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# SURGERY ON THE NOVIKOV COMPLEX

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# SURGERY ON THE NOVIKOV COMPLEX

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Let $M^n$ be a manifold, $n \geq 6$, $\xi$ be an epimorphism $\pi_1(M^n) \to \mathbf{Z}$. Suppose, that Ker$\xi$ is a finitely presented group. Denote by $\Lambda$ the group ring $\mathbf{Z}\pi_1 M$, by $\bar{\Lambda}_\xi$ - the Novikov completion of this ring. Let $C_*$ be a free based finitely generated complex over $\bar{\Lambda}_\xi$ of the same simple homotopy type, as the completed simplicial chain complex of the universal covering M. Suppose that $C_* \neq 0$ only for $2 \leq * \leq n-2$.

The main aim of this paper is to prove, that that $C_*$ can be realized (up to the terms in $\bar{\Lambda}_\xi$ of arbitrary high degree) as a Novikov complex of a Morse map $f: M \to S^1$, belonging to the homotopy class $\xi$ .The precise statement together with all the necessary definitions is contained in the Introduction.

The present text contains every detail of the proof. The author understands that sometimes it is overloaded. The reasonable way to read it would be to read the Introduction, which contains all the background material, as well as the main idea of the proof, and then consult the principal text for details, when necessary. The author will be grateful for every comment or remark.

A.V.Pazhitnov. Nantes, April 4,1992.

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A.V. PAZHITNOV

## 0. INTRODUCTION

The classical Morse theory is initiated by Morse and developed in 60s by differential topologists (Smale, Milnor and others) is based on a geometric construction of a chain complex, associated with each Morse function. This construction is due to Morse, Thom, Smale (see [Mi2], [Sm]) and it was recently renewed by Witten [Wi]. It is basic for our purposes and we begin by a short expository account.

Let $M^n$ be a smooth manifold, $f:M^n \longrightarrow \mathbf{R}$ be a Morse function, $v$ be a gradient-like vector field for $f$ (this means, we recall, that $df(v)$ is strictly positive except for the critical points of $f$ and near each critical point $p$ of index there exists a neighbourhood $U(p)$ with the coordinates $x_1,\dots,x_\lambda$ , $y_1,\dots,y_\mu$ , where $\lambda + \mu = n$, such that in these coordinates $f$ is $(-(x_1^2+\dots+x_\lambda^2) + (y_1^2+\dots+y_\mu^2)$ and $v$ has the coordinates $(-x_1 ,\dots,-x_\lambda ,\ y_1,\dots,y_u )$. We assume that the stable manifold $D(p,\ v)$ of any critical point $p$ is transversal to the unstable manifold $D(q,-v)$ of any other critical point $q$. Such vector fields will be called perfect and one can prove that in any neighbourhood of a gradient-like vector field there exists a perfect one (see, for example, [Pa1, App.B]). It is easy to see, that for a perfect vector field $v$ the number of $(-v)$-trajectories, joining $p$ and $q$, where $ind(p) = ind(q)+1$, is finite and if we choose the orientations for all the stable manifolds (these manifolds will be called descending discs from now on) each trajectory obtains a sign + or −. The sum of these signs is called the incidence coefficient $n(p, q)$.

Consider now the chain complex $C_*(f,v)$ of free abelian groups, such that $C_p(f,v)$ is generated by the critical points of $f$ of index $p$ and the differential is of the form $\partial p = \sum n(p,q)q$, where $q$ runs through the critical points of index $(ind(p) - 1)$.

<u>Theorem 0.1</u>. $\partial^2 = 0$ *and the homology* $H_*(C_*(f,v))$ *is isomorphic to* $H_*(M, Z)$.

(The proof can be extracted from [Mi2].)

From this theorem one deduces easily the classical Morse inequalities (as improved by Pitcher); that is the matter of elementary algebra:

<u>Corollary 0.2</u>. *The number* $m_p(f)$ *of critical points of index* $p$ *of any Morse function is not less than* $b_p(M) + q_p(M) + q_{p-1}(M)$, *where* $b_p(M)$ and $q_p(M)$ *are, respectively, rank and torsion number of* $H_p(M, Z)$.

The natural problem to pose is whether these inequalities are optimal. The answer is due to Smale.

<u>Theorem 0.3. (Smale [S])</u>. If $\pi_1 M = 0$, $n \geq 6$, *then there exists a Morse function* $f:M \to \mathbf{R}$, *such that* $m_p(f) = b_p(M) + q_p(M) + q_{p-1}(M)$ *for any* $p$.

The demonstration is essentially that one eliminates the "spare" critical points. One of the corollaries of this result is the h-cobordism theorem.

If the fundamental group $\pi_1(M)$ is non-zero, the inequalities above are not optimal in general and to improve the situation one considers the modified construction of the Morse complex, which takes into account the homotopy classes of $(-v)$-trajectories, joining the critical points, and, thus, the fundamental group.

We shall formulate the result for the Morse functions $f$ on the connected cobordisms $W$, $\partial W = V_0 \cup V_1$, such that $f \mid V_0 = \text{const} = \min f$, and $f \mid V_1 = \text{const} = \max f$, and the critical points of $f$ belong to $W \setminus (V_0 \cup V_1)$. We denote by $\widetilde{W}$ the universal covering of $W$, by $\widetilde{V}_0$ - the preimage of $V_0$ in $\widetilde{W}$.

<u>Theorem 0.4. [Mi1]</u> *There exists a chain complex* $C_*(f,v)$ *of free right finitely generated* $Z\pi_1 W$ *- modules, such that*

1) *the free generators of* $C_*(f,v)$ *are the critical points of* $f$,

2) $C_*(f,v)$ *is simply homotopy equivalent to* $C_*^{\Delta}(\widetilde{W}, \widetilde{V}_0)$.

Here $\Delta$ stands for any smooth triangulation of $W$, such that $V_0$ is a subcomplex (see [Mu]) and $C_*^{\Delta}$ - for the corresponding simplicial chain complex.

The analogue of the optimality of the Morse-Pitcher inequalities above is the following theorem, which is difficult to attribute precisely.

Theorem 0.5. *Assume that* $V_0$, $V_1$ *are connected and the inclusions* $V_0 \subset W^n \supset V_1$ *induce the isomorphisms in* $\pi_1$. *Assume that* $n > 6$.

*Then each free based right* $Z\pi_1 W$ - *complex* $\{0 \leftarrow D_2 \leftarrow \dots \leftarrow D_{n-2} \leftarrow 0\}$, *simply homotopy equivalent to* $C_*^\Delta(\widetilde{W}, \widetilde{V}_0)$, *is the Morse complex of some Morse function.*

One of the corollaries of this theorem is the s-cobordism theorem of Barden-Mazur-Stallings.

To deduce from this theorem the analogues of Morse-Pitcher inequalities one should be able to find in the given simple homotopy type $\{C_*^\Delta(\widetilde{W}, \widetilde{V}_0)\}$ the chain complex with the minimal possible number of generators in each dimension. This number will play the role of $b_p(M) + q_p(M) + q_{p-1}(M)$ above. This idea is due to V.V.Sharko [Sh1]; later he developed a general theory of such inequalities (see the book [Sh2]).

Having recollected these basic facts from standard Morse theory, we now pass to the Morse theory of maps $M \to S^1 = R/Z$, which is the subject of our paper. Assume that $f:M \to S^1$ is the Morse map. One can show that there exists always a gradient-like vector field $v$ for $f$, such that all the stable and unstable manifolds of critical points of $v$ are transversal. That is a version of Kupka-Smale theorem; for the proof see [Pa1] (this paper is actually the first part of the present work). Assume that $f_*: \pi_1 M \longrightarrow \pi_1 S^1 = Z$ is epimorphic and consider the cyclic covering $\bar{M} \longrightarrow M$, where $f$ resolves to a Morse function $\bar{f}:\bar{M} \longrightarrow R$, which is tequivariant, i.e.: $f(xt) = f(x)-1$ (Here $t$ is a generator of the structure group of the covering). The attempt to construct an analogue of Morse complex in this situation meets an obvious obstruction: the number of critical points of $f$ is infinite. The procedure of counting the critical points (and the resulting construction) is due to Novikov [No]. For each critical point $x$ of $f$ one chooses a lifting $\bar{x}$ to $\bar{M}$. One lifts also the field $v$ to $\bar{M}$. For $ind(x) = ind(y)+1$ the number of $(-v)$-trajectories, joining the critical points $\bar{x}$ and $t^i\bar{y}$ is finite for each fixed $i$. We count them (with the signs, due to orientations) and get the integer $n_i(x,y)$. Set by definition $n(x,y) = \sum n_i(x,y)t^i$ ; that is an element of the ring of the Laurent power series $\hat{\Lambda} = Z[[t]][t^{-1}]$. Consider now the free $\hat{\Lambda}$ - module $C_p(f,v)$ generated by critical points $x$ of index $p$ and set $\partial_p(x) = \sum n(x,y)y$, where $y$ runs through the set of critical points of index $(p-1)$. Denote by $\Lambda$ the group ring $Z[Z]$.

<u>Theorem 0.6.</u> $\partial_{p-1} \circ \partial_p = 0$ *and the homology* $H_p(C_*(f,v))$ *of the resulting complex is isomorphic to* $H_p(\bar{M},Z) \otimes_{\Lambda} \hat{\Lambda}$ *as a* $\hat{\Lambda}$*-module.*

This theorem was announced in [No]. Although the idea of proof is rather clear, the details were not clarified, and, for example, the papers [Frb] and [Pa2] used another proof of the Novikov inequalities. Now the full proof is written down (see [Pa1]).

Corollary 0.7. ([No]). *The Morse number* $m_p(f)$ *of a Morse map* $f:M \to S^1$ *is not less, than* $b_p(M,[f]) + q_p(M,[f]) + q_{p-1}(M,[f])$, *where* $[f]$ *is a homotopy class of* $f$ and $b_p(M,[f])$, $q_p(M,[f])$ *are, respectively, rank and torsion number of the* $\hat{\Lambda}$*-module* $H_*(\bar{M}) \otimes_{\Lambda} \hat{\Lambda}$.

(Note that $\bar{M}$ depends on $[f]$, so actually the good notation should be $M_{[f]}$.) These inequalities are optimal, as proved by Farber [Frb].

Theorem 0.8. [Frb]. *For any manifold* $M^n$, $n \geqslant 6$, $\pi_1(M^n) = Z$ *and any class* $0 \neq \gamma \in H^1(M,Z)$ *there exists a Morse map* $f:M \to S^1$, *such that* $f \in \gamma$ *and for every* $p$ *we have* $m_p(f) = b_p(M,\gamma) + q_p(M,\gamma) + q_{p-1}(M,\gamma)$.

This theorem is the direct analogue of the result of Smale, cited above. The interesting feature of Novikov theory is that these inequalities are optimal even in the case of the free abelian fundamental group for the cohomology class "in general position". This almost never happens to the Morse functions (see [Sh1]). The following theorem was proved in [Pa1] and [Pa3] under some restrictions on the homotopy type of M; these restrictions can be removed by means of the main theorem of the present paper.

Theorem 0.9. ([Pa1] + [Pa3] + present paper). *For a manifold* $M^n$, $n \geqslant 6$, $\pi_1(M) = Z^m$, *there exists a finite number of hyperplanes* $\Delta_i \subset H^1(M,Z) = Z^m$, *such that for* *each class* $0 \neq \gamma \in H^1(M,Z) \setminus \bigcup_i \Delta_i$ *there is a Morse map* f, *such that* $m_p(f) = b_p(M,\gamma) + q_p(M,\gamma) + q_{p-1}(M,\gamma)$. *The numbers* $b_p(M, \gamma)$, $q_p(M, \gamma)$ *do not depend on in every connected component of the complement* $H^1(M,R) \setminus \bigcup_i (\Delta_i \times R)$.

(We shall comment on the proof later on.)

Now we shall formulate the analogues of theorems 0.4 and 0.5 for the case of Morse maps $f: M \to S^1$. For that we need some more notations. We denote $\pi_1(M)$ by G. Let $\xi: G \to Z$ be an epimorphism and denote $\ker \xi$ by H. Denote ZG by $\Lambda$. Choose and fix some element $\theta \in G$, such that $\xi(\theta) = -1$. We denote by $\Lambda_\xi^-$ the ring of formal power series of the type $\{a_{-n}\theta^{-n} + \ldots + a_o + a_1\theta + \ldots\}$, where $a_i \in ZH$. (It seems at the first glance that this ring depends on the choice of $\theta$, but actually it does not and the invariant definition, which is valid also for the homomorphisms from G to R can be found in [Si]). Denote by $U_\xi^-$ the group of units of the ring $\Lambda_\xi^-$, which are of the form $(\pm g)(1+a_1\theta + \ldots)$, where $g \in G$, $a_i \in ZH$. Denote by $Wh(G, \xi)$ the group $K_1(\Lambda_\xi^-)/U_\xi^-$. The term "complex" from now on will be reserved for the finitely generated free right $\Lambda_\xi^-$-complexes with the fixed base. We say, that two complexes are simply homotopy equivalent, if there exists such a homotopy equivalence between them, that its torsion vanishes in $Wh(G, \xi)$. We say, that two elements $a_{-n}\theta^{-n} + \ldots + a_o + \ldots$ and $b_{-n}\theta^{-n} + \ldots + b_o + \ldots$ of the ring $\Lambda_\xi^-$ are N-equivalent, if $a_i = b_i$ for $i < N - 1$. We say, that two complexes $D_*$, $D_*'$ are N-equivalent, if there is a one-to-one correspondence between their bases and the elements $\partial_{ij}^{(p)}$, $\partial_{ij}'^{(p)}$ of the matrices of differentials $\partial^{(p)}$, $\partial'^{(p)}$ are N-equivalent. From now on we assume that our Morse maps induce the epimorphisms in $\pi_1$.

<u>Theorem 0.10.</u> <u>[Pa1]</u>. *For each Morse map* $f:M \to S^1$ *and a perfect gradient-like vector field* $v$ *for* $f$ *there exists a complex* $C_*(f,v)$ *over* $\bar{\Lambda}_\xi$ *, where* $\xi = [f]$*, such that*

1) $C_p(f,v)$ *is generated by the critical points of* $f$ *of index* $p$

2) $C_*(f,v)$ *is simply homotopy equivalent to* $C_*^\Delta(\tilde{M}) \underset{\Lambda}{\otimes} \bar{\Lambda}_\xi$

This complex is called Novikov complex.

The analogue of the theorem 0.5 (which presents in a sense a converse to the theorem 0.10) is the main aim of the present paper.

<u>Theorem 0.11.</u> *Assume that* $H = \ker(\xi : \pi_1(M) \to Z)$ *is a finitely presented group*; let $n \geq 6$. Let $D_* = \{0 \leftarrow D_2 \leftarrow \dots \leftarrow D_{n-2} \leftarrow 0\}$ *be a complex over* $\bar{\Lambda}_\xi$*, simply homotopy equivalent to* $C_*^\Delta(\tilde{M}) \underset{\Lambda}{\otimes} \bar{\Lambda}_\xi$ and let $N > 0$ *be a natural number.*

*Then there exists a Morse map* $f:M \to S^1$*, belonging to* $\xi$ *, and a perfect gradient-like vector field* $v$ *for* $f$*, such that* $C_*(f,v)$ is N-*equivalent to* $D_*$.

This theorem is exactly the theorem 2.1 of the present paper. The proof occupies the sections 3-8 of the present paper, the last step is contained in section 8.

Now we shall demonstrate the corollaries of this theorem, afterwards we present the main ideas of the proof, and we finish the introduction with the list of problems unsolved.

Corollary 0.12. *Let* $M^n$ *be a manifold,* $n \geqslant 6$, $\xi : \pi_1(M) \longrightarrow Z$ *be an epimorphism with a finitely presented kernel. Then the class* $\xi \in \mathrm{Hom}(\pi_1(M), Z)$ *can be realized by a fibration if and only if the complex* $C_*^{\Delta}(\tilde{M}) \otimes_{\Lambda} \Lambda_{\xi}^{-}$ *is simply homotopy equivalent to zero.*

For brevity we shall denote $C_*^{\Delta}(\tilde{M}) \otimes_{\Lambda} \Lambda_{\xi}^{-}$ by $C_*(\tilde{M}, \xi)$.

The condition of this corollary is equivalent to the following: the homology of the complex $C_*(\tilde{M}, \xi)$ vanishes, hence the torsion of this complex is defined and it vanishes in the group $Wh(G, \xi)$. To rewrite the second condition in more familiar terms one should be able to compute $Wh(G, \xi)$. We shall do it for the simplest case $G = H \times Z$. In this case $\Lambda_{\xi}^{-} = (ZH)[[t]][t^{-1}]$ and the exact sequence of localization in K-theory [Gr] gives us the following exact sequence:

$$K_1(\mathcal{C}) \longrightarrow K_1((ZH)[[t]]) \longrightarrow K_1(\Lambda_{\xi}^{-}) \longrightarrow K_0(\mathcal{C})$$

where $\mathcal{C}$ is the category of finitely generated $(ZH)[[t]]$-modules, which are annihilated by some power of $t$ and which have the projective resolution of length $\leqslant 1$. By means of algebra one deduces the following exact sequence:

$$Wh(H) \xrightarrow{\mu} Wh(G, \xi) \longrightarrow \widetilde{Nil}(ZH) \oplus \widetilde{K}_0(ZH).$$

So, for vanishing of the torsion $\tau = \tau(C_*(M, \xi))$ in $Wh(G, \xi)$ it is necessary, that $\theta(\tau) = 0$; in the latter case the element $\mu^{-1}(\tau)$ is defined in the group WhH and vanishing of these two obstructions is equivalent to vanishing of $\tau$. It is very probable, that these obstructions are the same as those of Farrell and Siebenmann, at least when they both are defined, i.e. the homology of $C*(M, \xi)$ vanishes <u>and</u> $\bar{M}$ is finitely dominated. (I do not know whether these two conditions are the same or different, it seems that they are the same for abelian groups.)

For the group $\pi_1(M) = Z^m$ the exact sequence above gives us $Wh(Z^m, \xi) = 0$, and this implies in turn that for the case of free abelian fundamental group every complex $D_* = \{0 \leftarrow D_2 \leftarrow \dots \leftarrow D_{n-2} \leftarrow 0\}$, which is simply homotopy equivalent to $C\ (M, \xi)$ can be realized as $C\ (f,v)$. So to prove the theorem 0.9 above it suffices to find the hyperplanes $\Delta_i$, such that for each 1-dimensional cohomology class , which does not belong to the union of $\Delta_i$, the complex $C_*(M, \xi)$ is simply homotopy equivalent to some complex $D_*$, such that $D_i = 0$ for $i = 0, 1, n-1, n$, and the number of generators of $D_p$ equals $b_p(M, \gamma) + q_p(M, \gamma) + q_{p-1}(M, \gamma)$. That is the matter of pure algebra and it is done in [Pa3]. This algebra is based on the beautiful theorem of J.-C. Sikorav, which states that the Novikov-type completion $\Lambda_\xi^-$ of the ring $\Lambda = Z[Z]$ with respect to a monomorphism $\xi : Z \longrightarrow R$ is euclidean (see [Pa2] for the exposition of this theorem).

These are the applications, which I had in mind, when I started to write this paper, and now I proceed as to explain the main ideas of the proof.

There exist already several theorems concerning the number of critical points of Morse maps $f:M \longrightarrow S^1$. One usually tries to get in a given homotopy class a function with the minimal possible number of critical points. This number in the cases considered by Browder-Levine [BrL], Farrell [Far], and Siebenmann [Si] is zero, and in the cases, considered by Farber [Frb] and myself [Pa2] is given by the Novikov numbers. The strategy, proposed by Browder and Levine is as follows. Oe constructs first some submanifold $N \subset M$, dual to the class $\xi = [f]$. We now think of the cyclic covering $\bar{M}$ as of the union of the similar bricks W, each of which has two components of the boundary, diffeomorphic to N (see pict. 0.1). Next one makes surgery over N inside M, so as to obtain at the end the cobordism (W, tN) with the desired homotopy properties, i.e. h-cobordism in the case of Browder-Levine and in the case of Farrell, the cobordism with Betti and torsion numbers, equal to the corresponding Novikov nubers in the case of Farber, and, finally, the cobordism, such that in homotopy type $C_*(W, tN)$ there is a chain complex with the number of generators equal to the corresponding Novikov numbers in the case of [Pa2]. To do this there is an obstruction in the case of Farrell, which belongs to Nil Z(H). Also in the case

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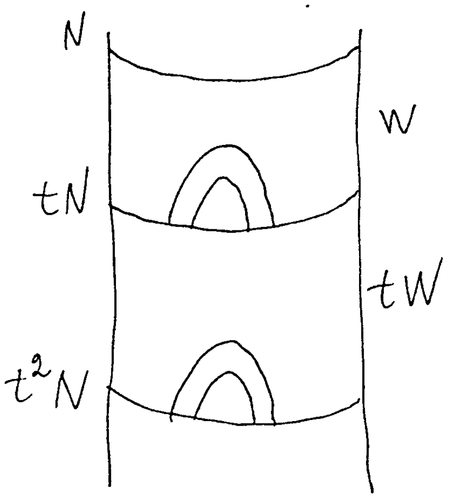



Pict. 0.1

The second stage is that one seeks for a Morse function on the cobordism W, constant on N and tN, with the desired number of critical points. In the case of Farrell, for example, this is possible if the Whitehead torsion of h-cobordism (W, tN) vanishes. In the cases of [BrL], [Frb] and [Pa2] that is always possible, mainly because of vanishing of all the K-theoretic groups.

Thus, the method breaks into two steps. First we improve N (which can be considered as a regular level of some Morse map $f:M \longrightarrow S^1$ in our 1-dimensional cohomology class, but we forget this for a while). Afterwards we seek for a Morse function with the desired number of critical points on the resulting cobordism.

