5. The Whitney trick

In this section we shall do the preparatory work, for cancellation of a trivial direct summand of the Novikov complex (Theorem 5.1). For that we shall need to realize geometrically the incidence coefficients, that is to cancel two homotopic trajectories of opposite sign, joining the same points (Theorem 5.2). That is the subject of ordinary Whitney trick, but we shall need a bit more than is the output in the usual situation. Namely we must cancel the pair of trajectories joining p and q in such a way that the set of trajectories, joining all the other pairs x, y with $\operatorname{ind} x = \operatorname{ind} y + 1$, essentially does not change.

Before we state the theorem 5.1 we need one more notation. For $x \in \widetilde{M}$ we denote by $\bar{x}$ its projection to $\overline{M}$.

<u>Theorem 5.1.</u> *Let* $\mathscr{X} = (f, \bar{f}, v, E)$ *be a regular quadruple, belonging to a regular class* $\xi \in H^1(M, Z)$. *Assume that the Novikov complex* $C_*(v, E)$ *is* N*-equivalent to the complex* $D_* \oplus \{0 \longleftarrow \Lambda_\xi^-(b) \xleftarrow{\partial} \Lambda_\xi^-(a) \longleftarrow 0\}$ *where* $a, b \in E$, $\operatorname{ind} a = \operatorname{ind} b + 1$, $\partial(a) = b$, $N > 1$.

Then there exists an admissible modification $(g, \bar{g}, v', E)$ of $\mathscr{X}$, such that

1) There are no $(-v')$-trajectories, joining $\bar{a}$ and $\bar{y}t^s$, where $y \in E$, $y \neq b$, $\operatorname{ind} y = \operatorname{ind} b$, $s < N$.

2) There are no $(-v')$-trajectories, joining $\bar{x}$ and $\bar{b}t^s$, where $x \in E$, $\operatorname{ind} x = \operatorname{ind} a$, $x \neq a$, $s < N$.

3) $\bar{g}(\bar{b}) < \bar{g}(\bar{a}) < \bar{g}(\bar{b}) + 1$ and there is only one $(-v')$-trajectory γ, joining a and b; $h(\gamma) = 1$, $\varepsilon(\gamma) = 1$.

4) The complex $C_*(v', E)$ is N-equivalent to $C_*(v, E)$.

For a Morse function φ we denote by $\mathrm{Cr}\,\varphi$ the set of all critical points of φ. For two points $\alpha, \beta \in \bar{M}$ we say that α is a t-shift of β, if $\alpha = t^s\beta$ for $s \in Z$.

<u>Theorem 5.2</u>. *Let* $(f, \bar{f}, v, E)$ *be an r-quadruple for a regular class* $\xi \in H^1(M, Z)$. *Let* $p, q \in \mathrm{Cr}\,\bar{f}$, $\mathrm{ind}\, p = \mathrm{ind}\, q + 1$, *such that for some regular value* c *of* $\bar{f}$ *we have* $c < \bar{f}(q) < \bar{f}(p) < c+1$. *Let* γ_1, γ_2 *be two trajectories of* $(-v)$, *joining* p *and* q, *such that* $h(\gamma_1) = h(\gamma_2)$, $\varepsilon(\gamma_1) = -\varepsilon(\gamma_2)$. *Let* $g: \bar{f}^{-1}([c, c+1]) \to [c, c+1]$ *be some enumerating function with the same g.-l.v. field* v, *such that for some interval* $(\mu, \nu) \subset (c, c+1)$ *the only critical points of* g *in* $g^{-1}((\mu, \nu))$ *are* p, q. *Let* $K \subset \mathrm{Cr}\,\bar{f}$ *be any finite set.*

Then there exists a good gradient-like vector field v' for g, such that:

1) $(g, \bar{g}, v', E)$ is an admissible modification of $(f, \bar{f}, v, E)$

2) If $x, y \in K$, $\mathrm{ind}\, x = \mathrm{ind}\, y + 1$ x is not a t-shift of p, y is not a t-shift of q then $N(x, y; v')$ is homotopically the same as $N(x, y; v)$.

3) Assume that $\mathrm{ind}\, p \geq 4$. If $y \in K$, $\mathrm{ind}\, y = \mathrm{ind}\, q$, y is not a t-shift of q, then $N(pt^s, y; v')$ is homotopically the same as $N(pt^s, y; v)$ for every $pt^s \in K$.

4) Assume that ind $p \leqslant n-3$. If $x \in K$, ind x = ind p, x is not a t-shift of p, then $N(x, qt^s; v')$ is homotopically the same as $N(x, qt^s; v)$ for every $qt^s \in K$.

5) For every $x,y \in K$, ind x = ind $y + 1$, the set $N(x,y; v')$ is homologically the same as $N(x,y; v)$.

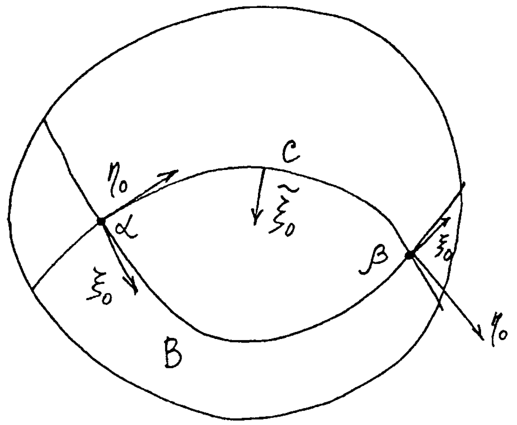
6) The set $N(p,q; v)$ is homotopically the same as $N(p,q; v) \setminus \{Q(\gamma_1), Q(\gamma_2)\}$.

Proof of theorem 5.2.

Consider the cobordism $g^{-1}([\mu, \nu])$ of our statement. We can assume that μ, ν are regular. We are giving to apply the Whitney trick to this cobordism. Since this cobordism projects bijectively to M this modification determines also a new vector field on M which will suit us as v'. The resulting vector field on M is of course the result of this Whitney modification, applied at once to all the cobordisms $g^{-1}([\mu+n, \nu+n])$, $n \in Z$.

The construction of Whitney trick which we use is classical and we refer to [Mi2, §6] for an excellent exposition of the procedure. If we apply just this procedure we shall get the properties 1) and 6) of our theorem. The properties 2)-5) will be achieved by carefully choosing the initial data for performing the Whitney trick. For the sake of completeness we reproduce here the construction from [Mi2, §6], so that the items 5.4-5.10 contain the same material as in [Mi2], a bit rearranged.

<u>Definition 5.3.</u> Let D^2 be the standard closed 2-dimensional disc in euclidean plane and B, C be any two smooth curves in D^2 in the mutual disposition as presented at the picture 5.1.



$$\varphi(B) = \varphi(D^2) \cap N$$
$$\varphi(C) = \varphi(D^2) \cap L$$

Pict. 5.1

<u>Lemma 5.4.</u> *Let* N^r, L^s *be two compact submanifolds of the compact manifold* V^{r-s}, $r+s \geqslant 5$. *Let* N *be oriented,* L *be cooriented, and let* $L \pitchfork N$. *Let* a,b *belong to* $L \cap N$ *and be of opposite signs (say* a *of* +, b *of* −). *Let* $\varphi : D^2 \longrightarrow V$ *be a smooth imbedding of a standard disc, such that* *) : $\varphi(\alpha) = a$, $\varphi(\beta) = b$; *the intersection* $\varphi(D^2) \cap N$ *is equal to* $\varphi(B)$ *and for every* $\gamma \in B$ *the intersection* $\varphi_*(T_\gamma D^2) \cap T_{\varphi(\gamma)}N = \varphi_*(T_\gamma B)$, *in intersection* $\varphi(D^2) \cap L = \varphi(C)$ *and for every* $\gamma \in C$ *we have* $\varphi_*(T_\gamma D^2) \cap T_{\varphi(\gamma)}L = \varphi_*(T_\gamma C)$.

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*) These a, b have nothing in common with a, b in the statement of th.5.1. We keep the present notation until the end of the proof of th.5.2.

Then there exists an embedding $\Phi : D^2 \times D^{r-1} \times D^{s-1} \to V$ (where ε is sufficiently small), such that $\Phi| D^2 \times \{0\} \times \{0\} =$ $= \varphi$; $\Phi^{-1}(L) = C \times \{0\} \times D_\varepsilon^{s-1}$; $\Phi^{-1}(N) = B \times D_\varepsilon^{r-1} \times \{0\}$.

Proof. It is made in several steps (lemmas 5.5-5.7). First we do the homotopy - theoretical stuff.

Lemma 5.5. *There exist the smooth vector fields* $\xi_1, \ldots, \xi_{r-1}$, $\eta_1, \ldots, \eta_{s-1}$, *defined on* $\varphi(D^2)$, *such that*

1) *for such point* $z \in \varphi(D^2)$ *the vectors* $\xi_i(z)$, $\eta_j(z)$ form a base of $T_z V \,/\, T_z\, \varphi(D^2)$.

2) *for* $z \in \varphi(B)$ *the vectors* $\xi_i(z)$ *form a base of* $T_z N \,/\, T_z\, \varphi(B)$

3) *for* $z \in \varphi(C)$ *the vectors* $\eta_j(z)$ *form a base of* $T_z L \,/\, T_z\, \varphi(C)$

Proof. First we construct ξ_i. Denote by ξ_0 (resp. η_0) the tangent vector fields along B (resp. C) (see the picture 5.1) and by abuse by notations also the image $\varphi_*(\xi_0)$, resp. $\varphi_*(\eta_0)$. Pick up any vector fields $\xi_1, \ldots, \xi_{r-1}$ along $\varphi(B)$, such that $\xi_0, \xi_1, \ldots, \xi_{r-1}$ form the positive base of $T_*(N)$ along $\varphi(B)$. Consider now the vector field $\tilde{\xi}_0$ along the curve C, which is tangent to D^2, transversal to C and $\tilde{\xi}_0(\alpha)$ $= \xi_0(\alpha)$, $\tilde{\xi}_0(\beta) = -\xi_0(\beta)$. The quotient $T_*(V) \,/\, T_*(L)$ is oriented by the assumption, the vectors ($\xi_0, \xi_1, \ldots, \xi_{r-1}$) form the positive base of it at the point α, since the intersection sign is + there, and $(-\xi_0, \xi_1, \ldots, \xi_{r-1})$ form the positive base in β (sign is -). The vector field $\tilde{\xi}_0$ is the smooth section of $T_*V \,/\, T_*L$ along C which implies that there exist the vectors $\xi_1, \dots, \xi_{r-1}$ along C, which form the base of $T_* V / (T_* L + T_* \varphi(D^2))$ and which coincide in the points α, β with the vector previously chosen. Now we have the vector fields $\xi_1, \dots, \xi_{r-1}$ on $\varphi(C) \cup \varphi(C')$, and we want to expand them to $\varphi(D^2)$. The obstruction for doing this belongs to $\pi_1(V(r-1, r+s-2)) = O(r+s-2)/O(s-1)$. If $s \geqslant 3$ this is zero, if $s \leqslant 2$ then $r \geqslant 3$, $\pi_1(O(r-1)) \to \pi_1(O(r+s-2))$ is epi and, having changed the fields $\xi_1, \dots, \xi_{r-1}$ on the small interval of C we get the zero obstruction.

To construct the vectors $\eta_1, \dots, \eta_{s-1}$ we consider the trivial bundle ξ over $\varphi(D^2)$, generated by the sections $\xi_1, \dots, \xi_{r-1}$ and denote by η the bundle $T_* V / \xi + T_* \varphi(D^2)$. This is an (s-1)-dimensional bundle η over D^2. If we choose a trivialization of $T_* L / T_* \varphi(C)$ along $\varphi(C)$ we get a trivialization of η over $\varphi(C)$ which expands to all the D^2 by homotopical triviality of the pair (D^2, C). Q.E.D.

The vectors $\xi_1, \dots, \xi_{r-1}, \eta_1, \dots, \eta_{r-1}$ form a base of the normal bundle to $\varphi(D^2)$, that is an infinitesimal variant of the tubular neighbourhood. The neighbourhood itself is constructed as usual with the help of geodesic exponent and to satisfy the conclusion of our theorem we shall need that L and N are totally geodesic submanifolds of V.

<u>Lemma 5.6</u>. *There exists a riemannian metric on* V, *such that* L *and* N *are totally geodesic.*

<u>Proof.</u> If K is a smooth compact submanifold of a manifold P denote by νK the normal bundle, $B_\varepsilon(\nu K)$ – the

open- ε-disc-bundle, associated with some metrics on ν. The map $\Phi_\varepsilon : B_\varepsilon(\nu K) \longrightarrow P$ will be called tubular if it is diffeomorphism onto some open set in P and $\Phi_\varepsilon(K, 0) = K$. The image of Φ_ε will be denoted T_ε. (Tubular maps exist, for example the exponential map for some metrics.) The map $(x, \xi) \longrightarrow (x, -\xi)$ of $B_\varepsilon(\nu K)$ will be denoted by J_ε.

Note that if g is a riemannian metric on P, such that $g|T_\varepsilon$ is invariant under J_ε, then K is totally geodesic. (Indeed, for a geodesic γ, such that $\gamma(0) \in K$, $\dot\gamma(0) \in K$ the curve $J_\varepsilon(\gamma)$ is again geodesic with the same origin and tangent vector, hence $\gamma = J_\varepsilon(\gamma) \Rightarrow \gamma \in K$).

Return to our L and N. We shall construct a riemannian metrics g on V and two tubular maps $\Phi_{\varepsilon,N} : B_\varepsilon(\nu L) \longrightarrow V$, $\Phi_{\varepsilon,N} : B_\varepsilon(\nu N) \longrightarrow V$, such that g is $J_{\varepsilon,L}$ and $J_{\varepsilon,N}$ -invariant. Near each points of intersection choose a chart $\Psi : R^s \times R^r \longrightarrow U(z) \subset V$, such that $N \cap U(z) = \Psi(R^s \times 0)$, $L \cap U(z) = \Psi(0 \times R^r)$ and pick up a metrics g_0 on V, such that $g_0 | \Psi(B_{2\varepsilon}(0) \times B_{2\varepsilon}(0))$ is standard euclidean. The tubular maps $\Phi_{\varepsilon,L}$ and $\Phi_{\varepsilon,K}$ will be constructed as the exponential maps for g_0. If ε is chosen small enough, the intersection $T_{\varepsilon,L} \cap T_{\varepsilon,N}$ is exactly the union of U(z) over all the intersection points. By our choice of metrics in U(z) the map $J_{\varepsilon,L}$ acts as $(x, y) \longmapsto (-x, y)$ and $J_{\varepsilon,N}$ – as $(x, y) \longmapsto (x, -y)$. Consider now the metrics $1/2\,(g_0 + (J_{\varepsilon,N})_* g_0)$ on $T_{\varepsilon,N}$ and the metrics $1/2\,(g_0 + (J_{\varepsilon,N})_* g_0)$ on $T_{\varepsilon,K}$. These metrics coinside on the intersection and give the $J_{\varepsilon,L}$ and $J_{\varepsilon,N}$ -invariant metric on $T_{\varepsilon,L} \cup T_{\varepsilon,N}$ which can be glued by partition of unity with arbitrary metrics on $V \setminus (T_{\varepsilon/2,L} \cup T_{\varepsilon/2,N})$ so as to give the metrics sought on $T_{\varepsilon/2,L} \cup T_{\varepsilon/2,N}$. Q.E.D.

Now we are ready to prove lemma 5.4. We pick up the metrics g as the satisfy lemma 5.6 and the vector fields $\xi_1, \dots, \xi_{r-1}, \eta_1, \dots, \eta_{s-1}$ as to satisfy lemma 5.5. Consider now the map $\Phi_\varepsilon : D^2 \times D_\varepsilon^{r-1} \times D_\varepsilon^{s-1} \longrightarrow V$ defined as $(x, t_1, \dots, t_{r-1}, u_1, \dots, u_{s-1}) \longmapsto \exp_{\varphi(x)}\left(\sum_{i=1}^{r-1} t_i \xi_i(\varphi(x)) + \sum_{j=1}^{s-1} u_j \eta_j(\varphi(x)) \right)$, where exp is defined with respect to the riemannian metric g. It follows from the standard theorems that for ε sufficiently small Φ_ε is a diffeomorphism onto a neighbourhood of $\Phi_\varepsilon(D^2)$ in V. This neighbourhood is arbitrarily small if ε is sufficiently small. Obviously $\Phi_\varepsilon(D^2 \times 0 \times 0) = \varphi$, and if $\delta < \varepsilon$, then $\Phi_\varepsilon|D^2 \times D_\delta^{r-1} \times D_\delta^{s-1} = \Phi_\delta$.

Further, since N is totally geodesic the set $\Phi_\varepsilon(B \times D_\varepsilon^{r-1} \times 0)$ is contained in N. We are going to show that $\Phi_\varepsilon(B \times D_\varepsilon^{r-1} \times 0) = N \cap \operatorname{Im} \Phi_\varepsilon$, $\Phi_\varepsilon(C \times 0 \times D_\varepsilon^{s-1}) = L \cap \operatorname{Im} \Phi_\varepsilon$, if only ε is sufficiently small. For that we fix some ε, such that Φ_ε is a diffeomorphism and suppose on the contrary that for every $n > 0$ there is a point $z_n \in N$, such that $z_n \in \operatorname{Im} \Phi_{\varepsilon/n}$, but $z_n \notin \Phi_{\varepsilon/n}(B \times D_{\varepsilon/n}^{r-1} \times 0)$. Note, that since $\Phi_\varepsilon(B \times D_\varepsilon^{r-1} \times 0) \cap \cap \operatorname{Im} \Phi_\delta = \Phi_\delta(B \times D_\delta^{r-1} \times 0)$ where $\delta < \varepsilon$, we have that $z_n \notin \notin \Phi_\varepsilon(B \times D_\varepsilon^{r-1} \times 0)$. By compactness of V we can suppose that $z_n \longrightarrow z$. Note that $z \in \Phi(D^2) \cap N = B$. The manifold N with our riemannian metrics g is compact, hence the geodesic distance $\rho(z_n , z) \to 0$, hence there exists a geodesic $N \supset \gamma_n$, starting from z with $|\dot\gamma(0)| = 1$ and arriving at z_n at

$\tau_n \to 0$. But by construction every geodesic γ, starting at $z \in B$ with the tangent vector $\gamma \in T_zN$, $|\dot\gamma(0)| = 1$ rests in $\Phi_\varepsilon(B \times D^{r-1} \times 0)$ for $\tau \leqslant \varepsilon/(\max_{1\leq i\leq r-1} |\xi_i|)$, and thus cannot reach z_n for this values of τ. In the same way one proves that $\Phi_\varepsilon(C \times 0 \times D_\varepsilon^{s-1}) = L \cap \mathrm{Im}\,\Phi_\varepsilon$ for ε sufficiently small.

Lemma 5.4 is proven.

Now we are going to define the isotopy h of V to itself which will cancel the intersection points a, b.

Lemma 5.7. *Assume that* V^{r+s}, L^s, N^r, φ *satisfy the hypotheses of lemma 5.4, and* Φ *satisfies the conclusions.*

Then there exists a smooth isotopy h, $0 \leqslant t \leqslant 1$ of V, such that

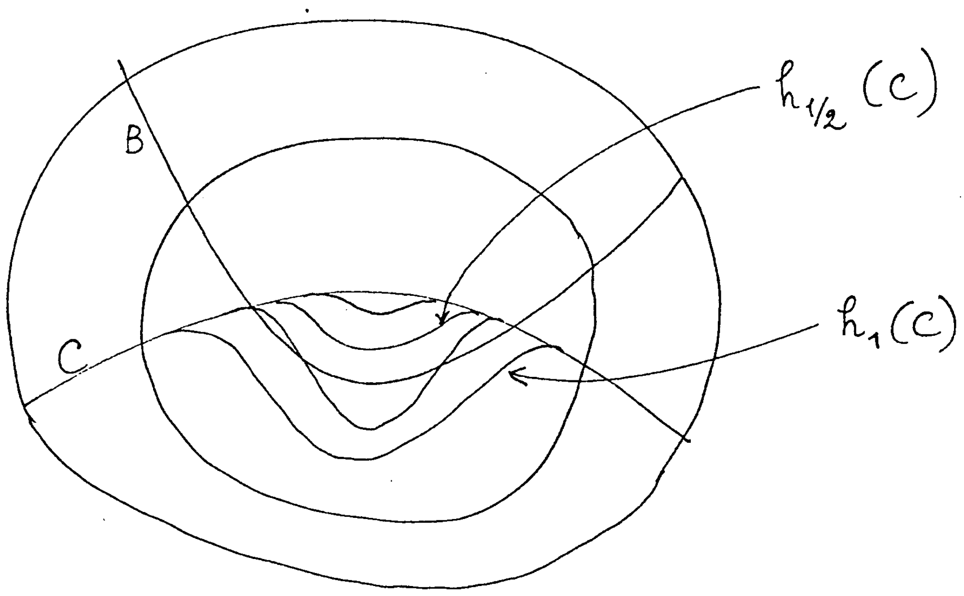
1) h_t = id in some compact neighbourhood of $V \setminus \mathrm{Im}$, h_t = id for t sufficiently small and h_t = id for t sufficiently close to 1.

2) h_0 = id.

3) $h_t(\mathrm{Im}\,\Phi) = \mathrm{Im}\,\Phi$ for all t.

4) $h_1(L)$ is transversal to N and the set of intersection points $h_1(L) \cap N$ equals $(L \cap N) \setminus \{a, b\}$, the intersection signs equal the old ones.

Proof. Consider first of all a smooth isotopy $d \to g(t, d)$ of D^2 to itself which is identical in some neighbourhood of the boundary and $g(1, d) \cap B = \emptyset$ (see pict. 5.2). We demand also that $g(t, d) = d$ for t small enough and $g(t, d) = g(1, d)$ for t sufficiently close to 1.



Pict. 5.2

Next let k_1, k_2 be the smooth non-negative function on R^{r-1}, resp. R^{s-1}, such that $k_1(x) = 1$ for $|x| \leqslant \varepsilon/2$, $k(x) = 0$ for $|x| \geqslant 3\varepsilon/4$ and $k_2(y) = 1$ for $|y| \leqslant \varepsilon/2$, $k_2(y) = 0$ for $|y| > 3\varepsilon/4$. Define the isotopy $h_t : V \longrightarrow V$ as following:

1) if $v \notin \operatorname{Im} \Phi$, then $h_t(v) = v$.

2) if $v = \Phi(d, x, y)$, then

$$h_t(v) = \Phi(g((k_1(x) + k_2(y))t, d), x, y).$$

Now we check that the conclusions of lemma hold. First of all we note that $h_t(v) = v$ if $v \in \operatorname{Im} \Phi$ and v is close enough to the boundary (indeed, in this case either $g(t, d) = d$ or $k_2(x) + k_2(y) = 0$). This implies that h_t is really a smooth map $V \to V$, and the properties 1), 2) hold. By definition $h_t(\operatorname{Im} \Phi) \subset \operatorname{Im} \Phi$. On the $\operatorname{Im} \Phi$ the map h_t is defined actually in terms of $D^2 \times D^{r-1}_\varepsilon \times D^{s-1}_\varepsilon$ and it is easy to verify that h_t above defines a smooth isotopy $D^2 \times D_\varepsilon^{r-1} \times D_\varepsilon^{s-1}$ to itself and, therefore h_t on Im Φ and on the complement glue together to give a smooth isotopy, satisfying 3).

To check up 4) let $x \in N \cap h_1(L)$ and distinguish two cases: a) $x \notin \operatorname{Im}\Phi$. If $x = h_1(\bar{x})$ then $\bar{x} \in \operatorname{Im}\Phi$ by 3), hence $h_1(x) = x$ and since Im Φ is compact the same holds for $x' \in h_1(L)$ close enough to x, hence $U(x) \cap (N \cap h_1(L)) = U(x) \cap (N \cap L)$ where $U(x)$ is a small neighbourhood of x. So in this case N and $h_1(L)$ are transversal at x and the sign is as before.

b) $x \in \operatorname{Im}\Phi$. The manifold $h(L) \cap \operatorname{Im}\Phi = h(L \cap \operatorname{Im}\Phi)$, by 3), and the latter is equal to Φ-image of the set $(g(k_1(x), d), x, 0)$. The manifold $N \cap \operatorname{Im}\Phi$ is the Φ-image of the set $(d, 0, y)$, where $d \in C$. These two sets do not intersect. Note finally, that $L \cap N \cap \operatorname{Im}\Phi = \{a,b\}$, and we have finished the proof of lemma 5.7.

<u>Remark 5.8.</u> We can assume that $h_t = \mathrm{id}$ for t small enough and that $h_t = h_1$ for t close enough to 1.

Let now $f: W \longrightarrow [a,b]$ be a Morse function on some cobordism W and v be a gradient-like field for f. Let $c, d \in (a,b)$ be regular values for f, such that $c < d$ and the segment (c, d) contains no critical values. The shift along $(-v)$-trajectories defines a diffeomorphism $T(v)$: $f^{-1}(c) \to f^{-1}(d)$. For a set $X \subset f^{-1}(d)$ we denote by $tr(X)$ the set of the points $z \in f^{-1}([c,d])$, lying on some $(-v)$-

trajectory, starting froma point $x \in X$. Note that $tr(X) \cap \cap f^{-1}(d) = T(X)$.

Denote $f^{-1}(d)$ by V and let $h_t : V \longrightarrow V$ be a smooth isotopy of $h_0 = id$ to some diffeomorphism h_1, such that $h_t = id$ everywhere except some set $K \subset V$, and $h_t = id$ for t small enough and $h_t = h_1$ for $(1-t)$ small enough.

Lemma 5.9. *In the above assumptions there exist a new gradient-like vector field* v' *which differs from* v *only in* $f^{-1}([c+\varepsilon, d-\varepsilon]) \cap tr(K)$ *and for which* $T(v') = h\ T(v)$.

Proof. We denote by $\Phi(t, v)$ the diffeomorphism of shift by t along the v-trajectories.

Choose $\lambda > 0$ such that the $\Phi(\lambda, -v)V \subset f^{-1}(c, d)$. Denote $\Phi(\lambda, -v)V$ by V' and the union of $\Phi(t, v)V$ for $0 \leqslant t \leqslant \lambda_2$ by W'. Then W' is a cobordism, $\partial W' = V \sqcup V'$. The shift along (+v)-trajectories defines the diffeomorphism $\varphi : f^{-1}(c) \longrightarrow V'$, such that $T(v) = \Phi(\lambda, -v) \circ \varphi$.

Thus it suffices to find a gradient-like vector field v' on W', such that $supp(v'-v) \in Int\ W' \cap tr\ K$ and that the shift along v'-trajectories from V' to V is the map $h_1 \circ \Phi(\lambda, -v)$.

The map $(x, -t) \longmapsto \gamma_t(x, -v)$ determines the diffeomorphism $\varkappa : V \times [-\lambda, 0] \longrightarrow W'$, such that the vector field $\partial/\partial t$ is carried to v.

Define a new vector field w on $V \times [-\lambda, 0]$ by $w(x,t) = (\frac{\partial}{\partial t}(h_{1+t/\lambda}(x)), 1)$. By our assumption on h we have $h_{1+t/\lambda}(x) = x$ for $t \in [-\lambda, 0]$, $t + \lambda$ is small, also

$h_{1+t/\lambda}(x) = h_1(x)$ for $t \in [-\lambda, 0]$ t is small, hence supp $(w(x, t) - v)$ is contained in $V \times (-\lambda, 0)$. The integral curves of $w(x, t)$ are given by the formula $\gamma_t(x, w) = (h_{1+t/\lambda}(x), t)$ hence w-shift determines the diffeomorphism $V \times (-\lambda) \to V \times (0)$ which equals to h_1. Note that if $h_\tau(x) = x$ for all then $w(x, t) = v$ for all $t \in [-\lambda, 0]$, which means that $\text{supp}(w,-v) \in \varkappa^{-1}(\text{tr } K)$.

Now we denote by v' the vector field $\varkappa_*(w)$ on W'. The above properties of w imply the properties of v' sought.

Definition 5.10. *Let $f: W \to [a,b]$ be a Morse function on a cobordism W^n, v be a gradient-like vector field for f, $c \in (a,b)$ be a regular value of f. Denote by V the level surface $f^{-1}(c)$. Assume that $n \geqslant 6$ and let N^r, L^s be two compact transversal submanifolds of V^{n-1}, $r+s = n-1$, and $\varphi: D^2 \to V$ be the imbedding of 2-disc into V, such that these data satisfy the hypotheses of lemma 5.4.*

Let $\Phi: D^2 \times D_\varepsilon^{r-1} \times D_\varepsilon^{s-1} \to V$ be the imbedding of the "fat disc D^2" to V, satisfying the conclusions of lemma 5.4.

Let h_t be a smooth isotopy, satisfying the conclusions of lemma 5.7 with respect to V, L, N, .

Choose some $\delta > 0$ small enough, so that the segment $[c-\delta, c]$ contains no critical values of f and let v' be a new gradient-like field for f, satisfying the conclusion of lemma 5.9 with respect to the segment $[c-\delta, c]$ and $K = \text{Im } \Phi$. This new vector field v' will be called the Whitney modification of v with respect to the data $(V, v, N, L, \varphi, \Phi, [c-\delta, c], h_t)$.

Note that $\sup(v' - v) \subset f^{-1}(c-\delta, c) \cap \mathrm{tr}(\mathrm{im}\,\Phi)$.

The result of the several applications of this operation to the vector field v with respect to different regular values c_i (and corresponding data) will be called the multiple Whitney modification (or, briefly, multiple W-modification).

Now, the preliminaries done, we proceed to the proof of theorem 5.2.

We expand the function g to the t-equivariant function $\bar{g}$ on $\bar{M}$ and glue from it the function $g: M \longrightarrow S^1$ which will be denoted by the same letter. Note that $f^{-1}(c + k) = g^{-1}(c+k)$ for $k \in Z$, as well as $f^{-1}([c+k, e+n]) = g^{-1}([c+k, c+n])$. Choose some regular value λ for $\bar{g}$, $\lambda \in (g(q), g(p))$.

Denote ind p by r, $g^{-1}(\lambda)$ by V and consider the manifolds $N = V \cap D(p, +v)$, $L = V \cap D(q, -v)$. Note that $\dim N = r-1$, $\dim L = n-r$, so $N \pitchfork L$ since v is good; $\dim N + \dim L = \dim V$. Since we have chosen the orientations of all the descending discs of v, the N is oriented, L is cooriented. Denote by a the point $\gamma_1 \cap V$, by b – the point $\gamma_2 \cap V$. The manifold N is diffeomorphic to S^{r-1}, which is simply connected, since ind $q = r-1 \geqslant 2$. The manifold L is diffeomorphic to S^{n-r}, which is simply connected since $r = $ ind $p \leqslant n-2$, hence $n-r \geqslant 2$. The signs of intersection points a, b are opposite. Choose the positive integer A so large that $K \subset \bar{f}^{-1}([c-A, c+1+A])$. Denote by W_1 the cobordism $\bar{f}^{-1}([c-A, c+1+A])$, and by W the cobordism $\bar{f}^{-1}([c-4A, c+1+4A])$.

Denote by K_λ^+ the intersection of the set $\{\bar{g}(x) \geqslant \lambda\}$ with the union $\bigcup_z D(z, v)$, where z runs through the critical points of $\bar{f}$, belonging to W, ind $z \leqslant r$, $z \neq p$. Similarly, K_λ^- will denote the intersection of $\{\bar{g}(x) \leqslant \lambda\}$ with the union $\bigcup_{z'} D(z', -v)$ where z' runs throgh the critical points of $\bar{f}$, $z' \in W$, ind $z' \geqslant r-1$, $z' \neq q$. Denote by $\bar{K}_\lambda^+$ the intersection of K_λ^+ with the union of all the discs $D(z,v)$ for $z \in W$, ind $z \leqslant n-3$. Thus for $r \leqslant n-3$ we have $\bar{K}_\lambda^+ = K_\lambda^+$.

Similarly let $\bar{K}_\lambda^-$ be the intersection of K_λ^- with the union of all the discs $D(z, v)$ for $z \in W$, ind $z \geqslant 3$. The following remark is a consequence of lemma 3.7 and will be used frequently in the sequel.

Remark 5.11. 1) The sets K_λ^+, $\bar{K}_\lambda^+$ are compact and do not intersect with the compact $D(p, v) \cap \{\bar{g}(x) \geqslant \lambda\}$.

2) The sets K_λ^-, $\bar{K}_\lambda^-$ are compact and do not intersect with the compact $D(q, -v) \cap \{\bar{g}(x) \leqslant \lambda\}$.

Lemma 5.12. There exists a smooth imbedding $\varphi: D^2 \to V$, satisfying the assumptions of lemma 5.4 with respect to N, L, a, b. This embedding can be chosen in such a way, that

1) *If* $z \in W$ *is a critical point of* $\bar{f}$, $z \neq p$, *and* ind $z \leqslant r$, ind $z \leqslant n-3$, *the* disc $D(z, +v)$ *does not intersect with* $\varphi(D^2)$,

2) *If* $z \in W$ *is a critical point of* $\bar{f}$, $z \neq q$, *and* $\operatorname{ind} z \geqslant r-1$, $\operatorname{ind} z \geqslant 3$ *then the* disc $D(z, v)$ *does not intersect with* $\varphi(D^2)$.

