pairs (f_1, v_1) and (f_2, v_2) are *ϵ-fitted* if:

1)$S(f_1) = S(f_2)$,

2)$\forall p \in S(f_1)$ we have $r_1(p) > \epsilon, r_2(p) > \epsilon$ and on $\Phi_1^{-1}(D^n(0,\epsilon))$ we have $\Phi_1 = \Phi_2, f_1 = f_2 + \mathrm{const}, v_1 = v_2$.

We say that two r-quads $(f, \overline{f}, v, E)$ and $(g, \overline{g}, w, E')$ are *ϵ-fitted*, if the pairs (f, v) and (g, w) are *ϵ-fitted*. We say that two r-quads are fitted, if they are *ϵ-fitted* for some ϵ. $\triangle$

We shall now describe the incidence coefficients in Novikov complex in terms of $\overline{M}$. Let $(f, \overline{f}, v, E)$ be an r-quad. We define a lifting of the set $S(\overline{f})$ to $\widetilde{M}$ by the following rule: if $e \in E$ then $\mathcal{Q}(e)t^s$ is lifted to $e\theta^s$. Let $x, y \in S(f), \mathrm{ind}\, x = \mathrm{ind}\, y + 1$, and let $\alpha, \beta \in \mathbf{R}$ be regular values of $\overline{f} : \overline{M} \to \mathbf{R}$ such that $f(\overline{x}), f(\overline{y}t^s) \in [\alpha, \beta]$. Denote $f^{-1}([\alpha, \beta])$ by $W_{(\alpha,\beta)}$. Since f is regular, we have $\pi_1(W_{(\alpha,\beta)}) = H$, and the restriction $\mathcal{Q}|\mathcal{Q}^{-1}(W_{(\alpha,\beta)}) \to W_{(\alpha,\beta)}$ is the universal covering. By 2.18 the incidence coefficient $\nu(\overline{x}, \overline{y}t^s; v) \in \mathbf{Z}H$ is defined, and obviously it does not depend on the choice of α, β. We denote it by $n_s(x, y; v)$.

For an r-quad $(f, \overline{f}, v, E)$ we denote by $|E; f|$ the number $\max\limits_{x,y \in \mathcal{Q}(E)} |\overline{f}(x) - \overline{f}(y)|$.

Lemma 3.10. *Let $\eta = (f, \overline{f}, v, E)$ be an r-quad. Then:*

1) *for $x, y \in S(f)$, $\mathrm{ind}\, x = \mathrm{ind}\, y + 1$ we have $n(x, y; v) = \sum\limits_{s \in \mathbf{Z}} \theta^s \cdot n_s(x, y; v)$.*

2) *Let N be an integer. Suppose that for an r-quad $\mu = (g, \overline{g}, w, E')$ we have $S(f) = S(g)$, and that $n_s(x, y; v) = n_s(x, y; w)$ for any $x, y \in S(f)$ such that $\mathrm{ind}\, x = \mathrm{ind}\, y + 1$, and for $N > s \geq -\max(|E; f|, |E'; g|)$. Then $C_*(v, E)$ and $C_*(w, E')$ are N-equivalent.*

3) *Let $\epsilon > 0$, N be an integer, w be a good f-gradient, ϵ-fitted to v. Then if w is sufficiently close to v, we have $C_*(v, E) \underset{N}{\sim} C_*(w, E)$.*

Proof. 1) Obvious. To prove 2) we are only to show that $n_s(x, y; v) = n_s(x, y; w)$ for all $s < -\max(|E; f|, |E'; g|)$. For these values of s we have $\overline{f}(\overline{x}) < \overline{f}(\overline{y}t^s)$, $\overline{g}(\overline{x}) < \overline{g}(\overline{y}t^s)$, therefore both terms are equal to zero. To prove 3) apply 2) and 2.19 to the cobordism $W_{(\alpha,\beta)}$ where α and β are chosen so that $W_{(\alpha,\beta)}$ contains all the $\overline{x}t^s$, where $x \in S(f)$ and $N > s \geq -|E; f|$. $\square$

§4. Elimination of trajectories, joining critical points.

Theorem 4.1. *Let $\kappa = (f, \overline{f}, v, E)$ be an r-quad, $N > 1$ an integer. Assume that the Novikov complex $C_*(v, E)$ is N-equivalent to $D_* \oplus \mathrm{Triv}_*(r-1, \Lambda_\xi^-)$, where $3 \leq r \leq n-2$. Denote by a and b the elements of E, corresponding to the generators of* $\mathrm{Triv}_*(r-1, \Lambda_\xi^-)$*; $\partial a = b$, $\mathrm{ind}\, b = r-1$, $\mathrm{ind}\, a = r$.*

Then there exists an r-quad $\eta = (g, \overline{g}, w, E)$, fitted to κ such that

1) *There are no $(-w)$-trajectories, joining $\overline{a}$ with $\overline{y}t^s$ in $\overline{M}$, if $y \in E$, $\mathrm{ind}\, y = r-1$, $y \neq b$, $s < N$.*

2) *There are no $(-w)$-trajectories, joining $\overline{x}$ with $\overline{b}t^s$ in $\overline{M}$, if $x \in E$, $\mathrm{ind}\, x = r$, $x \neq a$, $s < N$.*

3) $\overline{g}(\overline{b}) < \overline{g}(\overline{a}) < \overline{g}(\overline{b}) + 1$.

4) *There is (up to a parameter change $t \mapsto t + C$) only one $(-w)$-trajectory γ, joining $\overline{a}$ with $\overline{b}$, $g(\gamma) = \mathbf{1} \in H$, and $\varepsilon(\gamma) = 1$.*

5) *$C_*(w, E)$ is N-equivalent to $C_*(v, E)$.*

We shall deduce this theorem from Theorem 4.2.

Recall that $x \in \overline{M}$ is a *t-shift* of $y \in \overline{M}$, if there is $s \in \mathbf{Z}$ such that $x = yt^s$.

Theorem 4.2. *Let $\kappa = (f, \overline{f}, v, E)$ be an r-quad. Let $p, q \in \overline{M}$ be critical points of $\overline{f}$ such that $\overline{f}(q) < \overline{f}(p)$, $\mathrm{ind}\, p = r = \mathrm{ind}\, q + 1$ and $\overline{f}$ has no other critical values in $[\overline{f}(q), \overline{f}(p)]$. Assume that there are two $(-v)$-trajectories γ_1, γ_2 in $\overline{M}$, joining p with q, such that $g(\gamma_1) = g(\gamma_2)$, $\varepsilon(\gamma_1) = -\varepsilon(\gamma_2)$. Let K be any finite subset of $S(\overline{f})$, N be any integer.*

Then there exists a good gradient w for f such that

1) *$N(p, q; w)$ is homotopic to $N(p, q; v) \setminus \{(\gamma_1, \varepsilon(\gamma_1)), (\gamma_2, \varepsilon(\gamma_2))\}$.*

2) *If $x, y \in K$, $\mathrm{ind}\, x = \mathrm{ind}\, y + 1$, then $\nu(x, y; w) = \nu(x, y; v)$.*

3) *Moreover for $x, y \in K$, $\mathrm{ind}\, x = \mathrm{ind}\, y + 1$, the sets $N(x, y; v)$ and $N(x, y; w)$ are homotopic in the following cases:*

   A) *x is not a t-shift of p, y is not a t-shift of q.*

   B) *$r \geq 4$, y is not a t-shift of q.*

   C) *$r \leq n-3$, x is not a t-shift of p.*

4) *$C_*(w, E)$ is N-equivalent to $C_*(v, E)$.*

Proof of 4.2. Note that $\overline{f}(p) - \overline{f}(q) < 1$. Choose regular values c, λ such that $c < \overline{f}(q) < \lambda < \overline{f}(p) < c + 1$. We can multiply v by any positive t-invariant function φ on $\overline{M}$ such that $\mathrm{supp}\,(\varphi - 1)$ does not contain $S(\overline{f})$; that does not change neither the manifolds $D(p, v)$, nor the v-trajectories (up to a parameter change), nor the Novikov complex. Therefore we shall assume that for some $\mu > \lambda$ the function $d\overline{f}(v)$ is equal to 1 in $\overline{f}^{-1}([\lambda, \mu])$. Denote $\overline{f}^{-1}(\lambda)$ by V, $D(p, v) \cap V$ by P, $D(q, -v) \cap V$ by Q. The manifolds P, Q are smoothly embedded spheres in V; $\dim\, P = r - 1$, $\dim\, Q = n - r$. (Note that $2 \leq r - 1$, $n - r \leq n - 3$.) Since v is good, they are transversal; P is oriented, Q is cooriented. Denote $\gamma_1 \cap V$ by A, $\gamma_2 \cap V$ by B; these are points of intersection of P with Q of opposite signs. Choose a positive integer Z so large that $K \subset \overline{f}^{-1}([c - Z, c + Z + 1])$, and that for every $z \in E$ and for every integer s, satisfying $-|E, f| \leq s \leq N - 1$ we have:

$\overline{z}, \overline{z}t^s \in \overline{f}^{-1}([c-Z, c+Z+1])$. Set

$$W = \overline{f}^{-1}([c-Z, c+Z+1]), \quad W_1 = \overline{f}^{-1}([c-2Z-1, c+2Z+2]).$$

Denote by K_λ^+ the intersection of $\overline{f}^{-1}([\lambda, \infty[)$ with the union of all discs $D(z,v)$, where $z \in S(\overline{f})$, $z \in W_1$, $z \neq p$, $\operatorname{ind} z \leq r$ and $\operatorname{ind} z \leq n-3$. Similarly, denote by K_λ^- the intersection of $\overline{f}^{-1}(]-\infty, \lambda])$ with the union of all $D(z,-v)$, where $z \in S(\overline{f}) \cap W_1$, $z \neq q$, $\operatorname{ind} z \geq r-1$ and $\operatorname{ind} z \geq 3$. As a consequence of Lemma 2.11.2) (applied to $(-v)$ for the case of K_λ^-) we have the following:

Remark 4.3. 1) K_λ^+ is compact; it does not intersect $D(p,v)$.

2) K_λ^- is compact; it does not intersect $D(q,-v)$. $\triangle$

Lemma 4.4. *There is a smooth isotopy $h : V \times [0,1] \to V$ such that:*

1) $\operatorname{supp} h \cap K_\lambda^+ = \operatorname{supp} h \cap K_\lambda^- = \emptyset$.

2) $\operatorname{supp} h \cap (P \cap Q) = \{A, B\}$.

3)$h_1(Q) \pitchfork P$ *and* $h_1(Q) \cap P = (Q \cap P) \setminus \{A, B\}$.

4) *For an* $\epsilon > 0$ *we have:* $h_\tau = \mathrm{id}$ *for* $\tau < \epsilon$ *and* $h_\tau = h_1$ *for* $\tau > 1-\epsilon$.

In order not to interrupt the exposition we have placed the essential part of the proof of this lemma to §9. It differs from the standard Whitney elimination procedure (see [46],[18, §6]) mainly in what concerns the part 1) of our statement.

Proof of 4.4 from 9.1. We have to verify the applicability of 9.1. We keep the terminology, introduced before 4.3. Further, denote by $\mathbf{X}$ the family of submanifolds $D(z,v) \cap V$, where $z \in S(\overline{f})$, $z \in W_1$, $z \neq p$, $\operatorname{ind} z \leq r$ and $\operatorname{ind} z \leq n-3$; then $X = K_\lambda^+$. Denote by $\mathbf{Y}$ the family of submanifolds $D(z,-v) \cap V$, where $z \in S(\overline{f}) \cap W_1$, $z \neq q$, $\operatorname{ind} z \geq r-1$ and $\operatorname{ind} z \geq 3$; then $Y = K_\lambda^-$. Since v is good, we have $D(p,v) \pitchfork \mathbf{Y}$ and $D(q,-v) \pitchfork \mathbf{X}$. By Remark 4.3 we have $X \cap P = Y \cap Q = \emptyset$. Therefore the part of the hypotheses of 9.1, concerning $\mathbf{X}$ and $\mathbf{Y}$ is verified.

If $\dim P = \dim V - 2$ then $\operatorname{ind} p = n-2$. The inclusion $V \setminus P \hookrightarrow V$ induces an epimorphism in π_1. To show, that it is a monomorphism, let θ be a loop in $V \setminus P$, homotopic to zero in V, and, therefore, in $\overline{M}$. Consider the diffeomorphism $\widehat{(-v)}_{[\lambda, \lambda']}$, where λ' is a regular value of $\bar{f}$ such that $]\overline{f}(p), \lambda'[$ is regular. The loop $\widehat{(-v)}(\theta)$ is homotopic to 0 in $\overline{M}$, therefore in $\overline{f}^{-1}(\lambda')$, therefore in $\overline{f}^{-1}(\lambda') \setminus D(p,-v)$ (since $\dim D(p,-v) = 2$), therefore $\theta \sim 0$ in $V \setminus P$. The inclusion $V \setminus Q \hookrightarrow V$ is considered similarly.

The condition $g(\gamma_1) = g(\gamma_2)$ implies that the liftings to $\widetilde{M}$ of the paths γ_1 and γ_2, starting at the same point, end in the same point. It is easy to deduce from it that the liftings to $\widetilde{M}$ of a path, joining A with B in P, and of a path, joining A to B in Q, finish at the same point, if they start at the same point. Since the inclusion $V \hookrightarrow W$ induces an isomorphism in π_1, this finishes the verification of the hypotheses of 9.1. $\square$

Going on with the proof of 4.2, denote by u the vector field on $\overline{f}^{-1}([c, c+1])$, associated to v and to the isotopy h (see def. 2.31). Extend it to a t-invariant $\overline{f}$-gradient on $\overline{M}$, which will be denoted by the same letter u. Denote by Σ the union of $\big(\Theta(\operatorname{supp} h \times [0, \mu-\lambda])\big)t^s$, where $s \in \mathbf{Z}$; $\Sigma \subset \overline{M}$ (see the definition of Θ in 2.31). Note that $\operatorname{supp}(u-v) \subset \Sigma$.

Lemma 4.5. *The f-gradient u, constructed above, has the following properties*:

i) $u \mid W$ *is almost good.*

ii) $D(p,u) \pitchfork D(q,-u)$ *and* $N(p,q;u) = N(p,q;v) \setminus \{(\gamma_1, \varepsilon(\gamma_1)), (\gamma_2, \varepsilon(\gamma_2))\}$.

iii) *For* $x, y \in S(\overline{f}) \cap W$, $\operatorname{ind} x = \operatorname{ind} y + 1$, *we have* $\nu(x,y;u) = \nu(x,y;v)$.

iv) *Moreover, let* $x, y \in S(\overline{f}) \cap W$, $\operatorname{ind} x = \operatorname{ind} y + 1$. *Then in the cases* A), B), C) *below we have:* $D(x,u) \pitchfork D(y,-u)$ *and* $N(x,y;u) = N(x,y;v)$.

A) x *is not a* t*-shift of* p, *and* y *is not a* t*-shift of* q.

B) $r \geq 4$ *and* y *is not a* t*-shift of* q.

C) $r \leq n-3$ *and* x *is not a* t*-shift of* p.

Proof. We say that $x \in S(\overline{f})$ is of type $(*)$, if:

$$(x \text{ is not a } t\text{–shift of } p) \,\&\, (\operatorname{ind} x \leq r) \,\&\, (\operatorname{ind} x \leq n-3). \qquad (*)$$

We say that $y \in S(\overline{f})$ is of type $(**)$, if:

$$(y \text{ is not a } t\text{–shift of } q) \,\&\, (\operatorname{ind} y \geq r-1) \,\&\, (\operatorname{ind} y \geq 3). \qquad (**)$$

From 4.4.1) one deduces easily that if $x \in S(\overline{f}) \cap W$ is of type (*) then $D(x,v) \cap W \cap \Sigma = \emptyset$. Similarly, for $y \in S(\overline{f}) \cap W$ of type (**) we have $D(y,-v) \cap W \cap \Sigma = \emptyset$. Note that if $x, y \in S(\overline{f})$, $\operatorname{ind} x \leq \operatorname{ind} y$ then either x is of type (*), or y is of type (**). Apply now 2.20.1) and i) is proved. To prove iv) note that for $x, y \in S(\overline{f})$ the condition $\operatorname{ind} x = \operatorname{ind} y + 1$ together with (A) or together with (B), or together with (C) imply again that either x is of type (*), or y is of type (**), and applying 2.20.2),3) we obtain iv). The point ii) follows from the construction of u.

In order to prove iii), let k be an integer, $-Z \leq k \leq Z$, and denote by u_k the vector field on W, which is equal to u in $\overline{f}^{-1}([c-Z, c+k])$ and is equal to v in $\overline{f}^{-1}([c+k, c+Z+1])$. For each k the field u_k is an $(\overline{f}|W)$-gradient. The point iii) is a consequence of the following lemma.

Lemma 4.6. 1) u_k *is an almost good* $(\overline{f}|W)$*-gradient.*

2) *For* $x \in S(\overline{f}) \cap W$ *of type* (*) *we have:* $D(x,u_k) \cap W = D(x,v) \cap W$.

3) *For* $y \in S(\overline{f}) \cap W$ *of type* (**) *we have:* $D(y,-u_k) \cap W = D(y,-v) \cap W$.

4) $\nu(p,q;u_k) = \nu(p,q;v)$.

Proof. Induction in k. Assume that 4.6 is true for some $k \in [-Z, Z-1]$; we shall deduce 4.6 for $k+1$. The vector field u_{k+1} is associated to u_k and to the isotopy h' of $\overline{f}^{-1}(k+\lambda)$, where $h'_\tau = t^{-k} \circ h_\tau \circ t^k$. Note that $\operatorname{supp} h' \subset \Sigma$. One sees easily that the hypotheses of Corollary 2.32 hold for the Morse function $(\overline{f}|W) : W \to [c-Z, c+Z+1]$, the almost good $(\overline{f}|W)$-gradient u_k, the isotopy h' of $\overline{f}^{-1}(k+\lambda)$, and the pair of critical points $pt^{-k} \in S_r(\overline{f}), qt^{-k} \in S_{r-1}(\overline{f})$. Therefore 2.32 implies 1) and 4). To obtain 2) note that for x of type (*) we have $\big(D(x,u_k) \cap W\big) \cap \operatorname{supp}\,(u_k - u_{k+1}) = \emptyset$. 3) is proved similarly. $\square$

To finish the proof of 4.2, choose a good f-gradient w, sufficiently close to u, and apply 2.19.2) to get 1), 2), and 3) of our conclusions and apply 3.10.2) and 3.10.3) to obtain 4). $\square$

The proof of 4.1 will follow from Lemmas 4.8 - 4.14 below.

Definition 4.7. We say that a set $X \subset \overline{M}$ is *bounded from below*, resp. *bounded from above*, if for some equivariant continuous function (and then for any such function)

$F : \overline{M} \to \mathbf{R}$, the set $\overline{F}(X) \subset \mathbf{R}$ is bounded from below, resp. from above. A set $X \subset \overline{M}$ is called *positive*, resp. *negative*, if it is bounded from below (resp. from above) and closed under t^{-1} (resp. t). $\triangle$

Lemma 4.8. *Let $\kappa = (f, \overline{f}, v, E)$ be an r-quad, N be any integer, k an integer between 3 and $n-2$. Let $\alpha \in S_k(\overline{f}), \beta \in S_{k-1}(\overline{f})$. Let $\Pi \subset S_{k-1}(\overline{f})$ be a positive subset such that $\nu(\alpha, y; v) = 0$ for every $y \in \Pi$. Let $\Upsilon \subset S_k(\overline{f})$ be a negative subset, such that $\nu(x, \beta; v) = 0$ for every $x \in \Upsilon$.*

Then there exists an r-quad $\eta = (g, \overline{g}, w, E)$, fitted to κ, such that $C_*(w, E)$ is N-equivalent to $C_*(v, E)$ and $\nu(x, \beta; w) = 0$ for every $x \in \Upsilon$ and $\overline{g}(\alpha) < \overline{g}(y)$ for every $y \in \Pi$.

Proof. For any continuous equivariant function $F : \overline{M} \to \mathbf{R}$ denote by $\Pi(F)$ the finite set $\{y \in \Pi | F(y) \leq F(\alpha)\}$. Denote by $\mathcal{R}$ the set of all r-quads $(f', \overline{f'}, v', E')$, fitted to κ such that $E' = E$ and $C_*(v', E)$ is N-equivalent to $C_*(v, E)$ and $\nu(\alpha, y; v') = 0$ for every $y \in \Pi$, and $\nu(x, \beta; v') = 0$ for every $x \in \Upsilon$. Choose $\eta = (g, \overline{g}, w, E) \in \mathcal{R}$ with minimal possible card$(\Pi(\overline{g}))$. I claim that $\Pi(\overline{g}) = \emptyset$, and, thus, η satisfies the conclusions of our lemma. Indeed, if $\Pi(\overline{g}) \neq \emptyset$, choose such a regular value c of $\overline{g}$ that $c < \overline{g}(\alpha) < c+1$ and the interval $]\overline{g}(\alpha), c+1[$ is regular. Denote $\overline{g}^{-1}([c, c+1])$ by W. Since Π is closed under t^{-1}, we have $\Pi \cap W \neq \emptyset$. Let $z \in \Pi \cap W$ and choose (by 2.7) an ordered Morse function $A : W \to [c, c+1]$, adjusted to $(\overline{g}, w)$ such that for some regular values $\lambda < \mu$ of A the only critical points of A in $A^{-1}([\lambda, \mu])$ are z and α and that $A(z) < A(\alpha)$. Extend A to an equivariant function $\overline{h} : \overline{M} \to \mathbf{R}$; then $\eta_1 = (h, \overline{h}, w, E) \in \mathcal{R}$.

Among good h-gradients u such that $(h, \overline{h}, u, E) \in \mathcal{R}$ choose one with minimal possible value of card$(N(\alpha, z; u))$. Denote it by u_0. I claim that $N(\alpha, z; u_0) = 0$. Indeed, if not then there are two $(-u_0)$-trajectories γ_1, γ_2, joining α with z such that $\varepsilon(\gamma_1) = -\varepsilon(\gamma_2)$ and $g(\gamma_1) = g(\gamma_2)$ (because $\nu(\alpha, z; u_0) = 0$). Choose a finite subset K of $S(\overline{h})$ in such a way that 1) $\alpha, \beta \in K$, 2) every $y \in \Pi$, such that $\overline{h}(\alpha) > \overline{h}(y)$, is in K, 3) every $x \in \Upsilon$ such that $\overline{h}(x) > \overline{h}(\beta)$, is in K. Apply now 4.2 to the critical points α, z of $\overline{h}$ and to the set K and get a good gradient u_1 of h such that card$(N(\alpha, z; u_1)) =$ card$(N(\alpha, z; u_0)) - 2$, and $C_*(u_0, E)$ is N-equivalent to $C_*(u_1, E)$ and $\nu(\alpha, y; u_1) = \nu(\alpha, y; u_0) = 0$ for all $y \in \Pi$ and $\nu(x, \beta; u_1) = \nu(x, \beta; u_0) = 0$ for all $x \in \Upsilon$ (three last points follow from our choice of K.)

Denote $\overline{h}^{-1}([\lambda, \mu])$ by W_0. Apply Lemma 2.7 to $A | W_0 : W_0 \to [\lambda, \mu]$ and get a Morse function $A_0 : W_0 \to [\lambda, \mu]$, adjusted to $(A | W_0, u_0)$, and such that $A_0(z) > A_0(\alpha)$. By Lemma 2.15 H_0 is regular. Extend A_0 to W, setting $A_0(x) = A(x)$ for $x \in W \setminus W_0$ and extend the result to an equivariant function $\overline{h}_0$ on $\overline{M}$. It is easy to see that $(h_0, \overline{h}_0, u_0, E) \in \mathcal{R}$ and that card$(\Pi(\overline{h}_0)) =$ card$(\Pi(\overline{g})) - 1$. $\square$

The following lemma is a "dual version" of 4.8 and is proved similarly.

Lemma 4.9. *Let $\kappa = (f, \overline{f}, v, E)$ be an r-quad, N any integer, k be an integer between 3 and $n-2$. Let $\alpha \in S_k(\overline{f})$,$\beta \in S_{k-1}(\overline{f})$. Let $\Pi \subset S_{k-1}(\overline{f})$ be a positive set such that $\nu(\alpha, y; v) = 0$ for every $y \in \Pi$. Let $\Upsilon \subset S_k(\overline{f})$ be a negative subset such that $\nu(x, \beta; v) = 0$ for every $x \in \Upsilon$.*

Then there is an r-quad $\eta = (g, \overline{g}, w, E)$, fitted to κ such that $C_*(w, E)$ is N-equivalent to $C_*(v, E)$ and $\nu(\alpha, y; w) = 0$ for every $y \in \Pi$ and $\overline{g}(\beta) > \overline{g}(z)$ for every $z \in \Upsilon$. $\square$

Definition 4.10. For the rest of §4 we adopt the following terminology and conventions. In Lemmas 4.11 - 4.14 we assume the hypotheses of 4.1. We denote $\overline{a}$ by α, $\overline{b}$ by β. Define a positive set $\widehat{E}_{r-1}$ by

$$\widehat{E}_{r-1} = \{\overline{y}t^s | y \in E, y \neq b, \operatorname{ind} y = r-1, s < N\} \cup \{\beta t^s | s < 0\}.$$

Define a negative set $\widehat{E}_r$ by

$$\widehat{E}_r = \{\overline{x}t^s | x \in E, x \neq a, \operatorname{ind} x = r, s > -N\} \cup \{\alpha t^s | s > 0\}.$$

W stands always for $\overline{g}^{-1}([c, c+1])$. $\triangle$

Lemma 4.11. *Let $\Upsilon \subset S_r(\overline{f})$ be a negative set such that $\nu(x, \beta; v) = 0$ for each $x \in \Upsilon$.*

Then there is an r-quad $\eta = (g, \overline{g}, w, E)$, fitted to κ, satisfying 1), 3), 4) *and* 5) *of the conclusions of* 4.1 *and such that $\nu(x, \beta; w) = 0$ for each $x \in \Upsilon$.*

Proof. Denote by $\mathcal{S}$ the set of all r-quads $(f', \overline{f'}, v', E')$, fitted to κ, and satisfying the following condition:

$(\mathcal{S})$: $E' = E$; $C_*(v', E)$ is N-equivalent to $C_*(v, E)$; for every $y \in \widehat{E}_{r-1}$ we have $\overline{f'}(y) > \overline{f'}(\alpha)$; for every $x \in \Upsilon$ we have $\nu(x, \beta; v) = 0$.

It is obvious that if $(f', \overline{f'}, v', E')$ is in $\mathcal{S}$, then it satisfies the conditions 1),3),5) of 4.1. (Indeed, the only point to verify is $\overline{f'}(\alpha) > \overline{f'}(\beta)$; this follows because $N > 1$ and, therefore, we have $\nu(\alpha, \beta; v) = \mathbf{1} \in H$.) Note that $\mathcal{S} \neq \emptyset$. (Indeed, the hypotheses of 4.1 imply that $\nu(\alpha, y; v) = 0$ for all $y \in \widehat{E}_{r-1}$. Apply now 4.8 to the negative set Υ and the positive set $\widehat{E}_{r-1}$.)

In order to satisfy also 4) of 4.1, choose in $\mathcal{S}$ an r-quad $\eta = (g, \overline{g}, w, E)$ with the minimal possible value of $\operatorname{card}(N(\alpha, \beta; w))$. I claim that $\operatorname{card}(N(\alpha, \beta; w)) = 1$, and, therefore, $(g, \overline{g}, w, E)$ satisfies 4) of 4.1. Indeed, if not then choose a regular value c of $\overline{g}$ in such a way that $c < \overline{g}(\beta) < \overline{g}(\alpha) < c+1$, and that $]\overline{g}(\alpha), c+1[$ is regular. Then $\overline{g}(\overline{y}) > c+1$ for every $y \in \widehat{E}_{r-1}$. Choose an ordered Morse function $A : W \to [c, c+1]$, adjusted to $(\overline{g}, w)$ such that for some regular values $\lambda < \mu$ of A the only critical points of A in $A^{-1}([\lambda, \mu])$ are β and α. Extend A to an equivariant function $\overline{h} : \overline{M} \to \mathbf{R}$, and note that $(h, \overline{h}, w, E) \in \mathcal{S}$. Similarly to the proof of 4.8 we apply 4.2 to the critical points α, β and eliminate two $(-w)$ trajectories. $\square$

In the same way one proves the dual version:

Lemma 4.12. *Let $\Pi \subset S_{r-1}(\overline{f})$ be a positive set such that $\nu(\alpha, y; v) = 0$ for every $y \in \Pi$.*

Then there is an r-quad $\eta = (g, \overline{g}, w, E)$, fitted to κ, satisfying 2), 3), 4) 5) *of the conclusions of* 4.1, *and such that $\nu(\alpha, y; v) = 0$ for every $y \in \Pi$.* $\square$

Lemma 4.13. *Assume that $r = \operatorname{ind} a \leq n-3$. Let $\Upsilon \subset S_r(\overline{f})$ be a negative subset, containing no t-shift of α, and such that $\forall x \in \Upsilon$ we have $\nu(x, \beta; v) = 0$.*

Then there is an r-quad $\eta = (g, \overline{g}, w, E)$, fitted to κ such that η satisfies 1), 3), 4), *and* 5) *of the conclusions of* 4.1, *and such that $\overline{g}(\beta) > \overline{g}(x)$ for every $x \in \Upsilon$.*

Proof. For any continuous equivariant function $F : \overline{M} \to \mathbf{R}$ denote by $\Upsilon(F)$ the finite set $\{x \in \Upsilon | F(x) \geq F(\beta)\}$. Denote by $\mathcal{T}$ the set of all the r-quads $\theta = (f', \overline{f'}, v', E')$, such that $E = E'$, θ is fitted to κ, and θ satisfies 1), 3), 4) and 5) of the conclusions of 4.1, and $\nu(x, \beta; v') = 0$ for every $x \in \Upsilon$.

