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ON THE SHARPNESS OF NOVIKOV TYPE INEQUALITIES FOR MANIFOLDS WITH FREE ABELIAN FUNDAMENTAL GROUP

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ABSTRACT. For manifolds M^n, $n \geq 6$, with free abelian fundamental group and four-connected universal covering, the author proves the sharpness of Novikov's inequalities for rational cohomology classes $\xi \in H^1(M, \mathbf{Q})$ belonging to an open everywhere dense set $U \subset H^1(M, \mathbf{R})$.

Figures: 1. Bibliography: 20 titles.

Introduction

In [1] and [2] S. P. Novikov constructed an analogue of Morse theory for "multivalued Morse functions," i.e. for closed but (generally) nonexact Morse 1-forms on smooth manifolds. Recall that a closed 1-form ω on a smooth manifold M^n is called a *Morse form* if all the zeros of ω are nondegenerate, or, equivalently, if locally $\omega = dh$, where h is a Morse function. In this case the index $\operatorname{ind}_c \omega$ of each zero c of the form ω is defined. Denote by $m_p(\omega)$ the number of zeros of ω of index p.

One of the basic problems of this theory in the finite-dimensional case is the following: for a given cohomology class $\xi \in H^1(M, \mathbf{R})$ find the numbers $c_p(M, \xi)$, where $0 \leq p \leq n$, providing the lower estimates (sharp if possible) for the Morse numbers $m_p(\omega)$ of any form ω belonging to the class ξ:

$$m_p(\omega) \geq c_p(M, [\omega]), \qquad 0 \leq p \leq n. \tag{0.1}$$

Recall that the estimates (0.1) are said to be *sharp* for a manifold M and a class ξ if there exists a Morse 1-form ω, belonging to ξ, such that the inequalities (0.1) become equalities for all p.

For $\xi = 0$ the estimates (0.1) are provided by Morse inequalities; in this case $c_p(M^n, 0) = b_p(M) + q_p(M) + q_{p-1}(M)$, where $b_p(M)$ denotes the rank of $H_p(M)$ and $q_p(M)$ the torsion number of $H_p(M)$ (i.e. the minimal number of generators of the torsion subgroup $\operatorname{Tors} H_p(M)$).

For $\pi_1 M^n = 0$, $n \geq 6$, these estimates are sharp (Smale's theorem [3]).

For any cohomology class $\xi \in H^1(M, \mathbf{R})$ (and also for any form ω such that $[\omega] = \xi$) we define the *irrationality degree* of ξ to be the maximal number of $\mathbf{Q}$-linearly independent periods of (or, equivalently, the rank of $\operatorname{Im}(\xi : \pi_1 M \to \mathbf{R})$). The forms ω of irrationality degree 1 will be called *rational*.

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For a class $\xi \neq 0$ having irrationality degree 1 (these are exactly the multiples of the integer classes) the estimates (0.1) were suggested by Novikov [1], [2] (see below). The sharpness of these estimates for the case $\pi_1 M^n = \mathbf{Z}$, $n \geq 6$, has been proved by Farber [4].

For $\pi_1 M = \mathbf{Z}$ any Morse 1-form is up to a positive constant the differential of the Morse map $M^n \to S^1$; thus the sharpness problem is equivalent to the problem of constructing a Morse map $f: M \to S^1$ with a minimal number of critical points of all indices. A necessary and sufficient condition for the existence of a map $f: M \to S^1$ without critical points (i.e. a fibration) was supplied by Browder and Levine [5] (it is easy to check the equivalence of this condition to the condition arising from Novikov inequalities).

The main purpose of the present paper is to obtain the estimates of the type (0.1) for forms of arbitrary irrationality degree and to prove their sharpness for generic cohomology classes $[\omega]$ and manifolds M^n satisfying $\pi_1 M^n = \mathbf{Z}^m$, $n \geq 6$ (and some additional homotopical restrictions; for precise formulation see Theorem 0.1 below). These results were partly announced in [6].

We begin with definitions (cf. [1] and [2]).

Let G be a group, and $\xi: G \to \mathbf{R}$ a homomorphism. Denote by Λ the group ring $\mathbf{Z}[G]$. Consider the set of all linear combinations (infinite in general) $\lambda = \sum_{g\in G} n_g g$ such that the intersection of $\operatorname{supp}\lambda$ (where $\operatorname{supp}\lambda = \{g \mid n_g \neq 0\} \subset G$) with any set $\{g \mid \xi(g) \geq c\}$ is finite. It is easy to see that the resulting abelian group is a ring. We denote it by Λ_ξ^-. (Note that Λ_ξ^- is the completion of Λ with respect to the system of subrings, but not ideals.)

For $G = \mathbf{Z}^l$ the group ring $\Lambda = \mathbf{Z}[G] = \mathbf{Z}[t_1^{\pm 1}, \dots, t_l^{\pm 1}]$ is the ring of Laurent polynomials with integer coefficients; the completion Λ_ξ^- is the ring of all the power series $\lambda = \sum_I \lambda_I t^I$ (where $I = (i_1, \dots, i_l) \in \mathbf{Z}^l$, $\lambda_I \in \mathbf{Z}$) such that for any c the set $\operatorname{supp}\lambda$ contains only a finite number of indices I satisfying $\xi(I) \geq c$. The ring $\mathbf{Z}[\mathbf{Z}]$ will henceforth be denoted by L. For $l = 1$ and $\xi \neq 0$ the ring Λ_ξ^- (denoted henceforth by $\widehat{L}$) is the ring of all integer Laurent power series with finite negative part (we suppose that $\xi(1) < 0$).

Next we recall some results from [1] and [2].

Let ω be a Morse 1-form on a manifold M^n (having an arbitrary fundamental group); denote by $\xi \in H^1(M, \mathbf{R})$ its cohomology class. Consider the minimal covering $p: \overline{M}_\xi^n \to M$ for which the pullback of ω is exact: $p^*\omega = df$. This covering corresponds to the subgroup $\operatorname{Ker}(\xi: \pi_1 M \to \mathbf{R}) \subset \pi_1 M$ and is regular with the structure group $\mathbf{Z}^l$, where l is the irrationality degree of ω. The critical points of $f: \overline{M}_\xi^n \to \mathbf{R}$ and the paths of steepest descent of f give rise to the Novikov complex $C_*(\overline{M}_\xi^n, \omega)$. The latter is an analogue of the Morse complex of a single-valued function f on a compact manifold M. The Novikov complex is a free finitely generated complex over the ring Λ_ξ^-, where $\Lambda = \mathbf{Z}[\mathbf{Z}^l]$ (note that the homomorphism ξ factors through $\mathbf{Z}^l$ by definition). The number of free Λ_ξ^--generators of $C_p(\overline{M}_\xi^n, \omega)$ equals $m_p(\omega)$. The homology $H_*(C_*(\overline{M}_\xi^n, \omega))$ is isomorphic to $H_*(C_*(\overline{M}_\xi^n) \otimes_\Lambda \Lambda_\xi^-)$.

Consider for instance a cohomology class ξ of irrationality degree 1. The covering $M_\xi^n \to M^n$ is infinite cyclic. The ring Λ_ξ^- is the ring $\widehat{L} = \mathbf{Z}[[t]][t^{-1}]$, which is known to be a principal ideal domain. For any finitely generated module M over a principal ideal domain R the rank $b(M)$ and the torsion number $q(M)$ are defined; for any free finitely generated R-complex C_* the number of free generators $\mu(C_p)$ is not less than

$$b(H_p(C_*)) + q(H_p(C_*)) + q(H_{p-1}(C_*)).$$

Hence, for rational forms

$$m_p(\omega) \geq b(H_p(\overline{M}_\xi) \otimes_L \widehat{L}) + q(H_p(\overline{M}_\xi) \otimes_L \widehat{L}) + q(H_{p-1}(\overline{M}_\xi) \otimes_L \widehat{L}) \qquad (0.2)$$

(see [1] and [2]; another proof is given in [4]).

Jean-Claude Sikorav has proved (see §1, below) that for any k the completion Λ_ξ^- of the ring $\Lambda = \mathbf{Z}[\mathbf{Z}^k]$ with respect to a homomorphism $\xi\colon \mathbf{Z}^k \to \mathbf{R}$ of maximal irrationality degree k is a principal ideal domain. Therefore the same argument as above enables us to obtain the analogues of (0.2) for forms of arbitrary irrationality degree.

We will need still another variant of these inequalities, dealing with forms of arbitrary irrationality degree, but arising from a maximal free abelian covering. To produce it we need one more algebraic lemma.

Namely, let $m = \operatorname{rk} H_1(M)$ and consider a class $\xi \in H^1(M, \mathbf{R})$ of maximal irrationality degree m. Denote by $b_p(M, \xi)$ the rank and by $q_p(M, \xi)$ the torsion number of the module $H_p(C_*(\overline{M}) \otimes_\Lambda \Lambda_\xi^-)$, where $\overline{M} \to M$ is a $\mathbf{Z}^m$-covering corresponding to the homomorphism $\pi_1 M \to H_1 M/\operatorname{Tors} H_1 M$, $\Lambda = \mathbf{Z}[\mathbf{Z}^m]$. For $[\omega] = \xi$ we have

$$m_p(\omega) \geq b_p(M, \xi) + q_p(M, \xi) + q_{p-1}(M, \xi). \qquad (0.3)$$

One easily proves that $b_p(M, \xi)$ does not depend on ξ. Sikorav has also proven that $q_p(M, \xi)$ does not depend on ξ in any connected component of the complement in $H^1(M, \mathbf{R}) = \mathbf{R}^m$ to the finite union $\bigcup_i \Gamma_i$ of hyperplanes Γ_i, each of which is determined by a linear equation with integer coefficients (see §1, below). Note, by the way, that the definition implies

$$q_p(M, [\omega]) = q_p(M, c[\omega]), \qquad c > 0.$$

Now let ξ be any element of $H^1(M, \mathbf{R}) \setminus \bigcup_i \Gamma_i$. We set by definition $q_p(M, \xi) = q_p(M, \xi')$, where ξ' is an arbitrary maximally irrational class, sufficiently close to ξ. Suppose that ω is a Morse 1-form such that $[\omega] \in H^1(M, \mathbf{R}) \setminus \bigcup_i \Gamma_i$. Approximating ω with the maximally irrational forms, we obtain the inequalities (0.3) also for an arbitrary cohomology class $[\omega] \in H^1(M, \mathbf{R}) \setminus \bigcup_i \Gamma_i$.

Our notation differs here from that of [1] and [2]. In those papers $b_p(M, \xi)$ and $q_p(M, \xi)$ stand for numbers which we have denoted by $b(H_p(\overline{M}_\xi) \otimes_L \widehat{L})$ and $q(H_p(\overline{M}_\xi) \otimes_L \widehat{L})$. Still these notations correspond rather well to one another. Namely, Lemma 2.6 from §2 asserts that for rational cohomology classes ξ, belonging to some dense conical open set U, ([1]) we have

$$b_p(M, \xi) = b(H_p(\overline{M}_\xi) \otimes_L \widehat{L}), \qquad q_p(M, \xi) = q(H_p(\overline{M}_\xi) \otimes_L \widehat{L}).$$

Now we can state the main theorem of this paper.

([1]) A subset $U \subset \mathbf{R}^m$ is called *conical* if $x \in U \Rightarrow tx \in U$ for every $t > 0$.

THEOREM 0.1. *Let* M^n, $n \geq 6$, *be a smooth compact connected manifold without boundary, with* $\pi_1 M^n = \mathbf{Z}^m$. *Suppose that for some* r, $2 \leq r \leq n-4$, *the homology* $H_p(\widetilde{M}^n)$ *of the universal cover vanishes for* $r-1 \leq p \leq r+2$.

Then there exists an open dense conical subset $U \subset H^1(M, \mathbf{R}) = \mathbf{R}^m$ *such that any* $\gamma \in U$ *can be realized by a Morse* 1*-form* ω *with* $[\omega] = \gamma$, *which has the minimal possible number of zeros of any index* p *in the class* γ, *this number being equal to the right-hand side of* (0.3).

Applying Lemma 2.6 from §2, we immediately deduce from this the sharpness of the classical Novikov inequalities (0.1).

COROLLARY 0.2. *Under the assumptions of Theorem* 0.1 *there exists a conical dense open set* $V \subset H^1(M, \mathbf{R}) = \mathbf{R}^m$ *such that for any rational cohomology class* $\gamma \in V$ *the Novikov inequalities* (0.1) *are sharp* (*i.e. any rational* $\gamma \in V$ *can be realized by a Morse form* ω *with* $m_p(\omega)$ *equal to the right-hand side of* (0.1)).

COROLLARY 0.3. *For a smooth manifold* M^n *with* $\pi_1 M^n = \mathbf{Z}^m$, $n \geq 6$, *and* $\widetilde{M}^n$ *four-connected, there exists a conical dense open set* $U \subset H^1(M, \mathbf{R})$ *such that any integer class* $\xi \in U$ *can be realized by a Morse map* $f \colon M \to S^1$ *which has the minimal possible number of zeros of any index* p *in the class* ξ.

Denote by $m_p([\omega])$ the right-hand side of (0.3). In the proof of Theorem 0.1 we actually use not the vanishing of the universal cover homology, but the weaker condition $m_r(\gamma) = m_{r+1}(\gamma) = m_{r+2}(\gamma) = 0$. Hence we get

COROLLARY 0.4. *For a smooth manifold* M^n *with* $\pi_1 M^n = \mathbf{Z}^m$, $n \geq 6$, *the set of cohomology classes* $\gamma \in H^1(M, \mathbf{R})$ *realizable* (*up to a multiplicative constant*) *by a fibration* $M^n \to S^1$ *is contained in the open cone* $V \subset H^1(M, \mathbf{R})$, *which is determined by the condition* $m_*(\gamma) = 0$. *This cone contains an open dense subset* V_0, *any rational class of which is realized by a fibration.*

This corollary gives a partial generalization (for dimensions ≥ 6) of Thurston's result [7] concerning the fibrations of 3-manifolds over a circle.

It is natural to ask if one can weaken the assumptions of Theorem 0.1, keeping the conclusion. The Morse 1-form ω is called *minimal* if it has the minimal possible number of zeros of all indices in its cohomology class $[\omega]$. If $\pi_1 M^n = \mathbf{Z}^m$, $n \geq 6$, and $\gamma \in H^1(M, \mathbf{R})$ is an *arbitrary* cohomology class, the problem of existence of a minimal form in γ seems to be rather difficult. Indeed, if $\gamma = 0$ this problem is just the problem of existence of minimal Morse functions on manifolds with free abelian fundamental groups. The latter problem is not yet completely solved; a detailed treatment can be found in [8] and [9]. Taking Theorem 0.1 into account, we can describe the situation as follows. We treat Morse 1-forms as multivalued Morse functions; the monodromy of each function of this type is given by m real numbers. The condition of zero monodromy (corresponding to usual Morse functions) proves to be too rigid for the known methods to deform the function into a minimal one. If the class γ is in general position (actually it is sufficient that γ is close enough to some class γ_0 having $\mathbf{Q}$-linearly independent periods), the Morse form belonging to γ can be deformed into a minimal one.

It is now natural to state the conjecture that the set of cohomology classes $\gamma \in H^1(M, \mathbf{R}) = \mathbf{R}^m$ realizable by minimal Morse forms contains the complement to a finite union of hyperplanes determined by linear equations with integer coefficients. These hyperplanes correspond to "uncomfortable" monodromy conditions. The set of these hyperplanes must contain the hyperplanes Γ_i mentioned above, and perhaps others.

The restriction $m_r(\gamma) = m_{r+1}(\gamma) = m_{r+2}(\gamma) = 0$ is imposed for technical reasons. The author does not know if it is removable.

Now we sketch the main idea of the proof of Theorem 0.1 and the contents of the paper.

§§1 and 2 include the proof (not published before) of Jean-Claude Sikorav's theorems on the Euclidean property of Novikov ring and on the numbers $q_*(M, \xi)$. We also prove here that in many cases one can replace the Novikov ring Λ_ξ^- by a suitable localization $\Lambda_{(\xi)}$ of the ring Λ (this is essential for the proof of Theorem 0.1), and that for a generic rational class ξ the number $b_p(M, \xi)$ coincides with the rank of the module $H_p(\overline{M}_\xi) \otimes_L \widehat{L}$, while $q_p(M, \xi)$ coincides with its torsion number.

In §3 we prove the Poincaré duality formula for Novikov homology.

§§4 and 5 contain some auxiliary material. We recall results on Morse functions on cobordisms due to V. V. Sharko (§4), and prove some algebraic lemmas (§5).

In §6 we produce two other proofs of the inequalities (0.3). They use only Morse theory for the functions on the compact manifolds with boundary. These proofs are formally independent of the properties of the Novikov complex, which we discussed above to clarify the roots of this work. Thus this paper is self-contained. The first proof makes use of the algebraic lemmas of §2 and reduces the problem to the case of cyclic covering; then we refer to [4]. The second proof is independent of [4]. In §6 we also state an algebraic conjecture; if it holds, the second proof provides general Morse type estimates for rational forms, similar to [8].

§§7, 8, and 9 are devoted directly to the proof of Theorem 0.1.

It is sufficient to prove this theorem for rational cohomology classes in general position. We shall prove it for any rational class γ which is sufficiently close to the maximally irrational class γ'. The point is that in this case the modules $H_*(\widehat{M}^n, \Lambda_{(\gamma)})$ have resolutions of length two (although the ring $\Lambda_{(\gamma)}$ is not a principal ideal domain). This enables us to apply the scheme of proof of sharpness due to Browder, Levine, Farrell, and Farber (§§7 and 8). Instead of Smale's theorem on the minimal functions on simply-connected manifolds we use Sharko's corresponding result [8].

The Browder-Levine-Farrell-Farber scheme does not go through directly. We use non-simply-connected surgery, and while constructing the Morse form we come across an obstruction of Farrell type [10]. (To define this obstruction correctly we need the vanishing of $H_*(\widetilde{M}^n)$ in four successive dimensions.) The obstruction lies in a zero group $C(\mathbf{Z}[\mathbf{Z}^{m-1}])$ (see [10]), but we need to realize this vanishing geometrically, which requires some extra arguments (§9).

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§1. The Novikov ring is Euclidean

Denote $\mathbf{Z}[\mathbf{Z}^l] = \mathbf{Z}[t_1^{\pm 1}, \dots, t_l^{\pm 1}]$ by Λ, and let $\xi: \mathbf{Z}^l \to \mathbf{R}$ be a homomorphism. Recall from the Introduction that the Novikov ring Λ_ξ^- consists of power series

$\lambda = \sum a_i t^I$ (where $I = (i_1, \dots, i_l) \in \mathbf{Z}^l$) such that for every c the domain $\xi \geq c$ contains only a finite number of indices I belonging to $\operatorname{supp}\lambda$ (recall that $\operatorname{supp}\lambda = \{I \mid a_I \neq 0\}$).

THEOREM 1.1 (J.-C. Sikorav). *If $\xi: \mathbf{Z}^l \to \mathbf{R}$ is a monomorphism, then Λ_ξ^- is Euclidean.*

PROOF. First of all, note that injectivity of ξ implies that for any $c \in \mathbf{R}$ the set $\{\xi(x) = c\}$ contains at most one element. Therefore any element λ of the ring Λ_ξ^- contains a principal term $a_{I_0} t^{I_0}$, which is uniquely determined. For $\lambda \in \Lambda_\xi^-$ we define the height $h(\lambda) \in \mathbf{R}$ and the norm $\|\lambda\| \in \mathbf{Z}$, setting $h(\lambda) = \xi(I_0)$ and $\|\lambda\| = |a_{I_0}|$. We prove that Λ_ξ^- is Euclidean with respect to this norm.

Suppose that $A, B \in \Lambda_\xi^-$. We are to divide A by B with a remainder. Without losing generality we may assume that the leading terms of these power series are $a_0 \cdot \mathbf{1}$ and $b_0 \cdot \mathbf{1}$ ($\mathbf{1}$ stands for the unit of the group $\mathbf{Z}^l$).