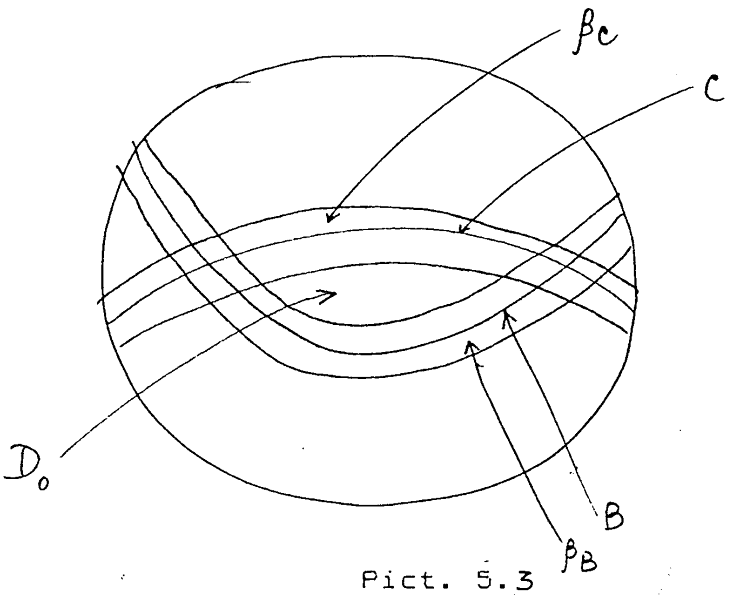
Proof. We construct first the embeddings of the curves C, B, drawn on the pict. 5.1. Note that the points a, b do not belong to $D(x, v)$ or $D(y, -v)$ for $x \neq p$, resp. $y \neq q$, since no two ascending (descending) discs intersect. Since v is good, the manifolds $D(z,-v)$ are transversal to $D(p, v)$ for $\operatorname{ind} z \geqslant r-1$, therefore $(D(z,-v) \cap V) \pitchfork (D(p,v) \cap V)$. That implies that $D(z,-v) \cap (D(p, v) \cap V)$ is a submanifold in N of codimension $n - \operatorname{ind} z \geqslant 2$. This manifold has a countable base, which implies that if $\operatorname{ind} z \geqslant r-1$ we can find an embedding $\varphi : B \hookrightarrow N$, such that $\varphi(\alpha) = a$, $\varphi(\beta) = b$ and $\varphi(B) \cap D(z, -v) = 0$. Note that the intersection $\varphi(B) \cap \cap D(z', v)$ is automatically empty for $z' \neq p$. In particular $\varphi(B) \cap K^{+}_{\lambda} = \varphi(B) \cap K^{-}_{\lambda} = \emptyset$.

Similarly we find an embedding $\varphi : C \longrightarrow D(q, -v) \cap V = = L$, such that $\varphi(\alpha) = a$, $\varphi(\beta) = b$, $\varphi(C)$ intersect no ascending disc starting at $z \neq q$ and $\varphi(C) \cap D(z, v) = \emptyset$. if $\operatorname{ind} z \leqslant r$, $z \neq p$. In particular $\varphi(C) \cap K^{+}_{\lambda} = \varphi(C) \cap K^{-}_{\lambda} = = \emptyset$.

Since $\dim N$, $\dim L \geqslant 2$ we can assume also that $\varphi(C)$, $\varphi(B)$ contain no intersection points of $L \cap N$ except a, b.

Now we choose the vector field ξ_0 along $\varphi(C)$ and η_0 along $\varphi(B)$ in V, such that ξ_0 is transversal to L, η_0 is transversal to N, $\xi_0(a)$ (resp. $\xi_0(b)$ coincides with tangent to $\varphi(B)$ in a (resp. with (-tangent) to $\varphi(B)$ in b), $\eta_0(a)$, resp. $\eta_o(b)$ coincides with tangent to $\varphi(C)$ in a (resp. (-tangent) to $\varphi(B)$ in b).

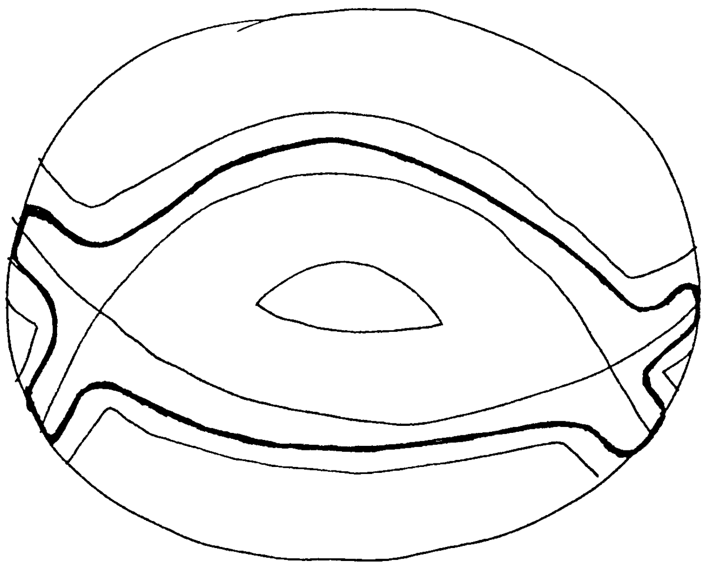
It is shown in [Mi2, §6] that we can expand a map φ from $C \cup B$ to a smooth embedding of some neighbourhood U of $C \cup B$ in a disc (see pict. 5.3), such that the neighbourhood U is the union of two narrow bands: β_C around C and β_B around B, the tangent space to $\varphi(\beta_C)$ is the direct sum of TC and $\{\xi_o\}$ and respectively $T(\beta_B) = TB \oplus \{\eta_o\}$. Since ξ_o is transversal to L, the band β_C intersects L by C if only φ_C is narrow enough (the same for β_B). Further, the intersection $\varphi(\beta_C) \cap N \subset B$ if only β_C is narrow enough, similarly $\varphi(\beta_B) \cap L \subset C$. That is to say, $\varphi(U) \cap N = B$, $\varphi(U) \cap L = C$.



Pict. 5.3

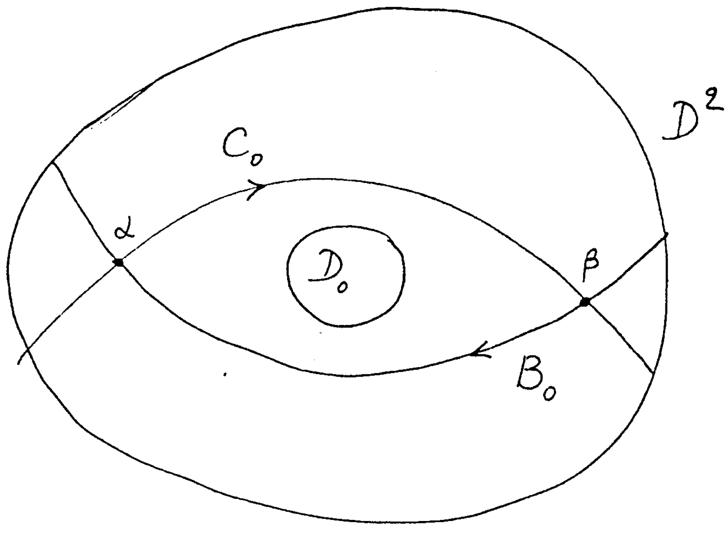
Recall now that $\varphi(c) \cup \varphi(B)$ does not intersect K_λ^+ and K_λ^-. Since all the three are compact, the image $\varphi(U)$ also does not intersect K_λ^+, K_λ^- if only U is small enough.

Note that we can assume that U contains all the disc D^2, except the central part D_o. Indeed, if not, we construct a diffeomorphism $a: D^2 \longrightarrow D^2$, which is identical on $C \cup B$ and in the central part D_o, and such that $\operatorname{im} a \subset N$. (see the pict. 5.4, where the black curve is the a-image of $S^1 = \partial D^2$).



Pict. 5.4.

Consider now the picture 5.5. We have constructed a smooth embedding of $D^2 \setminus D_o$ (where D_o is an open disc, centered in the centre of D^2), and we want to expand the embedding to the whole D^2.



Pict. 5.5.

Note that the boundary ∂D_o is homotopic to zero in V. Indeed it suffices to show that $C_o \circ B_o$ is homotopic to zero, where C_o is a piece of C between α and β, B is a piece of (-B) between β and α. For that it suffices to lift the loop $C_o \circ B_o$ to the loop in the universal covering $\widetilde{W} \longrightarrow W$ of W. Denote by γ_1^+, γ_2^+ the parts of the (-v)-trajectories γ_1, γ_2 from $\bar{p}$ up to $\bar{g}^{-1}(\lambda)$ and by $\tilde{\gamma}_1^+$, $\tilde{\gamma}_2^+$ - their liftings to w, starting from p. Similarly γ_1^-, γ_2^- - stand for the parts of γ_1, γ_2 from $g^{-1}(\lambda)$ to $\bar{q}$ and $\tilde{\gamma}_1^-$, $\tilde{\gamma}_2^-$ - their liftings to $\widetilde{W}$, finishing at q. Now we lift B_o to a curve β in $\widetilde{W}$, starting at the end of $\tilde{\gamma}_2^+$. Since D(p, v) is one-connected, β finishes at the end of $\tilde{\gamma}_1^+$. Next we lift C_o to a curve δ in $\widetilde{W}$, starting at the beginning of $\tilde{\gamma}_2^-$. Since D(q, -v) is one-connected, it finishes at the beginning of $\tilde{\gamma}_1^-$. Consider now the lifting $\delta \cdot h(\gamma_2)$ of B_o. This starts at the beginning of $\tilde{\gamma}_2^- \cdot h(\gamma_2)$ and finishes at the beginning of $\tilde{\gamma}_1^- \cdot h(\gamma_2)$. Note that the end of $\tilde{\gamma}_2^+$ is the beginning of $\tilde{\gamma}_2^- \cdot h(\gamma_2)$ by the definition of $h(\gamma_2)$. Further, the end of $\tilde{\gamma}_1^+$ is the beginning of $\tilde{\gamma}_1^- \cdot h(\gamma_1)$, which equals $\tilde{\gamma}_2^- \cdot h(\gamma_2)$ by the hypothesis. Hence $\delta \cdot h(\gamma_2)$ starts at the end of $\tilde{\gamma}_2^+$ and finishes at the end of $\tilde{\gamma}_1^+$, hence the composition $(\delta \cdot h(\gamma_2)) \circ \beta$ is a closed lifting of $C_o \circ B_o$, q.e.d.

Further, ∂D_o is homotopic to zero already in $V \setminus (L \cup N)$. To prove that we distinguish two cases: 1) $r \leqslant n-3$, 2) $n - (r-1) \leqslant n-3$ (since $n \geqslant 6$ one of these two necessarily holds). We do the first, the second is similar. The codimension of N in V is $\geqslant 3$, hence it suffices to prove that ∂D_o is zero homotopic in $V \setminus L$. The codimension of L in V equals

$r-1$. If $(r-1)$ 3 then again we are over. Let $r < 4$. The $(-v)$-shift defines the diffeomorphism α of $V \setminus L$ onto $g^{-1}(\mu) \setminus D(q, v)$, together with the homotopy of ∂D_o to its α-image; hence $\alpha(\partial D_o)$ is homotopic to zero in W, hence in $f^{-1}(c)$ by regularity of f, therefore $\alpha(\partial D_o)$ is homotopic to zero in $\bar{g}^{-1}(\mu) \setminus D(q,v)$, since $D(q,v)$ has the codimension $n - (r-1) = n-r+1 \geqslant n-2 \geqslant 4$. The $(-v)$-shift of this homotopy gives us the required homotopy from ∂D_o to zero in $\bar{g}^{-1}(\lambda)$. This homotopy together with the smooth embedding of $D^2 \setminus D_o$, constructed above, gives us a continuous map $\varkappa : D^2 \longrightarrow V$. Having expanded D_o a bit we can assume that this map is smooth on some compact neighbourhood of $D^2 \setminus D_o$.

Consider now the set $\bar{K}^+_\lambda \cap \bar{g}^{-1}(\lambda)$. That is a compact set which is a finite union of submanifolds $D(z,v) \cap \bar{g}^{-1}(\lambda)$ of V, each of which has the countable base and has the codimension $\geqslant 3$. The same holds for $\bar{K}^-_\lambda \cap \bar{g}^{-1}(\lambda)$. By the choice of φ the image $\varphi(D^2 \setminus D_o)$ does not intersect $\bar{K}^+_\lambda \cap V$ and $\bar{K}^-_\lambda \cap V$.

Now we apply the standard transversality argument and get a smooth embedding $k : D^2 \longrightarrow V$, which is as close as we want to $\varkappa$, coincides with $\varkappa$ on the compact set $D^2 \setminus D_o$, whose image does not intersect with $\bar{K}^+_\lambda \cap V$, $\bar{K}^-_\lambda \cap V$, and such, that the image $k(D_o)$ does not intersect $N \cup L$.

I claim that this k satisfies the conclusions of the lemma 5.12. The requirements 1), 2) have just been checked, so we must check only that the assumptions of lemma 5.4 hold for k.

The map k is smooth by definition, $k(\alpha) = a$ and

$k(\beta) = b$ since $k|B \cup C = \varphi|B \cup C$. The intersection $k(D^2) \cap N$ equals $(\varphi(D^2 \setminus D_o) \cap N) \cup (k(D_o) \cap N) = \varphi(D^2 \setminus D_o) \cap N$. We have chosen the φ above in such a way that this latter intersection is exactly B. Similarly, $k(D^2) \cap L = B$. For every $\gamma \in B$ the tangent space $T_\gamma k(D^2) = T_\gamma \varphi(\beta_B) = T_\gamma \varphi(B) \oplus + \{\eta_o\}$. Since η_o is transversal to N, we have $T_\gamma k(D^2) \cap \cap T_\gamma N = T_\gamma \varphi(B)$. Similarly for L.

Lemma 5.12 is proved.

Corollary 5.13. For a smooth embedding $\varphi : D^2 \to V$, satisfying the conclusions of lemma 5.12 the embedding $\Phi : D^2 \times D_\varepsilon^{r-2} \times D_\varepsilon^{n-r-1} \to V$, constructed in lemma 5.4, can be chosen so that

1) If $z \in W$ is a critical point of $\bar{f}$, $z \neq p$, $\operatorname{ind} z \leqslant r$, $\operatorname{ind} z \leqslant n-3$, then $D(z,v) \cap \operatorname{Im}\Phi = \emptyset$.

2) If $z \in W$ is a critical point of $\bar{f}$, $z \neq q$, $\operatorname{ind} z \geqslant r-1$, $\operatorname{ind} z \geqslant 3$, then $D(z,-v) \cap \operatorname{Im}\Phi = \emptyset$.

Proof. Just note that $\varphi(D^2) \cap \bar{K}_\lambda^+ = \emptyset = \bar{K}_\lambda^- \cap \varphi(D^2)$, that $\varphi(D^2)$, $\bar{K}_\lambda^+$, $\bar{K}_\lambda^-$ are compact, and that for δ small enough the map $\Phi : D^2 \times D_\varepsilon^{r-2} \times D_\varepsilon^{n-r-1} \to V$, restricted to $D^2 \times D_\delta^{r-2} \times D_\delta^{n-r-1}$ has its image arbitrarily close to $\varphi(D^2)$.

Consider now the gradient-like vector field w on the cobordism $W_0 = \bar{g}^{-1}([c, c+1])$, which is the result of the Whitney modification, applied to the vector field v on the cobordism W_o, with respect to data $(V = \bar{g}^{-1}(\lambda), v, N = D(p, v) \cap V, L = D(q, -v) \cap V, \varphi, \Phi, [\lambda-\delta, \lambda], h_t)$ where φ is the embedding constructed in lemma 5.12, Φ is the expansion of φ, satisfying the conclusions of corollary 5.13, h_t is any isotopy as constructed in lemma 5.7, δ is small enough.

Since $g^{-1}([\lambda-\delta, \lambda])$ projects injectively to M, the w determines the gradient-like vector field for g on M, which will be denoted by the same letter w, as well as its lifting to M; the latter, being restricted to a cobordism W is the result of multiple W-modification with respect to the levels $\lambda+k$, $k \in \mathbb{Z}$.

Now I claim that the vector field w is an almost good gradient-like vector field for $g:M \to S^1$, satisfying the following conditions.

a) Let $x,y \in W_1$ be the critical points of $\bar{g}$, ind x = ind $y + 1$, such that x is not a t-shift of p and y is not a t-shift of q. Then $(D(x, w) \cap W_1) \pitchfork (D(y,-w) \cap W_1)$, and $N(x,y;w)$ is the same as $N(x,y;v)$.

b) Let $y \in W_1$ be a critical point of $\bar{g}$, ind $y = r-1$, such that y is not a t-shift of q. Then if r = ind $p \geq 4$, then for every $s \in Z$ such that $pt^s \in W_1$ we have $(D(pt^s,w) \cap W_1) \pitchfork$ $(D(y,-w) \cap W_1)$ and $N(pt^s,y;w)$ is the same as $N(pt^s,y;v)$.

c) Let $x \in W_1$ be a critical point of $\bar{g}$, ind $x = r$, and x is not a t-shift of p. Then if r = ind $p \leq n-3$ we have for $qt^s \in W_1$: $(D(x\ \ ;w) \cap W_1) \pitchfork (D(qt^s;-w) \cap W_1)$ and $N(x, qt^s; w)$ is the same as $N(x,qt^s;v)$.

d) For every pair $x,y \in W_1$ of critical points of $\bar{g}$, ind x = ind $y + 1$ the incidence coefficient $\nu(x,y;w)$ equals $\nu(x,y;v)$.

e) The discs $D(p,w)$ and $D(q,-w)$ are transversal and the set $N(p,q;w)$ is the same as $N(p,q;v) \setminus \{Q(\gamma_1), Q(\gamma_2)$.

Lemma 5.14. 1) Let z be any critical point of f, $z \neq t^m p$ for any m and suppose that $\operatorname{ind} z \leqslant n-3$, $\operatorname{ind} z \leqslant r$.

Then $D(z,w) \cap \bar{f}^{-1}([\bar{f}(z) - 2A - 2, \bar{f}(z)]) = D(z,v) \cap \cap \bar{f}^{-1}([\bar{f}(z) - 2A - 2, \bar{f}(z)])$ and on this set $w = v$.

2) Let z be any critical point of $\bar{f}$, $z \neq t^m q$ for any m, and suppose that $\operatorname{ind} z \geqslant 3$, $\operatorname{ind} z \geqslant r-1$.

Then $D(z,-w) \cap \bar{f}^{-1}([\bar{f}(z), \bar{f}(z)+2A+2]) = D(z,-v) \cap \cap \bar{f}^{-1}([\bar{f}(z), \bar{f}(z)+2A+2])$ and on this set $w = v$.

Proof. We prove 1), the 2) is similar.

Since both w, v are invariant under t it is enough to verify the property for any t-shift z_o of z. Consider the t-shift z_o such that $c+2A+3 < \bar{f}(z_o) < c+2A+4$. By our choice of A this z_o belongs to W and $\bar{f}^{-1}([\bar{f}(z_o)-2A-2, \bar{f}(z_o)]) \subset \subset \{\bar{f}(x) \geqslant c+1\} \cap W$. Note that $D(z_o,v) \cap \bar{f}^{-1}([\bar{f}(z_o)-2A-2, \bar{f}(z_o)]) \subset \subset D(z_o,v) \cap \{\bar{f}(x) \geqslant c\}$ and that the latter set does not intersect the set $\operatorname{Im}\Phi \cdot t^s$ for any $s \in \mathbf{Z}$.

Indeed, this is obvious for $s > 0$ and for $s < -(2A+3)$. Now if $-(2A+3) \leqslant s \leqslant 0$, then $D(z_o,v) \cap \operatorname{Im}\Phi \cdot t^s = (D(z_o t^{-s}, v) \cap \operatorname{Im}\Phi$ The point $z_o t^{-s}$ belongs to $\{\bar{f}(x) > c\}$, hence to W and $z_o t^{-s} \neq p$, $\operatorname{ind} z_o t^{-s} \leqslant n-3, r$, by hypotheses. Hence, $D(z_o t^{-s}, v) \cap \operatorname{Im}\Phi = \emptyset$.

Note, further, that together with some point x the set $D(z_o,v) \cap \{\bar{f}(x) \geqslant c\}$ contains all the (-v)-trajectory, starting at x. That implies it intersects some A if and only if it intersect tr A. Therefore $\left[D(z_o,v) \cap \{\bar{f}(x) \geq c\}\right] \cap$ $\cap \left[\bigcup_{s\in\mathbf{Z}} tr(Im\Phi)\cdot t^s\right] = \emptyset$. The support of $(v-w)$ is contained in $\bigcup_{s\in\mathbf{Z}} tr(Im\Phi)\cdot t^s$ (that goes by construction, see lemma 5.9). So in the set $D(z_o,v) \cap \{\bar{f}(x) \geq c\}$ we have $v = w$ and 1) is proved.

Corollary 5.15. 1) Let $z \in W_1$ be a critical point of $\bar{f}$, $z \neq t^m p$ for any m. Assume that $\operatorname{ind} z \leq n-3$, $\operatorname{ind} z \leq r$.

Then $D(z,v) \cap W_1 = D(z,w) \cap W_1$ and $w = v$ on this set.

2) Let $z \in W_1$ be a critical point of $\bar{f}$, $z \neq t^m q$ for any m. Assume that $\operatorname{ind} z \geq 3$, $\operatorname{ind} z \geq r-1$.

Then $D(z,-v) \cap W_1 = D(z,-w) \cap W_1$ and $w = v$ on this set.

Proof. To prove 1) it is enough to note that $D(z,v) \cap W_1 \subset$ $\subset D(z,v) \cap \bar{f}^{-1}([c-A, z])$, but $\bar{f}(z) < c+A+1$, hence $\left(D(z,v) \cap W_1\right) \subset \left(D(z,v) \cap \bar{f}^{-1}([\bar{f}(z)-2A-2, \bar{f}(z)])\right)$, and we apply the previous lemma.

2) is similar.

Now we proceed as to prove a)-e). First of all I'll prove that w is almost good. Let $\alpha, \beta \in W_1$ be two critical points of $\bar{g}$, such that $\operatorname{ind}\alpha \leq \operatorname{ind}\beta$. We distinguish three cases:

A) $\alpha \neq t^n p$ for any $n \in \mathbf{Z}$, $\beta \neq t^m q$ for any $m \in \mathbf{Z}$. Note that either $(\operatorname{ind}\alpha \leq n-3, \operatorname{ind}\alpha \leq r)$ or $(\operatorname{ind}\beta \geq 3,$ $\operatorname{ind}\beta \geq r-1)$. (Indeed if $\operatorname{ind}\alpha = n-2$, then $\operatorname{ind}\beta \geq \operatorname{ind}\alpha \geq$ $n-2$. If $\operatorname{ind}\alpha > r$ then $\operatorname{ind}\beta \geq \operatorname{ind}\alpha \geq r+1 \geq 4$.) We consider the first possibility. By corollary 5.15 the set $D(\alpha,v)$ $\cap W_1 = D(\alpha,w) \cap W_1$ and $v = w$ on this set. This implies that any $(-w)$-trajectory, joining α and β is also $(-v)$-trajectory joining α and β, and these do not exist since v is good. The second possibility is done similarly.

B) $\alpha = t^n p$ for some $n \in \mathbb{Z}$. Then ind $\beta \geqslant r \geqslant 3$, the disc $D(\beta,-w) \cap W_1$ is the same as $D(\beta,-v) \cap W_1$ and we apply the same argument as in A).

C) $\beta = t^m q$ for some $m \in \mathbb{Z}$. Then ind $\alpha \leq$ ind $\beta = r-1 \leqslant \leqslant n-3$, the disc $D(\alpha,w) \cap W_1$ is the same as $D(\alpha,v) \cap W_1$ and we apply the same argument as in A).

Next we check up a). Note that again either (ind $x \leqslant n-3$, ind $x \leqslant r$) or (ind $y \geqslant 3$, ind $y \geqslant r-1$). (Indeed ind $x = n-2$ $\Longrightarrow$ ind $y = n-3 \geqslant 3$, since $n \geqslant 6$, and $n-3 \geqslant r-1$, since $n-2 \geqslant r$. Also ind $x \geqslant r+1$ implies ind $y \geqslant r \geqslant 3$.) We consider the first case, the second is similar. The disc $D(\alpha, +v) \cap$ $\cap W_1$ is the same as $D(\alpha, +w) \cap W_1$ and $v = w$ on this set, which implies that the $(-w)$-trajectories, joining α and β are the same as that of $(-v)$. Since v is good there is only finite number of them and we are now to show that for any such curve γ the discs $D(\alpha,w)$ and $D(\beta,-w)$ are transversal at γ. Consider the manifold $S = D(\beta,-w) \cap \bar{g}^{-1}(\beta+\delta)$, where δ is very small. This manifold is diffeomorphic to a sphere and the intersection $D(\alpha,w) \cap S$ is a finite set. Denote $\gamma \cap S$ by a. Note that the $(-w)$-trajectory θ, starting at a is γ^{-1}, which means that it does not intersect the supp$(w-v)$. If we fix some $t > 0$, then the sets $\theta|[0,t]$ and supp$(w-v) \cap W_1$ do not intersect, and since they are compact, all the $(-w)$-trajectories starting at some small neighbourhood $\mathcal{U}$ of a in S are $(-v)$-trajectories, which implies that the tangent space to $D(\beta,-w)$ at the points of $\gamma|[0,t]$ is the same as to $D(\beta,-v)$ and hence they are transversal.

Next we check up C). Since $r = \text{ind } p \leqslant n-3$ and $\text{ind } z \leqslant r$ we are again in the situation of corollary 5.15. 1) and again $D(x,v) \cap W_1 = D(x,w) \cap W_1$ and $v = w$ on this set. We apply now the same argument as for the case a).

To check up b) we note that since $\text{ind } y = r-1 \geqslant 3$ then we are in the assumptions of corollary 5.15. 2) and we proceed on the same lines as before.

To prove d) we introduce the vector fields w_k as the result of multiple Whitney modification, applied to the field v and the levels $\lambda+A$, $\lambda+A-1,\ldots,$ $\lambda+A-k$ (and the corresponding data which are the shifts of the critical data for w). The w_k is the gradient-like field for g, although it is not t-invariant. Note also that $w_k = w \,|\{g(x) \geqslant A-n\}$ and $w_k = v$ in $\{\bar{g}(x) \leqslant A-n\}$.

Note that since $\text{supp}(w_k-v) \subset \text{supp}(w-v)$ the results of the corollary 5.15 hold for w_k as well as for w. That implies as before that w_k is an almost good gradient-like field for $\bar{g}$. Now we shall show that if $x,y \in W_1$ are two critical points of $\bar{g}$, $\text{ind } x = \text{ind } y + 1$, then $\nu(x,y;w_k) = \nu(x,y;w_{k-1})$, which implies that these coefficients are the same with respect to $w_{-1} = v$ and w_{2A}, but since $w|W_1 = w_{2A}|W_1$, that suffices to get d).

The field w_{k-1} differs from w_k in the set contained in $\bar{g}^{-1}([\lambda+A-k-\delta,\ \lambda+A-k])$, so we are only to consider the case $\bar{g}(x) > \lambda+A-k > \bar{g}(y)$.

To calculate the incidence coeffcient $\nu(x,y;w_{k-1})$ we proceed a in § 3. Denote ind x by s, $\bar{g}(x)$ by d and $\bar{g}(y)$ by c.

We take $\delta > 0$ so small that the sets $V_{\leqslant s-1}(\delta; w_{k-1}) \cap W_1$ $V_{\geqslant s}(\delta; -w_{k-1})$ do not intersect. Note that the set $K_{\leqslant s-1}(w_{k-1}) \cap W_1 = K_{\leqslant s-1}(v) \cap W_1$ does not intersect supp(v-w), therefore, for the small δ we have $V_{\leqslant s-1}(\delta; w_{k-1}) \cap$ $\cap W_1 = V_{\leqslant s-1}(\delta; w_k) \cap W_1 = V_{\leqslant s-1}(\delta, v) \cap W_1$.

We demand also that $K_{\leqslant s}(w_{k-1}) \cap V_{\geqslant s}(\delta; -w_{k-1}) = \emptyset$. Note that similarly to the above $V_{\geqslant s}(\delta; -w_{k-1}) = V_{\geqslant s}(\delta; -w_k)$ for δ small enough.

We demand as well that $K_{\leqslant s-1}(w_{k-1}) \cap V_{\leqslant s-1}(\delta; -w_{k-1}) = \emptyset$.

We take $\mu = \lambda + A - k$ as a level with respect to which we are going to apply a definition from § 3.

Take $\beta < d$ very close to d and $c < \alpha$ very close to c, and consider the manifolds $S_\beta = D(x,v) \cap f^{-1}(\beta)$, $S_\alpha =$ $= D(y,-v) \cap f^{-1}(\alpha)$. Pick up the compact neighbourhood P with the smooth boundary ∂P of the set $S_\beta \setminus U(\delta; [\mu, \beta]; -w_{k-1})$ in the set $S_\beta \setminus K([\mu, \beta]; -w_{k-1})$. Pick up the compact neighbourhood Q with the smooth boundary ∂Q of the set $S_\alpha \setminus U(\delta; [\alpha, \mu]; +w_{k-1})$ in the set $S_\alpha \setminus K([\alpha, \mu]; w_{k-1})$.

We denote by P_μ and Q_μ the results of shits along $(-w_{k-1})$-trajectories (resp. w_{k-1}-trajectories) up to the level μ of the manifolds P and Q.

The boundary ∂P_μ belongs to the compact $X_{\mu,\delta} = f^{-1}(\mu) \cap$ $\cap V_{\leqslant s-1}(\delta; [\mu, c+A+1]; w_{k-1})$. The boundary ∂Q_μ belongs to $Y_{\mu,\delta} = f^{-1}(\mu) \cap V_{\leqslant s}(\delta; [c-A, \mu]; -w_{k-1})$.

By our assumption $P_\mu \cap Y_{\mu,\delta} = \emptyset = Q_\mu \cap X_{\mu,\delta}$. Now the incidence coefficient $(x,y; w_{k-1})$ is defined as an intersection number $i(P,Q) \in \mathbb{Z}H = \mathbb{Z}\pi_1 W$ with respect to a pair of compacta $X_{\mu,\delta}$, $Y_{\mu,\delta}$ (see lemma 3.9).

Now we are going to perform the same procedure for w_k. Choose the δ so small that the above conditions hold for w_k also. Since $w_k = w_{k-1}$ in the set $g^{-1}([\mu, A+C+1)$ the sets $U(\delta; [\mu, \beta]; -w_{k-1})$, $K([\mu, \beta]; -w_{k-1})$ are the same as that for w_k. So the manifold P, chosen for w_{k-1}, is good also for w_k, and, since the $(-w_{k-1})$-shift up to $f^{-1}(\mu)$ is the same as $(-w_k)$-shift, we can assume $P_\mu(w_{k-1}) = P_\mu(w_k)$. Similarly, $X_{\mu,\delta}(w_{k-1}) = X_{\mu,\delta}(w_k)$.

Consider now the other part. The compact $V_{\geqslant s}(\delta; [c-A, \mu]; -w_{k-1})$ is the same as that for w_k by the above; so $Y_{\mu,\delta}(w_k) = Y_{\mu,\delta}(w_{k-1})$. Next the set $U(\delta; [\alpha, \mu]; w_k)$ consists of all $(-w_k)$-trajectories, starting from the δ-discs around the critical points $z \in \bar{g}^{-1}([\alpha, \mu])$. Note though that these critical points lie below the $\mathrm{supp}(w_{k-1}-w_k)$ which belongs to a narrow strip $\bar{g}^{-1}(\mu-\delta_0, \mu)$. Hence for δ small enough this set, as well as $K([\alpha, \mu], w_k)$ are the same as these for w_{k-1}. So the manifold $Q(w_{k-1})$ can be chosen as $Q(w_k)$.

Now the coefficient $\nu(x,y;w_k)$ is by definition the intersection coefficient of $\mathcal{P}_u$ and $Q_\mu(w_k)$ with respect to the compacts $X_{\mu,\delta}$ and $Y_{\mu,\delta}$. The manifold $Q_\mu(w_k)$ is (by definition of Whitney modification) the result of the final map h_1 of some isotopy h_t of $V = f^{-1}(\mu)$, applied to Q. This isotopy differs from id on the set $\mathrm{Im}\,\Phi \cdot t^{A-k}$, which by our choice of Φ intersects neither $K_{\leqslant s-1}(w_{k-1}) \cap W_1$, nor $K_{\geqslant s}(w_k) \cap W_1$ and, therefore, for small δ, it does not intersect also the sets $X_{\mu,\delta}$ and $Y_{\mu,\delta}$. We finish now by applying lemma 3.9, 2).

To prove e) it is enough to notice that transversality holds by lemma 5.7, as well as the sign business. All the curves γ, joining a, b did not intersect supp(w-v) except γ_1, γ_2, so they rest the same, hence the coefficients $h(\gamma)$ are the same as before.

The properties a)-e) are proved.