The method of the present paper is, in a sense, more direct. We start with an arbitrary Morse function $f_0:M \rightarrow S^1$, which defines the Novikov complex $C_*(f_0, v_0)$. The complex $D_*$, which we want to realize as $C_*(f,v)$ (up to some N) is simply homotopy equivalent to $C_*(M, \xi)$ and, therefore, to $C_*(f_0, v_0)$. That implies, by the standard theory of simple homotopy type (we recall it in the section 1), that D can be obtained from $C_*(f_0, v_0)$ by a finite series of elementary transformations, such as: adding a trivial direct summand $\{0 \longleftarrow \Lambda_\xi^- \xleftarrow{id} \Lambda_\xi^- \longleftarrow 0\}$, cancelling such a summand, and changing the base by the elementary transformations. So we need to realize each of these operations (up to our N) on the level of functions f and fields v. This can be done and this occupies the sections 5-7 of the present paper. The only thing which we demand from our initial function, is that all the regular preimages $f_0^{-1}(c)$ are connected, $f_0^{-1}(c) \hookrightarrow M$ induces a monomorphism of $\pi_1(f_0^{-1}(c))$ onto Ker $\xi$ = H and f has no critical points of indices 0, 1, n-1, n. This is possible, since H is finitely presented (see [Far]). This property will be preserved in course of our transformations, since by §1 we can choose our elementary transformations in such a way, that all the intermediate complexes have no generators in dimensions 0, 1, n-1, n.

This is the method, which is usually employed for the proof of theorem 0.5.

The main part of the proof is of course the cancellation of the trivial direct summand. I shall explain the main lines of this process. The instruments, used here, are those of standard surgery theory: the Whitney trick for the functions on the cobordisms and the cancellation process as described in [Mi2]. What we have to do is to reduce our problem to the case when this instruments work and to perform the operations in such a way, that the other parts of Novikov complex are not disturbed.

We need some some definitions and conventions. We say, that a gradient-like vector field $v$ is almost good, if all the descending discs $D(p,v)$ are transversal to all the ascending discs $D(q,-v)$ for $ind(p) > ind(q)$. We say that $v$ is good, if this holds also if $ind(p) > ind(q) - 1$, and we say, that $v$ is perfect, if that holds for all $p$, $q$. By [Pa1] every gradient-like vector field $v$ for some Morse function $f:M \longrightarrow S^1$ can be approximated by a perfect one. For a Morse function $h$ on a manifold, cobordism etc. the $Cr\,h$ will denote the set of critical points of $h$.

We shall assume, that our initial Morse function $f:M \longrightarrow S^1$ is regular, i.e. there are no critical points of indices 0, 1, $n-1$, $n$, all the regular preimages $f^{-1}(c)$ are connected and $\pi_1(f^{-1}(c)) \longrightarrow \pi_1(M)$ is an epimorphism onto $Ker(\ \xi : \pi_1(M) \longrightarrow Z)$. All the operations below can be performed as to preserve this property.

If $f:M \longrightarrow S^1$ is a regular Morse function, $v$ is a good gradient-like vector field for $f$, and we have chosen and fixed the liftings of all the critical points of $f$ to $\bar{M}$ together with the orientations of all the descending discs, the Novikov complex $C_*(f,v)$ is defined (see [Pa1]; in this paper we defined it for the perfect vector field $v$, but it is easy to see, that for good vector fields the construction works as well).

If $x$, $y$ are some critical points of the lifting $\bar{f}:\bar{M} \longrightarrow R$, $\mathrm{ind}(x) = \mathrm{ind}(y) + 1$, then for a good gradient-like vector field $v$ the incidence coefficients $\vartheta(x,y;v) \in ZH$ are defined and one shows easily, that if $a,b \in Crf$, the incidence coefficient $n(a,b)$ in the Novikov complex is the sum $\sum(\bar{a}, t^s\bar{b})\cdot\theta^s$ , where $a$ and $b$ are the projections to $M$ of the fixed liftings $\bar{a},\bar{b} \in \bar{M}$. For $x,y \in Cr\bar{f}$ we denote by $N(x,y;v)$ the set of all $(-v)$-trajectories, joining $a$ and $y$ (the trajectories, which differ by a parameter shift, are identified). If $\mathrm{ind}(x) = \mathrm{ind}(y) + 1$ this set is finite. We say, that $N(x,y;v)$ and $N(x,y;v')$ are homotopically the same, if they are in bijective correspondence, which preserves the homotopy classes. In this case, obviously, $\vartheta(x,y;v) = \vartheta(x,y;v')$. Note, that if $v$, $v'$ are the good gradient-like fields for the same $f$, then for $n(a,b;v)$ be $N$-equivalent to $n(a,b;v')$ it suffices that $\vartheta(\bar{a},\bar{b}t^s;v) = \vartheta(\bar{a},\bar{b}t^s; v')$ for a finite set of $s$ (more precisely, for all $s$, such that $-(|f(a) - f(b)| + 2) \leqslant s \leqslant N-1$).

Step A. Whitney trick.

Theorem A (see th. 5.2 of §5). *Let* $g:M \longrightarrow S^1$ *be regular,* v *be good. Let* $\mu < \nu$ *be the regular values of* $\bar{g}:\bar{M} \to \mathbf{R}$, *such that the only critical points of* $\bar{g}$ *in* $\bar{g}^{-1}([\mu, \nu])$ *are* p, q, $\bar{g}(p) > \bar{g}(p)$, ind(p) = ind(q) + 1. *Let* $\gamma_1$ , $\gamma_2$ *be two trajectories, joining* p *and* q. *Let* $K \subset Cr\bar{g}$ *be any finite set.*

*Then there exists a good gradient-like vector field* v' *for* g, *such that:*

A1) *The coefficients* $\nu(x,y;v)$ *and* $\nu(x,y;v')$ *are the same for* ind(x) = ind(y) + 1, $x,y \in K$.

A2) *The set* N(p,q;v') *is homotopicallt the same as* N(p,q;v) \ { $\gamma_1$, $\gamma_2$ }.

A3) *For* $x,y \in K$, ind(x) = ind(y) + 1, *and* x *is not a* t-*shift of* p *and* y *is not a* t-*shift of* q, *the set* N(x,y; v') *is homotopically the same as* N(x,y; v).

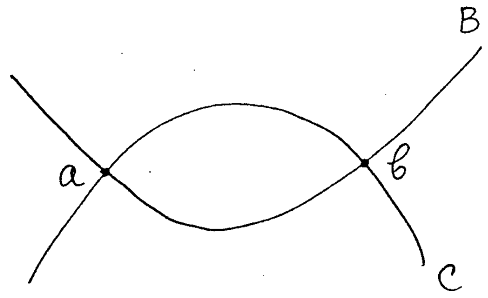
A4) *Assume that* ind(p) $\geqslant$ 4. *Then if* y *is a critical in* K *of index* ind(p) − 1, *and* y *is not a* t-*shift of* q, *then* $N(pt^s,y;\ v)$ *is homotopically the same as* $N(pt^s,y;\ v')$ *for* $pt^s \in K$.

A4') *Assume that* ind(p) $\leqslant$ n−3. *Then if* $x \in K$, ind(x) = ind(p), x *is not a* t-*shift of* p, *the set* $N(x,qt^s,\ v')$ *is homotopically the same as* $N(x,qt^s;\ v)$ *for* $qt^s \in K$.

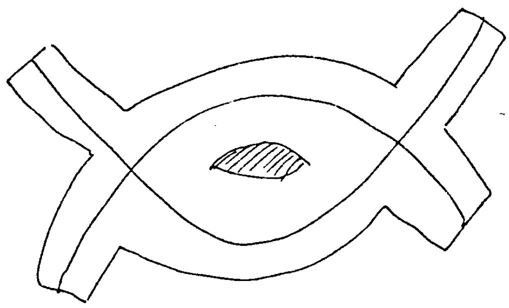
Sketch of proof. The basic idea is the following. We apply the classical Whitney trick, but we choose the embedding of 2-dimensional Whitney disc $D^2$ in such a way, that it is transversal to all the descending and ascending discs in consideration. We perform the Whitney modification and cancel our two trajectories of opposite sign. Consider now the incidence coefficient $n(x,y)$ for any two critical points $x,y$, $ind(x) = ind(y) + 1$. There are no critical points of indices 0, 1, $n$, $n-1$, therefore either the descending disc $D(x,v)$ or the ascending disc $D(y,-v)$ has the dimension $\leqslant (n-3)$, therefore does not intersect $D^2$. This implies, that either $D(x,v)$, or $D(y,-v)$ is the same as before, and, therefore, $n(x,y)$ has not changed.

Of course this idea does not work through directly because of various subtleties. (For example, the Whitney modification takes place in the neighbourhood $\mathcal{U}$ of $D^2$, so we should have $D(x,v) \cap \mathcal{U} = \emptyset$, or $D(y,-v) \cap \mathcal{U} = \emptyset$, which actually can not be achieved in all the cases, and that is the cause of the several subdivisions in the conclusions of the Theorem A.)

Now we can expose the proof. Choose an integer N, such that $K$ $g^{-1}([c,c+N])$ and denote by $W$ the cobordism $g^{-1}([\mu - 2N, \nu + 2N])$ and by $V$ the preimage of some regular value $\lambda \in (\mu, \nu)$. Denote $ind(p)$ by $r$ and denote by $N$ the $(r-1)$ - sphere $D(p,v) \cap V$, by $L$ - the $(n-r)$ - sphere $D(q,-v) \cap V$. Recall, that $r-1$ and $n-r$ are less than $n-3$ and not less, than 1. The intersections $\gamma_1 \cap V$ and $\gamma_2 \cap V$ are the points, say $a$ and $b$, which belong both to $D(p,v)$ and to $D(q,-v)$, which imply that they do not belong to any other descending or ascending disc of v in M. Now we choose a curve C in L, joining a and b such that C is transversal to all the descending discs D(x,v) of the points $x \in M$, ind(x) < r, $x \neq p$. (Note, that, therefore, C does not intersect these D(x,v).) That is possible, since, by the assumption D(x,v) is transversal to $L = V \cap D(q,-v)$ and D(x,v) is the submanifold of V of codimension $\geqslant 2$ (see lemma 3.1.p.2)). Similarly we choose the curve B in N, joining a and b, such that B does not intersect D(y,-v) for $y \in \bar{M}$, ind(y) $\geqslant r - 1$, $y \neq q$. We can assume that C and B are expanded a bit to give a classical picture of Whitney trick:




Pict. 02



Pict. 0.3

Note that C intersects no disc D(y,-v) for $y \neq q$ and, similarly, B intersects no D(x,v) for $x \neq p$, for trivial reasons. Hence $B \cup C$ does not intersect D(x,v) for ind(x) < r, $x \neq p$, and D(y,-v) for ind(y) > r-1, $y \neq q$.

Now (see the lemma 3.7 p.7) the union of all the D(x,v) for ind(x) $\leqslant r$ and $x \neq p$, intersected with $\bar{g}^{-1}([\lambda, \mu + 2N])$ is a compact set. Therefore, any small neighbourhood of B and C intersects with no disc $D(x,v)$ for $ind(x) \leq r$, $x \neq p$, $x \in W$. Similarly, it intersects no disc $D(y,-v)$ for $ind(y) \geq r-1$, $y \neq q$, $y \in W$.

By the standard Whitney argument (see [Mi2, §6]; we reproduce it in our paper for the sake of completeness, see p. 94-104 ) we construct a 2-dimensional thickening T of $B \cup C$ and, afterwards, the Whitney-type embedding $\varphi$ of $D^2$ to V (see pict. 0.3, all the details are given in § 5). We can assume that this embedding is transversal to all the $D(x,v)$ for $ind(x) \leq r$, $x \neq p$, $x \in W$ and to all the discs $D(y,-v)$ for $ind(y) \geq r-1$, $y \neq q$, $y \in W$, since T was transversal to them. This implies, that $\varphi(D^2)$ intersects neither $D(x,v)$ for $ind(x) \leq \min(r, n-3)$, $x \neq p$, $x \in W$, nor $D(y,-v)$ for $ind(y) \geq \max(r, 3)$, $y \neq p$, $y \in W$. Again by compactness the same is true for any small neighbourhood U of $\varphi(D^2)$. Now we consider the standard Whitney-type thickening of $D^2$, that is, an embedding $\Phi$ of $D^2 \times D^{r-1} \times D^{n-r}$ to V (see lemma 5.4), such that the image of $\Phi$ is contained in the small neighbourhood of the $Im\,\varphi$ . Next we perform the Whitney modification simultaneously in all the neighbourhoods $(Im\,\Phi \cdot t^s)$, $s \in Z$ , and get a new t-invariant gradient-like vector field v' for g on M, hence also a gradient-like field for g on M.

Note that from the very definition of U it follows that for any critical point $z \in \overline{M}$, which is not a t-shift of p, $ind(z) \leq r = ind(p)$, and $ind(z) \leq n-3$ the intersection $D(z,v) \cap \bar{g}^{-1}([z - 2N -1, z])$ is the same as $D(z,v') \cap \bar{g}^{-1}([z$

$- 2N - 1, z])$, and the vector fields $v$ and $v'$ coincide on these sets. Similarly, if a critical point $y \in \bar{M}$ is not a t-shift of $q$ and $ind(z) \geqslant r-1$, $ind(z) \geqslant 3$, the intersection $D(y,-v) \cap \bar{g}^{-1}([y, y + 2N])$ is the same as $D(y,-v') \cap \bar{g}^{-1}([y,y + 2N])$.

That implies immediately, that the restriction of $v'$ to the cobordism $W_0 = g^{-1}([\mu - N, \nu + N])$ is almost good. Indeed, if $x,y$ are critical points of $g$ in $W_0$, and $ind(x) \leqslant ind(y)$, then one checks up easily, that either

1) $ind(x) \leqslant \min(r, n-3)$ and $x$ is not a t-shift of $p$, or

2) $ind(y) \geqslant \max(r-1, 3)$ and $y$ is not a t-shift of $q$.

In the case 1) the disc $D(x,v) \cap W_0$ has not changed, therefore there are no $(-v)$-trajectories, joining $x$ and $y$; similarly for the case 2).

Similarly if $x,y \in W_0$, $ind(x) = ind(y) + 1$, and neither $x$ is a t-shift of $p$ nor $y$ is a t-shift of $q$, we deduce $N(x,y;v) = N(x,y;v')$ (just because either $ind(x) \leqslant n-3$, or $ind(y) \geqslant 3$).

By the same reason if $ind(p) \geqslant 4$ and $y \in Cr\bar{g} \cap W_0$, $ind(y)$ is less by 1 than $ind(p)$, and $y$ is not a t-shift of $q$, the sets $N(pt^s,y;v)$ and $N(pt^s,y;v')$ are the same for $pt^s \in W_0$.

The same argument applies for $ind(p) \leqslant n-3$.

Next we perturb $v'$ a bit to get a good gradient-like field $w$ for $g$ and the homotopical stability of the sets $N(x,y;v)$ under the small perturbations of $v$ (see lemma 3.12) gives us the items A2 -A4') of the theorem A.

It is left to prove A1) for the case, when $\mathrm{ind}(x) = \mathrm{ind}(p)$. For that we need a special technical instrument. That is, we define an incidence coefficient $\sqrt{}(x,y;v)$ for $\mathrm{ind}(x) = \mathrm{ind}(y) + 1$ and for $v$ being almost a good gradient-like vector field for our Morse function. That is done in §3 (see pages 60–80) and we prove that this coefficient is invariant under small perturbations of v, and coincides with the incidence coefficient for the case of good gradient-like vector fields (lemma 3.10). This coefficient is not changed, when we perform the Whitney trick. For details see §3; the basic idea is that this coefficient is defined as the intersection index of two certain manifolds with non-intersecting boundaries. It appears that our version of the Whitney trick changes one of these manifolds by the isotopy with the compact support, which does not intersect the boundaries. The other manifold does not change, hence the intersection index is preserved. The step A is done.

Step B. Preparations for cancelling.

Theorem B (see th. 5.1 of §5). *Let* $f:M \longrightarrow S^1$ *be regular,* $v$ *be good. Let* $N \geqslant 1$ *be an integer. Assume that* $C_*(f,v)$ *is N-equivalent to* $D_* \oplus \{0 \leftarrow \Lambda_\xi^- \xleftarrow{\mathrm{id}} \Lambda_\xi^- \leftarrow 0\}$, *where the first copy of* $\Lambda_\xi^-$ *is generated by* b, *the second – by* a, $\mathrm{ind}(a) = \mathrm{ind}(b) + 1$.

*Then there exists a regular* $g: M \longrightarrow S^1$ *with the same critical points as* $f$ *(always in our class* $\xi \in H^1(M, Z)$*), a lifting* $\bar{g}: \bar{M} \longrightarrow R$ *and a good gradient-like vector* $v'$ *for* $g$*, such that*

1) *There are no* $(-v')$*-trajectories, joining* $\bar{a}$ *and* $\bar{y}t^s$, *where* $y \in Crg$, $y \neq b$, $ind(y) = ind(b)$, $s < N$.

2) *There are no* $(-v')$*-trajectories, joining* $\bar{x}$ *and* $\bar{b}t^s$, *where* $x \in Crg$, $x \neq b$, $ind(x) = ind(a)$, $s < N$.

3) $\bar{g}(\bar{b}) < \bar{g}(\bar{a}) < \bar{g}(\bar{b}) + 1$ *and there is only one* $(-v')$*-trajectory, joining* $\bar{a}$ *and* $\bar{b}$. *Moreover,* $\nu(\bar{a},\bar{b}) = 1$.

4) *The complex* $C_*(g,v')$ *is* N*-equivalent to* $C_*(f,v)$.

Sketch of proof.

We must distinguish two cases: a) $ind(a) \leqslant n-3$, and b) $ind(a) \geqslant 4$. We present here the idea of the proof for the first case, the second case is similar.

Using the assumptions and employing the Whitney trick (step A), we push the critical point $\bar{a}$ below all the critical points $\bar{y}t^s$, where $\bar{y}$ runs through the critical points of $f$ of the same index as $b$, but different from $b$, and $s < N$. We push it lower than $\bar{b}t^{-1}$ as well. (The term "push" means the following. If there are two critical points $\alpha, \beta$ in $M$, $ind(\alpha) = ind(\beta) + 1$, $\bar{f}(\beta) < \bar{f}(\alpha) < \bar{f}(\beta)+1$ and $\nu(\alpha, \beta) = 0$, we apply the Whitney trick several times and afterwards there are no paths of steepest descent, joining $\alpha$ and $\beta$, so we can change the function $f$ on the cobordism $\bar{f}^{-1}([\bar{f}(\beta) - \varepsilon, \bar{f}(\beta) + \varepsilon])$ in such a way, that for the new function $\bar{g}$ . the point $\alpha$ is situated lower, than $\beta$ . We can keep the same vector field v as gradient-like vector field.) This gives automatically the item 1) of the conclusion of the theorem, as well as the first part of the item 3). Note that our step A guarantees, that $\nu(\bar{a}, \bar{b})$ is the same as before, so we can apply the Whitney trick to the pair a, b and get the second part of the item 3) as well. Note that this operation does not spoil the item 1) because the points $\bar{y}t^s$, $y \neq b$, $ind(y) = ind(b)$, $s < N$ are still higher up, than $\bar{a}$, $\bar{b}$.

Now we want to push the point $\bar{b}$ upwards to make it higher, than all the points $\bar{x}t^{-N}$, where x runs through the critical points of f the same index as a, but different from a. Each time we apply our Theorem A to the pair of critical points $(\bar{x}t^m, \bar{b})$, where $x \neq a$. Thus the set $N(\bar{a}, \bar{b})$ as well as the sets $N(\bar{a}, \bar{y}t^s)$ for $y \neq b$, $|s| < 2N$ can be preserved. Note that the sets $N(\bar{a}, \bar{y}t^s)$ are preserved automatically, since $a \neq x$, $y \neq b$, but we must apply A4') and thus use $ind(a) \leqslant n-3$ in order not to deform $N(a, b)$.

The step B is over.

Step C. Cancelling

Theorem C (see theorem 6.1 of §6). *Let* $f:M \longrightarrow S^1$ *be regular,* v *be good. Let* $N \geqslant 2$ *and assume, that the Novikov complex* $C_*(f,v)$ *is* N-*equivalent to the direct sum of some* $D_*$

*and of the complex* $\{0 \leftarrow \Lambda^-_\xi \xleftarrow{id} \Lambda^-_\xi \leftarrow 0\}$, *concentrated in dimensions* k, k+1.

*Then there exists a Morse function* $h: M \to S^1$ *(belonging to our class* $\xi$ *) and a good gradient-like vector field* w *for* h, *such that* $C_*(h,w)$ *is* N-*equivalent to* $D_*$.

Sketch of proof. We can assume that f and v satisfy the conclusions of the theorem B. Moreover, we can assume that there are the regular values $\mu$, $\nu$ of f, such that $\mu < \nu$, $|\nu - \mu| < 1$, and a, b are the only critical points of f in the cobordism $W_0 = \bar{f}^{-1}([\mu, \nu])$. We consider a small neighbourhood $U \subset W_0$ of the unique (−v)-trajectory $\gamma$, joining $\bar{a}$ and $\bar{b}$ and apply to it the cancellation procedure, described in [Mi2, §5]. (Of course, we apply it to all the t-shifts of U to get a t-invariant vector field.) The critical points a, b disappear. What we must show is that the incidence coefficients $n_i(x,y)$, $i \leqslant N$ have not changed for all the other generators x,y of $C_*(f,v)$.

Consider the cobordism $W = \bar{f}^{-1}([\mu - A, \nu + A])$, where A is sufficiently large positive integer. Similarly to the proof of the theorem A we can show, that if only U is sufficiently small, the modified vector field v′ will be almost good and the incidence coefficient $n_i(x,y)$, $i \leqslant N$ can change only if ind(x) = k+1 (see lemma 6.8, p.1), 2)). Here x,y are the critical points of f, $x \neq a$, $y \neq b$.