The set $\mathcal{T}$ is not empty by 4.11. In the set $\mathcal{T}$ choose an r-quad $(g, \overline{g}, w, E)$ with the minimal possible value of $\mathrm{card}(\Upsilon(\overline{g}))$. I claim that $\Upsilon(\overline{g}) = \emptyset$, and, therefore, $(g, \overline{g}, w, E)$ satisfies the conclusions of our lemma. Indeed, if not, choose a regular value c of $\overline{g}$ such that $c < \overline{g}(\beta) < c+1$, and that $[c, \overline{g}(\beta)[$ is regular. There is $z \in \Upsilon(\overline{g})$ such that $c < \overline{g}(\beta) < \overline{g}(z) < c+1$, and $c < \overline{g}(\alpha) < c+1$. Choose an ordered Morse function $A : W \to [c, c+1]$, adjusted to $(\overline{g}|W, w)$, such that for some regular values $\lambda < \mu$ of A the only critical points of A in $W_0 = A^{-1}([\lambda, \mu])$ are β and z. Extend A to an equivariant function $\overline{h} : \overline{M} \to \mathbf{R}$, and note that $(h, \overline{h}, w, E) \in \mathcal{T}$, and that $\Upsilon(\overline{h}) = \Upsilon(\overline{g})$. In a set of all good h-gradients u such that $(h, \overline{h}, u, E) \in \mathcal{T}$, choose one with the minimal possible value of $\mathrm{card}(N(z, \beta; u))$. Denote it by u_0. I claim that $N(z, \beta; u_0) = \emptyset$.

Indeed, if not, there are two $(-u_0)$-trajectories γ_1, γ_2, joining z with β, and such that $\varepsilon(\gamma_1) = -\varepsilon(\gamma_2)$ and $g(\gamma_1) = g(\gamma_2) \in H$. Let $K \subset S(\overline{f})$ be a finite set such that: K1) $\alpha, \beta \in K$, K2) every $x \in \Upsilon$ such that $\overline{h}(x) \geq \overline{h}(\beta)$, is in K, K3) if $y \in \widehat{E}_{r-1}$, and $\overline{h}(\alpha) \geq \overline{h}(\overline{y})$ then $y \in K$. Apply 4.2 to the $(-u_0)$-trajectories γ_1, γ_2, and the set K. Denote the resulting good h-gradient by u_1. We shall verify that $(h, \overline{h}, u_1, E) \in \mathcal{T}$. Indeed, $\nu(x, \beta; u_1) = 0$ for every $x \in \Upsilon$ by K2) and 4.2.2). The point 3) of the conclusions of 4.1 still hold, because we did not change h; 5) holds by 4.2.4). To verify 1), let $y \in E, \mathrm{ind}\, y = r-1, y \neq b$ and let $s < N$, and assume that $\overline{h}(\alpha) \geq \overline{h}(\overline{y}t^s)$. Then α and $\overline{y}t^s$ are in K, α is not a t-shift of z, $\overline{y}t^s$ is not a t-shift of β, therefore 4.2.3.A) applies to obtain $N(\alpha, \overline{y}t^s; u_1) \sim N(\alpha, \overline{y}t^s; u_0) = \emptyset$. To verify 4) note that $\alpha, \beta \in K$, and α is not a t-shift of z and $\mathrm{ind}\, z = \mathrm{ind}\, \alpha = r \leq n-3$, therefore 4.2.C) applies to obtain $N(\alpha, \beta; u_1) \sim N(\alpha, \beta; u_0)$. Therefore $(h, \overline{h}, u_1, E) \in \mathcal{T}$, but $\mathrm{card}(N(z, \beta; u_1))= \mathrm{card}(N(z, \beta; u_0)) - 2$.

Now apply Theorem 2.6 to the cobordism W_0 and get an r-quad $(h_1, \overline{h}_1, u_1, E) \in \mathcal{T}$ such that $\mathrm{card}(\Upsilon(\overline{h}_1))=\mathrm{card}(\Upsilon(\overline{h})) - 1 = \mathrm{card}(\Upsilon(\overline{g})) - 1$. $\square$

In the same way one proves the following dual version:

Lemma 4.14. *Assume that $r = \mathrm{ind}\, a \geq 4$. Let $\Pi \subset S_{r-1}(\overline{f})$ be a positive subset, containing no t-shift of β, and such that $\forall y \in \Pi$ we have $\nu(\alpha, y; v) = 0$.*

Then there is an r-quad $\eta = (g, \overline{g}, w, E)$, fitted to κ such that η satisfies 2), 3), 4), *and* 5) *of the conclusions of* 4.1, *and such that $\overline{g}(\alpha) < \overline{g}(y)$ for every $y \in \Pi$.* $\square$

Proof of 4.1 for the case $r \leq n-3$. Apply 4.13 to the negative subset $\Upsilon = \{\overline{z}t^s | z \in E, \mathrm{ind}\, z = r, z \neq a, s > -N\}$. For the resulting r-quad $(g, \overline{g}, w, E)$ we have 1),3),4),5) of 4.1, and, moreover,2) holds, since $\overline{g}(\overline{b}) > \overline{g}(x)$ for $x \in \Upsilon$, therefore there are no $(-w)$-trajectories, joining $\overline{z}$ to $\overline{b}t^r$, where $z \in E, z \neq a, \mathrm{ind}\, z = r, r < N$. $\square$

Proof of 4.1 for the case $r \geq 4$. Similar to the above (apply 4.14 to the positive subset $\Pi = \{\overline{z}t^s | z \in \widehat{E}, \mathrm{ind}\, z = r-1, z \neq b, s < N\}$.) $\square$

§5. Cancelling and adding of trivial direct summand.

A. Cancelling.

Theorem 5.1. *Let $\kappa = (f, \overline{f}, v, E)$ be an r-quad. Let $N \geq 1$, and assume that $C_*(v, E)$ is N-equivalent to $D_* \oplus \mathrm{Triv}_*(r-1, \Lambda_\xi^-)$, where D_* is a complex, $3 \leq r \leq n-2$.*

Then there exists an r-quad $\eta = (g, \overline{g}, w, E')$, such that $C_*(w, E') \underset{N}{\sim} D_*$.

Proof. We assume the terminology of Theorem 4.1. Let $\eta = (g, \overline{g}, w, E)$ be an r-quad, satisfying the conclusions of 4.1. Choose a regular value c of $\overline{g}$ such that $c < \overline{g}(\overline{b}) < \overline{g}(\overline{a}) < c+1$. Applying if necessary, Lemma 2.7 to the cobordism $\overline{g}^{-1}([c, c+1])$ we can assume that for some regular values $\lambda, \mu \in]c, c+1[, \lambda < \mu$ the only critical points of $\overline{g}$ in $\overline{g}^{-1}([\lambda, \mu])$ are $\overline{a}$ and $\overline{b}$. Denote by γ the unique (up to a parameter change $t \mapsto t + C$) $(-w)$-trajectory, joining $\overline{a}$ with $\overline{b}$, and denote by Γ the closure of the image of γ.

Choose a positive integer Z so large that the cobordism $W = \overline{g}^{-1}([c - Z, c + Z + 1])$ contains the projection $\mathcal{Q}(E)$ of E to $\overline{M}$ together with all t-shifts $\mathcal{Q}(E)t^s$ for $N > s \geq -|E, g|$. Denote by W_1 the cobordism $\overline{g}^{-1}([c - 2Z, c + 2Z + 1])$. Denote by K^+ the intersection with W_1 of the union of all the $D(p, w)$, where $p \in W_1 \cap S(\overline{g})$, $\operatorname{ind} p \leq r - 1$, and p is not a t-shift of $\overline{b}$. By 2.11.5) this set is compact. Denote by K^- the intersection with W_1 of the union of all the $D(q, -w)$, where $q \in W_1 \cap S(\overline{g})$, $\operatorname{ind} q \geq r$ and q is not a t-shift of $\overline{a}$; K^- is also compact. Note that $K^+ \cap \Gamma = K^- \cap \Gamma = \emptyset$. Choose an open neighbourhood U_1 of Γ such that $K^+ \cap U_1 = K^- \cap U_1 = \emptyset$, and that $\overline{U_1} \subset \overline{g}^{-1}([\lambda, \mu])$. By 2.8 there is an open neighbourhood U_2 of Γ such that: 1) if for some $p \in S_r(\overline{g}) \cap W_1$ there is a $(-w)$-trajectory, joining p with a point of U_2 then there is a $(-w)$-trajectory, joining p with $\overline{b}$, and 2) if for some $q \in S_{r-1}(\overline{g}) \cap W_1$ there is a $(-w)$-trajectory, joining a point of U_2 with q, then there is a $(-w)$-trajectory, joining $\overline{a}$ with q. Denote by k^+ the intersection with W of the union of all the $D(p, w)$, where $p \in W \cap S(\overline{g})$, $\operatorname{ind} p \leq r - 1$, and p is not a t-shift of $\overline{b}$. By 2.11.5) this set is compact. Denote by k^- the intersection with W of the union of all the $D(q, -w)$, where $q \in W \cap S(\overline{g})$, $\operatorname{ind} q \geq r$ and q is not a t-shift of $\overline{a}$; k^- is also compact. It follows from this construction that the intersection $U = U_1 \cap U_2$ satisfies the following condition:

($\mathcal{L}$) For every $s \in \mathbf{Z}, -Z \leq s \leq Z$ we have: $k^+ \cap Ut^s = k^- \cap Ut^s = \emptyset$. If $p \in S_r(\overline{g}) \cap W$ and for some $s \in \mathbf{Z}, -Z \leq s \leq Z$ there is a $(-w)$-trajectory, joining p with a point of $U \cdot t^s$ then there is a $(-w)$-trajectory, joining p with $\overline{b}t^s$. If $q \in S_{r-1}(\overline{g}) \cap W$ and for some $s \in \mathbf{Z}, -Z \leq s \leq Z$ there is a $(-w)$-trajectory, joining a point of $U \cdot t^s$ with q then there is a $(-w)$-trajectory, joining $\overline{a}t^s$ with q.

The following theorem is a reformulation of the fundamental cancellation theorem [18, Th.5.4].

Theorem 5.2 ([18, Th.5.4]). *Let $f: W \to [\alpha, \beta]$ be a Morse function on a cobordism W, v be an f-gradient. Assume that the only critical points of f are P, Q, $\operatorname{ind} P = \operatorname{ind} Q + 1$, the discs $D(P, v)$ and $D(Q, -v)$ are transversal and there is a unique (up to a parameter change $t \mapsto t + C$) $(-v)$-trajectory γ, joining P with Q. Let U be a neighbourhood of the closure of γ.*

Then there exists a Morse function $g: W \to [\alpha, \beta]$ without critical points and a g-gradient w, such that $\operatorname{supp}(v - w) \subset U$, and $g = f$ in a neighbourhood of ∂W. $\square$

Apply 5.2 to the function $\overline{g} \mid \overline{g}^{-1}([\lambda, \mu])$, the neighbourhood U of Γ and the $\overline{g}$-gradient w. Denote the resulting function by $A : \overline{g}^{-1}([\lambda, \mu]) \to [\lambda, \mu]$ and the resulting A-gradient by u. Extend them to $\overline{g}^{-1}([c, c+1])$, setting $A(z) = \overline{g}(z)$ and $u(z) = w(z)$ for $z \notin \overline{g}^{-1}([\lambda, \mu])$. Extend A to an equivariant function $\overline{h} : \overline{M} \to \mathbf{R}$, u to an invariant $\overline{h}$-gradient. Note that $S(h) = S(g) \setminus \{a, b\}$, and that for $x \in S(\overline{h})$ we have $\overline{h}(x) = \overline{g}(x)$. We have obviously: $\operatorname{supp}(u - w) \cap W \subset \bigcup_{s=-Z}^{Z} (U \cdot t^s)$.

Lemma 5.3. *The function $\overline{h}$ and the gradient u, introduced above, have the following properties:*

i) *$u \mid W$ is an almost good $(\overline{h} \mid W)$-gradient.*

ii) *If $x, y \in S(g) \setminus \{a, b\}$, and $\operatorname{ind} x = \operatorname{ind} y + 1$ then for $N > i$ we have: $D(\overline{x}, u) \pitchfork D(\overline{y}t^i, -u)$ and $N(\overline{x}, \overline{y}t^i; u) = N(\overline{x}, \overline{y}t^i; w)$.*

Proof. If p is a critical point of $\overline{h}|W$, $\operatorname{ind} p \leq r - 1$ then $D(p, w) \cap W \subset k^+$, and $(\mathcal{L})$ implies $D(p, w) \cap W \cap \operatorname{supp}(u - w) = \emptyset$. If q is a critical point of $\overline{h}|W$, $\operatorname{ind} q \geq r$ then $D(q, -w) \cap W \subset k^-$, and $(\mathcal{L})$ implies $D(q, -w) \cap W \cap \operatorname{supp}(u - w) = \emptyset$. Apply now 2.20 and deduce i) as well as ii) for the case $\operatorname{ind} x \leq r - 1$ or $\operatorname{ind} x \geq r + 1$.

To prove ii) for the case $\operatorname{ind} x = r$ note that if $\overline{h}(\overline{x}) < \overline{h}(\overline{y}t^i)$ then both sets are empty. Assume then, that $\overline{h}(\overline{x}) \geq \overline{h}(\overline{y}t^i)$; then both $\overline{x}$ and $\overline{y}t^i$ belong to W. Let ϖ be a $(-w)$-trajectory, joining $\overline{x}$ with $\overline{y}t^i$. Then $\operatorname{Im} \varpi$ does not intersect $\operatorname{supp}(u - w)$. (Indeed, if $\operatorname{Im} \varpi \cap \operatorname{supp}(u - w) \neq \emptyset$ then let s be the minimal integer in $[-Z, Z]$ such that $\operatorname{Im} \varpi \cap U \cdot t^s \neq \emptyset$, and s' be the maximal one. Then, by $(\mathcal{L})$, there is a $(-w)$-trajectory, joining $\overline{x}$ with $\overline{b}t^s$, and a $(-w)$-trajectory, joining $\overline{a}t^{s'}$ with $\overline{y}t^i$. By 4.1.1),2) we have $s \geq N, i - s' \geq N$, therefore $i \geq 2N$.) Let θ be a $(-u)$-trajectory, joining $\overline{x}$ with $\overline{y}t^i$. Then θ does not intersect $\operatorname{supp}(u - w)$. (Indeed, if $\operatorname{Im} \theta \cap \operatorname{supp}(u - w) \neq \emptyset$ then let $\tau = \min\{z \in \mathbf{R} | \theta(z) \in \operatorname{supp}(u - w)\}$ and $T = \max\{z \in \mathbf{R} | \theta(z) \in \operatorname{supp}(u - w)\}$. Let $\theta(\tau) \in Ut^s$, $\theta(T) \in Ut^{s'}$; then $s' \geq s$. On the intervals $]-\infty, \tau]$ and $[T, \infty[$ the curve θ is also a $(-w)$-trajectory, therefore the same reasoning as above applies to show that $s \geq N, i - s' \geq N$, and $I \geq 2N$. Now we apply 2.20 and ii) is proved. $\square$

For the critical points of h we choose the same liftings to $\widetilde{M}$ and orientations of descending discs, as in E. Denote this system by E'. Let $\tilde{u}$ be a good h-gradient, ϵ-fitted to u for some $\epsilon > 0$. I claim that for $N > i$ and $x, y \in S(h)$ we have $N(\overline{x}, \overline{y}t^i; u) = N(\overline{x}, \overline{y}t^i; \tilde{u})$, if only $\tilde{u}$ is sufficiently close to u. Indeed that is true, if $\overline{y}t^i \in W$ (by 2.19). If $\overline{y}t^i \notin W$ then either $i \geq N$, or $\overline{h}(\overline{y}t^i) > c + Z + 1$, and in the last case both sets are empty. Therefore $C_*(\tilde{u}, E') \underset{N}{\sim} D_*$, and the r-quad $(h, \overline{h}, \tilde{u}, E')$ satisfies the conclusions of 5.1. $\square$

B. Adding.

Theorem 5.4. *Let $\kappa = (f, \overline{f}, v, E)$ be an r-quad. Let k be an integer, $2 \leq k \leq n - 3$, N be any integer. Assume that $C_*(v, E)$ is N-equivalent to a complex D_*.*

Then there is an r-quad $\eta = (g, \overline{g}, w, E')$ such that $C_*(w, E')$ is N-equivalent to $D_* \oplus \operatorname{Triv}_*(k, \Lambda_{\xi}^{-})$.

Proof. The following lemma asserts the existence of a model Morse function. The part a) is quoted from [18, Lemma 8.2], the part b) is immediate from the construction there. Denote by $f_0 : \mathbf{R}^1 \times \mathbf{R}^{n-1} \to \mathbf{R}^1$ the projection onto the first factor, by v_0 the vector field on $\mathbf{R}^1 \times \mathbf{R}^{n-1}$ with coordinates $(1, 0)$.

Lemma 5.5. *In the above terminology we have:*

a) *For any integer* $\lambda, 0 \leq \lambda < n$, *there is a Morse function* g_0 *on* $\mathbf{R}^n$ *such that* $g_0 = f_0$ *outside of a compact set and such that* g_0 *has exactly two non-degenerate critical points* q_0, p_0 *of indices* λ, *resp.* $\lambda+1$, *and* $g_0(q_0) < g_0(p_0)$.

b) *For an* $\epsilon > 0$ *and* $r > 0$ *we can choose* g_0 *and* g_0*-gradient* w_0 *in such a way that* $\operatorname{supp}(g_0 - f_0)$, $\operatorname{supp}(w_0 - v_0) \subset]-2\epsilon/3, -\epsilon/3[\times B^{n-1}(0,r)$, *and there is only one (up to a parameter change* $t \mapsto t + C$*)* $(-w_0)$*-trajectory, joining* p_0 *with* q_0 *and* $D(p_0, w_0) \pitchfork D(q_0, -w_0)$ *along this trajectory.* $\square$

Let c be a regular value of $\overline{f}$, and let Z be a positive integer such that for $0 \leq s \leq N$ we have $\mathcal{Q}(E)t^s \subset \overline{f}^{-1}([c-Z, c]$. Denote $\overline{f}^{-1}([c-Z,c])$ by W, and $\overline{f}^{-1}(c)$ by V. We can assume that for some $\epsilon > 0$ we have $d\overline{f}(v) = 1$ in $\overline{f}^{-1}([c-\epsilon, c])$. Denote by K^+ (resp. K^-) the intersection with $\overline{f}^{-1}([c-Z, c+Z])$ of the union of $D(p,v)$ (resp. $D(q,-v)$) over $p \in S(\overline{f}) \cap \overline{f}^{-1}([c-Z, c+Z])$. The set $(K^+ \cup K^-) \cap V$ is a finite union of submanifolds of codimension ≥ 2, and is compact (by 2.11). Every v-trajectory (resp. $(-v)$-trajectory), starting at a point of $V \setminus (K^+ \cup K^-)$, reaches $\overline{f}^{-1}(c-Z)$(resp. $\overline{f}^{-1}(c+Z)$). For a subset X of $V \setminus (K^+ \cup K^-)$ denote by $\Gamma(X)$ the set $\{\gamma(x,\tau;v) | x \in X, \tau \in \mathbf{R}, \overline{f}(\gamma(x,\tau;v)) \in [c-Z, c+Z]\}$.

We shall say that a subset $Y \subset W$ is invariant with respect to a vector field $u \in Vect^\infty(W, \perp)$, if with every point $y \in Y$ the set Y contains all the points $\gamma(y,t;u), t \in \mathbf{R}$. If $Y \subset W$ is invariant then $W \setminus Y$ is also invariant. For example, the set $(\Gamma(X) \cdot t^s) \cap W$ is invariant with respect to $v \mid W$ for $0 \leq s \leq Z$.

During our construction we shall distinguish two cases: 1°) $S(f) = \emptyset$, 2°) $S(f) \neq \emptyset$. In the first case choose any point $\alpha \in V \setminus (K^+ \cup K^-)$. In the second case first choose a critical point ζ of $\overline{f}$ such that $]\overline{f}(\zeta), c[$ is regular, and let $I = \operatorname{ind} \zeta$. Then for any $\delta > 0$ the intersection of $B_\delta(\zeta, -v)$ with V is a nonempty open set in V. Choose $\delta > 0$ so small that $D_\delta(\leq I, [c-Z, c]; v) \cap D_\delta(\leq n-I, [c-Z, c]; -v) = \emptyset$, and choose $\alpha \in V$ belonging to $B_\delta(\zeta, -v) \setminus (K^+ \cup K^-)$. Denote $\big(V \cap B_\delta(\zeta, -v)\big) \setminus (K^+ \cup K^-)$ by U_0. It follows from our choice of δ that the sets $\big(\Gamma(U_0) \cdot t^s\big) \cap W$ are disjoint for $0 \leq s < Z$.

Choose a chart $\Psi : U(\alpha) \to B^{n-1}(0, 2r)$, defined in a neighbourhood $U(\alpha)$ of α in V, in such a way that: in the case 1°) $U(\alpha) \subset V \setminus (K^+ \cup K^-)$, in the case 2°) $U(\alpha) \subset U_0$. For $0 \leq s < Z$ denote by Δ_s the intersection of $\Gamma(\Psi^{-1}(B^{n-1}(0,r)) \cdot t^s$ with $\overline{f}^{-1}([c-Z, c])$. Denote the union of Δ_s over $0 \leq s < Z$ by Δ. Note the following property:

(Δ) The sets Δ_s are invariant with respect to $v \mid W$. In the case 2°) the sets Δ_s are disjoint.

Define a diffeomorphism Φ of $]-\epsilon, 0[\times B^{n-1}(0, 2r)$ onto an open subset of $\overline{f}^{-1}(]c-\epsilon, c[)$ by $\Phi(\tau, x) = \gamma(\Psi^{-1}(x), \tau; v)$. Define a new Morse function $\mathcal{G}$ on $\overline{f}^{-1}([c, c+1])$ and its gradient w by:

$w(x) = v(x)$ and $\mathcal{G}(x) = f(x)$ for $x \notin \Phi\big([-2\epsilon/3, -\epsilon/3] \times D^{n-1}(0,r)\big)$.

$w(x) = \Phi_*\big(w_0(\Phi^{-1}(x))\big)$ and $\mathcal{G}(x) = g_0(\Phi^{-1}(x))$ for $x \in \operatorname{Im}\Phi$.

Extend $\mathcal{G}$ to an equivariant Morse function $\overline{g}$ on $\overline{M}$ and w to a t-invariant $\overline{g}$-gradient. Note, that $\operatorname{supp}(w - v) \cap W \subset \Delta$. Denote $\Phi(p_0)$ by p, $\Phi(q_0)$ by q.

Lemma 5.6. 1) *There is only one (up to a parameter change* $t \mapsto t + C$*)* $(-w)$*-trajectory, joining* p *with* q, *and* $D(p, w) \pitchfork D(q, -w)$.

Furthermore, in the case 2°) we have:

2) *w is a good $(\overline{g}|W)$-gradient.*

3) *For every $x, y \in S(\overline{f}) \cap W, \operatorname{ind} x = \operatorname{ind} y + 1$ we have:*
$N(x, y; w) = N(x, y; v)$ and $N(x, qt^s; w) = N(pt^s, y; w) = \emptyset$ for $0 \leq s < Z$.

Proof. To prove 1) note that by 5.5.b) there is a $(-w)$-trajectory, joining p with q, and if there is another one, it must leave $\Phi(B^{n-1}(0, r) \times [-2\epsilon/3, -\epsilon/3])$, but it is easy to see that in this case it never returns. Therefore, $D(p, w) \pitchfork D(q, -w)$.

For the case 2°) note that the property (Δ) implies that for $x, y \in S(\overline{f}) \cap W$ the sets $D(x, v)$ and $D(y, -v)$ belong to $W \setminus \Delta$; apply now 2.20 and get that $D(x, w) \pitchfork D(y, -w)$ and $N(x, y; w) = N(x, y; v)$. The second part of 3) is obvious. Now to obtain 2) we have only to verify the transversality relations between $D(pt^s, \pm w)$ and $D(qt^{s'}, \pm w)$, where $s \neq s'$. But it follows from (Δ) that the intersections of these sets are empty. $\square$

Choose any lifting $\tilde{p}$ of p to $\widetilde{W}$ and lift q to $\widetilde{W}$ in such a way that the lifting to $\widetilde{W}$ of the unique trajectory, joining p with q, joins $\tilde{p}$ with $\tilde{q}$. Choose the orientations of $D(p, w)$ and $D(q, -w)$ in such a way, that the sign of intersection equals 1. Together with E this defines a system E' of liftings and orientations of descending discs for $S(\overline{g})$.

To finish the proof of 5.4 we consider first the case 2°). Let u be a good g-gradient, ϵ-fitted and sufficiently close to w. Then for every $x, y \in S(\overline{g}) \cap W$, $\operatorname{ind} x = \operatorname{ind} y + 1$ we have $N(x, y; u) \sim N(x, y; w)$. Let $d, e \in E, i < N$, $\operatorname{ind} d = \operatorname{ind} e + 1$. Then $\overline{d}, \overline{e} \in W$. If $\overline{f}(\overline{e}t^i) > c$ then $\overline{g}(\overline{e}t^i) > c$, and $\nu_i(d, e; v) = \nu_i(d, e; u) = 0$. If $\overline{f}(\overline{e}t^i) < c$ then $\overline{e}t^i \in W$, and it follows from the above that $\nu_i(d, e; v) = \nu_i(d, e; u)$. Similarly, we obtain for $e \in E$: $\nu_i(e, q; u) = 0$ for $\operatorname{ind} e = k + 1, i < N$; $\nu_i(p, q; u) = 0$ for $i < 0$ and $1 < i < N$; $\nu_0(p, q; u) = 1$. Therefore $(g, \overline{g}, u, E')$ satisfies the conclusions of the theorem.

In the case 1°) note that $S(g) = \{p, q\}$ and $\nu_0(p, q; w) = 1, \nu_i(p, q; w) = 0$ for $i < 0$. Consider a good g-gradient u, ϵ-fitted and sufficiently close to w. Then the Novikov complex $C_*(u, E)$ is of the form $\{0 \leftarrow \Lambda_\xi^- \xleftarrow{\partial} \Lambda_\xi^- \leftarrow 0\}$, where ∂ is the multiplication by an element of the form $1 + a_1\theta + \ldots$. Now we apply Theorem 6.1.2) from the following section (which is proved independently of Section 5), and we obtain the conclusion. $\square$

§6. Change of basis.

Theorem 6.1. *Let $\kappa = (f, \overline{f}, v, E)$ be an r-quad. Let N be an integer, and D_* be a complex, N-equivalent to $C_*(v, E)$. Then:*

1) If d is an element of the basis of D_*, $\lambda = \varepsilon g$, $g \in G$, $\varepsilon = \pm 1$, $|\xi(g)| = k$ then there is an r-quad $\eta = (g, \overline{g}, w, E)$, fitted to κ, and such that $C_*(w, E)$ is $(N - 2k)$-equivalent to $[d, \lambda]D_*$.

2) If d is an element of the basis of D_*, $\lambda = 1 + \mu$, $\operatorname{supp}\mu \subset G^{=}$ then there is an r-quad $\eta = (g, \overline{g}, w, E)$, fitted to κ, and such that $C_*(w, E)$ is N-equivalent to $[d; \lambda]D_*$.

3) If d, d' are elements of the basis of D_* of the same degree, $d \neq d'$, $a \in \Lambda_{\xi}^{-}$, $h(a) = k$ then there is an r-quad $\eta = (g, \overline{g}, w, E)$, fitted to κ, and such that $C_*(w, E)$ is $(N - k)$-equivalent to $[d, d'; a]D_*$.

We shall deduce this theorem from Theorem 6.2.

Theorem 6.2. *Let $\kappa = (f, \overline{f}, v, E)$ be an r-quad. Assume that there exists a regular value c of $\overline{f}$ such that the projection $\overline{E}$ of E to $\overline{M}$ belongs to $\overline{f}^{-1}([c-1, c])$. Let N be an integer. Then:*

1) If e, e' are two elements of E of the same index r, $e \neq e'$, $a \in \Lambda_{\xi}^{-}$, $\operatorname{supp} a \subset G^{=}$ then there is an r-quad $\eta = (g, \overline{g}, w, E)$, fitted to κ, and such that $C_*(w, E)$ is N-equivalent to $[e, e'; a]C_*(v, E)$.

2) If $e \in E$, $\lambda = 1 + \mu$, $\operatorname{supp}\mu \subset G^{=}$ then there is an r-quad $\eta = (g, \overline{g}, w, E)$, fitted to κ, and such that $C_*(w, E)$ is N-equivalent to $[e; \lambda]C_*(v, E)$.