Now we consider any good gradient-like vector field v' which is close to w and coincide with w in the small neighbourhoods of zeros of w. I claim that the vector field v' resctricted to W_1, satisfies again the condition d) above and as well the conditions a)-c) and e) if we change there the word "same" to "homotopically same". Indeed for the item d) it follows from lemma 3.10, and the others - from lemma 3.12.

Now I claim that this field satisfies all the conclusion 1)-6) of theorem 5.2.

Indeed, 1) is obvious, 6) is already done, and for all the rest it suffices to check up that $Q(K) \subset W_1$. But that was by definition. The theorem 5.2 is proven.

The proof of theorem 5.1. We must distinguish two cases: 1) $4 \leqslant \text{ind } a \leqslant n-2$, 2) $3 \leqslant \text{ind } a \leqslant n-3$.

We do the case 1) first.

Lemma 5.16. Let $\varkappa = (f, \bar{f}, v, E)$ be a regular quadruple, belonging to a regular class $\xi \in H^1(M, \mathbb{Z})$. Let $q \in E$, ind $q \leqslant n-3$. Let $N > 0$ be any number. For every $x \in E$, ind x = ind $q + 1$ let $m(x)$ be an integer, such that

$n_i(x,q) = 0$ for $i \leqslant m(x)$.

Then there exists an admissible modification $\varkappa' = (g, \bar{g}, v', E)$ of $\varkappa$, such that $C_*(v',E)$ is N-equivalent to $C_*(v,E)$ and $\bar{g}(\bar{q}) > \bar{g}(\bar{x}t^{-m(x)})$ for every $x \in E$, ind x = ind $q + 1$.

Proof. Note first of all that it suffices to construct such $\varkappa'$, that $C_*(v',E) \underset{N}{\sim} C_*(v,E)$ and $\bar{g}(\bar{q}) \geqslant \bar{g}(\bar{x}\cdot t^{-m(x)})$, since for a Morse function on a cobordism one can always lift a bit one of the critical points, belonging to a given level.

For brevity we shall call an r-quadruple θ suitable, if θ is an admissible modification of $\varkappa$ and $C_*(v',E)$ is M-equivalent to $C_*(v,E)$, where $M = N + \max\limits_{x \in E} m(x) + 2$.

For each suitable θ' and $x \in E$ we denote by $\vartheta(\theta, x)$ (not to be confused with the incidence coefficients $\nu(p, q)$) the non-negative integer $\max([\bar{g}(\bar{x}t^{-m(x)}) - \bar{g}(\bar{q})], 0)$, and by $S(\theta)$ the sum of $\vartheta(\theta,x)$ over all the $x \in E$.

From all the suitable θ choose the one $\theta_0 = (g, \bar{g}, w, E)$ with the minimal possible value $S(\theta)$. I claim that $S(\theta_0) = 0$. Indeed, suppose that $S(\theta_0) > 0$. Choose the regular value c of $\bar{g}$, such that f has no critical values in $(c, \bar{g}(q))$. Choose some $x \in E$, ind x = ind $q + 1$, such that $[\bar{g}(\bar{x}t^{-m(x)}) - \bar{g}(\bar{q})] = s > 0$. We have

$$\bar{g}(\bar{q}) + s + 1 > \bar{g}(\bar{x}\cdot t^{-m(x)}) \geqslant \bar{g}(\bar{q}) + s, \quad \text{hence}$$

$$\bar{g}(\bar{q}) + 1 > \bar{g}(\bar{x}\cdot t^{-m(x)+s}) \geqslant \bar{g}(\bar{q}) .$$

Choose now an enumerating Morse function on the cobordism $W_\theta = \bar{g}^{-1}([c, c+1])$ with the same vector field w. We can demand that the value of $\bar{g}$ on $\bar{q}$ does not change and the values of $\bar{g}$ on all the critical points of index, equal to ind q + 1 decrease. Furthermore, we can demand, that the critical point $\bar{x}t^{-m(x)+s} \in W_o$ is the nearest to q, so that for some $\mu, \nu \in (c, c+1)$ there are no critical points of $\bar{g}$ in $\bar{g}^{-1}(\mu, \nu)$ except q and $\bar{x}t^{-m(x)+s}$. The index $n_{m(x)-s}(x,q;w)$ equals zero, which implies that the incidence coefficient $\nu(\bar{x}t^{-m(x)+s}, \bar{q}) \in \mathbb{Z}H$ is zero. Now we apply theorem 5.2 to the enumerating function φ, pair of critical points $(\bar{x}t^{-m(x)+s}, \bar{q})$ and the finite set K, which contains $\bar{E}$ and for each $x \in E$ it contains all the $\bar{y}t^s$, where ind y = ind x - 1 $-ex(E,\varphi) - 2 \leq s \leq M$.

After that we diminish the number of trajectories joining $\bar{x}t^{-m(x)+s}$ and $\bar{q}$ by 2 and the choice of K guarantees that the resulting complex $C_*(w_1,E)$ is M-equivalent to $C_*(w,E)$ (see corollary 4.5).

Applying this procedure several times (the same φ, the same K), we finish with the vector field w_2 for φ such that there are no trajectories, joining $\bar{x}t^{-m(x)+s}$ and $\bar{q}$. Now by the standard argument we can change φ on the set $\varphi^{-1}(\mu,\nu)$ so that $\varphi_1(\bar{q}) > \varphi_1(\bar{x}t^{-m(x)+s})$, without changing w_2. By the lemma 4.3 φ_1 is again regular. Denote by θ_1 the quadruple $(\varphi_1, \bar{\varphi}_1, w_2, E)$.

Note that the integers $\nu(\theta_1, z)$ for ind z = ind q + 1, did not increase, since the values of our functions on the critical points of index, equal to ind q + 1, decreased, and the values on $\bar{q}$ - increased.

Further, $\varphi_1(\bar{x}t^{-m(x)}) < \varphi_1(\bar{q})+s$, hence $[\varphi_1(xt^{-m(x)}) - \varphi_1(\bar{q})] < s$. Thus, if $s > 0$, the number $\nu(\varphi_1, x)$ is less then $\nu(\theta_o,x) = s$, and θ_o can not have the minimal possible value $\Theta_o(s)$.

Consider now the suitable $(g, \bar{g}, w, E)$, such that $S(\Theta_o) = 0$.

This condition means exactly that for every $x \in E$, ind x = ind $q + 1$, all the points $\bar{x}t^{-m(x)}$ which lie above $\bar{q}$, lie below $\bar{q}+1$. Choose the suitable Θ_o with minimal possible number of such $\bar{x}$. I claim that this number is equal to zero.

Indeed, if there is at least one point x, such that $\bar{x}t^{-m(x)}$ lies above $\bar{q}$, we apply exactly the above procedure and get rid of it. Lemma 5.16 is proved.

Corollary 5.17. In the assumptions of the theorem 5.1 there exists an admissible modification $\Theta = (\varphi, \bar{\varphi}, w, E)$ of $\varkappa$, such that:

1) For every $x \in E$, ind x = ind $b + 1$, $x \neq a$, and every $s < N$ there are no $(-w)$-trajectories, joining $\bar{x}$ and $\bar{b}t^s$.

2) $\bar{\varphi}(\bar{b}) < \bar{\varphi}(\bar{a}) < \bar{\varphi}(\bar{b}) + 1$ and there is only one $(-w)$-trajectory γ, joining $\bar{a}$ and $\bar{b}$, $h(\gamma) = 1$, $\varepsilon(\gamma) = 1$.

3) The complex $C_*(w, E)$ is N-equivalent to $C_*(v, E)$.

Proof. We apply lemma 5.16 to the quadruple $\varkappa$ and the numbers $m(x) = N-1$ if $x \neq a$, and $m(a) = -1$. We get the quadruple $\varkappa' = (g, \bar{g}, v', E)$, such that $\bar{g}(\bar{b}) > \bar{g}(\bar{x}t^{-(N-1)})$ for $x \neq a$, and $\bar{g}(\bar{b}) > \bar{g}(\bar{a}t) = \bar{g}(\bar{a}) - 1$. Note that $\bar{g}(\bar{b}) < \bar{g}(\bar{a})$, because otherwise $n_o(a;b;w) = 0$ which contradicts the N-equivalence of $C_*(v,E)$ and $C_*(w,E)$.

Consider now the regular value c of $\bar{g}$, such that there are no critical values of $\bar{g}$ in $(c, \bar{g}(\bar{b}))$, and such that $\bar{g}(\bar{a}) < c+1$. Choose now the enumerating function φ_1 on the cobordism $\bar{g}^{-1}([c, c+1])$, such that for some interval $[\mu, \nu] \subset \subset (c, c+1)$ the only critical points in $\varphi_1^{-1}(\mu, \nu)$ are $\bar{b}$, $\bar{a}$. Note that all the conclusions of lemma 5.16 still hold for φ_1, since for every $x \in E$, ind x = ind $b + 1$ we have $c > \bar{\varphi}_1(\bar{x}t^{-m(x)})$.

Applying now several times the theorem 5.2 to the r-quadruple $(\varphi_1, \bar{\varphi}_1, w, E)$, the points a, b and the set K, containing $Q(E)$ and for every point $x \in E$ containing all the points $\bar{y}t^s$, $y \in E$, ind y = ind $x - 1$, such that $-\mathrm{ex}(\bar{\varphi}_1, E) - 2 \leq s \leq N$, we get finally the new vector field v' for the same function φ_1, such that there is only one $(-v')$-trajectory γ, joining $\bar{a}$ and $\bar{b}$, $h(\gamma) = 1$, $\varepsilon(\gamma) = 1$, and the complex $C_*(v', E)$ is N-equivalent to $C_*(w, E)$. q.e.d.

<u>Proof of theorem 5.1 for ind $a \geqslant 4$.</u>

We proceed in the same way, as while proving the lemma 5.16, starting with Θ of corollary 5.17 and lowering the level of the point $\bar{a}$, but this time taking care not to create new trajectories, joining $\bar{x}$ and $\bar{b}t^s$.

Namely we are going to prove the following assertion A:

A: there exists an admissible modification $\varkappa' = (\varphi, \bar{\varphi}, w, E)$ of $\varkappa$, such that the conclusions 1)-3) of corollary 5.17 hold for $\varkappa'$, and, in addition, the condition (β) below holds.

(β): for every $y \in E$, ind b = ind y, $y \neq b$, we have $\varphi(\bar{y} \cdot t^{N-1}) \geqslant \bar{\varphi}(\bar{a})$.

(Obviously the $\varkappa'$ satisfying 1)-3) of 5.17 and (3) satisfies the conclusions of 5.1.) Denote ind a by r.

We call an admissible modification $\alpha = (\varphi, \bar{\varphi}, w, E)$ of $(f, \bar{f}, v, E)$ suitable, if 1)-3) of 5.17 hold. For every suitable $\mathfrak{X}'$ and $y \in E$, ind $y = r-1$ denote by $\mu(\alpha, y)$ the non-negative integer $\max([-\bar{\varphi}(\bar{y}t^{N-1}) + \bar{\varphi}(\bar{a})], 0)$ and by $\mu(\alpha)$ the sum of $\mu(\alpha, y)$ over all the $y \neq b$. Consider the suitable $\alpha_0 = (\varphi, \bar{\varphi}, w, E)$, such that $\mu(\alpha_0)$ is the minimal possible. I claim that $\mu(\alpha_0) = 0$.

Indeed, if $\mu(\alpha_0) > 0$, there exist some $y \in E$, $y \neq b$, such that $\bar{\varphi}(\bar{y}t^{N-1})+s+1 > \bar{\varphi}(\bar{a}) > \bar{\varphi}(\bar{y}t^{N-1})+s$, where $s > 0$. Choose $c > \bar{\varphi}(\bar{a})$, such that there are no critical values in the interval $(\bar{\varphi}(\bar{a}), c]$ and such that $\bar{\varphi}(\bar{b}) > c-1$. There is exactly one t-shift of y in the cobordism $\bar{\varphi}^{-1}([c-1, c])$, namely $\bar{y}t^{N-1-s}$. Choose now the enumerating function φ_1 on the cobordism $\bar{\varphi}^{-1}([c-1, c])$, such that $\varphi_1(\bar{a}) = \bar{\varphi}(\bar{a})$, the values of $\bar{\varphi}$ on all critical points of index $r-1$ do not decrease, and there is an interval $[\alpha, \beta] \subset (c-1, c)$, such that only critical points of φ_1 in $\varphi_1^{-1}([\alpha, \beta])$ are $\bar{a}$ and $\bar{y}t^{N-1-s}$. Note that for a new r-quadruple $\alpha_1 = (\varphi_1, \bar{\varphi}_1, w, E)$ we have $\mu(\alpha_1) \leqslant \mu(\alpha_0)$. Note that by the condition 3) of 5.17 the incidence coefficient $\gamma(\bar{a}, \bar{y}t^{N-1-s}) = n_{N-1-s}(a, y) = 0$.

Now we choose the finite set $K \subset \bar{M}$ of critical points such that $K \supset \pi(E)$, and such that for every $x \in E$ the set K contains all the $\bar{y}t^m$ where $-ex(\bar{E}, \bar{\varphi}_1)-2 \leqslant m \leqslant N+2$ and apply several times the theorem 5.2 to the pair of critical points $(\bar{a}, \bar{y}t^{N-1-s})$ to get rid of all trajectories, joining $\bar{a}$ and $\bar{y}t^{N-1-s}$. The new r-quadruple $(\varphi_1, \bar{\varphi}_1, w_1, E)$ satisfies again 3) of 5.17 by the choice of K and 1) of 5.17 because of p.2) of 5.2 and because $\bar{b}$ is not a t-shift of $\bar{y}$, and $\bar{x}$ is not a t-shift of $\bar{a}$. The first part of condition 2) of 5.17 follows from the definition of φ_1. The second holds also because of p.3) of theorem 5.2, since $\bar{b}$ is not a t-shift of $\bar{y}$.

Here we use that ind a $\geqslant$ 4.

Now, since there are no $(-w_1)$-trajectories from a to $\bar{y}t^{N-1-s}$ we can exchange the values, that is to pick up a new function $\varphi_2 : \varphi_1^{-1}([\alpha, \beta]) \to [\alpha, \beta]$, such that $\varphi_2(\bar{a}) < \varphi_2(\bar{y}t^{N-1-s})$. Expand φ_2 to $\varphi_1^{-1}([\alpha, \beta])$, setting $\varphi_2 = \varphi_1$ for all the other points. That is obvious that for a quadruple $\alpha_2 = (\varphi_2, \bar{\varphi}_2, w_1, E)$ the conditions 1)-3) of 5.17 hold also. The values of the function on critical points of index r-1 did not decrease, $\varphi(a)$ has decreased, hence $\mu(\alpha_2, z) \leqslant \mu(\alpha_2, z)$ for all $z \in E$, ind z = r-1, and $\mu(\alpha_2, y) \leqslant s-1 < \mu(\alpha_o, y)$.

Thus $\mu(\alpha_o) = 0$. That implies that for the quadruple $\alpha_o = (\varphi, \bar{\varphi}, w, E)$ we have for every $y \neq b$, ind y = r-1 the following: $\bar{\varphi}(\bar{y}t^{N-1}) \geqslant \bar{\varphi}(\bar{a})-1$. For such a quadruple we denote by $\nu(\alpha_o)$ the number of $y \in E$, $y \neq q$, such that $\bar{\varphi}(\bar{y})t^{N-1} < \bar{\varphi}(\bar{a})$. From all the suitable α with $\mu(\alpha) = 0$ we choose the α_1 with minimal possible $\nu(\alpha_1)$. I claim that $\nu(\alpha_1) = 0$. Indeed, if there is at least one y, with $\bar{\varphi}(\bar{a}) > \bar{\varphi}(\bar{y})t^{N-1}$ we apply the same procedure as above to get rid of it.

(Note that if we had started to eliminate the trajectories from a to b after what we have done, we would not succeed, since this operation would create the trajectories joining something and b.)

Proof of theorem 5.1 for ind $a \leq n-3$.

It goes on the same lines, but the order is reversed. We give only the sketch of proof because the details are similar to above.

1. Similarly to lemma 5.16 and corollary 5.17 we obtain an admissible modification $\eta = (\psi, \bar{\psi}, u, E)$ of $\varkappa$, such that

1') For every $y \in E$, ind y = ind $a - 1$, $y \neq b$, $s < N$, there are no $(-u)$-trajectories, joining $\bar{a}$ and $\bar{y}t^s$.

2') $\bar{\psi}(\bar{b}) < \bar{\psi}(\bar{a}) < \bar{\psi}(\bar{b}) + 1$ and there is only one $(-w)$-trajectory γ, joining $\bar{a}$ and $\bar{b}$, $\varepsilon(\gamma) = 1$, $h(\gamma) = 1$.

3') The complex $C_*(u, E)$ is N-equivalent to $C_*(v, E)$. (This is done by descending gradually with the point $\bar{a}$, and after $\bar{a}$ is lower than all the $\bar{y}t^s$ for $s < N$ and also than $\bar{b}t^{-1}$, we get rid of the trajectories, joining $\bar{a}$ and $\bar{b}$.)

2. Similarly to the proof of 5.1 for ind $p \geq 4$ we find an admissible modification $\eta_0 = (\psi, \bar{\psi}, u, E)$, satisfying 1')-3') above and, in addition,

4') for every $x \in E$, ind x = ind $b + 1$, $x \neq a$ we have $\bar{\psi}(\bar{x} \cdot t^{N-1}) \leq \bar{\psi}(b)$.

(This is done in the following way. We start from the quadruple, satisfying 1')-3') above and we gradually rise the level of $\bar{b}$, interchanging it with the points of the type $\bar{x}t^s$, $s \leq N-1$, $x \neq a$. Each time we preserve the properties 1'), 3') by th. 5.2, 2) and, resp. 5). The property 2') is preserved by the theorem 5.2, 4)).

Theorem 5.1 is proved.

6. Cancelling and adding of the trivial direct summand

We recall from § 2 that we denote by $\Gamma_*^{(k)}$ the complex $0 \leftarrow \Lambda_\xi^- \xleftarrow{id} \Lambda_\xi^- \leftarrow 0$, concentrated in dimensions $k, k+1$.

Theorem 6.1. Let $\mathscr{X} = (f, \bar{f}, v, E)$ be a regular quadruple, belonging to $\xi \in H^1(M, \mathbb{Z})$, ξ is regular. Let $N \geqslant 2$, $2 \leqslant k \leqslant n-3$, and assume that the Novikov complex $C_*(v, E)$ is N-equivalent to $D_* \oplus \Gamma_*^{(k)}$.

Then there exists a new r-quadruple $\Theta = (h, \bar{h}, w, E')$ belonging to ξ , such that $C_*(w, E)$ is N-equivalent to D_*.

Proof. We shall need some preliminaries on good gradient-like vector fields. Let W be a cobordism, $f: W \longrightarrow [a,b]$ be a Morse function, and v be a good gradient-like vector field for f.

Lemma 6.2. Let $p, p', q' \in Cr f$, $\operatorname{ind} p = \operatorname{ind} p' = \operatorname{ind} q' + 1$, $p \neq p'$. Let γ be a (-v)-trajectory, joining p' and q'. Denote by $\bar{\gamma}$ the closure of γ ; $\bar{\gamma} = \gamma \cup p' \cup q'$.

Then there exists a neighbourhood U of $\bar{\gamma}$, such that the following condition holds:

(i) if there exists a (-v)-trajectory, joining p and some point of U, then there exists a (-v)-trajectory, joining p and q'.

Proof. Assume that the conclusion is false. This means that for every neighbourhood U of $\bar{\gamma}$ there exists a point $\lambda \in U$, and a (-v)-trajectory, joining p and λ, but there is no trajectory, joining p and q'.

Then, since ind p = ind q' + 1, we can choose a new Morse function $G:W \longrightarrow [a,b]$ with the same gradient v, such that $G(q') > G(p)$. Since γ joins p' and q' we have also that $G(\lambda) \geqslant G(q')$ for all the points $\lambda \in \bar{\gamma}$. Hence there exists a neighbourhood $U(\bar{\gamma})$, such that for all $\lambda \in U(\bar{\gamma})$ we have $G(\lambda) > G(p)$. Therefore there can exist no $(-v)$-trajectory, joining p and some $\lambda \in U(\bar{\gamma})$. Contradiction.

Lemma 6.3. Let $p,q,p',q' \in \mathrm{Cr}\, f$, ind p = ind p' = = ind q + 1 = ind q'+ 1, $p \neq p'$, $q \neq q'$. Let γ, γ' be two $(-v)$-trajectories, joining, respectively, p with q and p' with q'.

Then there exist two neighbourhoods U, U' of $\bar{\gamma}$ and $\bar{\gamma}'$ respectively, such that the following holds:

(ii) If there exist a $(-v)$-trajectory, joining some point of U with some point of U', then there exists a $(-v)$-trajectory, joining p and q'.

Proof. Assume that our conclusion is false. Then for every pair U, U' of the neighbourhoods of, respectively $\bar{\gamma}$ and $\bar{\gamma}'$ there is a $(-v)$-trajectory, joining $\lambda \in U$ with $\mu \in U'$, but there is no $(-v)$-trajectory, joining p and q'.

Then, since ind p = ind q' + 1, we can find a new Morse function G on W, such that v is still a gradient-like vector for G, and $G(p) < G(q')$. Since the $(-v)$-trajectories γ, γ', join, respectively, p with q and p' with q' we have $G(\lambda) \leqslant G(p) < G(q') \leqslant G(u)$ for every $\lambda \in \bar{\gamma}$, $\mu \in \bar{\gamma}'$. This implies that there exist neighbourhoods U of $\bar{\gamma}$ and U' of $\bar{\gamma}'$ such that if $\lambda \in U$, $\lambda' \in U'$ we have $G(\lambda) < G(\lambda')$. But then the existence of a (-v)-trajectory, joining λ and λ' is impossible.

Lemma 6.4. Let $p, q, q' \in \mathrm{Cr} f$, $\operatorname{ind} p = \operatorname{ind} q + 1 = \operatorname{ind} q' + 1$, $q \neq q'$. Let γ be a (-v)-trajectory, joining p and q.

Then there exists a neighbourhood U of $\bar{\gamma}$, such that the following condition holds:

(iii) if there exists a (-v)-trajectory, joining some point of U with q', then there exists a (-v)-trajectory, joining p and q'.

Proof. Similar to that of 6.3. If there is a (-v)-trajectory, joining p and q' there is nothing to prove.

If not, then we can choose a new Morse function G on the cobordism W with the same gradient-like vector field, such that $G(q') > G(p)$, and, therefore $G(q') > G(\lambda)$ for any point λ of $\bar{\gamma}$. Then there exists a neighbourhood U of $\bar{\gamma}$, such that $G(q') > G(u)$ for every $u \in U$, and the antecedent of (iii) is false, q.e.d.

Now we return to the proof of the theorem 6.1. Let $a, b \in E$ be the generators of $\Gamma_*^{(k)}$, $\operatorname{ind} b = k$, $\operatorname{ind} a = k+1$. We pick up an admissible modification $\varkappa_1 = (g, \bar{g}, v', E)$ of $\varkappa$, satisfying the conclusions of theorem 5.1. Pick up some regular value c of $\bar{g}$, such that $c < \bar{g}(\bar{b}) < \bar{g}(\bar{a}) < c+1$. We can assume that for some interval $[\lambda, \mu] \subset (c, c+1)$ the only critical points of $\bar{g}$, belonging to $\bar{g}^{-1}([\lambda, \mu])$ are $\bar{b}$, $\bar{a}$, and $\lambda < \bar{g}(\bar{b}) < \bar{g}(\bar{a}) < \mu$. Let A be a natural number, such that all the sets $\bar{E}t^s$ for $0 \leq s \leq N+1$ are contained in

$\bar{g}^{-1}([c-A,\ c+1+A])$. The latter will be denoted W.

Corollary 6.5. There exist a neighbourhood R of $\bar{\gamma}$, belonging to $\bar{g}^{-1}(\lambda, \mu)$, such that the following three conditions hold:

(i') If $p \in \mathrm{Cr}\bar{g}$, $p \in W$, ind $p = k+1$ and $s \in \mathbb{Z}$, $-A \leq s \leq A$, and $p \neq \bar{a}t^s$ there is a $(-v')$-trajectory, joining p and some point of Rt^s, then there is a $(-v')$-trajectory, joining p and $\bar{b}t^s$.

(ii') If $s, s' \in \mathbb{Z}$, $s \neq s'$, $-A \leq s, s' \leq A$, and there is a $(-v')$-trajectory, joining Rt^s and $Rt^{s'}$, then there is a $(-v')$-trajectory, joining $\bar{a}t^s$ and $\bar{b}t^{s'}$.

(iii') If $q \in \mathrm{Cr}\bar{g}$, $q \in W$, ind $q = k$, and $s \in \mathbb{Z}$, $-A \leq s \leq A$ and $q \neq bt^s$, and there is a $(-v')$-trajectory, joining some point in Rt^s with q, then there is a $(-v)$-trajectory, joining $\bar{a}t^s$ with q.

Proof. First we choose R so as to satisfy (i'). Fix for a moment $s \in \mathbb{Z}$, $-A \leq s \leq A$ and $p \in \mathrm{Cr}\bar{g}$, $p \neq \bar{a}t^s$, ind $p = k+1$. Then by lemma 6.2 there is a neighbourhood U satisfying the condition (i) for the triplet of points p, $\bar{a}t^s$, $\bar{b}t^s$. Note that if condition (i) holds for some U, then it holds also for any $U' \subset U$. So we can assume $U \subset \bar{g}^{-1}(\lambda + s, \mu + s)$.

Now for any point $p \in \mathrm{Cr}\bar{g}$ we choose the corresponding neighbourhood U_p of $\bar{\gamma} t^s$. The intersection $U = \bigcup_p U_p$ satisfies then the condition (i) of 6.2 for all the p, and will be denoted U_s.

Now we consider $R_s = U_s t^{-s}$, which is a neighbourhood of $\bar{\gamma}$, $R_s \subset \bar{g}^{-1}(\lambda, \mu)$. Denote by R the intersection of all the R_s for $s \in [-A, A]$. Then this R satisfies (i') for all s, p.

In the similar fashion one finds the neighbourhoods Q, S of $\bar{\gamma}$, satisfying (ii') and (iii'). Since the passage to the smaller neighbourhood preserves (ii') and (iii'), the neighbourhood $R \cap Q \cap S$ satisfies all the conditions of our corollary.

<u>Lemma 6.6</u>. There exists a neighbourhood R of $\bar{\gamma}$, belonging to $\bar{g}^{-1}(\lambda, \mu)$, satisfying the conclusions of corollary 6.5 and the following:

iv) If $p \in Cr\bar{g}$, $p \in W$, ind $p \geqslant k+1$, $s \in \mathbb{Z}$, $-A \leqslant s \leqslant A$, $p \neq \bar{a}t^s$ then $D(p,-v') \cap Rt^s = \emptyset$.

(v) If $q \in Cr\bar{g}$, $q \in W$, ind $p \leqslant k$, $s \in \mathbb{Z}$, $-A \leqslant s \leqslant A$, $q \neq \bar{b}t^s$ then $D(q,v') \cap Rt^s = \emptyset$.

<u>Proof</u>. First we fix s, satisfying the hypotheses of (iv) and find the corresponding R_s. Let p satisfy the hypotheses of (iv). Note that γt^s belongs to the $D(\bar{a}t^s, v')$ which does not intersect with $D(p,-v')$, since v is good. Further, $\bar{a}t^s \notin D(p,-v')$ since $p \neq \bar{a}t^s$ and $\bar{b}t^s \notin D(p,-v')$ by the same reason. Therefore $\bar{\gamma}t^s \cap D(p,-v') = \emptyset$, if $p \neq \bar{a}t^s$. Thus $\bar{\gamma}t^s \cap \left(\bigcup_p D(p,-v')\right) = \emptyset$, where p in the union runs through critical points of $\bar{g}$, such that ind $p \geqslant k+1$, $p \neq \bar{a}t^s$. This union is compact by lemma 3.8, hence for any sufficiently small neighbourhood R of $\bar{\gamma}$ we get $Rt^s \cap D(p,-v') = \emptyset$ if ind $p \geqslant k+1$, $p \neq \bar{a}t^s$.

We find such neighbourhoods R_s for each s and setting

$R = \bigcap_{-A \le s \le A} R_s$ we get finally the neighbourhood R of $\bar{\gamma}$, such that $Rt^s \cap D(p,-v') = \emptyset$ for all p, s, such that ind $p \geqslant k+1$, $p \neq \bar{a}t^s$, $-A \leq s \leq A$.

Similarly we find the neighbourhood Q of $\bar{\gamma}$ to satisfy (v) and then we intersect Q, R and the neighbourhood, satisfying the (i')-(iii') of corollary 6.5 to get the neighbourhood sought. Lemma 6.6 is proved.

Consider now the cobordism $W_0 = \bar{g}^{-1}([\lambda, \mu])$. The Morse function $\bar{g}|W_0$ has only two critical points $\bar{a}$, $\bar{b}$, which are joined by the single trajectory γ of the gradient-like field v'.

Theorem 6.7. [Mi2, §5] For any neighbourhood U of $\bar{\gamma}$ there exists a new vector field v_1 on W_0, and a Morse function $h: W_0 \to [\lambda, \mu]$, such that

1) $\mathrm{supp}(v'-v_1) \subset U$
2) $\mathrm{supp}(h-\bar{g}) \subset \bar{g}^{-1}((\lambda, \mu))$
3) v_1 is a gradient-like field for h
4) v_1 has no zeros and h has no critical points.

We do not reproduce the proof here because we apply the theorem just in the form presented in [Mi2].

We apply this theorem to our neighbourhood R, satisfying the conclusions of lemma 6.6. We set $v_1 = v'$ and $h = g$ on the set $\bar{g}^{-1}([c, c+1]) \setminus \bar{g}^{-1}([\lambda, \mu])$. Thus we get the new Morse function h and the gradient-like vector field v_1 for it on the cobordism $\bar{g}^{-1}([c, c+1])$, and we expand it equivariently the vector field v_1 and the function h on all the

M. Note that $\mathrm{Cr}\,\bar{h} = \mathrm{Cr}\,\bar{g} \setminus \{t\text{-shifts of } a, b\}$. Note that h is obviously regular.

Lemma 6.8. 1) The vector field v_1 is almost good, when restricted to W.

2) Let $p,q \in \mathrm{Cr}\,\bar{h} \cap W$, ind p = ind q + 1 and either ind p $>$ k+1 or ind q $<$ k. Then $D(p,v_1)$ is transversal to $D(q,-v_1)$ and $N(p,q;\, v')$ is the same as $N(p,q;\, v_1)$.

3) Let $x,y \in E$, $x \neq a$, $y \neq b$, ind x = k+1, ind y = k, $\bar{y}t^s \in W$ and $s \leqslant N-1$.

Then the discs $D(\bar{x},\, v_1)$ and $D(\bar{y}t^s,\, -v_1)$ are transversal and $N(\bar{x},\, \bar{y}t^s;\, v')$ is the same as $N(\bar{x},\, \bar{y}t^s;\, v_1)$.

Proof. 1) Let $p,q \in \mathrm{Cr}\,\bar{h} \cap W$, and ind p $\leqslant$ ind q. There are two possibilities: a) ind p $\leqslant$ k or b) ind q $\geqslant$ k+1. Consider first a). The point p is a critical point of $\bar{g}$ as well, and it is not a t-shift of $\bar{b}$. Hence $D(p,v')$ does not intersect the $\mathrm{supp}(v'-v_1) \cap W$ by lemma 6.6 (v), and thus $D(p,v') \cap W = D(p,v_1) \cap W$ and $v_1 = v'$ on this set. Therefore if there exists a $(-v_1)$-trajectory joining p with q it is also a $(-v')$-trajectory, and these do not exist, so $D(p,\, v_1) \cap D(q,\, -v_1) = \emptyset$.

The case b) is considered similarly. By the restriction (iv) the ascending disc $D(q,\, -v')$ does not intersect with $\mathrm{supp}(v'-v_1)$, hence any $(+v_1)$-trajectory, joining q with p is also a $(+v')$-trajectory, and these do not exist, so $D(q,\, -v_1) \cap D(p,\, v_1) = \emptyset$.

2) Consider first the case ind q $<$ k. That implies ind p $\leqslant$ k. We have already proved above that in this case

$D(p,v') \cap W = D(p, v_1) \cap W$ and $v' = v_1$ on this set. Thus the set of trajectories of $(-v_1)$, joining p and q, is the same as that of $(-v')$. We are to prove transversality. Note that in general for any gradient-like vector field v and two critical points p, q, the transversality $D(p,v) \pitchfork D(q,-v)$ is equivalent to transversality $D(p,v) \pitchfork (D(q,-v) \cap B_\varepsilon(q))$ where $B_\varepsilon(q)$ is a small ε-ball around q. Now $D(p,v_1) \cap D(q,-v_1) \subset W$, so to verify $D(p,v_1) \pitchfork (D(q_1-v_1) \cap B_\varepsilon(q))$ it is enough to verify $(D(p,v_1) \cap W) \pitchfork (D(q,-v_1) \cap B_\varepsilon(q))$. But the both terms are the same as $D(p,v') \cap W$ and, respectively, $D(q,-v') \cap B_\varepsilon(q)$; the first - by above, the second - since $v' = v_1$ in the small neighbourhood of q.