Divide a_0 by b_0 with a remainder: $a_0 = m_0 b_0 + q_0$.

If $q_0 \neq 0$, then $A = m_0 B + Q$, where $\|Q\| = |a_0 - m_0 b_0| = |q_0| < |b_0|$, and the division is over. If not, apply the same procedure to the power series Q. Going on in the same fashion, we construct the sequence of polynomials M_i and the sequence of power series Q_i such that $A = M_i B + Q_i$ and $M_{i+1} = M_i + \mu_{i+1}$, where μ_{i+1} is a monomial located lower (with respect to ξ), than any monomial of M_i; $h(\mu_{i+1}) = h(Q_i)$. If at some step we obtain $\|Q_i\| < |b_0| = \|B\|$, then our sequence stops and the division is over. If this never happens, consider the power series $M = \sum_0^\infty \mu_i$. I claim that $M \in \Lambda_\xi^-$.

Indeed, let $\varepsilon = |h(B - b_0 \cdot \mathbf{1})|$. Suppose we have already proved that only a finite number of monomials μ_i are located above the level $(\xi = -N\varepsilon)$, and let μ_{n+1} be the first monomial lying below this level. Denote by K the number of monomials of the power series Q_n lying in the stratum $-(N+1)\varepsilon < \xi \leq -N\varepsilon$. It is clear that μ_{n+1+K} lies below the level $\xi = -(N+1)\varepsilon$. Thus $M \in \Lambda_\xi^-$, and it is clear that $A = MB$.

REMARK 1.2. Consider $\mathbf{Z}^l$ as a lattice in $\mathbf{R}^l$ and extend ξ to a linear functional on $\mathbf{R}^l$. Consider the cone $C \subset \mathbf{R}^l$ formed by the intersection of a finite number of half-spaces and lying in the domain $(\xi \leq 0)$. One deduces easily from the proof of Theorem 1.1 that if all the monomials of a and b are contained in C, then the monomials of Q and M also are contained in C.

DEFINITION 1.3. Let $\gamma: \mathbf{Z}^l \to \mathbf{R}$ be any homomorphism (not necessary injective). Define the multiplicative subset $S_\gamma \subset \Lambda$ as follows: $S_\gamma = \{1 + P\}$, where all the monomials of P belong to the domain $\gamma < 0$. (If γ is injective, S_ν is just the set of polynomials with leading term equal to 1.) Set $\Lambda_{(\gamma)} = S^{-1}\Lambda$.

THEOREM 1.4. *Let $\xi: \mathbf{Z}^l \to \mathbf{R}$ be a monomorphism. Then $\Lambda_{(\xi)}$ is Euclidean.*

PROOF. We introduce some notation. Suppose that $e = \{e_1, \dots, e_l\}$ is a collection of independent integer vectors in $\mathbf{R}^l$ such that $\xi(e_i) < 0$. Denote by $M(e)$ the set of all linear combinations of the e_i with nonnegative integer coefficients, by $C(e)$ the cone in $\mathbf{R}^l$ generated by the vectors e_i, by $\widetilde{M}(e)$ the intersection of $\mathbf{Z}^l$ and $C(e)$, and by $\mathbf{Z}[M(e)]$ and $\mathbf{Z}[\widetilde{M}(e)]$ the subrings of Λ generated by the monomials whose exponents belong to $M(e)$ and $\widetilde{M}(e)$, respectively. The ring $\mathbf{Z}[M(e)]$ is isomorphic to the polynomial ring $\mathbf{Z}[u_1, \dots, u_l]$. The ring $\mathbf{Z}[\widetilde{M}(e)]$ is finitely generated as a module over its subring $\mathbf{Z}[M(e)]$; therefore it is Noetherian.

Consider the ideal $I \subset \mathbf{Z}[\widetilde{M}(e)]$ consisting of all Laurent polynomials having coefficient zero at $\mathbf{1}$. The ring of all power series in monomials belonging to $\widetilde{M}(e)$ coincides with the completion of $\mathbf{Z}[\widetilde{M}(e)]^\wedge$ with respect to I (and is contained in Λ_ξ^-). Consider the localization

$$S(e)^{-1}\mathbf{Z}[\widetilde{M}(e)], \quad \text{where } S(e) = \mathbf{1} + I \subset \mathbf{Z}[\widetilde{M}(e)].$$

The ring $\mathbf{Z}[\widetilde{M}(e)]^\wedge$ is a faithfully flat module over the ring $S(e)^{-1}\mathbf{Z}[\widetilde{M}(e)]$ (since the latter is a Zariski ring; see [11], Chapter III, §3). This implies that the equation $ax = b$, where $a, b \in S(e)^{-1}\mathbf{Z}[\widetilde{M}(e)]$, has a solution in $\mathbf{Z}[\widetilde{M}(e)]^\wedge$ if and only if it has a solution in $S(e)^{-1}\mathbf{Z}[\widetilde{M}(e)]$.

Now we can prove Theorem 1.4. The ring $\Lambda_{(\xi)}$ is contained in Λ_ξ^- and inherits from it the norm $\| \ \|$.

We show that $\Lambda_{(\xi)}$ is Euclidean with respect to this norm. It suffices to divide a by b, where $a, b \in \Lambda$ and $h(a) = h(b) = 0$. Apply now the division procedure described above. If it finishes after a finite number of steps, we have $a = mb + q$, where $m, q \in \Lambda$, and the division is over. If not, we have $a = bx$, where $x \in \Lambda_\xi^-$. Choose any collection $e = \{e_1, \dots, e_l\}$ of vectors for which $\operatorname{supp} a, \operatorname{supp} b \subset C(e)$. Then $a, b \in \mathbf{Z}[\widetilde{M}(e)]$ and Remark 1.2 implies that $x \in \mathbf{Z}[\widetilde{M}(e)]^\wedge$; hence

$$x \in S(e)^{-1}\mathbf{Z}[\widetilde{M}(e)] \subset \Lambda_{(\xi)}.$$

Q.E.D.

REMARK 1.5. The ring Λ_ξ^- is a faithfully flat module over its subring $\Lambda_{(\xi)}$, since they both are principal ideal domains and $a \in \Lambda_{(\xi)}$ is invertible in $\Lambda_{(\xi)}$ if and only if it is invertible in Λ_ξ^- (see [11], §1).

Thus Λ_ξ^- and $\Lambda_{(\xi)}$ have the same homological properties. The ring $\Lambda_{(\xi)}$ has some technical advantages over the ring Λ_ξ^-; the replacement of Λ_ξ^- by $\Lambda_{(\xi)}$ is used essentially in the proof of the sharpness theorem.

REMARK 1.6. For $l = 1$ the faithful flatness of $\Lambda_\xi^- = \mathbf{Z}[[t]][t^{-1}]$ over $\Lambda_{(\xi)} = S_\xi^{-1}\mathbf{Z}[t, t^{-1}]$ is well known (see [11], Chapter III). This property implies that the equation $Px = Q$, where $P, Q \in \mathbf{Z}[t, t^{-1}]$, is solvable in $\mathbf{Z}[[t]][t^{-1}]$ if and only if it is solvable in $S^{-1}\mathbf{Z}[t, t^{-1}]$ (where S is the multiplicative subset $\{1 + tP(t)\}$). This fact was known already to Hurwitz (see [12], problem 156).

§2. The numbers $b_*(\xi)$ and $q_*(\xi)$

Suppose that C_* is a free finitely generated complex over $\Lambda = \mathbf{Z}[\mathbf{Z}^l]$ and $\xi \colon \mathbf{Z}^l \to \mathbf{R}$ is a monomorphism. Consider the complex $S_\xi^{-1}C_* = C_* \otimes_\Lambda \Lambda_{(\xi)}$. The homology modules

$$H_*(C_* \otimes_\Lambda \Lambda_{(\xi)}) = H_*(C_*) \otimes_\Lambda \Lambda_{(\xi)}$$

are finitely generated over the principal ideal domain $\Lambda_{(\xi)}$; therefore the rank $b_p(C_*, \xi)$ and the torsion number $q_p(C_*, \xi)$ of the module $H_p(C_*) \otimes_\Lambda \Lambda_{(\xi)}$ are defined (the torsion number of a module is by definition the minimal number of generators in the submodule of torsion elements). For a manifold M^n and a maximally irrational $\xi \in H^1(M, \mathbf{R})$ we set $b_p(M, \xi) = b_p(C_*(\overline{M}), \xi)$ and $q_p(M, \xi) = q_p(C_*(\overline{M}), \xi)$, where $\overline{M} \to M$ is the maximal free abelian cover. Note that in the Introduction we defined these numbers using the modules $H_*(C_*(\overline{M}) \otimes_\Lambda \Lambda_\xi^-)$; these two definitions are the same. Indeed, the faithful flatness of Λ_ξ^- over $\Lambda_{(\xi)}$ implies

$$\begin{aligned} H_*(C_* \otimes_\Lambda \Lambda_\xi^-) &= H_*(C_* \otimes_\Lambda \Lambda_{(\xi)} \otimes_{\Lambda_{(\xi)}} \Lambda_\xi^-) \\ &= H_*(C_* \otimes_\Lambda \Lambda_{(\xi)}) \otimes_{\Lambda_{(\xi)}} \Lambda_\xi^- = (H_*(C_*) \otimes_\Lambda \Lambda_{(\xi)}) \otimes_{\Lambda_{(\xi)}} \Lambda_\xi^- . \end{aligned}$$

Next we study the behavior of $b_p(C_*, \xi)$ and $q_p(C_*, \xi)$ when C_* is fixed and ξ varies. For this purpose we need Lemma 2.1. Denote by M the set of all monomorphisms $\mathbf{Z}^l \to \mathbf{R}$, $M \subset \mathbf{R}^l$.

LEMMA 2.1. *Let $a_1, \dots, a_n \in \Lambda$. Then the set of ξ for which the greatest common divisor of elements $a_1, \dots, a_n$ (abbreviated* g.c.d.$(a_1, \dots, a_n)$) *in $\Lambda_{(\xi)}$ is equal to 1 is the intersection of M and several components (or possibly none at all) of the complement $U = \mathbf{R}^l \setminus \bigcup_i \Gamma_i$ in $\mathbf{R}^l$ to the finite union of some integer hyperplanes (i.e. hyperplanes determined by linear equations with integer coefficients) $\Gamma_i \subset \mathbf{R}^l$.*

PROOF. Set $A = \text{g.c.d.}(a_1, \dots, a_n)$ in the unique factorization domain Λ. Then the set sought consists of those ξ's for which the leading coefficient of the polynomial A with respect to ξ is equal to 1. Denote by $\langle A \rangle$ the convex hull in $\mathbf{R}^l$ of the subset $\operatorname{supp} A \subset \mathbf{Z}^l$. For an edge γ of the polyhedron $\langle A \rangle$ denote by Γ_γ the hyperplane in $\operatorname{Hom}(\mathbf{Z}^l, \mathbf{R}) = \mathbf{R}$ consisting of all the homomorphisms ξ that vanish on γ. The lemma is now obvious: the set U is $\mathbf{R}^l \setminus \bigcup_i \Gamma_i$.

THEOREM 2.2 (J.-C. Sikorav). 1. *The number $b_p(C_*, \xi)$ does not depend on ξ; it is equal to the rank of the module $H_p(C_*) \otimes_\Lambda \{\Lambda\}$ over the fraction field $\{\Lambda\}$ of Λ.*

2. *There exists a finite collection of integer hyperplanes $\Gamma_i \subset \operatorname{Hom}(\mathbf{Z}^l, \mathbf{R}) = \mathbf{R}$ such that $q_p(C_*, \xi)$ does not depend on ξ in any connected component of $\mathbf{R}^l \setminus \bigcup_i \Gamma_i$.*

PROOF. 1. The module

$$H_p(C_* \otimes_\Lambda \Lambda_{(\xi)}) = H_p(C_*) \otimes_\Lambda \Lambda_{(\xi)}$$

can be presented as a sum of a free module of rank $b_p(C_*, \xi)$ and a torsion module. When we pass to $\{\Lambda\}$, which means additional localization, the torsion module disappears and the free module of rank $b_p(C_*, \xi)$ survives.

2. Recall that the pth torsion number q_p for a complex

$$C_{p-1} \overset{\partial_{p-1}}{\leftarrow} C_p \overset{\partial_p}{\leftarrow} C_{p+1}$$

over a principal ideal domain can be calculated as follows. Consider the matrix D of the homomorphism ∂_p. Denote by d_r the g.c.d. of the r-minors of D, and by δ the greatest r for which $d_r = 1$. Then $q_p = \operatorname{rk} D - \delta$.

Now our assertion is easily deduced from Lemma 2.1.

REMARK 2.3. One can prove (similarly to [13] and [14]) that the number $b_p(C_*, \xi)$ is equal to the dimension of the pth homology of C_* with coefficients in a 1-dimensional local system, determined by a generic representation $\rho: t_i \to \rho(t_i) \in \mathbf{C}$. The numbers $b_p(C_*, \xi)$ can be computed in terms of usual homology with real coefficients and Massey operations; see [13].

Now let ξ be a homomorphism $\mathbf{Z}^l \to \mathbf{R}$ which is not contained in $\bigcup_i \Gamma_i$. Define the numbers $b_p(C_*, \xi) = b_p(C_*)$ and $q_p(C_*, \xi)$ by setting $b_p(C_*, \xi) = b_p(C_*, \xi')$ and $q_p(C_*, \xi) = q_p(C_*, \xi')$, where ξ' is any maximally irrational homomorphism sufficiently close to ξ.

The numbers $b_p(M)$ and $q_p(M, \xi)$ (2) are defined and calculated in terms of the maximal free abelian covering $\overline{M} \to M$. It appears, however, that for a generic class ξ they can be computed in terms of a cyclic covering $\overline{M}_\xi \to M$ and coincide with the corresponding classical Novikov numbers of [1] and [2].

To prove this we need a simple lemma (which we shall also use many times in the sequel).

LEMMA 2.4. *Let M and N be finitely generated modules over a Noetherian commutative ring W, and let S be a multiplicative subset of W. For $\sigma \in W$, denote by $S_{(\sigma)}$ the multiplicative subset generated by σ. Then the following assertions are true:*

1) *If $f: S^{-1}M \to S^{-1}N$ is a homomorphism of $S^{-1}W$-modules, then there exist $\sigma \in W$ and $f_{(\sigma)}: S^{-1}_{(\sigma)}M \to S^{-1}_{(\sigma)}N$ such that $S^{-1}f_{(\sigma)} = f$.*

2) *If $\theta \in W$ and $f_{(\theta)}, f'_{(\theta)}: S^{-1}_{(\theta)}M \to S^{-1}_{(\theta)}N$ are homomorphisms of $S^{-1}_{(\theta)}W$-modules such that $S^{-1}f_{(\theta)} = f = S^{-1}f'_{(\theta)}$, then there exists $\sigma' \in W$ such that $S^{-1}_{(\sigma'\theta)}f_{(\theta)} = S^{-1}_{(\sigma'\theta)}f'_{(\theta)}$.*

3) *If $f: S^{-1}M \to S^{-1}N$ is an isomorphism, then there exist $\sigma \in W$ and $f_{(\sigma)}: S^{-1}_{(\sigma)}M \to S^{-1}_{(\sigma)}N$ such that $f_{(\sigma)}$ is an isomorphism and $S^{-1}f_{(\sigma)} = f$.*

PROOF. 1) Denote by M_0 and N_0 the kernels of the localization maps $M \to S^{-1}M$ and $N \to S^{-1}N$; they are finitely generated, since W is Noetherian.

Consider any $\gamma \in S$ annihilating both M_0 and N_0. The localization maps $S^{-1}_{(\gamma)}M \to S^{-1}M$ and $S^{-1}_{(\gamma)}N \to S^{-1}N$ are injective. Pick any finite system of generators m_i of M. There exists $\gamma \in S$ such that $f(m_i) \in S^{-1}_{(\gamma)}N$. Thus $S^{-1}_{(\gamma\lambda)}M$ and $S^{-1}_{(\gamma\lambda)}N$ are submodules of $S^{-1}M$ and $S^{-1}N$, respectively, and f sends one of them into the other. Now we set $\sigma = \gamma\lambda$ and $f_{(\sigma)} = f \mid S^{-1}_{(\gamma\lambda)}M$.

The first claim of the lemma is proved.

2) The homomorphisms $S^{-1}_{(\gamma)}f_{(\theta)}$ and $S^{-1}_{(\gamma)}f'_{(\theta)}$ both are restrictions of the homomorphism $f: S^{-1}M \to S^{-1}N$ to the submodule $S^{-1}_{(\theta\gamma)}M \subset S^{-1}M$.

3) follows from the first two points.

COROLLARY 2.5. *Let C_* and D_* be finitely generated free complexes over a commutative Noetherian ring W. Suppose that $S^{-1}C_* \sim S^{-1}D_*$. Then, for some*

(2) Our definitions here differ from the standard ones: usually $b_p(M)$ denotes the pth Betti number of M.

$\sigma \in W$,

$$S_{(\sigma)}^{-1}C_* \sim S_{(\sigma)}^{-1}D_*.$$

Return now to the numbers b_* and q_*. Let C_* be a free finitely generated complex over the ring $\Lambda = \mathbf{Z}[\mathbf{Z}^k]$. Let $\xi\colon \mathbf{Z}^k \to \mathbf{R}$ be a homomorphism of irrationality degree 1. The image of ξ is isomorphic to $\mathbf{Z}$. Denote by (ξ) the homomorphism of the group rings $\Lambda \to \mathbf{Z}[\mathbf{Z}] = L$ obtained from the composition of $\xi\colon \mathbf{Z}^k \to \mathbf{Z}$ with $(-1)\colon \mathbf{Z} \to \mathbf{Z}$ by passing to group rings ((-1) is due to our sign conventions). Denote by $S \subset L$ the multiplicative subset consisting of Laurent polynomials with principal coefficient 1. Set

$$\beta_p(C_*, \xi) = \operatorname{rk}(H_p(C_* \otimes_\Lambda S^{-1}L)), \qquad \kappa_p(C_*, \xi) = q(H_p(C_* \otimes_\Lambda S^{-1}L)).$$

Here the structure of Λ-module on $S^{-1}L$ is defined via the homomorphism $(\xi)\colon \Lambda \to L \subset S^{-1}L$ (so the $S^{-1}L$-module $H_p(C_* \otimes_\Lambda S^{-1}L)$ depends on ξ).