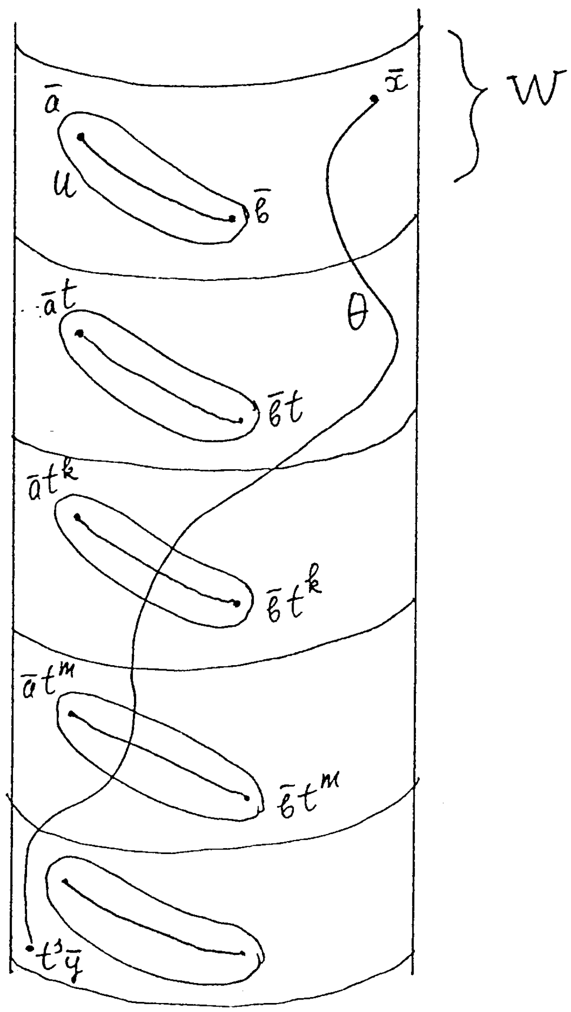
Now we shall explain why it does not change even in this case (see lemma 6.8, p.3)). For this we show that the trajectories of $(-v)$, joining $\bar{x}$ and $t^s\bar{y}$, where $\bar{x}$, $t^s\bar{y} \in W$, are the same as those of $(-v')$, and that they do not intersect $supp(v - v')$. There is only a finite number of $(-v)$-trajectories, joining $\bar{x}$ and $t^s\bar{y}$, and they obviously do not intersect $\gamma$, so they survive, if only $U$ is small enough. So we are to show, that we have not created a new trajectory. Assume that we have. Look at the picture 0.4. This new trajectory $\theta$ must intersect some of the neighbourhoods $t^iU$. Suppose that $t^kU$ and $t^mU$ are respectively the first and the last copies of $U$, intersected by $\theta$, $k \leqslant m$. That means, that there exists a $(-v)$-trajectory, joining $t^mU$ with $t^s\bar{y}$, and a $(-v)$-trajectory, joining $\bar{x}$ with $t^kU$. The simple compactness argument shows that if $U$ is sufficiently small, that implies the existence of the $(-v)$-trajectory $\theta_1$, joining $x$ and $t^kb$, and $\theta_2$, joining $t^ma$ and $t^s y$. (See corollary 6.5; of course these $\theta_1$, $\theta_2$ need not be the parts of $\theta$.) By our assumptions $k \geqslant N$, $s - m \geqslant N$, hence $s \geqslant 2N > N$, contradiction. The step C is over.

<u>Some open problems</u>.

We denote by $h(\xi)$ the homotopy type of $C_*^{\Delta}(\tilde{M}) \underset{\Lambda}{\otimes} \bar{\Lambda}_{\xi^-}$ and by $\eta(\xi)$ - its simple homotopy type.

<u>*Problem 1.*</u> *Clarify the connection between $\eta(\xi)$ and the obstructions of Farrell and Siebenmann. Is it true, that for* $\ker\xi$ *finitely presented* $h(\xi)$ *is zero if and only if* $\bar{M}$ *is finitely dominated (that can be verified for abelian groups)?*

The theorem 0.9 leads to the following problem, suggested by J.-C. Sikorav.




Pict. 0.4

<u>Problem 2 (conical properties).</u> *Assume that* $\ker(\xi)$ *is always finitely presented (for example,* G *is abelian). Is it true that the domain of vanishing of* $\eta(\xi)$ (or $h(\xi)$) *is*

*the union of polyhedral cones in the space* $H^1(M, Z)$? *What is the connection between vanishing of* $\eta(\xi)$ *and fibering properties for* $\dim M = 3$ ? (*It is known by Thurston* [Th], *that the union of the classes* $\xi \in H^1(M^3, Z)$, *represented by fibrations, is the union of polyhedral cones*).

The theorem 0.9 leads also to the optimality of Novikov-type inequalities for the open dense set of cohomology classes $\xi \in H^1(M, R)$ (rational or not), see [Pa2]. The argument is by perturbation of the result for the rational forms, it works because the corresponding algebraic properties are stable under small perturbations.

Problem 3. *Extend the results of the present paper to the case of the closed 1-forms with Morse singularities, belonging to an irrational cohomology class* $[\xi] \in H^1(M,R)$.

The next problem was posed by Novikov approximately 10 years ago and was discussed in literature (see [Ar]), though it seems not yet solved. It can be reformulated in terms of our paper as the problem of the realizability of the chain complexes as the Novikov complexes up to infinity.

Problem 4. (Novikov conjecture). *Show that the incidence coefficients* $n_k(x,y)$ *have no more than exponential growth in* $k$.

It is known that Farrell's theorem is true in dimension 5 in the topological category [Wei]. It seems that the theorem 0.9 holds as well in the top category and if the number $\mu_p(\gamma) = b_p(\gamma) + q_p(\gamma) + q_{p-1}(\gamma)$ is strictly greater than zero, one can even obtain a smooth Morse map $f:M \longrightarrow S^1$, $f \in \gamma$ with $m_p(f) = \mu_p(\gamma)$. (I am greatly indebted to M.Kreck for the discussions on this subject.) For the dimension 4 nothing is known, except that the Farrell's result is false even in topological category [Wei].

Problem 5. *What are the analogues of our results for the dimensions 5 and 4 ?*

We have seen that the absence of critical points of indices 0, 1 was essential for us. It would be very interesting to see what is going on when one can not do without them. For Morse functions on the closed compact manifolds that was done by V.V.Sharko [Sh2]; the answer is rather complicated and is given in terms of homotopy systems of J.H.C. Whitehead.

Problem 6. *What is going on when the critical points of index 1 are unavoidable? (The simplest case would be, probably, the case of free non-abelian fundamental group.)*

Acknowledgements. I am grateful to S.P.Novikov, J.-C. Sikorav, M.Kreck, P.Vogel, H.Ibisch for valuable discussions.

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Note for the readers. The text which follows contains every detail of the proof, therefore it contains lots of material which is not new. In particular I was to rewrite many passages of [Mi2] because I needed them in slightly different situation or I wanted a slightly more general result. The principle here was the same as in the first part [Pa1] of the present work: if I had a slightest doubt at some point, I included the full proof. It seems to me that the best way to read the paper is to look through the introduction, since the idea of the proof is contained there, and afterwards consult the text for details if necessary.

Nantes, February 19, 1992

## 1. Algebraic preliminaries

We will suppose that the free modules $R^n$, $R^m$ are not isomorphic if $m \neq n$.

Let $R$ be any associative ring with unit. Let $G$ be a subgroup in the group $R^*$ of invertible elements of $R$. There is the standard homomorphism $G \longrightarrow K_1(R)$ and the factorgroup of $K_1(R)$ by the subgroup generated by the image of $G$ and by $\{\pm 1\}$ will be denoted as $K_1(R;G)$. (If $R$ is $ZG$ and we choose $G$ itself to be the subgroup of $(ZG)^*$ then we get $Wh(G)$).

The following material is due essentially to J.H.C.Whitehead [Wh]. The statements of the propositions can be found, for example, in [Sh]. We reproduce it here for the sake of completeness. The basic reference for us in what is concerning the simple homotopy type is [Mi] (and also [C-S]) and we suppose, that the reader is familiar with it.

We shall consider the complexes $C_* = \{0 \leftarrow C_0 \leftarrow \dots \leftarrow C_s \leftarrow 0\}$ consisting of free finitely generated right $R$-modules. If $s \leqslant N$ we call $C_*$ an $N$-restricted complex. If for every $C_i$ there is chosen a free basis in it, we call $C_*$ a based complex.

The two free bases in a given free module $F$ will be called $G$-equivalent, if the transition matrix belongs to $K_1(R, G)$.

The isomorphism $f:C \longrightarrow D$ of two free finitely generated modules is called G-simple if $\tau(f)$ vanishes in $K_1(R,G)$. If $G = \{\pm 1\}$, $f$ is called simple.

The isomorphism $f:C_* \rightarrow D_*$ of two complexes is called G-simple if all the isomorphisms $f_n:C_n \rightarrow D_n$ are G-simple.

The homotopy equivalence $f:C_* \longrightarrow D_*$ is called a G-simple homotopy equivalence (abbreviation: G - s.h.e.) if the cone C(f) (which is an acyclic chain complex) has the torsion, vanishing in $K_1(R,G)$. The $\{\pm 1\}$-simple homotopy equivalence will be called simple homotopy equivalence.

The main technical instrument which we use working within simple homotopy theory is the following

Lemma 1.0 (Th. 3.1 of [Mi]). *Suppose that* $0 \rightarrow C''_* \rightarrow C_* \rightarrow C'_* \rightarrow 0$ *is an exact sequence of acyclic complexes, such that the bases* $c''$ and $c'$, *chosen for* $C''_*$ *and* $C'_*$ *compose together to give a base in* $C_*$, *equivalent to the chosen base* $c$ *in* $C_*$. Then $\tau(C_*) = \tau(C'_*) + \tau(C''_*)$.

Our main aim in this section is to prove that the complexes, G-simply homotopy equivalent to one another can be built one from another with the help of elementary operations, listed below.

We call elementary operations on the complex $C_*$ the operations of the following four types:

1) the multiplication of some element $e_i$ of basis of some $C_j$ by an element $g \in G$, or $g = -1$.

2) the transition from the basis $(e_1, \ldots, e_i, \ldots, e_j, \ldots, e_k)$ of some $C_s$ to the basis $(e_1, \ldots, e_i + e_j \cdot \lambda, \ldots, e_j, \ldots, e_k)$, where R.

3) Forming the direct sum with a complex of the type $\Gamma_*^{(s,k)} = \{0 \leftarrow \ldots \leftarrow R^k \xleftarrow{id} R^k \leftarrow \ldots 0\}$, where the two copies of $R^k$ have the dimensions s and s+1, respectively, and the bases in the two copies of $R^k$ are the same.

The complexes $\Gamma_*^{(s,k)}$ are called also the trivial acyclic complexes. We will omit the second index k if no confusion can occur.

4) For the complexes $D_*$ of the form $C_* \oplus \Gamma_*^{(s,k)}$ – cancelling the summand $\Gamma_*^{(s,k)}$.

If $C_*$ is obtained from $D_*$ by the series of elementary operations we say that $C_*$ is E-equivalent to $D_*$. That is obviously an equivalence relation.

Lemma 1.1. *Suppose that* $f: C_* \longrightarrow D_*$ *is an isomorphism of based N-restricted complexes, so that* $f_n: C_n \longrightarrow D_n$ *is an isomorphism of based free modules. Then* $\tau(f) = \tau(f_0) - \tau(f_1) + \ldots + (-1)^N \tau(f_N)$.

Proof. We proceed by induction in the length of the complex $C_*$ (where the length of $C_*$ is by definition the number of non-zero modules $C_i$). Indeed if the length of $C_*$ (and hence $D_*$) is 1, then the cone $C(f)$ is just $0 \leftarrow D_s \leftarrow C_s \leftarrow 0$ and $\tau(f) = (-1)^s \tau(f_s)$. If the assertion is proven for the length $s$ let $C_* = \{0 \leftarrow \ldots \leftarrow 0 \leftarrow C_k \leftarrow \ldots \leftarrow C_{k+s} \leftarrow 0 \leftarrow \ldots\}$, $D_* = \{0 \leftarrow \ldots \leftarrow D_k \leftarrow \ldots \leftarrow D_{k+s} \leftarrow \ldots\}$ be the complexes of length $s+1$ and $f: C_* \longrightarrow D_*$ be an isomorphism. We cut off the last terms of the complexes $C_*$ and $D_*$ and denote the results by $C'_* = \{0 \leftarrow \ldots C_k \leftarrow \ldots C_{k+s-1} \leftarrow 0 \leftarrow \ldots\}$ and $D'_*$ (which is similar). The map $f: C_* \longrightarrow D_*$ being restricted to $C'_*$ gives an isomorphism $f': C'_* \longrightarrow D'_*$ and the isomorphism in factorcomplexes $f'': C_*/C'_* \rightarrow D_*/D'_*$. From the definition of the cone it follows that there exists an exact sequence $0 \rightarrow C(f') \longrightarrow C(f) \longrightarrow C(f'') \longrightarrow 0$. Since all the three chain maps involved are isomorphisms, all the three complexes are acyclic and applying lemma 1.0 we get $\tau(f) = \tau(f') + \tau(f'')$ and we are over by the induction assumption.

The chain map $f: C_* \longrightarrow D_*$ of two based complexes will be called based embedding, if $f$ is injective and the chosen free generators of the $C_i$ are carried to those of $D_i$. Note that $D_*/f(C_*)$ is a based complex.

<u>Lemma 1.2</u>. *Let* $f: C_* \longrightarrow D_*$ *be a based embedding, which is a homotopy equivalence. Then the torsion* $\tau(f)$ *equals the torsion of* $D_*/f(C_*)$.

<u>Proof.</u> (See also the proof of lemma 7.6 in [Mi]). Consider the cone $C(f)$. The cone $C(id)$ of the identical map $id: C_* \longrightarrow C_*$ admits obviously the based embedding to $C(f)$ and

the factor is $D_*/f(C_*)$. The torsion of identity is zero and applying again the lemma 1.0 we get this lemma.

It is obvious also (by the same lemma 1.0) that the torsion of the direct sum of homotopy equivalences is equal to the sum of the torsions of the summands.

Note that the torsion of homotopy equivalences $0 \to \Gamma_*^{(s)}$; $\Gamma_*^{(s)} \to 0$ vanishes.

Hence, by the previous remarks we obtain that the result of any of elementary operations is G-simple homotopy equivalent to the original complex.

Remark 1.3. Let $C_*$, $D_*$ be acyclic based complexes and $f:C_* \to D_*$ be any map. Then $\tau(f) = \tau(C_*) - \tau(D_*)$.

(For the proof apply Th. 1.3 [Mi] to the exact sequence $0 \to D_* \to C(f_*) \to C_{*-1} - 0$ and note that $\tau(C_{*-1}) =$ $= -\tau(C_*)$).

Lemma 1.4. *The torsion* $\tau(f)$ *is a homotopy invariant of a chain map* f.

Proof. Let $f,g:C_* \to D_*$ be the homotopy equivalences of based complexes and H be a chain homotopy between them, $H_n : C_n \to D_{n+1}$. The cones $C(f)$, $C(g)$ are identical as graded complexes; $C(f)_n = C(g)_n = D_n \oplus C_{n-1}$ and differ by differentials. The simple calculation shows that the map $\Phi : C(f) \to C(g)$ given by $\Phi(x,y) = (x - H(y),y)$ is a chain isomorphism which is given in each dimension by the matrix $1 +$ (upper triangular matrix). So, by lemma

1.1, $\tau(\Phi) = 0$, and by the preceding remark we get $\tau(f) = \tau(g)$.

Lemma 1.5. *The torsion* (g,f) *of a composition of two homotopy equivalences* $f:C_* \to D_*$, $g:D_* \to E_*$ *is equal to* $\tau(g) + \tau(f)$.

Proof. Consider the auxiliary complex $G_*$, defined by $G_n = E_n \oplus D_n \oplus D_{n-1} \oplus C_{n-1}$ (which is naturally based), where the differential acts as: $\partial(e_n, d_n, d_{n-1}, c_{n-1}) = (\partial e_n - g(d_{n-1}), \partial d_n - d_{n-1} - f(c_{n-1}), -\partial d_{n-1}, -\partial c_{n-1})$. There is an embedding $\lambda$ of $C(gf)$ to $G_n$, defined by $(e_n, c_{n-1}) = (e_n, 0, f(c_{n-1}), -c_{n-1})$. The factorcomplex is easily seen to be the cone $C(id_D)$ of $id:D_* \to D_*$. Hence, $H_*(G_n) = 0$, and since the bases of $C(gf)$ and $C(id_D)$ compose to give the base, equivalent to that of $G_n$ we apply the lemma 1.0 to deduce $\tau(G_n) = \tau(gf)$.

On the other hand, there is a based embedding $I:C(f) \to G_*$, defined by $I(d_n, c_{n-1}) = (0, d_n, 0, c_{n-1})$ and the factor is exactly $C(g)$. Applying again the lemma 1.0, we are over.

Proposition 1.6. [Wh]. *Let* $f:C_* \to D_*$ *be a G-simple homotopy equivalence between two based N-restricted complexes. Then there is a sequence of elementary operations, starting at* $D_*$ *and finishing with* $C_*$, *such that all the intermediate complexes are N-restricted.*

Proof. The idea of the proof is as follows. We add to $D_*$ some number of trivial acyclic N-restricted complexes in order to get $D'_* \supset D_*$ and a based embedding $f':C_* \to D'_*$ which will be a G-s.h.e. The factor $D_*/f'(C_*)$ is an acyclic complex with zero torsion. The particular case of our proposition (lemma 1.9) shows that by a finite number of elementary operations one can reduce this factor to zero. Afterward we show that every elementary operation on the factor can be lifted to an elementary operation of the complex itself, preserving the embedded one, and this finishes the proof.

Now we pass to details.

Lemma 1.7. *The proposition* 1.6 *holds for the* G-*simple isomorphisms* $f:C_* \longrightarrow D_*$.

Proof. Consider the isomorphism $f_o:C_o \longrightarrow D_o$. Its torsion vanishes, hence there is a free module $R^k$, such that $f_o \oplus id: C_o \oplus R^k \to D_o \oplus R^k$ can be transformed to id by means of elementary changes of base of type 1) and 2).

So we add a trivial acyclic complex $\{0 \to R^k \to R^k \to 0\}$, concentrated in the dimensions (0, 1), to $C_*$ and $D_*$ and after applying some elementary changes of base of type 1), 2) we get $f:C_* \oplus \Gamma_*^{(o)} \longrightarrow D_* \oplus \Gamma_*^{(o)}$ which after the changes of base in dimension zero respects the bases in dimension zero. Proceeding further like that we get the map $F:C'_* \to D'_*$ which respects the bases (that is $C'_*$ and $D'_*$ are the same) and such that $C'_*$ is E-equivalent to $C_*$, $D'_*$ is E-equivalent to $D_*$. Q.E.D.

Lemma 1.8. *The proposition* 1.6 *holds for any isomorphism* $f: C_* \longrightarrow D_*$, *which is a* G-s.h.e.

Proof. Realize the element $(-\tau(f_o)) \in \overline{K_1(R)}$ by some isomorphism $\varphi_o : R^k \longrightarrow R^k$ and consider the isomorphism $\Phi_o : \Gamma_k^{(o,k)} \longrightarrow \Gamma_k^{(o,k)}$, defined as $\varphi_o$ in both dimensions; $\tau(\Phi_o) = 0$. Let $C_*^{(1)}$, $D_*^{(1)}$ denote respectively $C_* \oplus \Gamma_*^{(o,k)}$, $D_* \oplus \Gamma_*^{(o,k)}$, and $f^{(1)} : C_*^{(1)} \longrightarrow D_*^{(1)}$ an isomorphism $f \oplus \Phi_o$. Note that $\tau(f^{(1)}) = 0$, $\tau(f_o^{(1)}) = 0$, and $\tau(f_1^{(1)}) = \tau(f_1) - \tau(f_o)$.

Now realize $-\tau(f_1^{(1)})$ by some isomorphism $\varphi_1 : R^m \longrightarrow R^m$ and consider the isomorphism $\Phi_1 : \Gamma_*^{(1,m)} \longrightarrow \Gamma_*^{(1,m)}$ defined as $\varphi_1$ in both dimensions; $\tau(\Phi_1) = 0$. Add the $\Gamma_*^{(1,m)}$ to $C_*^{(1)}$ and $D_*^{(1)}$ to get $C_*^{(2)}$ and $D_*^{(2)}$ and an isomorphism $f_*^{(2)} = f_*^{(1)} \oplus \Phi_1 : C_*^{(2)} \longrightarrow D_*^{(2)}$. Observe that $\tau(f^{(2)}) = 0$, $\tau(f_o^{(2)}) = \tau(f_1^{(2)}) = 0$, $\tau(f_2^{(2)}) = \tau(f_2) - \tau(f_1) + \tau(f_o)$. We go on with that procedure and having added the complex $\Gamma_*^{(N-1)}$ to $C_*^{(N-1)}$ and to $D_*^{(N-1)}$ and constructed the isomorphism $f^{(N-1)}$ we note that $\tau(f_i^{(N-1)}) = 0$ for $0 < i < N-1$ and $\tau(f_N^{(N-1)}) = \tau(f_N) - \tau(f_{N-1}) + \dots \pm \tau(f_o)$ which equals $\tau(f)$ by lemma 1.1 and hence vanishes in $K_1(R,G)$. Note that $C_*^{(N-1)}$ is E-equivalent to $C_*$; $D_*^{(N-1)}$ is equivalent to $D_*$, and both are N-restricted. Now we apply, lemma 1.7 to conclude that $D_*^{(N-1)}$ is E-equivalent to $C_*^{(N-1)}$

Lemma 1.9. *The proposition 1.6 holds for the G-simple homotopy equivalences* $f: C_* \longrightarrow 0$.

Proof. It is obvious that every acyclic N-restricted complex is isomorphic to a sum of N-restricted complexes of the type $\{0 \longleftarrow L \xleftarrow{id} L \longleftarrow 0\}$, where $L$ is a stable free module. Therefore $C_*$ is E-equivalent to a complex $C'_*$ which is isomorphic to some direct sum $D_*$ of N-restricted complexes $\Gamma_*^{(s)}$. Both $C'_*$ and $D_*$ are acyclic, hence the torsion of this isomorphism equals $\tau(C'_*) - \tau(D_*)$, which, by assumption, vanishes in $K_1(R,G)$. Applying the lemma 1.8, we get that $C'_*$ is E-equivalent to $D_*$, and, therefore, to zero.