The case ind $p > k+1$ is similar. Namely, ind $q \geqslant k+1$, hence by (iv) of lemma 6.6 the disc $D(q,-v') \cap W$ is the same as $D(q,-v') \cap W$ and $v_1 = v'$ on this set. That implies that the $(-v_1)$-trajectories, joining p and q are the same as these of $(-v')$. The rest is the same as above.

We consider two cases. 1) $\bar{g}(\bar{x}) \leqslant \bar{g}(\bar{y}t^s)$ and 2) $\bar{g}(x) \geqslant \bar{g}(\bar{y}t^s)$. Since x, y do not coincide with a, b we have $\bar{g}(\bar{x}) = \bar{h}(\bar{x})$, $\bar{g}(\bar{y}) = \bar{h}(\bar{y})$. Thus in the first case there are neither $(-v)$-trajectories from $\bar{x}$ to $\bar{y}t^s$, nor $(-v')$-trajectories. So it is enough to consider the second case. Note that by the choice of A we have $A+c+1 > \bar{g}(\bar{x}) \geqslant \bar{g}(\bar{y}\bar{t}^s) \geqslant \bar{g}(\bar{y}t^{N-1}) \geqslant c-A$, which implies that both $\bar{x}$, $\bar{y}t^s$ belong to our cobordism W.

3) First of all we show that the $(-v_1)$-trajectories, joining $\bar{x}$ and $\bar{y}t^s$ are the same as those for $(-v')$. We denote by R_1 the set $\operatorname{supp}(v'-v_1) \cap R$. That is a compact neighbourhood of γ.

Let Θ be a $(-v_1)$-trajectory, joining $\bar{x}$ and $\bar{y}t^s$. There are two possibilities: a) Θ does not intersect any $R_1 \cdot t^q$, $-A \leq q \leq A$, b) Θ intersects some of these neighbourhoods. In the case a) Θ does not intersect $\mathrm{supp}(v_1 - v')$, hence it is also the v'-trajectory. So we are to check up that the second case can not occur.

The trajectory Θ is defined on $(-\infty, \infty)$. Choose L big enough, so that for $t < -L$ and for $t > L$ the trajectory Θ belongs to the neighbourhoods of $\bar{x}$, resp. $\bar{y}t^s$ where $v_1 = v'$. So all the intersections of Θ with $R_1 \cdot t^q$ occur for $t \in [-L, L]$.

Denote by $q_1 < q_2 < \dots < q_\ell$ the integers, such that $\Theta \cap$ $\cap R_1 \cdot t^q$ is not empty. Denote by K_i, $1 \leq i \leq \ell$, the Θ-preimage of $R_1 \cdot t^{q_i}$. These are the compacts in $[-L, L]$. Note that if $t_i \in K_i$, $t_j \in K_j$, $i < j$, then $t_i < t_j$. (Indeed, since Θ is $(-v_1)$-trajectory and v_1 is a gradient-like vector field for h, the inequality $t_j \leq t_i$ implies $h(\Theta(t_j)) \geq h(\Theta(t_i))$. But $h(R_1 \, t^{q_j}) \subset [\lambda - q_j, \mu - q_j]$ and $h(R_1 \, t^{q_i}) \subset$. Hence every number in $h(R_1 \, t^{q_i})$ is strictly greater than every number in $h(R_1 \, t^{q_j})$, contradiction). For each $1 \leq i \leq \ell$ denote by $I_i = [\alpha_i, \beta_i]$ the minimal segment, containing K_i. Note that $\alpha_i, \beta_i \in K_i$, since K_i is compact. Therefor, $\beta_i < \alpha_{i+1}$. Denote now by Θ_0 the trajectory $\Theta | (-\infty, \alpha_1]$, by Θ_1 - the trajectory $\Theta | [\beta_1, \alpha_2], \dots$ by Θ_i if $1 \leq i < \ell$ - the trajectory $\Theta | [\beta_i, \alpha_{i+1}]$, and by Θ_ℓ - the trajectory $\Theta | [\beta_\ell, \infty)$. Since $\Theta_i | (\beta_i, \alpha_{i+1})$ does not intersect $\mathrm{supp}(v' - v_1)$, the vector fields v' and v_1 coincide on the image of Θ_i. Therefore Θ_0 is a $(-v')$-trajectory which joins $\bar{x}$ with some point $y_1 \in R_1 t^q$, for $1 \leq i \leq \ell$ the Θ_i is a $(-v')$-trajectory, joining some point $x_i \in R_1 t^{q_i}$ with some point $y_{i+1} \in$ $\in R_1 t^{q_{i+1}}$, and θ_ℓ is a (-v')-trajectory, joining some points $x_\ell \in R_1 t^{q_\ell}$ with $\bar{y}t^s$. Now we recall that $R_1 \subset R$ and R satisfies the conclusions of corollary 6.5. That means that there is a (-v')-trajectory, joining $\bar{x}$ and bt^{q_1}; for each i, such that $1 \leq i \leq \ell - 1$, there is a (-v')-trajectory, joining $\bar{a}t^{q_i}$ with $\bar{b}t^{q_{i+1}}$, and, finally, there is a (-v')-trajectory, joining $\bar{a}t^{q_\ell}$ with $\bar{y}t^s$.

Now we recall that our field v' satisfies the conclusions of th. 5.1. Therefore $q_1 \geqslant N$ (by p.2) of this theorem) and $s - q_\ell \geqslant N$ (by p.1) of this theorem). If $\ell = 1$ we have $q_1 \geqslant N$, $s - q_1 \geqslant N$, hence $s \geqslant 2N$. But $N \geqslant 1$, hence $s \leqslant N-1$ contradicts that. If $\ell > 1$, then $q_\ell > q_1$ and $s \geqslant 2N + q_\ell - q_1 > 2N$ which again contradicts $s \leqslant N-1$.

So we have shown that every $(-v_1)$-trajectory θ, joining $\bar{x}$ and $\bar{y}t^s$ for $s \leqslant N-1$ does not intersect $\operatorname{supp}(v_1 - v')$, and hence is also the (-v')-trajectory. Now we must show that $D(\bar{x}, v_1)$ and $D(\bar{y}t^s, -v_1)$ are transversal at the points of θ. This can be verified at any point of θ, say at the point α, belonging to small standard neighbourhood of $\bar{y}t^s$, where $v' =$ $= v_1$. The tangent space $T_*(\alpha)(D(\bar{x}, v_1))$ is the $\Psi_{v_1}(T)'$-image of $T_*(\beta)(D(\bar{x}, v_1))$, where $\Psi_{v_1}(T)$ is a diffeomorphism of shift along the $(-v_1)$-trajectories by the time T, so that $\Psi_{v_1}($. $=$ $= \alpha$. We can take β very close to $\bar{x}$, so that $T_*(\beta)(D(\bar{x}, v_1)) =$
$= T_*(\beta)(D(\bar{x}, v'))$. Now, since $\gamma \cap \operatorname{supp}(v_1 - v') = \emptyset$, some neighbourhood of γ also does not intersect $\operatorname{supp}(v_1 - v')$ which implies that $\Psi_{v_1}(T)'(\alpha)$ equals $\Psi_{v'}(T)'(\alpha)$. Q.E.D.

Now the theorem 6.1 follows immediately:

Denote by E' the set $E \setminus \{a,b\}$. Pick up a good gradient-like vector field w for a function $h:M \to S^1$, such that w coincides with v_1 near the critical points and is close enough to v_1. I claim that the r-quadruple $(h, \bar{h}, w, E')$ satisfies the conclusions of the theorem 6.1. For that it is enough to show that for every $x,y \in E'$, $\operatorname{ind} x = \operatorname{ind} y + 1$ and every $s \geq N-1$ the incidence coefficients $n_s(x,y;w)$ are equal to $n_s(x,y;v')$.

We distinguish two cases: 1) $\bar{g}(\bar{x}) \leq \bar{g}(\bar{y}t^s)$, 2) $\bar{g}(\bar{x}) > \bar{g}(\bar{y}t^s)$. Since x, y do not coincide with a, b, we have $\bar{g}(\bar{x}) = \bar{h}(\bar{x})$, $\bar{g}(\bar{y}) = \bar{h}(\bar{y})$. So in the case 1) there are no (v')-trajectories from $\bar{x}$ to $\bar{y}t^s$, hence $n_s(x,y;v') = 0$ but there are also no $(-w)$-trajectories from $\bar{x}$ to $\bar{y}t^s$, hence $n(x,y;w) = 0$.

In the case 2) we have by the choice of A the following: $A+c+1 > \bar{g}(\bar{x}) > \bar{g}(\bar{y}t^s) \geq \bar{g}(\bar{y}t^{N-1}) > c-A$, hence both $\bar{x}$ and $\bar{y}t^s$ belong to W. Then, applying 2) or 3) of lemma 6.8, we see that the discs $D(\bar{x}, v_1)$ and $D(\bar{y}t^s, -v_1)$ are transversal and $N(\bar{x}, \bar{y}t^s; v_1)$ is the same as $N(\bar{x}, \bar{y}t^s; v')$. By lemma 3.12 the sets $N(\bar{x},\bar{y}t^s;w)$ and $N(\bar{x},\bar{y}t^s;v_1)$ are homotopically the same if only w is close enough to v_1, which implies that $N(\bar{x},\bar{y}t^s;v')$ is homotopically the same as $N(\bar{x},\bar{y}t^s;w)$, so $\nu(\bar{x},\bar{y}t^s;v') = \nu(\bar{x},\bar{y}t^s;w) \in \mathbb{Z}H$, and, therefore, $n_s(x,y;v') = n_s(x,y;w)$. Q.E.D.

Now we proceed as to add a trivial direct summand to the Novikov complex.

Let $\mathcal{X} = (f, \bar{f}, v, E)$ be an r-quadruple, belonging to a regular class $\xi \in H^1(M, \mathbb{Z})$. Let $L \geq 0$ be a natural number and

and c be a regular value of $\bar{f}$ and denote by W the cobordism $f^{-1}([c-L, c])$. By $K(v)$ we denote as usual the set of points in W, belonging to the $(-v)$-trajectories, starting or finishing at critical points in W. Recall from § 3 that $K(v)$ is compact.

For the technical reason we must distinguish two cases.

Lemma 6.9. Assume that f has no critical points.

Then there exists a new gradient-like vector field v_1 for $\bar{f}$ and a point $\alpha \in \bar{f}^{-1}((c-1, c))$, such that the $(-v_1)$-trajectory, starting at α, does not contain any point αt^s for $0 < s \leq L$.

Lemma 6.10. Assume that $\bar{f}$ has at least one critical point. Then there exist an admissible modification $(f, \bar{f}, w, \varepsilon)$ of $\mathscr{X}$ with w arbitrarily close to v, a regular point $\alpha \in \bar{f}^{-1}((c-1, c))$ on the regular level of $\bar{f}$, and a neighbourhood $U(\alpha) \subset \bar{f}^{-1}((c-1, c))$, such that

1) The t-shifts of $U(\alpha)$ do not intersect with $K(w)$.

2) If the $(-v)$-trajectory γ starts at some point of $U(\alpha)$, then it does not intersect $U(\alpha)t^s$ for $0 < s \leq L$.

Proof of the lemma 6.9.

For $a,b \in R$, $a < b$ we denote by $\Phi_{a,b}(v)$ the diffeomorphism of the shift along $(-v)$-trajectories from $\bar{f}^{-1}(b)$ to $\bar{f}^{-1}(a)$. Fix any $\lambda \in (c-1, c)$, and denote by $\varphi(v)$ the diffeomorphism $\Phi_{\lambda-1,\lambda}(v)$. Note that if $(-v)$-trajectory, starting from $\alpha \in \bar{f}^{-1}(\lambda)$, contains some αt^s for $0 \leq s \leq n$, then $\varphi^{n!}(v)(\alpha) = \alpha$. So it is enough to consider the case when

$\varphi^{L!}(v) = \mathrm{id}$.

By lemma 5.9 for each isotopy h_t of $V = \bar{f}^{-1}(\lambda)$ there exists a new gradient vector field v_1 in $\bar{f}^{-1}([\lambda, \lambda+\varepsilon])$, which differs from v only in $\bar{f}^{-1}((\lambda, \lambda+\varepsilon))$, such that $\Phi_{\lambda,\lambda+\varepsilon}(v_1) = h_1 \circ \Phi_{\lambda,\lambda+\varepsilon}(v)$. We need the following simple

Lemma 6.11. For any diffeomorphism $\varphi : V \to V$ of any compact manifold V, such that $\varphi^n = \mathrm{id}$ there exists a vector field ξ on V, such that the diffeomorphism $(\psi_\tau \varphi)^n \neq \mathrm{id}$ for some τ, where ψ_τ is a τ-shift along ξ.

Proof of lemma 6.11. Let $\alpha \in V$ be a point of maximal period. That is $n = pq$, $\varphi^p(\alpha) = \alpha$, and $(\varphi^p)_* = \mathrm{id}$ in the tangent space $T_*(p)$. The vector field $\frac{d}{dt}((\psi_\tau \varphi)^n)$ equals in the point α to the following:

$$\sum_{i=0}^{q-1} \varphi_*^{n-ip}(\xi(\alpha)) + \sum_{i=0}^{q-1} \varphi_*^{n-1-ip}(\xi(\varphi(\alpha)) + \dots$$

$$+ \sum_{i=0}^{q-1} \varphi_*^{n-(p-1)-ip}(\xi(\varphi^{p-1}(\alpha))).$$ The sth sum equals

$$\sum_{j=0}^{q-1} \varphi_*^{-sp}[\varphi_*^{n-s}(\xi(\varphi^s(\alpha)))] = q\cdot\varphi_*^{n-s}[\xi(\varphi^s(\alpha))].$$ Since all the points $\varphi^s(\alpha)$ are different for $0 \leq s \leq p-1$, we can set $\xi(\alpha) \neq 0$, $\xi(\varphi^s(\alpha)) = 0$ for $0 < s \leq p-1$. Then $\frac{d}{d\tau}((\psi_\tau \varphi)^n) \neq 0$, therefore the equality $(\psi_\tau \varphi)^n = \mathrm{id}$ cannot hold for all τ.

Now we expand the vector field v_1 to the cobordism $\bar{f}^{-1}([c-1, c])$ setting $v_1 = v$ in $\bar{f}^{-1}([c-1, c]) \setminus \bar{f}^{-1}((\lambda, \lambda+\varepsilon))$ and expand it further to get a t-equivariant vector field v_1 on $\bar{M}$. Note that $\Phi_{\lambda-1,\lambda}(v) = \psi_\tau \cdot \varphi$. Now pick up a point

$\alpha \in V = \bar{f}^{-1}(\lambda)$, such that $(\psi_\tau \cdot \varphi)^{L!}(\alpha) \neq \alpha$ and the proof is over.

Proof of lemma 6.10. We take as w the vector field, which equals v in the small neighbourhoods of the critical points and such that all the stable manifolds $D(p, w)$ are transversal to all the unstable ones $D(q, -w)$. Such fields exist and can be chosen arbitrarily close to v by a version of Kupka-Smale theorem (see [Pal, App. A]).

Consider now the value $\lambda \in (c-1, c)$ so close to c, that the segment $[\lambda, c]$ is regular, and let p be the lowest critical point in $\bar{f}^{-1}(c, \infty))$, so that the interval $[\lambda, \bar{f}^{-1}(p))$ is regular. Denote by S the intersection $D(p, w) \cap \bar{f}^{-1}(\lambda)$. That is a compact manifold, diffeomorphic to a sphere. Consider the compact $K([c-L, c], -w) \cap S$. By transversality it is the countable union of submanifolds of codimension 2, hence it is nowhere dense, hence $S \setminus S \cap K([c-L, c], -w)$ is non-empty. Therefore there exists a point $\beta \in S$, such that the $(-v)$-trajectory γ, starting at β reaches $\bar{f}^{-1}(c-L)$. Note that this trajectory does not contain any points βt^s for $s > 0$, since the v-trajectory, starting from βt^s finishes at the level $\lambda - s + (\bar{f}^{-1}(p) - \lambda) < \lambda$. That implies that there exists a neighbourhood $U(\beta)$, such that no trajectory, starting at a point of $U(\beta)$, intersects $U(\beta)t^s$ for $0 < s \leqslant N$.

Consider now the compact $K([c-L, c+L], w)$. Since all the critical points have the indices $\leqslant n-2$, this compact is the finite union of separable submanifolds of codimension $\geqslant 2$ in W, therefore it is nowhere dense. So it is possible to choose a neighbourhood $U \subset U(\beta)$, such that $U \cap K([c-L, c+L];w)$ is empty. That implies that all the t^s-shifts Ut^s for $0 \leqslant s \leqslant L$ do not intersect $D(p,w)$ for $p \in \tilde{f}^{-1}([c-L, c])$.

So the vector field w, the neighbourhood U and any point $\alpha \in U$ satisfy the conclusion of lemma 6.10.

The tool for adding a trivial direct summand is of course the procedure of creating two new critical points of adjacent indices as described in [Mi2, §8]. We cite the following lemma from that book:

<u>Lemma 6.12 (lemma 8.2 from [Mi2]</u>). For any λ, $0 \leqslant \lambda < n$ there exists a smooth function $h:\mathbb{R}^n \to \mathbb{R}$, satisfying $h(x_1, \ldots, x_n) = x_1$ outside of some compact and such that f has exactly 2 nondegenerate critical points p_1, p_2 of indices λ, $\lambda+1$, and $h(p_1) < h(p_2)$.

We shall need some additional information on this construnction, which is contained (explicitely or not) in the cited book. So we formulate these properties as a separate lemma. We need some more notations. We represent $\mathbb{R}^n$ as $\mathbb{R}^1 \times \mathbb{R}^\lambda \times \mathbb{R}^{n-\lambda-1}$ with the coordinates (x, y, z), so that x_1 above is x, $y \in \mathbb{R}$, $z \in \mathbb{R}^{n-\lambda-1}$.

<u>Lemma 6.13</u>. Fix any positive real numbers a, b, c.

Then the function h above can be chosen as to satisfy the following properties:

1) $\operatorname{supp}(h-x) \subset \{|x| < a, \ |y| < b, \ |z| < c\} = U_{a,b,c}$.

2) The vector field $\omega = \operatorname{grad} h$ with respect to the standard euclidean metrics is a gradient-like field for h.

3) The points p_2, p_1 have the coordinates $(x_o, 0, 0)$ and $(x_1, 0, 0)$ respectively, where $x_o < 0$, $x_1 > 0$.

4) There is only one (-u)-trajectory γ_o, joining p_2 and p_1, and it belongs to $[-a, a] \times 0 \times 0 \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-\lambda-1}$. The discs $D(p_2, +u)$ and $D(p_1, -u)$ are transversal along γ_o.

5) If some (-v)-trajectory γ starts at p_2, intersects the boundary $\partial U_{a,b,c}$, then it happens for the first time in the point $\theta \in \partial U_{a,b,c}$ for which $x = -a$ and $h(\theta) < h(p_1)$.

6) If some v-trajectory γ starts at p_1 and intersects the boundary $\partial U_{a,b,c}$, then it happens for the first time in the point $\nu \in \partial U_{a,b,c}$ for which $x = a$ and $h(\nu) > h(p_2)$.

7) $-3a \leq h(x) \leq 3a$.

<u>Proof</u>. Recall that $h(x,y,z)$ is given by the formula $h = x + s(x)\cdot \alpha(y^2+z^2) + \gamma(x)\cdot(z^2-y^2)\cdot\beta(y^2+z^2)$, where s and γ are functions $\mathbb{R} \to \mathbb{R}$, α, β are the functions $\mathbb{R}^+ \to \mathbb{R}$, and s, γ, α, β must satisfy the relations, given in [Mi2, lemma 8.2]. We shall see that the relation 1) from there is unnecessary and we shall not satisfy it.

I shall impose additional restrictions on these functions so as to satisfy our 1)-5).

Na ely we start with two positive numbers $a_2 < a_1 < a$, and $b_1 < b$, $c_1 < c$ and with the arbitrary $\gamma: \mathbb{R} \to \mathbb{R}$, $0 \leq \gamma \leq 1$, which equals zero for $x \geq a_1$, equals 1 for $0 \leq x \leq a_2$ and $\gamma(-x) = \gamma(x)$. Denote $\sup |\gamma'|$ by $\Gamma > 0$. Let a_3 be less than $1/\Gamma$ and less than $b_1^2 + c_1^2$, and $0 < a_4 < a_3$, and β be a smooth function $\mathbb{R}^+ \to \mathbb{R}$, such that $0 \leq \beta \leq 1$ and $\beta = 0$ for $x \geq a_3$, $\beta = 1$ for $x \leq a_4$. Then we have

$\max_{t\in\mathbb{R}^+} (t\ (t)) \leq a_3 < \frac{1}{\Gamma}$. Let $0 < a_5 < a_4$ and α be a smooth function $\mathbb{R}^+ \to \mathbb{R}$, such that $0 \leq \alpha \leq 1$, $\alpha = 1$ for $x \leq a_4$ and $\alpha = 0$ for $x \geqslant a_3$. Denote by A the $\sup |\alpha'|$; $A > 0$. Choose any positive real number $R > \max(2A,\ 1/a_5, 10)$. Pick up any real-valued smooth function $r_o(x)$ on $\mathbb{R}$, which satisfy the following conditions:

1) $r_o(x) = x$ for $|x| \geqslant 1$.

2) There are two roots of $r_o'(x)$: $\mu_o > 0$ and $-\mu_o < 0$, and $|\mu_o| < 1$.

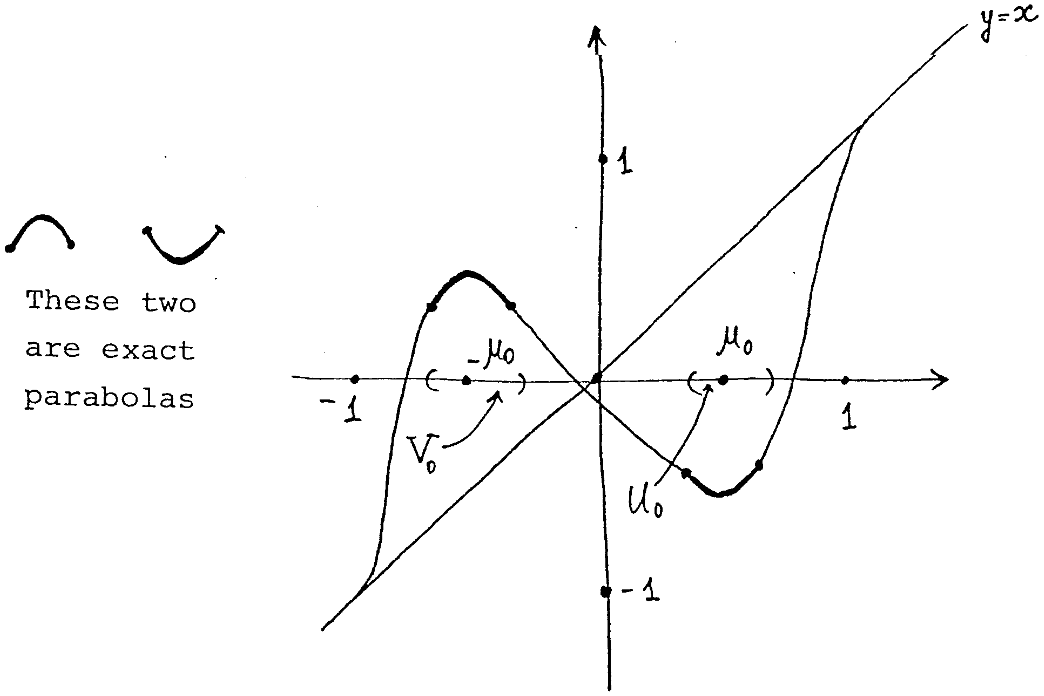
3) $r_o'(x) > 0$ for $x < -\mu_o$, $r_o'(x) < 0$ for $-\mu_o < x < \mu_o$, $r_o'(x) > 0$ for $x > \mu_o$.

4) $\sup_{|x|\leq 1} |r_o(x)| \leqslant 1$.

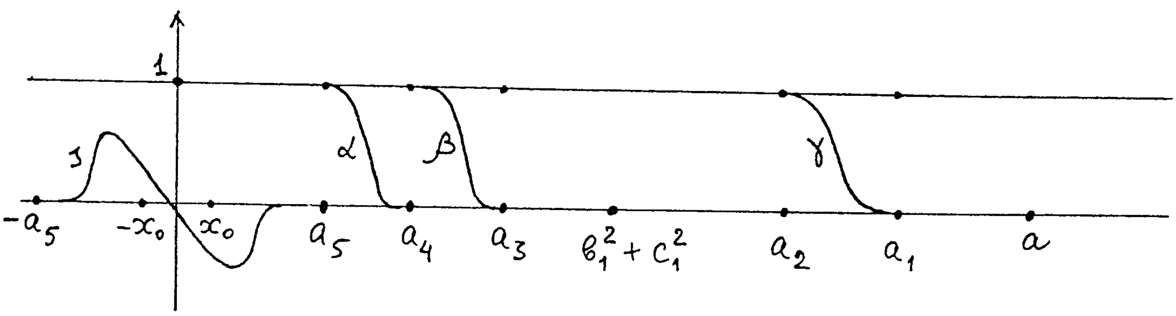
5) There are neighbourhoods U_o of μ_o and V_o of $-u_o$, such that $r_o(x) = \frac{(x-\mu_o)^2}{R} + r_o(\mu_o) < 0$ for $x \in U_o$ and $r_o(x) = -\frac{(x+\mu_o)^2}{R} + r_o(-\mu_o) > 0$ for $x \in V_o$.

(See picture 6.1).

Denote $r_o(x) - x$ by $s_o(x)$ and let $s(x) = \frac{1}{R} s_o(Rx)$, $r(x) = x+s(x)$. Then supp $s(x)$ is contained in $[-\frac{1}{R}, \frac{1}{R}] \subset \subset (-a_5, a_5)$. Further, $\max |s(x)| = \frac{1}{R} \max|s_o(x)| = \frac{2}{R} < \frac{1}{A}$. Denote by x_o the positive root of $r'(x)$, $x_o = \frac{\mu_o}{R}$. Note that in the small neighbourhood of x_o (resp. $(-x_o)$) the function $x + s(x) = \frac{1}{R}\ r_o(R\ x)$ equals $(x-x_o)^2 + r(x_o)$, $r(x_o) < 0$ (resp $-(x+x_o)^2 + r(-x_o)$, $r(-x_o) > 0$).



Pict. 6.1



Pict. 6.2

Now all the conditions 2)-5) from Milnor's book are satisfied by the formula above and we start to check the conclusions of lemma 6.12 and 6.13 for h. Lemma 6.12 first (we copy Milnor).

If $h(x) \neq x$, then $\gamma(x) \neq 0$ and $\beta(y^2 + z^2) \neq 0$ which implies $|x| < a$ and $y^2+z^2 < a_3 < b_1^2 + c_1^2 \Rightarrow |y| < b,\ |z| < c.$

To find the critical points of h we write: $\frac{\partial h}{\partial x} =$
$= 1 + s'(x)\cdot\alpha(y^2+z^2) + \gamma'(x)\cdot(z^2-y^2)\cdot\beta(y^2+z^2)$. By our choice of β the third term is strictly less than 1, hence the critical points exist only in the support of the second term, i.e. for $x \in [-a_5, a_5]$ and $y^2+z^2 < a_4$. But there $\frac{\partial h}{\partial y} =$
$= 2y(s(x)\cdot\alpha'(y^2+z^2) - 1)$ and $\frac{\partial h}{\partial z} = 2z(s(x)\cdot\alpha'(y^2+z^2) + 1)$. Both can vanish only for $y = 0$, $z = 0$ by our choice of s. Hence the critical points belong to $I = [-a_5, a_5] \times 0 \times 0$. Restricted to that segment h is equal to $x + s(x)$, hence the only critical points of h are $-x_0$, x_0. In the small neighbourhood of I h has the form $x+s(x) + z^2 - y^2$, hence the points are nondegenerate; index of $p_2 = (-x_0, 0, 0)$ is $\lambda+1$, index of $p_1 = (x_0, 0, 0)$ is λ.

We have checked up the conclusions of 6.12 and p.1), 3) of 6.13.

To check up 2) of 6.13 we note that $|dh(\text{grad } h)| > 0$ by definition outside of critical points and we are to verify the existence of standard coordinates nearby the critical points. Consider for example the point $p_1 = (x_0, 0, 0)$. In the small neighbourhood of this point the functions $\alpha(y^2+z^2)$, $\beta(y^2+z^2)$ and $\gamma(x)$ are equal to 1, and $x + s(x) = r(x) =$
$= (x-x_0)^2 + r(x_0)$. So $h(x,y,z) = r(x_0) + (x-x_0)^2 + z^2 - y^2$, and $\text{grad } h = (2x, -2y, 2z)$, which is required. The same for $(-x_0)$.

To prove 4) we note that the intersection of the ascending disc $D(p_1,-u)$ with the small disc around p_1 is contained in $[x_0-\varepsilon, x_0+\varepsilon] \times 0 \times B^{n-\lambda-1}_{\varepsilon}$, which means that each v-trajectory, starting at x_o, has the zero y-coordinate on some interval. Now the y-coordinate of the field v in any point (x,y,z) is $2y(s(x)\cdot\alpha'(y^2+z^2) - \gamma(x)\cdot\beta(y^2+z^2) + \gamma(x)\cdot(z^2-y^2)\cdot \cdot\beta'(y^2+z^2))$, which implies that the trajectory, intersecting the plane $(y = 0)$ rests there forever. Thus $D(p_1,-u) \subset \{y = 0\}$. Similarly $D(p_2,u) \subset \{z = 0\}$. Therefore $D(p_1-u) \cap D(p_2,u) \subset \subset \{x, 0, 0\}$. There can be no more than one trajectory, joining x_o and $-x_o$, belonging to $\mathbb{R} \times 0 \times 0$, because it must cross $(0 \times 0 \times 0)$ and since $\mathbb{R} \times 0 \times 0$ is tangent to v, there exists one.

To check transversality we note that the $D(p_2,u)$ and $D(p_1,-u)$ are submanifolds of $\mathbb{R}^n$ of dimensions $\lambda+1$, $n-\lambda$, and $D(p_2,u)$ (resp. $D(p_1,-u)$) is contained in the submanifold $\mathbb{R} \times \mathbb{R}^{\lambda} \times 0 \subset R^n$ (resp. $\mathbb{R} \times 0 \times \mathbb{R}^{n-\lambda-1} \subset \mathbb{R}^n$) of the same dimension, hence $D(p_2,u)$ is an open set in $\mathbb{R} \times \mathbb{R}^{\lambda} \times 0$, $D(p_1,-u)$ is an open set in $\mathbb{R} \times 0 \times \mathbb{R}^{n-\lambda-1}$. In every point of intersection $(\mathbb{R} \times \mathbb{R}^{\lambda} \times 0) \cap (\mathbb{R} \times 0 \times \mathbb{R}^{n-\lambda-1})$ these manifolds are transversal, hence the same is true for $D(p_2, u)$ and $D(p_1,-u)$.

To prove 5) we note that in $\partial U_{a,b,c}$ the vector field v equals $(x, 0, 0)$. The set $\partial U_{a,b,c}$ is the union of $D_1 = \{x = a, |y| \leq b, |z| \leq c\}$, $D_2 = \{|x| < a, |y| \leq b, |z| \leq c\}$ and $|y| = b$ or $|z| = c$, $D_3 = \{x = -a, |y| \leq b, |z| \leq c\}$. The first intersection q cannot belong to D_1, since $(+u)$ points outside U in D_1. It cannot belong either to D_2, because for every point $\lambda \in D_2$, the trajectory, starting from λ, rests some time in D_2. So only D_3 is left. Now $h(x)$ equals x in D_3, and $|h(p_1)| = |r(x_o)| = \frac{1}{R}|r_o(\mu_o)| < \frac{1}{R}$ hence $h(q) = -a$, $|h(p_1)| < a$.

The point 6) is done in the same manner, and 7) is obvious from the definition of h. Lemma 6.13 is proved.

Now we can add a trivial summand.

Theorem 6.14. Let $\mathcal{X} = (f, \bar{f}, v, E)$ be a regular quadruple, belonging to a regular class $\xi \in H^1(M, \mathbb{Z})$, $N \geq 1$ a natural number, $0 \leq k \leq n-1$.

Then there exists a new r-quadruple $\Theta = (g, \bar{g}, v', E')$, belonging to ξ, such that $C_*(v', E)$ is N-equivalent to $C_*(v, E) \oplus \Gamma_*^{(k)}$.

Proof. We distinguish two cases.

1) f has at least one critical point. Pick up some regular value γ for $\bar{f}$ and $L \geq N+1$, such that the sets $\bar{E} \cdot t^s$, $0 \leq s \leq N+1$ are contained in the cobordism $W = \bar{f}^{-1}([\gamma - L, \gamma])$. Pick up the new good gradient-like vector field w for f, a regular point $\alpha \in \bar{f}^{-1}((\gamma-1, \gamma))$ and a neighbourhood $U(\alpha)$ of α, satisfying the conclusions of lemma 6.10. We can demand that w is so close to v, that $C_*(w, E)$ is N-equivalent to $C_*(v, E)$. Thus it suffices to construct a new r-quadruple $\Theta = (g, \bar{g}, v', E)$ such that $C_*(v', E)$ is N-equivalent to $C_*(w, E) \oplus \Gamma_*^k$.