LEMMA 2.6. *Any maximally irrational homomorphism $\gamma\colon \mathbf{Z}^k \to \mathbf{R}$ possesses an open conical neighborhood $U(\gamma)$ in the set* $\operatorname{Hom}(\mathbf{Z}^k, \mathbf{R})$ *such that the following equalities hold for $\xi \in U(\gamma)$:*

$$b_p(C_*, \gamma) = \beta_p(C_*, \xi), \qquad q_p(C_*, \gamma) = \kappa_p(C_*, \xi).$$

PROOF. Consider for all p the isomorphisms

$$H_p(C_*) \otimes \Lambda_{(\gamma)} \approx \left(\bigoplus_{i=1}^{b_p(C_*, \gamma)} \Lambda_{(\gamma)} \right) \oplus \left(\bigoplus_{j=1}^{q_p(C_*, \gamma)} \Lambda_{(\gamma)}/a_j^{(p)}\Lambda_{(\gamma)} \right),$$

where $a_j^{(p)} \in \Lambda = \mathbf{Z}[\mathbf{Z}^k]$ is divisible in Λ by $a_{j-1}^{(p)}$ and the γ-principal terms of the Laurent polynomials $a_j^{(p)}$ are equal to $\alpha_j^{(p)} \cdot \mathbf{1}$, where $\alpha_j^{(p)} \in \mathbf{Z}$ and $\alpha_j^{(p)} \neq \pm 1$. Consider the free finitely generated Λ-complex D_*, defined as follows. The module D_p is the sum of free modules F_p, E_p, and B_p of ranks $b_p(C_*, \gamma)$, $q_p(C_*, \gamma)$, and $q_{p-1}(C_*, \gamma)$, respectively. The differential $d_p\colon D_p \to D_{p-1}$ vanishes on $F_p \oplus E_p$ and sends the kth free generator b_k of the module B_p to the element $a_k^{(p)} \cdot e_k$, where e_k is the kth free generator of the module E_p. It is known that any complex over a principal ideal domain is homotopy equivalent to a standard one like this; from this we easily deduce that $C_* \otimes \Lambda_{(\gamma)}$ is homotopy equivalent to $D_* \otimes \Lambda_{(\gamma)}$. Since Λ is Noetherian, C_* and D_* become equivalent when localized with respect to the multiplicative subset generated by a single element $\sigma \in S_\gamma$ (see Lemma 2.6). Consider now any rational homomorphism $\xi\colon \mathbf{Z}^k \to \mathbf{Q}$ which is close enough to γ so that 1) each polynomial $a_j^{(p)}$ has only one ξ-principal term, namely $\alpha_j^{(p)} \cdot \mathbf{1}$, and 2) $\sigma \in S_\xi$. The complexes $C_* \otimes_\Lambda \Lambda_{(\xi)}$ and $D_* \otimes_\Lambda \Lambda_{(\xi)}$ are homotopy equivalent over $\Lambda_{(\xi)}$. The homomorphism $(\xi)\colon \Lambda \to S^{-1}L$ can be factored through $S_\xi^{-1}\Lambda = \Lambda_{(\xi)}$, and to calculate the homology of $C_* \otimes S^{-1}L$ we can use the complex

$$D_* \otimes \Lambda_{(\xi)} \otimes S^{-1}L = D_* \otimes_\Lambda S^{-1}L.$$

The complex D_* is described above; using 1) we easily get

$$H_p(D_* \otimes S^{-1}L) \approx \left(\bigoplus_{i=1}^{b_p(C_*, \gamma)} S^{-1}L \right) \oplus \left(\bigoplus_{j=1}^{q_p(C_*, \gamma)} S^{-1}L/\tilde{a}_j^{(p)} S^{-1}L \right),$$

where $\tilde{a}_j^{(p)} \in S^{-1}L$ and $\tilde{a}_j^{(p)} \vdots \tilde{a}_{j-1}^{(p)}$; the γ-principal coefficient of $\tilde{a}_j^{(p)}$ is equal to $\alpha_j^{(p)} \neq \pm 1$. Now the lemma follows easily.

The argument applied here will often be used in the sequel.

REMARK 2.7. It is easy to see that for any connected manifold M and any maximally irrational $\xi \in H^1(M, \mathbf{R})$ the numbers $b_0(\xi)$ and $q_0(\xi)$ vanish. Indeed, the module $H_0(\overline{M}) \approx \mathbf{Z}$ is annihilated by any element of $\mathbf{Z}[\mathbf{Z}^l]$ of the type $1-t$, where $t \in \mathbf{Z}^l$. Choosing $t \in \mathbf{Z}^l$ with $\xi(t) < 0$, we get $S_\xi^{-1} H_0(\overline{M}) = 0$.

§3. Duality properties

Recall first the formulation of Poincaré duality for manifolds that are not simply connected.

Let M^n be a smooth manifold (not necessary orientable). The universal covering $\widetilde{M}^n$ is equipped with the fundamental n-cycle U (infinite, in general). The intersection $\bigcap U$ defines an isomorphism D between the cohomology $H_c^*(\widetilde{M}^n, \mathbf{Z})$ with compact support and the homology $H_{n-*}(\widetilde{M}^n, \mathbf{Z})$. Define the automorphism χ of the group ring $\mathbf{Z}[\pi_1 M]$ by setting $\chi(g) = \varepsilon(g) g^{-1}$, where $g \in \pi_1 M$, $\varepsilon(g) = -1$ if the orientation of M is changed along γ, and $\varepsilon(g) = 1$ if not.

To simplify the statements we suppose that $\pi_1 M$ is abelian (we need only this case below).

The homology and cohomology groups of $\widetilde{M}^n$ are the $\mathbf{Z}[\pi_1 M]$-modules, and the isomorphism D is subjected to the following commutativity relation: $D(gx) = \chi(g) D(x)$. Isomorphisms of that kind are called *χ-isomorphisms*. Thus we have the χ-isomorphism of $\mathbf{Z}[\pi_1 M]$-modules

$$D\colon H_c^*(\widetilde{M}^n, \mathbf{Z}) \to H_{n-*}(\widetilde{M}^n, \mathbf{Z}).$$

Note that there exists the natural isomorphism of $\mathbf{Z}[\pi_1 M]$-modules

$$H_c^*(\widetilde{M}^n, \mathbf{Z}) \to H_*(\mathrm{Hom}_{\mathbf{Z}[\pi_1 M]}(C_*(\widetilde{M}^n), \mathbf{Z}[\pi_1 M])).$$

For a $\mathbf{Z}[\pi_1 M]$-module G we denote the modules

$$H_*(\mathrm{Hom}_{\mathbf{Z}[\pi_1 M]}(C_*(\widetilde{M}^n), G)) \quad \text{and} \quad H_*(C_*(\widetilde{M}^n) \otimes_{\mathbf{Z}[\pi_1 M]} G)$$

by $H^*(M^n, G)$ and $H_*(M^n, G)$.

Thus we have the χ-isomorphism of $\mathbf{Z}[\pi_1 M]$-modules

$$D\colon H^*(M, \mathbf{Z}[\pi_1 M]) \to H_{n-*}(\widetilde{M}). \tag{3.1}$$

Now we turn to the case $\pi_1 M = \mathbf{Z}^l$. Set $\Lambda = \mathbf{Z}[\pi_1 M]$. Let γ be a homomorphism $\mathbf{Z}^l \to \mathbf{R}$ (not necessarily injective). It is clear that χ sends S_γ to $S_{-\gamma}$ and thus defines an isomorphism $\chi\colon \Lambda_{(\gamma)} \to \Lambda_{(-\gamma)}$. Localizing D, we get the following lemma.

LEMMA 3.1. *There exists a χ-isomorphism*

$$D_\gamma\colon H^p(M^n, \Lambda_{(\gamma)}) \to H_{n-p}(M^n, \Lambda_{(-\gamma)}) \tag{3.2}$$

of the (left) $\Lambda_{(\gamma)}$-module and (right) $\Lambda_{(-\gamma)}$-module.

A module G over a ring W is called *principal* if it is isomorphic to a direct sum of modules of the form W/aW (where $a \in W$). Suppose now that all the modules $H_i(M^n, \Lambda_{(\gamma)})$ are principal for $r \le p$, and fix the decompositions

$$H_r(M^n, \Lambda_{(\gamma)}) \cong \left(\bigoplus_{i=1}^{\beta_r} \Lambda_{(\gamma)}\right) \oplus \left(\bigoplus_{i=1}^{\kappa_r} \Lambda_{(\gamma)}/a_i^{(r)}\Lambda_{(\gamma)}\right), \qquad 0 \le r \le p.$$

Applying Lemma 5.1 from §5 (which provides the standard presentation for a complex with principal homology) to the complex $C_*(\widetilde{M}^n) \otimes \Lambda_{(\gamma)}$, we get

$$H^r(M^n, \Lambda_{(\gamma)}) \cong \left(\bigoplus_{i=1}^{\beta_r} \Lambda_{(\gamma)}\right) \oplus \left(\bigoplus_{i=1}^{\kappa_{r-1}} \Lambda_{(\gamma)}/a_i^{(r-1)}\Lambda_{(\gamma)}\right), \qquad 0 \le r \le p. \tag{3.3}$$

Now the Poincaré duality (3.2) implies

$$H_q(M^n, \Lambda_{(-\gamma)}) \cong \left(\bigoplus_{i=1}^{\beta_{n-q}} \Lambda_{(-\gamma)}\right) \oplus \left(\bigoplus_{i=1}^{\kappa_{n-q-1}} \Lambda_{(-\gamma)}/\chi(a_i^{n-q-1})\Lambda_{(-\gamma)}\right),$$

$$n - p \le q \le n. \tag{3.4}$$

COROLLARY 3.2. *For a monomorphism $\gamma\colon \mathbf{Z}^l \to \mathbf{R}$,*

$$b_k = b_{n-k}, \qquad q_k(\gamma) = q_{n-k-1}(-\gamma).$$

PROOF. The first equality follows immediately from the above. To prove the second we recall that for any module N over a principal ideal domain R the number $q_k(N)$ equals the number of nonzero ideals J in any decomposition $N = R/J_1 \oplus \cdots \oplus R/J_n$, where $J_i \subset J_{i+1}$. We can choose the decomposition (3.3) to be of this kind; then so is (3.4). Q.E.D.

§4. V. V. Sharko's results concerning Morse functions on cobordisms that are not simply connected

In this section we recall some results from [8] which we shall use later. Let $(W; V_0, V_1)$ be a manifold with boundary ∂W, consisting of two components V_0 and V_1. Any regular Morse function $f\colon W \to \mathbf{R}$, constant on V_0 and on V_1, together with the suitable gradient-like vector field gives rise to a Morse complex, defined over $\mathbf{Z}[\pi_1 W]$. This complex is simply homotopy equivalent to $C_*(\widetilde{W}^n, \widetilde{V}_0)$, where $\widetilde{W}^n$ is the universal covering of W^n. In some cases the converse also holds, i.e. a free finitely generated complex C_* over $\mathbf{Z}[\pi_1 W]$, simply homotopy equivalent to $C_*(\widetilde{W}^n, \widetilde{V}_0)$, can be realized as a Morse complex for some Morse function on $(W; V_0, V_1)$.

THEOREM 4.1 (see [8], Proposition 6.1). *Suppose that $\pi_1(V_0) \to \pi_1(W) \leftarrow \pi_1(V_1)$ are isomorphisms. Then any free finitely generated $\mathbf{Z}[\pi_1 W]$-complex of the form*

$$C_* = \{0 \leftarrow C_2 \leftarrow C_3 \leftarrow \cdots \leftarrow C_{n-2} \leftarrow 0\}, \tag{4.1}$$

simply homotopy equivalent to $C_*(\widetilde{W}_n, \widetilde{V}_0^{n-1})$, *can be realized as a Morse complex of some regular Morse function* $f: W \to \mathbf{R}$ *which is constant on* V_1 *and on* V_0.

The idea of the proof is the following (see [8]). First we choose a regular Morse function f_0 on the cobordism $(W; V_0, V_1)$ without critical points of indices $0, 1, n-1$, or n (this is possible since $\pi_1(V_0) \to \pi_1(W) \leftarrow \pi_1(V_1)$ are isomorphisms). Denote the corresponding Morse complex by $C_*(f_0)$. The complexes C_* and $C_*(f_0)$ are simply homotopy equivalent. By a result of Cockcroft and Swan [15], $C_*(f_0) \oplus D_1 \approx C_* \oplus D_2$, where the complexes D_i $(i = 1, 2)$ are direct sums of complexes of the type $0 \leftarrow F \overset{\text{id}}{\leftarrow} F \leftarrow 0$. The complexes D_1 and D_2 can be chosen so as to concentrate in dimensions $2 \leq * \leq n-2$ (as C_* and $C_*(f_0)$ do). The procedure of adding (or subtracting) the complex $0 \leftarrow F \overset{\text{id}}{\leftarrow} F \leftarrow 0$ can be realized by corresponding changes of the function f_0, so that we obtain in the end a function f with the Morse complex C_*. The details can be found in [8].

REMARK. We shall need this theorem for $\pi_1 W = \mathbf{Z}^l$. In this case the notions of homotopy equivalence and simple homotopy equivalence coincide. Therefore, any complex of the type (4.1), homotopy equivalent to $C_*(\widetilde{W}^n, \widetilde{V}_0^{n-1})$, can be realized as a Morse complex of some function.

The paper [8] also contains results concerning minimal Morse functions. We reproduce them here partially. We do not need them for the proof of Theorem 0.1, and will use them only in §6 and in Remark 2 of §7.

For some classes of fundamental groups any cobordism $(W; V_0, V_1)$, where $\pi_1(V_0) \to \pi_1(W) \leftarrow \pi_1(V_1)$ are isomorphisms, possesses a Morse function having a minimal possible (among all the Morse functions) number of critical points of all indices. Namely, let $\mathrm{Wh}(\pi_1 W) = 0$. Theorem 4.1 implies that existence of such a function is guaranteed if we can find among the free finitely generated $\mathbf{Z}[\pi_1 W]$-complexes of the type (4.1) a complex C_*^0 which has the minimal possible number of generators in each dimension. The following theorem can be proved purely algebraically.

THEOREM 4.2 (see [8], Proposition 4.8). *Let* Q *be an IBN-ring* (*which means that free modules* Q^n *and* Q^m *are not isomorphic if* $m \neq n$) *and also an* s*-ring* (*which means that for any finitely generated* Q*-module* N *the condition* $N \oplus Q^n \approx Q^l$ *implies* $N \approx Q^{l-n}$).

Then for any free finitely generated Q*-complex* C_* *there exists a free finitely generated* Q*-complex* C'_*, *homotopy equivalent to* C_*, *which has the minimal possible number of generators in each dimension among free finitely generated complexes homotopy equivalent to* C_*. *The complex* C'_* *is called minimal.*

There is also a simple criterion to decide whether a given complex is minimal (in its homotopy type) or not.

DEFINITION. A pair (N, M) of modules over a ring Q, where $N \subset M$, is called *irreducible* if there exists no module $F \subset N$ which is a direct summand of M. The pair (N, M) is called *strongly irreducible* if for each $n \geq 0$ the greatest possible rank of a free submodule F of $N \oplus Q^n$ which is a direct summand of $M \oplus Q^n$ equals n. Our terminology differs here from that of [8].

THEOREM 4.3 (see [8], Theorem 4.7). *A* Q*-complex*

$$C_* = \{0 \leftarrow C_0 \overset{\partial_1}{\leftarrow} \cdots \overset{\partial_n}{\leftarrow} C_n \leftarrow 0\}$$

(*here* Q *is an IBN, s-ring) is minimal if and only if for each* i *the pair* $(\partial_i(C_i), C_{i-1})$ *is strongly irreducible.*

If the ring Q *possesses a nontrivial homomorphism to a field, then the rank of a free module is well-defined. All the group rings* $\mathbf{Z}[G]$ *enjoy this property. The ring* $\mathbf{Z}[\mathbf{Z}^l]$ *is also an s-ring; see* [16] *and* [17].

§5. Algebraic lemmas

First of all we prove the lemma (already used in §3) on the standard presentation for complexes.

LEMMA 5.1. *Let* W *be a commutative ring, and let*

$$C_* = \{0 \leftarrow C_1 \leftarrow \cdots \overset{\partial_n}{\leftarrow} C_n \leftarrow 0\}$$

be a free finitely generated W*-complex. Suppose that for* $p \le k$ *the* W*-modules* $H_p(C_*)$ *have the free resolutions of length* 2

$$0 \leftarrow H_p(C_*) \overset{\pi_p}{\leftarrow} F_p \overset{\varphi_p}{\leftarrow} G_p \leftarrow 0.$$

Then C_* *is homotopy equivalent to a free finitely generated complex*

$$C'_* = \{0 \leftarrow C'_1 \leftarrow \cdots \overset{\partial'_n}{\leftarrow} C'_n \leftarrow 0\}$$

such that

1) $C'_* = C_*$ *if* $* \ge k+2$, *and*
2) $C'_p = G_{p-1} \oplus F_p$ *for* $p \le k$; *furthermore,*

$$\partial'_p \mid F_p = 0, \qquad (\partial'_p \mid G_{p-1}) = \varphi_{p-1}\colon G_{p-1} \to F_{p-1} \subset C'_{p-1},$$

and $\operatorname{Im}(\partial'_{p+1}\colon C'_{p+1} \to C'_p) \subset C'_p$ *coincides with* $\operatorname{Im}\varphi_p \subset F_p$.

The proof is by induction on p. For $p = 0$ the assertion is obvious. For the induction step we suppose that the assertion is proved for all $k < m$ and prove it for $k = m$.

Suppose that C_* satisfies the assumptions of our lemma for $k = m$. Find a complex C'_* which satisfies the conclusion of the lemma for $k = m-1$. The image

$$\operatorname{Im}(\partial'_m\colon C'_m \to C'_{m-1})$$

is a free module $\operatorname{Im}\varphi_{m-1} \approx G_{m-1}$; hence C'_m can be decomposed as $G_{m-1} \oplus K_m$, where $\partial'_m \mid G_{m-1} = \mathrm{id}$. We may assume that K_m is a free finitely generated W-module (having added if necessary a complex $0 \leftarrow G_{m-1} \overset{\mathrm{id}}{\leftarrow} G_{m-1} \leftarrow 0$, located in dimensions m and $m+1$, to C'_*). The complex C'_* splits into the sum of two complexes

$$E_* = \{0 \leftarrow C'_1 \leftarrow \cdots \leftarrow C'_{m-2} \leftarrow C'_{m-1} \leftarrow 0\}, \qquad D_* = \{0 \leftarrow \cdots \leftarrow 0 \leftarrow K_m \leftarrow \cdots\}$$

(D_* is located in dimensions $\ge m$). Here $H_m(D_*) \approx H_m(C'_*) \approx H_m(C_*)$.

Now add the complex $0 \leftarrow F_m \overset{\mathrm{id}}{\leftarrow} F_m \leftarrow 0$, located in dimensions m and $m+1$, to D_* and consider the resulting complex D'_*. There is an epimorphism

$$D'_m = K_m \oplus F_m \overset{P}{\to} H_m(D'_*) \approx H_m(C_*),$$

where $P \mid F_m = 0$ and $P \mid K_m$ is the natural projection.

The map $P': K_m \oplus F_m \to H_m(C_*)$, defined by $P' \mid K_m = 0$ and $P' \mid F_m = \pi_{m'}$ is another epimorphism onto the same module.

By [8], Lemma 1.10, there exists an isomorphism $\varphi: K_m \oplus F_m \to K_m \oplus F_m$ such that $P' \circ \varphi = P$. That means we can find another decomposition $D'_m = K_m \oplus F_m$ (i.e., choose another free basis) such that the projection $P: D'_m \to H_m(D'_*) = H_m(C_*)$ is given by $P \mid K_m = 0$ and $P \mid F_m = \pi_m$. It is clear now that we can split the complex $0 \leftarrow K_m \leftarrow K_m \leftarrow 0$ (located in dimensions m and $m+1$) off the D'_*. In the resulting complex

$$D''_m = \{0 \leftarrow \cdots \leftarrow F_m \xleftarrow{\partial''_{m+1}} \cdots\}$$

the image $\operatorname{Im} \partial''_{m+1}$ coincides with $\operatorname{Im} \varphi_m \subset F_m$, since the projection $D''_m \to H_m(D''_m) \approx H_m(C_*)$ coincides with π_m. It is clear now that the complex $E_* \oplus D''_*$ satisfies parts 1) and 2) of the conclusion for $k = m$.

REMARK. We do not prove (and do not use) the assertions concerning simple homotopy equivalence.

In order to state our next lemma, we introduce some notation.

Set $\Lambda = \mathbf{Z}[\mathbf{Z}^l]$ and let $\gamma: \mathbf{Z} \to \mathbf{R}$ be a *rational* homomorphism (i.e. $\operatorname{rk} \operatorname{Im} \gamma = 1$). We shall study the localizations $S^{-1}A$ of Λ-modules A. We can assume that there is chosen a system of generators $(t, t_1, \dots, t_{l-1})$ for a group $\mathbf{Z}^l$ such that γ is the projection of $\mathbf{Z}^l$ onto the direct summand $\mathbf{Z}$ generated by t, and that $\gamma(t) = -1$.