Lemma 1.10. *Suppose that* $0 \to A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0$ *is an exact sequence of* N-*restricted based complexes, such that the bases* (a) *and* (c) *for say* $A_i$ *and* $C_i$ *give together a base* (ac), G-*equivalent to the fixed base in* $B_i$. *Suppose that* $C_*$ *is* E-*equivalent to* $D_*$. *Then there exists an exact sequence* $0 \longrightarrow A_* \longrightarrow F_* \longrightarrow D_* \longrightarrow 0$ *such that* $F_*$ *is* E-*equivalent to* $B_*$ *and the bases for* $A_i$ , $D_i$ *compose to the base* G-*equivalent to that of* $F_i$.

Proof. By lemma 1.7 it suffices to prove our lemma for the case when the base in B is composed from that of $A_*$ and $C_*$ . It is enough obviously to consider the case of $D_*$ , obtained from $C_*$ by means of one of the elementary operations. For the operations 1), 2), 3) that is obvious. Consider the operation 4). The assumptions imply that the map $\partial : B_{s+1} \to B_s$ is of the following form. The $B_{s+1}$ and $B_s$ decompose as $A_{s+1} \oplus X \oplus R^k$ and $A_s \oplus Y \oplus R^k$ respectively, where X and Y are based modules, $R^k$ has the standard base and these bases compose together and with that of $A_{s+1}$ , $A_s$ to give the bases equivalent to the fixed bases in $B_{s+1}$, $B_s$. Further, $\partial(A_{s+1}) \subset A_s$ and the third coordinate of the map $\partial | R^k$ is identify. From the last property it follows that by an elementary operation 2), changing only the base elements of $R^k$ we can pass to the new base in $B_s$, such that $B_s$ decompose as $A_s \oplus Y \oplus R^k$ and differential $\bar{\partial} : B_{s+1} \longrightarrow B_s$ is identical on the third component. This implies that $\Gamma_*^{(s,k)}$ is a direct summand of B and can be split away with the help of an elementary operation 4).

Now we proceed as to prove proposition 1.6 in its generality. For each $0 \leqslant i \leqslant N-1$ we add to our $D_*$ the trivial acyclic complex $\{0 \longleftarrow C_i \xleftarrow{id} C_i \longleftarrow 0\}$ concentrated in dimensions i, i+1. The result is denoted $E_*$. Thus $E_0 = D_0 \oplus C_0$ , $E_i = D_i \oplus C_i \oplus C_{i-1}$ for $0 < i < N$ and $E_N = D_N \oplus C_N$ . The natural projection $E_* \longrightarrow D_*$ and embedding $D_* \subset E_*$ are the s.h.e. We define the map $\varphi : C_* \longrightarrow E_*$ as follows. For $c \in C_0$ let $\varphi(c) = (f(c), c)$, for $c \in C_i$ where $0 < i < N$ let $\varphi(c) = f(c), c, \partial c)$, for $c \in C_N$ we let $\varphi(c) = (f(c), \partial c)$. One checks easily that $\varphi$ is a chain map, and since $\pi \circ \varphi = f$, $\varphi$ is a homotopy equivalence. Moreover, since $i \circ \pi$ is homotopic to identity, $\varphi$ is homotopic to

$f$, hence is also the G-s.h.e. Note that $\varphi$ is injective. Indeed, that is obvious for $C_i$ if $i < N$ and if $x \in C_N$ is such that $\varphi(x) = 0$, that is $\partial x = 0$, $f(x) = 0$, then $x$ must be equal to $0$ since $f$ induces isomorphism in homology. Consider the factorcomplex $E_*/\varphi(C_*)$. For $0 < i < N-1$ the modules $E_i/\varphi(C_i)$ are isomorphic to $D_i \oplus C_{i-1}$, hence free. Since $E_*/\varphi(C_*)$ is acyclic, the last module $E_N/\varphi(C_N)$ is stably free. Adding to $E_*$ one more trivial acyclic complex, concentrated in dimensions $N-1$, $N$, we get $E'_*$ together with natural base and the embedding $\varphi : C_* \longrightarrow E'_*$ such that $E'_*$ is obtained from $D_*$ by adding several trivial acyclic complexes, and that the factorcomplex is free. Denote $E'_*/C_*$ by $K_*$ and consider the exact sequence $0 \leftarrow C_* \leftarrow \leftarrow E'_* \leftarrow K_* \longleftarrow 0$. Adding if necessary some trivial acyclic complexes to $K_*$ and choosing there appropriate bases we can assume that the base in $F_*$ and in $C_*$ give the base in $E'_*$, equivalent to the mentioned above. Note that $E'_*$ with this new base is E-equivalent to $E'_*$ with the former base by lemma 1.7. Note also that $E'_*$ with the former base is obviously E-equivalent to $D_*$. By lemma 1.2 we have $\tau(F_*) = \tau(\varphi)$, which vanishes in $K_1(R, G)$. By lemma 1.9 the complex $K_*$ is E-equivalent to zero and by lemma 1.10 there exists a G-simple isomorphism $0 \longleftarrow C_* \longleftarrow F_* \longleftarrow 0$, where $F_*$ is E-equivalent to $E'_*$. Hence $C_*$ is E-equivalent to $E'_*$ and proposition 1.6 is proved.

<u>Remark 1.11</u>. The proof could be simpler if we would not insist that all the intermediate complexes were N-

restricted: instead of $E'_*$ we could use $D_* \oplus \mathcal{C}(id)$ where $\mathcal{C}(id)$ stands for the cone of the identity $C_* \to C_*$ . But we will need precisely the above strong statement.

<u>Remark 1.12.</u> The above proof can be also simplified as to give a (comparatively) new proof of Cockroft-Swan theorem [C-S]: the homotopy equivalent complexes become isomorphic after adding some number of trivial acyclic complexes. The length of these isomorphic ones can be assumed to be not greater than the length of the original ones.

## 2. The statement of the main theorem

We recall first the result of [Pa1].

Let $M^n$ be a closed manifold (that means smooth component, connected and without boundary). Let $f: M \to S^1$ be a Morse map. Denote by $\xi$ the homotopy class of $f$ as belonging to $Hom(\pi_1 M, Z) = H^1(M, Z)$. Denote by $\Lambda$ the group ring $Z[G]$ and by $\Lambda_\xi^-$ the Novikov completion of $\Lambda$ (introduced by Novikov [No1] for the case of free abelian $G$ and by Sikorav [Si] for the general case). Let $v$ be a gradient-like vector field for $f$, such that all the stable manifolds of critical points are transversal to the unstable ones (the existence of such a vector field follows from an appropriate version of Kupka-Smale theorem, exposed in [Pa1, app. A]). We denote by $Wh(G, \xi)$ the group $K_1(G)/U_\xi^-$ where $U_\xi^-$ is the multiplicative group of units in $\Lambda_\xi^-$ of the type $\pm g + \lambda$, where $supp\, \lambda \in \{x \in G \mid \xi(x) < 0\}$. We fix the liftings of all the critical points of $f$ to $\tilde{M}$ and for each critical point we fix an orientation of stable manifold (descending disc).

To all these data we have associated in [Pa1] the complex $C_*(v, \tilde{M})$, which possesses the following properties

1) For each $p$ the module $C_p(v, \tilde{M})$ is a finitely generated free right module over $\Lambda_\xi^-$, and the number of generators equals the number of critical points of $f$ of index $p$.

2) The complex $C_*(v, M)$ is homotopy equivalent to $C_*(\tilde{M}) \underset{\Lambda}{\otimes} \bar{\Lambda}_\xi$ via a homotopy equivalence of which the torsion vanishes in $Wh(G, \xi)$.

The main aim of the present paper is to present a theorem, which is in a sense a converse to this result. For that we need some more definitions and notations.

The homomorphism $\xi : \pi_1 M \longrightarrow Z$ is called regular, if it is epimorphic and $\mathrm{Ker}\,\xi$ is a finitely presented group.

A Morse function $f: W \longrightarrow [a,b]$ where $W$ is a cobordism, $\partial W = V_0 \cup V_1$, $f^{-1}(a) = V_0$, $f^{-1}(b) = V_1$, and all the critical points of $f$ belong to $W_0 = W \setminus (V_0 \cup V_1)$, is called regular if there are no critical points of indices 0, 1, n-1, n all the regular preimages $f^{-1}(c)$ are connected and the inclusion induced homomorphism $\pi_1(f^{-1}(c)) \longrightarrow \pi_1(W)$ is isomorphism.

A Morse map $f: M \longrightarrow S^1$, belonging to a class $\xi \in H^1(M, \mathbf{Z})$, is called regular if there are no critical points of indices 0, 1, n-1, n all the regular preimages $f^{-1}(c)$ are connected and the inclusion induced homomorphism $\pi_1(f^{-1}(c)) \longrightarrow \pi_1(M)$ is an injection onto the subgroup $\mathrm{Ker}(\xi : \pi_1 M \rightarrow Z)$.

<u>Lemma 2.1.</u> *In a regular class* $\xi \in H^1(M, Z)$ *there exists always a regular Morse map* $f: M \rightarrow S^1$.

The proof is due essentially to Farrell [Far] and will be postponed until we state the main theorem.

The gradient-like vector field $v$ for $f$ is called perfect if all the stable manifolds of critical points are transversal to all the unstable ones. It is called good, if the stable manifold of a critical point $p$ is transfersal to the unstable manifold of a critical point $q$, if only ind $p$ < ind $q - 1$. The theorem from [Pa1], stated before, is actually valid for all the good gradient-like vector fields and the proof is just the same for this case as before.

We call a quadruple of objects $(f, \bar{f}, v, E)$ a regular Morse quadruple belonging to $\xi \in H^1(M, Z)$ (abbreviation: an r-quadruple.) if:

1) $f: M \longrightarrow S^1$ is a regular Morse map, belonging to a regular class $\xi \in H^1(M, Z)$.

2) $\bar{f}: \bar{M} \longrightarrow R$ is a Morse function on the cyclic covering $\bar{M} \longrightarrow M$, induced by $f$ from $S^1$, such that $f(xt) = f(x) - 1$, where $t$ is a generator of a group $Z$, acting on $M$, and quotient map of $\bar{f}$ by action of $Z$ is $f$.

3) $v$ is a good gradient-like vector field for $f$.

4) $E$ is a system of liftings of critical points of $f$ to the universal covering together with fixed orientations of stable manifolds of critical points.

(Sometimes when the orientations do not change during some section we speak simply of "system of liftings" $E$, as, for example, in §5, 6, 7.)

Note that the Novikov complex for a map $f: M \rightarrow S^1$ is defined in terms of the trajectories of the gradient-like field $v$ and actually depends only on $v$ and on the system

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Note that the Novikov complex for a map $f: M \to S^1$ is defined in terms of the trajectories of the gradient-like field $v$ and actually depends only on $v$ and on the system of liftings and orientations. That is why we will denote it $C_*(v, E)$ from now on, and suppress $f$ in the notation.

By a "complex" we mean a free finitely generated chain complex of right $\Lambda^-_\xi$-modules with a fixed base. Recall that $\Lambda^-_\xi$ consists of such power series $\lambda$ in the variables $g \in G$, that for any $c \in \mathbf{R}$ the intersection $\operatorname{supp}\lambda \cap \{\xi(g) \geqslant c\}$ is finite.

We say that two complexes $C_*$, $D_*$ over $\Lambda^-_\xi$ are simply homotopy equivalent, if there exists a homotopy equivalence $\varphi : C_* \to D_*$ such that the torsion $\tau(\varphi)$ vanishes in $Wh(G, \xi)$.

We denote by $\Lambda_{\xi,k}$ the abelian subgroup in $\Lambda^-_\xi$, consisting of all the power series $\lambda$, such that $\operatorname{supp}\lambda \in \{\xi(g) \leqslant -k\}$. (Note that $\Lambda_{\xi,k}$ is not a subring).

Two complexes $D_*$, $C_*$ will be called N-equivalent: $D_* \underset{N}{\sim} C$ if there is a bijective correspondence of their bases, such that the matrices of differentials in these bases are the same modulo $\Lambda^-_{\xi,N}$. We shall say that $D_*$, $C_*$ are the same if there exists an isomorphism between $D_*$ and $C_*$, preserving bases.

<u>Theorem 2.2</u>. *Let $M^n$ be a closed manifold, $n \geqslant 6$, $\xi : \pi_1 M \to Z$ be a regular homomorphism. Let $D_* = \{0 \leftarrow D_2 \leftarrow \dots \leftarrow D_{n-2} \leftarrow 0\}$ be a complex over $\Lambda^-_\xi$, simply homotopy equivalent to $C_*(\tilde{M}) \underset{\Lambda}{\otimes} \Lambda^-_\xi$.*

Then for any natural $N > 0$ there exists a Morse map $f : M \to S^1$, belonging to $\xi$, a good gradient-like vector field $v$ for $f$, and a system $E$ of liftings and orientations, such that $D_*$ and $C_*(v, E)$ are N-equivalent.

By lemma 2.1 there exist regular maps $f: M \longrightarrow S^1$, belonging to $\xi$. Therefore, to prove this theorem it suffices to prove the following.

<u>Theorem 2.3.</u> Let $M^n$ be a closed manifold, $n \geqslant 6$, $\xi: \pi_1 M \longrightarrow Z$, be a regular class, $\Theta = (f, \bar{f}, v, E)$ be an r-quadruple, belonging to $\xi$, $D_* = \{0 \leftarrow D_2 \leftarrow \dots \leftarrow D_{n-2} \leftarrow 0\}$ be a complex, simply homotopy equivalent to $C_*(v, E)$, and $N > 0$ a natural number.

Then there exists a new r-quadruple $(g, \bar{g}, w, E')$, belonging to $\xi$, such that $C_*(w, E')$ is N-equivalent to $D_*$.

The idea of the proof is as follows. By $\S 1$ we know that any complex, simply homotopy equivalent to $C_*(v, E)$ is obtained from $C_*(v, E)$ by a series of elementary operations like elementary change of base, adding and cancelling of trivial direct summand. We are, therefore, to show that each operation like that can be realized (up to some N) by changing the function and the vector field appropriately. This will occupy the rest of the paper.

We start with the proof of lemma 2.1. By the argument of Farrell [Far, p.325] there exists a smooth closed cooriented submanifold $i: N^{n-1} \subset M^n$, such that N is connected, $i_*: \pi_1 N^{n-1} \rightarrow \pi_1 M$ is the injection with the image $\ker \xi$ and the Pontryagin-Thom construction, applied to N gives a map $M \longrightarrow S^1$, homotopic to $\xi$. We cut M along N and get a connected cobordism W with two components of boundary $V_0$ and

$V_1$, such that $V_0 \approx V_1$ and the inclusions $V_0 \subset W \supset V_1$ induce the isomorphism in $_1$. Since $\dim W = n > 6$ there exists a Morse function on the cobordism W without critical points of indices 0, 1, n-1, n (see [K] or apply directly the argument from [Mi2, § 8].) We can suppose that for any pair x,y of the critical points ind x > ind y we have $f(x) > f(y)$. Now our lemma will follow from the following simple remark which will be frequently used in the sequel.

We call a Morse function $f: W \to [a, b]$ enumerating if for every pair of critical points x,y, such that ind x > ind y we have $f(x) > f(y)$.

<u>Remark 2.4.</u> Let $W^n$ be a cobordism, $\partial W = V_0 \cup V_1$, W, $V_0$, $V_1$ are connected, $n \geq 6$, $\pi_1(V_0) \to \pi_1(W) \leftarrow \pi_1(V_1)$ are isomorphisms, $f: W^n \longrightarrow [a,b]$ be an enumerating Morse function, such that $f^{-1}(a) = V_0$, $f^{-1}(b) = V_1$.

Then f is regular.

<u>Proof.</u> Let $a < c_1 < \ldots < c_N < b$ be a sequence of regular values of f, separating all the critical values one from another. Since there are no critical points of index n-1, all the $f^{-1}(c_i)$ are connected. Let $c_k$ (resp: $c_\ell$) denote the first (resp. the last) of the $c_i$, possessing the property that all the critical values of index 2 (resp. n-2) lie below (resp. above) $c_i$. If we prove for every i the isomorphisms $\pi_1(f^{-1}(c_i)) \to \pi_1(f^{-1}[c_i, c_{i+1}]) \leftarrow \pi_1(f^{-1}(c_{i+1}))$ then we are over. For i = 0 that is obvious since $\pi_1(f^{-1}(-1)) \to \pi_1(W)$ is monic and $f^{-1}[c_0, c_1]$ is obtained from $f^{-1}(c_0)$ by attaching the 2-handles. Similarly

it is obvious for all $i \leqslant k-1$ and for $i \geqslant \ell$. For the rest values of $i$ it is even more obvious.

Remark 2.4.A. If $f:M \longrightarrow S^1$ is a regular Morse function, and $c \in R$ is a regular value of $\bar{f}:\bar{M} \to R$, then the inclusion of the cobordism $W_n = f^{-1}([c, c+n]) \subset M$ induces iso in $\pi_1$.

Proof obvious.

Let $C_*$ be a based free complex of right modules over some ring $R$. Let $\Lambda$ be a function, defined on the elements of base of $C_*$ with the values in $R^*$. Then we denote by $\Lambda(C_*)$ the complex obtained from $C_*$ by the base change $e_i \mapsto e_i \cdot \Lambda(e_i)$.

Remark 2.5. Let $(f, \bar{f}, v, E)$ be a regular quadruple, belonging to a regular class $H^1(M, Z)$. Here $E$ is a function $q \longmapsto (\tilde{q}, \theta(q))$ where $q \in Crf$, $\tilde{q}$ is some lifting of $q$ to $\tilde{M}$ and $\theta$ is some orientation of descending disc $D(q, v)$. Let $\Lambda$ be a function on $Crf$, such that $\Lambda(q) = \mathcal{E}(q) \cdot g(q)$, where $\mathcal{E}(q) = +1$, $g(q) \in G$ for every $q \in Crf$. Denote by $\Lambda(E)$ the new system of liftings of critical points to $\tilde{M}$ and orientations of descending discs, such that $\tilde{q}' = \tilde{q} \cdot g(q)$, $\theta'(q) = \mathcal{E}(q) \cdot \theta(q)$.

Then $C_*(v, \Lambda(E))$ is isomorphic, preserving bases, to $\Lambda(C_*(v, E))$.

Proof. It is enough to prove the assertion for the case when $\Lambda$ is equal to 1 for all the elements $q \in Crf$, except one $q_0$ and $\Lambda(q_0) = g$ or $\Lambda(q_0) = -1$.

1) $\Lambda(q_0) = -1$. Recall that the differential in the Novikov complex $\partial_m : C_m(v, E) \to C_{m-1}(v, E)$ is defined on

the base elements by the formula $\partial e_i = \sum f_j \cdot n(e_i\ f_j)$, where $e_i$ is an element of Crf of index $m$ and $f_j$ runs through the elements of Crf of index $m-1$. The element $n(e_i, f_j)$ in turn is the sum $\sum_\gamma \varepsilon(\gamma)\, g(\gamma)$ where $\gamma$ runs through the $(-v)$-trajectories in $M$, joining $e_i$ with $f_j$, $\varepsilon(\gamma)$ is the sugn of intersection of the (oriented) disc $D(e_i, v)$ with the (cooriented) disc $D(f_j, -v)$

and $g(\gamma)$ is defined from the condition that the lifting $\tilde{\gamma}$ of $\gamma$ to $\tilde{M}$, starting at $\tilde{e}_i$, finishes at $\tilde{f}_j \cdot g(\gamma)$. If we change orientation of a generator $q_o$, then the value of $n(e_i, f_j)$ will change only when one of $e_i$, $f_j$ coincides with $q_o$, and then the new $n'$ differs from $n$ by sign. Thus the map $\varphi: C_*(v, E) \to C_*(v, \Lambda(E))$ which sends all $e_i$ to themselves, except $q_o$ and $\varphi(q_o) = -q_o$ commutes with differentials. That is, the complex $C_*(v, E)$ is isomorphic to $\Lambda^{-1}(C_*(v, \Lambda(E)))$ which is equivalent to our statement.

2) $\Lambda(q_o) = g$. It follows from the definition that the new $n'(e_i, f_j)$ equals the old one if $e_i \neq q_o$, $f_j \neq q_o$. If $e_i = q_o$, then $n'(e_i, f_j) = n(e_i, f_j)g$; if $f_j = q_o$, then $n'(e_i, f_j) = g^{-1}\, n(e_i, f_j)$. One easily checks up that the map $\varphi: C_*(v, E) \to C_*(v, \Lambda(E))$, which sends all $e_i$ to themselves, except $q_o$ which goes to $q_o \cdot g^{-1}$, commutes with differentials. That is, the complex $C_*(v, E)$ is isomorphic, preserving bases to $\Lambda^{-1}(C_*(v, \Lambda(E)))$, which is equivalent to our statement.

Now we list some definitions to be used in the sequel.

If $f$ is a Morse function on M (with values in **R** or in $S^1$) we denote by Crf the set of critical points of f.

If $f:M \longrightarrow S^1$ is a function, $[f] = \xi$ and E is some finite set in $\bar{M}$ we denote by ex(E, f) the number $\max_{x,y \in E} |\bar{f}(x) - \bar{f}(y)|$, where $\bar{f}:\bar{M} \longrightarrow R$ is some lifting of f. (It is obvious, that ex(E, f) does not depend on the particular choice of lifting $\bar{f}$.).

## 3. Preliminaries on Morse functions

This section contains some lemmas on stable and unstable manifolds of critical points of Morse functions.

First some notations.

We denote by $D^s_o(0)$ (resp. $B^s_o(0)$) the closed (resp. open) disc of radius r around the origin in $R^s$. If the dimension is clear from the context, we omit it.

Let $f:W \longrightarrow [a,b]$ be a Morse function on the compact cobordism W, $\partial W = V_o \cup V_1$, $V_o = f^{-1}(a)$, $V_1 = f^{-1}(b)$. We assume that all the critical points of f belong to $W_o = W \setminus (V_o \cup V_1)$. Let v be any gradient-like vector field for f. If p is a critical point of f, we denote by D(p,v) the union of all v-trajectories $\gamma$, such that $\gamma(t) \to p$, when $t \to +\infty$. We call this union the descending disc, although in general it is not a disc and even not a manifold. (Respectively, D(p,−v) stands for the union of all (−v)-trajectories , such that $\gamma(t) \to p$, when $t \to +\infty$).