Pick up a small open neighbourhood $U_1(\alpha)$ of α, $\overline{U_1(\alpha)} \subset U(\alpha)$ and a diffeomorphism $\Phi : B_\varepsilon^n(0) \to U_1(\alpha)$, such that $\Phi(0) = \alpha$ and $(\bar{f} \circ \Phi)(x_1, \ldots, x_n) = x_1 + \bar{f}(\alpha)$. Denote by ξ the vector field $\Phi_*(\frac{\partial}{\partial x_1})$ on $U_1(\alpha)$.

Note that $U(\alpha)$ projects bijectively to M. Consider a pair of smooth functions λ, μ on M, such that $0 \leq \lambda \leq 1$,

$\lambda = 1$ on $U_1(\alpha)$ and $\lambda = 0$ on $M \setminus U(\alpha)$; $\mu = 1-\lambda$. Set $w_1 = \lambda\xi + \mu w$. The vector field w_1 equals ξ on $U_1(\alpha)$ and equals w outside $U(\alpha)$. Both ξ, w are gradient-like fields for f, hence w_1 also. Note that it is good if lifted to $\bar{M}$ and restricted to W. (Indeed, by conclusion 1) of lemma 6.10, the ascending and descending discs of w, when intersected with W do not intersect t-shifts of $U(\alpha)$). Furthermore, for every $p,q \in \mathrm{Cr}(\bar{f}|W)$, $\mathrm{ind}\ p = \mathrm{ind}\ q + 1$, the sets $N(p,q;\ w)$ and $N(p,q;\ w_1)$ are the same (the same argument.)

Now we denote the coordinates in $B^n_\varepsilon(0)$ as (x,y,z), $x \in \mathbb{R}$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^{n-k-1}$ and choose $a,b,c > 0$ in such a way, that $[-3a,\ 3a] \times D^k_b(0) \times D^{n-k-1}_c \subset B^n_\varepsilon(0)$. Consider the function $h:\mathbb{R}^n \to \mathbb{R}$, and the field ω, constructed in lemma 6.13 with respect to the numbers a,b,c, and restrict them to $B^n_\varepsilon(0)$. Denote the image $\Phi([-a,a] \times D^k_b(0) \times D^{n-k-1}_c)$ by U_o. Now we define a new function $g:\bar{f}^{-1}([\gamma-1,\gamma]) \to [\gamma-1,\gamma]$ and a new gradient-like vector field w_2 for g as being $h + \bar{f}(\alpha)$, resp. $\Phi_* u$ in $U_1(\alpha)$ and being $\bar{f}$, resp. w_1 on the complement of U_o. Since $\bar{f}|U_1\ \ U_o = (h|U_1\ \ U_o) + \bar{f}(\alpha)$ and $\Phi_* u|(U_1-U_o) = w_2|(U_1-U_o) = \xi|(U_1-U_o)$, this operation is correct and since $\gamma > f(\alpha)+3a$, $f(\alpha)-3a > \gamma-1$, the function really takes its values in $[\gamma-1,\gamma]$. We expand g and w_2 to M as usual. We choose an arbitrary lifting $\tilde{p}_2$ of p_2 to $\tilde{M}$ and we choose a lifting $\tilde{p}_1$ of p_1 in such a way that the lifting of γ_o to $\tilde{M}$, which starts at $\tilde{p}_2$, finishes at p_1. The set $E \cup \{\tilde{p}_2, \tilde{p}_1\}$ will be denoted E'.

I claim that the following properties hold for g, w_2:

1) $\mathrm{Cr}\ g = \mathrm{Cr}\ f \cup \{\pi(\phi(p_1)), \pi(\phi(p_2))\}$, where $\pi:\bar{M} \to M$

is the projection.

2) If $x, y \in \operatorname{Cr} \bar{f} \cap W$, $\operatorname{ind} x = \operatorname{ind} y + 1$, then $D(x, w_2) \cap W$ is transversal to $D(y, -w_2) \cap W$ and $N(x,y; w_2)$ is the same as $N(x,y; w_1)$.

3) If $x \in \operatorname{Cr} \bar{f} \cap W$ there are no $(-w)$-trajectories, joining x and $p_1 t^s$ for $0 \leq s \leq L$, and no $(-w)$-trajectories, joining $p_2 t^s$, and y for $0 \leq s \leq L$.

4) The intersection $D(p_2, w_2) \cap D(p_1 t^s, -w_2)$ is empty for $0 < s \leq L$ and for $s = 0$ it is transversal and consists of one $(-w_2)$-trajectory.

5) $w_2 | W$ is a good gradient-like field for $g | W$.

Proof.

Verify 1). There are no critical points of $\bar{f}$ inside $U(\alpha)$, hence in $U_1 - U_0$ and there are only two critical points for $g = h$ in U_0.

Verify 2). Denote $\bar{f}^{-1}((\gamma - 1, \gamma))$ by W_0. Note that $\operatorname{supp}\big((w_2 - w_1) \,|\, W_0\big) \subset U_0$, $\operatorname{supp}\big((w - w_1) \,|\, W_0\big) \subset U_1$, hence $\operatorname{supp}\big((w_2 - w) | W_0\big) \subset$ $\subset U(\alpha)$, hence $\operatorname{supp}\big((w_2 - w) | W\big)$ is contained in the union of t-shifts of $U(\alpha)$, which does not intersect the $K(w)$ by construction of w. Hence the discs $D(x,w) \cap W$, $D(x,-w) \cap W$ are the same as $D(x,w_2) \cap W$, resp. $D(x,-w_2) \cap W$ for $x \in \operatorname{Cr} \bar{f} \cap W$. Therefore, since w was good, when restricted to W, the discs $D(x, w_2) \cap W$ and $D(y,-w_2) \cap W$ are transversal for $\operatorname{ind} x \leq \operatorname{ind} y + 1$ and $N(x,y;w_2) = N(x,y;w_1)$.

The same argument proves 3).

Now check up 4). Let $s = 0$. There is a $(-w_2)$-trajectory γ_0, joining p_2 and p_1, and $D(p_2,-w_2) \pitchfork D(p_1,w_2)$ along γ_0, by 4) of lemma 6.13. Assume that there is another $(-w_2)$-trajectory, joining p_2 and p_1. Then it must quit U_1 at some moment, by property 4) of 6.13, hence it must quit U_0 and by property 5) of 6.13 this happens in the point θ, for which $h(\theta) < h(p_1) = g(p_1)$ and after this there are no chances to get to p_1 again.

Now let $0 < s \leq L$. If there exists some $(-w_2)$-trajectory η, joining p_2 and $p_1 t^s$, then it must quit $\overline{U(\alpha)}$ at some moment and enter $\overline{U(\alpha)} t^s$ at other moment. Consider the last moment τ_0, when it belongs to $U(\alpha)$ and the first moment τ_1, when it enters $\overline{U(\alpha)} t^s$ for some $s > 0$. The trajectory η, restricted to $[\tau_0, \tau_1]$ is also the trajectory of $(-w)$, since $\operatorname{supp}\left((w_2 - w) \,|\, W\right) \subset \bigcup_{0 \leq s \leq L} U(\alpha) t^s$. But that is impossible by the property 2) of the lemma 6.10.

Note that our property 5) is already proved and all points 1)-5) are verified.

Now we consider a vector field v' which coincides with w_2 in the small neighbourhoods of zeros of w_2, and is a good gradient-like field for g. If we choose v' close enough to w_2 the properties 1)-5) above, together with the lemma 3.12, imply the following:

1') for $x, y \in \operatorname{Cr} \bar{f} \cap W$, $\operatorname{ind} x = \operatorname{ind} y + 1$, $N(x,y;\, v')$ is homotopically the same as $N(x,y;\, w)$.

2') If $x \in \operatorname{Cr} \bar{f} \cap W$, $\operatorname{ind} x = k+1$, then $N(x, p_1 t^s, v') = \emptyset$ for $0 \leq s \leq L$.

3') If $y \in \operatorname{Cr} \bar{f} \cap W$, $\operatorname{ind} y = k$, then $N(p_2, y t^s, v') = \emptyset$ for $0 \leq s \leq L$.

4) $N(p_2, p_1t^s; v') = \emptyset$ for $0 < s \leqslant L$, $n_o(p_2,p_1;v') = 1$.

Now I claim that $C_*(v',\mathcal{E}')$ is N-equivalent to $C_*(w,E) \oplus \Gamma_*^{(k)}$.

Indeed, let $p,q \in E$, ind p = ind $q + 1$, and $L \leqslant s$. We consider the numbers $n_s(p,q;v) = \nu(\bar{p}, \bar{q}t^s;v)$ and $n_s(p,q;v') = \nu(\bar{p}, \bar{q}t^s;v')$.

If $\bar{g}(\bar{q}t^s) \geqslant \bar{g}(\bar{p})$, then $\nu(\bar{p}, \bar{q}t^s;v') = 0$, but since we did not change the function $\bar{f}$ near the critical points of $\bar{f}$ to get $\bar{g}$, we have also $\bar{f}(\bar{q}t^s) \geqslant \bar{f}(\bar{p})$, hence $\nu(\bar{p},\bar{q}t^s;v) = 0$.

If $\bar{g}(\bar{q}t^s) < \bar{g}(\bar{p})$ and $s \leqslant N$ then both $\bar{p}$ and $\bar{q}t^s$ belong to W by the choice of L and we get $n_s(p,q;v) = n_s(p,q,v')$ by 1') above.

Now let $r \in E$, ind $r = k+1$. I claim that $n_s(r, \tilde{p}_1;v') = 0$ for $s \leqslant L$. Indeed, if $s < 0$, then $\bar{g}(p_1t^s) \geqslant \bar{g}(p_1)+1 > \gamma > \bar{g}(r) = \bar{f}(r)$, hence $n(\bar{r},p_1t^s;v') = 0$. If $s \geqslant 0$ we apply 2') above.

Next let $r \in E$, ind $r = k$. I claim that $n_s(\tilde{p}_2,r;\ v') = 0$. If $\bar{g}(\bar{r}t^s) > \lambda$, there is nothing to prove, if $\bar{g}(\bar{r}t^s) \leqslant \lambda$ and $s \leqslant N$, the point $\bar{r}t^s$ belongs to $\mathrm{Cr}\,\bar{f} \cap W$ and we apply 3').

The index $n_s(\tilde{p}_2,\tilde{p}_1;v')$ is zero for $s < 0$ obviously, $n_o(\tilde{p}_2,\tilde{p}_1;v') = 1$, and since $L \geqslant N+1$, the property 4) gives $n_s(\tilde{p}_2, \tilde{p}_1;\ v') = 0$ for $s \leqslant N$.

The case 2): "f has no critical points" is done similarly, using lemma 6.9 instead of lemma 6.10.

The theorem 6.14 is proved.

7. Change of base

In this section we show how to change the vector field v in order to make the elementary change of base in Novikov complex up to an arbitrary degree N, fixed beforehand. By the elementary change of base in a based complex we mean the replacement of one of the generators by $e+fa$, where f is another generator, $a \in \Lambda_\xi^-$ (change of type 1) or the replacement of one of the generators e by $e \cdot \lambda$, where λ is invertible in Λ_ξ^- (change of type 2). We shall perform in this section the changes of type 1 and the changes of type 2 with $\lambda = 1 + \mu$, $\operatorname{supp} \mu \subset \{x \in G \mid \xi(x) < 0\}$. The general case of type 1 change can be done easily afterwards with the help of remark 2.5 and we postpone it to § 8.

We introduce the corresponding notations.

Let C_* be a free based complex of right modules over some ring R. Let e_i, e_j be the free generators of C_* of the same degree and $a \in R$. The complex, obtained from C_* by replacing of e_i by $e_i + e_j a$, and beaving all the other generators as before is denoted $(i,j;a)C_*$ and called the (result of) change of type 1 with respect to $(i,j;a)$.

If e_i is an element of base and $\lambda \in R^*$, we denote by $(i;\lambda)C_*$ the complex, obtained from C_* by replacing the generator e_i by $e_i \cdot \lambda$ and leaving the others invariant. That is called the (result of) change of type 2 with respect to $(i;\lambda)$.

<u>Theorem 7.1</u>. Let $\theta = (f, \bar{f}, v, E)$ be a regular quadruple belonging to a regular $\xi \in H^1(M, \mathbb{Z})$, where v is per-

fect. Let $N \geqslant 1$. Let $p_1, \dots, p_k$ be all the elements of E of index m.

Then:

1) If $a \in \Lambda_\xi^-$ is any element and $1 \leqslant i, j \leqslant k$, then there exists an admissible modification $\varkappa = (g, \bar{g}, w, E)$ of Θ, such that w is perfect and $C_*(w,E)$ is N-equivalent to $(i,j;a)(C_*(v,E))$, that is to the result of the change of type 1 with respect to $(i,j;a)$.

2) If $\lambda \in \Lambda_\xi^-$ is an element of type $1 + \mu$, $\operatorname{supp} \mu \subset \{g \mid \xi(g) < 0\}$ and $1 \leqslant i \leqslant k$, then there exists an admissible modification $\varkappa = (g, \bar{g}, w, E)$ of Θ, such that w is perfect and $C_*(w,E)$ is N-equivalent to $(i,\lambda)(C_*(v,E))$.

The proof of this theorem occupies the rest of this chapter.

To perform these changes we shall recall from [Pa1] the description of Novikov complex in terms of filtrations, associated to auxiliary enumerating Morse functions on the cobordisms $\bar{f}^{-1}([c-N, c])$.

Namely, let $f:M \to S^1$ be a regular Morse function, belonging to a regular class $\xi \in H^1(M, \mathbb{Z})$, and $\bar{f}:\bar{M} \to \mathbb{R}$ - some lifting. Let v be an almost good gradient-like vector field for f. Let $N \geqslant 1$ be a natural number and c be a regular value for f. Denote by W the cobordism $\bar{f}^{-1}([c-N, c])$. We say that the Morse function $\varphi : W \to \mathbb{R}$ satisfies the condition (E) with respect to all these data if

(E1) φ is an admissible modification of $\bar{f}$ and v is a gradient-like vector field for φ ; $\varphi \mid \bar{f}^{-1}(c-N) = a_0$, $\varphi \mid \bar{f}^{-1}(c) = a_{n+1}$.

(E2) φ is enumerating; that is, there exists a sequence of regular values $a_o < a_2 < \dots < a_{n-1} < a_{n+1}$, such that all the critical points of index p belong to the domain $\varphi^{-1}([a_p, a_{p+1}])$.

(E3) if $x, tx \in W$, then $\varphi(tx) < \varphi(x)$.

The functions, satisfying (E) exist, see [Pal, §5]. (Actually we shall reproduce here a part of proof, because we shall need a refined version of the result. The statement of this result in Pal uses the assumption, that v is perfect, but the proof is valid without any changes for almost good ones.)

For a function φ, satisfying (E), we have a filtration $F(\varphi)$ in the pair $(W, \bar{f}^{-1}(c-N))$, given by $(X_p, Y_p) = (\varphi^{-1}([a_o, a_{p+1}]), \varphi^{-1}(a_o))$. The homology of the inversal covering $H_*(\tilde{X}_p, \tilde{Y}_p)$ vanishes for all $* \neq p$. For $* = p$ this homology is a free module over $\mathbb{Z}H$, where $H = \mathrm{Ker}(\xi : \pi_1 M \to \mathbb{Z})$. The base in this module is given by the liftings to $\tilde{W}$ of the pairs $(D(x,v) \cap \{\varphi(x) \geqslant a_p\}, D(x,v) \cap \{\varphi(x) = a_p\})$, where x is a critical point of index p. (This pair is diffeomorphic to (D^p, S^{p-1}).) This lifting is determined by the lifting $\tilde{x}$ of the critical point and will be denoted $\tilde{x}(v)$. The covering $\tilde{W} \xrightarrow{H} W$ is the restriction of the covering $Q: \tilde{M} \xrightarrow{H} M$. Note that the module $F_p^{gr}(\varphi)$ is isomorphic (by excision) to the homology $H_p(Q^{-1}[\varphi^{-1}([a_o, a_{p+1}]) \cup \bar{f}^{-1}((-\infty, c-N])]$, $Q^{-1}[\varphi^{-1}([a_o, a_p]) \cup \bar{f}^{-1}((-\infty, c-N])])$. The space $\varphi^{-1}([a_o, a_{p+1}]) \cup f^{-1}((-\infty, c-N])$ for any p is t-invariant by (E3), which implies that the Q-preimage of this space is G^--invariant, where $G^- = \{g \in G = \pi_1 M \mid \xi(g) \leq 0\}$. Therefore, there is a right $\mathbb{Z}G^-$-module structure on $F_p^{gr}(\varphi)$ (commuting with the differentials in the exact sequence of pair). Furthermore, it is a free $\mathbb{Z}G_N^-$-module, where $\mathbb{Z}G_N^- = \mathbb{Z}G^-/\{g \in \mathbb{Z}G^- \mid \text{supp } g \subset \{\xi \leqslant -N\}\}$. The generators can be chosen to be the discs $\tilde{x}(v)$, where x runs through the critical points of index p, belonging to $W_o = \bar{f}^{-1}([c-1,c])$. The ring $\mathbb{Z}G_N^-$ is a free bimodule over $\mathbb{Z}H$ with the base $1, \theta, \dots, \theta^{N-1}$, where θ is an element in G, such that $\xi(\theta) = -1$. Thus there exists a preferable $\mathbb{Z}H$-base in $F_p^{gr}(\varphi)$, namely, consisting of $\tilde{x}(v)\theta^i$, where $0 \leqslant i \leqslant N-1$, x runs through $\text{Cr } \bar{f} \cap W_o$. (We fix an element θ once and for good.) These elements will be denoted $\tilde{x}^{(i)}(v)$. The action of G^- on $F_p^{gr}(\varphi)$ is described in terms of this base as following. If $g \in G^-$ and $g = \theta^r h$, $h \in H$, then $[\tilde{x}^{(i)}(v)]\cdot g = \tilde{x}^{(i+r)}(v)\cdot h$. We imply that if $i+r \geqslant N$, then $\tilde{x}^{(i+r)}(v) = 0$.

To abbreviate the notations, we will say that two elements α, β of G^- (or of G_N^-) are m-equivalent if $\alpha - \beta$ vanishes in G_m^- (notation: $\alpha \equiv \beta \bmod m$). Two elements α, β in some G^--module F will be called m-equivalent (i.e. $\alpha - \beta \equiv 0 \pmod m$), if $\alpha - \beta$ vanishes in $F \underset{G^-}{\otimes} G_m^-$ (that is if $\alpha - \beta = \gamma \cdot g$, where $\xi(g) \leqslant -m$).

The Novikov complex $C_*(v,E)$ for an r-quadruple $(f, \bar{f}, v, E)$ is related to the above construction as follows.

We assume now that the regular value c for f, chosen above, and E satisfy the following restriction: the projection $\bar{E}$ of E to $\bar{M}$ belongs to $f^{-1}([c-1,c])$. Then the Novikov complex can be defined over the ring $\bar{\bar{\Lambda}}_\xi$ of all elements $\lambda \in \bar{\Lambda}_\xi$ with $\text{supp}\lambda \subset G^-$. (The definition of differential is the same). This complex will be denoted $C^{\circ}_*(v,E)$. Note that $C_*(v,E) = C^{\circ}_*(v,E) \otimes_{\Lambda^{=}_{\xi}} \Lambda^{-}_{\xi}$. For any $N \geqslant 1$ and φ as above consider the bijection of E onto the $\mathbb{Z}G^{-}_{N}$ -base of $F^{gr}_*(\varphi)$ defined as following: the critical point q is carried to the above generator $q(v)$. (Note that tilde $\sim$ is absent over q, since q, by definition, is the element of $E \subset \widetilde{M}$.) Note that $\mathbb{Z}G^{-}_{N} = \Lambda^{=}_{\xi}/\theta^N\Lambda^{=}_{\xi}$, hence it is naturally a $\Lambda^{=}_{\xi}$-bimodule. The bijection of bases, therefore, expands uniquely to the isomorphism $I(\varphi)$ of $\mathbb{Z}G^{-}_{N}$ -modules $C^{\circ}_*(v,E)/C^{\circ}_*(v,E)\theta^N \xrightarrow{I(\varphi)} F^{gr}_*(\varphi)$. The result, proved in [Pal] (see pages 27, 34 of this paper), states that this $I(\varphi)$ commutes with differential, hence is an isomorphism of chain complexes. (This fact was known since long ago except for the action of t; see [Mi2]; the condition (E3) provides all what is necessary to make the proof.) To get the complete notation we write this as following:

$$I(\varphi,v): C^{\circ}_*(v,E) \otimes_{\Lambda^{=}_{\xi}} \mathbb{Z}G^{-}_{N} \xrightarrow{\approx} (F^{gr}_*(\varphi), \{g(v)\}),$$

where $\{q(v)\}$ form the base of $F^{gr}_*(\varphi)$, $q \in E$. (Recall that $\mathbb{Z}G^{-}_{N}$ is a bimodule over $\Lambda^{=}_{\xi}$, so the tensor product is the right $\mathbb{Z}G^{-}_{N}$ -complex, the same as $C^{\circ}_*(v,E)/C^{\circ}_*(v,E)\theta^N$.

Now we can proceed to our base change.

For that we need some definitions.

Let $\bar{g}:\bar{M} \to \mathbb{R}$ be a Morse function $\bar{g}(xt) = \bar{g}(x) - 1$, and let $A \subset \mathrm{Cr}\,\bar{g}$ be a finite subset. The function $\bar{g}$ is called A-enumerating if for $\alpha, \beta \subset A$ with $\mathrm{ind}\,\alpha > \mathrm{ind}\,\beta$ we have $\bar{g}(\alpha) > \bar{g}(\beta)$.

The function $\bar{g}:\bar{M} \to R$ and the corresponding function $g:M \to S^1$ are called indexing, if $g \mid \mathrm{Cr}_i(g)$ is constant for every i and $g|Cr_i(g) \neq g|Cr_j(g)$ for $i \neq j$, and $g(Cr(g))$ contains of a finite number of points, equidistantly placed on S^1.

The function $\bar{g}$ is called K-indexing if it is K-enumerating and indexing.

For an indexing function $\bar{g}$ every critical value has its own index, correctly defined.

Let a, b be the regular values of an indexing function $\bar{g}:\bar{M} \to \mathbb{R}$, such that $0 < b-a < 1$, and there are only two critical values $x < y$ in $[a, b]$ of indices $\operatorname{ind} x > \operatorname{ind} y$. Then there is an operation, called elementary modification of the function $\bar{g}$ which changes it only in the sets $\bar{g}^{-1}([a+n, b+n])$ and inside these sets it interchanges x and y. The field v is still the gradient-like vector field after this operation. (Details see in [Pal, §5].)

We say that a path $\gamma : [\alpha, \beta] \to \bar{M}$ is normal with respect to a perfect field v, if

a) γ is an embedding, which does not contain any zeros of v.

b) there are two zeros p, q of v, $\operatorname{ind} p = \operatorname{ind} q$, such that $\gamma \cap D(p,v)$ consists of a single point, as well as $\gamma \cap D(q,-v)$.

c) γ do not intersect any other ascending or descending disc of v.

(The existence of normal path is shown in lemma 7.5.)

We say that a r-quadruple $(f, \bar{f}, v, E)$ with a perfect v is γ-good if γ is normal with respect to v and γ belongs to some regular level $\bar{f}^{-1}(\lambda)$.

We call an elementary modification γ-good, if both the starting and the resulting functions are γ-good.

<u>Lemma 7.2</u>. Let $\Theta = (f, \bar{f}, v, E)$ be an r-quadruple with v perfect and f indexing. Let a, d be the regular values of f, such that $0 < d-a < 1$, and there are only two critical values $b < c$ in $[a, d]$, ind $b >$ ind c. Assume that Θ is γ-good.

Then there exists a γ-good elementary modification, interchanging b and c.

<u>Proof</u>. By our assumption γ is contained in the $\bar{f}^{-1}(\lambda)$ for some regular λ. If λ is not contained in some $[a+n, d+n]$, then there is nothing to prove. If it belongs to some $[a+n, d+n]$, then we construct the elementary modification explicitely, following Milnor. For brevity we suppose $n = 0$.

We recall from Mi 2 that it $a < b < c < d$ are the real numbers, then there exists a C^∞-function $G: [a,d] \times [0, 1] \to a,d$, such that:

1) For some $\varepsilon > 0$ we have $G(x,y) = x$ for $a \leq x \leq a+\varepsilon$ or $d-\varepsilon \leq x \leq d$.

2) $\dfrac{\partial G(x,y)}{\partial x} > 0$ for all x, y.

3) For some $\delta > 0$ we have $G(x,y) = G(x,0)$ for $y \in [0, \delta]$, and $G(x,y) = G(x, 1)$ for $y \in [1-\delta, 1]$.

4) $\dfrac{\partial G(x,y)}{\partial y} \leq 0$ for all x, y.

5) For $x \in [c-\varepsilon, c+\varepsilon]$ we have $G(x, 1) = x + (b-c)$

For $x \in [b-\varepsilon, b+\varepsilon]$ we have $G(x, 0) = x + (c-b)$.

The existence of such a function is rather obvious. Denote by $p_1,\dots,p_n$ the critical points of index ind c, by $q_1,\dots,q$ the critical points of index ind b. Since $\bar{f}(p_i) = c > b = \bar{f}(q_j)$ and ind c < ind b and v is perfect the union $N_p = \left[\bigcup_i D(p_i,v)\right] \cap f^{-1}(a)$ is a compact manifold, diffeomorphic to the union of n spheres $S^{\text{ind } c-1}$. It does not intersect with the compact $N_q = \left[\bigcup_j D(q_j,v)\right] \cap \bar{f}^{-1}(b)$. For the $\delta > 0$ small enough the set $N_{\delta,p} = \left[\bigcup_i D_\delta(p_i,v)\right] \cap \bar{f}^{-1}(a)$ does not intersect $\left[\bigcup_j D_\delta(q_j,v)\right] \cap \bar{f}^{-1}(a) = N_{\delta,q}$. Consider a smooth function $\mu: \bar{f}^{-1}(a) \to [0, 1]$, which equals 1 in some neighbourhood of $N_{\delta,p}$ and 0 in some neighbourhood of $N_{\delta,q}$. Expand μ to the whole cobordism $W_0 = \bar{f}^{-1}([a,d])$, setting μ to be constant on every v-trajectory, and setting $\mu = 1$ on $\left[\bigcup_i D_\delta(p_i, \pm v)\right] \cap W_0$, $\mu = 0$ on $\left[\bigcup_j D_0(q_j, \pm v)\right] \cap W_0$. That gives a smooth function on W_0, constant on $(\pm v)$-trajectories. Now the new function $h(r)$ is defined as $G(\bar{f}(r), \mu(r))$.

We want h to be γ-good. For that it is enough to have $\mu(\gamma(t)) = \text{const}$, since $\bar{f}(\gamma(t))$ is constant by our assumption.

I claim that of all the discs $D(p_i,v)$, $D(p_i,-v)$, $D(q_i,v)$, $D(q_i,-v)$ the γ intersects at most one. Indeed, if γ intersects some $D(p_i,-v)$, and one of the rest, then this another must be $D(p_j,-v)$ or $D(q_i,-v)$ (since γ is contained in a regular level). That is forbidden by normality. The same for $D(q_i,v)$. Further, if γ intersects one of $D(p_i,v)$, then it cannot intersect $D(p_j,-v)$ (since $\gamma \in \bar{f}^{-1}(\lambda)$) and cannot also intersect $D(p_j, v)$, $D(q_j, v)$ by normality. Hence it can intersect only $D(q_i, -v)$. But that is impossible since ind $p_i >$ $>$ ind q_j. By the same reason, if γ intersect one of $D(q_j, -v)$, it cannot intersect any other disc of listed above.

This implies that there exists $\delta > 0$ such that γ intersects at most one of the sets $D_\delta(p_i, \pm v)$, $D_\delta(q_i, \pm v)$,

1) Suppose that γ does not intersect these sets at all. Then the diffeomorphism of shift along $(-v)$ from $\bar{f}^{-1}(\lambda)$ to $\bar{f}^{-1}(a)$ is defined on γ and sends it to a compact which intersects neither $N_{\delta,p}$ nor $N_{\delta,q}$. Then one can choose μ to be 1 on $N_{\delta,p} \cup \Phi(\gamma)$ and 0 on $N_{\delta,q}$, and since μ is constant on the $(\pm v)$-trajectories, we have $\mu(\gamma(t)) \equiv 1$.

2) Suppose that γ intersects one, for example, $\gamma \cap D_\delta(p_1, \pm v) \neq \emptyset$. Then $\gamma \cap D_\delta(p_i, \pm v) = \emptyset$ for $i \neq 1$ and $\gamma \cap D_\delta(q_j, \pm v) = \emptyset$ for all j. Denote by Q the compact $\gamma \setminus D_{\delta/2}(p_1, \pm v)$. The compact Q belongs to the domain of Φ and $\Phi(Q)$ does not intersect $N_{\delta,q}$. That means that one can choose a smooth function μ to be equal to 1 on $N_{\delta,p} \cup \Phi(Q)$ and zero on $N_{\delta,q}$. This function μ is, therefore, equal to 1 on Q and on $D_\delta(p_1, \pm v)$, hence on the whole γ. The case $\gamma \cap D_\delta(q_i, \pm v) \neq \emptyset$ is done similarly. In this case the value $\mu(\gamma)$ will be equal to 0.

We are only to check now that γ belongs again to the regular level of $h(r) = G(\bar{f}(r), \mu(r))$. Suppose, for example, that $\mu(\gamma(t)) \equiv 1$, then $h(\gamma(t)) = G(\bar{f}(\gamma(t)), 1)$.

Therefore, $h(\gamma)$ can be a critical value of h only if $G(\lambda, 1)$ equals b or c. Since $x \mapsto G(x, 1)$ is a bijection, $G(\lambda, 1) = b$ implies $\lambda = c$ which is impossible since λ was regular. So only $G(\lambda, 1) = c$ is possible. We know $G(x, 1) = x$, hence $G(\lambda, 1) = c \Rightarrow \lambda \geqslant c$. But we can choose the and function h_0 from lemma D.3 in such a way that $\lambda > c+\varepsilon$, and $h_0(\lambda) \neq c$, which will imply that $G(\lambda, 1) \neq c$. The case $\mu(\gamma(t)) = 0$ is done similarly. The lemma 7.2 is proven.

Lemma 7.3. Let $\Theta = (f, \bar{f}, v, E)$ be an r-quadruple, such that f is indexing, and v is perfect. Let $\gamma : [\alpha, \beta] \to \bar{M}$ be a normal path. Let $A \subset \mathrm{Cr}\,\bar{f}$ be any finite set. Assume that Θ is γ-good, that is $\gamma \subset \bar{f}^{-1}(\lambda)$ where λ is regular.

Then there exists series of γ-good elementary modifications of f, finishing with an A-indexing r-quadruple $\varkappa = (g, \bar{g}, v, E)$.

Proof. The proof repeats the proof of lemma 5.2 of [Pal], replacing the term "elementary modification" by "γ-good elementary modification". The fact that each elementary modification can be chosen as to be γ-good follows from the lemma 7.2. (We just recall that in our notations the lemma 5.2 says: for any finite subset $A \subset \mathrm{Cr}\,\bar{f}$ there exists a finite sequence of elementary modifications of $\bar{f}$, resulting with an A-indexing function g.)

Lemma 7.4. Let $\Theta = (f, \bar{f}, v, E)$ be an r-quadruple, such that f is indexing and v is perfect. Let $c \in \mathbb{R}$ be a regular value for $\bar{f}: \bar{M} \to \mathbb{R}$ and $N \geqslant 1$ - a natural number. Let γ be a normal path with respect to critical points $p, q \in W = \bar{f}^{-1}([c-N, c])$, ind p = ind $q = k$. Assume that belongs to a regular level λ of $\bar{f}$, such that $\lambda \not\equiv c \mod \mathbb{Z}$.

Then there exists a Morse function $\varphi: W \to \mathbb{R}$, such that

1) φ satisfies condition (E) for v.

2) Every t-shift of γ, which is contained in W belongs to some regular level surface $\varphi^{-1}(\lambda)$, where $a_k < \lambda < a_{k+1}$.

3) There exists a neighbourhood U of γ in W, such that all the t-shifts of U, contained in W, belong already to $\varphi^{-1}((a_k, a_{k+1}))$, φ is t-equivariant, when restricted to the union of these shifts, and $d\varphi(v)$ is a constant function on U.

<u>Proof</u>. Consider the cobordism $W_1 = \bar{f}^{-1}([c-2N, c+N])$ and the finite set $A = \mathrm{Cr}\, \bar{f} \cap W_1$. By the lemma 7.3 there is an A-indexing function $\bar{g}: \bar{M} \to R$, which is an admissible modification of f and such that g is γ-good. Therefore, there exists a sequence $a_2 < \dots < a_{n-1}$ of regular values of $\bar{g}$, such that all the critical points in A of index r belong to $\bar{g}^{-1}((a_r, a_{r+1}))$.

Note that if a t-shift γt^{ν} of γ belongs to W, then $-(N-1) \leqslant \nu \leqslant (N-1)$, which means that γt^{ν} intersects with $D(pt^{\nu}, v)$ and $D(qt^{\nu}, -v)$, therefore $\bar{g}(qt^{\nu}) < \bar{g}(\gamma t^{\nu}) < \bar{g}(pt^{\nu})$, therefore γt^{ν} is contained in $\bar{g}^{-1}((a_k, a_{k+1}))$, since $pt^{\nu}, qt^{\nu} \in W_1$.