Set

$$R = \mathbf{Z}[t_1^{\pm 1}, \dots, t_{l-1}^{\pm 1}] = \mathbf{Z}[\mathbf{Z}^{l-1}],$$
$$P = R[t], \qquad S = \{1 + tQ(t) \mid Q(t) \in R[t]\}, \tag{5.1}$$
$$K = S^{-1}P, \qquad S' = \{t^n \mid n \in \mathbf{N}\}, \qquad \Gamma = S'^{-1}S^{-1}P = S_\gamma^{-1}\Lambda.$$

LEMMA 5.2. *Let A be a finitely generated P-module such that $S_\gamma^{-1}A$ is a principal Γ-module. Fix a decomposition*

$$S_\gamma^{-1}A = \left(\bigoplus_{i=1}^{b} \Gamma(e_i)\right) \oplus \left(\bigoplus_{j=1}^{q} (\Gamma/a_j\Gamma)(f_j)\right),$$

where $a_j = a_{j,0} + a_{j,1}t + \cdots$, $a_{j,k} \in P$, and $a_{j,0} \neq 0$ is a noninvertible element of R.

Then there exists a finitely generated P-module $B \subset A$ such that

1) $S^{-1}B \approx (\bigoplus_{i=1}^{k} K(e'_i)) \oplus (\bigoplus_{j=1}^{q} (K/a_jK)(f'_j))$, and

2) $t^N A \subset B$ for some N.

PROOF. Note first that we may assume that e_i and f_i belong to A. Furthermore, having multiplied e_i and f_i by t^N, where N is large enough, we may assume that the module $P(e_i, f_j) \subset A$ is free of t-torsion.

We show that $P(e_i, f_j)$ satisfies 1).

Consider the K-submodule $K(e_i, f_j)$ of the module $S_\gamma^{-1}A$. It is easy to see that

$$K(e_i, f_j) \approx \left(\bigoplus_{i=1}^{b} (S^{-1}P)(e_i)\right) \oplus \left(\bigoplus_{j=1}^{q} (S^{-1}P/a_jS^{-1}P)(f_j)\right).$$

We show that $K(e_i, f_j)$ is the S-localization of the P-module $P(e_i, f_j)$. Inde $K(e_i, f_j)$ is an $S^{-1}P$-module; hence there exists an epimorphism

$$\varphi: S^{-1}(P(e_i, f_j)) \to K(e_i, f_j) \subset S_\gamma^{-1}A.$$

Now we show that φ is an isomorphism.

Denote by $\bar{e}_i$ and $\bar{f}_j$ the images of $e_i, f_j \in P(e_i, f_j)$ under the map of tl module to its localization; they generate over K the entire module $S^{-1}(P(e_i, f_j$ Consider any $x = \sum \alpha_i \bar{e}_i + \sum \beta_i \bar{f}_j$ belonging to $\operatorname{Ker}\varphi$, where $\alpha_i, \beta_j \in P$. Obser that $\alpha_i e_i = 0 = \beta_j f_j$ in the module $S_\gamma^{-1}A$, which means that

$$t^{N_i}(1 + tQ_i(t))\alpha_i e_i = 0 = t^{M_j}(1 + tW_j(t))\beta_j f_j$$

in A for some natural N_i, M_j and $Q_i(t)$, $W_j(t) \in P$. The module $P(e_i, f_j)$ is fr of t-torsion by construction; therefore

$$(1 + tW_j(t))\beta_j f_j = 0 = (1 + tQ_j(t))\alpha_i e_i$$

in A. It is clear now that $x = 0$ and thus, $\operatorname{Ker}\varphi = 0$ and $P(e_i, f_j)$ satisfi requirement 1) of the conclusion.

Observe next that the localization $S_\gamma^{-1}P(e_i, f_j)$ coincides with the entire modu $S_\gamma^{-1}A$, and A is finitely generated. Therefore there exists an element $t^N(1 + tQ(t$ of P such that $t^N(1 + tQ(t))a \in P(e_j, f_j)$ for all $a \in A$. Consider now the modu

$$B = \{a \in A \mid \exists Q(t) \in P\colon (1 + tQ(t))a \in P(e_i, f_j)\}.$$

The Noetherian property of P implies that B is finitely generated; obviousl $S^{-1}B \approx S^{-1}P(e_i, f_j)$. By construction, $t^N A \subset B$. The lemma is proved.

For a P-module M we denote by $\operatorname{Tor}_t M$ the submodule of elements annihilate by some power of t, by ${}^tM$ the submodule of elements annihilated by t, and b M_t the quotient module M/tM.

LEMMA 5.3. *For a P-module M the localization map $M \to S^{-1}M$ induces*
1) *an isomorphism $M/tM \xrightarrow{\approx} S^{-1}M/tS^{-1}M$,*
1a) *an isomorphism $M/t^qM \xrightarrow{\approx} S^{-1}M/t^qS^{-1}M$ for any natural q,*
2) *a monomorphism ${}^tM \to {}^t(S^{-1}M)$, and*
2a) *a monomorphism $\operatorname{Tor}_t M \to \operatorname{Tor}_t(S^{-1}M)$.*

PROOF. 1) *Injectivity.* Suppose that $m \in M$ and $m = t_n/(1 + tQ(t))$ in $S^{-1}M$ Then

$$(1 + tQ(t))m = tn + x,$$

where $(1 + tQ_1(t))x = 0$. Therefore $(1 + tQ_2(t))m = tn'$, hence $m \in tM$.

Surjectivity. We have

$$\frac{m}{1 + tQ(t)} = m - t\frac{Q(t)m}{1 + tQ(t)}.$$

1a) Note that if $n \in M$ and $Q(t) \in P$, then in the module $S^{-1}M/t^qS^{-1}M$ we have

$$\frac{n}{1 + tQ(t)} = (1 - tQ(t) + \cdots \pm (tQ(t))^{q-1})n.$$

Now the surjectivity is straightforward. Further, if $m = t^q n/(1 + tQ(t))$, where $n, m \in M$, then the equality

$$m = t^q(1 - tQ(t) + \cdots \pm (tQ(t))^{q-1})n$$

holds in $S^{-1}M/t^q S^{-1}M$, and injectivity is also proved.

2) If $tm = 0$ and $m \mapsto 0$ in $S^{-1}M$, then $(1 + tQ(t))m = 0 \Rightarrow m = 0$.

2a) If $t^N m = 0$ and $(1 + tQ(t))m = 0$, then

$$(t^{N-1} + t^N Q(t))m = 0 \Rightarrow t^{N-1}m = 0;$$

proceeding further in a similar way, we find that $m = 0$.

LEMMA 5.4. *Let $a = a_0 + a_1 t + \cdots + a_N t^N \in P$, where $a_i \in R$, $a_0 \neq 0$ is a noninvertible element of R, and q is a natural number. Then the following assertions are true:*

1) *The quotients $K_q = K/(a, t^q)$ and $P_q = P/(a, t^q)$ of the rings K and P by the ideals generated by a and t^q are isomorphic as rings and as P-modules.*

2) *The ring P_q is isomorphic as an R-module to the module $R^q/F(R^q)$, where $F: R^q \to R^q$ is injective and is given by the matrix*

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_q \\ & a_0 & a_1 & \cdots & a_{q-1} \\ & & & & \\ & & \ddots & & \vdots \\ 0 & & & & a_0 \end{pmatrix}. \tag{5.2}$$

Thus the R-module P_q has a free resolution of length 2.

3) *The multiplication map $t^l: P_q \to P_{l+q}$ (where $l > 0$) is injective and induces an epimorphism*

$$\mathrm{Ext}^1_R(P_{l+q}, R) \to \mathrm{Ext}^1_R(P_q, R).$$

The module $\mathrm{Hom}_R(P_q, R)$ is trivial.

PROOF. 1) The embedding $P \subset K$ induces a map $P_q \to K_q$ which is a ring homomorphism and a P-module homomorphism. The map $P/(t^q) \to K/(t^q)$ is an isomorphism (see Lemma 5.3); hence so is the map

$$(P/(t^q))/(a) \to (K/(t^q))/(a).$$

2) The ring $P/(t^q) = R[t]/(t^q)$ is a free R-module with basis $1, t, \ldots, t^{q-1}$. The multiplication by $a \in P$ is given in this basis by the matrix (5.2). Since $a_0 \neq 0$ and R has no zero divisors, F is a monomorphism.

3) Suppose that $x \in \mathrm{Ker}(t^l: P_q \to P_{l+q})$, i.e. $t^l x = Na + Mt^{l+q}$, where $x, N, M \in P$. The ring P is a unique factorization domain and t^l does not divide a; therefore $N = t^l N'$. Furthermore, since P has no zero divisors, we have $x = N'a + Mt^q$, and hence $x = 0$ in P_q.

Next observe that the map $t^l: P_q \to P_{l+q}$ lifts to resolutions in the following way:

$$\begin{array}{ccccccccc} 0 & \leftarrow & P_q & \leftarrow & R[t]/(t^q) & \xleftarrow{a} & R[t]/(t^q) & \leftarrow & 0 \\ & & \downarrow t^l & & \downarrow t^l & & \downarrow t^l & & \\ 0 & \leftarrow & P_{l+q} & \leftarrow & R[t]/(t^{q+l}) & \xleftarrow{a} & R[t]/(t^{q+l}) & \leftarrow & 0. \end{array}$$

The right arrow is a monomorphism onto the direct summand. This implies the surjectivity of the map of Ext's.

The equality $\mathrm{Hom}_R(P_q, R) = 0$ is obvious.

LEMMA 5.5. 1) *The ring* $K = S^{-1}P$ *is an IBN-ring and an s-ring.*

2) *A free finitely generated K-complex* $C_* = \{0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_l \leftarrow 0\}$ *is minimal if and only if the free finitely generated R-complex*

$$C_*/tC_* = \{0 \leftarrow C_1/tC_1 \leftarrow \cdots \leftarrow C_l/tC_l \leftarrow 0\}$$

is minimal.

PROOF. 1) We can embed K into its field of fractions; hence K is an IBN-ring. The ring $R = K/tK$ is an s-ring; therefore to establish the s-property for K it is enough to prove the following: for a finitely generated K-module N which is free of t-torsion, any collection $\{n_1, \dots, n_s\}$ of elements of N which forms a basis of N/tN over R is a basis of N over K.

To prove this we consider the free module $F(e_1, \dots, e_s)$ and the homomorphism $\varphi: F(e_1, \dots, e_s) \to N$ sending e_i to n_i. Denote by π the projection $N \to N/tN$. Since $\pi \circ \varphi$ is surjective, $N/\mathrm{Im}\,\varphi = t(N/\mathrm{Im}\,\varphi)$.

The element t belongs to the radical of K, and using Nakayama's lemma we get $N/\mathrm{Im}\,\varphi = 0$, i.e. φ is surjective. Next we show that φ is a monomorphism. Indeed, suppose that $\sum a_i n_i = 0$, where the a_i belong to K and some a_i is nonzero. We can assume that $a_i \in P$ and that there is a number i for which the free term of the polynomial a_i is not equal to zero (here we use that N is free of t-torsion). Reducing this equality modulo t, we obtain a contradiction.

2) By the minimality criterion (see Theorem 4.3) it suffices to prove that for a homomorphism $\varphi: F_1 \to F_2$, where F_1 and F_2 are finitely generated free K-modules, the pair $(\mathrm{Im}\,\varphi, F_2)$ is strongly irreducible if and only if the pair $(\mathrm{Im}\,\varphi_t, F_2/tF_2)$ is strongly irreducible (where $\varphi_t: F_1/tF_1 \to F_2/tF_2$ is the homomorphism φ reduced modulo t).

Suppose that the pair $(\mathrm{Im}\,\varphi, F_2)$ is irreducible. We show that $(\mathrm{Im}\,\varphi_t, F_2/tF_2)$ is irreducible. (The inverse implication is obvious.) Indeed suppose not, i.e. there exist elements $(e_1, \dots, e_k, f_1, \dots, f_n)$, such that $f_j \in \mathrm{Im}\,\varphi$ and the images $\bar{e}_i$ and $\bar{f}_j$ in F_2/tF_2 form the basis of this module. By 1) the elements $e_1, \dots, e_k, f_1, \dots, f_n$ form the basis of F_2; this is our contradiction.

The assertion concerning the strong irreducibility is proved on the same lines.

§6. Novikov type inequalities

The simplest way of proving (0.3) is the following. Let ω be a maximally irrational form. We can find a rational form ω', belonging to a very small neighborhood of ω, such that $m_p(\omega') = m_p(\omega)$. Now we apply Lemma 2.6 and reduce the problem to proving the Novikov inequalities (0.1); the latter are treated in [1] and [2] (see also [4]).

We give here one more proof of the Novikov type inequalities, which in our opinion clarifies the general homological reasons why these inequalities arise.

First we recall some well-known facts from Morse theory. Let $(W; V_0, V_1)$ be a compact manifold with boundary $\partial W = V_0 \sqcup V_1$, and let f be a Morse function on W, constant on V_0 and on V_1. The function f gives rise to a relative cell complex (K, V_0), homotopy equivalent to (W, V_0). The number of relative p-cells of (K, V_0) is equal to $m_p(f)$. Consider any regular covering $\overline{K}$ over K with a structure group G, and denote by $\overline{V}_0$ the preimage of V_0 in $\overline{K}$. The chain complex of the relative cell complex $(\overline{K}, \overline{V}_0)$ is a free $\mathbf{Z}[G]$-complex with $m_p(f)$ free $\mathbf{Z}[G]$-generators in dimension p. Any homotopy equivalence $\varphi: (K, V_0) \to (W, V_0)$ (which we assume to be cellular) induces an isomorphism of $\mathbf{Z}[G]$-modules $H_*(\overline{K}, \overline{V}_0) \to H_*(\overline{W}, \overline{V}_0)$. This implies that there exists a free finitely generated $\mathbf{Z}[G]$-complex $C'_* \sim C_*(\overline{W}, \overline{V}_0)$ such that $\mu(C'_p) = m_p(f)$.

Consider now an arbitrary rational Morse 1-form ω on M, and set $[\omega] = \lambda$.

We may assume (having multiplied ω by a constant) that $\omega = df$, where f is a map of M to S^1.

Consider the cyclic covering $\overline{M} \to M$, on which ω becomes exact: $\omega = d\bar{f}$, $\bar{f}: \overline{M} \to \mathbf{R}$. Denote by V the preimage in M of any regular value c of the map f (or, equivalently, the preimage in $\overline{M}$ of the corresponding regular value of $\bar{f}: \overline{M} \to R$).

Denote by t the generator of the deck transformation group of $\overline{M}$; set $V^- = \{x \in \overline{M} \mid \bar{f}(x) \le c\}$ and $V^+ = \{x \in \overline{M} \mid \bar{f}(x) \ge c\}$. We can assume that $tV^- \subset V^-$; set $W = tV^+ \cap V^-$. The manifold $\overline{M}$ is the countable union of "bricks" $t^k W$, $k \in \mathbf{Z}$. There is a Morse function f on the cobordism $(W; V, tV)$.

Consider now the regular covering $\widehat{M} \xrightarrow{p} M$ with structure group

$$H_1(M, \mathbf{Z})/\operatorname{Tors} H_1(M, \mathbf{Z}) = \mathbf{Z}^m.$$

It can be factored through the cyclic covering: $\widehat{M} \to \overline{M} \to M$. The composition $\widehat{M} \xrightarrow{p} \overline{M} \xrightarrow{\bar{f}} \mathbf{R}$ will be denoted by $\hat{f}$, and the p-preimages of W, V, V^+, and V^- by $\widehat{W}$, $\widehat{V}$, $\widehat{V}^+$, and $\widehat{V}^-$. Choose now a triangulation of W and extend it to a triangulation of M. Then V^+, V^-, $\widehat{V}^+$, and $\widehat{V}^-$ also get the triangulations. The chain complex $C_*(\widehat{V}^-)$ is a free finitely generated P-complex (recall from (5.1) that $R = \mathbf{Z}[\mathbf{Z}^{m-1}]$ and $P = R[t]$); the R-complex $C_*(\widehat{V}^-)/tC_*(\widehat{V}^-)$ is the same as $C_*(\widehat{W}, t\widehat{V})$. The manifold $\widehat{W}$ is a covering of W, and by the above we get the following:

There exists a free finitely generated P-complex $C_* = C_*(\widehat{V}^-)$, such that the R-complex C_*/tC_* is homotopy equivalent to an R-complex D_* having exactly $m_p(f)$ generators of dimension p.

Consider now the complex $S^{-1}C_*$ (we use the notation (5.1)). It is a free finitely generated K-complex, and since K is an IBN, s-ring (by Lemma 5.5), we can find a minimal K-complex C^0_* in the homotopy type of $S^{-1}C_*$ (see §4). By the same lemma the complex C^0_*/tC^0_* is minimal in the homotopy type of

$$S^{-1}C_*(\widehat{V}^-)/tS^{-1}C_*(\widehat{V}^-) \cong C_*(\widehat{V}^-)/tC_*(\widehat{V}^-) \cong C_*(\widehat{V}^-, t\widehat{V}^-) \cong C_*(\widehat{W}, t\widehat{V}).$$

Note that $m_p(\omega) = \mu(D_p)$. The R-complex D_* is homotopy equivalent to $C_*(\widehat{W}, t\widehat{V})$; this homotopy type contains a minimal complex C^0_*/tC^0_* with exactly $\mu(C^0_p)$ generators in each dimension p.

Note further that the localized complex $S'^{-1}C^0_*$ (recall that S'^{-1} denotes the localization with respect to t) is homotopy equivalent to

$$S'^{-1}S^{-1}C_* = S^{-1}S'^{-1}C_* = S^{-1}C_*(\widehat{M}).$$

Thus we have constructed a free finitely generated K-complex C^0_* such that

$$S'^{-1}C^0_* \sim S^{-1}C_*(\widehat{M}), \qquad m_p(\omega) \geq \mu_K(C^0_*). \tag{6.1}$$

This implies, of course, that

$$m_p(\omega) \geq \mu_\Gamma(C^1_*) \tag{6.2}$$

for some free finitely generated Γ-complex C^1_* homotopy equivalent to $S^{-1}C_*(\widehat{M})$.

CONJECTURE. The ring $\Gamma = S'^{-1}S^{-1}P$ is an s-ring, i.e. any finitely generated stably free module is free. ([3])

If this conjecture holds, then the homotopy type of any finitely generated Γ-complex N_* contains a minimal complex, the numbers of generators of which are the invariants of homotopy type of N_* (see §4). Denote these numbers by $m_p(N_*)$. Then we get

$$m_p(\omega) \geq m_p(S^{-1}_{[\omega]}C_*(\overline{M})).$$

Next we deduce (0.3) from (6.2). Let λ be a rational cohomology class, $\lambda \in H^1(M, \mathbf{Q})$, and let C^1_* be a free finitely generated $S'^{-1}S^{-1}P$-complex homotopy equivalent to $S^{-1}_\lambda C_*(\widehat{M})$. Note that C^1_* is an S_λ-localization of some free finitely generated Λ-complex C^2_*. The Λ-complexes $C_*(\widehat{M})$ and C^2_* become homotopy equivalent after localization with respect to some element $\sigma \in S_\lambda$ (see Corollary 2.5). For any maximally irrational class $\lambda' \in H^1(M, \mathbf{R})$ sufficiently close to λ, the element σ belongs also to $S_{\lambda'}$; hence

$$S^{-1}_{\lambda'}C_*(\widehat{M}) \sim S^{-1}_{\lambda'}C_*.$$

The ring $S^{-1}_{\lambda'}\Lambda$ is a principal ideal domain (see Theorem 1.4); therefore the number of generators of C^2_* in dimension p is not greater than

$$b_p(S^{-1}_{\lambda'}H_*(\widehat{M})) + q_p(S^{-1}_{\lambda'}H_*(\widehat{M})) + q_{p-1}(S^{-1}_{\lambda'}H_*(\widehat{M})),$$

and, recalling (6.2), we obtain (0.3).