A. Proof of 6.1 from 6.2.

Lemma 6.3. *Let $\kappa = (f, \overline{f}, v, E)$ be an r-quad. Let $p, q \in S(\overline{f})$, $\operatorname{ind} p = \operatorname{ind} q$, p is not a t-shift of q. Then there is an equivariant Morse function $\overline{\phi} : \overline{M} \to \mathbf{R}$ such that $(\phi, \overline{\phi}, v, E)$ is an r-quad, fitted to κ, and such that $\overline{\phi}(p) > \overline{\phi}(q)$.*

Proof. In the case $\overline{f}(p) \leq \overline{f}(q)$ we proceed by induction in $s = [\overline{f}(q) - \overline{f}(p)]$. We have $\overline{f}(p) \leq \overline{f}(qt^s) < \overline{f}(p) + 1$. Therefore for a regular value c of $\overline{f}$ both p and qt^s are contained in the cobordism $W_0 = \overline{f}^{-1}([c, c+1])$, and $p \neq qt^s$. Choose an ordered function $\varphi : W_0 \to [c, c+1]$, adjusted to $(\overline{f}, v)$ such that $\varphi(qt^s) < \varphi(p)$. Extend it to an equivariant function $\overline{\psi} : \overline{M} \to \mathbf{R}$. Note, that $(\psi, \overline{\psi}, v, E)$ is an r-quad, fitted to κ, and $\overline{\psi}(q) - \overline{\psi}(p) < s$. Apply the induction assumption. □

Definition 6.4. Denote by $\pm G \subset \mathbf{Z}G$ the subset of elements of the type $\pm g$, $g \in G$. Extend $\xi : G \to \mathbf{Z}$ to $\pm G$, setting $\xi(-g) = \xi(g)$. Let C_* be a complex, I be a basis of C_*, and $\Lambda : I \to \pm G$ be a map. Denote by $|\Lambda|$ the maximum of $|\xi(h)|$ where $h \in \pm G$. The function $e \mapsto (\Lambda(e))^{-1}$ will be denoted Λ^{-1}. Denote by $\Lambda(C_*)$ the complex which is obtained from C_* by the change of the basis $e \mapsto e \cdot \Lambda(e)$. If $\kappa = (f, \overline{f}, v, E)$ is an r-quad then any $\Lambda\colon E \to \pm G$ determines in the obvious way a new system of liftings and orientations which will be denoted $\Lambda(E)$.△

The following three lemmas are obvious and will be left without proof.

Lemma 6.5. *Let C_* be a complex, e, e' be elements of its basis. Let $a, \lambda \in \Lambda_{\xi}^{-}$ and assume that λ is invertible.*

Then 1) $[e \cdot \Lambda(e), e' \cdot \Lambda(e'); a]\Lambda(C_*) = \Lambda\big([e, e'; \Lambda(e') \cdot a \cdot \Lambda^{-1}(e)]C_*\big)$.

2) $[e \cdot \Lambda(e); \lambda]\Lambda(C_*) = \Lambda\big([e; \Lambda(e) \cdot \lambda \cdot \Lambda^{-1}(e)]C_*\big)$. □

Lemma 6.6. *Let N be an integer and C_*, D_* be two N-equivalent complexes. Denote by $\chi: I \to J$ the identifications of the bases of C_*, resp. D_*. Let $\Lambda: I \to \pm G$ be a function. We abbreviate $\Lambda \circ \chi$ as Λ. Let $e, e' \in I$ be of the same degree and $d, d' \in J$ be their χ-images. Then*:

1) *If $|\Lambda| = k$, then $\Lambda(C_*)$ is $(N-2k)$-equivalent to $\Lambda(D_*)$.*
2) *If $a \in \Lambda_\xi^-$, $h(a) = k$ then $[e, e'; a]C_*$ is $(N-k)$-equivalent to $[d, d'; a]D_*$.*
3) *If $\lambda \in \Lambda_\xi^-$, $\lambda = 1 + \mu$, $\operatorname{supp}\mu \subset G^=$ then $[e; \lambda]C_*$ is N-equivalent to $[d; \lambda]D_*$.* $\square$

Lemma 6.7. *Let $\kappa = (f, \overline{f}, v, E)$ be an r-quad, $\Lambda: E \to \pm G$ be a map. Then $C_*(v, \Lambda(E)) \cong \Lambda(C_*(v, E))$.* $\square$

Proof of 6.1. 1) Let e be the element of E, corresponding, via the identification of the bases, to d. Denote by $\Lambda : I \to \pm G$ the following function: $\Lambda(x) = 1$, if $x \neq e$, $\Lambda(e) = \lambda$. Then the r-quad $(f, \overline{f}, v, \Lambda(E))$ satisfies the conclusions of 1). Indeed, we have by the previous lemmas: $C_*(v, \Lambda(E)) \cong \Lambda(C_*(v, E)) \underset{N-2k}{\sim} \Lambda(D_*) = [d, \lambda]D_*$.

3) Let e, e' be the elements of E, corresponding to d, d' via the identification of the bases. By 6.3 we can assume that $\overline{f}(\overline{e}) > \overline{f}(\overline{e'}t^{-k-1})$. Choose a regular value c of $\overline{f}$ such that $c - 1 < \overline{f}(\overline{e}) < c$. Denote $\overline{f}^{-1}([c-1, c])$ by W_0. For $\alpha \in E$ denote by $m(\alpha)$ the unique $s \in \mathbf{Z}$ such that $\overline{\alpha}t^s \in W_0$; note that $m(e) = 0$. Denote $m(e')$ by m_0, the function $\alpha \mapsto \theta^{m(\alpha)}$ by Λ, and $|\Lambda|$ by K. Note that supp $(\theta^{-m_0} \cdot a) \subset G^=$. Apply 6.2.1) to the r-quad $\eta = (f, \overline{f}, v, \Lambda(E))$, the integer $N + 2K$, the basis elements $e \cdot \Lambda(e)$ and $e' \cdot \Lambda(e')$ and the element $a' = \theta^{-m_0} \cdot a$. We obtain an r-quad $\nu = (g, \overline{g}, w, \Lambda(E))$, fitted to η and to κ, and such that $C_*(w, \Lambda(E)) \underset{N+2K}{\sim} [e \cdot \Lambda(e), e' \cdot \Lambda(e'); a']C_*(v, \Lambda(E))$. Now an easy computation, using 6.5 and 6.6, gives $C_*(w, E) \underset{N-k}{\sim} [d, d'; a]D_*$.

2) is proved similarly to 3). $\square$

The proof of 6.2 will occupy the rest of this section.

B. Functions, satisfying $(\mathcal{R})$ and the Novikov complex.

In this subsection we shall make some reductions of 6.2, using the results of Section 5 of [27], which express the Novikov complex in terms of some auxiliary Morse functions.

Denote by $\Lambda_\xi^=$ the subring of Λ_ξ^-, consisting of all the $\lambda \in \Lambda_\xi^-$ such that $\operatorname{supp}\lambda \subset G^-$. For $n \geq 0$ denote by I_n the two-sided ideal of $\mathbf{Z}G^-$, generated (as a left or right ideal) by θ^n, and by $\mathbf{Z}G_n^-$ the quotient ring $\mathbf{Z}G^-/I_n$. For $m \geq 0$ denote by $I_{m,n}$ the two-sided ideal of $\mathbf{Z}G_n^-$, generated (as a left or right ideal) by θ^m. Denote by $\mathcal{I}_n$ the two-sided ideal of $\Lambda_\xi^=$, generated (as a left or right ideal) by θ^n; the inclusion map $\mathbf{Z}G^- \subset \Lambda_\xi^=$ induces an isomorphism $\mathbf{Z}G_n^- \cong \Lambda_\xi^=/\mathcal{I}_n$.

An r-quad $(f, \overline{f}, v, E)$ will be called an *rs-quad* (s for "small"), if there is a regular value c of $\overline{f}$ such that $\overline{E} \subset \overline{f}^{-1}([c-1, c])$. For rs-quads there is a refinement of the construction of Novikov complex. Namely, denote by $\widehat{C}_p^-$ the free $\Lambda_\xi^=$-module, generated by the critical points of f of index p. We have obviously $\widehat{C}_p = \widehat{C}_p^- \otimes_{\Lambda_\xi^=} \Lambda_\xi^-$. Using the condition $\overline{E} \subset \overline{f}^{-1}([c-1, c])$ it is not difficult to verify that the same formula as in 3.2 defines a $\Lambda_\xi^=$-homomorphism $\partial_p^- : \widehat{C}_p^- \to \widehat{C}_{p-1}^-$ such that the homomorphism

$$\partial_p^- \otimes \mathrm{id} : \widehat{C}_p^- \otimes_{\Lambda_\xi^=} \Lambda_\xi^- \to \widehat{C}_{p-1}^- \otimes_{\Lambda_\xi^=} \Lambda_\xi^-$$

equals ∂_p. Theorem 4.1 of [27] states that $\partial_p^- \circ \partial_{p+1}^- = 0$ for every p. We shall denote the resulting chain complex by $C_*^-(v,E)$; it is a complex over $\Lambda_\xi^=$. For an integer $n \geq 0$ denote by $C_*^-(v,E;n)$ the $\mathbf{Z}G_n^-$-complex $C_*^-(v,E) \underset{\Lambda_\xi^=}{\otimes} \mathbf{Z}G_n^- = C_*^-(v,E)/C_*^-(v,E)\cdot\theta^n$. In order to prove 6.2 it is obviously sufficient to prove the following proposition.

Proposition 6.8. *Let $(f,\overline{f},v,E)$ be an rs-quad, $N>1$ be an integer.*

Then 1) If e, e' are two elements of E of the same index r, $e \neq e'$, and $a \in I_{1,N}$ then there is an rs-quad $\eta = (g,\overline{g},w,E)$, fitted to κ, and such that $C_*^-(w,E;N)$ is identical to $[e,e';a]C_*^-(v,E;N)$.

2) If $e \in E$, $\lambda = 1+\mu, \mu \in I_{1,N}$ then there is an rs-quad $\eta = (g,\overline{g},w,E)$, fitted to κ, and such that $C_*^-(w,E;N)$ is identical to $[e;\lambda]C_*^-(v,E;N)$.

This proposition follows in turn from the following one.

Proposition 6.9. *Let $(f,\overline{f},v,E)$ be an rs-quad. Let $N>1$ be a integer. Let l be an integer, $1 \leq l \leq N$, let $\varepsilon = \pm 1, \rho \in G^-, \xi(\rho) = -l$. Then*

1) If e, e' are two elements of E of the same index r, $e \neq e'$ then there is an rs-quad $\eta = (g,\overline{g},w,E)$, fitted to κ, and such that $C_*^-(w,E;N)$ is identical to $[e,e';\varepsilon\rho + x]C_*^-(v,E;N)$, where $x \in I_{l+1,N}$.

2) If $e \in E$, then there is an rs-quad $\eta = (g,\overline{g},w,E)$, fitted to κ, and such that $C_*^-(w,E;N)$ is identical to $[e;1+\varepsilon\rho+x]C_*^-(v,E;N)$, where $x \in I_{l+1,N}$.

Proof of 6.8 from 6.9. 1) Let $\mathcal{A}$ be the set of $a \in I_{1,N}$ such that for every rs-quad $\kappa = (f,\overline{f},v,E)$ and for every pair e,e' of different elements of E of the same index there is an rs-quad $\eta = (g,\overline{g},w,E')$, fitted to κ and such that $C_*^-(g,w;N)$ is identical to $[e,e';a]C_*^-(v,E;N)$. It is easy to see that $\mathcal{A}$ is a subgroup of $I_{1,N}$. From this and from Proposition 6.9.1), applied to $l = N-1$, it follows that $\mathcal{A} \supset I_{N-1,N}$. Now it is easy to show, by induction in s and applying 6.9.1)), that $\mathcal{A} \supset I_{N-s,N}$, where $1 \leq s \leq N-1$.

2) Similar to the above. Let $\mathbf{Z}G^{**} = \{\tau \in \mathbf{Z}G_N^- | \tau = 1+\mu, \mu \in I_{1,N}\}$. It is a group with respect to multiplication, induced from the ring structure of $\mathbf{Z}G_N^-$. Let $\mathcal{G}$ be the set of all $\tau \in \mathbf{Z}G^{**}$ such that for every rs-quad $\kappa = (f,\overline{f},v,E)$ and for every element $e \in E$ there is an rs-quad $\eta = (g,\overline{g},w,E')$, fitted to κ and such that $C_*^-(g,w;N)$ is identical to $[e,e';\tau]C_*^-(v,E;N)$. By an induction argument, similar to that of the proof of 1) above, one shows easily that $\mathcal{G}$ equals $\mathbf{Z}G^{**}$. $\square$

To prove 6.9 we need to recollect some more results from [27].

Definition 6.10. Let $\overline{f}:\overline{M} \to \mathbf{R}$ be an equivariant Morse function, $f: M \to S^1$ its quotient, v be an f-gradient, c be a regular value of $\overline{f}$, N be a positive integer. Denote by W the cobordism $\overline{f}^{-1}([c-N,c])$. We say that a Morse function $\phi: W \to [\alpha,\beta]$ satisfies condition $(\mathcal{R})$ with respect to $(\overline{f},v,c,N)$ if:

$(\mathcal{R}1)$ ϕ is adjusted to $(\overline{f},v)$;

$(\mathcal{R}2)$ ϕ is ordered;

$(\mathcal{R}3)$ If $x, x\cdot t \in W$ then $\phi(x\cdot t) < \phi(x)$. $\triangle$

Normally the value of $(\overline{f},v,c,N)$ is clear from the context, and we shall often write "ϕ satisfies condition $(\mathcal{R})$".

Lemma 5.1 of [27] states the existence of functions, satisfying $(\mathcal{R}2)$ and $(\mathcal{R}3)$, and such that v is a ϕ-gradient. The proof actually produce a function, satisfying also $(\mathcal{R}1)$. We shall prove a stronger version of this lemma in Subsection D.

Let $(f, \overline{f}, v, E)$ be an rs-quad, c be a regular value of $\overline{f}$ such that $\overline{E} \subset \overline{f}^{-1}([c-1, c])$, $N > 1$ be an integer. Denote $\overline{f}^{-1}([c-N, c])$ by W, and let $\phi : W \to [\alpha, \beta]$ be a Morse function, satisfying $(\mathcal{R})$ with respect to $(\overline{f}, v, c, N)$. Let $\vec{a} = (a_0, \ldots, a_{n+1})$ be an ordering sequence for ϕ. Using $(\mathcal{R}3)$ one verifies easily that the chain complex $C_*(\phi, v, \vec{a})$ (in the terminology of §2.G) is actually a complex of $\mathbf{Z}G_N^-$-modules. Every critical point $p \in S(f)$ equals $\overline{e}t^s$, where $\overline{e} \in E, 0 \le s \le N-1$. Define the lifting $\widetilde{p}$ of $p = \overline{e} \cdot t^s$ to $\widetilde{M}$ by $\widetilde{p} = e \cdot \theta^s$. It is easy to verify the following lemma (the proof is implicit in [27, §5, p.26]):

Lemma 6.11. 1) *$C_*(\phi, v, \vec{a})$ is a complex over $\mathbf{Z}G_N^-$ with the basis $[e](v, \phi)$.*

2) *The map $i(v) : C_*^-(v, E; N) \to C_*(\phi, v, \vec{a})$, which sends $e \in E$ to $[e](v, \phi)$, is an identification of $\mathbf{Z}G_N^-$-complexes.* □

We need one more definition.

Definition 6.12. For a Morse map $f : M \to S^1$ we denote by $n(f)$ the cardinality of $f(S(f))$. A Morse map $f : M \to S^1$ is called *indexing*, if

1) for each i we have $f|S_i(f) = \text{const}$;

2) for $i \ne j$ we have $f(S_i(f)) \ne f(S_j(f))$;

3) if $S(f) \ne \emptyset$ then $f(S(f)) = \{m/n(f) \bmod \mathbf{Z} | m \in \mathbf{Z}\}$.

An equivariant Morse function $\overline{f} : \overline{M} \to \mathbf{R}$ is called *indexing*, if its quotient is indexing. For an indexing Morse map $f : M \to S^1$ (resp. indexing Morse function $\overline{f} : \overline{M} \to \mathbf{R}$) and a critical value $c \in S^1$ (resp. $c \in \mathbf{R}$) denote by $\operatorname{ind}_f(c)$ the index of any critical point in $f^{-1}(c)$ (resp. $\overline{f}^{-1}(c)$). △

Proposition 6.13. *Let $(f, \overline{f}, v, E)$ be an r-quad.*

(1) *Then there is an indexing Morse function $\overline{g} : \overline{M} \to \mathbf{R}$ such that $(g, \overline{g}, v, E)$ is an r-quad, fitted to κ.*

(2) *Moreover, if for a regular value c of $\overline{f}$ we have $\overline{E} \subset \overline{f}^{-1}([c-1, c])$, we can assume also $\overline{E} \subset \overline{g}^{-1}([c-1, c])$.*

Proof. Let c be a regular value of $\overline{f}$; denote $\overline{f}([c-1, c])$ by W. By 2.7 there is a Morse function $R : W \to [c-1, c]$, adjusted to $(\overline{f}|W, v)$ such that for every i we have: $R|S_i(R)$ is constant and if $i \ne j$ then $R|S_i(R) \ne R|S_j(R)$. Denote by $\overline{g}$ the equivariant extension of R to $\overline{M}$. Then $\overline{g}$ satisfies already 1) and 2) of 6.12. Composing g, if necessary, with a diffeomorphism of S^1 to itself, we can obtain also 3). The point (2) is proved similarly. □

Now we can state a proposition, which will be proved in the subsection E. It is not difficult to deduce from it (with the help of 6.11) Proposition 6.9.

Proposition 6.14. *Let $(f, \overline{f}, v, E)$ be an rs-quad, c be a regular value of $\overline{f}$ such that $\overline{E} \subset \overline{f}^{-1}([c-1, c])$. Assume that f is indexing, and that v is perfect. Let e, e' be two elements of E (not necessarily different), $\operatorname{ind} e = \operatorname{ind} e' = r$. Let $\varepsilon = \pm 1$, l be an integer, $2 \le l \le N$. Let $\rho \in G^-$, $\xi(\rho) = -(l-1)$. Denote $\overline{f}^{-1}([c-N, c])$ by W.*

Then there is a perfect f-gradient w, and a Morse function $\phi : W \to [\alpha', \beta']$ such that

(1) *ϕ satisfies $(\mathcal{R})$ with respect to $(\overline{f}, v, c, N)$ and with respect to $(\overline{f}, w, c, N)$.*

(2) *If $p \in E, p \ne e$ then $[p](v, \phi) = [p](w, \phi) \in C_*(\phi, \vec{a})$.*

(3) *$[\overline{e}](w, \phi) = [\overline{e}](v, \phi) + \varepsilon \cdot [\overline{e'}](v, \phi) \cdot \rho + x$, where $x \in C_r(\phi, \vec{a}) \cdot \theta^l$.*

In Subsection E we shall need a lemma, concerning the homology of $(\widetilde{\phi^{-1}([a_k, a_{k+1}])}, \widetilde{\phi^{-1}(a_k)})$. Similar to the terminology of §2.G for $p \in S_k(\phi)$ we shall denote by $(p)(-v,\phi)$ the manifold $D(p,-v) \cap W_k$. It is an embedded disc of dimension $n - \operatorname{ind} p$; its boundary is contained in $\phi^{-1}(a_{k+1})$. Let Q be a smooth oriented compact manifold of dimension k with boundary . Let $F : Q \to \overline{M}$ be a map such that $\operatorname{Im} F \subset \phi^{-1}([a_k, a_{k+1}[)$ and $F(\partial Q) \subset \phi^{-1}(a_k)$. Assume, that a lifting $\widetilde{F} : Q \to \widetilde{M}$ is fixed. By §2.H the intersection index $I\big(F, (p)(-v,\phi); \phi^{-1}(a_k), \phi^{-1}(a_{k+1})\big)$ is defined. We denote it by $\langle F, p\rangle \in \mathbf{Z}H$.

Lemma 6.15. *The $\widetilde{F}_*$-image in $H_*(\widetilde{\phi^{-1}([a_k, a_{k+1}])}, \widetilde{\phi^{-1}(a_k)})$ of the fundamental class of Q modulo ∂Q equals to the sum $\sum [p](v,\phi)\cdot\langle F,p\rangle$, where p ranges over $S_k(\phi)$.*

Proof. By the standard Morse theory the elements $\{[p](v,\phi)\}_{p\in S_k(\phi)}$ form a basis of the right $\mathbf{Z}H$-module $H_k(\widetilde{\phi^{-1}([a_k, a_{k+1}])}, \widetilde{\phi^{-1}(a_k)})$. Therefore, in view of linearity (2.1) of the intersection index and of Lemma 2.26 it is sufficient to verify this formula for the elements $[p](v,\phi)$, but for these elements the assertion is trivial. □

C.Bridges.

Definition 6.16. Let $\overline{f} : \overline{M} \to \mathbf{R}$ be an equivariant Morse function, v be an f-gradient, λ be a regular value of $\overline{f}$, $p, q \in S(\overline{f}), \operatorname{ind} p = \operatorname{ind} q$. We shall say that a C^∞ embedding $\gamma : [0,3] \to f^{-1}(\lambda)$ is *a bridge from p to q of height λ* (or simply *a bridge*, if there is no possibility of confusion), if:

1) $D(p,v) \cap \operatorname{Im}\gamma = \{\gamma(1)\}, \quad D(q,-v)\cap \operatorname{Im}\gamma = \{\gamma(2)\}$;

2) $\gamma'(1) \notin T_{\gamma(1)}D(p,v), \quad \gamma'(2) \notin T_{\gamma(2)}D(q,-v)$;

3) $\operatorname{Im}\gamma$ does not intersect any other ascending or descending disc.

If γ is a bridge from p to q, we denote by $\langle\langle\gamma\rangle\rangle$ the composition of the following paths: 1) $(-v)$-trajectory, joining p with $\gamma(1)$; 2) $\gamma|[1,2]$; 3) $(-v)$-trajectory, joining $\gamma(2)$ with q. The fixed-ends-homotopy class of the continuous extension of $\langle\langle\gamma\rangle\rangle$ to $\overline{\mathbf{R}}$, will be called *homotopy class of the bridge* and will be denoted by $\langle\gamma\rangle$; $\langle\gamma\rangle \in \pi(p,q)$.△

To prove the existence of bridges we shall need a (rather standard) transversality lemma. If V is a manifold, and $m, n \in V$ we shall denote by $\mathcal{C}$ the set of C^∞ maps $\phi : S^1_{(5)} \to V$, such that $\phi(1) = m, \phi(2) = n$. In the topology, induced from $C^\infty(S^1_{(5)}, V)$, $\mathcal{C}$ is a Baire space ([13, p.62]).

Lemma 6.17. *Let V be a closed manifold,* $\dim V \geq 3$. *Let $X \subset V$ and $m, n \in V \setminus X$. Denote by $\mathcal{B}$ the set of all C^∞ embeddings $\phi : S^1_{(5)} \hookrightarrow V$ such that $\phi \in \mathcal{C}$ and* $\operatorname{Im}\phi \subset V\setminus X$.

Then $\mathcal{B}$ is residual in $\mathcal{C}$ in the following cases: 1) X is an embedded closed disc of codimension ≥ 2; 2) X is a union of countable family of submanifolds $(R_i)_{i\in\mathbf{N}}$ of codimensions ≥ 2.

Proof. The case 2) follows from 1). To prove 1) note that $\mathcal{B}$ is obviously open. To prove its density, let X be the image of a C^∞ embedding $\phi : D^k(0,r) \hookrightarrow V$. There exists an extension of ϕ to a C^∞ embedding $\widetilde{\phi} : B^k(0, r+\epsilon) \hookrightarrow V$. By [13,Th.1.4, 2.13 of Ch.2 and Th.2.1 of Ch.3] the set of embeddings $S^1_{(5)} \hookrightarrow V$ such that their image does not intersect $\operatorname{Im}\widetilde{\phi}$, is residual in $C^\infty(S^1_{(5)}, V)$. Denote this set by $\mathcal{B}'$. Let $g : S^1_{(5)} \to V$ be a map in $\mathcal{C}$ and $\widetilde{g} : S^1_{(5)} \to V$ be a map in $\mathcal{B}'$, close to g. Then $\widetilde{g}(0)$ is close to $g(0)$, $\widetilde{g}(1)$ is close to $g(1)$, and there is an isotopy $I_t : V \to V$, close to the identity map such that $\operatorname{supp} I \cap \operatorname{Im} \phi = \emptyset$ and $I_1(\widetilde{g}(0)) = g(0), I_1(\widetilde{g}(1)) = g(1)$. Now $h = I_1 \circ \widetilde{g}$ is close to g and is in $\mathcal{B}$. $\square$

Proposition 6.18. *Let $(f, \overline{f}, v, E)$ be an r-quad. Assume that v is perfect. Let $p, q \in S(\overline{f}), \operatorname{ind} p = \operatorname{ind} q$. Let $\theta \in \pi(p, q)$.*

Then there is a bridge from p to q such that its homotopy class equals to θ.

Proof. Denote $f^{-1}(\lambda)$ by V, $D(p, v) \cap V$ by P, $D(q, -v) \cap V$ by Q. It follows from 2.16 that P and Q are nonempty. Denote by R_+ the intersection with V of the union of $D(x, v)$, where $x \in \overline{f}^{-1}([\lambda, \infty[)$. Denote $R_+ \setminus D(p, v)$ by R_+^0. R_+ is a countable union of submanifolds of V, each of which is of codimension ≥ 2 in V, and is transversal to Q. Therefore, $Q \setminus R_+ \neq \emptyset$. Denote by r_+ the intersection with V of the union of $D(x, v)$, where $\lambda < \overline{f}(x) \leq \overline{f}(p), \operatorname{ind} x < \operatorname{ind} p$. It follows from 2.11 that r_+ and $r_+ \cup P$ are compact.

The intersection with V of the union of $D(x, -v)$, where $x \in \overline{f}^{-1}([-\infty, \lambda[)$ will be denoted by R_-. Denote $R_- \setminus D(q, -v)$ by R_-^0. R_- is a countable union of submanifolds of V, each of which is of codimension ≥ 2 in V, and is transversal to P. Therefore, $P \setminus R_- \neq \emptyset$. Denote by r_- the intersection with V of the union of $D(x, -v)$, where $\overline{f}(q) \leq \overline{f}(x) < \lambda, \operatorname{ind} x < \operatorname{ind} p$. It follows from 2.11 that r_- and $r_- \cup Q$ are compact.

Since v is perfect, we have $(r_+ \cup P) \cap (r_- \cup Q) = \emptyset$.

Choose $m \in P \setminus R_-$ and $n \in Q \setminus R_+$. Denote by $\mathcal{C}$ the subset of $C^\infty(S^1_{(5)}, V)$, consisting of all maps, sending 1 to m and 2 to n. For a given fixed-ends-homotopy class h of maps $[1, 2] \to V$ such that $h(1) = m, h(2) = n$, denote by $\mathcal{V}_h$ the set of all $\Gamma \in \mathcal{C}$, satisfying the following properties: 1)$\operatorname{Im} \Gamma \cap (r_+ \cup P \cup r_- \cup Q) = \{m, n\}$; 2) $\Gamma'(1)$ is not tangent to P, $\Gamma'(2)$ is not tangent to Q; 3)$\Gamma|[1, 2] \in h$.

To prove our proposition, it suffices to show that for every h the set $\mathcal{V}_h$ contains a C^∞ embedding Γ, whose image does not intersect $R_-^0 \cup R_+^0$ (indeed, the restriction of Γ to $[0, 3]$ will be a bridge belonging to θ for a proper choice of h). It is easy to show that for every h the set $\mathcal{V}_h$ is a non empty open subset of $\mathcal{C}$. Now our proposition follows from 6.17 (applied to $X = R_-^0 \cup R_+^0$). $\square$

D. Existence of functions, satisfying $(\mathcal{R})$.

Definition 6.19. We shall say that a septuple $(f, \overline{f}, v, E; p, q, \gamma)$ is *a BRI-quad* (B for "bridge", R for "regular", I for "indexing"), if

1) $(f, \overline{f}, v, E)$ is an r-quad, v is perfect and f is indexing,

2) $p, q \in S(f), \operatorname{ind} p = \operatorname{ind} q$ and γ is a bridge from p to q.$\triangle$

Definition 6.20. Let $(f, \overline{f}, v, E; p, q, \gamma)$ be a BRI-quad. Suppose that an integer $N > 0$, and real numbers c and $\epsilon > 0$ are such that $[c - \epsilon, c + \epsilon]$ is a regular interval of $\overline{f}$, and that no t-shift of $\operatorname{Im} \gamma$ is in $\overline{f}^{-1}([c - \epsilon, c + \epsilon])$, and that $c - N < \overline{f}(p), \overline{f}(q) < c$. Denote $\overline{f}^{-1}([c - N, c])$ by W, and $\operatorname{ind} p = \operatorname{ind} q$ by r. Denote $\overline{f}^{-1}([c - \epsilon, c])$ by U_1, $\overline{f}^{-1}([c - N, c + \epsilon - N])$ by U_0. We shall say that a Morse function $\phi : W \to [\alpha, \beta]$ satisfies $(\mathcal{RF})$ with respect to $(\overline{f}, v, c, N, \gamma, \epsilon)$ if it satisfies condition $(\mathcal{R})$ with respect to $(\overline{f}, v, c, N)$, and also the following condition $(\mathcal{R}4)$:

$(\mathcal{R}4)$ If x and xt are in $W \setminus (U_0 \cup U_1)$, then $\phi(xt) = \phi(x) - 1$. There is a neighbourhood U of $\operatorname{Im} \gamma$ such that no t-shift of U intersects with $U_0 \cup U_1$ and that $\phi - \overline{f}$ is constant in U. There is an ordering sequence $(a_0, \ldots, a_{n+1})$ for ϕ such that every t-shift of U, contained in W, is in $\phi^{-1}(]a_r, a_{r+1}[)$.$\triangle$

Proposition 6.21. *Under the assumptions of the previous definition there is a Morse function $\phi : W \to [\alpha, \beta]$, satisfying condition $(\mathcal{RF})$.*

The proof of 6.21 occupies the rest of Subsection D and is the result of several lemmas.