Consider now the regular value c of $\bar{g}$, such that γ belongs to $g^{-1}(c)$. Let ε be so small that the interval $[c-\varepsilon, c+\varepsilon]$ is regular for g. Then we can pick up a function $g_o: \bar{g}^{-1}([c-\varepsilon, c+\varepsilon]) \to [c-2\varepsilon, c+2\varepsilon]$, such that g_o coincides with $\bar{g}$ in the neighbourhoods of the boundary and $dg_o(v) = \text{const}$ in $g_o^{-1}([c-\varepsilon_o, c+\varepsilon_o])$, where $0 < \varepsilon_o < \varepsilon$. (The explicit construction can be read off from [Mi2, §5]. Namely, there exists $\varepsilon_1 > 0$, such that all the $(\pm v)$-trajectories, starting at $V = \bar{g}^{-1}(c)$ stay at $\bar{g}^{-1}((c-\varepsilon, c+\varepsilon))$ for the values of parameter $\leq \varepsilon_1$. Consider now the embedding $\Phi : [-\varepsilon_1, +\varepsilon_1] \times V \to$

$\to \bar{M}$, sending the vector field $\frac{\partial}{\partial t}$ to v. It suffices to construct a function ξ on $Z = [-\varepsilon_1, +\varepsilon_1] \times V$, such that $\frac{\partial}{\partial t}$ is a gradient-like vector field for ξ, ξ coincides with $\eta = \bar{g} \circ \Phi$ in the neighbourhoods of $[-\varepsilon_1 \times V, +\varepsilon_1 \times V]$ and $\frac{\partial \xi}{\partial t}$ = const for $t \in [-\varepsilon_2, \varepsilon_2]$, where $0 < \varepsilon_2 < \varepsilon_1$. Fix ε_2 and take a function $\lambda : [-\varepsilon_1, \varepsilon_1] \to [0, 1]$, such that λ equals 1 in the neighbourhoods of $-\varepsilon_1, \varepsilon_1$ and equals 0 on $[-\varepsilon_2, \varepsilon_2]$. For $q \in V$ denote by $k(q)$ the number $\left[\eta(\varepsilon_1, q) - \eta(-\varepsilon_1, q) - \int_{-\varepsilon_1}^{\varepsilon_1} \lambda(t) \cdot \frac{\partial}{\partial t} \eta(t, q) \cdot dt\right] \Big/ \int_0^1 (1 - \lambda(t))\, dt$. This defines a smooth positive function on $[-\varepsilon_1, \varepsilon_1] \times V$. Define now the function $\xi(t, q)$ by $\xi(t, q) =$

$$= \int_{-\varepsilon_1}^{t} \left[\lambda(\tau) \cdot \frac{\partial}{\partial \tau} \eta(\tau, q) + (1 - \lambda(\tau))\, k(q)\right] d\tau + \eta(-\varepsilon_1, q)$$

is smooth, coincides with $\eta(t, q)$ for t close to $-\varepsilon_1$ and coincides with $\eta(t, q) + C(q)$ for t close to ε_1, where the $C(q)$ equals 0 by our choice of $k(q)$. The derivative $\frac{\partial \xi}{\partial t}(t, q)$ equals $k(q)$ for $t \in [-\varepsilon_2, \varepsilon_2]$).

We change $\bar{g}$ to g_0 in the set $\bar{g}^{-1}([c-\varepsilon, c+\varepsilon])$ and expand g_0 to a t-equivariant function $\bar{g}_0$ on $\bar{M}$. We can assume that $a_i \notin \operatorname{supp}(\bar{g}_0 - \bar{g})$, so that again all the points in $\operatorname{Cr} \bar{f} \cap W_1$ of index k belong to $\bar{g}_0^{-1}((a_k, a_{k+1}))$. Note that $\bar{g}_0$ is γ-good and if ε is small enough the shift γt^{ν} for $-(N-1) \leq \nu \leq$ $\leq N-1$ is contained in $\bar{g}_0^{-1}((a_k, a_{k+1}))$, together with the set

$\bar{g}_o^{-1}([\bar{g}_o(\gamma t^\nu)-\varepsilon,\ \bar{g}_o(\gamma t^\nu)+\varepsilon]) = \bar{g}^{-1}([\bar{g}(\gamma t^\nu)-\varepsilon,\ \bar{g}(\gamma t^\nu)+\varepsilon])$.

Now take the neighbourhood U of γ so small that all the t-shift Ut^ν for $-(N-1) \leq \nu \leq N-1$ are contained in $\bar{g}_o^{-1}((a_k, a_{k+1}))$. that U belongs to $\bar{g}_o^{-1}([c-\varepsilon_1, c+\varepsilon_1])$ and that U does not intersect with $\bar{f}^{-1}([c-\varepsilon_3, c]) \cup \bar{f}^{-1}([c-N, c-N+\varepsilon_3])$ for some ε .

Now we apply to g_o the damping procedure as described in [Pal , §5] so as to get the function φ, which differs from $\bar{g}_o$ only in $\bar{f}^{-1}([c-\varepsilon, c]) \cup \bar{f}^{-1}([c-N, c-N+\varepsilon_3])$, has the same gradient-like vector field v and is constant on $\bar{f}^{-1}(c)$ and on $f^{-1}(c-N)$. This φ satisfies all the conclusions 1)-3).

<u>Lemma 7.5</u>. Let $\Theta = (f, \bar{f}, v, E)$ be an r-quadruple, where v is a perfect vector field. Let p, q be the critical points of $\bar{f}$, ind p = ind q. Let λ be a regular value of $\bar{f}$, $\bar{f}(p) < \lambda < \bar{f}(q)$. Let ρ be a class of homotopy of paths, starting at p and finishing at q.

Then there exist the smooth embedding $\gamma : [0, 3] \to \bar{f}^{-1}(\lambda)$, such that

1) γ intersect $D(p, v) \cap \bar{f}^{-1}(\lambda)$ (resp. $D(q, -v) \cap \bar{f}^{-1}(\lambda)$) at a single point $\gamma(1)$ (resp. $\gamma(2)$), and the vector $\dot{\gamma}(1)$ (resp. $\dot{\gamma}(2)$) does not belong to $T_*(D(p, v) \cap \bar{f}^{-1}(\lambda))$ (resp. $T_*(D(q, -v) \cap \bar{f}^{-1}(\lambda))$).

2) If $\alpha \neq p$ (resp. $\beta \neq q$) the path γ does not intersect the disc $D(\alpha, v)$ (resp. $D(\beta, -v)$).

3) The class of homotopy of the path, which is formed by a $(-v)$-trajectory from p to $\gamma(1)$, then by $\gamma | [1, 2]$, then

by a $(-v)$-trajectory from $\gamma(2)$ to q, equals ρ.

Proof. We recall that $N_p = D(p, v) \cap \bar{f}^{-1}(\lambda)$ (resp. $L_q = D(q, -v) \cap \bar{f}^{-1}(\lambda)$) is a smooth submanifold of $\bar{f}^{-1}(\lambda)$. Since v is perfect, N_p and L_q do not intersect. For each (resp. β) the intersection $L_q \cap D(\alpha, v)$ (resp. $N_p \cap D(\beta, -v)$) is transversal, hence the union $\bigcup_{\alpha} (L_q \cap D(\alpha, v))$ (resp. $\bigcup_{\beta} (N_p \cap D(\beta, -v))$ is the countable union of separable submanifolds of codimension $\geqslant 2$, hence nowhere dense in L_q (resp. N_p). We denote the union $\bigcup_{\alpha \neq p} D(\alpha, v)$ by Q^d and $\bigcup_{\beta \neq q} D(\beta, -v)$ by Q^{up}. Now we choose the points $a \in N_p \setminus Q^{up}$ and $b \in L_q \setminus Q^d$. We choose the neighbourhoods U of a and and V of b so small that there exists a chart diffeomorphism $\Phi: B_{2\varepsilon}(0) \to U$ (resp. $\Psi: B'_{2\varepsilon}(0) \to V$), such that $\Phi(0) = a$ (resp. $\Psi(0) = b$) and $\Phi^{-1}(N_p)$ is given by the equations $x_{\text{ind } p} = x_{\text{ind } p+1} = \ldots = x_{n-1} = 0$ (resp. $y_{n-\text{ind } q} = y_{n-\text{ind } q+1} = \ldots = y_{n-1} = 0$) where x_i, resp. y_j are the coordinates in $B_{2\varepsilon}(0)$, $B'_{2\varepsilon}(0)$.

Denote by K_p (resp. K_q) the intersection with $\bar{f}^{-1}(\lambda)$ of the union of all the discs $D(\alpha, v)$ (resp. $D(\beta, -v)$) where α (resp. β) runs through the critical points of $\bar{f}$, belonging to $\bar{f}^{-1}(\lambda, \bar{f}(p))$ (resp. $\bar{f}^{-1}(\bar{f}(q), \lambda)$), such that $\text{ind}\,\alpha < \text{ind}\,\beta$ (resp. $\text{ind}\,\beta > \text{ind}\,q$). The lemma 3.7 implies that K_p and $K_p \cup N_p$ are compacts (resp. K_q, $K_q \cup L_q$ are compacts). Note that $N_p \cap K_p = \emptyset = K_q \cap L_q$. Therefore it is possible to choose U and V so small that $U \cap K_p = \emptyset$, $V \cap K_q = \emptyset$. The Φ_i-images of $B_{\varepsilon}(0)$ and $B_{\varepsilon/2}(0)$ will be denoted by U^0, U^1. The Ψ-images of $B'_{\varepsilon}(0)$, $B'_{\varepsilon/2}(0)$ will be denoted by V^0, V^1.

Furthermore, since v is perfect, we have $(N_p \cup K_p) \cap$ $\cap (L_q \cup K_q) = \emptyset$. This implies that we can assume also $U \cap$ $\cap (L_q \cup K_q) = \emptyset$, $V \cap (N_p \cup K_p) = \emptyset$, and $U \cap V = \emptyset$.

Choose now the path γ_o, which is defined on $[-\varepsilon, 1+\varepsilon]$ and has the following properties

i) $\gamma_o | [-\varepsilon, \varepsilon]$ is a Φ-image of a radial curve $(0, \ldots, t, \ldots, 0)$ in $B_\varepsilon(0)$, where t belongs to a coordinate with the number ind p. Similarly, $\gamma_o | [1-\varepsilon, 1+\varepsilon]$ is a Ψ-image of a radial curve $(0, \ldots, t-1, \ldots, 0)$ where $t-1$ belongs to a coordinate with number $n-$ ind q.

ii) γ_o is an embedding.

iii) The homotopy class of the composition $[\gamma_o(1), q] \circ \gamma_o | [0, 1] \circ [p, \gamma_o(0)]$ equals ρ. *)

(That is possible since $\pi_1(\bar{f}^{-1}(\lambda)) \to \pi_1(\bar{M})$ is epi by regularity of f).

Note now that $\gamma_o | [\varepsilon/2, \varepsilon]$ and $\gamma_o | [1-\varepsilon, 1-\varepsilon/2]$ do not intersect neither $N_p \cup K_p$, nor $L_q \cup K_q$. Hence by transversality, we can perturb $\gamma_o | [\varepsilon/2, 1-\varepsilon/2]$ arbitrarily small so as to get $\gamma_1 : [\varepsilon/2, 1-\varepsilon/2] \to \bar{f}^{-1}(\lambda)$ which coincides with γ_o on $[\varepsilon/2, \varepsilon]$ and $[1-\varepsilon, 1-\varepsilon/2]$ and do not intersect $N_p \cup K_q$, neither $L_q \cup K_q$. The result of glueing γ_1 with $\gamma_o | [-\varepsilon, \varepsilon]$ at the beginning and $\gamma_o | [1-\varepsilon, 1+\varepsilon]$ at the end will be denoted γ_2. Note that the properties i)-iii)

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*) For brevity we denote by $[p,s]$ (resp. $[w,q]$) the $(-v)$-trajectory starting at p and finishing at $s \in N_p$ (resp. $(-v)$-trajectory, starting at $w \in L_q$ and finishing at q).

still hold for γ_2.

Recall now that by the choice of x, y, we have,that neither x nor y do not belong to $D(\alpha, v)$ or $D(\beta, -v)$ if $\alpha \neq p$, $\beta \neq q$. Then by transversality we can perturb γ_2, $\gamma_2(0)$ and $\gamma_2(1)$ fixed, in such a way that a new path γ_3 does not intersect $D(\alpha, v)$ for $\alpha \neq p$ neither $D(\beta, -v)$ for $\beta \neq q$. If the perturbation is small enough the properties ii), iii) above still hold for γ_3. Furthermore, $\gamma_3 | [-\varepsilon, \varepsilon]$ has the single intersection point with N_p (resp. $\gamma_3 | [1-\varepsilon, 1+\varepsilon]$ has the single intersection point with L_q) and $\dot{\gamma}_3(0) \notin T_* N_p$, resp. $\dot{\gamma}_3(1) \notin T_* L_q$. Furthermore, $\gamma_3 | [\varepsilon, 1-\varepsilon]$ does not intersect $(N_p \cup K_p) \cup (L_q \cup K_q)$.

After the reparametrization of γ_3 we get γ, satisfying all the conditions 1)-4).

Now we describe the construction of pulling one descending disc onto another. Next lemmas follow Mi with minor changes.

Lemma 7.6. Let V^m be a compact connected manifold, and N^r, L^s be submanifolds of V^m, $r+s = m-1$, $m-r \geqslant 2$, such that the closures N^r, L^s do not intersect. Assume that N is oriented, L is cooriented. Let $\gamma : [0, 3] \to V^m$ be an embedding, satisfying: (C) $\gamma \cap \overline{N^r} = \gamma(1)$ and $\dot{\gamma}(1) \notin TN$; $\gamma \in \overline{L^s} = \gamma(2)$ and $\dot{\gamma}(2) \notin TL$.

Let ρ be 1 or -1. Let A be any open neighbourhood of γ in V.

Then there exists $\delta > 0$ and an embedding $\Phi : [0, 3] \times$ $\times D_0^r(0) \times D_0^s(0) \to V^m$, such that:

0) $\operatorname{Im}\Phi \subset A$.

1) $\operatorname{Im}\Phi \cap \bar{N} = \Phi(\{1\} \times D^r_\delta(0) \times \{0\})$; $\operatorname{Im}\Phi \cap \bar{L} = \Phi(\{2\} \times \{0\} \times D^s_\delta(0))$; $\qquad \Phi|([0, 3] \times 0 \times 0) = \gamma$.

2) $\Phi([0, 3] \times D^r_o(0) \times 0)$ intersects $\bar{L}$ in a single point $\Phi(\{2\} \times 0 \times 0)$ and the intersection is transversal of the sign ρ.

Proof. The proof consists of two steps. First we construct the embedding "infinitesimally along the curve γ" (lemma 7.7) and then we build from it the real embedding (as in the proof of lemma 5.4 from lemma 5.5).

Lemma 7.7. In the assumptions of lemma 7.6 there exist the vector fields $\xi_1, \dots, \xi_r, \eta_1, \dots, \eta_s$ along the curve γ, such that

1) $\dot{\gamma}(t), \xi_1(t), \dots, \xi_r(t), \eta_1(t), \dots, \eta_s(t)$ form a base of $T_{\gamma(t)}V$.

2) $\xi_1(1), \dots, \xi_r(1)$ form a positively oriented base of $T_{\gamma(1)}N$, and $\dot{\gamma}(2), \xi_1(2), \dots, \xi_r(2)$ form a base of $T_{\gamma(2)}V / T_{\gamma(2)}L$ of the sign ρ.

3) $\eta_1(2), \dots, \eta_s(2)$ form a base of $T_{\gamma(2)}L$.

Proof. Consider the bundle $T_*V/T_*\gamma$ over γ. It is trivial. Let $\alpha_1, \dots, \alpha_r \in T_{\gamma(1)}N$ be the vectors, forming a positive base of $T_{\gamma(1)}N$. Let $\beta_1, \dots, \beta_r$ be the vectors in $T_{\gamma(2)}(V)$, such that $\gamma(2), \beta_1, \dots, \beta_r$ form a base of $T_{\gamma(2)}V/T_{\gamma(2)}L$ of the sign ρ. The vectors α_i (resp. β_i) form the section of the bundle of r-repers of $T_*V/T_*\gamma$. The latter is trivial, and the set of r-repers in $\mathbb{R}^{m-1}$ is connected since

$m-1-r \geq 1$. Therefore there exists a family $\xi_1(t), \dots, \xi_r(t)$ of vector fields along γ, such that $\dot{\gamma}(t), \xi_1(t), \dots, \xi_r(t)$ are linearly independent and $\xi_i(1) = \alpha_i$, $\xi_i(2) = \beta_i$. Consider now that $(r+1)$-bundle ν over γ, generated by $\dot{\gamma}(t), \xi_1(t), \dots, \xi_r(t)$, and the bundle T_*V/ν. The latter is trivial, hence every trivialisation of it at every point can be expanded to a trivialisation over the whole γ. Pick up any vectors $\lambda_1, \dots, \lambda_s$ forming the base of $T_{\gamma(2)}L$. They form the trivialisation of T_*V/ν in $\gamma(2)$, hence there exist the vectors $\eta_i(t)$ along γ, such that $\dot{\gamma}, \xi_i, \eta_j$ form the base of T_*V and $\eta_i(2) = \lambda_i$. Q.E.D.

Proof of the lemma 7.6. We denote by $B_{2\varepsilon}(0)$ the open ball in the euclidean space $\mathbb{R}^m$ around 0 of the radius 2ε. The coordinates will be called $x_0, x_1, \dots, x_r, x_{r+1}, \dots, x_m$. It is convenient to have another copy of it: $B'_2(0)$ with the coordinates $y_0, y_1, \dots, y_r, y_{r+1}, \dots, y_m$.

It follows from the condition (C) that for ε small enough there exist a diffeomorphism Γ of $B_{2\varepsilon}(0)$ onto a neighbourhood U of $\gamma(1)$, such that

1) the curve $\gamma \mid (1-2\varepsilon, 1+2\varepsilon)$ equals the curve $\Gamma(t-1, 0, 0, \dots)$.

2) the intersection $\bar{N} \cap \Gamma(B_{2\varepsilon}(0))$ equals the Γ-image of $\{x_0 = 0, x_{r+1} = 0, \dots, x_m = 0\}$.

Similarly we have a diffeomorphism $\Psi: B'_{2\varepsilon}(0) \to V \ni \gamma(2)$ such that

1) $\gamma \mid (2-2\varepsilon, 2+2\varepsilon)$ equals $\Gamma(t-2, 0, \dots, 0, \dots)$

2) $\bar{L} \cap \Gamma(B'_2(0)) = \Gamma(\{x_0 = 0, x_1 = 0, \dots, x_r = 0\})$.

We assume that $U \cap V = \emptyset$.

Choose a riemannian metric g on V, such that in the balls $\Gamma(B_\varepsilon(0))$ and $\Psi(B'_\varepsilon(0))$ it is euclidean in the coordinates x_i, resp. y_j. Denote by exp the exponential map, corresponding to this metric and by $\Phi_\delta: [0, 3] \times D^r_\delta(0) \times D^s_\delta(0) \to$ $\to V$ the map, defined by $\Phi_\delta(\tau, \theta_1, \dots, \theta_r, \nu_1, \dots, \nu_s) =$ $= \exp_{\gamma(\tau)}(\theta_1 \xi_1(\tau) + \dots + \theta_r \xi_r(\tau) + \nu_1 \eta_1(\tau) + \dots +$ $+ \nu_s \eta_s(\tau))$. I claim that for δ sufficiently small the map Φ_δ satisfies the conclusions of lemma 7.6. Indeed, 0) is obvious. Now we check that for small δ the first, for example, part of 1) holds.

Note that $\gamma \setminus U$ does not intersect N. This implies that for δ small enough the intersection $\mathrm{Im}\,\Phi_\delta \cap \bar{N} \subset U$. (Indeed, if there is a sequence of points $y_n \in [\mathrm{Im}\,\Phi_\delta \cap \bar{N}] \setminus U$, $\delta_n \to 0$, then we choose a convergent subsequence, which must converge to $y \in (\gamma \cap \bar{N}) \cap (W \setminus U)$, which is impossible.) Note further, that $\Phi_\delta(1 \times D^r_0(0) \times 0)$ consists of values at the parameter, equal to 1, of g-geodesics, starting at γ with a tangent vector, which is a linear combination $\theta_1 \xi_1(1) + \dots +$ $+ \theta_r \xi_r(1)$, and $\sqrt{\sum \theta_i^2} \leq \delta$. For δ small enough this geodesic is a straight line in coordinates x_i, which belongs to $\{x_0 = 0,\ x_{r+1} = x_{r+2} = \dots = 0\}$. Therefore, $\mathrm{Im}\,\Phi_\delta \cap \bar{N} \supset$ $\Phi_\delta(\{1\} \times D^r_\delta(0) \times \{0\})$ for δ sufficiently small. To prove that actually the equality holds, assume that for $\delta_n \to 0$ there exist a convergent sequence $\alpha_n \to \alpha$, such that α_n $\notin \mathrm{Im}\,\Phi_{\delta_n} \cap \bar{N}$, $\alpha_n \notin \Phi_{\delta_n}(\{1\} \times D^{(r)}_{\delta_n} \times \{0\})$. Each α_n is $\exp_{\gamma(\tau_n)}$ of some linear combination λ_n of $\xi_i(\tau_n)$, $\eta_i(\tau_n)$. We can assume that $\tau_n \to \tau$ and that the linear combination converge also, so that $\alpha = \exp_{\gamma(\tau_0)}(\theta_1 \xi_1(\tau_0) + \ldots + \theta_r \xi_r(\tau_0) + \nu_1 \eta_1(\tau_0) + \ldots + \nu_s \eta_s(\tau_0))$. Since $\delta_n \to 0$ all the θ_i, ν_j are zeros, and $\alpha = \gamma(\tau_0)$. Note that since $\alpha_n \in \bar{N}$ we have $\tau_0 = 0$. We can assume therefore that all the α_n belong to $\Gamma(B_\varepsilon(0)) \subset U$ and the $\exp_{\gamma(\tau_n)}(\mu \lambda_n)$ belongs to $\Gamma(B_\varepsilon(0))$ for $0 \le \mu \le 1$. Now we consider the coordinates (x_i) on the $\Gamma(B_\varepsilon(0))$. In this coordinates the path $\exp_{\gamma(\tau_n)}(\mu \lambda_n)$, $\mu \in [0, 1]$ is the straight line starting at some point $(\tau_n - 1, 0, \ldots, 0)$ with the tangent vector λ_n with the first coordinate zero. That implies that this path can intersect the hyperplane $(x_0 = 0)$ only if $\tau_n - 1 = 0$. But by condition $\exp_{\gamma(\tau_n)}(\mu \lambda_n) \in \Gamma(B_\varepsilon(0))$ for $\mu \in [0, 1]$ we know that the intersection exists, hence $\tau_n = 1$, hence $\gamma(\tau_n) = \gamma(1)$ for every n. Now our x_n is the end of the small segment of a straight line (in the coordinates x_i), starting at zero. If x belongs to the plane $(x_{r+1} = \ldots = x_m = 0)$, that implies that the vector to this segment belongs to this plane, $\Rightarrow \nu_1 = \ldots = \nu_s = 0$, and we have proved the property 1).

To prove 2) we note that by the second part of 1) we have $\Phi(\{1\} \times D^r_0(0) \times \{0\}) \cap \bar{L} = \Phi(\{2\} \times D^r_\delta(0) \times \{0\}) \cap \Phi(\{2\} \times \{0\} \times D_0(0)) = \{2\} \times \{0\} \times \{0\}$. The sign is obviously the sign of the reper $(\dot{\gamma}(2), \xi_1(2), \ldots, \xi_r(2))$ in the space $T_{\gamma(2)}V/T_{\gamma(2)}L$ which is ρ by the choice of ξ_i, the lemma 7.6 is proven.

We shall need one more notation. If $h_t : V \to V$ is an isotopy of V and $N \subset V$ is a submanifold of V, we shall denote by N_h the submanifold of $V \times [0, 1]$, consisting of all the pairs $\{h_t(x), t\}$, where $x \in V$. That is the manifold with boundary $\partial N_h =$ $= V \times \{0\} \cup h_1(V) \times \{1\}$.

Lemma 7.8. Under the assumptions of the lemma 7.6 and for any $\beta > 0$, $\beta < \delta$ there exists a smooth isotopy h_t of V, possessing the following properties

0) $h_t = \mathrm{id}$ only for $\varkappa \leq t \leq 1-\varkappa$, where $\varkappa$ is some positive integer.

1) $h_t \neq \mathrm{id}$ only in $\Phi([0, 3] \times D^r_\beta(0) \times D^s_\beta(0))$.

2) In the manifold $V \times [0, 1]$ the intersection of the manifold N_h with $L \times [0, 1]$ consists of the single point $a = \gamma(2) \times 2/3$. The sign of intersection is ρ and the path $(\gamma(t), t) \in N_h$ joins $\gamma(1) \times 0 \in V \times 0$ with $a = \gamma(2) \times 2/3 \in L \times [0, 1]$. *)

Proof. We shall denote this time the coordinates in $[0, 3] \times D^r_o(0) \times D^s_o(0)$ by $\tau, x_1, \dots, x_r, y_1, \dots, y_s$. Choose $\beta > 0$ such that $2\beta < \delta$. Choose a C^∞-function $\nu : \mathbb{R} \to \mathbb{R}$, such that $\nu(t) = 1$ for $t \leq \beta^2/4$, $\nu(t) = 0$ for $t \geqslant \beta^2$ and $\nu'(t) \leqslant 0$.

Pick up a smooth function $\theta_o : [0, 3] \to [0, 3]$, such that:

1) $\theta_o(x) = x$ for x close to zero and x close to 3.

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*) The coorientation of $L \times [0, 1]$ is obvious; the orientation of N_h is given by the following convention: if $e_1, \dots, e_r$ form a positive base of $T_\alpha N$, then the vectors $(\frac{\partial}{\partial t}(h_t(\alpha)), 1); ((h_t)_*(e_1), 0)), \dots; ((h_t)_*(e_r), 0)$ form the positive base of $T_{(h_t(\alpha), t)} N_h$.

2) $\theta_o'(x) > 0$

3) $\theta_o(x) \leq x$

4) $\theta_o(2{,}5) = 1$ and $\theta_o(x) = x - 1{,}5$ for x close to $2{,}5$.

Denote by θ the inverse function. Then:

1') $\theta(x) = x$ for x close to 0 or 3.

2') $\theta'(x) > 0$

3') $\theta(x) \geqslant x$

4') $\theta(1) = 2{,}5$ and $\theta(x) = x + 1{,}5$ for x close to 1.

a smooth function

: $[0, 1] \to [0, 1]$, posessing the following properties

1^o) $\chi(t) = 0$ for $t \leqslant \varkappa$; $\chi(t) = 1$ for $t \geqslant 1-\varkappa$; $\chi(t) = t$ for $2\varkappa \leq t \leq 1-2\varkappa$.

2^o) $\chi' \geqslant 0$; for $t \in [\varkappa, 2\varkappa]$ we have $\chi(t) \leq t$ and for $t \in [1-2\varkappa, 1-\varkappa]$ we have $\chi(t) \geqslant t$.

We choose $\varkappa$ so small, that $2\varkappa < \frac{2}{3} < 1-2\varkappa$. In this case the only solution of the equation $\chi(t) = 2/3$ is $t = 2/3$.

Now we define an isotopy H_t of $W = [0, 3] \times D^r_\delta(0) \times \mathcal{D}^s_\delta(0)$ by

$$H_t(\tau, \vec{x}, \vec{y}) = (\tau + \nu(|\vec{x}|^2)\cdot\nu(|\vec{y}|^2)\cdot\chi(t)\cdot(\theta(\tau)-\tau), \vec{x}, \vec{y}).$$

First of all we check up that for each t this map is really the homeomorphism $W \to W$. Continuity is obvious. Injectivity: if $H_t(\tau, \vec{x}, \vec{y}) = H_t(\tau', \vec{x}', \vec{y}')$, then $\vec{x} = \vec{x}'$, $\vec{y} = \vec{y}'$ and $\tau + \alpha\cdot(\theta(\tau) - \tau) = \tau' + \alpha\cdot(\theta(\tau') - \tau')$, where $\alpha = \nu(|\vec{x}|^2)\cdot\nu(|\vec{y}|^2)\cdot\chi(t)$, $0 \leqslant \alpha \leqslant 1$. Suppose $\tau > \tau'$. We have $\alpha(\theta(\tau) - \theta(\tau')) + (1-\alpha)(\tau - \tau') = 0$. But θ and id are strictly increasing, hence that is impossible. To find $(\tau_o, \vec{x}_o, \vec{y}_o)$, such that $H_t(\tau_o, \vec{x}_o, \vec{y}_o)$ equals the given triplet $(\tau, \vec{x}, \vec{y})$, we set $\vec{x}_o = \vec{x}$, $\vec{y}_o = \vec{y}$ and seek τ_o, such that $\tau_o + \alpha(\theta(\tau_o) - \tau_o) = \tau$, where $\alpha = \chi(t)\cdot\nu(|\vec{x}|^2)\cdot\nu(|\vec{y}|^2)$, $0 \le \alpha \le 1$. The function $\alpha\theta(x) + (1-\alpha)x$ satisfies 1')-3') above, hence is a bijection, therefore such τ_o exists.

Note that $H_t \ne id$ only in the compact, belonging to $W_o = (0, 3) \times B^r_o(0) \times B^s_o(0)$ it preserves W_o and is smooth on W_o. Therefore it can be expanded to the isotopy h_t of V, satisfying the properties 0), 1). *Note also:* $H_t \ne id \Rightarrow t \in [\varkappa, 1-\varkappa]$.

Now we prove 2). The manifold N_h is by definition the set of pairs $(h_t(\alpha), t)$, $\alpha \in N$. $L \times [0, 1]$ is by definition, the set of pairs (ℓ, t), $\ell \in L$. The intersection is the set of pairs $(h_t(\alpha), t)$, where $\alpha \in N$ and $h_t(\alpha) \in L$. For that it is necessary, that $h_t(\alpha) \ne \alpha$, therefore $\alpha \in W_o$. The isotopy h_t preserves W_o, hence it is enough to consider the Φ-preimage. α has the coordinates $(1, \vec{x}_o, 0)$; $h_t(\alpha)$ has the coordinates $(1 + \nu(|\vec{x}_o|^2)\cdot t\cdot(\theta(1) - 1), \vec{x}_o, 0)$. If this point belongs to $L \cap \operatorname{Im}\Phi$, then by p.1) of lemma 7.6 we have $\vec{x}_o = 0$ and $1 + \chi(t)\cdot(\theta(1) - 1) = 2$; $t = 2/3$. Thus, the point $\Phi(2, 0, 0)$ is the only point of intersection of $h_t(N)$, $t \in [0, 1]$ with L.

Now we prove transversality. The tangent space of N_h at a point $(h_t(\alpha), t)$ where $t \in (0, 1)$ is a direct sum of $((h_t)_*(T_\alpha N), 0)$ and a 1-dimensional space, generated by $(\frac{\partial}{\partial t} h_t(\alpha), 1)$. The tangent space to $L \times (0, 1)$ at a point (ℓ, t) is a direct sum of $(T L, 0)$ and a space, generated by $(0, 1)$. Note that $\dim N_h + \dim\big(L \times [0, 1]\big) = r + s + 2 = m + 1 = \dim\big(V \times [0, 1]\big)$, hence to prove transversality it suffices to show that these tangent spaces have trivial intersection. Suppose that $(A, 0) + \lambda(\frac{\partial}{\partial t} h_t(\alpha), 1) = (B, 0) + \mu(0, 1)$, where $A \in (h_t)_*(T_\alpha N)$, $B \in T_{\mathfrak{x}} L$. Then $\lambda = \mu$ and $A + \lambda \frac{\partial}{\partial t} h_t(\alpha) = B$. To calculate the intersection of $((h_t)_* T_\alpha N) + \{\frac{\partial}{\partial t} h_t(\alpha)\}$ with $T_{\mathfrak{x}} L$ we pass to the inverse image of Φ. Recall that $\alpha = \Phi(1, 0, 0)$. In this point $T_\alpha N$ is exactly given by the equation $(x_0 = 0, x_{r+1} = \ldots = x_m = 0)$ and in the small neighbourhood of the point h_t is given by $(\tau, \vec{x}, \vec{y}) \mapsto (\tau + \chi(t)\cdot(\theta(\tau) - \tau), \vec{x}, \vec{y})$, which implies that the $(h_t)_* T_\alpha N$ is again given by the equations $(x_0 = 0, x_{r+1} = \ldots = x_m = 0)$ and the vector $\frac{\partial}{\partial t} h_t(\alpha) = (\chi'(t)\cdot(\theta(\tau) - \tau), 0, 0)$.