§7. Reduction to a surgery problem

Let γ be a maximally irrational cohomology class. Then $\Lambda_{(\gamma)}$ is a principal ideal domain (Theorem 1.1); hence

$$S^{-1}_\gamma H_p(\widetilde{M}, \mathbf{Z}) \cong H_p(M, \Lambda_{(\gamma)}) = \left(\bigoplus_{i=1}^{b_p(\gamma)} \Lambda_{(\gamma)}\right) \oplus \left(\bigoplus_{j=1}^{q_p(\gamma)} \Lambda_{(\gamma)}/a^{(p)}_j\Lambda_{(\gamma)}\right).$$

We may assume that $a^{(p)}_j \in \Lambda$, thus we get

$$S^{-1}_\gamma H_p(\widetilde{M}, \mathbf{Z}) \cong S^{-1}_\gamma \left[\left(\bigoplus_{i=1}^{b_p(\gamma)} \Lambda\right) \oplus \left(\bigoplus_{j=1}^{q_p(\gamma)} \Lambda/a^{(p)}_j\Lambda\right)\right].$$

([3]) The validity of the s-property was analyzed for rings of a similar type in [18]. However, for Γ itself the conjecture seems to be not yet settled.

Together with Lemma 2.4 this implies that the modules $H_p(\widetilde{M}, \mathbf{Z})$ and

$$\left(\bigoplus_{i=1}^{b_p(\gamma)} \Lambda\right) \oplus \left(\bigoplus_{j=1}^{q_p(\gamma)} \Lambda / a_j^{(p)} \Lambda\right)$$

become isomorphic after localization with respect to multiplicative set, generated by some element $\sigma \in S_\lambda$. This σ belongs also to the set S_λ for every λ (in particular, the rational ones) sufficiently close to γ.

Thus for the elements λ of some open dense set $U \subset H^1(M, \mathbf{R}) = \mathbf{R}^m$ we get

$$S_\lambda^{-1} H_p(\widetilde{M}, \mathbf{Z}) \cong \left(\bigoplus_{i=1}^{b_p} S_\lambda^{-1} \Lambda\right) + \left(\bigoplus_{j=1}^{q_p} S_\lambda^{-1} \Lambda / a_j^{(p)} S_\lambda^{-1} \Lambda\right). \tag{7.1}$$

The number q_p depends of course on λ; in the complement to Γ_i in $H^1(M, \mathbf{R})$ the number q_p equals $q_p(\gamma)$ (see the Introduction). To obtain the proof of Theorem 0.1 it suffices to prove the following result.

THEOREM 7.1. *Let $\pi_1 M^n = \mathbf{Z}^m$, $n \geq 6$, and suppose that λ is a rational cohomology class, $\lambda \in H^1(M, \mathbf{Q})$, such that* (7.1) *holds and $m_r = m_{r+1} = m_{r+2} = 0$ for some r, $2 \leq r \leq n-4$. Then there exists a Morse map $f: M^n \to S^1$, inducing from the fundamental class i of the circle the element $f^* i$, which is a multiple of λ, such that $m_p(f) = m_p = b_p + q_p + q_{p-1}$ (for $0 \leq p \leq n$).*

It is sufficient to prove this theorem under the additional assumption that λ is an integer cohomology class defined by the projection of $H_1(M, \mathbf{Z}) = \mathbf{Z}^m$ onto the first direct summand $\mathbf{Z}$. In the course of this and the two subsequent sections we suppose that this condition holds, without further explicit mention.

Let $V^{n-1} \subset M^n$ be a connected smooth submanifold of codimension 1, and let ν be a normal vector field on V. The pair (V^{n-1}, ν) is called an *admissible splitting* (4) if

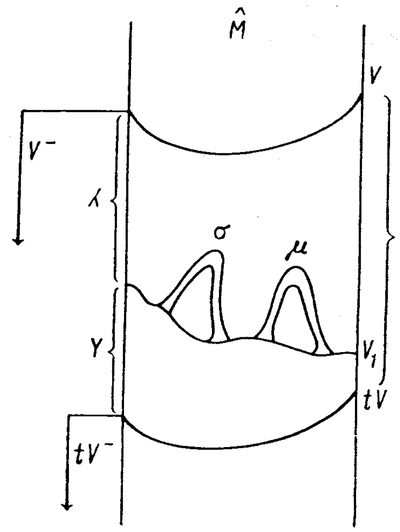
1) $\pi_1(V^{n-1}) \to \pi_1(M^n)$ is a monomorphism onto the subgroup $\operatorname{Ker}\lambda \approx \mathbf{Z}^{m-1}$, and

2) the Pontryagin-Thom construction with respect to ν determines a map $M^n \to S^1$ representing the class $\lambda \in H^1(M, \mathbf{Z})$.

The existence of admissible splittings is proved in [10] under the assumption that the homotopy fiber of $\lambda: M \to S^1$ has finite type (one can show that for $\pi_1 M = \mathbf{Z}^m$ this is equivalent to the following: the class λ satisfies (7.1) with $b_p = q_p = 0$ for all p). The same proof is valid for arbitrary cohomology classes. (Note also that the results of [19] imply that for any Morse map $f: M^n \to S^1$ which represents λ and has no critical points of index 0 or n the level surfaces $f^{-1}(c)$ are connected and $\pi_1(f^{-1}(c)) \to \operatorname{Ker}\lambda$ is an epimorphism.)

Consider the infinite cyclic covering $p: \widehat{M}^n \to M^n$ corresponding to λ. For an admissible splitting V^{n-1} the preimage $p^{-1}(V^{n-1})$ consists of a countably infinite number of copies of V^{n-1}, which divide $\widehat{M}^n$ into a countably infinite number of "bricks" W^n, with $\partial W^n \approx V^{n-1} \cup tV^{n-1}$ (where t is a generator of the structure group of the covering); see the picture below. Now fix any copy of $V^{n-1} \subset \widehat{M}^n$; it

(4) We will omit ν in the notation when no confusion is possible.



divides $\widehat{M}^n$ into two parts, V^+ and V^-. (We assume here and elsewhere that fo every admissible splitting V the lifting of V into $\widehat{M}^n$ is chosen and fixed. In thi case the notations V^+, V^-, etc. make sense.) The intersection $V^+ \cap V^-$ equal V^{n-1}, the vector ν points into V^+, and $tV^- \subset V^-$. Furthermore,

$$\pi_1(W^n) \approx \pi_1(V^{n-1}) \approx \pi_1(\widehat{M}^n) \approx \mathbf{Z}^{m-1},$$

and the universal covering $\widetilde{M}^n$ is divided by $\widetilde{V}^{n-1}$ into two parts, $\widetilde{V}^+$ and $\widetilde{V}^-$ (al this can be found in [10]).

One easily sees that any triangulation of M for which V^{n-1} is a subcomple: determines a free finitely generated P-complex $C_*(\widetilde{V}^-)$ (in the notation (5.1)); th complexes $S'^{-1}C_*(\widetilde{V}^-)$ and $C_*(\widetilde{M}^n)$ coincide, and, therefore

$$H_*(\widetilde{M}^n) \approx S'^{-1}H_*(\widetilde{V}^-).$$

The quotient complex $C_*(\widetilde{V}^-)/tC_*(\widetilde{V}^-)$ is the chain complex of the triangulatior of the pair $(\widetilde{W}^n, t\widetilde{V}^{n-1})$.

Suppose now that λ satisfies (7.1) for $p \leq k$, and that

$$a_j^{(p)} = a_{j,0}^{(p)} + a_{j,1}^{(p)}t + \cdots,$$

where the $a_{j,0}^{(p)}$ are nonzero, noninvertible elements of R. An admissible splitting V will be called *k-regular* if for $p \leq k$

$$S^{-1}H_p(\widetilde{V}^-, \mathbf{Z}) \approx \left(\bigoplus_{i=1}^{b_p} S^{-1}P\right) \oplus \left(\bigoplus_{j=1}^{q_p} S^{-1}P/a_j^{(p)}S^{-1}P\right). \tag{7.2}$$

LEMMA 7.2. *Suppose that $\pi_1 M^n = \mathbf{Z}^m$, $n \geq 6$, and let $\lambda \in H^1(M, \mathbf{Z})$ satisfy (7.1) for $p \leq k$, where $k \leq n-4$. Let V be an admissible splitting. Then there exists a k-regular splitting V_0, obtained from V by surgical modifications of indices $\leq k+1$ carried out inside $V^+ \subset \widehat{M}^n$.*

The proof of the lemma will be given in §8 (the main ideas were outlined in [6]).

Now we deduce from this lemma the existence of a Morse form in a class λ with the required number of zeros of indices $\leq k$.

First of all, note that, given an admissible splitting V and a Morse function f on the cobordism $(W; V, tV)$, we can produce from this data a Morse map $\tilde{f}: M \to S^1$ belonging to λ. (For this purpose we change f in a small neighborhood of the boundary $V^{n-1} \cup tV^{n-1}$ so that it becomes a projection on the second factor of the collar: $V^{n-1} \times [0, \varepsilon] \to [0, \varepsilon]$; then we glue together V and tV.) The map $\tilde{f}$ has the same critical points as f.

So we proceed to construct a Morse function with the required Morse numbers on the cobordism $(W_0; V_0, tV_0)$, where W_0 is the part of $\widehat{M}$ lying between V_0 and tV_0.

First we calculate the homology $H_p(\widetilde{W}_0, t\widetilde{V}_0; \mathbf{Z})$:

$$H_p(\widetilde{W}_0, t\widetilde{V}_0) \approx H_p(\widetilde{V}_0^-, t\widetilde{V}_0^-) \approx H_p(\widetilde{V}_0^-)/tH_p(\widetilde{V}_0^-).$$

(The first identification is due to excision axiom. To get the second we observe that the map $H_p(t\widetilde{V}_0^-) \to H_p(\widetilde{V}_0^-)$ of the exact sequence of the pair $(\widetilde{V}_0^-, t\widetilde{V}_0^-)$ coincides with multiplication by t, and the latter is injective since the module $B_p = H_p(\widetilde{V}_0^-)$ is free of t-torsion.)

Furthermore,

$$H_p(\widetilde{V}_0^-)/tH_p(\widetilde{V}_0^-) \approx B_p/tB_p \approx \left(\bigoplus_{i=1}^{b_p} R\right) \oplus \left(\bigoplus_{j=1}^{q_p} R/a_{j,0}^{(p)}R\right), \qquad p \leq k$$

(the last identification follows from Lemma 5.3). Also, $b_0 = q_0 = 0$ (Remark 2.7), and $b_1 = q_1 = 0$ since $\widetilde{M}$ is simply connected; hence from Corollary 3.2 we conclude that $b_n = q_n = b_{n-1} = q_{n-1} = 0$. Thus the homology modules $H_*(\widetilde{W}_0, t\widetilde{V}_0)$ vanish outside the dimensions $2 \leq * \leq n-2$ and are principal $\mathbf{Z}[\pi_1(W_0)]$-modules for $p \leq k$. According to Lemma 5.1 we can find a complex $C_* = \{0 \leftarrow C_2 \leftarrow \cdots \leftarrow C_{n-2} \leftarrow 0\}$ which is homotopy equivalent to $C_*(\widetilde{W}_0, t\widetilde{V}_0)$ and has $b_i + q_i + q_{i-1}$ free generators in dimension i (where $2 \leq i \leq k \leq n-4$).

According to Theorem 4.1 and the remark following it we can realize C_* as the Morse complex of some Morse function f on the cobordism (W_0, tV_0). This function gives rise to a 1-form which satisfies the requirements. Thus we have proved the following assertion.

THEOREM 7.3 (see [6]). ([5]) *Suppose that $\pi_1 M^n = \mathbf{Z}^m$, $n \geq 6$. Then any element γ of some dense open conical set $U \subset H^1(M, \mathbf{R})$ can be realized by a Morse form $\omega \in \Omega^1(M)$ which has the least possible number of zeros of indices p where $0 \leq p \leq n-4$ among all the forms of the class γ. This number equals $b_p + q_p(\bar{\gamma}) + q_{p-1}(\bar{\gamma})$, where $\bar{\gamma}$ is any maximally irrational cohomology class sufficiently close to γ.*

Now we use duality and do surgery also "from another end". In addition to (5.1) we introduce some new notations:

$$\overline{P} = R[t^{-1}], \qquad \overline{S} = \{1 + t^{-1}Q(t^{-1}) \mid Q(t^{-1}) \in R[t^{-1}]\},$$
$$\overline{K} = \overline{S}^{-1}\overline{P}, \qquad \overline{S}' = \{t^{-n} \mid n \in \mathbf{N}\}, \qquad \overline{\Gamma} = \overline{S}'^{-1}\overline{S}^{-1}\overline{P} = S_{(-\gamma)}^{-1}\Lambda.$$

([5]) The assumptions here are weaker than in Theorem 0.1, but the result is concerned only with the indices p, $0 \leq p \leq n-4$.

Note that if (V, ν) is an admissible splitting with respect to λ, then $(V, -$ an admissible splitting with respect to $(-\lambda)$ and all the above results hold for (with t replaced by t^{-1}). Note that $(V, \nu)^- = (V, -\nu)^+$.

If the class λ satisfies (7.1) for $p \le k$, then, according to (3.4),

$$S^{-1}_{(-\lambda)} H_s(\widetilde{M}^n, \mathbf{Z}) \cong \left(\bigoplus_{i=1}^{b_s} S^{-1}_{(-\lambda)}\Lambda\right) \oplus \left(\bigoplus_{j=1}^{q_{n-s-1}} (S^{-1}_{(-\lambda)}\Lambda)/(\chi(a_j^{(n-s-1)})S^{-1}_{(-\lambda)}\Lambda)\right)$$

for $s \ge n-k$ (recall that $b_s = b_{n-s}$).

Suppose now that $1 \le r \le n-4$ and (V, ν) is an admissible splitting. App Lemma 7.2 (where $k = r$), we get an r-regular splitting (V_0, ν_0). Applying Le 7.2 to the splitting $(V_0, -\nu_0)$, the cohomology class $(-\lambda)$, and $k = n-r-3$ get an $(n-r-3)$-regular splitting, say $(V_1, -\nu_1)$, corresponding to $(-\lambda)$. that V_1 is obtained from V_0 by a sequence of surgical modifications of in $\le n-r-2$; consequently, the homology of $\widetilde{(V_1, \nu_1)^-}$ coincides with that of $\widetilde{(V_0,}$ in dimensions r. Hence (V_1, ν_1) is also an r-regular splitting.

An admissible splitting (V, ν) is called *r-biregular* if (V, ν) is r-regular respect to λ and $(V, -\nu)$ is $(n-r-3)$-regular with respect to $(-\lambda)$.

We have proved that r-biregular splittings exist for $1 \le r \le n-4$.

REMARKS. 1. In §9 we shall deduce from the above that under the assumptior Theorem 7.3 every element γ of some dense open conical $U \subset H^1(M, \mathbf{R})$ ca realized by a Morse form ω having minimal Morse numbers of all indices ex two adjacent ones, say r and $r+1$, where $1 \le r \le n-4$.

2. Here we show that any $(n-3)$-regular splitting V is also n-regular. Cons the free finitely generated complex $S^{-1}C_*(\widetilde{V}^-)$. The ring $K = S^{-1}P$ is an IBN ring (Lemma 5.5); hence the homotopy type of this complex contains a minimal finitely generated K-complex C^0_*. Note that $C^0_0 = C^0_1 = C^0_{n-1} = C^0_n = 0$. Ind the complex C^0_*/tC^0_* is R-minimal (see Lemma 5.5) and belongs to the homot type of

$$S^{-1}C_*(\widetilde{V}^-)/tS^{-1}C_*(\widetilde{V}^-) = C_*(\widetilde{V}^-, t\widetilde{V}^-).$$

There exists a Morse function f on the cobordism $(W; V, tV)$, which has critical points of indices $0, 1, n-1$, or n. The corresponding complex C_* has no generators in dimension $0, 1, n-1$, or n, and since C^0_*/tC^0_* is mini and $C^0_*/tC^0_* \sim C_*(f) \sim C_*(\widetilde{W}, t\widetilde{V})$, we get $C^0_i/tC^0_i = 0$ for $i = 0, 1, n-1,$ Therefore, $C^0_i = 0$ for $i = 0, 1, n-1, n$.

The homology modules $H_s(C^0_*)$ have resolutions of length two for $s \le n-$ Therefore there exists a complex C^1_*, in the homotopy type of C^0_*, which is stand in dimensions $* \le n-3$, and moreover $C^1_n = 0$ and $C^1_{n-1} = B \oplus Z$, where $\partial \mid B$ injective and $\partial \mid Z = 0$. Therefore

$$Z \approx H_{n-2}(C^1_*) \approx H_{n-2}(S^{-1}C_*(\widetilde{V}^-)), \qquad \mu(Z) = b_{n-2}.$$

Now we realize C^1_*/tC^1_* by a Morse function and get our assertion.

For $\pi_1 M^n = \mathbf{Z}$ the $(n-3)$-regular splittings always exist (see [4]). The auth does not know if the same holds for $\pi_1 M = \mathbf{Z}^m$, $m > 1$, and any λ.

To cope with the remaining dimensions we are to impose the restriction m_r $m_{r+1} = m_{r+2} = 0$, where $m_s = b_s + q_s + q_{s-1}$. This restriction is an analogue (f those three particular dimensions) of the Farrell condition for existence of a map $f: M^n \to S^1$ which realizes a given cohomology class and has no critical points at all.

LEMMA 7.4. *Suppose $\pi_1 M^n = \mathbf{Z}^m$, $n \geq 6$, the integer cohomology class λ satisfies* (7.1), *and (V, ν) is an $(r-1)$-biregular splitting (here $1 \leq r \leq n-2$). Suppose further that $m_r = m_{r+1} = 0$. Then*

1) $H_r(\widetilde{M}^n, \widetilde{V}^-) = 0$, *and*

2) $H_{r+1}(\widetilde{M}^n, \widetilde{V}^-)$ *is a free finitely generated R-module and t is its nilpotent endomorphism.*

Thus the module $Q = H_{r+1}(\widetilde{M}^n, \widetilde{V}^-)$ together with the endomorphism t is an object of the category $\mathscr{C}(\mathbf{Z}[\mathbf{Z}^{m-1}], \mathrm{id})$ which consists of free finitely generated $\mathbf{Z}[\mathbf{Z}^{m-1}]$-modules and their nilpotent endomorphisms (see [10]). The corresponding Grothendieck group vanishes (see [10]); therefore our pair (Q, t) is equal to zero modulo relations in this group. To realize this equivalence in a geometric setting we need the third "critical point free" dimension.

LEMMA 7.5. *Suppose that the assumptions of Lemma 7.4 hold, and, moreover, $m_{r+2} = 0$, $2 \leq r \leq n-4$. Then there exists an $(r-1)$-biregular splitting (V, ν) such that*

$$H_r(\widetilde{M}, \widetilde{V}^-) = 0, \qquad H_{r+1}(\widetilde{M}, \widetilde{V}^-) = 0.$$

The proofs of Lemmas 7.4 and 7.5 will be presented in §9. Now we deduce Theorem 7.1 from them. Let (V, ν) be an admissible splitting constructed in Lemma 7.5, and $(W; V, tV)$ a corresponding brick in $\widehat{M}$. Compute the homology $H_*(\widetilde{W}, t\widetilde{V})$. For $p \leq r-1$

$$H_p(\widetilde{W}, t\widetilde{V}) \approx \left(\bigoplus_{i=1}^{b_p} R\right) \oplus \left(\bigoplus_{j=1}^{q_p} R/a_{j,0}^{(p)} R\right)$$

(see 7.2). Since $(V, -\nu)$ is $(n-r-2)$-regular,

$$H_s(\widetilde{W}, \widetilde{V}) \approx \left(\bigoplus_{i=1}^{b_s} R\right) \oplus \left(\bigoplus_{j=1}^{q_{n-s-1}} R/\chi(a_{j,0}^{(n-s-1)}) R\right)$$

for $s \leq n-r-2$. Now apply Poincaré duality to the manifold W with two components V, tV of the boundary.