Definition 6.22. Let $\mathcal{A} = (f, \overline{f}, v, E; p, q, \gamma)$ be a BRI-quad. We say that a BRI-quad $\mathcal{B}$ is a *flip* of $\mathcal{A}$, if the function $\overline{f} - \overline{g}$ is constant in a neighbourhood U of $\operatorname{Im}\gamma$ as well as in a neighbourhood of any critical point of $\overline{f}$, and there is a critical value λ of $\overline{f}$ such that:

1) $\operatorname{ind}_f(\lambda) > \operatorname{ind}_f(\lambda + 1/n(f))$;

2) if $i \neq \operatorname{ind}_f(\lambda), i \neq \operatorname{ind}_f(\lambda + 1/n(f))$ then $\overline{f}|S_i(v) = \overline{g}|S_i(v)$;

3) if $i = \operatorname{ind}_f(\lambda)$ then $\overline{g}|S_i(v) = \big(\overline{f}|S_i(v)\big) + 1/n(f)$;

4) if $i = \operatorname{ind}_f(\lambda + 1/n(f))$ then $\overline{g}|S_i(v) = \big(\overline{f}|S_i(v)\big) - 1/n(f)$.

Let $\mathcal{B} = (g, \overline{g}, v, E; p, q, \gamma)$ be a flip of $\mathcal{A} = (f, \overline{f}, v, E; p, q, \gamma)$, and $c \in \mathbf{R}$. We say that $\mathcal{B}$ is a *c-flip* of $\mathcal{A}$, if c is a regular value of $\overline{f}$, and $\overline{f}^{-1}(c) = \overline{g}^{-1}(c)$. (It is easy to see that in this case we have also $\overline{f}^{-1}([c-1, c]) = \overline{g}^{-1}([c-1, c])$.) Let $\mathcal{A}, \mathcal{B}$ be BRI-quads. We say that $\mathcal{B}$ is a *flop* of $\mathcal{A}$, resp. a *c-flop of* $\mathcal{A}$, if there is a finite sequence $\mathcal{A}_1, \dots, \mathcal{A}_k$ of BRI-quads, such that $\mathcal{A} = \mathcal{A}_1, \mathcal{B} = \mathcal{A}_k$ and for every $1 < i \leq k$ the BRI-quad $\mathcal{A}_i$ is a flip (resp. a c-flip) of $\mathcal{A}_{i-1}$.$\triangle$

Lemma 6.23. *Let $\mathcal{A} = (f, \overline{f}, v, E; p, q, \gamma)$ be a BRI-quad. Assume that for a critical value λ of $\overline{f}$ we have* $\operatorname{ind}_f(\lambda) > \operatorname{ind}_f(\lambda + 1/n(f))$. *Let c be a regular value of $\overline{f}$ such that $\forall s \in \mathbf{Z}$ we have $s + c \notin]\lambda, \lambda + 1/n(f)[$.*

Then there is a c-flip $\mathcal{B}$ of $\mathcal{A}$.

Proof. Note that $n(f) \geq 2$, otherwise the indices of all the critical points are the same. Consider two cases:

1) $\overline{f}(\operatorname{Im}\gamma) \notin]\lambda, \lambda + 1/n(f)[$. Then there are regular values c_1, c_2 of $\overline{f}$ such that $c_1 < \lambda < \lambda + 1/n(f) < c_2 < c_1 + 1$, and that $\forall s \in \mathbf{Z}$ we have $s + c \notin [c_1, c_2]$, and that λ and $\lambda + 1/n(f)$ are the only critical values of $\overline{f}$ in $\overline{f}^{-1}([c_1, c_2])$, and that $\overline{f}(\operatorname{Im}\gamma) \notin \overline{f}^{-1}([c_1, c_2])$. Denote $\overline{f}^{-1}([c_1, c_2])$ by W_0. Since v is perfect, the family of sets $\{(D(p, v) \cup D(p, -v)) \cap W_0\}$, where $p \in S(\overline{f}) \cap W_0$, is disjoint. Apply to $\overline{f}|W_0$ the generalized rearrangement lemma [18, 4.2] and obtain a Morse function $\phi : W_0 \to [c_1, c_2]$, adjusted to $(\overline{f}|W_0, v)$, and such that:

1) $\phi(p) = \lambda$, if $p \in S(\overline{f}|W_0)$ and $\overline{f}(p) = \lambda + 1/n(f)$,

2) $\phi(p) = \lambda + 1/n(f)$, if $p \in S(\overline{f}|W_0)$ and $\overline{f}(p) = \lambda$.

Set $\overline{g}(x) = \overline{f}(x)$ for $x \in \overline{f}^{-1}([c_1, c_1 + 1]) \setminus W$ and $\overline{g}(x) = \phi(x)$ for $x \in W$ and extend $\overline{g} : \overline{f}^{-1}([c_1, c_1 + 1]) \to [c_1, c_1 + 1]$ to an equivariant function $\overline{M} \to \mathbf{R}$. Then $(g, \overline{g}, v, E; p, q, \gamma)$ is the flip sought.

2) $\overline{f}(\operatorname{Im}\gamma) \in]\lambda, \lambda + 1/n(f)[$. Here the argument is the same with the only difference that the function ϕ must be chosen in such a way that in a neighbourhood of $\operatorname{Im}\gamma$ the function $\phi - \overline{f}$ is constant. Checking through the proof of the rearrangement lemma [18,4.1] shows, that it is possible, since of the sets $\{(D(p, v) \cup D(p, -v)) \cap W_0\}_{p \in S(\overline{f}) \cap W_0}$ the set $\operatorname{Im}\gamma$ can intersect at most one (this follows since γ is a bridge). $\square$

Notation 6.24. Let $\overline{f} : \overline{M} \to \mathbf{R}$ be an equivariant Morse function. Denote by $\operatorname{Irr}(\overline{f})$ the subset of $S(\overline{f}) \times S(\overline{f})$, consisting of all such pairs (x, y), that $\operatorname{ind} x < \operatorname{ind} y$ and

$\overline{f}(x) > \overline{f}(y)$. $\triangle$

Note that $\mathrm{Irr}(\overline{f})$ is always infinite. The following lemma is obvious.

Lemma 6.25. *Let $\mathcal{B} = (g, \overline{g}, v, E; p, q, \gamma)$ be a flop of $\mathcal{A} = (f, \overline{f}, v, E; p, q, \gamma)$.*

Then $\mathrm{Irr}(\overline{g}) \subset \mathrm{Irr}(\overline{f})$. $\square$

Lemma 6.26. *Let $\mathcal{A} = (f, \overline{f}, v, E; p, q, \gamma)$ be a BRI-quad. Let c be a regular value of $\overline{f}$. Denote $\overline{f}^{-1}([c-1, c])$ by W_0 and $S(\overline{f}) \cap W_0$ by K.*

Then there is a c-flop $\mathcal{B} = (g, \overline{g}, v, E; p, q, \gamma)$ of $\mathcal{A}$ such that $(K \times K) \cap \mathrm{Irr}(\overline{g}) = \emptyset$.

Proof. Denote by $\mathcal{X}$ the set of all c-flops $\mathcal{B}$ of $\mathcal{A}$. Let $\mathcal{B}_0 = (g_0, \overline{g}_0, v, E; p, q, \gamma)$ be an element of $\mathcal{X}$ with minimal possible $\mathrm{card}((K \times K) \cap \mathrm{Irr}(\overline{g}_0))$. I claim that $(K \times K) \cap \mathrm{Irr}(\overline{g}_0) = \emptyset$. Indeed, if not, there is a critical value λ of $\overline{g}_0$ such that $\lambda, \lambda + 1/n(f) \in]c-1, c[$ and $\mathrm{ind}_{g_0}(\lambda) > \mathrm{ind}_{g_0}(\lambda + 1/n(f))$. Apply now 6.23 and get a contradiction. $\square$

Corollary 6.27. *Let $\mathcal{A} = (f, \overline{f}, v, E; p, q, \gamma)$ be a BRI-quad. Let $(x, y) \in \mathrm{Irr}(\overline{f})$.*

Then there is a flop $\mathcal{B} = (g, \overline{g}, v, E; p, q, \gamma)$ of $\mathcal{A}$ such that $\overline{g}(x) - \overline{g}(y) < \overline{f}(x) - \overline{f}(y)$.

Proof. Denote $[\overline{f}(x) - \overline{f}(y)]$ by s; then $\overline{f}(y) < \overline{f}(xt^s) < \overline{f}(y) + 1$. Choose a regular value c of $\overline{f}$ such that $c < \overline{f}(y) < \overline{f}(xt^s) < c + 1$, and apply the previous lemma to get a c-flop $\mathcal{B} = (g, \overline{g}, v, E; p, q, \gamma)$ of $\mathcal{A}$ such that $\overline{g}^{-1}(c) = \overline{f}^{-1}(c)$ and $\overline{g}(xt^s) < \overline{g}(y)$. $\square$

Corollary 6.28. *Let $\mathcal{A} = (f, \overline{f}, v, E; p, q, \gamma)$ be a BRI-quad. Let Q be a finite subset of $S(\overline{f})$.*

Then there is a flop $\mathcal{B} = (g, \overline{g}, v, E; p, q, \gamma)$ of $\mathcal{A}$ such that $(Q \times Q) \cap \mathrm{Irr}(\overline{g}) = \emptyset$.

Proof. In view of 6.25 it suffices for every $(x, y) \in \mathrm{Irr}(\overline{f})$ to construct a flop $\mathcal{B} = (g, \overline{g}, v, E; p, q, \gamma)$ of $\mathcal{A}$ such that $\overline{g}(x) < \overline{g}(y)$. This is done by successive applying 6.27. $\square$

To finish the proof of 6.21, we need an easy lemma, which is a reformulation of [27, Prop.A.10].

Lemma 6.29. *Let U be a compact cobordism, $\partial U = V_0 \sqcup V_1$, $f : U \to \mathbf{R}$ be a Morse function (not necessarily constant on V_i), v be an f-gradient such that v points inward on V_0, and outward in V_1. Let S_i be neighbourhoods of V_i in W $(i = 0, 1)$.*

Then there is a Morse function $\check{f} : U \to [\alpha, \beta]$, where $\beta = \max_{x \in V_1} f(x), \alpha = \min_{x \in V_0} f(x)$ such that

1)$\check{f}^{-1}(\beta) = V_1, \check{f}^{-1}(\alpha) = V_0$.

2) v is an $\check{f}$-gradient.

3)$\check{f}(x) = f(x)$ for $x \notin S_0 \cup S_1$; $\check{f}(x) \geq f(x)$ for $x \in S_1$, $\check{f}(x) \leq f(x)$ for $x \in S_0$.

Moreover, if $f|V_0$ is constant then $\check{f}$ can be chosen in such a way that $\check{f}(x) = f(x)$ for $x \notin S_1$, and if $f|V_1$ is constant then $\check{f}$ can be chosen in such a way that $\check{f}(x) = f(x)$ for $x \notin S_0$. $\square$

The function $\check{f}$ will be called *damping* of f with respect to S_0, S_1.

Proof of 6.21. Denote by W' the cobordism $\overline{f}^{-1}([c - 2N, c + N])$. By Corollary 6.28 there is an BRI-quad $(g, \overline{g}, v, E; p, q, \gamma)$, which is a flop of $\mathcal{A}$, and such that for every $x, y \in S(\overline{g}|W')$ we have $(\mathrm{ind}\, x < \mathrm{ind}\, y) \Rightarrow (\overline{g}(x) < \overline{g}(y))$. Choose $\delta > 0$ so small that $\epsilon > \delta, 1/2 > \delta$, and $\overline{f}^{-1}([c - N, c - N + \delta]) \cup \overline{f}^{-1}([c - \delta, c])$ does not intersect neither $S(\overline{f})$ nor any t-shift of $\mathrm{Im}\, \gamma$. Let ψ be the damping of $\overline{g}|W$ with respect to the neighbourhood $S_0 = \overline{f}^{-1}([c-N, c-N+\delta[)$ of $V_0 = \overline{f}^{-1}(c-N)$ and the neighbourhood $S_1 = \overline{f}^{-1}(]c-\delta, c[)$ of

$V_1 = \overline{f}^{-1}(c)$. Denote $\psi(V_0)$ by R_0, and $\psi(V_1)$ by R_1. Apply 8.14 to the function $\psi : W \to [R_0, R_1]$ and to the neighbourhoods S_0, S_1 of V_0, V_1 and to the functions C_0, C_1, defined in some neighbourhoods of V_0, V_1 by the formulas $C_1(y) = df(v(y)), C_0(y) = df(v(y))$ and to $\nu = 1/2$. Denote the resulting function by $\phi : W \to [R_0, R_1]$.

I claim, that ϕ satisfies $(\mathcal{RF})$ with respect to $(\overline{f}, c, v, N, \epsilon, \gamma)$. Indeed, $(\mathcal{R}1)$ is immediate from the construction and from 8.14, 6.29. Since $\operatorname{supp}(\phi - \overline{g})$ does not intersect $S(\phi) = S(\overline{g})$, and $(\overline{g} \mid W_1)$ is ordered, ϕ is also ordered, so $(\mathcal{R}2)$ also holds.

To verify $(\mathcal{R}3)$ note that if $x, xt \notin S_0 \cup S_1$ we have $-\phi(xt) + \phi(x) = 1$. Therefore we have to verify $(\mathcal{R}3)$ only if either x or xt is in $S_0 \cup S_1$. Assume for example that $xt \in S_0$. Then $x \notin S_0$ and we have $\mid \psi(xt) - \phi(xt) \mid \leq 1/2$ by 8.14.4) and $\phi(xt) \leq \overline{g}(x) - 1$ by 6.29.3), which implies $\phi(xt) \leq \phi(x) - 1/2$.

Proceeding to $(\mathcal{R}4)$ note that first two properties of $(\mathcal{R}4)$ follow immediately from the construction of ϕ and from the definition of the flop. Furthermore, let $\vec{b} = (b_0, ..., b_{n+1})$ be an ordering sequence for $\overline{g} \mid W'$. Set $\mu = \max(b_r, R_0), \nu = \min(b_{r+1}, R_1)$. It is easy to show that $S(\phi) \cap \phi^{-1}(]\mu, \nu[) = S_r(\phi)$, therefore we can choose an ordering sequence $\vec{a} = (a_0, ..., a_{n+1})$ for ϕ in such a way that $a_r = \mu, a_{r+1} = \nu$.

Further, every t-shift of $\operatorname{Im}\gamma$ for $-N \leq s \leq N$ satisfies

$$b_r < \overline{g}(qt^s) < \overline{g} \mid (\operatorname{Im}\gamma)t^s < \overline{g}(pt^s) < b_{r+1}$$

If, moreover, $(\operatorname{Im}\gamma)t^s$ is in W, then $R_0 < \phi \mid (\operatorname{Im}\gamma)t^s = \overline{g} \mid (\operatorname{Im}\gamma)t^s < R_1$ and the last part of $(\mathcal{R}4)$ follows. □

Corollary 6.30. *Assume the hypotheses of* 6.20, *and let $\phi : W \to [\alpha, \beta]$ be a Morse function, satisfying the conclusions of* 6.21. *Let l be an integer such that $0 < l \leq N$. Denote $\overline{f}^{-1}([c - l, c])$ by W_1.*

Then there is a Morse function $\phi_1 : W_1 \to [\alpha', \beta']$ such that:

(1) *If $x \notin \left(\overline{f}^{-1}([c - l, c - l + \epsilon]) \cup \overline{f}^{-1}([c - \epsilon, c])\right)$ then $\phi_1(x) = \phi(x)$.*

(2) *ϕ_1 satisfies $(\mathcal{RF})$ with respect to $(\overline{f}, v, c, l, \gamma, \epsilon)$, and one can choose the corresponding ordering sequence $\vec{a'} = (a'_0, \dots, a'_{n+1})$ for ϕ_1 in such a way that $a'_{n+1} = a_{n+1}$, and $a'_i \geq a_i$ for every i.*

(3) *There is an identification of $\mathbf{Z}G^-_{(l)}$-complexes* $C_*(\phi, v, \vec{a}) \underset{\mathbf{Z}G^-_{(N)}}{\otimes} \mathbf{Z}G^-_{(l)} \to C_*(\phi_1, v, \vec{a'})$.

Proof. For $\nu > 0$ define a sequence $\overrightarrow{a(\nu)} = (a_0(\nu), ..., a_{n+1}(\nu))$ by the following formulas: 1)$a_0(\nu) = a_0$, 2) if $a_i \neq a_{n+1}$, then $a_i(\nu) = a_i + \nu$, 3) if $a_i = a_{n+1}$, then $a_i(\nu) = a_i$. It is obvious, that if ν is small enough, then $\overrightarrow{a(\nu)}$ is still an ordering sequence for ϕ, and every t-shift of $\operatorname{Im}\gamma$ which is in W is already in $\phi^{-1}(]a_r(\nu), a_{r+1}(\nu)[)$.

Choose $\delta > 0$ so small, that $\delta < \epsilon, \delta < 1/2$ and $\delta < \nu$. Let ϕ_0 be the result of the damping construction, applied to $\phi \mid W_1$ and to the neighbourhoods $\overline{f}^{-1}([c - l, c - l + \delta]), \overline{f}^{-1}([c - \delta, c])$. Since ϕ is constant on $\overline{f}^{-1}(c)$ we can assume (by 6.29), that $\operatorname{supp}(\phi - \phi_0) \subset \overline{f}^{-1}([c - l, c - l + \delta])$ and that $\phi_0(x) \leq \phi(x)$ for every $x \in W_1$. Denote $\phi_0 \mid \overline{f}^{-1}(c) =$

$\phi \mid \overline{f}^{-1}(c)$ by R_1 and $\phi_0 \mid \overline{f}^{-1}(c-l) = \min_{x\in W_1} \phi(x)$ by R'_0. Applying 8.14 in the same way, as we did in the proof of 6.21 we obtain a function $\phi_1 : W_1 \to [R'_0, R_1]$, adjusted to $(\overline{f}, v)$ and such that $\operatorname{supp}(\phi_1 - \phi_0), \operatorname{supp}(\phi - \phi_0) \subset \overline{f}^{-1}([c-l, c-l+\delta])$ and that $\max_{x\in W_1} \mid \phi_1(x) - \phi_0(x) \mid \leq \min(\nu, 1/2)$. I claim that ϕ_1 satisfies the conclusions of 6.30. Indeed, the same reasoning as in the proof of 6.21 gives that ϕ_1 satisfies $(\mathcal{R})$ with respect to $(\overline{f}, v, c, l)$ and also that the first two conditions of $(\mathcal{R}4)$ (with respect to $(\overline{f}, v, c, l, \gamma, \epsilon)$) hold.

It is also easy to verify, that the sequence $\overrightarrow{a'}$, defined by $a'_i = \max(R'_0, a_i(\nu))$ is an ordering sequence for ϕ_1 and that every t-shift of $\operatorname{Im}\gamma$ which is in W_1 is already in $\phi_1^{-1}(]a_r(\nu), a_{r+1}(\nu)[)$.

Proceeding to (3) note that $\forall x \in W_1$ we have $\phi_1(x) \leq \phi(x) + \nu$. Therefore if we denote $\phi^{-1}([a_0, a_i])$ by X_i, $\phi_1^{-1}([a'_0, a'_i]) \cup (W \setminus W_1)$ by Y_i, $\phi_1^{-1}([a'_0, a'_i])$ by Z_i, we obtain $X_i \subset Y_i$. There are homomorphisms of homology groups of pairs $H_*(\widetilde{X_i}, \widetilde{X_{i-1}}) \to H_*(\widetilde{Y_i}, \widetilde{Y_{i-1}}) \to H_*(\widetilde{Z_i}, \widetilde{Z_{i-1}})$, the first of which is induced by the inclusions, and the second is the excision isomorphism. These are homomorphisms of $\mathbf{Z}G^-$-modules; by functoriality of exact sequences of pairs they define a homomorphism of chain complexes over $\mathbf{Z}G^-$ and it is not difficult to see that their composition sends $[p](v, \phi)$ to $[p](v, \phi_1)$, which proves (3). $\square$

E.Proof of 6.14.

The proof of the following lemma is contained essentially in [18] (Lemma 7.7 and the end of the proof of 7.6.)

Lemma 6.31. *Let V be a closed manifold, Q be an oriented submanifold of V, L be a cooriented submanifold of V such that the closures $\overline{Q}$ of Q and $\overline{L}$ of L are disjoint, and $\dim Q + \dim L = \dim V - 1$, and $\dim Q, \dim L \leq \dim V - 2$. Let $\gamma : [0,3] \to V$ be an embedding such that $\operatorname{Im}\gamma \cap \overline{Q} = \gamma(1) \in Q, \operatorname{Im}\gamma \cap \overline{L} = \gamma(2) \in L$, and $\gamma'(1) \notin T_{\gamma(1)}Q$, and $\gamma'(2) \notin T_{\gamma(2)}L$. Let $\varepsilon = \pm 1$, and let S be a neighbourhood of $\operatorname{Im}\gamma$ in V.*

Then there is a C^∞ isotopy h_t of V such that:

(1) *There is an $\epsilon > 0$ such that $h_t(x) = x$ for $t < \epsilon$ and $h_t(x) = h_1(x)$ for $t > 1 - \epsilon$.*

(2) $\operatorname{supp} h_t \subset S$.

(3) *$h_t(\gamma(1)) = \gamma(\chi(t))$, where $\chi : [0,1] \to [1, 5/2]$ is a monotonous C^∞ function.*

(4) *In the manifold $V \times [0,1]$ the intersection of the manifold $\mathcal{N} = \{(h_t(x), t) | x \in Q, t \in [0,1]\}$ with $L \times [0,1]$ is transversal and consists of the unique point $(\gamma(2), 2/3)$. The sign of the intersection equals ε.* $\square$

Definition 6.32. Choose a regular value λ of $\overline{f}$ such that $\overline{f}(\overline{e}) - (l-1) < \lambda < c - (l-1)$, and that $]\overline{f}(\overline{e}) - (l-1), \lambda[$ is regular. Denote by R the fixed-ends-homotopy class of paths in $\pi(\overline{e}, \overline{e'} \cdot t^{-(l-1)})$ such that the lifting of a path from R, starting at e, finishes at $e' \cdot \rho$. Let $\gamma : [0,3] \to \overline{f}^{-1}(\lambda)$ be a bridge from $\overline{e}$ to $\overline{e'} \cdot t^{-(l-1)}$ belonging to R. The septuple $(f, \overline{f}, v, E; \overline{e}, \overline{e'}, \gamma)$ is a BRI-quad. Let $\epsilon > 0$ be sufficiently small and let $\phi : W \to [\alpha, \beta]$ be a Morse function with an ordering sequence $(a_0, \ldots, a_{n+1})$, satisfying $(\mathcal{RF})$ with respect to $(\overline{f}, c, v, N, \epsilon, \gamma)$; and let $U \subset \phi^{-1}(]a_r, a_{r+1}[)$ be the corresponding neighbourhood of $\operatorname{Im}\gamma$.

Denote $\overline{f}^{-1}([c-l, c])$ by W_1, and let $\phi_1 : W_1 \to [\alpha, \beta]$ be a Morse function, satisfying the conclusions of Corollary 6.30.

Let S be a neighbourhood of $\operatorname{Im}\gamma$ in $\overline{f}^{-1}(\lambda)$ such that $\overline{S} \subset U$ and for every $p \in$

$S_r(\overline{f}|W), p \neq \overline{e}$ we have $D(p,v) \cap S = \emptyset$, and for every $q \in S_r(\overline{f}|W), q \neq \overline{e'}$ we have $D(q,-v) \cap S = \emptyset$. Let μ be a regular value of $\overline{f}$ such that $\mu > \lambda$, and $]\lambda, \mu[$ is regular, and for every $x \in S$ the part of v-trajectory $\gamma(x,t;v)$, contained in $\overline{f}^{-1}([\lambda,\mu])$ is contained already in U. Let h_t be the isotopy of $\overline{f}^{-1}(\lambda)$, associated by 6.31 to the submanifolds $Q = D(\overline{e}, v) \cap \overline{f}^{-1}(\lambda)$ and $L = D(\overline{e'} \cdot t^{-(l-1)}, -v) \cap \overline{f}^{-1}(\lambda)$ and the path γ, and such that $\operatorname{supp} h_t \subset S$. Denote h_{1-t} by k_t. Let u be the vector field on $\overline{f}^{-1}([c-l, c-l+1])$, associated to the vector field v, the interval $[\lambda, \mu]$ and the isotopy k_t (in the sense of the definition 2.31).

Lemma 6.33. *The vector field u has the following properties:*

(1) *u is an $\overline{f}$-gradient;* $\operatorname{supp}(u-v)$ *is contained in the union of t-shifts of U.*

(2) *If $p \in S_r(\overline{f}|W)$, and p is not a t-shift of $\overline{e}$ then* $\operatorname{supp}(u-v) \cap \big(D(p,v) \cap W\big) = \emptyset$.

(3) *If $q \in S_r(\overline{f}|W)$, and q is not a t-shift of $\overline{e'}$ then* $\operatorname{supp}(u-v) \cap \big(D(q,-v) \cap W\big) = \emptyset$.

(4) $\big(D(\overline{e}, v) \cap \overline{f}^{-1}([c-l+1, c])\big) \cap \operatorname{supp}(u-v) = \emptyset$.

(5) *$u|W$ is a ϕ-gradient; $u|W_1$ is a ϕ_1-gradient.*

(6) *If* $\operatorname{ind} p \neq r$ *then* $(p)(v,\phi) \cap \operatorname{supp}(u-v) = \emptyset$.

(7) *$u|W_1$ is an almost good $(\overline{f}|W_1)$-gradient.*

(8) *If $p \in S_r(\phi_1)$, p is not a t-shift of $\overline{e}$ then* $[p](u,\phi_1) = [p](v,\phi_1)$.

(9) $[\overline{e}](u,\phi_1) = [\overline{e}](v,\phi_1) + \varepsilon \cdot [\overline{e'}](v,\phi_1) \cdot \rho$.

Proof. (1) - (4) follow from the definition of u. The property (5) is not immediate only for the points belonging to the union of t-shifts of U. But in each of these t-shifts the functions $\overline{f} - \phi$ and $\overline{f} - \phi_1$ are constant, and u is an $\overline{f}$-gradient. The property (6) follows from $(\mathcal{R}4)$. To prove (7) note that in view of (6) it is sufficient to verify only that $D(p,v) \cap D(q,-v) = \emptyset$ for $\operatorname{ind} p = \operatorname{ind} q = r$, and this (in view of (2) and (3)) needs to be verified for the case when p is a t-shift of $\overline{e}$ and q is a t-shift of $\overline{e'}$. In view of (4) the only case to verify is $p = \overline{e}, q = \overline{e'} \cdot t^{l-1}$. It suffices to show that $D(\overline{e}, u) \cap D(\overline{e'} \cdot t^{l-1}, -u) \cap \overline{f}^{-1}(\mu) = \emptyset$, but it follows immediately from the condition (4) of Lemma 6.31.

To prove (8) note that if $\operatorname{ind} p = s$ and $s \neq r$ then $(p)(v,\phi_1) \subset \phi_1^{-1}([a'_s, a'_{s+1}])$, and therefore, $(p)(v,\phi_1) \cap \operatorname{supp}(v-u) = \emptyset$. Therefore it suffices to verify (8) only for $\operatorname{ind} p = r$, and in this case it follows from (2). To prove (9) it is sufficient, by Lemma 6.15, to verify that the intersection indices of $[\overline{e}](u,\phi_1)$ and of $[\overline{e}](v,\phi_1) + \varepsilon \cdot [\overline{e'}](v,\phi_1) \cdot \rho$ with every $(p)(-v,\phi)$ (where $\operatorname{ind} p = r$) are equal. It is true for $p \in \overline{f}^{-1}([c-l+1, c])$, by (4). Further, if $p \in S_r(\overline{f}) \cap \overline{f}^{-1}([c-l, c-l+1]), p \neq \overline{e'} \cdot t^{l-1}$, both intersection indices are equal to zero. (The second one equals to zero, since v is almost good, and the first equals to zero, since $D(\overline{e}, u) \cap \overline{f}^{-1}(\mu) \subset \big(D(\overline{e}, v) \cap \overline{f}^{-1}(\mu)\big) \cup S$, and $D(p,-v) \cap \overline{f}^{-1}([-\infty, \mu]) \cap S = \emptyset$. Now the last intersection index to calculate is that of $[\overline{e}](u,\phi_1)$ with $[\overline{e'}](-v,\phi_1)$, but it is easy to use (3) and (4) of the conclusions of 6.31 to verify that this index equals $\varepsilon\rho$. $\square$

Lemma 6.34. *Let $\delta > 0$ and let w be a perfect f-gradient, δ-fitted to u. Then if w is sufficiently close to u, we have:*

(1) *w is also ϕ-gradient and ϕ_1-gradient.*

(2) $[p](v,\phi) = [p](w,\phi)$, *if p is not a t-shift of $\overline{e}$*

(3) $[\overline{e}](w,\phi) = [\overline{e}](v,\phi) + \varepsilon \cdot [\overline{e'}](v,\phi) \cdot \rho + x \cdot \theta^l$ *(where $x \in C_r(\phi, v, \vec{a})$)*

Proof. (1) is obvious. For the critical points p of index not equal to r,the property

(2) follows from 6.33.(6) and 2.23. If $\operatorname{ind} p = r$ then it follows from 6.33.(2) and 2.23. The property (3) follows from 6.30 (3) and 6.33.(9). $\square$

Now Proposition 6.14 is a direct consequence of 6.34. $\square$

§7. Applications.