But we know from 7.6. 1) that $T_{\mathfrak{x}} L$ is given by the equations $(x_0 = 0, x_1 = 0, \ldots, x_r = 0)$, which implies that $B = 0$ and $A = 0$, $\lambda \frac{\partial}{\partial t} h_t(\alpha) = 0$. Note now that $\frac{\partial}{\partial t} h_t(\alpha)$ in the point of intersection equals $(\chi'(t)(\theta(\tau) - \tau), 0, 0)$, where $t = 2/3$, $\tau = 1$, therefore $\frac{\partial}{\partial t} h_t(\alpha) \neq 0$, hence $\lambda = 0$, $\mu = 0$ and $T_* N_h \cap T_*(L \times [0, 1]) = \{0\}$.

Now we calculate the sign. We take the vectors $\xi_1(1), \ldots, \xi_r(1)$ at α (see lemma 7.7), which form the positive base of $T_\alpha N$. Then, by convention, the vectors $\left(\frac{\partial}{\partial t}(h_t(\alpha)), 1\right)$; $(h_t)_*(\xi_1(1), 0); \ldots ; (h_t)_*(\xi_r(1), 0))$ form the positive base of N_h at $(h_t(\alpha), t)$. We must calculate the sign of this reper in $T_{(h_t(\alpha), t)}(V \times [0, 1]) / T_{(h_t(\alpha), t)}(L \times [0, 1])$. This space is isomorphic with positive orientation to $T_{h_t(\alpha)} V / T_{h_t(\alpha)} L$.

The image of our vectors under that isomorphism is just $(\frac{\partial}{\partial t}(h_t(\alpha)), (h_t)_*(\xi_1(1)), \ldots, (h_t)_*(\xi_r(1))$. By definition of the map Φ the vectors $\xi_i(\theta)$ are for every $\theta \in [0, 3]$ the tangent vectors to coordinates x_i, and we have already seen that they are invariant under h_t (recall that $t = 2/3$). The coordinates of $\frac{\partial}{\partial t}(h_t(\alpha))$ are $(\chi'(t)(\theta(\tau) - \tau), 0,\dots, 0) = (1,5; 0,\dots, 0)$, hence $\frac{\partial}{\partial t}(h_t(\alpha)) = 1,5\cdot\dot{\gamma}(2)$. Therefore our base have the same sign in $T_{h_t(\alpha)}V/T_{h_t(\alpha)}L$ as $(\dot{\gamma}(2), \xi_1(2),\dots, \xi_r(2))$ which is, by definition, ρ. The last conclusion of p.2) of lemma 7.8 is trivial.

Now we describe the modification of the vector field v, which leads to the change of base.

Lemma 7.9. Let $\theta = (f, \bar{f}, v, E)$ be an r-quadruple, belonging to a regular class $\xi \in H^1(M, \mathbb{Z})$, where f is indexing and v perfect. Assume that there exists a regular value c of $\bar{f}$, such that the projection $\bar{E}$ of E belongs to $W_0 = \bar{f}^{-1}([c-1, c])$. Let $N \geq 1$ be an integer; denote $f^{-1}([c-N, c])$ by W. Denote by $p_1,\dots,p_k \in E$ all the generators of index m. Let $1 \leq i,j \leq k$, and $\rho = \pm 1$. Let $g \in G$ and $\xi(g) = -\ell$, where $\ell > 0$.

Then there exists a new perfect gradient-like field w for f, such that:

1) There exists a function $\varphi: W \to \mathbb{R}$, such that it satisfies the condition (E) both for v and w.

2) The base elements $q(v)$ and $q(w)$ in the associated complex $F_*^{(gr)}(\varphi)$ are the same for $q \in E$, $q \neq p_i$. For $q = p_i$ we have $p_i(w) = (p_i(v) + \rho\cdot p_j(v)\cdot g) \bmod(\ell + 1)$. *)

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*) Note that in the notation $p_i(v)$ the tilda $\sim$ is absent, because by definition the p_i is a lifting of $\bar{p}_i = Q(p_i) \in \bar{M}$ to the covering $\tilde{M}$.

Proof. We can assume $\ell \leq N-1$, otherwise we can set $w = v$. Denote the critical value of $\bar{f}\,|W_o$ of index $(m+1)$ by a, and the critical value of index m by a'; $a > a'$. Choose the regular values b, b', such that $a' < b' < b < a$. Consider the vector field v_o on W_o, which is a result of multiplication of v by a positive function ξ, such that $\mathrm{supp}(\xi - 1)$ is contained in $\bar{f}^{-1}((a', a))$ and that $d\bar{f}(v_o) = 1$ in $\bar{f}^{-1}([b', \ell])$.

Expand v_o to a t-invariant vector field on $\bar{M}$. Since $D(p, \pm v_o)$ are the same as $D(p, \pm v)$, v_o is a perfect gradient-like vector field for $\bar{f}$.

Consider the class Γ of homotopy of paths, joining p_i and $\bar{p}_j t^{+\ell}$, such that the lifting of any curve of Γ to $\widetilde{M}$ joins p_i and $p_j g$. (This exists, since $\xi(g) = -\ell$).

Apply now the lemma 7.5 to the r-quadruple $(f, \bar{f}, v_o, E)$, critical points $\bar{p}_i$ and $\bar{p}_j t^{\ell}$, class of homotopy of paths Γ, and a regular value $b-\ell$. We get a path $\gamma : [0, 3] \to \bar{f}^{-1}(b-\ell)$, satisfying 1)-3) of this lemma. In particular, this γ is a normal path for v and belongs to a regular level $(b-\ell)$ of $\bar{f}$. We denote $\bar{f}^{-1}(b-\ell)$ by V.

Apply now the lemma 7.4 to the r-quadruple $(f, \bar{f}, v_o, E)$, pair of critical points $(\bar{p}_i, \bar{p}_j t^{\ell})$ and the path γ. We get a function $\varphi : W \to \mathbb{R}$, satisfying (E) and a neighbourhood U of γ in W, such that $d\varphi(v_o)$ is constant in U. Choose the open neighbourhood A of γ in $\bar{f}^{-1}(b-\ell)$ and a number $\varepsilon > 0$ so small that $\bar{A} \subset U$ and every $(-v_o)$-trajectory, starting at $\bar{A}$ rests in U for the values of parameter $\in [0, \varepsilon]$.

One more notation: if Y is a manifold and v - a vector field on Y and $[\alpha, \beta]$ - the interval in $\mathbb{R}$ and X is a subset in Y we denote by $X([\alpha, \beta], v)$ the set of all the points in Y, belonging to some v-trajectory, starting at some point of X and evaluated at some $\tau \in [\alpha, \beta]$ If $\alpha = \beta$, we use the notation $X(\alpha, v)$. The similar notation $X((\alpha, \beta), v)$ for the open interval (α, β) will also be used.

We choose $\varepsilon > 0$ to be so small that $V([0,\varepsilon], -v_0)$ is contained in $\bar{f}^{-1}([b'-\ell, b-\ell])$, which implies in particular, that $d\bar{f}(v_0) = 1$ on $V([0,\varepsilon], -v_0)$. On the set $\bar{A}([0,\varepsilon], -v_0)$ the function $d\varphi(v_0)$ is also constant, say $C > 0$.

The curve γ intersects no descending disc, except $D(\bar{p}_i, v_0)$. Consider the union K_0 of all descending discs, starting from the critical points of index $\leq m$, belonging to $\bar{f}^{-1}([c-N, c+N])$, except $\bar{p}_i$. Let $K = K_0 \cap \bar{f}^{-1}([c-N, c+N])$. Since v_0 is perfect, K is compact by the lemma 3.7. This implies that we can choose the neighbourhood A of γ so small that $\bar{A} \cap K = \emptyset$, which implies that $\bar{A} \cap D(\bar{p}_s t^{\nu}, v_0) = \emptyset$, $s \neq i$, $-N \leq \nu \leq N-1$, therefore $At^{\nu} \cap D(\bar{p}_s, v_0) = \emptyset$ for $s \neq i$, $-(N-1) \leq \nu \leq N$. Therefore, every t-shift of A, which is contained in W, does not intersect $D(\bar{p}_s, v_0)$ for $s \neq i$. That implies of course that no t-shift of $\bar{A}([0,\varepsilon], -v_0)$, contained in W, intersects $D(\bar{p}_s, v_0)$ for $s \neq i$. Furthermore, we can assume that $\bar{A} \cap D(\bar{p}_i t^{\nu}, v_0) = \emptyset$ for $0 < \nu \leq N-1$. Hence no negative t-shift of $\bar{A}$, i.e. $\bar{A}t^{\nu}$, $\nu < 0$, intersects $D(\bar{p}_i, v_0)$, and, of course, this implies that $\bar{A}([0,\varepsilon], -v_0) \cdot t^{\nu} \cap$ $\cap D(p_i, v_0) = \emptyset$.

Now we fix A and ε, satisfying the properties above and proceed as to construct the new vector field w.

Consider two submanifolds of V, namely $N \overset{m-1}{=} D(\bar{p}_i, v_0) \cap V$

and $L^{n-m-1} = \bigcup_{s=1}^{k} D(\bar{p}_s t^{\ell}, -v_o)$. The first is (generally) non-compact, the second is the union of k spheres. They do not intersect, since v_o is perfect; moreover the closure of the first does not intersect the second, because the closure of $D(p_i, v_o)$ in W is the subset of the union of the descending discs of indices $\leq m$ and that intersect no ascending disc of index m. Our curve γ satisfies the conclusions of lemma 7.5, hence it intersects N^{m-1} in a single point $\gamma(1)$ and $T_{\gamma(1)}N \not\ni \dot{\gamma}(1)$. Since γ do not intersect any descending disc of index $\leq m-1$, the $\overline{N^{m-1}} \cap \gamma(1) = N^{m-1} \cap \gamma_1$. The curve γ intersect L in a single point $\gamma(2)$, belonging to a disc $D(\bar{p}_j t^{\ell}, -v_o)$ and $\dot{\gamma}(2) \notin T_{\gamma(2)}L$.

Now we apply lemma 7.6 to the pair N^{m-1}, L^{n-m-1} of submanifolds of V^{n-1}, the path γ, the neighbourhood A and the sign ρ. We get the corresponding imbedding $\Phi: [0, 3] \times D_{\delta}^{m-1}(0) \times D_{\delta}^{n-m-1}(0) \longrightarrow V$, $\operatorname{Im}\Phi \subset A$. Denote the domain $[0, 3] \times D_{\delta}^{m-1}(0) \times D_{\delta}^{n-m-1}(0)$ by X.

Consider the manifold $Z = V \times [0, \varepsilon]$, and the diffeomorphism $\Psi: Z \to \bar{f}^{-1}([b-\ell-\varepsilon, b-\ell])$, defined by the convention: the point $(q, u) \in Z$ is carried to a $(-v_o)$-trajectory, starting at q, and evaluated at u. The function $\bar{f}\circ\Psi$ is given by $(\bar{f}\circ\Psi)(q, u) = (b-\ell) - u$. The function $\varphi\circ\Psi$, restricted to $A \times [0, \varepsilon]$ is given by $(\varphi\circ\Psi)(q, u) = (\varphi\circ\Psi)(q, 0) - C\cdot u$. The vector field v_o is carried by Ψ^{-1} to the vector field $\xi = (0, -1)$. Note also that since $\varphi|\gamma = \text{const}$, the function $\varphi\circ\Psi$ is constant when restricted to (γ, t_o) for any $u_o \in [0, \varepsilon]$.

Now we consider the embedding $\Phi \times \mathrm{id}$ of $X \times [0, \varepsilon]$ into

$V \times [0,\varepsilon]$. For brevity we keep the notation Φ for $\Phi \times \mathrm{id}$. Consider the function $\eta = \varphi \circ \Psi \circ \Phi$, defined now on $X \times [0,\varepsilon] = [0, 3] \times D^{m-1}_{\delta}(0) \times D^{n-m-1}_{\delta}(0) \times [0,\varepsilon]$. Coordinates in this domain are, respectively, τ, $\vec{x}$, $\vec{y}$, u. Recall that $\eta(\tau, \vec{x}, \vec{y}, u) =$

$= \eta(\tau, \vec{x}, \vec{y}, 0) - C\cdot u$. The imbedding $t \mapsto (t, 0, 0, 0)$ of $[0, 3]$ to $X \times 0$ gives after composition with $\Psi \circ \Phi$ the curve γ, and we know that $\varphi|\gamma$ is constant. Therefore, $\frac{\partial\eta}{\partial\tau}(\tau, 0, 0, 0) = 0$, and therefore $\frac{\partial\eta}{\partial\tau}(\tau, 0, 0, u) = 0$ for every τ, u. That implies that for some $\beta > 0$, $\beta < \delta$ we have $\left|\frac{\partial\eta}{\partial\tau}(\tau, \vec{x}, \vec{y}, u)\right| \leqslant \frac{C\cdot\varepsilon}{6\cdot\sup\limits_{t\in[0,1]}|\chi'(t)|}$ for $|\vec{x}| \leqslant \beta$, $|\vec{y}| \leqslant \beta$, where $\chi'(t)$ is the auxiliary function from the proof of lemma 7.8. In other words, for every point $(\tau, \vec{x}, \vec{y}) \in [0, \beta] \times D^{m-1}_{\beta}(0) \times D^{n-m-1}_{\beta}(0)$ and any $u \in [0,\varepsilon]$ we have $|(d\eta)(1, 0, 0, 0)| \leqslant \frac{C\cdot\varepsilon}{6\cdot\sup\limits_{t\in[0,1]}|\chi'(t)|}$.

Now we apply the lemma 7.8 and get the isotopy h_t of V to itself, where $t \in [0, 1]$, the isotopy differs from the identity only in $\Phi([0, 3] \times D^{m-1}_{\beta}(0) \times D^{n-m-1}_{\beta}(0))$. Denote by s_u, $u \in [0,\varepsilon]$ the map $h_{(u/\varepsilon)}$; then s_u is the isotopy defined on the segment $[0,\varepsilon]$. We introduce a new vector field w_0 on $Z = V \times [0,\varepsilon]$, setting $w_0 = (-\frac{\partial}{\partial u}(s_u(q)), -1)$. Since the isotopy s_u differs from id only for $u \in [\varkappa\varepsilon, (1-\varkappa)\varepsilon]$, this vector field equals ξ in the neighbourhood of the boundaries $V \times 0 \cup V \times \varepsilon$. Note that w_0 is a gradient-like vector field for $\bar{f}\circ\Psi$, since $d(\bar{f}\circ\Psi)$ vanishes on the vectors $(\mu, 0)$, hence $d(\bar{f}\circ\Psi)(w_0) = d(\bar{f}\circ\Psi)(v_0)$.

I claim that this vector field is also a gradient-like vector field for $\eta = \varphi \circ \Psi \circ \Phi$. Indeed, we are to verify that only in $\Phi([0, 3] \times D_\beta^{m-1}(0) \times D_\beta^{n-m-1}(0) \times [0,\varepsilon])$ because in all the other points $w_o = \xi$, and $\xi = (0, \vec{0}, \vec{0}, 1)$ is a gradient-like field for η. For that we need to calculate $\frac{\partial s_u}{\partial u}$ explicitely. By definition $s_u(\tau, \vec{x}, \vec{y}) = h_{u/\varepsilon}(\tau, \vec{x}, \vec{y}) =$ $= (\tau + \vartheta(|\vec{x}|^2) \cdot \vartheta(|\vec{y}|^2) \cdot \chi(u/\varepsilon) \cdot (\theta(\tau) - \tau), \vec{x}, \vec{y})$. Therefore $\frac{\partial s_u}{\partial u}(\tau, \vec{x}, \vec{y}) = (\frac{1}{\varepsilon}\chi'(u/\varepsilon) \cdot \vartheta(|\vec{x}|^2) \cdot \vartheta(|\vec{y}|^2) \cdot (\theta(\tau) - \tau), 0, 0) =$

$$= R(\tau, \vec{x}, \vec{y}) \cdot (1, \vec{0}, \vec{0}), \text{ where } |R(\tau, \vec{x}, \vec{y})| \leq$$

$\leq \frac{3}{\varepsilon} \sup_{v \in [0,1]} |\chi'(v)|$. Now the derivative of η along w_o in the point $(\tau, \vec{x}, \vec{y}, u)$ equals $(d\eta)(-\frac{\partial}{\partial u}(s_u(q)), 0) + (d\eta)(0,-1)$. The second term equals C, the first is $R(\tau, \vec{x}, \vec{y}) \cdot d\eta(1, 0, 0, 0)$.

By the above the module of this vector is less than $\frac{C}{2}$. Therefore $(d\varphi)(\eta) \geqslant \frac{C}{2} > 0$.

Now we glue the vector field $\Psi(w_o)$ on $\bar{f}^{-1}([b - \ell - \varepsilon, b - \ell])$ with the vector field v_o in $W_o t^\ell \setminus \bar{f}^{-1}([b - \ell - \varepsilon, b - \ell])$, which is possible, since $\Psi(w_o) = v_o$ in the neighbourhoods of $\bar{f}^{-1}(b - \ell - \varepsilon)$ and $\bar{f}^{-1}(b - \ell)$. Expand that t-invariantly to M and denote the result by w_1.

I claim that the vector field w_1 satisfies the following properties:

1°) w_1 is a gradient-like vector field for $\bar{f}$ and for φ.

2°) $w_1 | W$ is almost good.

3°) The descending discs $D(\bar{p}_s, w_1) \cap W$ and $D(\bar{p}_s, v_o) \cap W$ coincide for $s \neq i$.

4°) The descending disc $D(\bar{p}_i, w_1) \cap \bar{f}^{-1}([b - \ell, c])$ equals $D(\bar{p}_i, v_o) \cap \bar{f}^{-1}([b - \ell, c])$.

5°) The descending disc $D(\bar{p}_i, w_1)$ does not intersect the discs $D(\bar{p}_s t, -v_o)$ for $s \neq j$, and it intersects $D(\bar{p}_j t^{\ell}, -v_o)$ transversally at a single point q, belonging to $\bar{f}^{-1}([b-\ell-\varepsilon, b-\ell])$.

The sign of intersection is ρ and the path joining $\bar{p}_i$ and q along $(-w_1)$ and then q and $\bar{p}_j t^{\ell}$ along $(-v_o)$ belongs to Γ.

The point 1°) was just proved. For 2°) we just note that φ is an enumerating function on W. To prove 3°) we recall that the $W \cap \operatorname{supp}(v_o - w_1)$ is contained in the union of $\Phi(X \times [0, \varepsilon]) \cdot t^{\nu}$ for $-\ell \leq \nu \leq (N-1) - \ell$. But $\Phi(X \times [0, \varepsilon]) \subset \subset A([0, \varepsilon], -v_o)$, and by our choice of A the disc $D(\bar{p}_s, v_o)$ does not intersect $\bar{A}([0,\varepsilon], -v_o)$ for $s \neq i$. The same argument proves 4°).

To prove 5°) we note that the intersection $D(\bar{p}_i, w_1) \cap D(\bar{p}_s t^{\ell}, -v_o)$ is concentrated in $\bar{f}^{-1}([a'-\ell, b-\ell])$, because $D(\bar{p}_i, w_1) \cap \bar{f}^{-1}([b-\ell, c]) = D(\bar{p}_i, v_o) \cap \bar{f}^{-1}([b-\ell, c])$, which does not intersect $D(\bar{p}_s t^{\ell}, -v_o)$, since v_o is perfect. We study first the intersection $D(\bar{p}_i, w_1) \cap D(\bar{p}_s t^{\ell}, -v_o) \cap \cap \bar{f}^{-1}([b-\ell-\varepsilon, b-\ell])$. Consider Ψ- preimage of this intersection. The Ψ- preimage of $D(\bar{p}_i, w_1)$ is the union of trajectories of $(-w_o)$, starting at $D(\bar{p}_i, w_1) \cap \bar{f}^{-1}(b-\ell) = = D(\bar{p}_i, v_o) \cap V = N^{m-1}$. The trajectory of $(-w_o)$ is just the path $\gamma(u) = (s_u(x), u)$, where $x \in N$, $u \in [0, \varepsilon]$, The Ψ-preimage of $D(\bar{p}_s t^{\ell}, -v_o)$ is just the set $[D(\bar{p}_s t^{\ell}, -v_o) \cap V] \times$ $\times [0, \varepsilon]$. Consider now the diffeomorphism $\Delta : V \times [0, 1] \to$ $\to V \times [0, \varepsilon]$, $\Delta = \mathrm{id} \times (\frac{1}{\varepsilon})$. The Δ-preimage of set of points $(h_u(x), u)$, where $u \in N$, is the set N_h in the notations of lemma 7.8. The Δ-preimage of $\bigcup_{s=1}^{k} [D(\bar{p}_s t^{\ell}, -v_o) \cap V] \times [0, \varepsilon]$ is just $L \times [0, 1]$. We know from lemma 7.8 that they intersect transversally in the single point $\gamma(2) \times 2/3$ with the sign ρ. This implies (after applying Δ) that $N_s \cap (L \times [0, \varepsilon])$ consists of a single point, $\gamma(2) \times \frac{2\varepsilon}{3}$ and the intersection is transversal of the sign ρ. Now we note that the $(-w_o)$-trajectory, finishing at $\gamma(2) \times \frac{2\varepsilon}{3}$ is just the trajectory $(s_u(\gamma(1)), u)$ where $u \in [0, \frac{2\varepsilon}{3}]$. To evaluate its homotopy class we note that it is homotopic to a trajectory, which is a composition of $(\gamma | [1, 2], 0)$ and a $[0, \tau] | \tau \in [0, \frac{2\varepsilon}{3}]$. Thus the path which goes along w_1 from $\bar{p}_i$ until the intersection with $D(\bar{p}_j t^{\ell}, -v_o)$ and afterwards along v_o to $\bar{p}_j$ is homotopic to Γ by our choice of γ.

Now we note that since $D(\bar{p}_i, w_1) \cap \bar{f}^{-1}(b - \ell - \varepsilon)$ does not intersect with $D(\bar{p}_s, -v_o)$, the $D(\bar{p}_i, w_1)$ does not intersect $D(\bar{p}_s, -v_o)$ also in the domain $\bar{f}^{-1}(a' - \ell, b - \ell - \varepsilon)$, since in this domain $w_1 = v_o$.

The properties $1^o)$-$5^o)$ are proved.

Next we want to verify the property 2) for w_1 instead of w. For the elements $q(w_1)$ where $q \neq p_i$ this property is clear. Indeed, recall that $\mathrm{supp}(v_o - w_1) \cap W$ is concentrated in the union of t-shifts of U, belonging to W. But this union belongs to $\varphi^{-1}([a_m, a_{m+1}])$, by the property 3) of lemma 7.4.

This implies that $w_1 = v$ in the domain $\varphi^{-1}([a_s, a_{s+1}])$ for every $s \neq m$, therefore $q(v_0) = q(w_1)$ for $\operatorname{ind} q \neq m$. If $\operatorname{ind} q = m$, $q \neq p_i$, we have by property 3^o) that $D(\bar{q}, w_1) \cap W = D(\bar{q}, v_0) \cap W$, hence $q(v_0) = q(w_1)$ also for $\operatorname{ind} q = m$, $q = p_i$. Note that for all critical points q we have $D(q, v_0) = D(q, v)$, so that the equality $q(v_0) = q(w_1)$ implies always $q(v) = q(w_1)$.

So we are only to check the conclusion 2) for the disc $p_i(w_1)$. That is the consequence of Poincare duality and the properties 3^o)-5^o). The simplest way to do this is to calculate intersection indices. We recall here the definition and the basic properties of these indices for the simplest situation. Let X^n be a connected compact manifold, ∂X is a disjoint union of V_0 and V_1. Assume that V_0 is connected and fix a point $q \in V_0$. Let $L^\ell \subset X$ be a compact simply connected submanifold, $L \cap V_0 = \emptyset$, $\partial L \subset V_1$. Assume that L is cooriented. Fix some point $p \in L$ and a path ξ, joining q and p.

Suppose that $\varphi : (D^k, S^{k-1}, x_0) \to (X^n, V_0, q)$ is a smooth map, $\operatorname{Im} \varphi \cap V_1 = \emptyset$. Assume that $k + \ell = n$. We can perturb φ a little, keeping $\varphi | S^{k-1}$ the same, to get φ_0, transversal to L. For each point z_0 of intersection $\varphi(D^k) \cap L$ we choose a path γ_0 in D^k, joining x_0 and z and a path γ_1 in L, joining z and p. The composition $\xi \circ \gamma_1 \circ \gamma_0$ determines a class in $\pi_1(X, q)$ which we denote by $g(z)$. Since D^k is oriented, each point of intersection has its sign, denoted $\varepsilon(z)$. The sum of $\varepsilon(z) g(z)$ over all the intersection points is denoted $i(\varphi_0, L) \in \mathbb{Z}\pi_1 L$. The standard transversality arguments imply the following lemma.

Lemma 7.10. a) The index $i(\varphi_o, L)$ does not depend on particular choice of the perturbation φ_o, thus gives a correctly defined index $i(\varphi, L)$.

b) The index $i(\varphi, L)$ depend only on the homotopy class of φ in $\pi_k(X, V_o; q)$, and the map $i(\cdot, L): \pi_k(X,V_o;q) \to \to \mathbb{Z}\,\pi_1 X$ is a homomorphism.

c) $i(\cdot, L)$ is linear with respect to the right action of $\pi_1(V_o)$ on $\pi_k(X, V_o; q)$, i.e. $i(\varphi\cdot g, L) = j(g)\cdot i(\varphi, L)$, where $j: \pi_1(V_o) \to \pi_1(X)$ - the homomorphism, induced by embedding $V_o \subset X$.

Suppose now, that inclusion $V_o \subset X$ induces an isomorphism $\pi_1 V_o \to \pi_1 X = H = \ker(\xi : \pi_1 M \to \mathbb{Z})$. Fix some point $q \in V_o$, and fix some lifting $\tilde{q}$ of q to $\tilde{V}_o \subset \tilde{X}$. Suppose that $r \geqslant 2$ and $\psi:(D^r, S^{r-1}) \to (X, V_o)$ is some map, which do not in general preserve basepoints, but for which the lifting $\tilde{\psi}$ of ψ to the universal covering $\tilde{X} \xrightarrow{H} X$ is fixed. Adding to ψ "the tail", starting from q and finishing at $x_o \in S^{r-1}$ we get an element in $\pi_r(X, V_o)$ and we can change it in order to get an element $\hat{\psi}$ in $\pi_r(X, V_o)$ which after lifting to $\tilde{X}$ coincides with $\tilde{\psi}$ up to free homotopy of maps $(D^r, S^{r-1}) \to (\tilde{X}, \tilde{V}_o)$. In other words, adding the right tail to ψ, we get an element $\hat{\psi}$, such that $H(\hat{\psi}) = \tilde{\psi}$, where $H: \pi_r(\tilde{X}, \tilde{V}_o) \to H_r(\tilde{X}, \tilde{V}_o)$ is the Hurewicz homomorphism.

Now we set $X = \varphi^{-1}([a_m, a_{m+1}])$, $V_o = \varphi^{-1}(a_m)$, $V_1 = \varphi^{-1}(a_{m+1})$. Recall that since X is obtained from V_o by attaching m-cells, $m \geqslant 2$, the Hurewicz homomorphism

$H: \pi_m(X, V_0) = \pi_m(\tilde{X}, \tilde{V}_0) \longrightarrow H_m(\tilde{X}, \tilde{V}_0)$ is an isomorphism, which commutes with the right action of $\mathbb{Z}H$, and the module $H_m(\tilde{X}, \tilde{V}_0)$ is a free right ZH-module. The base elements are enumerated by pair of integers (s, r) where $1 \leqslant s \leqslant k$, $1 \leqslant r \leqslant N$ and to each pair s, r corresponds the pair $(D(\bar{p}_s t^r; v) \cap \varphi^{-1}([a_m, a_{m+1}], D(\bar{p}_s t^r; v) \cap \varphi^{-1}(a_m))$ which is lifted to $\tilde{X} = \tilde{W}$ in such a way that $\bar{p}_s t^r$ lifts to $p_s\theta^r$. We denote these elements by $e(s, r)$. Choose and fix some point $q_0 \in V_0$ and the lifting $\tilde{q}_0 \in \tilde{V}_0$. Adding tails, we get the base $\hat{e}(s, r)$ in $\pi_m(X, V_0)$, such that $H(\hat{e}(s, r)) = e(s, r)$. In our older notations $e(s, r) = p_s(v)\theta^r$. Recall that the monoid G^- acts on the set of $e(s, r)$ in the following way: if $g \in G^-$, $g = \theta^\nu h$, $\nu \geqslant 0$, then $e(s, r)g = e(s, r+\nu)h$. In particular $e(s, r) = e(s, 0)\theta^r$. For each pair s, r we consider also the manifold $L(s, r) = D(\bar{p}_s t^r, -v) \cap \varphi^{-1}([a_m, a_{m+1}])$. That is a manifold of dimension $n-m$, the boundary $\partial L(s, r) = D(\bar{p}_s t^r, -v) \cap \varphi^{-1}(a_{m+1}) \subset V_1$ and the pair $(L, \partial L)$ is diffeomorphic to (D^{n-m}, S^{n-m-1}). We fix some point $p(s, r) \in \partial L(s, r)$ and choose a path $\xi(s, r)$, joining $p(s, r)$ and q_0 in such a way, that if λ is an arbitrary path in $L(s, r)$, joining $p(s, r)$ and $\bar{p}_s t^r$, then the composition $\lambda \circ \xi^{-1}(s, r)$, lifted to X as to start at q_0, finishes at $p_s\theta^r$. Recall that the orientation of $D(\bar{p}_s t^r; v)$ was fixed in the very beginning and the coorientation of $D(\bar{p}_s t^r, -v)$ is determined so as to give positive intersection with $D(\bar{p}_s t^r; v)$.

For these choices we have:

$$i(\hat{e}\ (s, r), L(s', r')) = \begin{cases} 1 & \text{if } (s, r) = (s', r') \\ 0 & \text{if } (s, r) \neq (s', r') \end{cases}$$

(Indeed, since v is perfect, the discs $D(\bar{p}_s t^r, +v)$ do not intersect with $D(\bar{p}_{s'} t^{r'}, -v)$ for $(s, r) \neq (s', r')$. For $(s, r) = (s', r')$ the intersection $D(\bar{p}_s t^r, +v) \cap D(\bar{p}_s t^r, -v) = \{\bar{p}_s t^r\}$ and our choices imply that the coefficient of intersection is $= 1$).

Now we consider the elements $f_s = p_s(w_1)$ in $H_m(\ ^{-1}([a_m, a_{m+1}])^\sim, \varphi^{-1}(a_m)^\sim\ ) = H_m(\tilde{X}, \tilde{V}_o)$. This element is represented by an embedding of pair $(D(\bar{p}_s, w_1) \cap X,\ D(\bar{p}_s, w_1) \cap V_o)$, lifted to $\tilde{X}, \tilde{V}_o$ in such a way, that $\bar{p}_s$ lifts to p_s. We can add a tail and get the element $\hat{f}_s$ in $\pi_m(X, V_o)$ such that $H(\hat{f}_s) = f_s$.

We calculate now $i(\hat{f}_i, L(s, r))$. The disc $D(\bar{p}_i, w_1) \cap \bar{f}^{-1}([b-\ell, c])$ equals $D(\bar{p}_i, v_o) \cap \bar{f}^{-1}([b-\ell, c])$. That implies $i(\hat{f}_i, L(i, 0)) = 1$ and $i(\hat{f}_i, L(s, r)) = 0$ for $0 < r < \ell$ and for $r = 0$, $s \neq i$. From the p.5°) we have also $i(\hat{f}_i, L(s, r)) = 0$ for $r = \ell$ and $s \neq j$. Now we calculate $i(\hat{f}_i, L(j, \ell))$. There is one point of intersection q of the sign ρ. To calculate the element in H we consider the path $\Delta \cdot \gamma \cdot E$ where E is a path in $\operatorname{Im} \hat{f}_i$, joining q_o and $\bar{p}_i$, the lifting $\tilde{E}$ to $\tilde{X}$ joins $\tilde{q}_o$ and p_i; γ is a path from $\bar{p}_i$ to q along $(-w_1)$ and then from q to $\bar{p}_j t^\ell$ along $(-v_o)$. By property 5°) it belongs to Γ, that is, being lifted to $\tilde{X}$, it joins p_i and $p_j g$. The last one, Δ is just $\xi(j, \ell) \cdot \lambda^{-1}$, as defined above, thus, lifted to X, it joins $p_j \cdot \theta^\ell$ and $\tilde{q}_o$. Thus the homotopy class of $\Delta \cdot \gamma \cdot E$ is $(\theta^{-\ell} \cdot g) \in H$.

That suffices to decompose $\hat{f}_i$ in terms of the base $\hat{e}(s, r)$. Indeed, let $\hat{f}_i = \sum_{s,r} \hat{e}(s, r) A_{s,r}$ where $A_{s,r} \in \mathbb{Z}H$. Then $i(\hat{f}_i, L(s_0, r_0)) = \sum_{s,r} i\, (\hat{e}(s, r)A_{s,r}, L(s_0, r_0)) =$

$= \sum_{s,r} A_{s,r}\, i(\hat{e}(s, r), L(s_0, r_0)) = A_{s_0, r_0}$. We have just calculated the indices $i(\hat{f}_i, L(s, r))$ for all (s, r) with $r \leqslant \ell$. They are zero except for $(i, 0)$ and (j, ℓ); for $(i, 0)$ this is 1, for (j, ℓ) this is $\rho\cdot(\theta^- \cdot g)$. That implies: $\hat{f}_i = \hat{e}(i, 0) + \hat{e}(j, \ell)\cdot(\rho\cdot\theta^-\cdot g) + \sum_{r \geqslant \ell+1} \hat{e}(s, r)\cdot A_{s,r}$. Applying the Hurewicz homomorphism we get $f_i = e(i, 0) +$ $+ \rho\cdot e(j, \ell)\cdot(\theta^- g) + \sum_{r \geqslant \ell+1} e(s, r) A_{s,r}$; $A_{s,r} \in \mathbb{Z}h$. Using the G^--action on the system of $e(r, s)$ we get the equality: $f_i = e(i, 0) + \rho\cdot e(j, 0)\cdot g + \sum_{r \geqslant \ell+1} e(s, 0)(\theta^r A_{s,r})$, which is exactly what we were seeking for.