The R-modules $H_s(\widetilde{W}, \widetilde{V})$ are principal for $s \leq n-r-2$, and it is easy to calculate the cohomology $H^s(\widetilde{W}, \widetilde{V})$ for $s \leq n-r-2$. Applying the Poincaré duality arguments from §3, we get

$$H_p(\widetilde{W}, t\widetilde{V}) \approx \left(\bigoplus_{i=1}^{b_p} R\right) \oplus \left(\bigoplus_{j=1}^{q_p} R/a_{j,0}^{(p)} R\right), \qquad p \geq r+2.$$

Since $m_{r+2} = 0$,

$$H_{r+2}(\widetilde{W}, t\widetilde{V}) = H_{r+2}(\widetilde{V}^-, t\widetilde{V}^-) = 0.$$

From the exact sequence of the triple $(t^{-1}\widetilde{V}^-, \widetilde{V}^-, t\widetilde{V}^-)$ we get $H_{r+2}(t^{-1}\widetilde{V}^-, t\widetilde{V}) = 0$ and (by induction) $H_{r+2}(t^{-n}\widetilde{V}^-, t\widetilde{V}^-) = 0$. Taking the direct limit as $n \to \infty$, we get $H_{r+2}(\widetilde{M}, t\widetilde{V}^-) = 0$

From the exact sequence of the triple $(\widetilde{M}, \widetilde{V}^-, t\widetilde{V}^-)$ (using the equality $H_*(\widetilde{M}, \widetilde{V}^-) = H_*(\widetilde{M}, t\widetilde{V}^-) = 0$ for $* = r, r+1, r = 2$) we get

$$H_{r+1}(\widetilde{V}^-, t\widetilde{V}^-) = H_r(\widetilde{V}^-, t\widetilde{V}^-) = 0.$$

Now we collect the results of our computations and see that $H_p(\widetilde{W}, t\widetilde{V})$ is isomorphic to

$$\left(\bigoplus_{i=1}^{b_p} R\right) \oplus \left(\bigoplus_{j=1}^{q_p} R/a_{j,0}^{(p)}R\right)$$

for all p. Now we apply the same argument as in proof of Theorem 7.3, and Theorem 7.1 is proved.

§8. Construction of a p-regular splitting

This section is devoted to the proof of Lemma 7.2. We prove it by induction in k.

Note first that every admissible splitting V is 1-regular. Indeed, since $\widetilde{V}^-$ is connected, the map $t: \widetilde{V}^- \to \widetilde{V}^-$ induces the identity homomorphism in the group $H_0(\widetilde{V}^-)$; hence $S^{-1}H_0(\widetilde{V}^-) = 0$. The same argument proves that $b_0 = q_0 = 0$. Furthermore, the commutativity of $\pi_1 M^n$ implies that t induces the identity homomorphism also in the group $H_1(\widetilde{V}^-)$ (which is isomorphic to $\pi_1(\widetilde{V}^-)$); therefore $S^{-1}H_1(\widetilde{V}^-) = 0$.

The induction step will proceed by means of Lemmas 8.1 and 8.2. In both lemmas we assume that $\pi_1 M^n = \mathbf{Z}^m$, $n \geq 6$, and that λ satisfies (7.1) for $p \leq k$.

LEMMA 8.1. *Let $k \leq n-4$ and suppose that an admissible splitting V is $(k-1)$-regular (i.e. V satisfies (7.2) for $p \leq k-1$). Then there exists an admissible splitting V_0, obtained from V by a sequence of surgical modifications inside V^+ of indices $\leq k+1$, such that*

1) *the P-modules $H_p(\widetilde{V}_0^-)$ and $H_p(\widetilde{V}^-)$ are isomorphic for $p \leq k-1$, and*

2) *the P-module $H_k(\widetilde{V}_0^-)$ is isomorphic to $H_k(\widetilde{V}^-)/\operatorname{Tor}_t H_k(\widetilde{V}^-)$ (recall that $\operatorname{Tor}_t M$ denotes the submodule of all elements of M annihilated by some power of t, and ${}^tM$ denotes the submodule of elements annihilated by t).*

LEMMA 8.2. *Let $k \leq n-3$, and suppose that V is a $(k-1)$-regular splitting such that $A = H_k(\widetilde{V}^-)$ is free of. t-torsion. Suppose that B is a P-submodule of A such that $tA \subset B \subset A$ and B/tA is a cyclic R-module.*

Then there exists an admissible splitting V_0, obtained from V by a sequence of surgical modifications inside V^+ of indices $\leq k+1$, such that

1) *the P-modules $H_p(\widetilde{V}_0^-)$ and $H_p(\widetilde{V}^-)$ are isomorphic for $p \leq k-1$, and*

2) *there is an epimorphism of P-modules $H_k(\widetilde{V}_0^-) \to B$ with kernel ${}^tH_k(\widetilde{V}_0^-) = \operatorname{Tor}_t H_k(\widetilde{V}_0^-)$.*

Once both lemmas are proved, the induction step proceeds as follows. Let V be a $(k-1)$-regular splitting, $k \leq n-4$. Having applied Lemma 8.1, we may assume that the P-module $H_k(\widetilde{V}^-)$ has no t-torsion. Consider the P-module $A_k = H_k(\widetilde{V}^-)$. Condition (7.1) holds for $p \leq k$; hence by Lemma 5.3 there exists a submodule

$B_k \subset A_k$ such that

$$\begin{cases} t^r A_k \subset B_k \subset A_k \quad \text{for some } r, \\ S^{-1}B_k \approx \left(\bigoplus_{i=1}^{b_k} S^{-1}P\right) \oplus \left(\bigoplus_{j=1}^{q_k} S^{-1}P/a_j^{(k)}S^{-1}P\right). \end{cases} \tag{8.1}$$

Choose the filtration of B_k

$$t^r A_k = B_k^{(0)} \subset B_k^{(1)} \subset \cdots \subset B_k^{(m)} = B_k,$$

such that the $B_k^{(i)}$ are P-submodules in B_k, and the quotient modules $B_k^{(i)}/B_k^{(i-1)}$ are cyclic R-modules with $tB_k^{(i)} \subset B_k^{(i-1)}$.

Since A_k has no t-torsion, the module $B_k^{(1)}$ is contained in $t^{r-1}A_k$. Applying Lemma 8.2 to the admissible splitting $t^{r-1}V$ and the modules $t^rA_k \subset B_k^{(1)} \subset t^{r-1}A_k$ and then killing the t-torsion in the resulting homology $H_k(\widetilde{V}_0^-)$ (by means of Lemma 8.1), we get an admissible splitting V_1 such that $H_s(\widetilde{V}_1^-) \approx H_s(\widetilde{V}^-)$ for $s \leq k-1$ and $H_k(\widetilde{V}_1^-) \approx B_k^{(1)}$. Applying Lemma 8.2 now to the manifold V_1 and the modules $tB_k^{(1)} \subset tB_k^{(2)} \subset B_k^{(1)}$ (and again killing the t-torsion by means of Lemma 8.1), we get an admissible splitting V_2 such that $H_s(\widetilde{V}_2^-) \approx H_s(\widetilde{V}_1^-)$ for $s \leq k-1$ and $H_k(\widetilde{V}_2^-) \approx tB_k^{(2)}$. Since the P-module A_k has no t-torsion, $tB_k^{(2)} \approx B_k^{(2)}$. We continue in a similar way and in m steps get an admissible splitting V_k satisfying the conclusion of Lemma 7.2.

PROOF OF LEMMA 8.1. 0. First we note that since P is Noetherian and $\operatorname{Tor}_t H_k(\widetilde{V}^-)$ is a finitely generated module, it suffices to construct for any $\alpha \in {}^tH_k(\widetilde{V}^-)$ a new admissible splitting V_0 satisfying requirement 1) and the requirement

$$H_k(\widetilde{V}_0^-) \approx H_k(\widetilde{V}^-)/(\alpha).$$

Indeed, suppose that $t^N \operatorname{Tor}_t H_k(\widetilde{V}^-) = 0$. Performing the construction several times, we get a manifold V' satisfying $t^{N-1}\operatorname{Tor}_t H_k(\widetilde{V}'^-) = 0$, and we end by induction on N.

1. By repeating the argument in the proof of Theorem 7.3, we find first of all that

$$H_p(\widetilde{W}, t\widetilde{V}) \approx H_p(\widetilde{V}^-, t\widetilde{V}^-) \approx \left(\bigoplus_{i=1}^{b_p} R\right) \oplus \left(\bigoplus_{j=1}^{q_p} R/a_{j,0}^{(p)}R\right), \qquad p \leq k-1. \tag{8.2}$$

Furthermore, the complex $C_*(\widetilde{W}, t\widetilde{V}) = C_*(\widetilde{V}^-, t\widetilde{V}^-)$ is homotopy equivalent to a free finitely generated R-complex

$$C_* = \left\{0 \overset{\partial_2}{\leftarrow} C_2 \leftarrow \cdots \overset{\partial_{n-2}}{\leftarrow} C_{n-2} \leftarrow 0\right\},$$

which through dimensions $\leq k-1$ is of the standard type, corresponding to the representation (8.2). This means that for $r \leq k-1$ we have $C_r = (R)^{b_p} \oplus (R)^{q_p} \oplus (R)^{q_{p-1}}$, $\partial \,|\, (R)^{b_p} = 0$, $\partial \,|\, (R)^{q_p} = 0$, and the differential $\partial \,|\, (R)^{q_{p-1}} \colon (R)^{q_{p-1}} \to (R)^{q_{p-1}} \subset C_{r-1}$ is given by a diagonal matrix with diagonal entries $a_{j,0}^{(r-1)}$. Besides, the image

$$\partial_k \colon C_k \to C_{k-1} \approx (R)^{b_{k-1}} \oplus (R)^{q_{k-1}} \oplus (R)^{q_{k-2}}$$

coincides with the submodule $(\bigoplus_{j=1}^{q_{k-1}} a_{j,0}^{(k-1)}R) \subset (R)^{q_{k-1}}$ (see Lemma 5.1). This implies that $\operatorname{Ker}\partial_k$ splits off as a direct summand, $C_k = \operatorname{Ker}\partial_k \oplus (R)^{q_{k-1}}$, and that $\partial_k | (R)^{q_{k-1}}$ is a diagonal injective operator. Having added to C_* if necessary the complex $0 \leftarrow (R)^{q_{k-1}} \leftarrow (R)^{q_{k-1}} \leftarrow 0$, located in dimensions $(k, k+1)$, we may assume that $\operatorname{Ker}\partial_k$ is a free R-module.

According to Theorem 4.1 and the remark following it we can realize C_* by a regular Morse function on the cobordism $(W; V, tV)$. This function gives rise to a handle decomposition of the pair (W, tV). Denote by Y the result of attaching to tV all the handles of indices $\leq k-1$ and those q_{k-1} handles of index k which correspond to the direct summand $(R)^{q_{k-1}} \subset C_k$.

The upper boundary of Y (which forms the result of corresponding surgical modification of tV) will be denoted by V_1 (see the picture). We have attached only the handles of indices i, where $2 \leq i \leq n-4$; hence $\pi_1(Y) \approx \pi_1(V_1) \approx \mathbf{Z}^{m-1}$ and V_1 is again an admissible splitting.

LEMMA 8.1.1. 1) *The embedding $\widetilde{V}_1^- \subset \widetilde{V}^-$ induces an isomorphism $H_s(\widetilde{V}_1^-) \to H_s(\widetilde{V}^-)$ for $s \leq k-1$.*

2) *The embedding $(t\widetilde{V})^- \subset \widetilde{V}_1^-$ induces an isomorphism $H_s(t\widetilde{V}^-) \to H_s(\widetilde{V}_1^-)$ for $s \geq k$.*

PROOF OF LEMMA 8.1.1. 1) Consider the segment of the exact sequence of the couple $(\widetilde{V}^-, \widetilde{V}_1^-)$

$$H_{s+1}(\widetilde{V}^-, \widetilde{V}_1^-) \to H_s(\widetilde{V}_1^-) \to H_s(\widetilde{V}^-) \to H_s(\widetilde{V}^-, \widetilde{V}_1^-).$$

The manifold W is obtained from Y by attaching handles of indices $\geq k$; hence for $s \leq k-1$ we have $H_s(\widetilde{W}, \widetilde{Y}) \approx H_s(\widetilde{V}^-, \widetilde{V}_1^-) = 0$, and 1) is proved for $s \leq k-2$.

Further, note that the boundary operator

$$\partial\colon H_k(\widetilde{W}, \widetilde{Y}) \to H_{k-1}(\widetilde{Y}, t\widetilde{V})$$

vanishes by construction (recall that the cellular decomposition of $(\widetilde{W}, \widetilde{Y})$ starts with k-dimensional cells having zero boundary in the cell complex of $(\widetilde{W}, t\widetilde{V})$; the k-dimensional cells with nonzero boundary are included into Y). Therefore the image of the differential $H_k(\widetilde{V}^-, \widetilde{V}_1^-) \to H_{k-1}(\widetilde{V}_1^-)$ is contained in

$$\operatorname{Im}(H_{k-1}(t\widetilde{V}^-) \to H_{k-1}(\widetilde{V}_1^-))$$

and also (for obvious reasons) in

$$\operatorname{Ker}(H_k(\widetilde{V}_1^-) \to H_{k-1}(\widetilde{V}^-)).$$

But V is $(k-1)$-regular; hence the homology $H_{k-1}(\widetilde{V}^-)$ has no t-torsion and the map $H_{k-1}(t\widetilde{V}^-) \to H_{k-1}(\widetilde{V}^-)$ has no kernel. Consequently

$$\operatorname{Im}(\partial\colon H_k(\widetilde{V}^-, \widetilde{V}_1^-) \to H_{k-1}(\widetilde{V}_1^-)) = 0,$$

and this proves 1).

2) Consider a segment of the exact sequence of the pair $(\widetilde{V}_1^-, t\widetilde{V}^-)$:

$$H_{s+1}(\widetilde{V}_1^-, t\widetilde{V}^-) \to H_s(t\widetilde{V}^-) \to H_s(\widetilde{V}_1^-) \to H_s(\widetilde{V}_1^-, t\widetilde{V}^-).$$

By construction the cell decomposition of the pair $(\widetilde{V}_1^-, t\widetilde{V}^-) = (\widetilde{Y}, t\widetilde{V})$ contains only cells of dimensions $\geq k$. Furthermore, the boundary operator in dimension k is injective; thus $H_s(\widetilde{V}_1^-, t\widetilde{V}^-) = 0$ for $s \geq k$. This implies our assertion.

2. Now we turn directly to proof of Lemma 8.1.

We consider the admissible splitting tV instead of V, and kill the element $\alpha \in {}'H_k(t\widetilde{V}^-)$, $\alpha \in \operatorname{Ker}(H_k(t\widetilde{V}^-) \to H_k(\widetilde{V}^-))$.

Consider the embedding of manifolds $V_1 \to X$, where $X = W \setminus \operatorname{Int} Y = V^- \cap V_1^+$ (see the picture). By excision, $H_*(\widetilde{X}, \widetilde{V}_1) \approx H_*(\widetilde{V}^-, \widetilde{V}_1^-)$. The first nonzero homology $H_*(\widetilde{V}^-, \widetilde{V}_1^-)$ appears in dimension k; therefore the strong Hurewicz theorem for simply-connected pairs implies that the Hurewicz map

$$H\colon \pi_{k+1}(\widetilde{X}, \widetilde{V}_1) \to H_{k+1}(\widetilde{X}, \widetilde{V}_1)$$

is surjective. Pick an element a of $H_{k+1}(\widetilde{X}, \widetilde{V}_1)$ such that $\partial a \in H_k(\widetilde{V}_1)$ is homologous in $\widetilde{V}_1^-$ to the element α, and an element A of $\pi_{k+1}(X, V_1) \approx \pi_{k+1}(\widetilde{X}, \widetilde{V}_1)$ such that $H(A) = a$.

According to Corollary 1.1 of [20] (Siebenmann's theorem), any element of $\pi_i(Q^q, P)$, where Q is a manifold and P is a component of ∂Q, can be realized by a smooth embedding of the disc $(D^i, S^{i-1}) \to (Q^q, P)$, provided $i \leq q-3$, and $\pi_j(Q^q, P) = 0$ for $j \leq 2i - q + 1$.

For our purposes we set $Q^q = X$, $P = V_1$, $q = n$, and $i = k+1$. The groups $\pi_*(X, V_1) = \pi_*(\widetilde{X}, \widetilde{V}_1)$ vanish for $* \leq k-1$. Observe now that $2i - q + 1 = 2(k+1) - n + 1 = k + (k-n) + 3 \leq k - 1$, and thus the assumptions of Siebenmann's theorem hold. Now realize the element $A \in \pi_{k+1}(X, V_1)$ by a smoothly embedded disc, consider a small tubular neighborhood of this disc, and attach it to V_1 (see the picture). Denote by V_0 the boundary of the manifold thus obtained; V_0 is the result of a surgical modification of V_1 with respect to the sphere ∂A.

The embedding $V^- \subset V_0^-$ induces an isomorphism $H_s(\widetilde{V}^-) \overset{\approx}{\to} H_s(\widetilde{V}_0^-)$ for $s \leq k-1$, and for $s = k$ we have $H_k(\widetilde{V}_0^-) = H_k(\widetilde{V}^-)/(\alpha)$ (here (α) stands for the R-submodule in $H_k(\widetilde{V}^-)$ generated by α). Recall now that for $s \leq k-1$ the homology modules $H_s(\widetilde{V}_1^-)$ are isomorphic to $H_s(\widetilde{V}^-) \approx H_s(t\widetilde{V}^-)$, and if $s = k$ then $H_k(\widetilde{V}_1^-) \approx H_k(t\widetilde{V}^-)$; consequently $H_k(\widetilde{V}_0^-) \approx H_k(t\widetilde{V}^-)/(\alpha)$, and, using step 0 of this proof, we get the desired manifold. Lemma 8.1 is proved.

Proof of Lemma 8.2. Consider the admissible splitting $V_1 \subset W$, constructed in step 1 of the proof of Lemma 8.1. We have the embeddings

$$H_k(t\widetilde{V}^-) = tA \subset B \subset A = H_k(\widetilde{V}^-);$$

here $A/tA \approx H_k(\widetilde{V}^-, t\widetilde{V}^-)$. Consider a generator m of the R-module $B/tA \subset A/tA$ and the image m' of m in the module $H_k(\widetilde{V}^-, \widetilde{V}_1^-) \approx H_k(\widetilde{X}, \widetilde{V}_1)$. We have shown in the course of proving Lemma 8.1 that $H_k(\widetilde{X}, \widetilde{V}_1)$ is a first nontrivial homology module of the pair $(\widetilde{X}, \widetilde{V}_1)$ and that it is isomorphic to $\pi_k(\widetilde{X}, \widetilde{V}_1) \approx \pi_k(X, V_1)$. By an argument similar to that in the proof of Lemma 8.1 we can realize $m' \in \pi_k(X, V_1)$ by a smoothly embedded disc (D^k, S^{k-1}). Now attach the corresponding handle μ (see the picture). Denote by V_0 the upper boundary of the manifold obtained; V_0 is the result of the surgical modification of V_1 with respect to the sphere $\partial m'$. Now we show that V_0 satisfies the conclusion of Lemma 8.2.