For our applications we shall need product and duality formulas for Novikov complexes. During this section we denote $C_*^\Delta(\widetilde M)\underset{\Lambda}{\otimes}\Lambda_\xi^-$ by $C_*(M,\xi)$. The tensor product over $\mathbf{Z}$ will be denoted by $\otimes$, the tensor product over a ring R will be denoted by $\underset{R}{\otimes}$.

A. Product formula.

Let G,K be groups, $\xi : G\to\mathbf{Z}$ and $\eta : K\to\mathbf{Z}$ be nonzero homomorphisms. Denote by $\xi\oplus\eta$ the homomorphism $G\times K\to\mathbf{Z}$, defined by $(\xi\oplus\eta)(g,k)=\xi(g)+\eta(k)$. Denote by Λ, Γ and Δ the group rings $\mathbf{Z}G$, resp. $\mathbf{Z}K$, and $\mathbf{Z}(G\times K)$. Denote by Λ_ξ^-, Γ_η^-, $\Delta_{\xi\oplus\eta}^-$ their Novikov completions. It is easy to verify that the following formula defines a ring homomorphism $\mu:\Lambda_\xi^-\otimes\Gamma_\eta^-\to\Delta_{\xi\oplus\eta}^-$.

$$\mu\big((\sum_{g\in G} n_g g)\otimes(\sum_{k\in K} n_k k)\big)=\sum_{g\in G,k\in K} n_g\ n_k\cdot(g,k)$$

This homomorphism endows $\Delta_{\xi\oplus\eta}^-$ with a structure of a bimodule over $\Lambda_\xi^-\otimes\Gamma_\eta^-$. If L,S are right modules over Λ_ξ^-, and, respectively, Γ_η^- then the module $L\otimes S$ is a right $\Lambda_\xi^-\otimes\Gamma_\eta^-$-module, and we denote by $L\mathbin{\widetilde\otimes} S$ the right $\Delta_{\xi\oplus\eta}^-$-module $(L\otimes S)\underset{\Lambda_\xi^-\otimes\Gamma_\eta^-}{\otimes}\Delta_{\xi\oplus\eta}^-$. The natural inclusions of G and K to $G\times K$ will be denoted by i_G, i_K; we keep the same terminology for the corresponding inclusions $\Lambda\hookrightarrow\Delta$, resp. $\Gamma\hookrightarrow\Delta$, and $\Lambda_\xi^-\hookrightarrow\Delta_{\xi\oplus\eta}^-$, resp. $\Gamma_\eta^-\hookrightarrow\Delta_{\xi\oplus\eta}^-$.

Now let M, N be closed connected manifolds, $\xi\in H^1(M),\eta\in H^1(N)$ be nonzero cohomology classes. Let $(f,v,E),(g,w,E')$ be M-triples belonging, respectively, to ξ and to η. Denote by $h:M\times N\to S^1$ the map, defined by $h(m,n)=f(m)+g(n)$. Denote by u the vector field on $M\times N$ with components (v,w). We have $S(h)=S(f)\times S(g)$ and $D((p,p'),u)=D(p,v)\times D(p',w)$, therefore, u is a good h-gradient. The orientations of $D(p,v)$ and $D(p',w)$ determine an orientation of $D((p,p'),u)$. The liftings of p,p' to $\widetilde M$, resp. $\widetilde N$, determine a lifting of (p,p') to $\widetilde{M\times N}$. Therefore a system E'' of liftings of critical points of h together with orientations of the corresponding descending discs is defined, and an M-triple (h,u,E'') is formed.

Proposition 7.1. *The complex $C_*(u,E'')$ is identical to $C_*(v,E)\mathbin{\widetilde\otimes} C_*(w,E')$.*

Proof. The generators of $C_*(u,E'')$ and of $C_*(v,E)\mathbin{\widetilde\otimes} C_*(w,E')$ are in natural one-to-one correspondence. Extend this correspondence to an isomorphism I of $\Delta_{\xi\oplus\eta}^-$-modules. It is sufficient to show that I commutes with the differentials. Let $(p_1,p_2)\in S_m(f+g)$, where $p_1\in S_{k_1}(f), p_2\in S_{k_2}(g), k_1+k_2=m$, and $(q_1,q_2)\in S_{m-1}(f+g)$, where $q_1\in S_{r_1}(f), q_2\in S_{r_2}(g), r_1+r_2=m-1$. Let $\gamma=(\gamma_1,\gamma_2)$ be a $(-u)$-trajectory, joining (p_1,p_2) with (q_1,q_2), where γ_1 is a $(-v)$-trajectory, joining p_1 with q_1, and γ_2 is a $(-w)$-trajectory, joining p_2 with q_2. Since v and w are good, we have (for $i=1,2$): $k_i\geq r_i$ and either $p_1=q_1$ and $\gamma_1=const$, or $p_2=q_2$, and $\gamma_2=const$. In the second case it is easy to see hat $\varepsilon(\gamma)=\varepsilon(\gamma_1)$, and $g(\gamma)=i_G(g(\gamma_1))$. In the first case it is easy to see that $\varepsilon(\gamma)=(-1)^{k_1}\varepsilon(\gamma_2)$, and $g(\gamma)=i_K(g(\gamma_2))$. Therefore the resulting formula for the differential of (p_1,p_2) is

$$\partial(p_1,p_2)=\sum_{\mathrm{ind}\, q_1=\mathrm{ind}\, p_1-1}(q_1\cdot n(p_1,q_1))\otimes p_2+(-1)^{\deg p_1}\sum_{\mathrm{ind}\, q_2=\mathrm{ind}\, p_2-1}p_1\otimes(q_2\cdot n(p_2,q_2))$$

That implies easily that I commutes with differentials. $\square$

Corollary 7.2. *Let M, N be closed connected manifolds, $\xi\in H^1(M),\eta\in H^1(N)$ be nonzero cohomology classes,*

Then $C_*(M\times N,\xi\oplus\eta)\sim C_*(M,\xi)\,\widetilde{\otimes}\,C_*(N,\eta)$ $\square$

B. Duality formula.

Let Q,R be rings with unity and let $\chi:Q\to R$ be an antiisomorphism (i.e. a bijection, which preserves the addition and the unity and such that $\chi(ab)=\chi(b)\chi(a)$. Let $A_*=\{0\leftarrow A_0\leftarrow\ldots\leftarrow A_n\leftarrow 0\}$ be a complex over Q. Set $A'_*=\mathrm{Hom}_Q(A_{n-*},Q)$; A'_* has a natural structure of a chain complex of f.g. *left* Q-modules. Define on A'_* a structure of a chain complex of right modules over R, setting $x\cdot\lambda=\chi^{-1}(\lambda)\cdot x$, where $\lambda\in R$. The dual basis for the basis of A_* turns A'_* to a based chain complex. The element of this dual basis, which corresponds to an element e of the basis of A_* will be denoted $\widehat{e}$. The resulting complex will be denoted $\widehat{A}_*$. If e is an element of the basis in A_{i+1}, and f is an element of the basis in A_i, denote by $n(e,f)$ the corresponding entry of the matrix of the differential $\partial:A_{i+1}\to A_i$. Note that $n(\widehat{f},\widehat{e})=\chi(n(e,f))$.

If, furthermore, U and W are two groups of units of Q and, respectively, R such that $\chi(U)=W$ then χ induces obviously an isomorphism $\chi_*:K_U(Q)\to K_W(R)$. The following lemma is obvious.

Lemma 7.3. 1) *A_* is acyclic if and only if $\widehat{A}_*$ is.*

2) *If A_* is acyclic then $\tau(\widehat{A}_*)=(-1)^{n-1}\chi_*(\tau(A_*))$.* $\square$

For example, let G be a group, $\xi:G\to\mathbf{Z}$ be a nonzero homomorphism, Λ_ξ^- be the corresponding Novikov ring, $\varepsilon:G\to\{\pm1\}$ be a homomorphism. In this case it is easy to see that the formula $\chi(\sum_{g\in G}n_g g)=(\sum_{g\in G}n_g\varepsilon(g)g^{-1})$ determines correctly an antiisomorphism $\chi:\Lambda_\xi^-\to\Lambda_{(-\xi)}^-$, which sends U_ξ^- to $U_{(-\xi)}^-$, and therefore, an isomorphism $\chi_*:Wh(G,\xi)\to Wh(G,-\xi)$.

Now let M be a closed connected manifold, $\xi:\pi_1(M)\to\mathbf{Z}$ be a nonzero homomorphism, and (f,v,E) be an M-triple belonging to ξ. We denote as usual $\mathbf{Z}G$ by Λ, and the corresponding Novikov completion by Λ_ξ^-. Consider the map $(-f)\colon M\to S^1$ belonging to the class $(-\xi)$. It is a Morse map, for which $(-v)$ is a good gradient. We fix the same liftings $p\mapsto\widetilde{p}$ of the points $p\in S(-f)$ to $\widetilde{M}$, as given by E. Choose an orientation of $\widetilde{M}$. It induces an orientation in T_pM where $p\in S(f)$, and therefore, an orientation of ascending discs of critical points. We obtain, thus, an M-triple $(-f,-v,\widehat{E})$, and the corresponding Novikov complex $C_*(-v,\widehat{E})$ is defined.

Proposition 7.4. *Under the above assumptions $C_*(-v,\widehat{E})$ is identical to $[C_*(v,E)]^\wedge$.*

Proof. The generators of $C_*(-v,\widehat{E})$ and of $C_{n-*}(v,E)$ are in 1–1 correspondence. For generators $p,q\in S(f)$, $\mathrm{ind}\,p=\mathrm{ind}\,q+1$ a $(-v)$-trajectory γ, joining p with q, determines obviously a (v)-trajectory $\widehat{\gamma}$, joining q with p (and vice versa). We have obviously $g(\widehat{\gamma})=$

$[g(\gamma)]^{-1}$. Computing the signs we obtain $\varepsilon(\widehat{\gamma}) = (-1)^{r(n-r-1)}\varepsilon(\gamma)$, where $r = \operatorname{ind} p$. Now the formula follows. □

The following corollary is an immediate consequence of 7.4 and 7.3.

Corollary 7.5. *Let M be a closed connected manifold, $\xi : \pi_1(M) \to \mathbf{Z}$ be an epimorphism. Then*

1) *$C_*(M, \xi)$ is acyclic if and only if $C_*(M, -\xi)$ is.*

2) *If $C_*(M, \xi)$ is acyclic then $\chi\big(\tau(C_*(M, \xi))\big) = (-1)^{n-1}\tau(C_*(M, -\xi))$.* □

C.Calculation of $Wh(G, \xi)$.

In this subsection H stands for a group, $G = H \times \mathbf{Z}$, and $\xi : G \to \mathbf{Z}$ is the projection onto the second factor. We compute here the group $Wh(G, \xi)$. This computation was begun in [26], where an exact sequence $WhH \to Wh(G, \xi) \to \operatorname{Nil} H$ was obtained. The full computation was done in [31]. We give here a proof of weaker result, using a theorem of Weibel [45]. I am grateful to A.Ranicki for pointing at the paper of Weibel.

For a ring R and a multiplicative set of central nonzerodivisors $S \subset R$ we denote by $\mathbf{H}^1_S(R)$ the category of right R-modules L such that L has a resolution of length ≤ 1 by finitely generated projective modules, and L is an S-torsion R-module.

Choose a generator t of $\mathbf{Z}$ in such a way that $\xi(t) = -1$. Denote $\mathbf{Z}H$ by A then $\mathbf{Z}G = A[t^{\pm 1}]$, and $(\mathbf{Z}G)^-_\xi = A[[t]][t^{-1}]$. Denote by i the inclusion of $A[t]$ into $A[[t]]$ and by I the inclusion of $A[t^{-1}]$ into $A[[t]][t^{-1}]$. Denote by S the multiplicative subset $\{t^k\}_{k\in\mathbf{Z}}$ of $A[t]$, and by S' its i-image.

Lemma 7.6. *There is a commutative diagram*

$$\begin{array}{ccccccc}
K_1(\mathbf{H}^1_S(A[t])) & \xrightarrow{\alpha_1} & K_1(A[t]) & \xrightarrow{\alpha_2} & K_1(A[t^{\pm 1}]) & \xrightarrow{\alpha_3} \\
\big\downarrow{\scriptstyle \phi_1} & & \big\downarrow{\scriptstyle \phi_2} & & \big\downarrow{\scriptstyle \phi_3} & \\
K_1(\mathbf{H}^1_S(A[[t]])) & \xrightarrow{a_1} & K_1(A[[t]]) & \xrightarrow{a_2} & K_1(A[[t]][t^{-1}]) & \xrightarrow{a_3}
\end{array}$$

$$\begin{array}{ccccccc}
\xrightarrow{\alpha_3} & K_0(\mathbf{H}^1_S(A[t])) & \xrightarrow{\alpha_4} & K_0(A[t]) & \xrightarrow{\alpha_5} & K_0(A[t^{\pm 1}]) \\
& \big\downarrow{\scriptstyle \phi_4} & & \big\downarrow{\scriptstyle \phi_5} & & \big\downarrow{\scriptstyle \phi_6} \\
\xrightarrow{a_3} & K_0(\mathbf{H}^1_S(A[[t]])) & \xrightarrow{a_4} & K_0(A[[t]]) & \xrightarrow{a_5} & K_0(A[[t]][t^{-1}]).
\end{array} \tag{7.1}$$

where $\alpha_2, \alpha_5, a_2, a_5, \phi_2, \phi_3, \phi_5, \phi_6$ are induced by the (obvious) inclusions of rings, and ϕ_1, ϕ_4 are isomorphisms.

Proof. The lines of this diagram are the long exact sequences of localization with respect to S and to S' (see [30], [12,p.233]). The inclusion $A[t] \hookrightarrow A[[t]]$ is analytic along S in the sense of [45] (that is, S is a multiplicatively closed set of central nonzerodivisors in $A[t]$ and its image $i(S)$ consists of central nonzerodivisors of $A[[t]]$ and for every $s \in S$ the map i induces an isomorphism $A[t]/(s) \cong A[[t]]/(i(s))$). Therefore, Corollary 1.2 of [45] implies the existence of a map of these exact sequences, and Theorem 1.1 there (due to Karoubi [14,p.400]) implies that ϕ_1 and ϕ_4 are isomorphisms. □

We denote by $\operatorname{Nil}(H)$ the Grothendieck group of the category $\mathbf{Nil}(H)$ of projective $\mathbf{Z}H$-modules, endowed with a nilpotent endomorphism. There is a natural epimorphism

$\mathrm{Nil}\,(H) \xrightarrow{\varepsilon} K_0(\mathbf{Z}H)$, which splits canonically. Therefore there is a canonical isomorphism $\mathrm{Nil}\,(H) \cong \widetilde{\mathrm{Nil}\,(H)} \oplus \widetilde{K}_0(\mathbf{Z}H) \oplus \mathbf{Z}$, where $\widetilde{\mathrm{Nil}\,(H)} = \mathrm{Ker}\,\varepsilon$. The third component $\mathbf{Z}$ will be called *dimension subgroup.* The exact categories $\mathbf{Nil}(H)$ and $\mathbf{H}^1_S(A[t])$ are equivalent by [12,p.236], therefore we have $K_0(\mathbf{H}^1_S(A[t])) \cong \mathrm{Nil}\,(H)$.

Proposition 7.7. *Let H be a group, $G = H \times \mathbf{Z}$, $\xi: G \to \mathbf{Z}$ be the projection of G onto the second factor. Then there is a short exact sequence $0 \to \mathrm{Wh}\,H \to \mathrm{Wh}\,(G,\xi) \to \widetilde{\mathrm{Nil}}\,(H) \oplus \widetilde{K}_0(\mathbf{Z}H) \to 0$.*

Proof. Consider the diagram (7.1). It is well known that α_2 and α_5 are monomorphisms (it follows for example from the exact sequence

$$0 \to K_qA \to K_qA[t] \oplus K_qA[t^{-1}] \to K_qA[t^{\pm 1}] \to K_{q-1}A \to 0$$

from [12, p.237]). Denote by $\bar{t}$ the image of the unit t of the ring $A[t^{\pm 1}]$ in $K_1(A[t^{\pm 1}])$. Denote by $\bar{\bar{t}}$ the image of t in $K_1(A[[t]][t^{-1}])$. It is easy to verify that $\alpha_3(\bar{t})$ is a generator of the dimension subgroup $\mathbf{Z} \subset K_0(\mathbf{Z}H) \subset \mathrm{Nil}\,(H)$. Therefore the same holds for $\bar{\bar{t}}$, and we have an exact sequence

$$0 \to K_1(A[[t]]) \to K_1(A[[t]][t^{-1}])/\{\bar{\bar{t}}\} \to \widetilde{\mathrm{Nil}\,(H)} \oplus \widetilde{K_0(\mathbf{Z}H)} \to 0$$

Let $U^=_\xi = \{\lambda \in (\mathbf{Z}G)^-_\xi \,|\, \lambda = (\pm h)(1 + a_1t + a_2t^2 + \ldots),$ where $h \in H, a_i \in \mathbf{Z}H\}$; $U^=_\xi$ is a subgroup of the group of units of $(\mathbf{Z}G)^-_\xi$. Denote by $\overline{U}_\xi$ its image in $K_1(A[[t]])$ and by $\overline{\overline{U}}_\xi$ its image in $K_1(A[[t]][t^{-1}])/\{\bar{\bar{t}}\}$. Since $U^=_\xi$ and t generate the group U^-_ξ, we have $K_1(A[[t]][t^{-1}])/\{\bar{\bar{t}}, \overline{\overline{U}}_\xi\} = Wh(G,\xi)$. Therefore there is an exact sequence

$$0 \to K_1(A[[t]])/\overline{U}_\xi \to \mathrm{Wh}\,(G,\xi) \to \widetilde{\mathrm{Nil}}\,(H) \oplus \widetilde{K}_0(\mathbf{Z}H) \to 0.$$

Now we have $K_1(A[[t]])/\overline{U}_\xi \cong \mathrm{Wh}\,H$ by [27, Lemma 1.1]. $\square$

D. Fibering obstructions.

The following corollary of our main theorem is immediate. (Recall that $C_*(M,\xi)$ stands for $C^\triangle_*(\widetilde{M}) \underset{\Lambda}{\otimes} \Lambda^-_\xi$).

Corollary 7.8. *Let M be a closed connected manifold, $\dim M \geq 6$, $\xi: \pi_1(M) \to \mathbf{Z}$ be an epimorphism, such that $\mathrm{Ker}\,\xi$ is a finitely presented group.*

Then ξ can be represented by a smooth fibration $M \to S^1$ if and only if the complex $C_*(M,\xi)$ is simply homotopy equivalent to zero. $\square$

The problem of fibering a manifold over a circle was solved by Farrell [10]. By [34] the resulting obstruction can be defined as an element of the Whitehead group of the manifold:

Theorem 7.9 [10], [34]. *Let M be a closed connected manifold, $\dim M \geq 6$, $\xi: \pi_1(M) \to \mathbf{Z}$ be an epimorphism. Assume that the cyclic covering $\overline{M}_\xi$, corresponding to ξ, is finitely dominated.*

Then there is an element $\theta(M,\xi) \in \mathrm{Wh}\,(\pi_1(M))$ such that $\theta(M,\xi) = 0$ if and only if ξ contains a smooth fibration of M over S^1. This element θ is an invariant of simple homotopy equivalence, respecting the cohomology class ξ. $\square$

The results of A.Ranicki [31], [33] provide a purely algebraic identification of these two obstructions (at least in the case $G = H \times \mathbf{Z}$). To state here these results we formulate first some notions and results of algebraic fibering obstruction theory of A.Ranicki [32,§20], [31,§3]. We accept here the terminology of the previous subsection, and we denote, moreover, $A[[t]][t^{-1}]$ by $\widehat{A}_-$ and $A[[t^{-1}]][t]$ by $\widehat{A}_+$. A complex over $A[t^{\pm 1}]$ is called *band* if it is finitely dominated over A. For a band C_* two elements $\Phi^+(C_*)$ and $\Phi^-(C_*)$ in $\mathrm{Wh}(G)$ are defined (see def.3.9 of [31]), which are called fibering obstructions of the band C_*. These elements have the following properties. Let M be a closed manifold, $\pi_1(M) = G$. By a theorem of C.T.C.Wall [43] the cyclic covering $\overline{M}_\xi$ is finitely dominated if and only if $C_*^\Delta(\widetilde{M})$ (which is a complex over $A[t^{\pm 1}]$) is a band. Therefore, if $\overline{M}_\xi$ is finitely dominated, the two elements $\Phi^+(C_*^\Delta(\widetilde{M}))$ and $\Phi^-(C_*^\Delta(\widetilde{M}))$ are defined. We shall denote them by $\Phi^+(M), \Phi^-(M)$. One can prove a duality formula for these elements. Namely, in the terminology of subsection B we have $\Phi^+(M) = (-1)^{n-1}\chi_*(\Phi^-(M))$, where $\chi : \mathbf{Z}G \to \mathbf{Z}G$ is the standard involution on the group ring of a manifold. The following proposition is a result of Farrell-Siebenmann fibering obstruction theory, and Ranicki chain complex fibering obstruction theory.

Proposition 7.10 [31, Prop.2.11]. *Let M be a manifold, $\pi_1(M) = H \times \mathbf{Z}, \xi : G \to \mathbf{Z}$ be the projection onto the second factor. Assume that $C_*^\Delta(\widetilde{M})$ is a band.*

Then there is a fibering $f : M \to S^1$ such that $f_* : \pi_1(M) \to \mathbf{Z}$ equals ξ, if and only if $\Phi^+(M) = 0 = \Phi^-(M)$. $\square$

Now the identification of the obstruction theory, coming from 7.8, and of the Farrell-Siebenmann obstruction theory follows immediately from the following theorems (together with the lemmas of subsections B and C).

Theorem 7.11 [33, §5]. *A complex C_* over $A[t^{\pm 1}]$ is a band if and only if $C_* \underset{A[t^{\pm 1}]}{\otimes} \widehat{A}_+$ and $C_* \underset{A[t^{\pm 1}]}{\otimes} \widehat{A}_-$ are acyclic.* $\square$

This theorem imply that for a band C_* a torsion $\tau_+ = \tau\big(C_* \underset{A[t^{\pm 1}]}{\otimes} \widehat{A}_+\big) \in Wh(G, -\xi)$ and a torsion $\tau_- = \tau\big(C_* \underset{A[t^{\pm 1}]}{\otimes} \widehat{A}_-\big) \in Wh(G, \xi)$ are defined.

Theorem 7.12 [31, §14]. *Let C_* be a band. Then $\Phi^+(C_*) = \Phi^-(C_*) = 0$ if and only if $\tau_+ = \tau_- = 0$.* $\square$

E.Minimal number of critical points.

The main theorem allows us to obtain in algebraic terms the minimal possible number of critical points of a Morse map belonging to a regular class.

Let R be a ring with unity. We assume that the number of elements in a basis for a free module L over R does not depend on the choice of basis. We denote this number by $m(L)$. Let U be a subgroup of R^*, n be a positive integer. For a complex C_* over R we denote $\sum m(C_p)$ by $m(C_*)$, and we denote by $\mu(C_*)$ the minimal possible number $m(D_*)$, where D_* is a complex, U-simple homotopy equivalent to C_*.

Let k, l be positive integers. We say that a complex C_* is (k, l)-bounded, if $C_i = 0$ for $i < l$ and for $i > k$.

Proposition 7.13. *Let C_* be a (k, l)-bounded complex, where $k - l \geq 1$.*

Then $\mu(C_*)$ equals to the minimal possible number $m(D_*)$, where D_* is a (k, l)-bounded *complex, U-simple homotopy equivalent to C_*.*

This proposition is an immediate consequence of the following one.

Proposition 7.14. *Let $0 \le l \le k-1$ and let C_* be a complex such that $H_i(C_*) = 0$ for $i < l$ and $H^i(C_*, R) = 0$ for $i > k$.*

Then there is a (k,l)-bounded complex D_* such that D_* is U-simple homotopy equivalent to C_* and such that $m(C_*) = m(D_*)$.

Proposition 7.14 follows in turn by an easy induction argument from the following lemmas.

Lemma 7.15. *Let $k \ge 2$ and C_* be a complex such that $C_i = 0$ for $i > k$ and $H^k(C_*, R) = 0$.*

Then there is a complex D_*, such that D_* is U-simple homotopy equivalent to C_*, and $D_i = 0$ for $i \ge k$, $D_i = C_i$ for $i < k-2$, and $m(D_*) = m(C_*)$.

Lemma 7.16. *Let $l \ge 0$ and C_* be a complex such that $C_i = 0$ for $i < l$ and $H_l(C_*) = 0$.*

Then there is a complex D_*, such that D_* is U-simple homotopy equivalent to C_*, and $D_i = 0$ for $i \le l$, $D_i = C_i$ for $i > l+2$, and $m(D_*) = m(C_*)$.

We shall prove Lemma 7.15, the proof of 7.16 is similar.

Proof of 7.15. Denote the basis in C_k by $\{e_i\}_{i\in I}$ and the basis in C_{k-1} by $\{f_s\}_{s\in S}$. Then we have $\partial_k e_i = \sum\limits_{s\in S} f_s\lambda_{si}$, where $\lambda_{si} \in R$. The condition $H^k(C_*, R) = 0$ implies easily that the differential $\partial_k : C_k \to C_{k-1}$ is a split monomorphism. Therefore there are elements $(\mu_{rs})_{r\in I, s\in S}$ such that $\sum\limits_{s\in S} \mu_{rs}\lambda_{si} = \delta_{ri}$. Consider the complex $C'_* = C_* \oplus \mathrm{Triv}_*(k-2, C_k)$; then $C'_{k-1} = C_{k-1} \oplus C_k$, with the basis $\mathcal{B} = \{f'_s\}_{s\in S} \cup \{e'_i\}_{i\in I}$, where $e'_i = (0, e_i) \in C_{k-1} \oplus C_k$ and $f'_s = (f_s, 0) \in C_{k-1} \oplus C_k$. Denote $f'_s - \sum\limits_{i\in I} e'_i\mu_{is}$ by a_s, and $\sum\limits_{s\in S} f'_s\lambda_{si}$ by b_i. Using the identity $b_i - \sum\limits_{s\in S} a_s\lambda_{si} = e'_i$, it is easy to see that the family $\mathcal{B}' = \{a_s\}_{s\in S} \cup \{b_i\}_{i\in I}$ is a basis of $C_{k-1} \oplus C_k$, and this basis is U-simply equivalent to $\mathcal{B}$. We have, therefore, $\partial'_k(e_i) = b_i$ for $i \in I$, which implies that C'_* is identical to $D_* \oplus \mathrm{Triv}_*\ (k-1, C_k)$, where $D_i = C_i$ for $i < k-2$, $D_{k-2} = C_{k-2} \oplus C_k, D_{k-1} \approx C_{k-1}$ and $D_i = 0$ for $i \ge k$. This implies our conclusion. □

For a complex C_* and $k \ge 0$ we denote by S_kC_* and call *k-th suspension of* C_* the chain complex $(S_kC_*)_n = C_{n+k}$ with the boundary operator $(S_kC_*)_n \to (S_kC_*)_{n-1}$, which equals the boundary operator $C_{n+k} \to C_{n+k-1}$ in C_*.

Corollary 7.17. *Let C_* be a complex and $k \ge 0$.*

Then $\mu(C_*) = \mu(S_kC_*)$.

Proof. It is obvious that $\mu(S_k(C_*)) \le \mu(C_*)$. To prove the inverse inequality let D_* be a complex, U-simple homotopy equivalent to S_kC_*. One deduces easily from 7.16 that there is a complex F_* such that F_* is U-simple homotopy equivalent to S_kC_*, and $F_i = 0$ for $i < k$, and $m(F_*) = m(D_*)$. Then $F_* = S_kC_*$ and the corollary follows. □

The following lemma is obvious.

Lemma 7.18. *Let R' be a ring with unity, U' be a subgroup of units in $(R')^*$ and let $\phi : R \to R'$ be a ring homomorphism such that $\phi(U) \subset U'$. Let C_* be a complex over R.*

Then $\mu(C_* \underset{R}{\otimes} R') \le \mu(C_*)$. □

To apply these results to Novikov complexes, note that if G is a group and $\xi : G \to \mathbf{Z}$ is an epimorphism then there is a natural epimorphism $\xi : (\mathbf{Z}G)^-_\xi \to (\mathbf{ZZ})^-_{\mathrm{id}} \cong \mathbf{Z}[[t]][t^{-1}]$, induced by ξ. Therefore the number of elements in a basis for a free module over $(\mathbf{Z}G)^-_\xi$ does not depend on the choice of the basis.

