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AN ANALYTIC PROOF OF THE REAL PART OF NOVIKOV'S INEQUALITIES

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In this note we propose an interpretation of the Novikov numbers (see [1] and [2]) in terms of de Rham cohomology. We present the corresponding method of obtaining the torsion-free Novikov inequalities, based on Witten's approach to Morse theory.

Let ξ be a closed (but not, in general, exact) 1-form on a smooth compact manifold M. The form ξ is assumed to be Morse, i.e., in a neighborhood of any point $\xi = dh$, where h is a Morse function. Such a form gives a multiple-valued Morse function (see [1]). The zeros of ξ are isolated, and the notion of index is defined for each zero. The number of zeros of ξ of index p is denoted by $m_p(\xi)$.

We give a representation ρ_t of the fundamental group $\pi_1(M)$ into the group $\mathbf{C}^* = \mathrm{GL}(1, \mathbf{C})$ by setting $\rho_t(\gamma) = \exp(t\int_\gamma \xi)$, where t is an arbitrary complex number. The representation ρ_t defines a local system of groups $\mathbf{C}$ on the manifold M. We denote by $H^*(M, \rho_t)$ the cohomology of M with coefficients in this local system, and we let $\dim H^k(M, \rho_t) = \beta_k(\xi, t)$.

THEOREM 1. *For sufficiently large real numbers t,*

$$m_p(\xi) \geqslant \beta_p(\xi, t) \quad \text{for all } p.$$

THEOREM 2. *Assume that all the periods of the form ξ are commensurable. If* Re t *is sufficiently large, then $\beta_p(\xi, t) = b_p(\xi)$, where $b_p(\xi)$ is a Novikov number.*

PROOF OF THEOREM 1. We need another definition of the numbers $\beta_k(\xi, t)$. Namely, in the de Rham complex $\Omega^*(M)$ of M we define a new differential d_t by the formula $d_t\omega = d\omega + t\xi\wedge\omega$ where $t \in \mathbf{C}$. The complex $(\Omega^*(M), d_t)$ is elliptic, its cohomology $H_t^*(M)$ is finite dimensional and the cohomology is isomorphic to the space of harmonic forms of the corresponding Laplacian $\Delta_t = d_t^* d_t + d_t d_t^*$.

LEMMA 1. *$dim H_t^p(M) = \beta_p(\xi, t)$.*

PROOF. Let $p : \widetilde{M} \to M$ be the universal cover of M. The cohomology groups $H^p(M, \rho_t)$ are the cohomology groups of the complex $\Omega^*(M, \rho_t)$ consisting of the ρ_t-equivariant forms on $\widetilde{M}$ relative to the usual differential (the proof is analogous to the sheaf-theoretic proof of de Rham theorem). Let h be a function on $\widetilde{M}$ such that $dh = p^*\xi$. We give a mapping $F : \Omega^*(M) \to \Omega^*(M, \rho_t)$ by the formula $F(\omega) = \exp(th)\cdot p^*\omega$. It is easy to see that F is one-to-one and commutes with the differentials. This proves the lemma.

The rest of the proof follows the scheme presented in [3]. In that paper the case $\xi = df$ is investigated; the proofs carry over without change to the general case. Namely, let $\omega \in \Omega^p(M)$. Then $\Delta_t\omega = \Delta\omega + |t|^2|\xi|^2\omega + tG\omega$ where the operator G does not depend on t and is given by a tensor field of type (p, p). Fix a number $A > 0$. As $t \to \infty$ the eigen-p-forms of the Laplacian Δ_t with eigenvalues at most A are concentrated in a neighborhood of the zeros of ξ. In a neighborhood of these points the operator Δ_t

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can be written out explicitly in local coordinates and it turns out that to each zero of index p there corresponds exactly one eigen-p-form of Δ_t, with eigenvalue less than A. These forms constitute a subcomplex $A^*(M)$ of $(\Omega^*(M), d_t)$ whose cohomology is equal to $H^*_t(M)$; from this we find that

$$m_p(\xi) = \dim A^p(M) \geqslant \dim H^p_t(M) = \beta_p(\xi, t).$$

PROOF OF THEOREM 2. We may assume that the form ξ is integral, i.e. $\xi = f^*(d\phi)$ where f is a mapping from M to S^1.

We consider the homology $H_k(\widehat{M}, \mathbf{Q})$ of the correzponding cyclic covering with coefficients in $\mathbf{Q}$. This is a finitely generated module over the principal ideal ring $\Lambda = \mathbf{Q}[z, z^{-1}]$; it is represented in the form

$$H_k(\widehat{M}, \mathbf{Q}) = \sum_i F_i \oplus \sum_j F_k / P_j F_j \tag{1}$$

where the F_i and F_j are free Λ-modules of rank 1, and the P_j are certain polynomials.

LEMMA 2. *If the number e^t is transcendental, then $\beta_k(\xi, t)$ is equal to the number of free direct summands F_i in the module $H_k(\widehat{M}, \mathbf{Q})$ and is equal to the Novikov number $b_k(\xi)$. If e^t is algebraic, then $\beta_k(\xi, t) = b_k(\xi) + \varkappa_k(\xi, t) + \varkappa_{k-1}(\xi, t)$ where $\varkappa_k(\xi, t)$ is the number of polynomials P_j in the representation (1) that vanish at e^t.*

PROOF. In this case the representation ρ_t is induced from a representation $\mathbf{Z} \to \mathbf{C}^*$ so that in order to calculate $H^*(M, \rho_t)$ it is sufficient to consider ρ_t-equivariant cochains (or forms) on $\widehat{M}$. We have an exact sequence

$$0 \to \mathrm{Ext}_\Lambda(H_{k-1}(\widehat{M}), \mathbf{C}) \to H^k(M, \rho_t) \to \mathrm{Hom}_\Lambda(H_k(\widehat{M}), \mathbf{C}) \to 0$$

where $H_{k-1}(\widehat{M})$ and $\mathbf{C}$ are considered as modules over $\Lambda = \mathbf{Q}[z, z^{-1}]$. If e^t is transcendental, then $\mathbf{C}$ is a flat Λ-module and it is clear that β_k is equal to the number of free generators and to the Novikov number $b_k(\xi)$ (see [2]). If e^t is algebraic, then the dimensions of the vector spaces $\mathrm{Hom}_\Lambda(H_k(\widehat{M}), \mathbf{C})$ and $Ext_\Lambda(H_{k-1}(\widehat{M}), \mathbf{C})$ are increased by $\varkappa_k(\xi, t)$ and $\varkappa_{k-1}(\xi, t)$ respectively. This proves the lemma.

Thus, the equality $\beta_k(\xi, t) = b_k(\xi)$ holds for all $t \in \mathbf{C}$ except for numbers of the form $t = 2\pi n i + \alpha$ where $n \in \mathbf{Z}$ and α runs through some finite set of complex numbers.

REMARKS. 1) For the case of forms ξ with incommensurable periods the numbers $\beta_k(\xi, t)$ provide an alternative variant of the definition of the Novikov numbers.

2) In a paper to appear, S.P.Novikov has computed the numbers $\beta_p(\xi, t)$ for the case of numbers t that are in general position in terms of ordinary cohomology of the manifold M and the actions on it of the Massey operations associated with the form ξ.

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