Observe that $\partial m' \in H_{k+1}(\widetilde{V}_1^-)$ vanishes (otherwise the map $H_{k-1}(\widetilde{V}_1^-) \to H_{k-1}(\widetilde{V}^-)$ would have a nontrivial kernel, and this is impossible since the embedding $\widetilde{V}_1^- \to \widetilde{V}^-$ induces an isomorphism in $(k-1)$-homology and $H_{k-1}(\widetilde{V}^-)$ has no t-torsion). Thus from the exact sequence of the pair $(\widetilde{V}_0^-, \widetilde{V}_1^-)$ we get that

1) $H_s(\widetilde{V}_0^-) \approx H_s(\widetilde{V}^-)$ for $s \leq k-1$, and

2) $H_k(\widetilde{V}_0^-) \approx H_k(\widetilde{V}_1^-) \oplus R(\bar{m})$, where $R(\bar{m})$ is the free module generated by an element $\bar{m}$ which is sent by the composition

$$H_k(\widetilde{V}_0^-) \to H_k(\widetilde{V}^-) \to H_k(\widetilde{V}^-, \widetilde{V}_1^-)$$

to the element m'.

Compute now the image of $H_k(\widetilde{V}_0^-)$ under the map $\widetilde{V}_0^- \to \widetilde{V}^-$. Since $H_k(t\widetilde{V}^-) \to H_k(\widetilde{V}_1^-)$ is an isomorphism, the image of $H_k(\widetilde{V}_1^-) \to H_k(\widetilde{V}_0^-)$ equals the image of $H_k(t\widetilde{V}^-)$ in $H_k(\widetilde{V}^-) = A$, i.e. the submodule $tA \subset A$.

The homology $H_k(\widetilde{V}_1^-, t\widetilde{V}^-)$ vanishes by construction (see the proof of Lemma 8.1); thus the injectivity of $H_k(\widetilde{V}^-, t\widetilde{V}^-) \to H_k(\widetilde{V}^-, \widetilde{V}_1^-)$ follows immediately from the exact sequence of the triple $(\widetilde{V}^-, \widetilde{V}_1^-, t\widetilde{V}^-)$. Therefore the projection of $H_k(\widetilde{V}^-) \approx A$ onto $H_k(\widetilde{V}^-, t\widetilde{V}^-) \approx A/tA$ sends the image of $\bar{m}$ in $H_k(\widetilde{V}^-)$ to the element $m \in A/tA$.

Therefore the image of $H_k(\widetilde{V}_0^-)$ in $A = H_k(\widetilde{V}^-)$ equals $tA + (m) = B$.

Now we compute the kernel of $H_k(\widetilde{V}_0^-) \to A$. If $x \in H_k(\widetilde{V}_0^-)$ goes to zero via the map $\widetilde{V}_0^- \to \widetilde{V}^-$, then $tx \in H_k(\widetilde{V}_0^-)$ goes to zero via the map $t\widetilde{V}_0^- \to t\widetilde{V}^- \hookrightarrow \widetilde{V}_0^-$; that means $tx = 0$. On the other hand, A lacks t-torsion, and hence

$$\operatorname{Tor}_t H_k(\widetilde{V}_0^-) \subset \operatorname{Ker}(H_k(\widetilde{V}_0^-) \to H_k(\widetilde{V}^-)) \subset {}^t H_k(\widetilde{V}_0^-).$$

Lemma 8.2 is proved.

§9. The surgery in the remaining dimensions

The proof of Lemma 7.4 will be split into several lemmas.

LEMMA 9.1. *Let M^n be a manifold, with $\pi_1 M^n = \mathbf{Z}^m$. Suppose that a class $\gamma \in H^1(M^n, \mathbf{Z})$ is represented by an epimorphism $\mathbf{Z}^m \to \mathbf{Z}$, and that $S_\gamma^{-1} H_q(\widetilde{M}^n, \mathbf{Z}) = 0$.*

Then for any admissible splitting V the module $H_q(\widetilde{M}^n, \widetilde{V}^-)$ is a finitely generated R-module (in the notation (5.1)) isomorphic to $\operatorname{Ker}(H_{q-1}(\widetilde{V}^-) \to H_{q-1}(\widetilde{M}))$. If V is $(q-1)$-regular, then $H_q(\widetilde{M}^n, \widetilde{V}^-) = 0$.

PROOF OF LEMMA 9.1. The condition $S^{-1} H_q(\widetilde{M}^n) = 0$ means that every cohomology class $x \in H_q(\widetilde{M}^n)$ is annihilated by some polynomial $1 + tQ(t)$, $Q(t) \in R[t] = \mathbf{Z}[\mathbf{Z}^{m-1}][t]$; hence $x = -tQ(t)x$.

Hence $H_q(\widetilde{V}^-) \to H_q(\widetilde{M}^n)$ is an epimorphism. Furthermore,

$$\operatorname{Ker}(H_{q-1}(\widetilde{V}^-) \to H_{q-1}(\widetilde{M})) \approx \operatorname{Tor}_t H_{q-1}(\widetilde{V}^-),$$

and since $H_{q-1}(\widetilde{V}^-)$ is finitely generated over P, $\operatorname{Tor}_t H_q(\widetilde{V}^-)$ is finitely generated over R. Now the lemma follows directly from the exact sequence of the pair $(\widetilde{M}^n, \widetilde{V}^-)$.

For an $(r-1)$-regular splitting V the map $H_{r-1}(\widetilde{V}^-) \to H_{r-1}(\widetilde{M})$ is injective; hence $H_r(\widetilde{M}, \widetilde{V}^-) = 0$ and we get part 1) of Lemma 7.4.

LEMMA 9.2. *Suppose that M^n is a manifold, $\pi_1 M^n = \mathbf{Z}^m$, $\gamma \in H^1(M, \mathbf{Z})$ is an epimorphism $\mathbf{Z}^m \to \mathbf{Z}$, the modules $S^{-1}H_p(\widetilde{M}^n)$ satisfy* (7.1) *for $p \le k$, and (V, ν) is k-regular. Then, for all natural q and l,*

1) *the R-modules $H_p(\widetilde{V}^-, t^q\widetilde{V}^-)$ have resolutions of length 2 for $p \le k$, and*
2) *the homomorphism*

$$H^p(\widetilde{V}^-, t^{q+1}\widetilde{V}^-) \to H^p(t^q\widetilde{V}^-, t^{q+1}\widetilde{V}^-),$$

induced by embedding of the pairs is an epimorphism for $p \le k$. [6]

PROOF. 1) Compute first the homology $H_p(\widetilde{V}^-, t^p\widetilde{V}^-)$. Since V is k-regular, the embedding $t^q\widetilde{V}^- \subset \widetilde{V}^-$ induces in p-homology a monomorphism, which equals $t^q\colon H_p(\widetilde{V}^-) \to H_p(\widetilde{V}^-)$. Hence

$$H_p(\widetilde{V}^-, t^q\widetilde{V}^-) \approx H_p(\widetilde{V}^-)/t^q H_p(\widetilde{V}^-).$$

Since (7.2) holds for $H_p(\widetilde{V}^-)$, we can apply Lemma 5.4 to compute the above factor. We get

$$\begin{aligned} H_p(\widetilde{V}^-, t^q\widetilde{V}^-) &\approx \left(\bigoplus_{i=1}^{b_p} P/t^qP\right) \oplus \left(\bigoplus_{j=1}^{q_p} P/(a_j^{(p)}, t^q)P\right) \\ &\approx \left(\bigoplus_{i=1}^{b_p} R^q\right) \oplus \left(\bigoplus_{j=1}^{q_p} R^q/F_j^{(p)}(R^q)\right) \end{aligned} \tag{9.1}$$

where $F_j^{(p)}\colon R^q \to R^q$ is a monomorphism given by the matrix

$$\begin{pmatrix} a_{j,0}^{(p)}, & a_{j,1}^{(p)}, & \dots, & a_{j,q}^{(p)} \\ & a_{j,0}^{(p)}, & \dots, & a_{j,q-1}^{(p)} \\ & & \dots & \dots \\ 0 & & & a_{j,0}^{(p)} \end{pmatrix}.$$

Assertion 1) is proved.

Next we note that for a free finitely generated complex C_*, such that for $p \le k$ the module $H_n(C_*)$ has a free resolution of length 2, the spectral sequence

$$E_2^{p,s} = \operatorname{Ext}^p(H_s(C_*), R) \Rightarrow H_{p+s}(\operatorname{Hom}_R(C_*, R)) = H^{p+s}(C_*, R)$$

degenerates in E_2 for $p \le k$, and there exists a functional exact sequence

$$0 \to \operatorname{Ext}^1(H_{p-1}(C_*), R) \to H^p(C_*, R) \to \operatorname{Hom}(H_p(C_*), R) \to 0.$$

The embedding $(t^q\widetilde{V}^-, t^{q+l}\widetilde{V}^-) \subset (\widetilde{V}^-, t^{q+l}\widetilde{V}^-)$ induces a homomorphism of these exact sequences which is an epimorphism on the left (by virtue of Lemma 5.4.3) and an epimorphism on the right (obviously); hence the middle map

$$H^p(\widetilde{V}^-, t^{q+l}\widetilde{V}^-) \to H^p(t^q\widetilde{V}^-, t^{q+l}\widetilde{V}^-)$$

is also an epimorphism. Lemma 9.2 is proved.

[6] We mean the cohomology of corresponding universal coverings with coefficients in the $R = \mathbf{Z}[\pi_1 V^-]$-module R, or equivalently the cohomology with compact supports.

LEMMA 9.3. *Let* $\pi_1 M^n = \mathbf{Z}^m$, $n \geq 6$, *let* $\lambda \in H^1(M, \mathbf{Z})$ *be an epimorph* $\mathbf{Z}^m \to \mathbf{Z}$, $1 \leq r \leq n-2$, $m_r = m_{r+1} = 0$, *and let* (V, ν) *be an* $(r-1)$*-bireg splitting. Then there exists a natural number* q_0 *such that for* $q > q_0$ *the embedd* $(\widetilde{V}^-, t^q\widetilde{V}^-) \to (\widetilde{M}, t^q\widetilde{V}^-)$ *induces an isomorphism*

$$H_{r+1}(\widetilde{V}^-, t^q\widetilde{V}^-) \overset{\approx}{\to} H_{r+1}(\widetilde{M}, t^q\widetilde{V}^-),$$

and the differential of the exact sequence of the triple $(\widetilde{M}, \widetilde{V}^-, t^q\widetilde{V}^-)$ *induces isomorphism*

$$H_{r+1}(\widetilde{M}, \widetilde{V}^-) \approx H_r(\widetilde{V}^-, t^q\widetilde{V}^-).$$

PROOF. Consider the exact sequence of the triple $(\widetilde{M}, \widetilde{V}^-, t^q\widetilde{V}^-)$:

$$\to H_{r+2}(\widetilde{M}, t^q\widetilde{V}^-) \to H_{r+2}(\widetilde{M}, \widetilde{V}^-) \overset{\partial_{r+1}}{\to} H_{r+1}(\widetilde{V}^-, t^q\widetilde{V}^-)$$
$$\to H_{r+1}(\widetilde{M}, t^q\widetilde{V}^-) \to H_{r+1}(\widetilde{M}, \widetilde{V}^-) \overset{\partial_r}{\to} H_r(\widetilde{V}^-, t^q\widetilde{V}^-) \to H_r(\widetilde{M}, t^q\widetilde{V}^-).$$

According to Lemma 9.1, $H_r(\widetilde{M}, t^q\widetilde{V}^-) = 0$ and $H_{r+1}(\widetilde{M}, \widetilde{V}^-) \approx H_{r+1}(\widetilde{M}, t^q\widetilde{V}$ is finitely generated over R; hence for q sufficiently large the embedding $(\widetilde{M}, t^q\widetilde{V}$ $\subset (\widetilde{M}, \widetilde{V}^-)$ induces the zero map in $(r+1)$-homology. That implies the seco statement of the lemma.

To prove the first it suffices to verify the surjectivity of the map $H_{r+2}(\widetilde{M}, t^q\widetilde{V}^-)$ $H_{r+2}(\widetilde{M}, \widetilde{V}^-)$. For this it suffices in turn to verify the surjectivity of the m $H_{r+2}(t^{-l}\widetilde{V}^-, t^q\widetilde{V}^-) \to H_{r+2}(t^{-l}\widetilde{V}^-, \widetilde{V}^-)$ for all natural l and q.

Consider now the manifold $Z = t^qV^+ \cap t^{-l}V^-$ with the boundary $t^qV \cup t^{-l}$ the manifold $Z_0 = V^+ \cap t^{-l}V^-$ with the boundary $V \cup t^{-l}V$, and the embedding pairs $(Z, t^qV) \subset (Z, V^- \cap t^qV^+)$. This embedding is a map of degree 1, i.e. it sen the infinite fundamental cycle $U_Z \in H_n^{\inf}(\widetilde{Z}, t^q\widetilde{V} \cup t^{-1}V)$ to the infinite fundament cycle $U_{Z_0} \in H_n^{\inf}(\widetilde{Z}_0, \widetilde{V} \cup t^{-l}\widetilde{V})$. Thus we obtain a commutative diagram

$$\begin{array}{ccc} H_{r+2}(t^{-l}\widetilde{V}^-, t^q\widetilde{V}^-) & \xleftarrow{\cap U_Z} & H_c^{n-r-2}(t^q\widetilde{V}^+, t^{-l}\widetilde{V}^+) \\ \downarrow & & \downarrow \\ H_{r+2}(t^{-l}\widetilde{V}^-, \widetilde{V}^-) & \xleftarrow{\cap U_{Z_0}} & H_c^{n-r-2}(\widetilde{V}^+, t^{-l}\widetilde{V}^+). \end{array}$$

The horizontal arrows are isomorphisms by Poincaré duality, and it suffices to sho that the right-hand arrow is an epimorphism. Since (V, ν) is $(r-1)$-biregula $(V, -\nu)$ is $(n-r-2)$-regular and we deduce the required assertion from part 2 o Lemma 9.2.

LEMMA 9.4. *Under the assumptions of Lemma* 9.3, *for any* q *there exists a Mors function* φ *on the cobordism* $(V^- \cap t^qV^+, t^qV)$ *such that the differentials* $\partial_r : C_r$ – C_{r-1} *and* $\partial_{r+2}: C_{r+2} \to C_{r+1}$ *of the corresponding complex* $C_*(\varphi) = \{0 \leftarrow C_2 \leftarrow \cdots \leftarrow C_{n-2} \leftarrow 0\}$ *vanish.*

PROOF. We proceed similarly to the proof of Lemma 8.1. Denote by W_q the manifold $V^- \cap t^qV^+$ with boundary $V \cup t^qV$. By Lemma 9.2 the R-modules $H_p(\widetilde{W}_q, t^q\widetilde{V})$ have resolutions of length 2 for $p \leq r-1$. Represent the complex

$C_*(\widetilde{W}_q, t^q\widetilde{V})$ up to homotopy by a standard one C_*^0 (Lemma 5.1) and realize C_*^0 as a Morse complex of a regular function f on the cobordism W_q. Denote by V_q a level surface $f^{-1}(c)$ separating the critical points of indices $\leq r-1$ from critical points of indices $\geq r$. V_q is the result of a surgical modification of indices $\leq r-1$ of the manifold t^qV. Set $Y_q = V_q^- \cap t^qV^+$ and $X_q = V_q^+ \cap V^-$ (the notation is similar to that of Lemma 8.1; after necessary corrections the picture illustrates the present situation as well). The Morse function $(-f)$ gives rise to a handle decomposition of the pair (W_q, V), and X_q is precisely the result of attaching all the handles of indices $\leq n-r$ to V. The Morse complex of $(-f)|(X_q, V)$ looks like $\{0 \leftarrow D_2 \leftarrow \cdots \leftarrow D_{n-r} \leftarrow 0\}$. Furthermore, the embedding $(X_q, V) \to (W_q, V)$ induces an isomorphism in the homology $H_s(\widetilde{X}_q, \widetilde{V}) \to H_s(\widetilde{W}_q, \widetilde{V})$ for $s \leq n-r$. Now apply Lemma 7.2 to the $(n-r-2)$-regular splitting $(V, -\nu)$ to get a Morse function $\tilde{f}$ on the cobordism (X_q, V) such that its Morse complex $\{0 \leftarrow \widetilde{D}_2 \leftarrow \cdots \leftarrow \widetilde{D}_{n-r} \leftarrow 0\}$ is standard in dimensions $\leq n-r-2$. Since $q_{n-r-2} = q_{r+1} = 0$, the differential $\partial_{n-r-1}: \widetilde{D}_{n-r-1} \to \widetilde{D}_{n-r-2}$ vanishes.

Now define a Morse function φ by setting $\varphi \mid Y_q = f$ and $\varphi \mid X_q = -f$ (V_q is a level surface for f and for $\tilde{f}$, and we may assume that $f \mid V_q = -\tilde{f} \mid V_q$ and that in a neighborhood of V_q both f and $(-\tilde{f})$ coincide with the coordinate normal to V_q, so that the definition makes sense).

The Morse complex of this function coincides with $C_*(f)$ for $* \leq r-1$ and with $C_*(-\tilde{f})$ for $* \geq r+1$. Hence the differential ∂_{n-r-1} vanishes. By construction

$$C_{r-1}(f) = R^{b_{r-1}\cdot q} \oplus B,$$

where $\partial \mid R^{b_{r-1}\cdot q} = 0$, $\partial \mid B$ is injective, and

$$R^{b_{r-1}\cdot q} \to H_{r-1}(C_*(f)) \approx H_{r-1}(\widetilde{W}_q, t^q\widetilde{V})$$

is an isomorphism. (Here we use $q_{r-1} = 0$.) Therefore the differential ∂_r must vanish. Lemma 9.4 is proved.

Proof of Lemma 7.4. We still need to prove part 2 of the statement (part 1 was proved in Lemma 9.1). For this we construct a free (maybe infinitely generated) $\mathbf{Z}[\mathbf{Z}^{m-1}]$-complex D_* whose homology is isomorphic to $H_*(\widetilde{M}, \widetilde{V}^-)$ and whose differentials ∂_r and ∂_{r+2} vanish. From this the lemma follows easily (similarly to [10], Lemma 3.5). Indeed, $H_r(D_*) = H_r(\widetilde{M}, \widetilde{V}^-) = 0$ implies $H_{r+1}(\widetilde{M}, \widetilde{V}^-) \oplus D_r = D_{r+1}$, i.e. $H_{r+1}(\widetilde{M}, \widetilde{V}^-)$ is projective. By Lemma 9.3 this module is finitely generated, hence (by the Suslin-Quillen theorem) free.

To construct D_* we pick a number $N > q_0$ and consider the embeddings

$$(t^{-N}V^-, V^-) \subset (t^{-2N}V^-, V^-) \subset \cdots. \tag{9.2}$$

The union of this sequence is $(\widetilde{M}, \widetilde{V}^-)$; the direct limit commutes with homology, and we get

$$\varinjlim H_*(t^{-N}\widetilde{V}^-, \widetilde{V}^-) = H_*(\widetilde{M}, \widetilde{V}^-).$$

Choose a triangulation of these pairs in such a way that every pair is a subcomplex of the next one. The quotient complex of the kth pair by the $(k-1)$th is $(t^{-2kN}V^-, t^{-2(k-1)N}V^-)$. We have proved in Lemma 9.3 that the differential maps

$H_{r+1}(\widetilde{M}, \widetilde{V}^-)$ isomorphically onto $H_r(\widetilde{V}^-, t^N\widetilde{V}^-)$; hence the differential $\partial_r: H_r(\widetilde{V}^-, t^N\widetilde{V}^-) \to H_{r-1}(t^N\widetilde{V}^-)$ vanishes. In the same lemma we proved that $H_{r+2}(t^{-2kN}\widetilde{V}^-, \widetilde{V}^-)$ is mapped surjectively onto $H_{r+2}(t^{-2kN}\widetilde{V}^-, t^{-2(k-1)N}\widetilde{V}^-)$. This implies that the differentials ∂_r and ∂_{r+2} of the exact sequence of the triple $(t^{-2kN}\widetilde{V}^-, t^{-2(k-1)N}\widetilde{V}^-, \widetilde{V}^-)$ vanish.

We need the following purely algebraic fact.