Proposition 7.19. *Let M be a closed connected manifold, $\xi \in H^1(M)$ be a regular class, $f : M \to S^1$ be a Morse map such that $f_* : \pi_1(M) \to \mathbf{Z}$ equals ξ.*

Then the minimal possible number of critical points of f equals $\mu(C_*(M,\xi))$.

Proof. By Corollary 3.6 there is a complex D_* in the U^-_ξ-simple homotopy type of $C_*(M,\xi)$ such that $D_i = 0$ for $i = 0, 1, n-1, n$. By Proposition 7.14 there is a complex F_* in the U^-_ξ-simple homotopy type of $C_*(M,\xi)$ such that $F_i = 0$ for $i = 0, 1, n-1, n$ and such that $m(F_*) = \mu(C_*(M,\xi))$. Applying now Theorem 3.7 we get the conclusion. □

F. The case of free abelian fundamental group.

Let $\pi_1(M)$ be free abelian and let $\xi : \pi_1(M) \to \mathbf{Z}$ be an epimorphism. Then ξ is regular. Denote $\operatorname{Ker}\xi$ by H; then $\widetilde{\mathrm{Nil}}(\mathbf{Z}H) = \widetilde{K}(\mathbf{Z}H) = Wh(H) = 0$ by [10] and [2]. Therefore we obtain from 3.7 the following:

Proposition 7.20. *Let M be a closed connected manifold,* $\dim M \geq 6, \pi_1(M) = \mathbf{Z}^m$, *$\xi\colon \pi_1(M) \to \mathbf{Z}$ be an epimorphism. Let D_* be a complex over Λ^-_ξ, homotopy equivalent to $C^\Delta_*(\widetilde{M}) \underset{\Lambda}{\otimes} \Lambda^-_\xi$, and such that $D_i = 0$ for $i \leq 1$ or $i \geq n-1$.*

Then there is a Morse map $f\colon M \to S^1$ belonging to ξ such that $m_p(f)$ equals the number of free generators of D_p. □

This proposition implies easily the following Farber exactness theorem.

Proposition 7.21 (Farber [9]). *Let M be a closed connected manifold,* $\dim M \geq 6, \pi_1(M) = \mathbf{Z}, \xi \in H^1(M)$ *be a non-zero cohomology class.*

Then there is a Morse map $f : M \to S^1$, belonging to ξ, and such that $m_p(f) = b_p(M,\xi) + q_p(M,\xi) + q_{p-1}(M,\xi)$.

Proof. It is sufficient to consider the case, when ξ is an epimorphism. (If ξ is not epimorphic, let $\xi = k\eta, k \in \mathbf{Z}, k > 0$ and η is an epimorphism. Then $\overline{M}_\eta = \overline{M}_\xi$ and $\Lambda^-_\xi = \Lambda^-_\eta$. Then pick a Morse map $g : M \to S^1$ belonging to η and satisfying the conclusions of our proposition with respect to η. Then $g = kg_1$, where g_1 satisfies the conclusions of our proposition with respect to ξ.) In our case the cyclic covering $\overline{M}_\xi$ is the universal covering space for M. Since $\widehat{L}$ is a principal ideal domain, any complex D_* in the homotopy type of $C_*(M,\xi) = C^\Delta_*(\overline{M}_\xi) \underset{L}{\otimes} \widehat{L}$ satisfies

$$m(D_p) \geq b_p(M,\xi) + q_p(M,\xi) + q_{p-1}(M,\xi) \quad \text{for } 0 \leq p \leq n,$$

and there is a complex D^0_* in this homotopy type such that

$$m(D^0_p) = b_p(M,\xi) + q_p(M,\xi) + q_{p-1}(M,\xi) \quad \text{for } 0 \leq p \leq n$$

From Corollary 3.6 it follows that $m(D^0_p) = 0$ for $p \leq 1$ and for $p \geq n-1$. Now apply Proposition 7.20. □

Proceeding now to the case $\pi_1(M) = \mathbf{Z}^m$, where m is an arbitrary natural number, we need some more terminology. We denote $\pi_1(M)$ always by G, and $\mathbf{Z}G$ by Λ. For a nonzero homomorphism $\xi : G \to \mathbf{Z}$ we denote by S_ξ the set of all $\lambda \in \mathbf{Z}G$ of the form $\lambda = (\pm g)(1+P)$, where $g \in G, P \in \mathbf{Z}G, \operatorname{supp} P \subset \xi^{-1}(]-\infty,-1])$. We denote $S_\xi^{-1}\Lambda$ by $\Lambda_{(\xi)}$. The embedding $\Lambda \subset \Lambda_\xi^-$ is extended naturally to an embedding $\Lambda_{(\xi)} \subset \Lambda_\xi^-$. The algebraic properties of the ring Λ_ξ^- and of the modules over this ring were first studied by J.C.Sikorav. His results were further developed in [23], [24]; the resulting algebraic properties are sufficient to prove Theorem 0.9 from the Introduction.

For the case $m = 1$ we have denoted Λ by L and we denote $\Lambda_{(\xi)}$ by $S^{-1}L$. It is not difficult to prove that: 1) The inclusion $S^{-1}L \subset \widehat{L}$ is a faithfully flat ring homomorphism. 2) $S^{-1}L$ is a euclidean ring, as well as $\widehat{L}$. 3) For any finitely generated L-module N the rank and the torsion number over $\widehat{L}$ of $N \underset{L}{\otimes} \widehat{L}$ are equal to, respectively, rank and torsion number over $S^{-1}L$ of $S^{-1}N = N \underset{L}{\otimes} S^{-1}L$.

Let $\xi : \mathbf{Z}^m \to \mathbf{Z}$ be a homomorphism. It induces a homomorphism $\Lambda \to L$. For a Λ-complex C_* we denote by $b_p(C_*, \xi)$ and by $q_p(C_*, \xi)$ the rank and torsion number of the $S^{-1}L$-module $H_p(C_* \underset{\Lambda}{\otimes} S^{-1}L)$. In view of the previous paragraph these numbers coincide with, respectively, rank and torsion numbers of $H_p(C_* \underset{\Lambda}{\otimes} \widehat{L})$. Therefore $b_p(C_*^\Delta(\widetilde{M}), \xi) = b_p(M, \xi)$ and $q_p(C_*^\Delta(\widetilde{M}), \xi) = q_p(M, \xi)$.

The following proposition is an immediate corollary of Theorem 0.1 of [24] and of Lemmas 2.2 and 2.6 of [23]. (I would like to note that actually the proof of 0.1 in [24] uses power series rings.)

Proposition 7.23. *Let C_* be a complex over $\Lambda = \mathbf{Z}[\mathbf{Z}^m]$.*

Then there is a finite set of hyperplanes $\{\Delta_i\}_{i\in I}$ in $\operatorname{Hom}(\mathbf{Z}^m, \mathbf{Z})$ such that for every nonzero homomorphism $\gamma : \mathbf{Z}^m \to \mathbf{Z}$ such that $\gamma \notin \bigcup_i \Delta_i$ we have:

$$S_\xi^{-1} H_p(C_*) \cong \Big(\bigoplus_{i=1}^{b_p(C_*,\xi)} \Lambda_{(\xi)} \Big) \oplus \Big(\bigoplus_{j=1}^{q_p(C_*,\xi)} \Lambda_{(\xi)} / a_j^{(p)} \Lambda_{(\xi)} \Big)$$

where $a_j^{(p)} \in \Lambda_{(\xi)}$, and $0 \le p \le n$. The numbers $b_p(C_*, \xi)$ and $q_p(C_*, \xi)$ depend only on C_* and the connected component of ξ in $\operatorname{Hom}(\mathbf{Z}^m, \mathbf{Z}) \otimes \mathbf{R} \setminus \bigcup_i (\Delta_i \otimes \mathbf{R})$.□

The following lemma is a particular case of Lemma 5.1 of [23].

Lemma 7.24. *Let C_* be a complex over a commutative ring R. Assume that for all p the R-module $H_p(C_*)$ has a free resolution of length 2: $0 \leftarrow H_p(C_*) \leftarrow F_p \leftarrow G_p \leftarrow 0$, where F_p and G_p are free modules with a finite basis, and $F_p = G_p = 0$ for $p > n$.*

Then C_* is homotopy equivalent to a complex C'_*, such that $m(C'_p) = m(G_{p-1}) + m(F_p)$ for every p. □

Theorem 7.25. *Let M be a closed connected manifold, $\dim M \ge 6$, $\pi_1(M) \cong \mathbf{Z}^m$.*

Then there is a finite set of hyperplanes $\{\Delta_i\}_{i\in I}$ in $H^1(M, \mathbf{Z})$ such that for a nonzero $\xi \in H^1(M, \mathbf{Z}) \setminus \bigcup_i \Delta_i$ there is a Morse map $f : M \to S^1$ such that

$$m_p(f) = b_p(M, \xi) + q_p(M, \xi) + q_{p-1}(M, \xi) \quad \text{for every } 0 \le p \le n.$$

The numbers $b_p(M,\xi), q_p(M,\xi)$ depend only on the homotopy type of M and the connected component of ξ in $H^1(M,\mathbf{R}) \setminus \bigcup_i (\Delta_i \otimes \mathbf{R})$.

Proof. In the proof of 7.21 we have shown that $b_i(M,\xi) = 0$ for $n-1 \le i$ and $i \le 1$, and that $q_i(M,\xi) = 0$ for $i \le 1$ and $i \ge n-2$. Therefore Propositions 7.20, 7.23, 7.24 imply our theorem. $\square$

Corollary 0.10 follow immediately.

G.Lagrangian intersections.

In this subsection we prove the conjecture (C) from the Introduction, assuming the hypotheses (A). We say that a closed 1-form $\omega \in \Omega^1(M)$ is a *Morse form*, if for any $p \in M$ there is a neighbourhood U of p and a Morse function $f : U \to \mathbf{R}$ such that $df = \omega | U$. Note that a Morse form ω is the differential of a Morse map to the circle $\mathbf{R}/\mathbf{Z}$ if and only if its deRham cohomology class $[\omega] \in H^1(M,\mathbf{R})$ is the image of a class in $H^1(M,\mathbf{Z})$. We recall first some results of [36], which allow to reduce (C) to a question on Morse numbers of Morse maps to a circle. In [36] for every natural number $k \ge 1$ it is constructed a closed manifold V_{4k} of dimension $4k$ together with Morse map $p_{4k} : V_{4k} \to S^1$ such that $(p_{4k})_* : \pi_1(V_{4k}) \to \mathbf{Z}$ is an isomorphism, and p_{4k} has a unique non degenerate critical point of index $2k$. Furthermore, under the assumption (C) of the Introduction for some integer $k \ge 1$, and for every sufficiently big real number A, there is a Morse form Ω_A on $M \times V_{4k}$ such that Ω_A is cohomologous to $\alpha_1 \oplus A \cdot p_{4k}^*(d\theta)$ and the number of the zeros of Ω_A equals the number of points of intersection $\phi_1(M) \cap M$ (see [36,sect.1]).

Let $\lambda \ne 0$ be a real number such that the cohomology class $[\lambda\alpha_1] \in H^1(M,\mathbf{R})$ is the image of a indivisible cohomology class in $H^1(M,\mathbf{Z})$, and choose A in such a way that λA is an integer. Then the number of points in $\phi_1(M) \cap M$ equals the number of zeros of a Morse map to the circle $\mathbf{R}/\mathbf{Z}$ belonging to the class $\lambda\alpha_1 + \lambda A p_{4k}^*(d\theta)$. Denote $\pi_1(M)$ by G, $\lambda\alpha_1 : G \to \mathbf{Z}$ by ξ, $\lambda A(p_{4k})_* : \mathbf{Z} \to \mathbf{Z}$ by η, $\xi \oplus \eta : G \times \mathbf{Z} \to \mathbf{Z}$ by ζ. Denote $\mathbf{Z}G$ by Λ, $\mathbf{Z}(G \oplus \mathbf{Z})$ by Γ, and denote the corresponding Novikov completions by $\Lambda_\xi^-, \Gamma_\zeta^-$. Then $\sharp(\phi_1(M) \cap M) \ge \mu(C_*(M \times V_{4k}, \zeta))$. In view of Proposition 7.19 in order to prove the conjecture (C) it is sufficient to prove the following lemma.

Lemma 7.26. *In the above assumptions we have: $\mu(C_*(M \times V_{4k}, \xi)) = \mu(C_*(M), \eta)$.*

Proof. By Corollary 7.2 we have $C_*(M \times V_{4k}, \zeta) \sim C_*(M,\xi) \tilde{\otimes} C_*(V_{4k}, \eta)$. Since $C_*(V_{4k}, \eta)$ is a complex with one generator and zero differentials, we have:

$$C_*(M \times V_{4k}, \xi) \sim S_{2k} C_*(M,\xi) \underset{\Lambda_\xi^-}{\otimes} \Gamma_\zeta^-$$

From our hypotheses it follows that there is a homomorphism $\phi : \mathbf{Z} \to G$ such that $\xi \circ \phi = \mathrm{id} : \mathbf{Z} \to \mathbf{Z}$. Define homomorphisms $a : G \to G \times \mathbf{Z}$ and $b : G \times \mathbf{Z} \to G$ by $a(g) = (g, 0)$ and $b(g,n) = g + \phi(\eta(n))$. Then $\xi \circ b = \xi \oplus \eta$ and $(\xi \oplus \eta) \circ a = \xi$. Therefore there are homomorphisms $\Lambda_\xi^- \to \Gamma_\zeta^- \to \Lambda_\xi^-$,such that their composition is identity. Applying now twice Lemma 7.18 we get $\mu(C_*(M \times V_{4k}, \xi)) = \mu(S_{2k} C_*(M,\eta)) = \mu(C_*(M,\eta))$. $\square$

§8. Appendix 1.

Vector fields and their integral curves on manifolds with boundary.

A. Manifolds without boundary.

Let M be a compact C^∞ manifold without boundary and $r \geq 1$ an integer. Denote by $Vect^r(M)$ the space of C^r vector fields on M. It is a separable Banach space ([1, Th. 12.2]). We denote by $Vect^\infty(M)$ the space of all C^∞ vector fields and we endow it with the topology, which is the union of the topologies in $Vect^r(M)$. For $r \geq 1$ and each vector field $v \in Vect^r(M)$ there is a flow $\phi(v) : M \times \mathbf{R} \to M$, where the map $\phi(v)$ is of class C^r ([1, Th.21.4]). The evaluation map $Ev : Vect^r(M) \times M \to TM$ is of class C^r, for $1 \leq r \leq \infty$ ([1, Th.21.3]), therefore ([1, Th.21.4]) there is a unique C^r parameterized flow of Ev that is a unique C^r map $\mathcal{E} : Vect^r(M) \times M \times \mathbf{R} \to M$ such that $\mathcal{E}(v, x, t)$ is the value at t of the integral curve of v, passing by x at $t = 0$.

Notation 8.1. *The value at the moment t of the integral curve of v passing through x at $t = 0$ will be denoted by $\gamma(x, t; v)$.* $\triangle$

Proposition 8.2. *Let $r \geq 1$ be an integer. Let $v \in Vect^r(M)$, and $m_0, m_1 \in M$ such that $\gamma(m_0, t_0; v) = m_1$. Let E be a submanifold without boundary of M,* codim $E = 1$ *such that $m_1 \in E$. Assume that $v(m_1) \notin T_{m_1}E$. Let $U \subset M$ be an open set, containing $\gamma(m_0, \tau; v)$ for every $\tau \in [0, t_0]$, and let R be an open neighbourhood of m_1.*

Then there is $\epsilon > 0$ and there are neighbourhoods: $A(m_0)$ of m_0 in M, $B(m_1) \subset R$ of m_1 in M, $\mathcal{U}(v)$ of v in $Vect^r(M)$ such that:

1) $\forall m \in A(m_0), \forall w \in \mathcal{U}(v)$ there is a unique $t \in [t_0 - \epsilon, t_0 + \epsilon]$ such that $\gamma(m, t; w) \in E \cap B(m_1)$. This $t = t(m, w)$ is a C^r function of $(m, w) \in A(m_0) \times \mathcal{U}(v)$.

2) $\forall m \in A(m_0), \forall w \in \mathcal{U}(v), \forall \tau \in [-\epsilon, t_0 + \epsilon]$ we have $\gamma(m, \tau; w) \in U$.

3) If E is compact and t_0 is the unique τ from $[0, t_0]$ such that $\gamma(m_0, t; v_0) \in E$, then $\forall m \in A(m_0)$ the number $t(m, v)$ is the unique τ from $[-\epsilon, t_0 + \epsilon]$ such that $\gamma(m, \tau; w) \in E$.

Proof. Denote $Vect^r(M) \times M \times \mathbf{R}$ by $\mathcal{V}$ and consider the C^r flow $\mathcal{E} : \mathcal{V} \to M$, introduced above. Denote by ξ the C^∞ vector field on $\mathcal{V}$, defined by $\xi(v, m, t) = (0, 0, 1)$. Note that $\mathcal{E}_*(\xi(v, m, t)) = v(\gamma(m, t; v))$. Since $v(m_1) \notin T_{m_1}E$, there is a neighbourhood $\widetilde{B}(m_1) \subset R$ of m_1 in M such that $v(x) \notin T_xE$ for $x \in E \cap \widetilde{B}(m_1)$. Denote $E \cap \widetilde{B}(m_1)$ by E_0. By [1, 17.2] the preimage $\mathcal{E}^{-1}(E_0)$ is a C^r submanifold of $\mathcal{V}$ such that for $x \in \mathcal{E}^{-1}(E_0)$ the tangent space $T_x\mathcal{V}$ is a direct sum of the closed subspaces $T_x\mathcal{E}^{-1}(E_0)$ and $\{\xi(x)\}$. Denote by π the projection of $\mathcal{V}$ onto $Vect^r(M) \times M$ and define $H : \mathcal{E}^{-1}(E_0) \times \mathbf{R} \to \mathcal{V}$ by $H(x, t) = (\pi(x), t)$; H is a C^r map. By the above the differential of H is a linear homeomorphism. Note that $\mu_0 = (v_0, m_0, t_0) \in \mathcal{E}^{-1}(E_0)$ and $H(\mu_0, t_0) = \mu_0$. By the inverse function theorem ([7, Thm.10.2.5]) H is a local homeomorphism, therefore if a neighbourhood $B(m_1)$ of m_1 in M and $\epsilon > 0$ are small enough, there is a neighbourhood $A(m_0)$ of m_0 in M and a neighbourhood $\mathcal{U}(v)$ of v in $\mathcal{V}$ such that for every $m \in A(m_0), w \in \mathcal{U}(v)$ there is a unique $t \in]t_0 - \epsilon, t_0 + \epsilon[$ such that $\gamma(m, t; w) \in E_0$, and this t is a C^r function of w and m. This proves 1) and an easy compactness argument shows that passing to some smaller neighbourhoods of m_0, v and $[0, t_0]$ we can achieve 2).

To prove 3), note that compactness of E implies that there is an open neighbourhood S of the set $\{\gamma(m_0, \tau; v) | \tau \in [0, t_0]\}$ such that $S \cap E \subset B(m_1) \cap E$. Applying the preceding result to the neighbourhood $U \cap S$, we get that for $(m, w, \tau) \in A(m_0) \times \mathcal{U}(v) \times [-\epsilon, t_0 + \epsilon]$ the condition $\gamma(m, \tau; w) \in S$ implies $\gamma(m, \tau; w) \in E \cap B(m_1)$, and this, by 1), implies the uniqueness of such τ for given m, w. □

B. Topology on the set of vector fields on a manifold with boundary.

Let M be a C^∞ manifold without boundary (not necessarily compact), dim $M = n$, $\pi : E \to M$ a C^∞ vector bundle with fiber $\mathbf{R}^N$. We shall call *chart* of π a triple $(U, \phi : U \to \mathbf{R}^n, \Phi : \pi^{-1}(U) \to \phi(U) \times \mathbf{R}^N)$, where $\phi : U \to \mathbf{R}^n$ is a chart of M, Φ is a C^∞ map, which is a linear isomorphism on each fiber, and $\phi \circ \pi : \pi^{-1}(U) \to \phi(U)$ equals $p_1 \circ \Phi$, where $p_1 : \phi(U) \times \mathbf{R}^N \to \phi(U)$ is the projection onto the first factor. A vector space of all C^r sections of E will be denoted by $\Gamma^{(r)}(E)$.

We shall introduce a topology on the vector space $\Gamma^{(r)}(E)$ of all C^r sections of E. † Let $\tau = (U, \phi, \Phi)$ be a chart of E, let $K \subset U$ and $\epsilon > 0$. For a C^r map $F : U \to \mathbf{R}^N$, where U is an open set in $\mathbf{R}^N$, and for $x \in U$ we set

$$\langle F \rangle_r(x) = \sum_{i,I} |\partial^I F_i(x)/\partial x^I|$$

where i ranges over integers from 1 to N, and I ranges over multiindices such that $|I| \leq r$. A *subbasic neighbourhood* $\mathcal{U}(K, \tau, \epsilon, r)$ of 0 in $\Gamma^{(r)}(E)$ is the set of such sections s of E that for the map $\sigma = p_2 \circ \Phi \circ s \circ \phi^{-1} : \phi(U) \to \mathbf{R}^N$ (where p_2 is the projection of $\phi(U) \times \mathbf{R}^N$ onto the second factor), we have $\langle \sigma \rangle_r(x) < \epsilon$ for $x \in \phi(K)$. Now an open neighbourhood of f in $\Gamma^{(r)}(E)$ is a union of finite intersections of the sets of the form $f + \mathcal{U}(K, \tau, \epsilon, r)$. The space $\Gamma^{(r)}(E)$ with this topology is a locally convex vector space. If $V \subset M$ is an open set, we denote by $\Gamma^{(r)}(V, E)$ the vector space $\Gamma^{(r)}(i^*(E))$, where $i : V \hookrightarrow M$ is the inclusion map. If $V_1 \subset V_2$ are open then, obviously, the restriction map $\Gamma^{(r)}(V_2, E) \to \Gamma^{(r)}(V_1, E)$ is continuous, with respect to these topologies. Now let $A \subset V \subset M$, where A is compact, V is open. Denote by $\Gamma^{(r)}(A, V; E)$ the set of all sections s of E over A, which admit an extension to a C^r section over V. Using the partition of unity it is easy to see that the vector space $\Gamma^{(r)}(A, V; E)$ depends only on r, A and E. We shall denote it by $\Gamma^{(r)}(A, E)$. For any open V, containing A, we have an epimorphism of vector spaces $R(A, V) : \Gamma^{(r)}(V, E) \to \Gamma^{(r)}(A, E)$. Its kernel is closed, therefore $\Gamma^{(r)}(A, E)$ with the induced topology is a locally convex topological vector space. We denote this topology by $\mathcal{T}(A, V)$.

Proposition 8.3. *Let $V_1 \supset A \subset V_2$, where A is compact, V_1, V_2 are open. Then $\mathcal{T}(A, V_1) = \mathcal{T}(A, V_2)$.*

Proof. It suffices to prove the lemma for the case $\overline{V}_1 \subset V_2$. In this case it is obvious that $\mathcal{T}(A, V_1) \subset \mathcal{T}(A, V_2)$. To prove the inverse inclusion it is sufficient to show that for

† One can verify easily that this topology is the same as that of [8,ch.XVII, §2]

every subbasic neighbourhood $\mathcal{U} = \mathcal{U}(K, \tau, \epsilon, r)$ of 0 in $\Gamma^{(r)}(V_2, E)$ its $R(A, V_2)$-image in $\Gamma^{(r)}(A, E)$ contains an $R(A, V_1)$-image of a subbasic neighbourhood of 0 in $\Gamma^{(r)}(V_1, E)$. Let $\tau = (U, \phi, \Phi)$ and choose two compacts K_1 and K_2 such that $K_1 \cup K_2 = K$ and $K_1 \subset V_1, K_2 \cap A = \emptyset$. Let S be an open neighbourhood of $A \cup K_1$ such that $\overline{S} \subset V_1$ and let h be a C^∞ function on M such that supp $h \subset S$ and $h(x) = 1$ for $x \in K_1 \cup A$. Denote $h \circ \phi^{-1}$ by $\tilde{h}$. There is $C > 0$ such that for every C^r function F on an open neighbourhood of $\phi(K \cap \overline{S})$ we have:

$$\Big(\langle F\rangle_r(x) < \alpha \quad \forall x \in \phi(\overline{S} \cap K)\Big) \Rightarrow \Big(\langle F \cdot \tilde{h}\rangle_r(x) < C\alpha \quad \forall x \in \phi(\overline{S} \cap K)\Big).$$

Now we define a subbasic neighbourhood $\mathcal{U}' = \mathcal{U}(K', \tau', \epsilon', r)$ of 0 in $\Gamma^{(r)}(V_1, E)$, setting

$$K' = \overline{S} \cap K, \quad \tau' = \big(U \cap V_1, \phi | U \cap V_1, \Phi | \pi^{-1}(U \cap V_1)\big), \quad \epsilon' = \epsilon / C.$$

I claim that $R(A, V_1)(\mathcal{U}') \subset R(A, V_2)(\mathcal{U})$. Indeed, let $s \in \mathcal{U}'$. Extend the section $h \cdot s$ by 0 outside S to get a section u over M such that $u | K_1 \cup A = s | K_1 \cup A$. We shall verify that $u \in \mathcal{U}$. Indeed, denote the map $\phi(U \cap V_1) \to \mathbf{R}^N$, corresponding to s, by F. The function $\widetilde{F} : \phi(U) \to \mathbf{R}^N$, corresponding to u, equals $\tilde{h} \cdot F$ on $\phi(S)$ and equals 0 on $\phi(U) \setminus \phi(S)$. Furthermore, for $X \in \phi(\overline{S} \cap K)$ we have $\langle \widetilde{F}\rangle_r(x) = \langle \tilde{h} \cdot F\rangle_r(x) < \epsilon$, therefore $\langle F\rangle_r(x) < \epsilon$, everywhere in K, and $u \in \mathcal{U}$. □

We proceed to the case of manifolds with boundary.

Definition 8.4. Denote the halfspace $\{(x_1, \ldots, x_n) \in \mathbf{R}^n | x_n \geq 0\}$ by H. A map $f : U \to \mathbf{R}^m$, where $U \subset H$ is open, is called C^r, if for every $x \in U$ there is an open neighbourhood $V(x)$ of x in $\mathbf{R}^n$ and a C^r function ϕ on $V(x)$, extending f. A vector field v on a C^∞ manifold with boundary W is called C^r, if for every chart $\phi : U \to \mathbf{R}^n$ of W, (where dim $W = n$) the induced vector field on $\phi(U) \subset \mathbf{R}^n$ is of class C^r. The set of all C^r vector fields on W will be denoted $C^r(W)$. △

Let W be a compact manifold with boundary. Choose and fix a C^∞ embedding $I : W \hookrightarrow M$, where M is a compact C^∞ manifold without boundary and dim$M$ = dim$W$ (for example, the double of W, see [20, def.5.10]). Using the partition of unity it is easy to see that every C^r vector field on $I(W)$ can be extended to a C^rvector field in an open neighbourhood U of $I(W)$ in M, therefore there is an isomorphism of vector spaces $C^r(W) \cong \Gamma^{(r)}(I(W), TM)$, which defines a topology on $C^r(W)$. This topology will be denoted by $\mathcal{T}(I)$, and the resulting topological vector space by $Vect^r(W, I)$.

Proposition 8.5. *The topology $\mathcal{T}(I)$ does not depend on I.*

Proof. By the preceding proposition $Vect^r(W, I) \cong \Gamma^{(r)}(I(W), V; E)$ for any open neighbourhood V of $I(W)$ in M. Therefore to prove the proposition it is sufficient to prove that for any two embeddings $I_1 : W \hookrightarrow M_1, I_2 : W \hookrightarrow M_2$, where dim M_1 = dim M_2 = dim W, the diffeomorphism $I_2 \circ I_1^{-1}$ can be extended to a diffeomorphism of an open neighbourhood V_1 of $I_1(W)$ in M_1 onto an open neighbourhood V_2 of $I_2(W)$ in M_2. Consider a C^∞ vector field v on W such that $v(x) \notin T(\partial W)$ for $x \in \partial W$ and choose two C^∞ vector fields w_1 and w_2 on M_1, resp. M_2 such that $w_i | I_i(W) = (I_i)_*(v), (i = 1, 2)$.

Since ∂W is compact, there is an $\epsilon > 0$ such that the map $\Phi : (a,t) \longmapsto \gamma(I_1(a), t; w_1)$ is a C^∞ diffeomorphism of $\partial W \times]-\epsilon, \epsilon[$ on a neighbourhood of $I_1(\partial W)$ and $\Phi(a,t) \in I(W)$ iff $t \leq 0$. Now for $x \in W \cup \operatorname{Im}\Phi$ we set $J(x) = I_2 \circ I_1^{-1}(x)$ if $x \in W$ and $J(x) = J(\Phi(a,t)) = \gamma(I_2 \circ I_1^{-1}(a), t; w_2)$ for $(a,t) \in \partial W \times]-\epsilon, \epsilon[$. It is obvious that J is a diffeomorphic extension of $I_2 \circ I_1^{-1}$ over an open neighbourhood of $I_1(W)$. $\square$

C.Trajectories of vector fields.

Let W be a compact C^∞ manifold with the boundary ∂W.