If p is a critical point of f then (by abuse of notations) we denote by $D_\delta(p)$, resp. $B_\delta(p)$ the image of closed (resp. open) $\delta$-disc under the standard coordinate system (assuming that $\delta$ is so small, that $D_\delta(p)$ belongs to the domain of the mapping, together with its open neighbourhood). We denote $\partial D_\delta(p)$ by $S_\delta(p)$. We assume that $\delta$ is so small that $D_\delta(p) \cap (V_o \cup V_1) = \emptyset$. We denote by $D^-_\delta(p)$, $B^-_\delta(p)$ (resp. $D^+_\delta(0)$, $B^+_\delta(0)$) the images of the negative (resp. positive) closed and open o-disc.

Underline: Lemma 3.1. 1) *The intersection* $D(p,v) \cap W_0$ *is a smooth submanifold of* $W_0$.

2) *The intersection* $D(p,v) \cap V_0$ *is a smooth submanifold of* $V_0$; *the closure* $\overline{D(p,v) \cap V_0}$ *equals* $\overline{D(p,v)} \cap V_0$.

Proof. 1) Let $x \in W_0$, $x \in D(p,v)$. That implies that there exists a point $x_0 \in D_\delta(p)$ and a (+v)-trajectory, starting at $x$ and finishing at $x_0$ at a moment $t_0$. Consider a small ball $D_\beta(x_0)$, contained in $D_\delta(p)$, and a diffeomorphism $\Phi$ of shift by along (-v). We can assume that $\Phi(D_\beta(x_0)) \subset W_0$. Note that a point $z$ belongs to $D(p,v)$ if and only if $\Phi(z)$ belongs to $D(p,v)$. Note also that $D(p,v) \cap D_\beta(x_0) = D_\beta(x_0) \cap D_0^-(0)$. (See App. 1, corollary A.2). That implies that $\Phi$ gives a diffeomorphism $(D_\beta(x_0), D_\beta(x_0) \cap D_\delta^-(0)) \longrightarrow (U, U \cap D(p,v))$ where $U$ is a neighbourhood $\Phi(D_\beta(x_0))$ of $\Phi(x_0)$, which is a definition of a submanifold.

The first part of p. 2) is proved similarly. To prove the second we note that $\overline{D(p,v) \cap V_0} \subset \overline{D(p.v)} \cap V_0$ obviously and to prove the inverse inclusion consider any sequence $\alpha_i \in D(p,v)$ converging to $\alpha \in V_0$. Note that $a_i = f(\alpha_i) \longrightarrow a$ and therefore tere are no critical points of $f$ in $[a, a_i]$, which means that the <df, v> is bounded from below by, say, $\delta$ in some domain, containing all the $\alpha_i$. This implies that the (-v)-trajectory, starting at $\alpha_i$, reaches the boundary $V_0$ at the moment $\tau_i \leqslant \frac{a_i - a}{\delta} \longrightarrow 0$, hence the ends $\beta_i$ of these trajectories belong to $V_o$ and converge to a, Q.E.D.

Now we need more notations. Let $c \in R$ be a regular value of $f = W \to [a,b]$ and assume that $\delta > 0$ is so small, that if a critical point p belongs to $f^{-1}([a,c))$, then $D_\delta(p)$ belongs also to $f^{-1}([a, c))$. We denote by $V_c(\delta,v)$, resp. $U_c(\delta,v)$ the union of all (-v)-trajectories, starting at some point z of some disc $D_\delta(p)$ (resp. $B_\delta(p)$), where p is a critical point in $f^{-1}([a,c])$. The notations $V_c(\delta,-v)$, $U_c(\delta,-v)$ are now clear without explanations. If c = a, then we abbreviate $V_c(\delta,v)$ to $V(\delta,v)$ (resp. $U_c(\delta,v)$ to $U(\delta,v)$). The union of all D(p,v) for $p \in f^{-1}([a,c])$ will be denoted $K_c(v)$, and if c = a, then simply K(v). The notations $K_c(-v)$, K(-v) are now clear without explanation.

<u>Lemma 3.2.</u> *Let* $c \in [a,b]$ *be a regular value and* $\delta > 0$ *be small enough. Then*:

1) $U_c(\delta,v)$ *is open.*

2) $V_c(\delta,v)$ *is compact and* $\overline{U_c(\delta,v)} = V_c(\delta,v)$.

3) $K_c(v)$ *is compact.*

4) $V_c(\delta,v)$ *equals* $\bigcap_{\delta'>\delta} U_c(\delta',v)$.

5) $K(v)$ *equals* $\bigcap_{\delta'>0} U_c(\delta',v)$.

6) *For every open neighbourhood* U *of* $V_c(\delta,v)$ *there exists* $\delta' > \delta$, *such that* $U_c(\delta',v) \subset U$.

7) *For every open neighbourhood* $U$ *of* $K_c(v)$ *there exists* $\delta' > 0$, *such that* $U_c(\delta', v) \subset U$.

Proof. 1) Let $z \in U_c(\delta,v)$. We must find an open neighbourhood $S$ of $z$, belonging to $U_c(\delta, v)$. We do it for $z \in W_0$, for $z \in V_0$ the argument is similar. Find a critical point $p \in f^{-1}([a,c])$ and a point $z_0 \in B_\delta(p)$, such that the $(-v)$-trajectory $\gamma$, starting at $z_0$, arrives at $z$ at a moment $\tau$. Take a neighbourhood $S'$ of $z_0$ so small, that $S' \subset B(p)$ and the diffeomorphism $\Phi$ of shift by $\tau$ along $(-v)$ carries $S'$ to a set, belonging to $W_0$. We can set $S = \Phi(S')$.

4) The inclusion $V_c(\delta,v) \subset \bigcap_{\delta'>\delta} U(\delta',v)$ is obvious. On the other hand, let $x \in \bigcap_{\delta'>\delta} U_c(\delta',v)$. Consider the $v$-trajectory $\theta$, starting at $x$. There are two possibilities: a) either $\theta(\tau) \to p$ where $\tau \to \infty$, $p$ is some critical point, $f(p) < c$, b) $\theta$ reaches the level $f^{-1}(c)$ at some moment $\tau_0 > 0$. In the first case $x \in K_c(v) \subset V_c(\delta,v)$. In the second the intersection of $\theta$ with $f^{-1}([a,c])$ is a compact, which intersects, by our assumption, with the $\bigcap_{\delta'>\delta} \bigcup_{p\in f^{-1}([a,c])} B_{\delta'}(p) = \bigcup_p \bigcap_{\delta'>\delta} B_{\delta'}(p) = \bigcup_p D_\delta(p)$, q.e.d.

5) The proof is the same as that of 4).

Note that 4) and 5) imply $V_c(\delta,v) = \bigcap_{\delta'>\delta} V_c(\delta',v)$ and $K_c(v) = \bigcap_{\delta'>\delta} V_c(\delta',v)$.

2) Denote by $p_1,\dots,p_k$ the critical points in $f^{-1}([a,c])$. We choose $\delta$ so small that $p_i$ are contained in

$f^{-1}((a,c))$ together with $D_o(p_i)$. Denote by $P$ the union of $D_\delta(p_i)$. Denote by $Q$ the set of such points $x \in W$, that $x$ does not belong to any of $B_\delta(p_i)$ and the v-trajectory $\gamma$, starting at $x$ intersects the union $\bigcup_i S_\delta(p_i)$. Note that $V_c(o,v) = P \cup Q$; $P$ is obviously compact and we are only to prove that $Q$ is compact. Suppose that $x_n \in Q$. Denote by $\gamma_n$ the v-trajectory, starting at $x_n$ and by $\tau_n$ the first moment when $\gamma_n$ intersects $\bigcup_i S_\delta(p_i)$. Note that the sequence $\tau_n$ is bounded from above. Indeed, if we denote by $\varepsilon$ the infimum of $\langle df, v\rangle$ on the domain $f^{-1}([a,c]) \setminus (\bigcup_i B_o(p_i))$, then $\tau_n < 2 \cdot \frac{a-c}{\varepsilon}$ (because $\gamma_n \setminus [0, \tau_n]$ does not intersect $B_\delta(p_i)$. Denote by $y_n$ the points of intersection $\gamma_n(\tau_n)$ and denote by $\theta_n$ the $(-v)$-trajectory, starting from $\gamma_n(\tau_n)$. The union $\bigcup_i S_\delta(p_i)$ is compact, hence, passing to a subsequence we can assume that $y_n \to y \in \bigcup_i S_\delta(p_i)$, $\tau_n \to \tau \geqslant 0$. Consider now the $(-v)$-trajectory $\theta$, starting at $y$. I claim that it is defined on the segment $[0,\tau]$. (We can assume $\tau > 0$.) Indeed, the opposite would mean that for some $\nu$, $0 < \nu < \tau$ the trajectory $\theta$ is defined on $[0, \nu]$ and $\theta(\nu) \in V_o$. From some number N on we have $\tau_n > \nu + \frac{\tau+\nu}{2}$. The trajectories $\theta_n$ are defined on $[0,\tau_n]$, hence $\theta_n(\nu)$ is defined and converge to $\theta(\nu)$; hence $f(\theta_n(\nu)) \to a$. Note now that the points $\theta_n(t)$ belong to $f^{-1}([a,c]) \setminus \bigcup_i D_\delta(p_i)$ for $0 < t < \tau_n$. Therefore $f(\theta_n(\nu + \frac{\tau-\nu}{2})) < f(\theta_n(\nu)) - \frac{\tau-\nu}{2}\cdot\varepsilon$ which contradicts $f(\theta_n(\nu)) \to 0$. Then by the standard theorems we have $\theta_n(\tau_n) \to \theta(\tau)$, but $\theta(\tau)$ obviously belongs to $Q$.

To finish with 2) we are to show, that $\overline{U_c(\delta,v)} \supset V_c(\delta,v)$ (the inverse inclusion is due to the compactness of $V_c(\delta,v)$). Let $x \in V_c(\delta,v)$. Then there exists a point y, belonging to a disc $D_\delta(p_i)$, such that the (-v)-trajectory starting at y, reaches x at some moment $\tau$. We take a sequence $y_n \in B_\delta(p_i)$, converging to y. Then one easily shows, that the sequence $\gamma_n(t_n)$ converge to x, where $\gamma_n$ is the (-v)-trajectory, starting at $y_n$, and $t_n$ is the minimum of $\tau$ and the moment, when $\gamma_n$ crosses $V_0$.

3) follows from $K_c = \bigcap_{\delta'>0} V_c(\delta',v)$.

The properties 6) and 7) follow from the following obvious general statement: If $K \subset M$ are compacts and U is an open neighbourhood of K in M and $V_1 \supset V_2 \supset \ldots \supset K$ is a sequence of compacts, such that $\bigcap_{i=1}^{\infty} V_i = K$, then there exists n, such, that $V_n \subset U$.

Next we study the behaviour of the sets $V_c(\delta,v)$, $U_c(\delta,v)$ under small perturbations of v. We call the vector field $v'$ an $\varepsilon$-regular perturbation of v, if supp$(v' - v)$ does not intersect with $D_\varepsilon(p_i)$ for evry critical point $p_i$.

We shall need the standard lemma, stating that the trajectories of the vector field depend continuously on the vector field and the initial value. We state it here without proof.

Lemma 3.3. *Let M be a $C^\infty$ -manifold without boundary, v be a $C^\infty$ -vector field on M, $\gamma : [0, t] \longrightarrow M$ be a trajectory of $\gamma$, A be a neighbourhood of the image $\Gamma = \gamma([0,t])$, B be a neighbourhood of $\gamma(t)$.*

Then there exists a neighbourhood C of $\gamma(0)$ and a neighbourhood $\mathcal{V}$ of v in the space of $C^\infty$-vector fields, such that for any $x \in C$ and $v' \in \mathcal{V}$ the trajectory $\gamma'$ of $v'$, starting at x is defined on $[0, t]$, the image $\Gamma' = \gamma'([0, t])$ belongs to A, the end $\gamma'(t)$ belongs to B.

Lemma 3.4. *Let $\varepsilon > 0$ be small enough and let $c \in [a,b]$ be a regular value of f. Then for all sufficiently small $\delta$ the following holds:*

1) *If U is some neighbourhood of $V_c(\delta, v)$, then for every $\varepsilon$-regular perturbation $v'$ of v, close enough to v, the set $V_c(\delta,v')$ is contained in U.*

2) If $Q$ is some compact, belonging to $U_c(\delta, v)$, then for every $\varepsilon$-regular perturbation $v'$ of $v$, close enough to $v$, the set $U_c(\delta, v')$ contains $Q$.

Proof. We can assume $c < b$, since $V_b(\delta, v) = V_c(\delta, v)$ for $b-c > 0$ sufficiently small and for $\delta > 0$ sufficiently small. We demand $\delta < \varepsilon$. By lemma 3.2. 6) it suffices to prove our assertion for $U = U_c(\delta', v)$ where $\delta < \delta' < \varepsilon$. Assume that it is not true. Then there exists a sequence $v_n \to v$ of $\varepsilon$-regular perturbations of $v$, and for each $v_n$ there exists $x_n \in V_c(\delta, v_n)$, $x_n \notin U_c(\delta', v)$. Let $p_1, \dots, p_k$ denote the critical points of $f$ in $f^{-1}([a, c])$. We have $x_n \notin B_{\delta'}(p_i)$. Since $v_n \to v$ there exists an $\mathcal{E} > 0$, such that $df(v_n) \geq \mathcal{E}$ in the complement $f^{-1}([a, c]) \setminus \bigcup_{i=1}^{k} D_\delta(p_i)$. Passing to a subsequence if necessary we can assume that $x_n \to x$. Obviously $x \notin V_1$, and we can assume $x \notin V_o$. Since $U_c(\delta', v)$ is open we have $x \notin U_c(\delta', v)$. That implies, that the $(-v)$-trajectory $\gamma$, starting at $x$, does not intersect the balls $B_{\delta'}(p_i)$ and since $p_i$ are the only critical points in $f^{-1}([a, c])$, it arrives at $f^{-1}(c)$ at a moment $\tau \leq \frac{c-a}{\mathcal{E}}$. The image $\Gamma = \gamma([0, \tau])$ is a compact, which does not intersect $\bigcup_i D_\delta(p_i)$. Hence there is an open neighbourhood $A$ of $\Gamma$ in $W$, such that $A \cap (\bigcup_i D_\delta(p_i)) = \emptyset$. Choose the open neighbourhood $B$ of $\gamma(\tau)$ in $W$ so small that $f \mid B$ is strictly greater, that $\max_{x \in D_\delta(p_i)} f(x)$. By our assumptions $\Gamma \subset W_o$. Therefore we can apply our lemma 3.3 and deduce that for $n$ sufficiently big the trajectory $\gamma_n$

of $v_n$, starting at $x_n$ is defined on $[0, \tau]$, belongs to $A$, and $\gamma_n(\tau) \in B$. That implies that the trajectory $\gamma_n$ will never intersect any of $D_\delta(p_i)$, contradiction.

2) Pick up $a' > a$ such that the segment $[a,a']$ is regular. It suffices to prove our assertion for two cases: a) $Q \in f^{-1}([a',c])$, b) $Q \in f^{-1}([a,a']))$.

a) By definition for each $x \in Q$ the $(+v)$-trajectory $\gamma$, starting from $x$ is at some moment $\tau$ in some $B_\varepsilon(p_i)$. Denote $\gamma(\tau)$ by $y$ and choose the neighbourhood $U(x)$ of $x$ and the neighbourhood $\mathcal{V}_x$ of $v$ so small, that the diffeomorphism of $\tau$-shift along $(+w)$ is defined for every $z \in \overline{U(x)}$ and $w \in \mathcal{V}$ and carries $\overline{U(x)}$ to $B_\varepsilon(p_i)$. Choose a finite covering of $K$ from $\{U(x)\}$. The intersection of corresponding $\mathcal{V}_x$ is a neighbourhood of $v$, which satisfies the conclusion.

b) Pick up $a'' > a'$ such that $[a, a'']$ is regular. Pick up a $C^\infty$-function $h$ on $\mathbf{R}$, such that $0 \leqslant h \leqslant 1$, $h(t) = 1$ for $t \leqslant a'$ and $h(t) = 0$ for $t \geqslant a''$. For any gradient-like vector field $v$ for $f$ define a new gradient-like vector field $\bar{v}$ by $\bar{v} = \frac{h(f(x))}{df(v)} v + (1 - h(f(x)))v$. The vector field $\bar{v}$ satisfied $df(\bar{v}) = 1$ for $x \in f^{-1}([a,a'])$ and $\bar{v} = v$ for $x \in f^{-1}([a'',b])$. If $v$ and $w$ are close to each other, then $\bar{v}$ and $w$ are close. Note that the set $U_c(\delta,v)$ is the same as $U_c(\delta,\bar{v})$. This implies, that it suffices to prove our assertion for the vector fields $v$, such that $df(v) = 1$ in $f^{-1}([a, a'])$. Consider the diffeomorphism $\Phi : V_0\ [a, a'] - f^{-1}([a, a'])$, such that $\Phi^{-1}(v) = (0, 1)$, $f \circ \Phi = \pi_2$ (i.e.

the projection onto the second coordinate). We need an easy lemma.

<u>Lemma.</u> *For any point* $x \in V_0$ *and any neighbourhood* $U$ *of* $x$ *in* $V_0$ *there exists a neighbourhood* $V \subset U$ *of* $x$ *in* $V_0$, *and a neighbourhood* $\mathcal{V}$ *of* $v$, *such that if* $w \in \mathcal{V}$ *and the second coordinate of* $w$ *is* 1 *and* $y \in V \times [a,a']$, *then the w-trajectories, starting at* $V \times [0, 1]$ *stay in* $U \times [0, 1]$.

(For the proof choose the coordinate system near $x$. Then the first coordinate of w-trajectory is given by a non-autonomic differential equation in $R^n$ with small righthand part.)

Now we consider the compact $K'$, which consists of all points $z \in f^{-1}([a,a'])$, lying on some v-trajectory, intersecting with $K$. Obviously $K \subset K' \subset U_c(\delta, v)$. Denote $U_c(\delta, v) \cap f^{-1}(a')$ by $U_0$, that is an open set in $f^{-1}(a')$. Now we identify $f^{-1}([a,a'])$ with $V_0 \times [a,a']$ by means of $\Phi$. The compact $K'$ is a product $\bar{K} \times [a,a']$ with $\bar{K}$ compact, $\bar{K} \subset U_0$. For each $x \in \bar{K}$ we choose a neighbourhood $U(x)$ of $x$ in $V_0$ such that $\overline{U(x)} \subset U_0$ and a neighbourhood $V(x)$ such that for every field $w$ with the second coordinate 1, close enough to $v$, the w-trajectories starting at $V(x) \times [a,a']$ rest in $U(x) \times [0, 1]$. Choose from $\{V(x)\}$ a finite covering $\{V(x_i)\}$, and denote by $\mathcal{V}$ the corresponding finite intersection of neighbourhoods of vector field $v$. For each $w \in \mathcal{V}$ with the second coordinate 1 we have that the w-trajectories starting at $\bar{K} \times [a,a']$ rest at $\bigcup_i U(x_i) \times [a,a']$, hence the intersections of these trajectories with $V_0 \times a'$ rest in $\bigcup_i U(x_i) \times a'$. Denote by $\bar{\bar{K}}$ the compact $\overline{\bigcup_i U(x_i)} \times a' \subset f^{-1}(a')$.

For every $w$ close enough to $v$ we have $\bar{\bar{K}} \subset U_c(\delta, w)$ by the part a) of our lemma. So, diminishing our neighbourhood further we get that for some $\mathcal{V}'$ and for all $w \in \mathcal{V}'$ with the second coordinate 1, all the w-trajectories, starting at K' intersect $f^{-1}(a')$ at the point, belonging to $U_c(\delta, w)$. q.e.d.

We shall need the definition of incidence coefficients n(x, y) for the gradient-like vector fields, which are a bit more general than good gradient-like f's.

<u>Definition 3.5.</u> *A gradient-like vector field* v *for* $f:W \longrightarrow [a,b]$ *is called almost good gradient-like vector field (abbreviation:* a.g.g.-l.v.f.) *if for every pair* (p, q) *of critical points, such that* $p \neq q$, ind $p \leqslant$ ind q, *the descending* disc D(p,v) *does not intersect the ascending* disc D(q, −v).

Note that if v is an agglvf for f, then (−v) is an agglvf for (−f).

<u>Lemma 3.6.</u> *Let* $f:W \longrightarrow [a,b]$ *be a Morse function and* v *be an almost good gradient-like vector field for* f.

*Then there exists an enumerating function* $\varphi:W \longrightarrow [a,b]$ *for which*

1) v *is still a gradient-like vector field for* $\varphi$.

2) $\varphi$ *coincides with* f *up to a constant in the small neighbourhoods of critical points.*

3) $\varphi$ *induces a given order on the set of all critical points of a fixed index.*

*If* f *is regular, then* $\varphi$ *can be chosen regular.*

Proof. The ordinary proof, given in [Mi] th. 4.1, works as well in our situation, since if there is a pair of critical points $p, q$, $f(p) > f(q)$, ind $p \leqslant$ ind $q$ and there are no critical values in $(f(q), f(p))$, then $D(p,v) \cap D(q,-v) = \emptyset$ by assumption, and the usual trick works.

If f is regular then $\varphi$ is automatically regular as shown in §2, remark 2.4.

We need more notations. Let $\alpha, \beta \in [a,b]$ be the regular values of f. Let $\delta > 0$ be small enough, so that the closed $\delta$-ball $D_\delta(p)$ belongs to $f^{-1}((\alpha, \beta))$, if $p$ is a critical point in $f^{-1}((\alpha, \beta))$.

For a critical point $p \in f^{-1}([\alpha, \beta])$ we denote by $D_\delta(p;[\alpha, \beta]; v)$ the set of points $z \in f^{-1}([\alpha, \beta])$, belonging to some $(-v)$-trajectory, starting at some point of $D_\delta(p)$. Similarly we denote by $B_O(p;[\alpha, \beta]; v)$ the set of points $z \in f^{-1}([\alpha, \beta])$, belonging to some $(-v)$-trajectory, starting at some point of $B_\delta(p)$. We denote by $D(p;[\alpha, \beta]; v)$ the intersection of descending disc $D(p,v)$ with $f^{-1}([\alpha, \beta])$.