LEMMA 9.5. *Let $X_* \subset Y_*$ be free finitely generated complexes such that the X_n are direct summands of the Y_n. Suppose that X_* and $Z_* = Y_*/X_*$ are homotopy equivalent to the free finitely generated complexes X'_* and Z'_*, and such that the rth and $(r+2)$th differentials of X'_* and Z'_* vanish. Suppose further that the differentials*

$$\delta_r: H_r(Z_*) \to H_{r-1}(X_*), \qquad \delta_{r+2}: H_{r+2}(Z_*) \to H_{r+1}(X_*)$$

in the exact sequence of the pair (Y_*, X_*) vanish.

Then there exists a pair of free finitely generated complexes $\widetilde{X}_* \subset \widetilde{Y}_*$, homotopy equivalent to the pair $X_* \subset Y_*$, and such that the modules X_* are direct summands in Y_*, and the differentials ∂_r and ∂_{r+2} of the complex $\widetilde{Y}_*$ vanish.

PROOF OF LEMMA 9.5. According to Cockcroft and Swan [15] the complexes X_* and X'_* (as well as Z_* and Z'_*) can be made isomorphic by adding to them several complexes of the type $0 \leftarrow F \overset{\mathrm{id}}{\leftarrow} F \leftarrow 0$, where F is some free module. From this we easily deduce that there exists a pair of complexes $X'_* \subset Y'_*$ (the modules X'_n being direct summands of the Y'_n) such that $Y'_*/X'_* \approx Z'_*$. Consider now the free generators z_i of the module Z'_r. Our assumptions imply $\partial z_i \in X'_{r-1}$; furthermore, these elements are homologous to zero in this complex, hence they are zero itself (since $\partial_r \mid X'_r = 0$). The same argument proves that ∂_{r+2} vanishes.

Return now to the sequence (9.2). According to Lemma 9.5 there exists a sequence $X_*^{(1)} \subset X_*^{(2)} \subset \cdots$, where the $X_*^{(i)}$ are free finitely generated complexes, $X_n^{(i)}$ being a direct summand in $X_n^{(i+1)}$, and the homotopy equivalences

$$\begin{array}{ccc} C_*(t^{-N}V^-, V^-) & \subset & C_*(t^{-2N}V^-, V^-) \subset \cdots \\ \wr & & \wr \\ X_*^{(1)} & \subset & X_*^{(2)} \subset \cdots . \end{array}$$

Now we set $D_* = \bigcup_i X_*^{(i)} = \varinjlim X_*^{(i)}$, and Lemma 7.4 is proved.

PROOF OF LEMMA 7.5. This proof will occupy the rest of §9. We obtain it by reproducing the argument of [10] in our setting. We exhibit the argument here, elaborating on the parts which need modification.

Denote by $\mathscr{C}(R)$ the category formed by pairs (F, f) where F is a free finitely generated R-module and f is a nilpotent endomorphism of F. (As always, $R = \mathbf{Z}[\mathbf{Z}^{m-1}]$. The definition is somewhat simplified in comparison with [10]: we omit the automorphism α since fundamental groups are abelian, and we use the free modules since all R-projectives are R-free.) Denote by $C(R)$ the set of equivalence classes of isomorphism classes of objects from $\mathscr{C}(R)$ with respect to equivalence relation, generated by two relations:

1) $(F, f) \sim (F \oplus F', f \oplus 0)$;

2) if the sequence $0 \to (F_2, f_2) \to (F_1, f_1) \to (F_0, f_0) \to 0$ is exact, then $(F_1, f_1) \sim (F_2, f_2) \oplus (F_0, f_0)$.

It is proved in [10] that $C(R) = 0$.

Suppose now that the assumptions of Lemma 7.4 hold. Then the module $H_{r+1}(\widetilde{M}, \widetilde{V}^-)$ together with the endomorphism t gives rise to an object of $\mathscr{C}(R)$, which we denote by $c(V, \nu)$. It vanishes when we pass to $C(R)$. We will now prove (following [10]) that this equivalence to zero can be realized geometrically, so that we can construct an $(r-1)$-biregular splitting (V', ν') with $c(V', \nu') = 0$ (i.e. satisfying the conclusion of Lemma 7.5).

The ring R is a subring of $\mathbf{Z}[\pi_1 M]$, stable with respect to the automorphism χ (see §3). Denote by $\overline{\mathrm{Hom}}(M, N)$ the set of all χ-homomorphisms of the R-module N into the R-module M. For any $\mathscr{C}(R)$-object $X = (F, f)$ we define the dual $X^* = (\mathrm{Hom}(F, R), \pm f^*)$, where we choose $(+)$ if $t \in \pi_1 M$ preserves the orientation and $(-)$ if not.

LEMMA 9.6. *Under the assumptions of Lemma* 7.4,

$$c(V, \nu)^* = c(V, -\nu).$$

PROOF. The manifold $(V, -\nu)$ is an $(n-r-2)$-regular splitting, and

$$c(V, -\nu) = (H_{n-r}(\widetilde{M}, \widetilde{V}^+), t^{-1}).$$

We know from Lemma 9.3 that $H_{r+1}(\widetilde{M}, \widetilde{V}^-) \approx H_r(\widetilde{V}^-, t^q \widetilde{V}^-)$ for q sufficiently large; hence

$$c(V, \nu) \approx (H_r(\widetilde{V}^-, t^q \widetilde{V}^-), t).$$

The same lemma implies that

$$c(V, -\nu) \approx (H_{n-r}(t^q \widetilde{V}^+, \widetilde{V}^+), t^{-1})$$

for q sufficiently large. Poincaré duality implies that there is a χ-isomorphism

$$(H_c^r(\widetilde{V}^-, t^q \widetilde{V}^-), \pm t) \approx (H_{n-r}(t^q \widetilde{V}^+, \widetilde{V}^+), t)$$

(the sign $(+)$ appears if t is orientation preserving; otherwise $(-)$ appears). Lemma 9.4 implies that we can choose a cell decomposition of a pair $(\widetilde{V}^-, t^q \widetilde{V}^-)$ such that in the resulting chain complex $C_*(\widetilde{V}^-, t^q \widetilde{V}^-)$ the differential ∂_{r+2} vanishes. Moreover, Lemma 9.3 and part 2 of Lemma 7.4 imply that $H_r(\widetilde{V}^-, t^q \widetilde{V}^-)$ is a free module. Hence we obtain

$$(H_c^r(\widetilde{V}^-, t^q \widetilde{V}^-), \pm t) \approx (\mathrm{Hom}(H_r(\widetilde{V}^-, t^q \widetilde{V}^-), R), \pm t)$$

and since $\mathrm{Hom}(M, R)$ is χ-isomorphic to $\overline{\mathrm{Hom}}(M, R)$, the lemma follows.

A triangular object of $\mathscr{C}(R)$ is by definition an object (F, f) together with a filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_n = F$ such that all the factors F_{i+1}/F_i are free modules of rank 1 and $f(F_{i+1}) \subset F_i$.

The basic role in realizing the relations geometrically is played by the following lemma.

LEMMA 9.7. *Suppose that all the assumptions of Lemma* 7.5 *hold and* $c(V, \nu) = (P, f)$. *Suppose given an exact sequence*

$$0 \to (Q, \psi) \to (F, \varphi) \xrightarrow{p} (P, f) \to 0$$

of the objects of $\mathscr{C}(R)$, *where* (F, φ) *is a triangular object.*

Then there exists an r-biregular splitting (V_0, ν_0), *such that*

$$c(V_0, \nu_0) = (Q, \psi).$$

PROOF. Denote by P_i the image of F_i in P (where $\{F_i\}$ is the filtration of F mentioned above)

LEMMA 9.8. *Let $i \leq n$. There exists an admissible splitting (V_i, ν_i), obtaine from (V, ν) by a sequence of surgical modifications of indices $\leq r-1$ and of inde $(r+1)$ (all the modifications take place inside V^+), such that $H_s(\widetilde{V}_i^-) \approx H_s(\widetilde{V}^-)$ fo $s \leq r-1$ and the map $H_{r+1}(\widetilde{V}_i^-, \widetilde{V}^-) \to H_{r+1}(\widetilde{M}, \widetilde{V}^-)$ is isomorphic to the ma $p_i = p \mid F_i\colon F_i \to P$.*

First we deduce Lemma 9.7 from Lemma 9.8.

Set $i = n$. We show that (V_n, ν_n) is an r-biregular splitting. Consider the exac sequence of the triple $(\widetilde{M}, \widetilde{V}_n^-, \widetilde{V}^-)$:

$$H_{r+1}(\widetilde{V}_n^-, \widetilde{V}^-) \to H_{r+1}(\widetilde{M}, \widetilde{V}^-) \to H_{r+1}(\widetilde{M}, \widetilde{V}_n^-) \to H_r(\widetilde{V}_n^-, \widetilde{V}^-).$$

The left arrow is the epimorphism $F_i \to P_i$; the right group vanishes. Therefore $H_{r+1}(\widetilde{M}, \widetilde{V}_n^-) = 0$. Since V_n is $(r-1)$-regular and $m_r = 0$, we deduce from Lemm 9.1 that $H_r(\widetilde{M}, \widetilde{V}_n^-) = 0$. Hence $H_r(\widetilde{V}_n^-, t\widetilde{V}_n^-) = 0$, and, consequently, V_n is r-regular. Further, $\widetilde{V}_n^-$ is obtained from $\widetilde{V}^-$ by attaching handles of indices $r+1$ hence the homology of $\widetilde{V}^+$ did not change through the dimensions $\leq n-r-3$, and consequently, V_n is r biregular. Furthermore, $m_{r+2} = 0$, and Lemma 9.1 implies that the R-module $H_{r+2}(\widetilde{M}, \widetilde{V}^-)$ is finitely generated, which implies that for sufficiently large N there exists an epimorphism $H_{r+2}(t^{-N}\widetilde{V}^-, \widetilde{V}^-) \to H_{r+2}(\widetilde{M}, \widetilde{V}^-)$. But $(V, -\nu)$ is $(n-r-2)$-regular and $m_{n-r-2}(-\lambda) = 0$; hence (by Lemma 9.2) $H_{r+2}(t^{-N}\widetilde{V}^-, \widetilde{V}^-) = 0$. Now our conclusion follows from the exact sequence

$$H_{r+2}(\widetilde{M}, \widetilde{V}^-) \to H_{r+2}(\widetilde{M}, \widetilde{V}_n^-) \to H_{r+1}(\widetilde{V}_n^-, \widetilde{V}^-) \to H_{r+1}(\widetilde{M}, \widetilde{V}^-).$$

PROOF OF LEMMA 9.8 (Induction on i). Suppose that we have constructed an admissible splitting V_i with the required properties. We identify F_i with $H_{r+1}(\widetilde{V}_i^-, \widetilde{V}^-)$, the homomorphism $H_{r+1}(\widetilde{V}_i^-, \widetilde{V}^-) \to H_{r+1}(\widetilde{M}, \widetilde{V}^-)$ induced by embedding of the pairs with $p \mid F_i$, and the image $p(H_{r+1}(\widetilde{V}_i^-, \widetilde{V}^-)) \subset H_{r+1}(\widetilde{M}, \widetilde{V}^-)$ with P_i. Choose an element $e_{i+1} \in F_{i+1}$ such that $F_{i+1} = F_i \oplus R(e_{i+1})$. Denote te_{i+1} by $x \in H_{r+1}(\widetilde{V}_i^-, \widetilde{V}^-)$ and $p(e_{i+1})$ by $y \in H_{r+1}(\widetilde{M}, \widetilde{V}^-)$. Note that $p(x) = ty$.

We prove first that there is an element $\sigma \in H_{r+1}(t^{-1}\widetilde{V}_i^-, \widetilde{V}^-)$ such that $t\sigma \in H_{r+1}(\widetilde{V}_i^-, \widetilde{V}^-)$ equals x and its image $j_*\sigma$ in $H_{r+1}(\widetilde{M}, \widetilde{V}^-)$ equals y.

Indeed, choose chains $\bar{y} \in C_{r+1}(\widetilde{M})$ and $\bar{x} \in C_{r+1}(\widetilde{V}_i^-)$ which represent y and x. The chain $t\bar{y}$ is homologous to $p(\bar{x})$ modulo $\widetilde{V}^-$, i.e. $t\bar{y} = p(\bar{x}) + v + \partial u$, where $v \in C_{r+1})(\widetilde{V}^-)$ and $u \in C_{r+1}(\widetilde{M})$.

The chain $\bar{\sigma} = t^{-1}(\bar{x} + v)$ is a relative cycle in $C_{r+1}(t^{-1}\widetilde{V}_i^-, \widetilde{V}^-)$, and the homology class σ of $\bar{\sigma}$ satisfies the requirement.

Consider now the admissible splitting $t^{-1}V_i$ and apply to it the procedure described in step 1 of the proof of Lemma 8.1 (where $k = r$). We get the manifold $V_1 \subset t^{-1}V_i^-$.

Note that V_1^- is obtained from V^- by attaching handles of indices $\leq r-1$, since $m_r = 0$. The spaces $\widetilde{V}_1^-$ and $t^{-1}\widetilde{V}_1^-$ are simply connected, and the first nonzero homology group of $(t^{-1}\widetilde{V}_i^-, \widetilde{V}_1^-)$ sits in dimension r. Therefore the Hurewicz homomorphism

$$\pi_{r+1}(t^{-1}V_i^-, V_1^-) = \pi_{r+1}(t^{-1}\widetilde{V}_i^-, \widetilde{V}_1^-) \xrightarrow{H} H_{r+1}(t^{-1}\widetilde{V}_i^-, \widetilde{V}_1^-)$$

is surjective. Consider the element $\sigma' \in \pi_{r+1}(t^{-1}V_i^-, V_1^-)$ such that $H(\sigma')$ equals σ, reduced modulo $\widetilde{V}_1^-$. By the same argument as in the proof of Lemma 8.1 we realize σ' by the embedded disc $D = (D^{r+1}, S^r)$ and attached to V_1^- a small tubular neighborhood of this disc. The upper boundary of the manifold thus obtained will be denoted by V_{i+1}. We claim that this manifold satisfies the conclusion of Lemma 9.8 for the number $(i+1)$.

Indeed, if $s \leq r-1$, then

$$H_s(\widetilde{V}_{i+1}^-) \approx H_s(\widetilde{V}_1^-) \approx H_s(t^{-1}\widetilde{V}_i^-) \approx H_s(\widetilde{V}_i^-).$$

Note further that the R-module $H_{r+1}(\widetilde{V}_{i+1}^-, \widetilde{V}^-)$ contains an element S' whose image in $H_{r+1}(\widetilde{V}_{i+1}^-, \widetilde{V}_1^-)$ equals $H(\sigma')$ and whose image in $H_{r+1}(t^{-1}\widetilde{V}_i^-, \widetilde{V}^-)$ equals σ. Indeed, the chains $D \in Z_{r+1}(t^{-1}\widetilde{V}^-, \widetilde{V}_1^-)$ and $\bar{\sigma} \in Z_{r+1}(t^{-1}\widetilde{V}_1^-, \widetilde{V}^-)$ are homologous modulo $\widetilde{V}_1^-$, i.e. $D = \bar{\sigma} + R + \partial u$, where $R \in C_{r+1}(\widetilde{V}_1^-)$. Now set $S' = D - R$.

Consider the exact sequence of the triple $(\widetilde{V}_{i+1}^-, \widetilde{V}_1^-, \widetilde{V}^-)$:

$$\begin{aligned} \cdots \to H_{r+2}(\widetilde{V}_{i+1}^-, \widetilde{V}_1^-) \to H_{r+1}(\widetilde{V}_1^-, \widetilde{V}^-) \to H_{r+1}(\widetilde{V}_{i+1}^-, \widetilde{V}^-) \\ \xrightarrow{p} H_{r+1}(\widetilde{V}_{i+1}^-, \widetilde{V}_1^-) \to H_r(\widetilde{V}_1^-, \widetilde{V}^-) \to \cdots \\ \wr \qquad\qquad\qquad \\ R(\sigma) \qquad\qquad\qquad \end{aligned}$$

The first modules on the right and on the left vanish. Thus

$$H_{r+1}(\widetilde{V}_{i+1}^-, \widetilde{V}^-) \approx H_{r+1}(\widetilde{V}_1^-, \widetilde{V}^-) \oplus R(\sigma) \approx H_{r+1}(\widetilde{V}_i^-, \widetilde{V}^-) \oplus R(\sigma).$$

Now we extend the identification $F_i = H_{r+1}(\widetilde{V}_1^-, \widetilde{V}^-)$ to an isomorphism of R-modules $\varphi: F_{i+1} \to H_{r+1}(\widetilde{V}_{i+1}^-, \widetilde{V}^-)$ by setting $\varphi(e_{i+1}) = S$. Since $tS = x$ in $H_{r+1}(\widetilde{V}_i^-, \widetilde{V}^-) = F_i$, the map φ commutes with the action of t; since the cycle S is homologous to y modulo $\widetilde{V}^-$, the composition of φ and the map $H_{r+1}(\widetilde{V}_{i+1}^-, \widetilde{V}^-) \to H_{r+1}(\widetilde{M}, \widetilde{V}^-)$ equals $p_{i+1}: F_{i+1} \to P_{i+1}$.

The induction step is over.

Now we show how to realize relation 2) of the definition of $C(R)$ geometrically. More precisely, suppose that the assumptions of Lemma 7.5 hold. Let (V, ν) be an $(r-1)$-biregular splitting such that $c(V, \nu) = (F_1, f_1)$. Let

$$0 \to (F_2, f_2) \to (F_1, f_1) \to (F_0, f_0) \to 0$$

be an exact sequence of objects from $\mathscr{C}(R)$. We show that there exists an $(r-1)$-biregular splitting (V', ν') such that $c(V', \nu') = (F_2 \oplus F_0, f_2 \oplus f_0)$.

Let

$$0 \to (P_0, \varphi_0) \to (Q_0, \psi_0) \to (F_0, f_0) \to 0,$$
$$0 \to (P_2, \varphi_2) \to (Q_2, \psi_2) \to (F_2, f_2) \to 0$$

be the exact sequences of objects from $\mathscr{C}(R)$, where (Q_0, ψ_0) and (Q_1, ψ_1) triangular (such sequences exist; see [10], Lemma 1.2). Since F_0 and F_2 are f we have that $F_1 = F_0 \oplus F_2$ and f_1 is given in this representation by the matrix

$$\begin{pmatrix} f_0 & g \\ 0 & f_2 \end{pmatrix},$$

where g is a homomorphism $F_2 \to F_0$. This enables us to construct the follow exact sequences of objects of $\mathscr{C}(R)$:

$$0 \to (P_2 \oplus P_0, \varphi_2 \oplus \varphi_0) \to (Q_2 \oplus Q_0, \psi_2 \oplus \psi_0) \to (F_2 \oplus F_0, f_2 \oplus f_0) \to 0,$$
$$0 \to (P_2 \oplus P_0, \varphi_2 \oplus \varphi_0) \to (Q_2 \oplus Q_0, \gamma) \to (F_2 \oplus F_0, f_1) \to 0; \quad (9$$

where the middle objects are both triangular. The details (in slightly different nc tion) can be found in [10], p. 338. Next we apply Lemma 9.7 and find an r-biregu manifold (V_0, ν_0) such that $c(V_0, \nu_0) = (P_2 \oplus P_0, \varphi_2 \oplus \varphi_0)$. The manifold $(V_0, -$ is an $(n-r-3)$-biregular splitting, and

$$c(V_0, -\nu_0) = (P_2^* \oplus P_0^*, \pm(\varphi_2 \oplus \varphi_0)^*)$$

(by Lemma 9.6). Since $r \geq 2$ and $(n-r-3)+1 \leq n-4$, we can apply Lem 9.7 again, this time to the exact sequence dual to the first sequence from (9.3). N we get a manifold (V_1, ν_1) such that $c(V_1, \nu_1) = (F_2 \oplus F_0, f_2 \oplus f_0)^*$, and one m application of Lemma 9.6 completes the proof.

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