Definition 8.6. A vector field v on W is called *transversal to the boundary* (or a *tb-vector field*), if $\forall x \in \partial W$ we have $v(x) \notin T_x \partial W$. The set of C^r tb-vector fields will be denoted by $Vect^r(W, \perp)$. Denote by $V_0(v)$ (resp.$V_1(v)$) the set of all $x \in \partial W$, such that $v(x)$ points inward (resp. outward) W; $V_0(v)$ and $V_1(v)$ are submanifolds of ∂W. We shall abbreviate $V_i(v)$ to V_i if there is no possibility of confusion. $\triangle$

Remark 8.7. Since W is compact any C^0 vector field on W sufficiently close in the C^0-topology to a given C^0 tb-vector field is also transversal to the boundary and $V_i(w) = V_i(v)$. $\triangle$

Definition 8.8. Let $v \in Vect^1(W), x \in W, a \in \mathbf{R}$. Let I be an interval (closed, open or semi-open, finite or not) of $\mathbf{R}$. Let $a \in I$. A C^1 map $\gamma : I \to W$ is called *partial integral curve* of v, passing through x at the moment a, if $\gamma(a) = x$ and $\forall t \in I$ we have $\gamma'(t) = v(\gamma(t))$. (If $t \in I$ is one of the extremities of I, $\gamma'(t)$ means the corresponding one-side derivative). A partial integral curve $\gamma : I \to W$ is called *integral curve* of v, passing through x at the moment a, if $\gamma(a) = x$, and the domain of any other partial integral curve of v, passing through x at a, is contained in I. $\triangle$

The following proposition is easily proved by choosing an embedding $\mathcal{E} : W \hookrightarrow M$, where M is compact, of the same dimension as W, and $\partial M = \emptyset$, and extending $\mathcal{E}_*(v)$ to a C^r vector field on M.

Proposition 8.9. *Let $a \in \mathbf{R}$, $x \in W$, $v \in Vect^r(W, \perp)$, $1 \leq r \leq \infty$. Then:*

1) *There is a partial integral curve of v which passes through x at the moment a.*

2) *Any two such partial integral curves coincide on the intersection of their intervals of definition.*

3) *There is an integral curve γ, passing through x at the moment a, defined on an interval I, and*

$\big(I =]-\infty, b]$, *where* $b < \infty\big) \Leftrightarrow \big(\gamma(t) \in \overset{\circ}{W}$ *for every* $t \in]-\infty, b[$ *and* $\gamma(b) \in V_1\big)$

$\big(I =]-\infty, \infty[\big) \Leftrightarrow \big(\gamma(t) \in \overset{\circ}{W}$ *for every* $t \in \mathbf{R}\big)$

$\big(I = [b, \infty[$, *where* $b > -\infty\big) \Leftrightarrow \big(\gamma(t) \in \overset{\circ}{W}$ *for every* $t \in]b, \infty[$ *and* $\gamma(b) \in V_0\big)$

$\big(I = [b, c]$, *where* $-\infty < b < c < \infty\big) \Leftrightarrow \big(\gamma(t) \in \overset{\circ}{W}$ *for every* $t \in]b, c[$ *and* $\gamma(b) \in V_0$ *and* $\gamma(c) \in V_1\big)$. $\square$

As in 8.1 we denote by $\gamma(x, t; v)$ the value of the integral curve of $v \in Vect^r(W, \perp)$, passing through x at $t = 0$.

The term *v-trajectory* for us is identical to *integral curve of v*.

The restriction of the integral curve $\gamma(x,t;v)$ to the subset of positive t will be called *v-trajectory, starting at x* (it is, obviously, a partial integral curve of v).

For $T<\infty$ we shall say that the v-trajectory, passing through x at the moment $t=a$ reaches the boundary at T, if the right extremity of the interval of definition equals T.

Proposition 8.10. *Let $v \in Vect^r(W,\perp), 1 \leq r \leq \infty, x \in W$. Assume that the v-trajectory $t \mapsto \gamma(x,t;v)$ reaches the boundary at T. Let $U \subset W$ be an open set, containing $\gamma(x,t;v)$ for every $t \in [0,T]$. Let $R \subset W$ be an open neighbourhood of $\gamma(x,T;v)$.*

Then there exists $\epsilon > 0$ and neighbourhoods $A(x)$ of x in W and $\mathcal{V}(v)$ of v in $Vect^r(W)$ such that for every $y \in A(x)$ and $w \in \mathcal{V}(v)$ the curve $t \mapsto \gamma(y,t;w)$ reaches the boundary at $T(y,w) \in [T-\epsilon, T+\epsilon]$, the image of this curve belongs to U, and $\gamma(y,T(y,w);w) \in R$. Moreover, the function $T(y,w)$ is of class C^r in (y,w).

Proof. We consider W as a submanifold of a manifold M without boundary and we choose $w \in Vect^r(M)$ such that $w|W = v$. Choose an open set U_0 in M such that $U_0 \cap W = U$. By 8.2 there is a neighbourhood $\widetilde{A}(x)$ of x in M and a neighbourhood $\mathcal{U}(w)$ of w in $Vect^r(M)$, and a neighbourhood $B(z) \subset R$ of $z = \gamma(x,T;v)$ in M such that $\forall y \in \widetilde{A}(x), \forall w' \in \mathcal{U}(w)$ there is a unique $T(y,w') \in [T-\epsilon, T+\epsilon]$ such that $\gamma(y,T(y,w');w') \in V_1 \cap B(z)$, and $\gamma(y,\tau;w') \in U_0$ for every $\tau \in [-\epsilon, T+\epsilon]$. Now we set $A(x) = \widetilde{A}(x) \cap W$, $\mathcal{V}(v) = R(W,M)(\mathcal{U}(w))$ (see Subsection B for the terminology) and we are over. $\square$

Proposition 8.11. *Let $v \in Vect^r(W,\perp), 1 \leq r \leq \infty, x \in W$. Assume that for a $T>0$ we have $\gamma(x,T;v) \in \overset{\circ}{W}$. Let $U \subset W$ be an open set, containing $\gamma(x,t;v)$ for every $t \in [0,T]$. Let R be an open neighbourhood of $\gamma(x,T;v)$.*

Then there exists $\epsilon > 0$ and neighbourhoods: $A(x)$ of x in W, $\mathcal{V}(v)$ of v in $Vect^r(W)$ such that $\forall y \in A(x), \forall w \in \mathcal{V}(v)$ the curve $t \mapsto \gamma(y,t;w)$ is defined on $[0,T+\epsilon]$ and its image is in U and $\gamma(y,T;w) \in R$.

Proof. Similar to that of 8.10. $\square$

Corollary 8.12. *Let $v \in Vect^r(W,\perp), 1 \leq r \leq \infty, x \in W$. Denote by L_1 the set of all $x \in V_1$ such that the $(-v)$-trajectory $\gamma(x,t;-v)$ reaches V_0, and denote by L_0 the set of all $y \in V_0$ such that the v-trajectory $\gamma(x,t;v)$ reaches V_1.*

Then L_i is open in $V_i (i=0,1)$ and the map, which sends $x \in L_1$ to the point of intersection of $\gamma(x,t;-v)$ with V_0 is a C^r diffeomorphism of L_1 onto L_0. $\square$

Corollary 8.13. *Let $v \in Vect^r(W,\perp), 1 \leq r \leq \infty$.*

Then there are open neighbourhoods U_1 of V_1 and U_0 of V_0 such that $\forall x \in U_1$ (resp. U_0) the v-trajectory $\gamma(x,t;v)$ (resp. $\gamma(x,t;-v)$) reaches the boundary at a moment T, which we denote by $T_1(x,v)$, resp. $T_0(x,v)$. The functions T_1, T_0 are of class C^r. $\square$

Corollary 8.14. *Let $f: W \to [a,b]$ be a Morse function on a compact cobordism W, $f^{-1}(b) = V_1, f^{-1}(a) = V_0$, and v be an f-gradient. Let A_0 be a neighbourhood of V_0, A_1 be a neighbourhood of V_1. Let C_1, C_0 be smooth strictly positive functions, defined, respectively, in A_1, A_0. Let $\nu > 0$.*

Then there are Morse functions $g_0, g_1, g : W \to [a,b]$, such that

1) $\operatorname{supp}(f-g_0) \subset A_0, \operatorname{supp}(f-g_1) \subset A_1, \operatorname{supp}(f-g) \subset A_0 \cup A_1$,

2) *v is a gradient for g_0, g_1, g,*

3) *for* $y \in W$ *in a neighbourhood of* V_0 *we have* $dg(v)(y) = dg_0(v)(y) = C_0(y)$ *and for* $y \in W$ *in a neighbourhood of* V_1 *we have* $dg(v)(y) = dg_1(v)(y) = C_1(y)$.

4) $\max\limits_{y\in W} \mid f(y) - g_0(y) \mid \leq \nu$, $\max\limits_{y\in W} \mid f(y) - g(y) \mid \leq \nu$, $\max\limits_{y\in W} \mid f(y) - g_1(y) \mid \leq \nu$.

Proof. We shall prove the existence of g_0, the other points are proved similarly. Let $U_0 \subset A_0$ be an open neighbourhood of V_0, satisfying the conclusions of 8.13.

Let $\epsilon > 0$ be so small, that the map $(x, \tau) \mapsto \gamma(x, \tau; v)$ is defined for every $x \in V_0$ and every $\tau \in [0, \epsilon]$, and that the image of this map is contained in A_0. Denote this map by $\Phi : V_0 \times [0, \epsilon] \to W$. Let $\mu : \mathbf{R} \to \mathbf{R}$ be a positive C^∞ function such that $\operatorname{supp} \mu \subset]\epsilon/2, \frac{3}{4}\epsilon[$ and $\int\limits_{\epsilon/2}^{\epsilon} \mu(t)dt = 1$; denote $\max \mu(t)$ by $M > 0$.

Set $\alpha = \min\Big(\nu/2, \ \frac{1}{2M} \min\limits_{x\in V_0, \tau\in[0,\epsilon]} C_0(\gamma(x,\tau;v)), \ \frac{1}{2M} \min\limits_{x\in V_0, \tau\in[0,\epsilon]} df(\gamma'(x,\tau;v))\Big)$. Set $\beta = \max\limits_{x\in V_0, \tau\in[0,\epsilon]} C_0(\gamma(x,\tau;v))$.

Choose $\delta \in]0, \epsilon/4[$ so small, that $f(\Phi(x, 2\delta)) - a + 2\delta\beta < \alpha$. Let $\theta : \mathbf{R} \to [0, 1]$ be a C^∞ function, such that $\operatorname{supp} \theta \subset [0, 2\delta[$, and $\theta(t) = 1$ for $t \in [0, \delta]$. For $x \in V_0$ set $s(x) = f(\gamma(x, \epsilon; v)) - a - \int\limits_0^{\epsilon} [(1 - \theta(\tau))df(\gamma'(x, \tau; v)) + \theta(\tau)C_0(\gamma(x, \tau; v))]d\tau$. It follows from our choice of parameters that $\mid s(x) \mid < \alpha$ for every $x \in V_0$. For $t \in [0, \epsilon]$ and $x \in V_0$ set

$$\phi(x,t) = \int\limits_0^t [\theta(\tau)C_0(\gamma(x,\tau;v)) + (1-\theta(\tau)) \cdot df(\gamma'(x,\tau;v)) + s(x) \cdot \mu(\tau)]d\tau + a$$

It is easy to verify that for our choices of parameters we have $d\phi(x,t)/dt > 0$, which implies that v is a $(\phi \circ \Phi^{-1})$-gradient. It follows from the definition of $s(x)$, that for $t \in [0, \delta]$ we have $\partial\phi(x,t)/\partial t = C_0(\gamma(x,t;v))$ and for $t \in [\frac{3}{4}\epsilon, \epsilon]$ we have $\phi(x,t) = (f \circ \Phi)(x,t)$. It follows also that $\phi(x,t) - f \circ \Phi(x,t) \neq 0$ only for $t \in [0, \frac{3}{4}\epsilon]$ and there $\mid \phi(x,t) - f \circ \Phi(x,t) \mid < \nu$.

Now define a function g_0 on W, setting $g_0(y) = f(y)$ for $y \in W \setminus \Phi(V \times [0, \epsilon[)$ and $g_0(y) = \phi(\Phi^{-1}(y))$ for $y \in \Phi(V \times [0, \epsilon[)$.

It is easy to verify that g_0 satisfies our conclusions. □

§9. Appendix 2. Whitney trick.

A. Statement of the theorem.

Let V be a C^∞ manifold without boundary. By boldface latin letters we shall denote finite families of submanifolds of V, e.g. $\mathbf{X} = (X_1, ..., X_k)$. The union of the family is denoted by the corresponding simple letter, here X. The maximum of $\dim X_k$ (resp. the minimum of codim X_k) is called *dimension* of $\mathbf{X}$, resp. *codimension* of $\mathbf{X}$; notation: $\dim \mathbf{X}$(resp. codim $\mathbf{X}$). If A is a submanifold of V, we say that A is *transversal* to $\mathbf{X}$ (notation: $A \pitchfork X$) if $A \pitchfork X_k$ for any k. In this case we denote by $\mathbf{X}_A$ the family $(X_1 \cap A, \ldots, X_k \cap A)$; the codimension of $\mathbf{X}_A$ in A is equal to the codimension of $\mathbf{X}$ in V. We say that $\mathbf{X}$ is *compact*, if X is compact. $\triangle$

The proof of the following theorem, which is essentially due to Whitney [46] will occupy the rest of the Appendix.

Theorem 9.1. *Let V be a closed connected manifold, $m = \dim V \geq 5$, N, L be embedded spheres in $V, \dim N = k, \dim L = l$. Assume that $k + l = m$, $2 \leq k, l \leq m-2$, $N \pitchfork L$, N is oriented, L is cooriented. Let $A, B \in N \cap L$, and assume that the signs $\varepsilon(A)$ and $\varepsilon(B)$ of intersection are opposite. Assume that for some path from A to B in N and for some path from B to A in L the composite loop is homotopic to zero in V. If $k = m-2$ assume further that the inclusion $V \setminus N \hookrightarrow V$ induces an isomorphism in π_1. If $l = m-2$ assume further that the inclusion $V \setminus L \hookrightarrow V$ induces an isomorphism in π_1.*

Let $\mathbf{X}$,$\mathbf{Y}$ *be compact families of submanifolds of* V *such that* codim $\mathbf{X} \geq 3$,codim $\mathbf{Y} \geq 3$. *Assume that $X \cap N = Y \cap L = \emptyset$ and that $\mathbf{X} \pitchfork L, \mathbf{Y} \pitchfork N$. Then there is a C^∞ isotopy $h : V \times [0,1] \to V$ such that:*

1) $\operatorname{supp} h \cap Y = \operatorname{supp} h \cap X = \emptyset$
2) $\operatorname{supp} h \cap (N \cap L) = \{A, B\}$
3) *$h_1(L) \pitchfork N$ and $h_1(L) \cap N = (N \cap L) \setminus \{A, B\}$*
4) *For an $\epsilon > 0$ we have: $h_t = \mathrm{id}$ for $t < \epsilon$ and $h_t = h_1$ for $t > 1 - \epsilon$.*

B. Constructions in $\mathbf{R}^2$.

Let $\varpi : [0,2] \to [0,1]$ be a C^∞ function such that: $\varpi(0) = 0, \varpi(2) = 1; \varpi'(0) = 1, \varpi'(2) = 0$ and $\forall n > 1 \quad \varpi^{(n)}(0) = 0, \quad \varpi^{(n)}(2) = 0$, and $\forall t \in [0,2[\quad \varpi'(t) > 0$ and $\varpi''(t) < 0$. The inverse function will be denoted $\varpi^{-1} : [0,1] \to [0,2]$.

Denote by $\chi : [0,2] \to \mathbf{R}^2$ a C^∞ curve, defined by $\chi(t) = (\varpi(t), \varpi(2-t))$. Denote by $\widehat{\chi} : [0,1] \to [0,1]$ the function $\widehat{\chi}(x) = \varpi(2 - \varpi^{-1}(x))$; the graph of $\widehat{\chi}$ is the image of χ. Denote by Ω the domain $\{(x,y) \in \mathbf{R}^2 | x \in [0,1], y \in [0,1], \text{ and } y \leq \widehat{\chi}(x)\}$. Note that $\widehat{\chi} \circ \widehat{\chi} = \mathrm{id}$ and that Ω is symmetric with respect to the diagonal $\{(x,y) \in \mathbf{R}^2 | x = y\}$ (see Fig.1 a)).

For a subset $A \subset \mathbf{R}^2, v \in \mathbf{R}^2$ and $c \in \mathbf{R}, c \neq 0$ we shall denote by $\underline{A}$ the set $\{(x,y) \in \mathbf{R}^2 | (x,-y) \in A\}$, by $A\langle c \rangle$ the set $\{(x,y) \in \mathbf{R}^2 | (x/c, y/c) \in A\}$, by $A + v$ the set $\{(x,y) \in \mathbf{R}^2 | (x,y) - v \in A\}$.

Denote by Δ_0 the union $\Omega\langle 2 \rangle \cup \underline{\Omega}\langle -2 \rangle \cup \big([-2,2] \times [-3/2, 0]\big) \cup \big([-3/2, 3/2] \times [-2, 0]\big) \cup \big(\underline{\Omega}\langle 1/2 \rangle + (3/2, -3/2)\big) \cup \big(\Omega\langle -1/2 \rangle + (-3/2, -3/2)\big)$ (see Fig.1). Δ_0 is a C^∞ embedded disc in $\mathbf{R}^2$. Denote $\Delta_0\langle 3/2 \rangle$ by Δ_{supp}, $\Delta\langle 2 \rangle$ by Δ, $\Delta\langle 5/2 \rangle$ by Δ_+ (see Fig.1 b)). The interiors of these sets will be denoted by adding $\circ$ above.

Denote by $\gamma : [-7,7] \to \mathbf{R}^2$ the C^∞ curve, defined by: $\gamma(t) = \chi(t)$ for $t \in [0,2]$; $\gamma(t) = (1, t-2)$ for $t \in [2,7]$, $\gamma(t) = (-\gamma_1(-t), \gamma_2(-t))$ for $t \in [-7,0]$. Denote by γ' the tangent vector field along γ and by η the unit normal vector field along γ such that $\eta(0) = (0,-1)$. For $\rho > 0$ denote by $\gamma[\rho]$ the set $\{\gamma(t) + \tau \cdot \eta(t) | t \in [-7,7], \tau \in]-\rho, \rho[\}$. For ρ sufficiently small this is a tubular neighbourhood of $\mathrm{Im}\,\gamma$. Denote by $\theta : [-5,5] \to \mathbf{R}^2$ the curve $\theta(t) = (t,-1)$. Denote by θ' its tangent vector field, by ξ the unit normal vector field $\xi(t) = (0,1)$ and by $\theta[\epsilon]$ the corresponding tubular neighbourhood $\theta[\epsilon] = [-5,5] \times]1-\epsilon, 1+\epsilon[$ (see Fig. 1b)).

C. Isotopy of the standard model.

Let $\omega : \mathbf{R}^2 \to [0,12]$ be a C^∞ function such that $\mathrm{supp}\,\omega \subset \overset{\circ}{\Delta}_{supp}$ and $\omega(x) = 12$ for $x \in \Delta_0$. Let $\kappa : \mathbf{R} \to [0,1]$ be a C^∞ function such that $\kappa(x) = 1$ for $x \in [1/3, 2/3]$ and such that $\kappa(x) = 0$ for $x \le 1/6$ and $x \ge 5/6$. Let $\rho > 0$ and $\zeta : [0,\rho] \to [0,1]$ be a C^∞ function such that $\zeta(x) = 1$ for $x \in [0, \rho/2]$ and $\zeta(x) = 0$ for $x \in [3\rho/4, \rho]$. Let $v(t), t \in [0,1]$ be time-dependent C^∞ vector field on $\mathbf{R}^2 \times \mathbf{R}^{k-1} \times \mathbf{R}^{l-1}$, defined by

$$v(t)(x,y,z_1,z_2) = (0,\ -\kappa(t) \cdot \omega(x,y) \cdot \zeta(|z_1|) \cdot \zeta(|z_2|),\ 0,\ 0)$$

where $(x,y) \in \mathbf{R}^2, z_1 \in \mathbf{R}^{k-1}, z_2 \in \mathbf{R}^{l-1}$. For the integral curves of this time-dependent vector field v we shall adopt the notation 8.1 and thus for $z \in \mathbf{R}^2 \times \mathbf{R}^{k-1} \times \mathbf{R}^{l-1}$ we shall denote by $\gamma(z,t;v)$ the value of the integral curve of v, which passes through z at the moment $t = 0$. (Note that since $\mathrm{supp}\,v$ is compact, these curves are defined on the whole real line.) Set

$$U_\rho = \overset{\circ}{\Delta} \times B^{k-1}(0,\rho) \times B^{l-1}(0,\rho); \quad W_\rho = \overset{\circ}{\Delta}_{supp} \times B^{k-1}(0, 3\rho/4) \times B^{l-1}(0, 3\rho/4).$$

The images of subsets in $\mathbf{R}^2$ under the embedding $\mathbf{R}^2 \cong \mathbf{R}^2 \times \{0\} \times \{0\} \hookrightarrow \mathbf{R}^2 \times \mathbf{R}^{k-1} \times \mathbf{R}^{l-1}$ will be denoted by the same letters as the subsets themselves.

Lemma 9.2. 1) *If $z = (x,y,z_1,z_2)$ then $\gamma(z,t;v) = (x, y(t), z_1, z_2)$, where $y(t)$ is a decreasing function of t.*

2) *For $t \in [0,1/6]$ we have $\gamma(z,t;v) = z$; for $t \in [5/6,1]$ we have $\gamma(z,t;v) = \gamma(z,1;v)$.*

3) *If $z \in W_\rho$, then for every t we have $\gamma(z,t;v) \in W_\rho$.*

4) *If $z \notin W_\rho$ then $\gamma(z,t;v) = z$ for every t.*

5) *If $z = (x,y,0,0)$, where $(x,y) \in \Delta_0$ then $y(t) \le -3/2$ for $t \ge 2/3$.*

Proof. 1) - 4) follow from the definition of v. To prove 5), note that the contrary would mean that for $t \in [1/3, 2/3]$ we have $y'(t) = -12$, therefore $y(2/3) - y(1/3) = -4$, contradiction. □

Define a C^∞ isotopy $\widehat{h} : U_\rho \times [0,1] \to U_\rho$ by $\widehat{h}(z,t) = \gamma(z,t;v)$.

Lemma 9.3. Is1) $\widehat{h}_t = \mathrm{id}$ *for $t \le 1/6$ and $\widehat{h}_t = \widehat{h}_1$ for $t \ge 5/6$.*

Is2) $\mathrm{supp}\,\widehat{h} \subset W_\rho$.

Is3) *For a subset X of $\overset{\circ}{\Delta}$ and $t \in [0,1]$ we have $\widehat{h}_t(X \times B^{k-1}(0,\rho) \times B^{l-1}(0,\rho)) = \widehat{h}_t(X) \times B^{k-1}(0,\rho) \times B^{l-1}(0,\rho))$.*

Is4) $\widehat{h}_1(\mathrm{Im}\,\gamma) \cap \overset{\circ}{\Delta} \cap \mathrm{Im}\,\theta = \emptyset$.

Proof. Is1) - Is3) follow from 1) - 3) of the preceding lemma. To prove Is4) , denote by Λ the set $\{(x,y) \in \mathbf{R}^2 | y \leq -3/2\}$. Then $\widehat{h}_1(\operatorname{Im}\gamma \cap \mathring{\Delta} \cap \Lambda) \subset \Lambda$, and $\widehat{h}_1(\operatorname{Im}\gamma \setminus \Lambda) \subset \Lambda$ by 9.2.5). $\square$

D. Theorem 9.1 follows from the existence of a thick Whitney disc.

Definition 9.4. Assume the hypotheses of 9.1. We say that a C^∞ embedding $\phi : \Delta_+ \to V$ is a *Whitney disc*, if:

Wh1) $\phi(\alpha) = A, \phi(\beta) = B$.

Wh2) $\phi(\Delta_+) \cap N = \phi(\operatorname{Im}\theta), \phi(\Delta_+) \cap L = \phi(\operatorname{Im}\gamma)$.

Wh3) $\forall t \in [-5,5]$ we have: $\phi_*(\xi(t))$ is not tangent to N; $\forall t \in [-7,7]$ we have: $\phi_*(\eta(t))$ is not tangent to L.

Wh4) $\phi(\Delta_+) \cap X = \phi(\Delta_+) \cap Y = \emptyset$. $\triangle$

Definition 9.5. Assume the hypotheses of 9.1. Let $\rho > 0$. We say that a C^∞ map $\Phi_+ : \Delta_+ \times B^{k-1}(0,\rho) \times B^{l-1}(0,\rho) \to V$ is a *thick Whitney disc*, if:

WH1) $\Phi_+|(\Delta_+ \times \{0\} \times \{0\})$ is a Whitney disc.

WH2) $\operatorname{Im}\Phi_+ \cap X = \operatorname{Im}\Phi_+ \cap Y = \emptyset$.

WH3) The map $\Phi = \Phi_+|U_\rho$ is a diffeomorphism onto an open subset of V, and

WH4) $\Phi^{-1}(L) = (\operatorname{Im}\gamma \cap \mathring{\Delta}) \times \{0\} \times B^{l-1}(0,\rho)$.

WH5) $\Phi^{-1}(N) = (\operatorname{Im}\theta \cap \mathring{\Delta}) \times B^{k-1}(0,\rho) \times \{0\}$.$\triangle$

Proposition 9.6. *If for some $\rho > 0$ there is a thick Whitney disc $\Phi_+ : \Delta_+ \times B^{k-1}(0,\rho) \times B^{l-1}(0,\rho) \to V$ then 9.1 is true.*

Proof. Let $\Phi_+ : \Delta_+ \times B^{k-1}(0,\rho) \times B^{l-1}(0,\rho) \to V$ be a thick Whitney disc. Define an isotopy $h : V \times [0,1] \to V$ as follows:

1) $h_t(x) = \Phi_+(\widehat{h}_t(x))$ for $x \in \Phi_+(U_\rho)$

2) $h_t(x) = x$ for $x \in V \setminus \Phi_+(\overline{W}_\rho)$.

(Since $\overline{W}_\rho \subset U_\rho$ and supp $\widehat{h} \subset W_\rho$, these formulas define correctly a C^∞ isotopy $V \to V$ with supp $h \subset \Phi(W_\rho)$.). This isotopy h satisfies all the conclusions of 9.1. Indeed, 1) of 9.1 follows from WH2); 4) follows from Is1); 2) follows from WH4) and WH5). To prove 3) note that

$$h_1(L) = h_1\big(L \setminus \Phi(\overline{W}_\rho)\big) \cup h_1\big(L \cap \Phi(U_\rho)\big) = \big(L \setminus \Phi(\overline{W}_\rho)\big) \cup \Phi\big(h_1(\Phi^{-1}(L))\big) =$$

$$= \big(L \setminus \Phi(\overline{W}_\rho)\big) \cup \Phi\big((h_1(\operatorname{Im}\gamma) \cap \mathring{\Delta}) \times \{0\} \times B^{l-1}(0,\rho)\big).$$

Thus $h_1(L)$ is the union of two submanifolds. The first one is transversal to N and its intersection with N equals $(N \cap L) \setminus \{A, B\}$. The second one does not intersect N by WH4) Is3) and Is4). $\square$

E. Special normal vector fields to a Whitney disc.

Suppose that P and Q are transversally intersecting submanifolds of a manifold M, and that $\dim P + \dim Q = \dim M$. Using the canonical local charts in the points of $P \cap Q$, it is easy to see that the tangent bundle TP of P and the algebraic normal bundle $T_Q M/TQ$

of Q in M glue together to give a vector bundle over $P \cup Q$, which we shall denote by $T(P, Q; M)$, $\dim T(P, Q; M) = \dim P$. Assume that P is oriented, Q is cooriented. Let $a, b \in P \cap Q$. Let λ be a curve from a to b in P, and μ be a curve from b to a in Q such that $\operatorname{Im}\lambda \cap Q = \operatorname{Im}\mu \cap P = \{a, b\}$. Then the restriction of $T(P, Q; M)$ to $\operatorname{Im}\lambda \cup \operatorname{Im}\mu$ is trivial if and only if the intersection signs in a and b are the same.