We denote by $\mathcal{V}_{\leqslant s}(\delta;[\alpha, \beta]; v)$ the union of $D_\delta(p;[\alpha, \beta]; v)$ over all critical points $p$ in $f^{-1}([\alpha, \beta])$ of index $\leqslant s$. We denote by $K_{\leqslant s}[\alpha, \beta]; v)$ the union of $D(p;[\alpha, \beta]; v)$ over all critical points $p$ in $f^{-1}([\alpha, \beta])$

of index $\leq s$. Similarly we form the sets $U_{\leq s}(\delta;[\alpha,\beta];v)$.
If $s = \dim W$ we omit it from notations.

Similarly we form the sets $V_{\geq s}(\delta;[\alpha,\beta];v)$, $U_{\geq s}(\delta;[\alpha,\beta];v)$ and $[\alpha,\beta];v)$.

If $[\alpha,\beta] = [a, b]$, we omit it from notations.

Similarly we form the sets $V_{\leq s}(\delta;[\alpha,\beta];-v)$ etc.

Lemma 3.7. Let $f:W \to [a, b]$ be a Morse function, $v$ be an almost good gradient-like vector field for $f$, $\alpha < \beta$ be the regular values for $f$. Then for all $\delta$, sufficiently small:

1) $U_{\leq s}(\delta;[\alpha,\beta];v)$ and $U_{\geq s}(\delta;[\alpha,\beta];v)$ are open.

2) $V_{s}(\delta;[\alpha,\beta];v)$ is compact and $\overline{U_{\leq s}(\delta;[\alpha,\beta];v)} = V_{\leq s}(\delta;[\alpha,\beta]; v)$.

4) For any critical point $p$ in $f^{-1}([\alpha,\beta])$ of index $(s+1)$ the set $V_{\leq s}(\delta;[\alpha,\beta];v) \cup D_{\delta}(p;[\alpha,\beta];v)$ is compact.

5) The set $V_{s}(\delta;[\alpha,\beta];v)$ equals the intersection of all $U_{s}(\delta';[\alpha,\beta];v)$ over all $\delta' > \delta$.

6) For every open neighbourhood $U$ of $V_{\leq s}(\delta;[\alpha,\beta];v)$ there exists $\delta' > \delta$, such that $V_{\leq s}(\delta';[\alpha,\beta];v) \subset U$.

7) $K_{\leq s}([\alpha,\beta];v)$ is compact; for every critical point $p$ in $f^{-1}([\alpha,\beta])$ of index $s+1$ the set $K_{\leq s}([\alpha,\beta];v) \cup D(p;[\alpha,\beta];v)$ is compact; $K_{\leq s}([\alpha,\beta];v) = \bigcap_{\delta>0} U_{\leq s}(\delta;[\alpha,\beta];v)$.

8) Let $U$ be a neighbourhood of $V_{\leq s}(\delta;[\alpha,\beta],v)$. Then for any $\varepsilon$-regular perturbation of $v'$, close enough to $v$, the set $V_{\leq s}(\delta;[\alpha,\beta]];v')$ is contained in $U$. The same holds

if instead of $V_{\leq s}(\delta;[\alpha,\beta];v')$ we substitute $K_{\leq s}([\alpha,\beta];v)$ or $K_{\leq s}([\alpha,\beta];v) \cup D(p;[\alpha,\beta];v)$ or $V_{\leq s}(\delta;[\alpha,\beta];v) \cup D_\delta(p;[\alpha,\beta];v)$,

where $p \in f^{-1}([\alpha,\beta])$ is a critical point of index $s+1$.

9) Let $K$ be any compact in $U_{\leq s}(\delta;[\alpha,\beta];v)$ (resp. $U_{\geq s}(\delta;[\alpha,\beta];v)$). Then for any $\varepsilon$-regular pertubation $v'$ of $v$, sufficiently close to $v$ we have $K \subset U_{\leq s}(\delta;[\alpha,\beta];v')$ (resp. $U_{\geq s}(\delta;[\alpha,\beta];v')$).

Proof. If suffices to prove our lemma for $[\alpha,\beta] = [a,b]$. We prove 4) for example. We choose an enumerating function $\varphi$ on $W$ in such a way that there exist a regular values $c_1, c_2 \in [a,b]$, such that $c_1 < c_2$, all the critical points of index $\leq s$ lie below $c_1$ and there is only one critical point among $c_1$ and $c_2$, namely $p$. For such a $\varphi$ and $\delta$ sufficiently small the set $V_{\leq s}(o;[a,b];v) \cup D_\delta(p;[\alpha,\beta];v)$ is exactly the set $V_{c_2}(\delta;v)$ in the notation of lemma 3.2 with respect to the function $\varphi$. This set is compact.

The only assertion which does not follow directly in this manner from 3.2 and 3.4 is the part of 9), concerning $U_{\geq s}(\delta;[\alpha,\beta];v)$. To do this we note that lemma 3.4. 2) is valid with $B_\delta(p,v)$ instead of $U_c(\delta,v)$ (the demonstration is the same). Afterwards the compact $K$ can be presented as a finite union of compacts $K_i$, each of which belongs to some $B_\delta(p,v)$ for some critical point of index $\leq s$ and we are over.

Now we proceed to the definitions of the incidence coefficient.

Let $f:W \longrightarrow [a,b]$ be a regular Morse function, let $v$ be an almost good gradient-like vector field for $f$, and $p,q$ be the critical points of $f$, $\operatorname{ind}(p) = \operatorname{ind}(q)+1$. Fix the liftings $\tilde{p}$, $\tilde{q}$ of $p$, $q$ to the universal covering $W$ and the orientations of $D(p)$, $D(q)$. We now define the incidence coefficient $\vartheta(p,q) \in Z\pi$ (where $\pi = \pi_1(W)$) with respect to these data.

Pick up a regular enumerating Morse function $\varphi$ on $W$ with the same g.-l.v.f.v. Denote by $c$ any regular value of $\varphi$ which separates the critical points of index $\leqslant \operatorname{ind}(q)$ from that of index $\geqslant \operatorname{ind}(q) + 1$. The intersections $D(p,v) \cap f^{-1}(c)$; $D(q,-v) \cap f^{-1}(c)$ are smooth spheres of complementary dimensions in the manifold $f^{-1}(c)$. First one is oriented, second one - cooriented. Both can be lifted to the universal covering $[f^{-1}(c)]^{\sim}$, since both are contractible in $W$ and $\varphi$ is regular. The particular liftings will be chosen by lifting to $W$ the discs $D(p,+v)$, $D(q,-v)$ as starting at $p$, $q$. Now the standard definition gives us the intersection coefficient $\vartheta(p,q) \in Z\pi$ which will be called incidence coefficient of $p$, $q$. If we want to stress the dependence on $v$ we write $\vartheta(p,q;v)$.

<u>Lemma 3.8</u>. 1) *The coefficient* $\vartheta(p,q)$ *does not depend on the choice of* $\varphi$ .

2) *If* $v$ *is good, then* $\vartheta(p,q;v)$ *coincide with the standard incidence coefficient as defined ,for example, in* [Pa1, p.45].

3) *A small* $\varepsilon$*-regular perturbation* $v'$ *of any almost good gradient-like vector* $v$ *for* $f$ *is again an* a.g.g.-l.v.f. *for* $f$ *and* $\nu(p,q;v) = \nu(p,q;v')$.

Proof. The property 2) is obvious. To prove 3) note that $v'$ is still a gradient-like vector field for $\varphi$, hence $v'$ is almost good. Then 3) follows from standard properties of intersection coefficient. To get 1) we perturb $v$ to get a good g.-l.v.f. $v'$ and then apply 2) and 3). q.e.d.

We shall need one more definition of $\nu(p,q)$ in terms of our original function $f$. For that we recall some standard facts on intersection coefficients.

Suppose that $N$ is a connected compact manifold without boundary, $X,Y \subset N$ are compacts, $X \cap Y = \emptyset$; $L$ is a compact connected submanifold of $N$, $\partial L \subset Y$, $L \cap X = \emptyset$; $X$ is a compact manifold with boundary $\partial K$, $\varphi : K \to N$ is a smooth map such that $\varphi(K) \cap Y = \emptyset$, $\varphi(\partial K) \subset X$; $K$ and $L$ have complementary dimensions: $\dim K + \dim L = \dim N$. Moreover, we assume that $\varphi_*(\pi_1 K) = \{1\}$, $\mathrm{im}(\pi_1 L \to \pi_1 N) = \{1\}$, that $K$ is oriented and $L$ cooriented ( or vice versa) and that there are fixed the liftings of $\varphi : K \to N$ and $id : L \to N$ to $\tilde{N}$.

Let $\varphi' : K \to N$ be any smooth small perturbation of $\varphi : K \to N$, such that $\varphi' | \partial K = \varphi | \partial K$, $\mathrm{Im}\varphi' \cap Y = \emptyset$, $\varphi'$ is transversal to $L$. Then a usual $\pi_1 M$-counting procedure defines for us an intersection number $i(\varphi', L) \in Z\pi_1 M$. The proof of the following is standard.

Lemma 3.9. *The index* $i(\varphi', L)$ *satisfies the following properties: 1) it does not depend on the particular choice of a small perturbation* $\varphi'$, *and thus is denoted* $i(\varphi, L)$.

2) *if* $\varphi, \psi : K \to N$ *are two maps, homotopic via* $h: K \times I \to N$, *such that* $\operatorname{Im} h \cap Y = \emptyset$, $\operatorname{Im}(h \mid \partial K \times I) \subset X$, *then* $i(\varphi, L) = i(\psi, L)$.

3) *if* $\varphi : K \to N$ *is a smooth embedding, then* $i(\varphi, L) = i(id: L \to N, \varphi(K))$. *This indices are denoted then by* $i(K, L) = i(L, K)$ *).

4) *The intersection index* $i(\varphi, L)$ *does not depend on the pair* $X, Y$, *in the sense that if* $X', Y'$ *are other compacta, such that* $X' \cap Y' = \emptyset = K \cap Y' = X' \cap L$, $\partial K \subset X'$, $\partial L \subset Y'$, *then the index, defined with respect to* $X'$, $Y'$ *coincides with that corresponding to* $X$, $Y$.

5) *If* $K \subset K'$ *are two manifolds,* $K \subset \operatorname{Int} K'$, $\varphi : K' \to N$ *a map, satisfying the above properties and such that* $\varphi \mid (K' \setminus K) \subset X$ *then* $i(\varphi, L) = i(\varphi \mid K, L)$.

Now we return to Morse theory. As before let $p, q$ be the critical points of a regular Morse function $f: W \to [a,b]$, $\operatorname{ind}(p) = \operatorname{ind}(q) + 1$ and $v$ be an agglvf for $f$. Fix the liftings $\tilde{p}$, $\tilde{q}$ to $\tilde{W}$ and the orientations of $D(p)$, $D(q)$. This gives as the coorientation of $D(q, -v)$, as well as the orientation of $D(p, v) \cap f^{-1}(\lambda)$ and coorientation of

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*) The absence of usual sign in this formula is due to the fact that one of our manifolds is oriented and other is cooriented.

$D(q,-v) \cap f^{-1}(\lambda)$ for any regular $\lambda$.

If $f(p) \leqslant f(q)$ we set naturally $\nu(p,q) = 0$. Otherwise we proceed as following. Denote $f(p)$ by $d$, $f(q)$ by $c$, ind $p$ by $s$.

Since $v$ is almost good the sets $K_{\leqslant s-1}(v)$ and $K_{\geqslant s}(-v)$ do not intersect, which by lemma 3.7, p. 2), 3) implies that for all $\delta$ sufficiently small the sets $V_{\leqslant s-1}(\delta;\ v)$, $V_{\geqslant s}(\delta;-v)$ do not intersect.

Furthermore, $K_{\leqslant s}(v) \cap K_{\geqslant s}(-v) = \emptyset$, hence by the same argument, $K_{\leqslant s}(v) \cap V_{\geqslant s}(\delta;\ -v) = \emptyset$ for all $\delta$ sufficiently small; in particular, $D(p,\ v) \cap V_{\geqslant s}(\delta;\ -v) = \emptyset$.

Similarly, $K_{\leqslant s-1}(v) \cap K_{\geqslant s-1}(-v) = \emptyset$, hence for $\delta$ sufficiently small, $D(q,-v) \cap V_{\leqslant s-1}(\delta;\ v) = \emptyset$.

Fix now any regular value $\lambda \in (c,\ d)$.

The above implies that for $\delta$ sufficiently small $D(p,v) \cap \cap V(\delta;[\lambda,\ b];-v) = D(p,v) \cap V_{\leqslant s-1}(\delta;[\lambda,b];-v)$, and $D(p,v) \cap \cap K([\lambda,\ b];-v) = D(p,v) \cap K_{\leqslant s-1}([\lambda, b];-v)$, and the same for the sets of type U. Similarly, $D(q,-v) \cap V(\delta;[a,\ \lambda];v) = D(q,-v) \cap V_{\geqslant s}(\delta;[a,\ \lambda];v)$, and $D(q,-v) \cap K([a,\ \lambda];v) = D(q,-v) \cap K_{\geqslant s}([a,\lambda];v)$ and the same for the sets of type U. Now we fix $\delta > 0$ sufficiently small so that the above conditions hold.

Let $\beta \in (\lambda,\ d)$ be a regular value of f, so close to d, that $(\beta,\ d)$ contains no critical values. Denote by $S_\beta$ the intersection of $D(p,\ v)$ with $f^{-1}(\beta)$. That is the sphere of dimension $s-1$. The intersection $S_\beta \cap K([\lambda,\ b];-v)$ is a compact set and the set $S_\beta \cap U(\delta;[\lambda,\ b];-v)$ is the open neighbourhood of it (arbitrary small if $\delta \to 0$, by lemma 3.7, p.4). So the set $S_\beta \setminus U(\delta;[\lambda, \beta];-v)$ is a compact inside the open set $S_\beta \setminus K([\lambda, \beta];-v)$. By the smooth partition of unity theorem there exists a compact neighbourhood $P$ of $S_\beta \setminus U(\delta;[\lambda,\beta];-v)$ in $S_\beta \setminus K([\lambda,\beta];-v)$ with a smooth boundary $\partial P$. Thus $\partial P \subset S_\beta \cap U(\delta;[\lambda,\beta];-v)$.

Recall now that for any $x \in f^{-1}((\lambda, b))$, $x \notin K([\lambda,\beta];-v)$ there exists a unique $(-v)$-trajectory, starting at $x$ and finishing at some point $y \in f^{-1}(\lambda)$. The correspodence $x \longmapsto y$ determines a smooth map $\Psi_\lambda : f^{-1}((\lambda, b]) \setminus K([\lambda,\beta],-v) \longrightarrow f^{-1}(\lambda)$ which is obviously injective when restricted to the level surface $f^{-1}(\gamma) \setminus K([\lambda,b];-v)$ for $\gamma$ regular, $\gamma \in (\lambda, b)$. Note that by definition of the set $V_{\leq s-1}(\delta;[\lambda,b];-v)$ it belongs to $f^{-1}((\lambda,b])$ and its image $\Psi_\lambda[V_{\leq s-1}(\delta;[\lambda, b];-v) \setminus K([\lambda, b];-v)]$ belongs to $V_{\leq s-1}(\delta;[\lambda,b];+v) \cap f^{-1}(\lambda)$. By prop. 2) of lemma 3.7 this set is compact. We denote it by $X_{\lambda,\delta}$.

Note that the manifold $P$ belongs to the domain of $\Psi_\lambda$, and $\Psi_\lambda$ maps it smoothly and injectively to $f^{-1}(\lambda)$. Moreover, $\partial P \subset S_\beta \cap U(\delta;[\lambda,\beta];-v) = S_\beta \cap U_{\leq s-1}(\delta;[\lambda,\beta];-v)$, since $S_\beta \subset D(p, v)$. By the above the image $\Psi_\lambda(\partial P)$ belongs to $X_{\lambda,\delta}$. The image $\Psi_\lambda(P)$ will be denoted $P_\lambda$.

Now we perform the similar procedure from the other end. Namely, let $\alpha \in (c, \lambda)$ be a regular value of $f$, so close to $c$, that there are no critical values of $f$ in $(c, \alpha)$. Denote by $S_\alpha$ the intersection $D(q, -v) \cap f^{-1}(\alpha)$; that is a sphere of dimension $n-s$. The set $S_\alpha \cap K([a, \lambda];v)$ is a compact set and the set $S_\alpha \cap U(\delta;[a,\lambda];v)$ is a neighbourhood of it. Consider a compact neighbourhood Q of a compact set $S_\alpha \setminus U(\delta;[a,\lambda];v)$ in the set $S_\alpha \setminus K([a,\lambda];v)$, such that Q is a smooth manifold with boundary $\partial Q \subset U(\delta,[a,\lambda];v) \cap S_\alpha$.

The shift along the v-trajectories determines a smooth map $\Phi_\lambda : f^{-1}([a,\lambda)) \setminus K([a,\lambda];v) \to f^{-1}(\lambda)$; $\Phi_\lambda$ is injective, when restricted to $f^{-1}(\gamma) \setminus K([a,\lambda];v)$, for $\gamma$ regular. By definition the $\Phi_\lambda$-image of $V_{\geqslant s}(\delta;[a,\lambda];v) \setminus K([a,\lambda];v)$ belongs to $V_{\geqslant s}(\delta;[a,\lambda];-v) \cap f^{-1}(\lambda)$ This set is compact and will be denoted by $Y_{\lambda,\delta}$.

The manifold Q belongs to the domain of $\Phi_\lambda$ and $\Phi_\lambda$ maps it smoothy and injectively to $f^{-1}(\lambda)$. Moreover, $\partial Q \subset S_\alpha \cap U(\delta,[a,\lambda];v) = S_\alpha \cap U_{\geqslant s}(\delta;[a;\lambda];v)$, hence $\Phi_\lambda(\partial Q) \subset Y_{\lambda,\delta}$. The image $\Phi_\lambda(Q)$ will be denoted $Q_\lambda$.

Note also that since $D(p,v) \cap V_{\geqslant s}(\delta,-v) = \emptyset$, $P_\lambda \cap Y_{\lambda,\delta} = \emptyset$, and similarly $Q_\lambda \cap X_{\lambda,\delta} = \emptyset$.

Now the manifolds $P_\lambda$, $Q_\lambda$ are submanifolds of complementary dimensions $s-1$, $n-s$ in the manifold $f^{-1}(\lambda)$; $P_\lambda$ is oriented, $Q_\lambda$ is cooriented (recall that we have chosen the orientations of the descending discs). The liftings of p, q to $\widetilde{W}$ determine the liftings to $\widetilde{W}$ of $D(p, v)$, $D(q, -v)$ which determine in turn the liftings of P, Q to $\widetilde{f^{-1}(\lambda)}$, since the latter is just the preimage of $f^{-1}(\lambda)$ in $\widetilde{W}$.

Now we can define the intersection index $i(P_\lambda, Q_\lambda) \in Z\pi_1 M$ and we call it by definition the incidence coefficient $\nu(p, q)$. (To image all the stuff, look at the picture 3.2.)

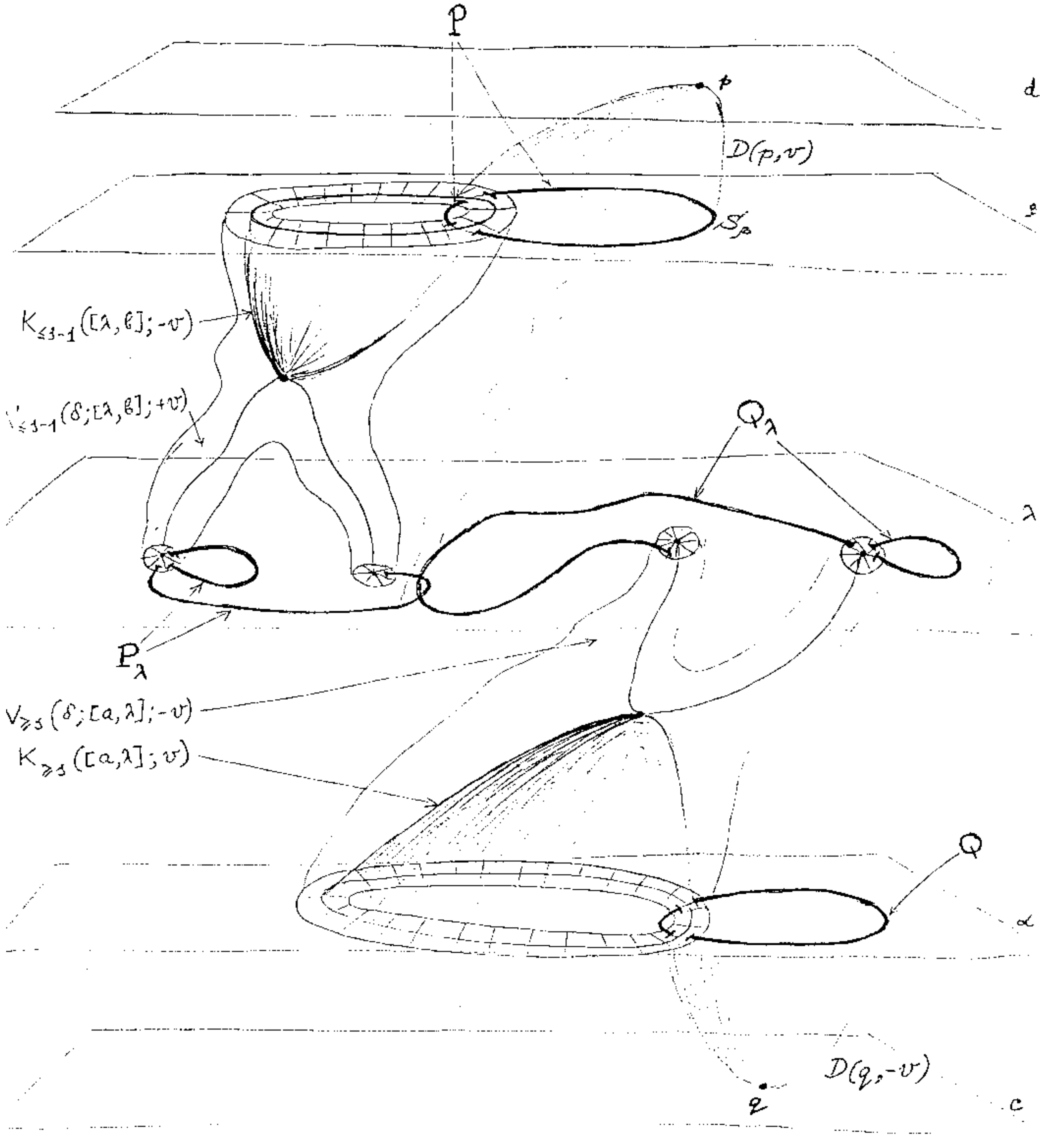
This index depends on a number of choices, namely, $\lambda$, $\delta$, $\alpha$, $\beta$, P, Q. If we want to underline this dependence we include the corresponding parameter in the notations, like $\nu_\lambda(p, q)$. The vector field v can also be included into the notations like $\nu_\lambda(p, q; v)$. We now proceed as to show that it actually does not depend on that ambiquity.