Lemma 9.7. *Assume the hypotheses of* 9.1. *Let $\phi : \Delta_+ \to V$ be a Whitney disc. Then there are C^∞ vector fields $\eta_1, \dots, \eta_{k-1}$ over $\phi(\Delta_+)$ such that:*

1) *For every $x \in \phi(\Delta_+)$ the vectors $\eta_1(x), \dots, \eta_{k-1}(x)$ are linearly independent in $T_xV/T_x(\phi_*(\Delta_+))$.*

2) *For $x = \phi(y), y \in \operatorname{Im}\theta$ the vectors $\eta_1(x), \dots, \eta_{k-1}(x)$ form a basis in $T_xN/\{\phi_*(\theta'(y))\}$.*

3) *For $x = \phi(y), y \in \operatorname{Im}\gamma$ the vectors $\eta_1(x), \dots, \eta_{k-1}(x)$ form a basis in $T_xV/\big(T_xL \oplus \{\phi_*(\eta(y))\}\big)$.*

Proof. The restriction of $T(N, L; V)$ to $\phi(\operatorname{Im}\theta) \cup \phi(\operatorname{Im}\gamma)$ is non trivial. Consider the 1-dimensional subbundle of $T(N, L; V)|\big(\phi(\operatorname{Im}\theta) \cup \phi(\operatorname{Im}\gamma)\big)$, generated by the vector field $\phi_*(\theta')$ on $\phi(\operatorname{Im}\theta)$ and by the vector field $\phi_*(\eta)$ on $\phi(\operatorname{Im}\gamma)$. It is isomorphic to $T(\operatorname{Im}\theta, \operatorname{Im}\gamma; \Delta_+)$, therefore non trivial. It follows that the quotient bundle is trivial, therefore there is an $(k-1)$-frame $\eta_1, \dots, \eta_{k-1}$ over $\phi(\operatorname{Im}\theta) \cup \phi(\operatorname{Im}\gamma)$ such that over $\phi(\operatorname{Im}\theta)$ the vectors $\eta_1, \dots, \eta_{k-1}$ form a basis of $TN/\{\phi_*(\theta')\}$, and over $\phi(\operatorname{Im}\gamma)$ they form a basis of $TV/\big(\{\phi_*(\eta)\} \oplus TL\big)$. From Wh3) it follows that for every $z \in \operatorname{Im}\theta \cup \operatorname{Im}\gamma$ the vectors $\eta_1, \dots, \eta_{k-1}$ are linearly independent in $T_{\phi(z)}V/T_{\phi(z)}(\phi(\Delta_+))$. Applying the Gram - Schmidt orthonormalization procedure we can assume that $\eta_1, \dots, \eta_{k-1}$ is a section over $\phi(\operatorname{Im}\gamma) \cup \phi(\operatorname{Im}\theta)$ of the bundle of $(k-1)$-orthonormal frames of the orthogonal complement in TV to $T(\phi_*(\Delta_+))$. The obstruction to extend this section to the whole disc lies in $\pi_1(V(k-1, m-2))$. For $m - k > 2$ this group is zero, therefore the section extends. For $k = m-2$ we have $V(k-1, m-2) \cong SO(m-2)$ and the inclusion $SO(k-1) \hookrightarrow SO(m-2)$ induces an epimorphism in π_1, therefore the section extends after changing it if necessary on a small interval by the action of $SO(k-1)$, which does not change the vector space, generated by $\eta_1, \dots, \eta_{k-1}$, therefore it does not change 2) and 3). □

Corollary 9.8. *Let $\phi : \Delta_+ \to V$ be a Whitney disc. Then there are C^∞ vector fields $\xi_1, \dots, \xi_{l-1}$ and $\eta_1, \dots, \eta_{k-1}$, defined on $\phi(\Delta_+)$ such that:*

1) *$\forall y \in \operatorname{Im}\gamma$ the vectors $\xi_1(\phi(y)), \dots, \xi_{l-1}(\phi(y))$ form a basis of $T_{\phi(y)}L/T_{\phi(y)}(\phi(\operatorname{Im}\gamma))$*

2) *$\forall y \in \operatorname{Im}\theta$ the vectors $\eta_1(\phi(y)), \dots, \eta_{k-1}(\phi(y))$ form a basis of $T_{\phi(y)}N/T_{\phi(y)}(\phi(\operatorname{Im}\theta))$*

3) *For each $y \in \Delta_+$ the vectors $\xi_1(\phi(y)), \dots, \xi_{l-1}(\phi(y)), \eta_1(\phi(y)), \dots, \eta_{k-1}(\phi(y))$ form a basis of $T_{\phi(y)}V/T_{\phi(y)}(\phi(\Delta))$.*

Proof. Denote by F the subbundle of $T_{\phi(\Delta_+)}V/T(\phi(\Delta_+))$, generated by $\eta_1, \dots, \eta_{k-1}$. Choose some vector fields $\xi_1, \dots, \xi_{l-1}$ along γ such that they form a trivialization of $T_{\phi(\operatorname{Im}\gamma)}L/T(\phi(\operatorname{Im}\gamma))$. Then the fields $\xi_1, \dots, \xi_{l-1}$ form an $(l-1)$-frame over $\phi(\operatorname{Im}\gamma)$ of the bundle $T_{\phi(\Delta_+)}V/(T(\phi(\Delta_+)) \oplus F)$, which can be extended to $\phi(\Delta_+)$ since $\pi_i(\Delta_+, \operatorname{Im}\gamma) = 0$. □

F.Appropriate riemannian metric on V.

For constructions of Whitney disc and of thick Whitney disc we shall use the following lemma, the proof of which is essentially contained in that of [18, Lemma 6.8].

Lemma 9.9. *Let N and L be closed submanifolds of a closed manifold $V, N \pitchfork L$, $\dim N = k, \dim L = l, k + l = \dim V = m$. Then there is a riemannian metric $\mathcal{G}$ on V such that:*

1) N and L are totally geodesic submanifolds.

2) For every $p \in L \cap N$ there is an open neighbourhood $U(p)$ of p and a chart $\mathcal{L}_p$: $U(p) \to B^m(0, r) \subset \mathbf{R}^m$, such that $\mathcal{G}|U(p)$ is induced by $\mathcal{L}_p$ from the euclidean metric on $\mathbf{R}^m$ and

$$\mathcal{L}_p(U(p) \cap N) = (\mathbf{R}^k \times \{0\}) \cap B^m(0, r); \quad \mathcal{L}_p(U(p) \cap L) = (\{0\} \times \mathbf{R}^l) \cap B^m(0, r).$$

□

G. Existence of thick Whitney disc follows from the existence of Whitney disc.

Lemma 9.10. *Assume the hypotheses of* 9.1. *If there is a Whitney disc $\phi : \Delta_+ \to V$ then for some $\rho > 0$ there exists also a thick Whitney disc $\Phi_+ : \Delta_+ \times B^{k-1}(0, \rho) \times B^{l-1}(0, \rho) \to V$.*

Proof. Pick some $\eta_1, \dots, \eta_{k-1}$ and $\xi_1, \dots, \xi_{l-1}$, satisfying the conclusions of 9.7, and a riemannian metric, satisfying the conclusions of 9.9. For any $r > 0$ define a map $\Psi : \Delta_+ \times B^{k-1}(0, r) \times B^{l-1}(0, r) \to V$ as follows: $\Psi(d, \alpha_1, \dots, \alpha_{k-1}, \beta_1, \dots, \beta_{l-1}) =$

$$= \exp_{\phi(d)}\big(\alpha_1 \cdot \eta_1(\phi(d)) + \ldots + \alpha_{k-1} \cdot \eta_{k-1}(\phi(d)) + \beta_1 \cdot \xi_1(\phi(d)) + \ldots + \beta_{l-1} \cdot \xi_{l-1}(\phi(d))\big).$$

Here $\alpha_1, \dots, \alpha_{k-1} \in \mathbf{R}^{k-1}$, and $\beta_1, \dots, \beta_{l-1} \in \mathbf{R}^{l-1}$, and $d \in \Delta_+$. We shall verify that for all $\rho > 0$ sufficiently small the restriction of Ψ to $\Delta_+ \times B^{k-1}(0, \rho) \times B^{l-1}(0, \rho)$ is a Whitney disc. The property WH1) is obviously true for every ρ, and WH2) for every $\rho > 0$ sufficiently small. It is easy to see that for every $d \in \Delta_+$ the differential $d(\Psi)$ at $(d, 0, 0)$ is an isomorphism. Since N and L are totally geodesic submanifolds of V,for any z from the domain of Ψ we have $\Psi(\operatorname{Im} \gamma \times \{0\} \times \mathbf{R}^{l-1}) \subset L$, and $\Psi(\operatorname{Im} \theta \times \mathbf{R}^{k-1}\{0\}) \subset N$. From that it is easy to deduce the following local version of WH3) - WH5):

(L) For every $d \in \mathring{\Delta}_+$ there is an $\epsilon(d) > 0$ and $\rho(d) > 0$ such that $\Psi|\big(B^2(d, \epsilon(d)) \times B^{k-1}(0, \rho(d)) \times B^{l-1}(0, \rho(d))$ is a diffeomorphism on its image and the preimage of L under this diffeomorphism equals to $\big(\operatorname{Im} \gamma \cap B^2(d, \epsilon(d))\big) \times \{0\} \times B^{l-1}(0, \rho(d))$ and the preimage of N under this diffeomorphism equals to $\big(\operatorname{Im} \theta \cap B^2(d, \epsilon(d))\big) \times B^{k-1}(0, \rho(d)) \times \{0\}$.

Let $\{d_1, \dots, d_s\}$ be a finite set of points of Δ such that $\{B^2(d_i, \epsilon(d_i))\}$ is a covering of Δ. Then for every $\rho < \rho_0 = \min \rho(d_i)$ the map $\Psi_\rho = \Psi| \mathring{\Delta} \times B^{k-1}(0, \rho) \times B^{l-1}(0, \rho)$ is a local diffeomorphism. To show that it is injective for every small ρ, suppose that there are two sequences μ_n, ν_n of points in $\Delta \times B^{k-1}(0, 1/n) \times B^{l-1}(0, 1/n)$ such that $\Psi(\mu_n) = \Psi(\nu_n)$. We can assume that $\mu_n \to \mu, \nu_n \to \nu$. Since $\Psi|\Delta = \phi|\Delta$ is injective, we have $\mu = \nu$, and therefore for some large r and for some $1 \leq i \leq s$ we have $\mu_n, \nu_n \in B^2(d_i, \epsilon(d_i)) \times B^{k-1}(0, \rho) \times B^{l-1}(0, \rho)$, which contradicts (L). The properties WH4) and WH5) are proved similarly. □

H. Some standard approximation, embedding and transversality results.

In this subsection we prove four lemmas on approximation, embedding and transversality "modulo a compact part of the manifold". They are very standard and the reader, willing to believe in 9.14 and 9.15 or to deduce them from [47] and [18, Lemmas 6.11 and 6.12], can skip the whole subsection. I included them, because I could not find their full proofs in the literature.

Definition 9.11. For $0 < r < R$ we denote by $Ann^n(r,R)$ the subset $D^n(0,R) \setminus B^n(0,r)$ of $\mathbf{R}^n$, by $\overset{\circ}{A}nn^n(r,R)$ its interior. Let $\vartheta_{r,R} : Ann^n(r,R) \to [0,1]$ be a C^∞ function such that $\vartheta_{r,R}(x) = 0$ for $|x| \le r + (R-r)/3$, $\vartheta_{r,R}(x) = 1$ for $|x| \ge r + 2(R-r)/3$.$\triangle$

Lemma 9.12. *Let $0 < r < \rho < R$. Let V be either $\mathbf{R}^m$ or a closed m-dimensional manifold. Let $0 \le n \le m$ and $\phi : D^n(0,R) \to V$ be a continuous map such that $\phi|Ann^n(r,R)$ is a C^∞ map. Let $\mathcal{U}$ be a neighbourhood of ϕ in $C^0(D^n(0,R),V)$.*

Then there is a C^∞ map $\psi : D^n(0,R) \to V$ such that $\psi|Ann^n(\rho,R) = \phi$ and $\psi \in \mathcal{U}$.

Proof. 1) $V = \mathbf{R}^m$. Let ϕ_0 be a polynomial approximation of ϕ. Define $\psi : D^n(0,R) \to \mathbf{R}^m$ as follows: $\psi(x) = \phi(x)$ for $x \in Ann^n(\rho,R)$; $\psi(x) = \phi_0(x)$ for $x \in D^n(0,r)$; $\psi(x) = (1-\vartheta_{r,\rho}(x)) \cdot \phi_0(x) + \vartheta_{r,\rho}(x) \cdot \phi(x)$ for $x \in Ann^n(r,\rho)$. It is easy to see that if ϕ_0 was chosen sufficiently close to ϕ,then ψ satisfies the conclusion.

2) V is a closed manifold. Let $j : V \to \mathbf{R}^N$ be a C^∞ embedding of V to $\mathbf{R}^N$, T be a tubular neighbourhood of $j(V)$ and $p : T \to V$ the corresponding projection. For a map $j \circ \phi : D^n(0,R) \to \mathbf{R}^N$ choose a C^∞ approximation χ (by 1) of our lemma), which equals $j \circ \phi$ over $Ann^n(\rho,R)$ then $p \circ \chi$ is the map sought.□

Lemma 9.13. *Let $0 < r < \rho < R$. Let V be a closed manifold,* $\dim V = m$. *Let $2n+1 \le m$ and let $\phi : D^n(0,R) \to V$ be a C^∞ map such that $\phi|Ann^n(r,R)$ is an embedding. Let $\mathcal{U}$ be a neighbourhood of ϕ in $C^0(D^n(0,R),V)$.*

Then there is a C^∞ embedding $\psi : D^n(0,R) \to V$ such that $\psi|Ann^n(\rho,R) = \phi|Ann^n(\rho,R)$ and $\psi \in \mathcal{U}$.

Proof. Note that the normal bundle ν to $\phi(Ann^n(r,R))$ is trivial. (Indeed, it suffices to verify that $\nu|\phi(S^{n-1}(0,\rho))$ is trivial, but this follows easily, since its dimension is in stable range and $\phi^*(TV)|S^{n-1}(0,\rho)$ is trivial). Therefore, for some $r' < r, R' < R$, and $\epsilon > 0$ there is a diffeomorphism Φ of $\overset{\circ}{A}nn^n(r',R') \times B^{m-n}(0,\epsilon)$ onto an open subset of V such that $\Phi|\overset{\circ}{A}nn^n(r,R) \times \{0\}$ equals ϕ. Denote $\overset{\circ}{A}nn^n(r',R') \times B^{m-n}(0,\epsilon)$ by $U \subset \mathbf{R}^m$.

For a C^∞ map $f : Ann^n(r,\rho) \to \mathbf{R}^m$ define a C^∞ map $\widehat{f} : Ann^n(r,\rho) \to \mathbf{R}^m$ as follows: $\widehat{f}(x) = \vartheta_{r,\rho}(x) \cdot x + (1-\vartheta_{r,\rho}(x)) \cdot f(x)$. It is easy to see that if f is sufficiently close to the identity map in C^∞-topology then $\widehat{f}$ is an embedding of $Ann^n(r,\rho)$ to U, close to the identity map in C^∞-topology.

By the embedding theorem ([11, p.235]), there is a C^∞ embedding $\phi_0 : D^n(0,\rho) \to V$, arbitrarily close to $\phi|D^n(0,\rho)$. Define a C^∞ map $\psi : D^n(0,R) \to V$ as follows:

$\psi(x) = \phi(x)$ for $(r+2\rho)/3 \le |x| \le R$;

$\psi(x) = \phi_0(x)$ for $|x| \le (2r+\rho)/3$;

$\psi(x) = \Phi\big((\widehat{\Phi^{-1} \circ \phi_0})(x)\big)$ for $r \le |x| \le \rho$.

This ψ is a well defined C^∞ map, which extends $\phi|Ann^n(\rho,R)$ and $\phi_0|D^n(0,r)$, and its restriction to $Ann^n(r,\rho)$ is a C^∞ embedding, close to $\phi|Ann^n(r,\rho)$. Now it is easy to see that for ϕ_0 sufficiently close to $\phi|D^n(0,\rho)$ the map ψ is a C^∞ embedding, C^∞ close to ϕ. $\square$

Corollary 9.14. *Let V be a closed C^∞ manifold, $\dim V = m \geq 5$. Let $0 < r_1 < r_2 < 1$. Let $\psi : \Delta_+ \hookrightarrow V$ be a continuous map such that $\psi|\Delta_+ \setminus \overset{\circ}{\Delta}_+\langle r_1\rangle$ is a C^∞ embedding. Let $\mathcal{U}$ be a neighbourhood of ψ in $C^0(\Delta_+, V)$.*

Then there is a C^∞ embedding $\phi : \Delta_+ \hookrightarrow V$ such that $\phi \in \mathcal{U}$ and $\phi|\Delta_+ \setminus \overset{\circ}{\Delta}_+\langle r_2\rangle = \psi|\Delta_+ \setminus \overset{\circ}{\Delta}_+\langle r_2\rangle$.

Proof. There is a diffeomorphism $\mathcal{D}$ of $D^2(0,1)$ onto Δ_+, which sends $D^2(0,r_i)$ to $\Delta_+\langle r_i\rangle, i = 1,2$. Apply now 9.12 to $\psi \circ \mathcal{D}$ and to $r = r_1, \rho = (r_1 + r_2)/2, R = 1$. Apply then 9.13 to the resulting C^∞ map and to $r = (r_1 + r_2)/2, \rho = r_2, R = 1$. $\square$

Lemma 9.15. *Let V be a C^∞ manifold, $\dim V = m$, $0 \leq n \leq \dim V - 1$. Let $\mathbf{X}$ be a compact family of submanifolds of V,* codim $\mathbf{X} > n$. *Let $\psi : D^n(0,r) \hookrightarrow V$ be a C^∞ embedding of the closed disc of radius $r > 0$ to V. Let $A \subset D^n(0,r)$ be a compact, such that $\psi(A) \cap X = \emptyset$. Let $\mathcal{U}$ be a neighbourhood of ψ in $C^0(D^n(0,r), V)$.*

Then there is a C^∞ embedding $\phi : D^n(0,r) \hookrightarrow V$ such that $\phi|A = \psi|A$, $\phi \in \mathcal{U}$, and $\operatorname{Im}\phi \cap X = \emptyset$.

Proof. There are $\rho > r$ and $\epsilon > 0$ and a smooth diffeomorphism Φ of $B^n(0,\rho) \times B^{m-n}(0,\epsilon)$ onto an open subset of V such that $\Phi|D^n(0,r)\times\{0\}$ equals to ψ. The preimages $\Phi^{-1}(X_k)$ are submanifolds of $B^n(0,\rho) \times B^{m-n}(0,\epsilon)$ of dimension $\leq m - n - 1$. Since X is compact, we can choose an open neighbourhood B of A in $B^n(0,\rho)$ and a $\delta \in]0,\epsilon[$ such that $\Phi(B \times B^{m-n}(0,\delta)) \cap X = \emptyset$. By the Sard lemma there is $v \in B^{m-n}(0,\delta)$ such that $B^n(0,\rho) \times \{v\}$ does not intersect the union of $\Phi^{-1}(X_k)$. Let $h : \mathbf{R}^n \to [0,1]$ be a C^∞ function such that $h(x) = 0 \ \ \forall x \in A$ and $h(x) = 1 \ \ \forall x \notin B$. Denote by i the map $D^n(0,r) \to B^n(0,\rho)\times B^{m-n}(0,\epsilon)$, defined by $i(x) = (x, h(x)v)$. Now the map $\Phi \circ i$ satisfies the conclusions of our lemma. $\square$

K.Existence of Whitney disc.

Proposition 9.16. *Assume the hypotheses of* 9.1. *Then there is a Whitney disc.*

Proof. 1) An embedding of $\operatorname{Im}\gamma$ to L.

Choose a unit vector a in $\mathbf{R}^l$. Choose $\epsilon > 0$ so small that the image of the curve $t \mapsto t\cdot a$, defined on $[-\epsilon,\epsilon]$ is contained in the image $B^m(0,r)$ of the coordinate neighbourhoods $U(A)$ and $U(B)$ from 9.9. Denote by $\widehat{\Gamma}^{(0)} : [-\epsilon,\epsilon] \to L$, $\widehat{\Gamma}^{(1)} : [-\epsilon,\epsilon] \to L$ the compositions of this curve with $\mathcal{L}_A^{-1}$ and, respectively, $\mathcal{L}_B^{-1}$. Add -1, resp. 1 to the parameters to obtain the curves $\Gamma^{(0)} : [-1-\epsilon, -1+\epsilon] \to L$, $\Gamma^{(1)} : [1-\epsilon, 1+\epsilon] \to L$. Since A and B do not belong to $X \cup Y$, the curves $\Gamma^{(0)}, \Gamma^{(1)}$ do not intersect $X \cup Y$either, if only ϵ is sufficiently small. Since $L \cong S^l$ and $l \geq 2$ we can join $\Gamma^{(0)}(\epsilon - 1)$ with $\Gamma^{(1)}(1-\epsilon)$ by a curve in $L \setminus (N \cup L)$. Denote the composite C^0 curve, defined on $[-1-\epsilon, 1+\epsilon]$ by Γ_0. Let $R : [-7,7] \to [-1-\epsilon, 1+\epsilon]$ be a C^∞ diffeomorphism such that $-7 \mapsto -1-\epsilon$, $7 \mapsto 1+\epsilon$, $-3 \mapsto -1$, $3 \mapsto 1$, and on the intervals $[-3-\epsilon/2, -3+\epsilon/2]$, $[+3-\epsilon/2, +3+\epsilon/2]$ the derivative of R is equal to 1. The composite $\Gamma_0 \circ R$ will be denoted by Γ_1. If $l \geq 3$ then, applying 9.14 and 9.15 we obtain a C^∞ embedding $\Gamma : [-7,7] \to L$ with the following properties:

Properties of Γ: $\Gamma(-3) = A$; $\Gamma(3) = B$; $\operatorname{Im}\Gamma \cap N = \{A, B\}$; $\operatorname{Im}\Gamma \cap X = \emptyset$; the images of the restrictions $\Gamma|[-3-\epsilon/2,-3+\epsilon/2], \Gamma|[3-\epsilon/2,3+\epsilon/2]$ belong to $U(A)$, resp. to $U(B)$, and are straight line segments in the charts $\mathcal{L}_A, \mathcal{L}_B$.

Note that codim $\mathbf{X}_L$ in L is ≥ 3.

If $l = 2$ then $L \cong S^2$, and it is easy to construct such a curve directly. (In this case $X \cap L = \emptyset$).

Now we denote by $\phi_0 : \operatorname{Im}\gamma \to L$ the embedding, defined by $\phi_0(\gamma(t)) = \Gamma(t)$.

2) An embedding of $\operatorname{Im}\theta$ to N.

Similarly we construct a C^∞ embedding $\Theta : [-5,5] \to N$ with the following properties:
Properties of Θ: $\Theta(-1) = A$; $\Theta(1) = B$; $\operatorname{Im}\Theta \cap L = \{A, B\}$; $\operatorname{Im}\Theta \cap Y = \emptyset$; the images of the restrictions $\Theta|[-1-\epsilon/2,-1+\epsilon/2]$, $\Theta|[1-\epsilon/2,1+\epsilon/2]$ belong to $U(A)$, resp. $U(B)$ and are straight line segments in the charts $\mathcal{L}_A, \mathcal{L}_B$.

We denote by $\phi_1 : \operatorname{Im}\theta \to N$ the embedding, defined by $\phi_1(\theta(t)) = \Theta(t)$.

3) An embedding ϕ_ρ of $\gamma[\rho] \cup \theta[\rho]$ to V.

Consider now the vector field along $\Gamma|[-3-\epsilon/2,-3+\epsilon/2]$, resp. along $\Gamma|[3-\epsilon/2,3+\epsilon/2]$, which in the chart $\mathcal{L}_A$ is constant and equal to (Θ') (resp. in the chart $\mathcal{L}_B$ is constant and equal to $(-\Theta')$). Since the normal bundle to L is of dimension ≥ 2, there is a C^∞ vector field $\widehat{\eta}$ along Γ such that $\widehat{\eta}$ is orthogonal to L, $|\widehat{\eta}| = 1$, and $\widehat{\eta}$ extends the local vector fields above.

Similarly, there is a C^∞ vector field $\widehat{\xi}$ along Θ, such that $\widehat{\xi}$ is orthogonal to N, $|\widehat{\xi}| = 1$ and along $\Theta|[-1-\epsilon/2,-1+\epsilon/2]$, (resp. along $\Theta|[1-\epsilon/2,1+\epsilon/2]$) in the chart $\mathcal{L}_A$, resp. $\mathcal{L}_B$ the field $\widehat{\xi}$ equals Γ' (resp. $(-\Gamma')$).

For a $\rho > 0$ sufficiently small we extend the map ϕ_0 to a map of the tubular neighbourhood $\gamma[\rho]$ to V as follows: $\phi_0\big(\gamma(t) + \tau \cdot \eta(t)\big) = \exp_{\Gamma(t)}(\tau \cdot \widehat{\eta}(t))$. Similarly we extend the map ϕ_1 to a map of the tubular neighbourhood $\theta[\rho]$ to V as follows: $\phi_1\big(\gamma(t) + \tau \cdot \xi(t)\big) = \exp_{\Theta(t)}(\tau \cdot \widehat{\xi}(t))$. Note that since the metric over $U(A)$ and $U(B)$ is flat and $\widehat{\xi}, \widehat{\eta}$ are constant in the charts $\mathcal{L}_A, \mathcal{L}_B$, these two maps glue together to give a C^∞ map $\phi_\rho : \gamma[\rho] \cup \theta[\rho] \to V$, which equals $\Gamma \circ \gamma^{-1}$ over $\operatorname{Im}\gamma$ and $\Theta \circ \theta^{-1}$ over $\operatorname{Im}\theta$. It is easy to see that for $\rho > 0$ sufficiently small this map is a C^∞ embedding, satisfying the following properties:

wh1) $\phi_\rho(\alpha) = A, \phi_\rho(\beta) = B$.

wh2) $\phi_\rho^{-1}(N) = \operatorname{Im}\theta, \phi_\rho^{-1}(L) = \operatorname{Im}\gamma$.

wh3) $\forall t$ the vector $(\phi_\rho)_*(\xi(t))$ is not tangent to N, and $(\phi_\rho)_*(\eta(t))$ is not tangent to L.

wh4) $\operatorname{Im}\phi_\rho \cap (X \cup Y) = \emptyset$.

4) Whitney disc.

In order to extend ϕ_ρ to Δ_+, let $\mathcal{D} : \Delta_+ \setminus \mathring{\Delta}\langle 1/4\rangle \to \mathbf{R}^2$ be a C^∞ map such that: 1) $\mathcal{D}$ is a diffeomorphism on its image; 2) $\mathcal{D}(x) = x$ for every $x \in \gamma[\rho/2] \cup \theta[\rho/2]$; 3) $\operatorname{Im}\mathcal{D} \subset \gamma[\rho] \cup \theta[\rho]$. Denote $\phi_\rho \circ \mathcal{D}$ by ψ; this map is a C^∞ embedding of $\Delta_+ \setminus \mathring{\Delta}\langle 1/4\rangle$ to V. The simple connectedness of N and L together with the hypotheses of 9.1 imply that $\phi_\rho|\operatorname{Im}\gamma \cup \operatorname{Im}\theta$ is homotopic to 0. Denote $\psi|\partial\Delta\langle 1/4\rangle$ by ψ_0. It is a C^∞ embedding to $V \setminus (N \cup L \cup X \cup Y)$, which is homotopic to 0 in V.

Lemma 9.17. *$\psi_0 : \partial\Delta\langle 1/4\rangle \to V \setminus (N \cup L \cup X \cup Y)$ is homotopic to zero.*

Proof. We distinguish two cases.

1) codim N, codim $L \geq 3$. Consider a C^0 map $\Psi : \Delta\langle 1/4\rangle \to V$, extending ψ_0. By [11, p.216, 235], there is a C^∞ embedding $\Psi_1 : \Delta\langle 1/4\rangle \to V$, C^0 close to Ψ, and, therefore mapping $\partial\Delta\langle 1/4\rangle$ to $V \setminus (N \cup L \cup X \cup Y)$. The map $\Psi_1|\partial\Delta\langle 1/4\rangle$ is homotopic to zero in V. Now note that the pair $(\Delta\langle 1/4\rangle, \partial\Delta\langle 1/4\rangle)$ is diffeomorphic to $(D^2(0,r), \partial D^2(0,r))$, and therefore, by 9.15, applied to the family $\{N\} \cup \{L\} \cup \mathbf{X} \cup \mathbf{Y}$ of submanifolds we deduce our conclusion.

2) One of the codimensions equals 2. Assume, for example, that codim $N = 2$. By the hypotheses of 9.1 there is a C^0 map $\Psi : \Delta\langle 1/4\rangle \to V$, extending ψ, and such that $\operatorname{Im}\Psi \cap N = \emptyset$. Now the rest of the proof is the same as for 1) (we apply 9.15 to $\mathbf{X} \cup \mathbf{Y} \cup \{L\}$, which has codimension ≥ 3.) $\square$

Denote by $\widehat{\phi}$ the C^0 map $\Delta_+ \to V$, which equals ψ on $\Delta_+ \setminus \mathring{\Delta}\langle 1/4\rangle$ and maps $\Delta\langle 1/4\rangle$ to $V \setminus (N \cup L \cup X \cup Y)$. It has the following properties:

$\widehat{\phi}$1) $\widehat{\phi}|\gamma[\rho/2] \cup \theta[\rho/2] = \phi_\rho$.

$\widehat{\phi}$2) $\widehat{\phi}|\Delta_+ \setminus \mathring{\Delta}\langle 1/4\rangle$ is a C^∞ embedding.

$\widehat{\phi}$3) $\operatorname{Im}\widehat{\phi} \cap (X \cup Y) = \emptyset$.

$\widehat{\phi}$4) $\widehat{\phi}(\Delta\langle 1/2\rangle)$ does not intersect $N \cup L$.

Now we apply 9.14 and get a C^∞ embedding $\phi : \Delta_+ \hookrightarrow V$ such that $\phi|\gamma[\rho/2] \cup \theta[\rho/2] = \phi_\rho$. If ϕ is close enough to $\widehat{\phi}$, the properties $\widehat{\phi}$3) and $\widehat{\phi}$4) hold for ϕ also, therefore ϕ is a Whitney disc. $\square$

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