For a while we fix $\lambda$ (the independence on $\lambda$ is a matter of lemma 3.10).

First of all we fix $\delta, \alpha, \beta$ and show that the index does not depend on particular choice of P, Q. It suffices to consider P. For the given $P_1, P_2 \subset S_\beta \setminus K([\lambda, b]; -v)$ consider the compact $R = P_1 \cup P_2 \cup (S_\beta \setminus U(\delta,[\lambda, b];-v))$, belonging to $S_\beta \setminus K([\lambda, b];-v)$. By the smooth partition of unity there exists a compact neighbourhood $P \subset S_\beta \setminus K([\lambda, b];-v)$, int(P) containing R, with the smooth boundary $\partial P \subset U(\delta;[\lambda, b];-v)$.

Now the manifolds $\Psi_\lambda(P_1)$, $\Psi_\lambda(P_2)$ are contained in $\Psi_\lambda(P)$, and applying the lemma 3.9, 5) we get the independence on P, Q.

Now we show that for the fixed $\delta$ the index does not depend on $\alpha$, $\beta$. It suffices to consider $\beta$. Let $\beta < \beta'$ be two regular values of f, such that $\beta' < d$ and there are no critical values of f in the interval $(\beta, d)$. The shift along (-v)-trajectories gives a diffeomorphism $\Psi_0 : f^{-1}(\beta') \to f^{-1}(\beta)$ which carries $S_{\beta'}$ to $S_\beta$, $K([\lambda, b];-v) \cap f^{-1}(\beta')$ to $K([\lambda, b]; v) \cap$ $\cap f^{-1}(\beta)$ and $U(\delta;[\lambda, b]; -v) \cap f^{-1}(\beta')$ to




Pict. 3.2

$U(\delta;[\lambda, b];-v) \cap f^{-1}(\beta)$. Moreover, the map $\Psi_\lambda$, introduced above, being restricted to $f^{-1}(\beta')$ is a composition of $\Psi_0$ and $\Psi_\lambda \mid f^{-1}(\beta)$. So if we choose any $P'$, satifying the conditions above for $\beta'$, the manifold $P = \Psi_0(P')$ satisfies the conditions above for $\beta$, $\Psi_\lambda(P') = \Psi_\lambda(P)$. This proves the independence of $\alpha, \beta$.

Now let $0 < \delta' < \delta$. We fix $\alpha, \beta$ and the manifolds $P \subset S_\beta$, $\partial P \subset S_\beta \cap U(\delta';[\lambda, b];-v)$ and $Q \subset S_\alpha$, $\partial Q \subset S_\alpha \cap U(\delta';[a,\lambda];v)$ so as to compute the incidence coefficient with respect to $\delta'$. By the above we can use P, Q also for $\delta$ and the only difference in definitions of indices is that $i_{\delta'}(P, Q)$ is computed with respect to the compacts $X_{\delta',\lambda}$; $Y_{\delta',\lambda}$, and $i_\delta(P, Q)$ is computed with respect to the compacts $X_{\delta,\lambda}$; $Y_{\delta,\lambda}$. But by lemma 3.9, p.4) these indices are the same.

Now we are to show that our indices do not depend on $\lambda$.

<u>Lemma 3.10.</u> 1) *For any perturbation* $v'$ *of* $v$ *small enough the incidence coefficients* $\nu_\lambda(p,q;v)$ *and* $\nu_\lambda(p,q;v')$ *coincide.*

2) *If the vector field* $v$ *is a good gradient-like vector field for* $f$, *then* $\nu_\lambda(p,q;v)$ *is an ordinary incidence coefficient as defined for example in* [Pa1, app.B].

3) *For an almost good gradient-like vector field* $v$ *the incidence coefficient* $\nu_\lambda(p,q;v)$ *does not depend on* .

<u>Proof.</u> First of all note that 3) follows immediately from 1) and 2) because in the equality $\nu_\lambda(p,q;v) = \nu(p,q;v')$

(where $v$ is an agglvf and $v'$ a small perturbation of $v$, being a gglvf) the lefthand side does not depend on $v'$, and the righthand side on $\lambda$.

To prove 2) we choose $\delta, \alpha, \beta$, P, Q to define $\nu_\lambda(p,q;v)$.

Consider now the set of $(-v)$-trajectories , such that $\gamma(-\infty) = p$, $\gamma(+\infty) = q$. This set is in one-to-one correspondence with the set $f^{-1}(\lambda) \cap D(p, v) \cap D(q, -v)$, hence the latter is finite. The vector field $v$ is good, hence $D(p, v) \pitchfork D(q, -v)$ and since both are transversal to $f^{-1}(\lambda)$, we have $(D(p, v) \cap f^{-1}(\lambda)) \pitchfork (D(q, -v) \cap f^{-1}(\lambda))$. If $\delta > 0$, P, Q, $\alpha$, $\beta$ satisfy the conditions above, necessary for definition of $\nu_\lambda(p,q; v)$, the manifolds $\Psi_\lambda(P)$, $\Phi_\lambda(Q)$ are submanifolds of the same dimension of $D(p,v) \cap f^{-1}(\lambda)$, $D(q,-v) \cap$
$\cap f^{-1}(\lambda)$ and such that $f^{-1}(\lambda) \cap D(p, v) \cap D(q, -v)$ $= \Psi_\lambda(P) \cap \Phi_\lambda(Q)$. This implies our assertion.

Now we proceed as to prove 1). Note first of all that if $v'$ is a perturbation of $v$, close enough to $v$, then $v'$ is again an agglvf for $f$, by lemma 3.8, p.3). Fix some regular $\lambda \in (c, d)$.

We choose some $\delta > 0$ so small that the sets $V(\delta; [\lambda, b]; \pm v)$ b]; $\pm v)$, $V(\delta; [a, \lambda]; \pm v)$ are well defined and the sets $V_{\leq s-1}(\delta; v)$, $V_{\geq s}(\delta; -v)$ do not intersect, and also $K_{\leq s}(v) \cap V_{\geq s}(\delta; -v) = \emptyset = K_{\geq s-1}(-v) \cap V_{\leq s-1}(\delta; v)$.

Now we fix any $\delta_0 > 0$, such that $\delta_0 < \delta$. For $v' - v$ small enough the set $V_{\leq s-1}(\delta_0; v')$ is contained in $V_{\leq s-1}(\delta; v)$ as well as $V_{\leq s-1}(\delta_0; [\lambda, b]; v')$ is contained in $V_{\leq s-1}(\delta_0; [\lambda, b]; v)$ (by lemma 3.7, 4), 5)) and $V_{\geq s}(\delta_0; -v')$ is contained

in $V_{\geqslant s}$ $(\delta;\ v)$ (as well as $V_{\geqslant s}(\delta_0\ ;[a,\lambda\ ];-v')$ is containec $V_{\geqslant s}$ $(\delta;[a,\lambda\ ];v))$. That implies that $V_{\leqslant s-1}(\delta_0;\ v')\cap V_{\geqslant s}$ (
$=\emptyset$ .

Further, for $v'-v$ small enough the set $K_{\leqslant s}$ $(v'$ still contained in $W\setminus V_{\geqslant s}$ $(\delta;\ -v)$, and $K_{\leqslant s-1}(-v')$ is s contained in $W\setminus V_{\leqslant s-1}(\delta;\ -v)$. Therefore the descending $D(p,\ v')$ does not intersect the set $V_{\geqslant s}(\delta_0;[a,\ \lambda\ ];-v')$ the ascending disc $D(q,-v')$ does not intersect $V_{\leqslant s-1}(\delta_0;$

If we denote the set $V_{\leqslant s-1}(\delta_0;[\ \lambda,b];+v')\cap f^{-1}\ (\ \lambda\ )$ $X_{\lambda,\delta_0}(v')$ and, respectively $V_{\geqslant s}(\delta_0;[a,\ \lambda\ ];-v')\cap f^{-1}(\ \lambda\ )$ $Y_{\lambda,\delta_0}(v')$ we can rewrite the above as following: 1) $D(p,$ $X_{\lambda,\delta_0}$ $(v')$ does not intersect $Y_{\lambda,\delta_0}(v')$ and 2) $X_{\lambda,\delta_0}$ $(v')$ not intersect $D(q,-v')\cup Y_{\lambda,\delta_0}(v')$.

Next we take $\alpha$ (resp. $\beta$) so close to d, that th $D(p,v)\ \cap\ f^{-1}(\ \alpha\ )$ (resp. $D(q,v)\ \cap\ f^{-1}(\alpha))$ is contain the neighbourhood of p (resp. q) where $v=v'$.

Next we fix a compact neighbourhood P of $S_\beta\setminus U(\delta_0$ in $S_\beta\setminus K([\ \lambda\ ,b];-v)$, such that P is a smooth fold with the boundary $\partial P$. Note that 1) $K([\ \lambda,\ b\ ];-v)$ and 2) $S_\beta\setminus \text{int}\ P\subset U(\delta_0\ ;[\ \lambda\ ,\ b\ ];-v)$. Similarly we compact neighbourhood $Q\subset S_\alpha$ with the smooth boundary that 1) $K([a,\ \lambda\ ];v)\subset S_\alpha\setminus Q$, 2) $S_\alpha\setminus \text{int}\ Q\subset U(\delta_0;[a,\lambda$

For $v'-v$ small enough these four propertie preserved by lemma 3.7, 5), 6). This implies tha manifolds P, Q can be taken as the initial data to ne $\mathcal{V}_\lambda(P,Q;v)$ as well as $\mathcal{V}_\lambda(P,Q;v')$ with respect to a

$\delta_0$ and the levels $\alpha, \beta$. The first is the intersection index of $\Psi_\lambda(v)(P)$ and $\Phi_\lambda(v)(Q)$, calculated with respect to the pair of compacts $X_{\lambda,\delta_0}(v)$, $Y_{\lambda,\delta_0}(v)$. The second is that of $\Psi_\lambda(v')P$ and $\Phi_\lambda(v')Q$ calculated with respect to the pair of compacts $X_{\lambda,\delta_0}(v')$, $Y_{\lambda,\delta_0}(v')$.

Consider now the pair of compacts $X_{\lambda,\delta}(v)$, $Y_{\lambda,\delta}(v)$. From the above we know that $X_{\lambda,\delta}(v)$ contains $X_{\lambda,\delta_0}(v')$ (and of course it contains $X_{\lambda,\delta_0}(v)$); similarly $Y_{\lambda,\delta}(v) \supset Y_{\lambda,\delta_0}(v') \cup Y_{\lambda,\delta_0}(v)$ Recall that $\Psi_\lambda(v)(P) \cap Y_{\lambda,\delta}(v) = \emptyset = \Phi_\lambda(v)(Q) \cap X_{\lambda,\delta}(v)$, and the boundaries $\partial P$, $\partial Q$ satisfy the relations $\Psi_\lambda(v)(\partial P) \subset X_{\lambda,\delta_0}(v)$, $\Phi_\lambda(v)(\partial Q) \subset Y_{\lambda,\delta_0}(v)$.

Recall now that the set $U(\delta_0;[\lambda,b];-v) \cap S_\beta = U_{\leq s-1}(\delta_0;[\lambda,b];-v) \cap S_\beta$ is open in $S_\beta$ and, therefore, the set $U_{\leq s-1}(\delta_0;[\lambda,\beta];-v) \cap S_\beta) \setminus (K_{\leq s-1}(\delta_0;[\lambda,\beta];-v) \cap S_\beta)$ is open in $S_\beta \setminus K_{\leq s-1}(\delta_0;[\lambda,\beta];-v)$, thus its image under $\Psi_\lambda(v)$ is open and so $\Psi_\lambda(v)(\partial P)$ is contained in the interior of $X_{\lambda,\delta_0}$. (Similarly, $\Psi_\lambda(v)(\partial Q)$ is contained in the interior of $Y_{\lambda,\delta_0}$.)

This implies that for $v'-v$ sufficiently small the manifold $\Psi_\lambda(v')(P)$ does not intersect $Y_{\lambda,\delta}(v)$ and the boundary $\Psi_\lambda(v')(\partial P)$ belongs to $X_{\lambda,\delta_0}$ (similarly, $\Phi_\lambda(v')(Q) \cap X_{\lambda,\delta}(v) = \emptyset$ and $\Phi_\lambda(v')(\partial P) \subset Y_{\lambda,\delta_0}$). Moreover $\Psi_\lambda(v')(P)$ is homotopic to $\Psi_\lambda(v)(P)$ in the class of maps, satisfying these restrictions (same for Q). Now we apply lemma 3.9.2) and finish the proof.

Remark 3.11. If the vector field v is good, then the compact $S_\beta \cap K([\lambda,b];-v)$ is nowhere dense in $S_\beta$. In general, however, it is not, and it well can happen for example, that

$S_\beta \subset K([\lambda, b]; -v)$. In this case the only choice for P will be $\emptyset$, and hence, by lemma 3.10, $\nu(p,q) = 0$. That implies, for example, that in this case the index $\nu(p,q;v')$ vanishes for all the agglvf $v'$ close enough to v. One checks this up like following. If $S_\beta \subset K([\lambda,b];-v)$, then by the above, $S_\beta \subset$ $\subset K_{\leq s-1}([\lambda;b];-v)$. We choose $\beta$ close to d so that $S_\beta(v)$ $= S_\beta(v')$. Fix some $\delta$ which satisfies the conditions of the definition of $\nu(p,q)$, and some $\delta_0 < \delta$. By the lemma 3.7 $S_\beta(v) = S_\beta(v')$ belongs to $V_{\leq s-1}(\delta_0;[\lambda, \beta];-v')$ for all $v'$ close enough to v and this set in turn is contained in $V_{\leq s-1}(\delta; [\lambda,b]; -v)$ for $v'-v$ small enough. That implies $P_\lambda(v') \subset V_{\leq s-1}(\delta;[\lambda,b]; v)$. But $D(q,-v) \subset V_{\leq s-1}(\delta_0;[\lambda,b];-v')$ which belongs to $V_{\geq s-1}(\delta;[\lambda,b]; -v)$ for $v-v'$ small and this implies $Q(v') \subset V_{\geq s-1}(\delta;[\lambda,b]; -v)$ thus $D(p,v) \cap D(q,-v) = \emptyset$,

q.e.d.

From time to time we shall need to consider all the set of trajectories, joining two given points p, q and not only the index $\nu(p, q)$.

Let $f:W \longrightarrow [a,b]$ be a Morse function on a cobordism W, and let p, q be two critical points, ind p = ind q. Let v be any gradient-like vector field for f. Assume that the orientations of D(p, v), D(q, v) are chosen. Assume that the discs D(p, v) and D(q,-v) are transversal.

Then we denote by N(p,q; v) the set of all pairs $(\gamma, \varepsilon(\gamma))$ where $\gamma$ is a (-v)-trajectory, joining p and q and $\varepsilon(\gamma)$ is the sign of intersection of D(p,v) and D(q,-v) along $\gamma$. Here we imply that the trajectories $\gamma_1$, $\gamma_2$, which differ one from another by a change of time $t \to t + C$, are identified.

Each trajectory $\gamma$ is uniquely expanded to a map $\bar{\gamma}: \bar{R} \to M$, where $\bar{R}$ is the compactification of R by two points: $-\infty$ and $+\infty$. The $(-\infty)$ is carried to p, $(+\infty)$ – to q. The class of homotopy with fixed ends of $\bar{\gamma}$ does not depend of the choice of parameter and will be denoted $[\gamma]$.

If $g: W \to [c,d]$ is another Morse function with a gradient-like field $v'$, such that in some neighbourhoods $U_1(p)$ and $U_2(q)$ the fields v and $v'$ coincide and f, g coincide up to constants, and $D(p, v') \pitchfork D(q, -v')$, we say that the sets $N(p,q;v)$, $N(p,q;v')$ are the same if they coincide as sets.

Suppose now that M, M′ are two sets of pairs $(\gamma, \varepsilon(\gamma))$, where $\gamma$ is a map $R \to M$, which can be expanded to $\bar{\gamma}: \bar{R} \to M$, such that $\bar{\gamma}(-\infty) = p$, $\bar{\gamma}(+\infty) = q$, and $\varepsilon = \pm 1$. We say that M and M′ are homotopically the same if there is a bijection $\Phi: M \to M'$, such that $\varepsilon(\gamma) = \varepsilon(\Phi(\gamma))$ and $\bar{\gamma}$ and $\Phi(\bar{\gamma})$ are homotopic with the fixed ends.

The basic example is $M = N(p,q;v)$, $M' = N(p,q;v')$.

Lemma 3.12. *Assume that* v *is almost good, and* $D(p,v) \pitchfork D(q,-v)$, *where* $\operatorname{ind}(p) = \operatorname{ind}(q) + 1$. *Then the set* $N(p,q;v)$ *is finite and for every* $\varepsilon$*-regular perturbation* $v'$ *and* v, *close enough to* v *we have again* $D(p,v') \pitchfork D(q,-v')$ *and the sets* $N(p,q;v)$, $N(p,q;v')$ *are homotopically the same.*

Proof. Since $v$ is almost good, there exists a Morse function $\varphi : W \longrightarrow [c,d]$, such that $v$ is a gradient-like vector field for $\varphi$, $\varphi$ equals $f$ up to a constant in the neighbourhoods of critical points, $\varphi(p) > \varphi(q)$ and there exist regular $\lambda, \mu \in (c,d)$, such that in $\varphi^{-1}([\lambda, \mu])$ there are only two critical points: p and q. We denote by $W_o$ the cobordism $\varphi^{-1}([\lambda, u])$ and choose a regular value $\theta \in (\lambda, \mu)$ such that $\varphi(q) < \theta < \varphi(p)$. The intersection $D(p,v) \cap \varphi^{-1}(\theta)$ is a smooth compact submanifold $P(v)$ of $\varphi^{-1}(\theta)$, diffeomorphic to a sphere $S^{\text{ind } p-1}$. The intersection $D(q, -v) \cap \varphi^{-1}(\theta)$ is a smooth compact submanifold $Q(v)$ of $\varphi^{-1}(\theta)$, diffeomorphic to a sphere $S^{n-\text{ind } p}$. Since $v$ is transversal to $\varphi^{-1}(\theta)$ we have $P(v) \pitchfork Q(v)$ and the points of intersection are in one-to-one correspondence with the elements of $N(p,q;v)$ (the signs are also the same). Therefore, $N(p,q;v)$ is finite.

We choose $\delta$ $\varepsilon$ so small, that $D_\delta(p)$ and $D_\delta(q)$ are both contained in $f^{-1}((\lambda, \mu))$. We demand that $v'$ is so close to $v$, that it is also a gradient-like field for $\varphi$ and for $x \in W_o \setminus (D_\delta(p) \cup D_\delta(q))$ we have $d\varphi(v) \geqslant \theta > 0$, $d\varphi(v') \geqslant \theta > 0$.

The standard transversality arguments imply that for $v'$ close enough to $v$ the set $A(v) = D(p,v) \cap D(q,-v)$ is in one-to-one correspondence $\varphi$ with $A(v') = D(p,v') \cap D(q,-v')$ such that $x$ and $\varphi(x)$ are arbitrarily close. For each $x \in A(v)$, $\varphi(x) \in A(v')$ we choose the $(-v)$-trajectory $\gamma$, joining p and q and passing through $x$ in such a way, that $\gamma(0) = x$. Similarly we choose the parameter on the

$(-v')$-trajectory $\gamma'$, joining p and q and passing through $\varphi(x)$ so that $\gamma'(0) = \varphi(x)$. We denote by $\eta$ the v-trajectory, starting at x and by $\eta'$ - the $(v')$-trajectory, starting at $\varphi(x)$. Note that $\gamma, \gamma', \eta, \eta'$ are defined on $(-\infty, \infty)$. Note also that $\eta(\frac{\mu-\lambda}{\theta})$ and $\eta'(\frac{\mu-\lambda}{\theta})$ belong to $B_\delta(p)$; similarly $\gamma(\frac{\mu-\lambda}{\theta})$ and $\gamma(\frac{\mu-\lambda}{\theta})$ belong to $B_\delta(q)$ (by corollary A.2 of App. A and by the choice of $\theta$).

We choose the riemannian metric on W, such that in the neighbourhoods of critical points it is euclidean in the standard coordinate system.

For all $v'$ close enough to v we have, therefore, that the curves $\gamma$ and $\gamma'$, restricted to $[-\frac{\mu-\lambda}{\theta}, \frac{\mu-\lambda}{\theta}]$ are close to each other and the starting points (resp. the endpoints) belong to $B_\theta(p)$ (resp. $B_\theta(q)$). That implies that $\gamma, \gamma'$ are homotopic and the homotopy leaves the starting points at $B_\delta(p)$ (resp. $B_\delta(q)$). After that the homotopy is expanded to the homotopy of $\gamma|\bar{R}$ to $\gamma'|\bar{R}$ in the standard way. Therefore $\bar{\gamma} \sim \bar{\gamma}'$, q.e.d.

Assume now thatW is connected and denote $\pi_1 W$ by H. Fix some lifting $\tilde{p}$, $\tilde{q}$ of p and q to the universal covering $\tilde{W}$. Then each $(-v)$-trajectory $\gamma$, joining p and q, determines an element $h(\gamma)$ in H, namely the lifting of $\tilde{\gamma}$ to W, starting at $\tilde{p}$, finishes at $\tilde{q} \cdot h(\gamma)$. So if $D(p,v) \pitchfork D(q,-v)$ and $N(p,q;v)$ is finite we define $\nu(p,q;v)$ to be $\sum \varepsilon(\gamma) h(\gamma) \in ZH$, where $\gamma$ runs through the $(-v)$-trajectories, joining p and q. Note that if v is good then the assumptions are satisfied and this $\nu$ is a standard incidence coefficient.

We say that $N(p,q;v)$ and $N(p,q;v')$ are homologically the same if $\nu(p,q;v) = \nu(p,q;v')$. (Note that this notion does not depend on the particular choice of liftings $p$ and $q$).

Corollary 3.13. *Let* $f: W \to [a,b]$ *be a Morse function on a connected cobordism* $W$, $v$ *is an almost good gradient-like vector field for* $f$, $p$ *and* $q$ *are critical points of* $f$, $\operatorname{ind}(p) = \operatorname{ind}(q) + 1$. *Assume that* $D(p,v) \pitchfork D(q,-v)$.

Then for every $\varepsilon$-regular perturbation $v'$ of $v$ we have again $D(p,v') \pitchfork D(q,-v)$; $\nu(p,q;v)$ and $\nu(p,q;v')$ are well defined and are equal.

## Appendix A.

Lemma A.1. *Let* $R^n = R^m \oplus R^k$ *with coordinates* $(\vec{x}, \vec{y})$. *Denote by* f *the function* $-|\vec{x}|^2 + |\vec{y}|^2 + c$, *where* c *is a constant. Denote by* $v = (-2\vec{x}, +2\vec{y})$ *the gradient vector field in the standard metrics. Assume that* $a = (\vec{x}_0, \vec{y}_0)$ *belongs to* $\bar{D}_\delta(0)$. *Denote by* $\gamma$ *the* v*-trajectory, starting at* a.

*Then either* 1) $\vec{y}_0 = 0$ *and* $\gamma(\tau)$ *stays forever in* $D_\delta(0) \cap \{\vec{y} = 0\}$ *and* $\gamma(\tau) \to 0$ *when* $\tau \to \infty$,

*or* 2) $\vec{y}_0 \neq 0$, *and there exists* $\alpha \geqslant 0$ *such that* $\gamma(\tau)$ *belongs to* $D_\delta(0)$ *if and only if* $\tau \in [0, \alpha]$, *and* $f(\gamma(\alpha)) \geqslant c$.

Proof. The trajectory $\gamma(t)$ has the coordinates $(\vec{x}_0 \cdot e^{-2t}, \vec{y}_0 \cdot e^{2t})$. If $\vec{y}_0 = 0$, then the assertions of the part 1) are obvious. If not, then the condition $|\gamma(\tau)| \leqslant \delta$ is equivalent to $\mu^2 - \frac{\delta^2}{|\vec{y}_0|^2}\mu + \frac{|\vec{x}_0|^2}{|\vec{y}_0|^2} \leqslant 0$, where $\mu = e^{4\tau}$. This condition holds for $\mu = 1$ (i.e. $\tau = 0$), which implies that there exists $\alpha' \geqslant 1$ such that for $\mu \geqslant 1$ it holds if and only if $\mu \leqslant \alpha'$. Set $\alpha = (\ln \alpha')/4$. It is left only to prove that $f(\gamma(\alpha)) \geqslant c$. But $\gamma(\alpha) = (\vec{x}_0 \cdot e^{-2\alpha}, \vec{y}_0\, e^{2\alpha})$, and $f(\gamma(\alpha)) = -|\vec{x}_0|^2 e^{-4\alpha} + |\vec{y}_0|^2 e^{4\alpha} + c$ $= c - \frac{|\vec{x}_0|^2}{\alpha'} + |\vec{y}_0|^2 \alpha'$, where $\alpha'$ is the biggest root of the equation $\mu^2 - \frac{\delta^2}{|\vec{y}_0|^2}\mu + \frac{|\vec{x}_0|^2}{|\vec{y}_0|^2} = 0$. To prove $f(\gamma(\alpha)) \geqslant c$ it suffices to show $\alpha' \geqslant \frac{|\vec{x}_0|}{|\vec{y}_0|}$. For that it is enough to show that the polynom $t^2 - \frac{\delta^2}{|\vec{y}_0|^2} t + \frac{|\vec{x}_0|^2}{|\vec{y}_0|^2}$ is nonpositive when evaluated at $t = \frac{|\vec{x}_0|}{|\vec{y}_0|}$. This follows from $\delta^2 \geq |\vec{x}_0|^2 + |\vec{y}_0|^2$. Q.E.D.

Corollary A.2. *In the notations of lemma* 3.1 *we have* $D(p,v) \cap D_\delta(p) = D_\delta^-(p)$.

Proof. We assume that for some $\delta' > \delta > 0$ there is a diffeomorphism $\Psi$ of $D_{\delta'}(0)$ onto a neighbourhood of p, such that $f \cdot \Psi$ is equal to $c - |\vec{x}|^2 + |\vec{y}|^2$, and $\Psi^{-1}(v)$ is the euclidean gradient $(-2\vec{x}, 2\vec{y})$. Now the inclusion $D_\delta^-(p) \subset D(p,v) \cap D_\delta(p)$ follows from lemma A.1. 1). To prove the inverse let $z \in D_\delta(p) \setminus D_\delta^-(p)$. That means that $z = (\vec{x}_0, \vec{y}_0)$, $|\vec{y}_0| \neq 0$. Then by lemma A.1 there exists a segment $[0, \alpha]$, $\alpha \geq 0$ such that the v-trajectory $\gamma(t)$, starting at z, stays at $D_\delta(p)$ when $t \in [0, \alpha]$ and $f(\gamma(\alpha)) \geq c$. Since $\dot{\gamma}(\alpha) \neq 0$ and v is a gradient-like field for f there exists $\alpha' > \alpha$ such that $f(\gamma(\alpha')) > c$. Hence the equality $\lim_{\tau \to +\infty} \gamma(\tau) = p$ is impossible and $z \notin D(p,v)$, q.e.d.

Corollary A.3. *Let* p *be a critical point of* f *(we imply the notations of* §3*). Suppose that a point* $z \in W$ *does not belong to* $D_\delta(p)$*. Denote by* $\gamma$ *the* v*-trajectory, starting at* z.

*Then one and only one of the following alternatives hold:*

1) $\gamma(\tau) \cap D_\delta(p) = \emptyset$.

2) *There exist numbers* $\tau_0, \tau_1$ *such that* $0 < \tau_0 < \tau_1$ *and* $\gamma(\tau) \in D_\delta(p)$ *if and only if* $\tau \in [\tau_0, \tau_1]$.

3) *There exist* $\alpha > 0$ *such that* $\gamma(\tau) \in D_\delta(p)$ *if and only if* $\tau \geqslant \alpha$.

*The condition* $z \in D(p, v)$ *is equivalent to* 3).

Proof. Assume that $\gamma(\tau) \cap D_\delta(p) \neq \emptyset$. Consider the minimal possible $\tau_0 > 0$ such that $\gamma(\tau_0) \in D_\delta(p)$. (This exists since $D_\delta(p)$ is compact and since $z \in D_\delta(p)$.) Now we distinguish two cases. A) $\gamma(\tau_0) \in D_\delta^-(p)$. Then by lemma A.1 case 1) the trajectory $\gamma$ stays forever in $D_\delta^-(p)$ after $\tau_0$ and our case 3) holds.

B) $\gamma(\tau_0) \notin D_\delta^-(p)$. Then by lemma A.1 case 2) there exists $\alpha > 0$, such that $\gamma(\tau)$ belongs to $D_\delta(p)$ when $\tau \in [\tau_0, \tau_0 + \alpha]$ and $f(\gamma(\tau_0 + \alpha)) \geqslant c$. Since $D_\delta(p)$ belongs to the standard coordinate system together with some neighbourhood of $D_\delta(p)$ the case 2) of lemma A.1 implies also that there exists some $\varepsilon > 0$, such that $\gamma(\tau) \notin D_\delta$ for $\tau \in (\tau_0 + \alpha, \tau_0 + \alpha + \varepsilon)$.

Now I set $\tau_1 = \tau_0 + \alpha$ and I claim that the alternative 2) holds. Indeed, suppose that there exists $\nu > \tau_0 + \alpha$, such that $\gamma(\nu) \in D_\delta(p)$. Note that $\nu \geqslant \tau_0 + \alpha + \varepsilon$. Choose the minimal possible $\nu_0$, satisfying this condition (which is possible, since $D_\delta(p)$ is compact). Again $\nu_0 \geqslant \tau_0 + \alpha + \varepsilon$. On the interval $[\tau_0 + \alpha, \tau_0 + \alpha + \varepsilon]$ the derivative $\langle df, \dot\gamma \rangle$ is

hence $f(\gamma(\nu_0)) > c$. The point $\gamma(\nu_0) = (\vec{x}_0, \vec{y}_0)$ belongs to the boundary $\partial D_\delta(p)$, and the vector $\dot\gamma(\nu_0)$ points inside $D_\delta(p)$ or is tangent to $\partial D_\delta(p)$. But that is impossible, since the scalar product of $\dot{\gamma}(\nu_0)$ with the outward pointing normal to $\partial D_\delta(p)$ equals $\langle v(\gamma(\nu_0)), (\vec{x}_0, \vec{y}_0)\rangle = \langle(-2\vec{x}_0, 2\vec{y}_0), (\vec{x}_0, \vec{y}_0)\rangle = -2|\vec{x}_0|^2 + 2|\vec{y}_0|^2 = f(\gamma(\nu_0)) - c > 0$.

Now we are only to note that 1)-3) exclude one another obviously and that $z \in D(p,v)$ contradicts 1) and 2). Q.E.D.

The same argument as used while considering the case 2) of the above corollary proves the following.

<u>Corollary A.4.</u> Suppose that $p_1, \ldots, p_k$ are the critical points of f, belonging to one critical level $f^{-1}(\beta)$. Suppose that $z \in W$ does not belong to $D_\delta(p_1) \cup \ldots \cup D_\delta(p_k)$. Denote by $\gamma(\tau)$ the v-trajectory, starting at z.

Then $\gamma$ intersects at most one of $D(p_i)$.

## 4. Preliminaries on regular Morse maps $M \to S^1$

In this section we return to the framework of § 2 and collect here some definition and simple lemmas, which will be used in what follows.

We fix a regular class $\xi \in H^1(M, Z)$ once and for good.

The universal covering $\widetilde{M} \to M$ is factored as $\widetilde{M} \to \overline{M} \to M$ where $\overline{M}$ is an infinite cyclic covering, corresponding to $\operatorname{Ker} \xi = H$.

We shall change the Morse maps $M \to S^1$ in such a way that in the neighbourhoods of critical points nothing changes. More precisely:

Definition 4.1. *Let* $(f, \bar{f}, v, E)$ *be an r-quadruple. An r-quadruple* $(g, \bar{g}, w, E')$ *is called an admissible modification of* $(f, \bar{f}, v, E)$ *if*

1) $\operatorname{Cr} f = \operatorname{Cr} g$

2) *for every critical point* $x \in M$ *of* $f$ *there exists a neighbourhood* $U(x)$ *of* $x$, *such that in this neighbourhood* $v = w$ *and* $f - g = \text{const}$.

3) $E' = E$.

The simplest example of an admissible modification in given by the following procedure. It will be of permanent use.

Let $(f, \bar{f}, v, E)$ be an r-quadruple and $c \in R$ be a regular value for $\bar{f}$. Let $d \in R$ be another regular value such that $0 < d-c < 1$. Consider the cobordism $W = \bar{f}^{-1}([c, d])$.

Let $h:W \longrightarrow [c,d]$ be another regular function on the cobordism $W$ with the same gradient-like field $v$, and such that $h = \bar{f}$ in the neighbourhoods of $\bar{f}^{-1}(c)$, $\bar{f}^{-1}(d)$ and $h - \bar{f}$ is constant in the neighbourhoods of critical points. Then we glue from $h$ and $\bar{f}$ a new Morse function on $f^{-1}([c,c+1])$ . expand it equivariantly to $M$ and factor it to get a new Morse map $M \longrightarrow S^1$ , which is called a result of rearranging procedure.

Most often we apply this procedure when there are only two critical points $x, y$ in $\bar{f}^{-1}([c,d])$ and $h$ changes the order, that is $h(x) = f(y)$, $h(y) = f(x)$. Recall that a Morse function $f$ on a cobordism $W$ is called enumerating if for every pair of critical points $x, y$ such that $\operatorname{ind} x < \operatorname{ind} y$ we have $f(x) < f(y)$. If for the rearranging we take the enumerating function $h:W_c \to [c, c+1]$ (which exists by [Mi1, § 3] and is automatically regular by remark 2.4), we call the procedure renumerating.

The example of the renumerating procedure is given by the following lemma, which will be useful in § 7.

<u>Lemma 4.2</u>. *Let* $\mathcal{X} = (f, \bar{f}, v, E)$ *be an r-quadruple. Let* $a, b$ *be two critical points of* $\bar{f}$ *in* $\bar{M}$, $\operatorname{ind} a = \operatorname{ind} b$ *and* $\bar{f}(a) \leqslant \bar{f}(b)$. *Then there exists an admissible modification* $(g, \bar{g}, v, E)$ of $\mathcal{X}$, *such that* $\bar{g}(a) > \bar{g}(b)$.

<u>Proof.</u> Induction in $s = [\bar{f}(b) - \bar{f}(a)]$.

1) $s = 0$. That implies $\bar{f}(a) \leqslant f(b) < \bar{f}(a) + 1$. Choose a regular $c$, such that $c < \bar{f}(a) \leqslant \bar{f}(b) < c+1$ and apply to $f$

a renumerating procedure with respect to c. We demand that $\bar{h}(b) > \bar{h}(a)$ (this can be arranged since v is almost good). By the above the $\tau$-quadruple $(h, \bar{h}, v, E)$ satisfies the conclusion.

2) $s > 0$. Note that $\bar{f}(b) \geqslant \bar{f}(t^{-s}a)$, and $[\bar{f}(b) - \bar{f}(t^{-s} a)] = 0$. Apply the above and get the renumerating $\bar{\mathcal{X}} = (h, \bar{h}, v, E)$, such that $\bar{h}(b) < \bar{h}(t^{-s}a)$. If $\bar{h}(a) > \bar{h}(b)$ we are over, if not, note that $\bar{h}(b) - \bar{h}(a) < s$ and apply the induction assumption to get an admissible modification $(g, \bar{g}, v, E)$ of $\bar{\mathcal{X}}$, such that $\bar{g}(a) > \bar{g}(b)$, q.e.d.

In § 5 we shall need to interchange the values of two critical points x, y, ind x = ind y + 1, $\bar{f}(x) > \bar{f}(y)$ under the assumption that there are no (-v)-trajectories, joining x and y. For that we shall need a following lemma

Lemma 4.3. *Suppose that* $\varphi$ *is a Morse function on the cobordism* Y, $\partial Y = \partial_0 Y \sqcup \partial_1 Y$, *such that:*

1) $\pi_0(Y) = \pi_0(\partial_0 Y) = \pi_0(\partial_1 Y) = \{1\}$, *the inclusions* $\partial_0 Y \hookrightarrow Y \hookleftarrow \partial_1 Y$ *induce the isomorphisms in* $\pi_1$.

2) $n = \dim Y \geqslant 6$.

3) $\varphi$ *has only two critical points* x, y *such that* $n-2 \geqslant \text{ind } x, \text{ind } y \geqslant 2$, $\text{ind } x = \text{ind } y - 1$, *and* $\varphi(x) > \varphi(y)$.

*Then* $\varphi$ *is regular.*

Proof. The only case which is to be considered is that of $\varphi(x) > \varphi(y)$, ind x < ind y (and this is exactly what we need). Denote $\varphi(\partial_0 Y)$ by a, $\varphi(\partial_1 Y)$ by b, $\varphi(y) = \alpha$, $\varphi(x) = \beta$ and let c belong to $(\alpha, \beta)$. We are to prove that $\varphi^{-1}(c)$ is connected and the inclusion $\varphi^{-1}(c) \subset Y$ induces the isomorphism in $\pi_1$. The first is obvious since ind y > 1.

For the second denote $\varphi^{-1}([a,c])$ by $Y_0$, $\varphi^{-1}([c,b])$ by $Y$. Since ind y > ind x $\geqslant 2$, the inclusion $\partial_0 Y \subset Y_0$ induces the iso in $\pi_1$. Since ind x $\geqslant 2$, the inclusion $Y_0 \subset Y$ induces epimorphism in $\pi_1$. But $\pi_1(\partial_0 Y) \to \pi_1 Y$ is iso, hence $\pi_1 Y_0 \to \pi_1 Y$ is also iso. Applying the same argument to the function $(-\varphi)$ we get that $\pi_1 Y_1 \to \pi_1 Y$ is an iso. Now since ind x + (n - ind y) = n-1 $\geqslant 5$, one of these numbers is >2 which implies that either $f^{-1}(c) \subset Y_1$, or $f^{-1}(c) \subset Y_0$ induces the iso in $\pi_1$ and we are over.

Now we want to rewrite the formula for differential in the Novikov complex in terms of $\bar{M}$ rather then $\tilde{M}$.

Let $(f, \bar{f}, v, E)$ be an r-quadruple, belonging to $\xi$. If p, q are critical points of $\bar{f}$ in $\bar{M}$ we consider any regular value c of $\bar{f}$, such that $p, q \in \bar{f}^{-1}([c, c+n])$ for some $n \in N$. By remark 2.4.A the $\pi_1(f^{-1}([c, c+n])) \to \pi_1(\bar{M})$ is an isomorphism. Following the end of § 3 we define the set $N(p,q;v)$.

It is obvious that this set does not depend on c or n.

Now the set of liftings E determines the liftings $p \to \tilde{p}$ of all critical points $p \in Cr\,\bar{f}$ to $\tilde{M}$ in the following way. The projection Q(e) are lifting to e, where $e \in E$ and the element $Q(e)t^s$ is lifted to $e\theta^s$. So for each curve $\gamma$ in $\bar{M}$ which joins p,q Cr $\bar{f}$ the index $h(\gamma) \in H$ is determined.

For our specific choice of liftings we have $h(\gamma t^s) = \theta^{-s} h(\gamma)\theta^s$ (obvious).

Let now $x$, $y$ be any two critical points of $f$ in $M$, $\mathrm{ind}(x) = \mathrm{ind}(y) + 1$. Denote by $\hat{x}$, $\tilde{y}$ their liftings to $\tilde{M}$, determined by $E$, and by $\bar{x}$, $\bar{y}$ the projections of $\tilde{x}$, $\tilde{y}$ to $M$. Denote by $n_s(x, y)$ the element $\nu(\bar{x}, \bar{y}t^s; v) \in ZH$. Recall that $n(x, y) \in \Lambda^-_\xi$ stands for an incidence coefficient in Novikov complex $C_*(v, E)$.

Lemma 4.4. $n(x, y) = \sum_{s\in\mathbf{Z}} \theta^s n_s(x, y)$.

Proof. The incidence coefficient $n(x, y)$ as defined in [Pa1] is the formal sum $\sum_{g\in G} n_g(x, y)g$. Here $n_g(x, y) \in Z$ and is the sum of orientation signs, counted over all the $(-v)$-trajectories in $M$, joining $\tilde{x}$ and $\tilde{y}g$.

Now $\sum_{s\in N} \theta^s n_s(x, y)$ is equal to $\sum_{s\in N} (\sum_{\gamma} \varepsilon(\gamma)\cdot\theta^s\cdot h(\gamma))$ where $\gamma$ runs through the $(-v)$-trajectories on $\bar{M}$, joining $\bar{x}$ and $\bar{y}t^s$. The internal sum equals the sum $\sum_{\gamma} \varepsilon(\gamma)\cdot g(\gamma)$ where $\gamma$ runs through all the trajectories in $M$ joining $\tilde{x}$ and $\tilde{y}g(\gamma)$, $\xi(g(\gamma)) = -s$. The resulting sum is equal, obviously, to $\sum_{g\in G} n_g(x, y)\cdot g$.

Corollary 4.5. *Let* $x, y \in Cr\ f$, $\mathrm{ind}\ x = \mathrm{ind}\ y + 1$, *and let* $v$, $v'$ *be two good gradient-like fields for* $f$. *Assume that* $N(\bar{x}, \bar{y}t^s; v)$ *and* $N(\bar{x}, \bar{y}t^s, -v')$ *are homotopically the same for all* $s \in Z$, *such that* $N \geqslant s \geqslant -ex(E, f) - 2$.

*Then* $n(x,y; v)$ *is* N-*equivalent to* $n(x,y; v')$.

Proof. By the definition of ex(E, f) (see the end of §2) $|f(\bar{x}) - f(\bar{y})| \leqslant ex(E, f) + 1$, therefore if $s \leqslant -ex(E, f) - 2$, we have $f(\bar{x}) < f(\bar{y}t^s)$ and therefore for these values of s we have $n_s(x,y; v) = n_s(x,y; v') = 0$. For $N \geqslant s \geqslant -ex(E, f) - 2$ we have $n_s(x, y; v) = n_s(x,y; v')$ by the condition; therefore $n(x,y; v) = n(x,y; v')$ for all $-\infty < s \leqslant N$, q.e.d.

Corollary 4.6. *Let* N *be any natural number. Let* $(f, \bar{f}, v, E)$ *be a regular quadruple, belonging to* $\xi$ *. Then for every good gradient-like vector field* $v'$ *for* f *which is an* $\varepsilon$*-regular perturbation of* v*, sufficiently close to* v*, we have* $C_*(v, E) \underset{N}{\sim} C_*(v', E)$.

Proof. We apply corollary 4.5 and corollary 3.13 to the cobordism $W = \bar{f}^{-1}([c, c+n])$ where c is a regular value of $\bar{f}$, $n \in N$ and c and n are chosen in such a way, that W contains all the $\bar{x}t^s$ where $x \in Cr\ f$, and $N \geqslant s \geqslant ex(E, f)$. q.e.d.