To finish the proof we consider an arbitrary vector field w, which is perfect and sufficiently ose to w_1, so that w is a gradient-like field also for φ, f and so that $q(w) = q(w_1)$ for every $q \in W \cap \mathrm{Cr}\, \bar{f}$. Lemma 7.9 is proved.

<u>Corollary 7.10</u>. Let $(f, \bar{f}, v, E)$ be an r-quadruple, belonging to a regular class $\xi \in H^1(M, \mathbb{Z})$, where f is indexing and v perfect. Assume that there exists a regular value $c \in \mathbb{R}$ of f, such that the projection $\bar{E}$ of E to $\bar{M}$ belongs to the domain $W_0 = \bar{f}^{-1}([c-1, c])$. Let $p_1, \dots, p_k$ be all the elements of E of degree m. Let $N \geqslant 1$ be any natural number.

Then

1) If $a \in \Lambda_\xi^=$, supp $a \in \{x \in G \mid \xi(x) < 0\}$ and $1 \leq i, j \leq k$, then there exists a perfect gradient-like vector field w for f, such that the Novikov complex $C_*(w, E)$ is N-equivalent to $(i,j;\, a)\, C_*(v, E)$.

2) If $\lambda \in \Lambda_\xi^=$, $\lambda = 1 + \mu$, where supp $\mu \subset \{x \in G \mid \xi(x) < 0\}$ and $1 \leq i \leq k$, then there exists a perfect gradient-like vector field w for f, such that the Novikov complex $C_*(w, E)$ is N-equivalent to $(i;\, \lambda)\, C_*(v, E)$.

<u>Proof</u>. 1) Since $a \in \Lambda_\xi^=$ it suffices to construct w, such that $C_*^0(w, E)$ is N-equivalent to $(i,j;\, a)\, C_*^0(v, E)$.

We apply the lemma 7.9 several times to i, j and the components of a (increasing ℓ gradually) to get a perfect gradient-like vector field w for f and a function $\varphi : W \to \mathbb{R}$, such that φ satisfies (E) for v and for w, and such that the base elements $q(w)$ in the graded complex $F_*^{(gr)}(\varphi)$ (where $q \in E$) are the same as $q(v)$, if $q \neq p_i$ and the element $p_i(w)$ is equal to $p_i(v) + p_j(v) \cdot [a]^{N-1}$, where $[a]^{N-1}$ stands for the image in $\mathbb{Z}G_N^- = \Lambda_\xi^= / \Lambda_\xi^= \theta^N$ of a. That means, that the complex $F_*^{(gr)}(\varphi)$ with the base $q(w)$ is the result of the change of type $(i,j;\, [a]^{N-1})$, applied to the complex $F_*^{(gr)}(\varphi)$ with the base $q(v)$.

We recall from the beginning of this section that there are $\mathbb{Z}G_N^-$-isomorphisms, preserving the bases:

$$I(\varphi, v) : C_*^0(v, E) \underset{\Lambda_\xi^=}{\otimes} \mathbb{Z}\, G_N^- \longrightarrow (F_*^{(gr)}(\varphi),\ \{q(v)\})$$

$I(\varphi, w) : C^{\circ}_{*}(w, E) \underset{\Lambda^{=}_{\xi}}{\otimes} \mathbb{Z} G_N^{-} \longrightarrow (F^{(gr)}_{*}(\varphi), \{q(w)\})$.

The first of them provides us with the base preserving isomorphism $I'(\varphi, v) : [(i,j;\ a)\ C^{\circ}_{*}(v, E)] \underset{\Lambda^{=}_{\xi}}{\otimes} \mathbb{Z}G_N^{-} \longrightarrow$ $\rightarrow (i,j;\ [a]^{N-1}) (F^{(gr)}_{*}(\varphi), \{q(v)\})$. But the righthand side is just $(F^{(gr)}_{*}(\varphi), \{q(w)\})$, which implies that $C^{\circ}_{*}(w, E) \underset{\Lambda^{=}_{\xi}}{\otimes} \mathbb{Z}G_N^{-}$ and $[(i,j;\ a)\ C_{*}(v, E)] \underset{\Lambda^{=}_{\xi}}{\otimes} \mathbb{Z}G_N^{-}$ are isomorphic, preserving bases, which implies the conclusion 1) of our corollary.

The item 2) is proved similarly; we apply lemma 7.9 to $i = j$ and (successively) to the components of μ.

Now we want to cope also with system of generators E, which do not belong to the domain $f^{-1}([c-1, c])$ for any c. For that we need some notations and a simple lemma.

Let C_{*} be a free based complex of right modules over some ring R and let Λ be a function on the set of all free generators of C_{*} to $R^{*} = \{r \in R \mid r$ is invertible$\}$. By $\Lambda(C)_{*}$ we denote the based complex which is obtained from C_{*} by the change of bases: instead of base element e_i we consider $e_i \cdot \Lambda(e_i)$. The function $e_i \to (\Lambda(e_i))^{-1}$ will be denoted Λ^{-1}. (Note that the change of type 2 is a particular case of this operation, but we have reasons to consider it separately.)

Lemma 7.11. a) For any free based complex C_{*} of right modules over R we have $\Lambda^{-1}((i,j;\ a)(\Lambda(C_{*}))) =$ $= (i,j;\ \Lambda(e_j)\cdot a \cdot \Lambda^{-1}(e_j))\ (C_{*})$.

b) Under the same assumptions we have
$\Lambda^{-1}((i;\mu)(\Lambda(C_*))) = (i;\Lambda(e_i)\cdot\mu\cdot\Lambda^{-1}(e_i))\ (C_*)$.

c) Under the same assumptions denote by (f_k) the base in C_{m-1} and by (e_j) the base in C_m. Then the matrix element Δ_{kj} of the matrix Δ of differential $\delta\colon \Lambda(C)_m \to \Lambda(C)_{m-1}$ equals $\Lambda^{-1}(f_k)\cdot d_{kj}\cdot\Lambda(e_j)$, where d_{kj} is the matrix element of $d\colon C_m \to C_{m-1}$.

Proof. Obvious.

Lemma 7.12. Let C_*, D_* be free finite based complexes of right Λ_ξ^--modules. Suppose that for some $N \in \mathbb{Z}$ they are N-equivalent. Let the equivalence be given by $e_i \mapsto f_i$ where $\{e_i\}$, $\{f_i\}$ are bases of respectively C_*, D_*, $i \in I$.

Then 1) Let $\Lambda : I \to \{\varepsilon g \mid \varepsilon = \pm 1,\ g \in G\}$ be a function, such that $|\xi(\Lambda(i))| \leq k$ for every i. Then $\Lambda(C_*)$ and $\Lambda(D_*)$ are $(N-2k)$-equivalent.

2) Let e_i, e_j be the generators of C_* of the same dimension. Let $a \in \Lambda_\xi^-$ and assume that $\operatorname{supp} a \subset \{x \in G \mid \xi(x) \leq \leq k\}$, where $k \geqslant 0$. Then $C_*' = (i,j;\ a)C_*$ is $(N-k)$-equivalent to $D_*' = (i,j;\ a)D_*$.

3) Let e_i be a generator of C_* and $\lambda \in \Lambda_\xi^-$, $\lambda = 1+\mu$, $\operatorname{supp}\mu \subset \{x \in G \mid \xi(x) < 0\}$. Then $(i;\lambda)C_*$ is N-equivalent to $(i;\lambda)D_*$.

Proof. We prove, for example 2). The bijection between bases of C_*' and D_*' is naturally $e_k \mapsto f_k$ for $k \in I$, $k \neq i$ and $e_i + e_j a \to f_i + f_j a$. The matrix Δ_m of differential $C_*' \to C_{m-1}'$ is the same as matrix δ_m of differential

$C_m \to C_{m-1}$, except $m = \dim e_i$, $\dim e_i + 1$. For dimension $m = \dim e_i$ Δ_m is obtained from δ_m by adding the j-th row, multiplied by a from the right, to the i-th one. The same for the differential $D'_m \to D'_{m-1}$. Now our assertion follows from the fact that if $x, y \in \Lambda^-_{\xi,N}$, $\xi(a) \le k$, then $x + ya \in \Lambda^-_{\xi,N_0}$, where $N_0 = \min(N, N-k)$. Same in the dimension $\dim e_i + 1$.

The assertions 1) and 2) are proved similarly. In p.1) we use the p. c) from lemma 7.11.

Proof of the theorem 7.1. 1) Denote by A the integer $\max_{g \in \operatorname{supp} a}(\xi(g))$. By the lemma 4.2, applied to the critical points $\bar{p}_i$, $\bar{p}_j \cdot t^{-A-1}$, there exists an admissible modification $\varkappa = (g, \bar{g}, v, E)$, such that $\bar{g}(\bar{p}_i) > \bar{g}(\bar{p}_j \cdot t^{-A-1})$, which means $\bar{g}(\bar{p}_i) > \bar{g}(\bar{p}_j) + A + 1$. Consider any regular value c of $\bar{g}$, such that $\bar{p}_i \in \bar{g}^{-1}([c-1, c])$. Note that $c > \bar{g}(\bar{p}_i) > c-1 > \bar{g}(\bar{p}_j) + A$. We can apply a renumerating procedure to the cobordism $W_0 = \bar{g}^{-1}([c-1, c])$ and the function g to get an indexing function $h: W_0 \to [c-1, c]$ with the same gradient-like vector field v, and such that we still have $c > \bar{h}(\bar{p}_i) > c-1 > \bar{h}(\bar{p}_j) + A$.

Now for each element $q \in E$ we pick up an element $\Lambda(q) \in G$, such that $q \cdot \Lambda(q) \in \bar{h}^{-1}([c-1, c])$. The value $\xi(\Lambda(q))$ is determined by q. From the inequality above we deduce $\xi(\Lambda(p_j)) > A$, hence $\operatorname{supp}(\Lambda(p_j)^{-1} \cdot a) \in \{x \in G \mid \xi(x) < 0\}$.

Denote by K the number $\max_{q \in E} |\xi(\Lambda(q))|$. We choose element $\Lambda(p_i)$ to be 1. The new system of liftings $\{q \cdot \Lambda(q)\}$

will be denoted E' and the generator $q\cdot\Lambda(q)$ - by q'.

Now we apply the corollary 7.10 to the r-quadruple $(h, \bar{h}, v, E')$, the base elements $p_i', p_j' \in E'$, the element $a' = (\Lambda^{-1}(p_j)^{-1} a) \in \Lambda_{\xi}^{=}$, supp $a' \subset \{x \in G \mid \xi(x) < 0\}$, and the natural number $N+2K$. We get a perfect gradient-like vector field w for h, such that $C_*(w, E')$ is $(N+2K)$-equivalent to $(i,j;\ a')\ C_*(v, E')$.

Now by the remark 2.5 we know that $C_*(v, E') = \Lambda(C_*(v,E))$ and $C_*(w, E) = \Lambda^{-1}(C_*(w, E'))$. Thus, by lemma 7.11, d) the complex $C_*(w, E)$ is N-equivalent to $\Lambda^{-1}((i,j;a')(C_*(v,E'))=$

$$= \Lambda^{-1}((i,j;a')(\Lambda(C_*(v, E))) = (i,j;\Lambda(p_j)\cdot a'\cdot\Lambda^{-1}(p_i))\cdot$$

$\cdot(C_*(v, E)) = (i,j;\ a)(C_*(v, E))$.

Point 1) is proved.

Point 2) is done similarly, except that we do not need to seek for g and can apply the renumerating procedure directly to $\bar{f}$ and any regular value c of $\bar{f}$, such that $p_i \in \bar{f}^{-1}([c-1, c])$. Afterwards we find the elements $\Lambda(q) \in G$, such that the system of generators $\{q\Lambda(q)\} = E'$ projects to $\bar{M}$ to $\bar{f}^{-1}([c-1, c])$, $\Lambda(p_i) = 1$ and apply the corollary 7.10, 2) to $N + 2\sup_q |\xi(\Lambda(q))|$ and the element μ itself. Theorem 7.1 is proved.

8. The proof of the main theorem

Now we are only to combine the results from §6, §7 to get the proof of the main theorem.

It is convenient for us to distinguish the following 4 types of elementary simple homotopy equivalence. We fix a natural number $n \geqslant 6$. We say that a chain complex $A_0 \leftarrow \dots \leftarrow \leftarrow A_k \leftarrow \dots$ is restricted, if $A_i = 0$ for $i < 2$ and for $i > n-2$. In this section "complex" means "finite free based restricted chain complex of right Λ_ξ^--modules. We introduce the following notations for brevity: $G^- = \{g \in G \mid \xi(g) \leq 0\}$, $G^= = \{g \in G \mid \xi(g) < 0\}$, $(\pm G) = \{\varepsilon g \mid \varepsilon = \pm 1,\ g \in G\}$. For an element $a \in \Lambda_\xi^-$ we denote by $h(a)$ the $\max(0, \max_{g \in \operatorname{supp} a} \xi(g))$. For a function $\Lambda : X \longrightarrow (\pm G)$ we denote by $\|\Lambda\|$ the $\max_{x \in X} |\xi(\Lambda(x))|$. The term "r-quadruples" will mean "r-quadruple, belonging to the regular class $\xi \in H^1(M, \mathbb{Z})$. We say, that a complex D_* is obtained from a complex C_* by an elementary operation of type, respectively, 1°)-5°) if:

1°) $D_* = \Lambda(C_*)$, *) where Λ is a function on the set of generators of C_*, with values in the set $(\pm G)$.

2°) $D_* = (i,j;\ a)(C_*)$, where e_i, e_j are the different generators of C_* of the same degree $m, a \in \Lambda_\xi^-$.

3°) $D_* = (i;\ \lambda)(C_*)$, where e_i is a generator of C_*, $\lambda \in \Lambda_\xi^-$, $\lambda = 1 + \mu$, where $\operatorname{supp} \mu \subset G^=$.

4°) $C_* = D_* \oplus \Gamma_*^{(k)'}$, where $2 \leqslant k \leqslant n-3$.

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*) The sign = here stands for "isomorphic, preserving bases".

5°) $D_* = C_* \oplus \Gamma_*^{(k)}$, where $2 \leq k \leq n-3$.

By the theory of J.H.C. Whitehead (exposed in our §1) we know that if A_*, B_* are two finite free restricted based complexes of right modules over Λ_ξ^-, which are homotopy equivalent via a homotopy equivalence $\varphi: A_* \to B_*$, such that the torsion $\tau(\varphi)$ vanishes in $Wh(G, \xi) = K_1(\Lambda_\xi^-)/\, U_\xi^-$, where $U_\xi^- = \{\pm g + \lambda \mid g \in G,\ \lambda \in \Lambda_\xi^-,\ \operatorname{supp}\lambda \subset G^=\}$, then B_* can be obtained from A_* by the finite series of elementary operations of type 1)-4). (The only difference with our definition of elementary operation from 1 is that we presented the element $\pm g + \lambda$ as $\pm g(1 + g^{-1}\lambda')$ and decomposed the elementary operation 1) from §1 to a composition of 1°) and 3°).)

Lemma 8.1. Let $\theta = (f, \bar{f}, v, E)$ be a regular quadruple, belonging to a regular class $\xi \in H^1(M, \mathbb{Z})$. Assume that v is perfect. Let $N \geqslant 2$ be natural. Let e_i, $i \in I$, be the generators of $C_*(v, E)$, i.e. the points in Cr f.

1) Let $\Lambda : I \to (\pm G)$ be a function, $\|\Lambda\| = k$. Let C_* be a complex, (N+2k)-equivalent to $C_*(v, E)$. Then there exists a r-quadruple $(g, \bar{g}, w, E')$ with w perfect, such that $C_*(w, E')$ is N-equivalent to the result of 1°), applied to C_*.

2) Let e_i, e_j be the different generators of $C_*(v, E)$ of the same degree. Let $a \in \Lambda_\xi^-$, $h(a) = k$. Let C_* be a complex, (N+k)-equivalent to $C_*(v, E)$. Then there exists an r-quadruple $(g, \bar{g}, w, E)$ with w perfect, such that $C_*(w, E)$ is N-equivalent to the result of 2°), applied to C_*.

3) Let e_i be a generator of $C_*(v, E)$. Let $\lambda = 1 + \mu$,

supp $\mu \subset G^{=}$. Let C_* be a complex, N-equivalent to $C_*(v, E)$. Then there exists an r-quadruple $(g, \bar{g}, w, E)$, with w perfect, such that $C_*(w, E)$ is N-equivalent to the result of 3°), applied to C_*.

4) Let $C_*(v, E)$ be N-equivalent to $D_* \oplus \Gamma_*^{(k)}$, where D_* is a complex, $2 \leq k \leq n-3$. Then there exists an r-quadruple $(g, \bar{g}, w, E')$, w perfect, such that $C_*(w, E)$ is N-equivalent to D_*, i.e. to the result of 4°), applied to C_*.

5) Let $C_*(v, E)$ be N-equivalent to a complex C_*. Then there exists an r-quadruple $(g, \bar{g}, w, E')$, w perfect, such that $C_*(w, E')$ is N-equivalent to $C_* \oplus \Gamma_*^{(k)}$, where $2 \leq k \leq n-3$, that is to the result of 5°) applied to C_*.

Proof. 1) We take an r-quadruple $(f, \bar{f}, v, E')$, where $E' = \Lambda(E)$. We know from remark 2.5 that $C_*(v, \Lambda(E)) = \Lambda(C_*(v, E))$. Also we know from lemma 7.12. 1) that $\Lambda(C_*(v, E))$ and $\Lambda(C_*)$ are N-equivalent (here we imply as usual that Λ on the set of generators of C_* is defined via the identification of this set with I, provided by N-equivalence). Therefore $C_*(v, E')$ is N-equivalent to $\Lambda(C_*)$.

2) We apply theorem 7.1, p.1) to the quadruple $(f, \bar{f}, v, E)$ and the number $N+k$. We get a quadruple $(g, \bar{g}, w, E)$, such that $C_*(w, E)$ is $(N+k)$-equivalent to $(i,j;a)(C_*(v, E))$. Then by lemma 7.12. 2) we have that $(i,j;a)(C_*(v, E))$ is N-equivalent to $(i,j;a)(C_*)$, that is, $C_*(w, E) \underset{N}{\sim} (i,j;a)(C_*)$.

3) We apply theorem 7.1. p2) to $(f, \bar{f}, v, E)$ and the number N. We get a quadruple $(g, \bar{g}, w, E)$, such that $C_*(w, E)$ is N-equivalent to $(i, \lambda)\ C_*(v, E)$, which is, by 7.12. 3),

N-equivalent to $(i, \lambda)C_*$.

4) That is exactly the theorem 6.1, with the only difference that for the new quadruple $(h, \bar{h}, w, E')$ the field w is not necessary perfect. But we perturb it a little to get a new quadruple $(h, \bar{h}, w', E')$, such that w' is perfect and $C_*(w', E')$ is N-equivalent to D_* (corollary 4.6).

5) By theorem 6.14 there exists a new r-quadruple $(g, \bar{g}, v', E')$, such that $C_*(v', E)$ is N-equivalent to $C_*(v, E) \oplus \oplus \Gamma_*^{(k)}$, hence, N-equivalent to $C_* \oplus \Gamma_*^{(k)}$. Having perturbed v' a little we can assume that it is perfect.

Lemma 8.1 is proved.

Proof of theorem 2.4. So let $(f, \bar{f}, v, E)$ be a regular quadruple, where v is a good gradient-like field and let C_* be a complex over Λ_{ξ}^{-}, which is homotopy equivalent to $C_*(v, E)$. This implies that there exists a finite sequence $C_*^{(o)} = C_*(v, E)$, $C_*^{(1)}, \dots, C_*^{(r)}$ of complexes, such that $C_*^{(r)}$ equals C_*, and that for every i, $1 \leqslant i \leqslant r$, the complex $C_*^{(i)}$ is the result of one of the operations $1^o)$-$5^o)$, applied to $C_*^{(i-1)}$. There is a finite number of operations of type $2^o)$. Denote by A the maximum of $h(a_i)$ for $a_i \in \Lambda_{\xi}^{-}$, corresponding to these operations. There is a finite number of operations of type $1^o)$. Denote by B the maximum of $\|\Lambda\|$ for corresponding to these operations. Denote A+2B by C. We suppose that $N \geqslant 2$.

I claim that for $0 \leqslant s \leqslant r$ there exists an r-quadruple $(g_s, \bar{g}_s, v_s, E_s)$ with v_s perfect, such that $C_*(v_s, E_s)$ is

$N + (A+2B)(r-s)$ - equivalent to $C_*^{(s)}$.

We proceed by induction in s.

1. $s = 0$. We take $(g_0, \bar{g}_0, v_0, E_0)$ to be $(f, \bar{f}, v_0, E)$, where v_0 is perfect and close enough to v for $C_*(v_0, E)$ to be $(N + (A+2B)\,r)$-equivalent to $C_*(v, E)$.

2. Induction step. Suppose that $0 \leqslant s < r$ and that we have already the r-quadruple $(g_s, \bar{g}_s, v_s, E_s)$, such that v_s is perfect and that $C_*(v_s, E_s)$ is $(N + (A+2B)(r-s))$ - equivalent to $C_*^{(s)}$. If $C_*^{(s+1)}$ is obtained from $C_*^{(s)}$ by an operation of the type $3^o)$, $4^o)$ or $5^o)$ we find by lemma 8.1 recall that $N \geqslant 2$ the new r-quadruple $(g_{s+1}, \bar{g}_{s+1}, v_{s+1}, E_{s+1})$ with v_{s+1} perfect, such that $C_*(v_{s+1}, E_{s+1})$ is $(N + (A+2B)(r-s))$ - equivalent to $C_*^{(s+1)}$ and $N + (A+2B)(r-s) \geqslant N + (A+2B)(r-(s+1))$.

Suppose that $C_*^{(s+1)}$ is obtained from $C_*^{(s)}$ by an operation of type 2) with an element $a \in \Lambda_{\xi}^{-}$. Note that $C_*^{(s)}$ is $[N + (A+2B)(r-(s+1)) + A]$ - equivalent to $C_*(v_s, E_s)$, and $h(a) = k \leqslant A$. Therefore by lemma 8.1. 2) there exists a new r-quadruple $(g_{s+1}, \bar{g}_{s+1}, v_{s+1}, E_{s+1})$ with v_{s+1} perfect, such that $C_*(v_{s+1}, E_{s+1})$ is $[N + (A+2B)(r-(s+1))]$ - equivalent to $C_*^{(s+1)}$.

The operation of type $1^o)$ is realized similarly.

The theorem 2.4 and hence the main theorem are proved.

9. Applications

1. The first case to consider is of course the case when one do not expect the critical points. The main theorem of [Pal], together with the main theorem of the present paper imply immediately.

Theorem 9.1. Let M^n be a closed manifold, $n \geqslant 6$. Let $\xi : \pi_1 M \to \mathbb{Z}$ be an epimorphism with a finitely presented kernel. Denote by Λ the group ring $\mathbb{Z}\,\pi_1 M$. Then there exists a smooth fibration $f : M^n \to S^1$, belonging to ξ if and only if the chain complex $C_*(\tilde{M}^n) \underset{\Lambda}{\otimes} \Lambda_\xi^-$ is homotopy equivalent to zero and the torsion $\tau(C_*(\tilde{M}^n) \underset{\Lambda}{\otimes} \Lambda_\xi^-)$ of this complex vanishes in the group $Wh(G, \xi)$.

We recall that the group $Wh(G,\xi)$ is by definition the factor-group of $K_1(\Lambda_\xi^-)$ by the image of group of units $U_\xi^- = \{\mu \in \Lambda_\xi^- \mid \mu = (\pm g)(1+\lambda)\}$, where $g \in G$, supp $\lambda \subset$ $\subset G^= = \{x \in G \mid \xi(x) < 0\}$.

Corollary 9.2. Under the assumptions of the theorem 9.1 the property of being fibered is an invariant of the simple homotopy equivalence, preserving ξ.

Now we want to identify in more familiar terms the group $Wh(G,\xi) = K_1(\Lambda_\xi^-)/\, U_\xi^-$.

We restrict ourselves to the case $G = H \times \mathbb{Z}$, equals the projection on the second coordinate. Denote $\mathbb{Z}[H]$ by R. In this case Λ_ξ^- is the localization of the ring $R[[t]]$ with respect to the multiplicative set $S = \{t^n, n > 0\}$. Denote by

$U_\xi^{=}$ the set of units of the type $\{\pm h + \lambda\}$ where $h \in H$ and $\operatorname{supp}\lambda \subset G^{=}$.

Proposition 9.3. There exists an exact sequence $Wh(H) \to$ $\to Wh(G, \xi) \to \widetilde{Nil}(\mathbb{Z}H) \oplus \widetilde{K}_0(\mathbb{Z}H)$.

Proof. We apply the standard theorem on the localization in algebraic K-theory (see [Gr]), to the S-localization of the ring $R[[t]]$ and we get an exact sequence $K_1(R[[t]]) \xrightarrow{\alpha}$ $\to K_1(S^{-1}R[[t]]) \xrightarrow{\beta} K_0 N$, where N is the category of finitely generated $R[[t]]$-modules of projective dimension ≤ 1, annihilated by some power of t. In the same manner as for the case of the ring $R[t]$ we see that the category N consists exactly of finitely generated projective R modules, endowed with a nilpotent endomorphism. (Indeed, if $0 \to P \xrightarrow{\varphi} Q \xrightarrow{\psi} M \to 0$ is a projective resolution of M over $R[[t]]$, then we have an exact sequence

$$0 \to \underset{\substack{\wr\wr \\ M}}{t^nQ/t^n \operatorname{Im}\varphi} \to \underset{\substack{\wr\wr \\ P/t^nP}}{\operatorname{Im}\varphi/t^n\operatorname{Im}\varphi} \to \operatorname{Im}\varphi/t^nQ \to 0,$$

where the righthand term has projective dimension ≤ 1 over R and the middle one is R-projective. Hence M is R-projective.

Inversely, if M is an projective f.g. R-module with a nilpotent endomorphism f the characteristic sequence $0 \to$ $\to M[[t]] \xrightarrow{t-f} M[[t]] \to M \to 0$ is well defined and shows that the projective dimension of M over $R[[t]]$ is ≤ 1.

Therefore $K_0N = \operatorname{Nil} R$.

One can show that the p-image of t is a free module over R with the zero nilpotent endomorphism. Therefore we get the exact sequence

$$K_1(R[[t]]) \to K_1(S^{-1} R[[t]]) \;/\; \{t\} \to \widetilde{\mathrm{Nil}}\, R \oplus \widetilde{K}_o(R)$$

where $\widetilde{K}_o(R)$ stands for $\ker (K_o(R) \to K_o(\mathbb{Z}))$, induced by augmentation homomorphism. Now we factor two left terms by the image in K_1 of $U^{=}_{\xi}$ and we get the formula sought, since the group of units U^{-}_{ξ} is generated, by t^n and $U^{=}_{\xi}$, and by lemma 4.1 of [Pal] we get
$K_1(R[[t]]) \;/\; U^{=}_{\xi} \approx \mathrm{Wh}\, H.$

Therefore our main theorem implies the following.

Corollary 9.4. If $H_*(C_*(\tilde{M}^n) \underset{\Lambda}{\otimes} \Lambda^{-}_{\xi}) = 0$, then there is defined an element $\theta(\xi) \in \widetilde{\mathrm{Nil}}(\mathbb{Z}H) \oplus \widetilde{K}_o(\mathbb{Z}H)$ and if $\theta(\xi)$ vanishes then there is defined an element $\tau(\xi) \in \mathrm{Wh}\, H$.

The conditions $H_*(C_*(\tilde{M}^n) \underset{\Lambda}{\otimes} \Lambda^{-}_{\xi}) = 0$, $\theta(\xi) = 0$, $T(\xi) = 0$ are necessary and sufficient for ξ to be represented by a fibration.

We note that the situation here resembles Farrell's obstruction, although the condition $H_*(C_*(\tilde{M}^n) \underset{\Lambda}{} \Lambda^{-}_{\xi}) = 0$ is not equivalent to the finiteness of $\bar{M}^n$. For the case of $\pi_1 M^n$ abelian one can show that $H_*(C_*(\tilde{M}) \underset{\Lambda}{\otimes} \Lambda^{-}_{\xi}) = 0$ if and only if $\tilde{M}^n$ is finitely dominated. The following conjecture seems to be very likely to be true.

Conjecture 9.5. Assume that M^n is homotopy equivalent to the finite complex and $H_*(C_*(\tilde{M}^n) \underset{\Lambda}{\otimes} \Lambda^{-}_{\xi}) = 0$. Then the obstructions, $\theta(\xi)$ and $\tau(\xi)$, introduced above, coincide with those of Farrell [Far].

2. The second case is $\pi_1 M = Z$, $\xi : \mathbb{Z} \to \mathbb{Z}$ is an

morphism. The ring Λ_ξ^- is just the $\mathbb{Z}[[t]][t^{-1}]$, which is the principal ideal domain, hence $K_1(\Lambda_\xi^-)$ equals to the group of all units of Λ_ξ^-, and all of them are of the form $\pm t^{-n} + a_{-(n-1)}t^{-(n-1)} + \ldots \in U$ That means that $Wh(\mathbb{Z}, \xi) = 0$.

This implies that each complex over Λ_ξ^-, which is homotopy equivalent to $C_*(\tilde{M})$ Λ_ξ^- and which has no generators in dimensions $0, 1, n-1, n$ can be realized up to an arbitrary dimension N as a Novikov complex of some Morse form.

Since any free complex C_* over a principal ideal domain R is homotopy equivalent to the complex, having in dimension r exactly $b_r(C_*) + q_r(C_*) + q_{r-1}(C_*)$ generators, where $b_i(C_*) = rk_R\, H_i(C_*)$, $q_i(C_*) = t.n._R\, H_i(C_*)$, we obtain as a corollary, a following theorem of Farber.

<u>Theorem [Frb]</u>. For a manifold M^n, $n \geqslant 6$, $\pi_1 M = \mathbb{Z}$ there exists a Morse map $f: M^n \longrightarrow S^1$, such that $m_p(f) = b_p(M, \xi) + q_p(M, \xi) + q_{p-1}(M, \xi)$ (where $\xi: \pi_1 M \rightarrow \mathbb{Z}$ is any epimorphism, b_p, q_p are the Novikov numbers.

3. The third case is $\pi_1 M = \mathbb{Z}^m = G$. From the proposition 9.3 it follows immediately, that $Wh(G, \xi) = 0$. Again it implies that each chain complex D_* with $D_0 = D_1 = D_{n-1} = D_n = 0$ over Λ_ξ^-, homotopy equivalent to $C_*(\tilde{M}) \underset{\Lambda}{\otimes} \Lambda_\xi^-$ can be realized (up to N) as a Morse complex of some map $f: M^n \longrightarrow S^1$, belonging to ξ. So the problem of finding the optimal estimates for $m_p(f)$ is again reduced to an algebraic one. We have an algebraic lemma from [Pa2], which is based actually on lemma, due to J.-C. Sikorav [Si].

Lemma 9.6 [Pa2]. There exists a finite number of hyperplanes $\Delta_i \subset \mathrm{Hom}(\pi_1 M, \mathbb{Z}) \approx \mathbb{Z}^m$, such that for any $\gamma \in \bigcup \Delta_i$, $\gamma \neq 0$ the chain complex $C_*(\tilde{M}) \otimes_\Lambda \Lambda^-_\xi$ is homotopy equivalent to a free complex D_*, having in each dimension, the number of generators equal to $b_p(M, \xi) + q_p(M, \xi) + q_{p-1}(M, \xi)$, where $b_*(M, \xi)$ and $q_*(M, \xi)$ are the Novikov numbers, associated with the cyclic covering $\bar{M} \to M$, corresponding to a subgroup $\mathrm{Ker}\,\xi \subset \mathbb{Z}^m$. The numbers $b_*(M, \gamma)$, $q_*(M, \gamma)$ are in each connected component of the complement in $H^1(M, \mathbb{R})$ to $\bigcup_i \Delta_i$.

Now the main theorem of the present paper implies the following.

Theorem 9.5. Let M^n be a closed manifold, $n \geqslant 6$, $\pi_1 M^n = \mathbb{Z}^m$. Then there exists a finite number of hyperplanes $\Delta_i \subset H^1(M, \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z}) \approx \mathbb{Z}^m$, such that each nonzero class $\gamma \in H^1(M, \mathbb{Z})$, $\gamma \notin \bigcup_i \Delta_i$ contain a Morse map $M^n \to S^1$, which has the minimal possible number of zeros of all indices in its cohomology class, and that number $m_p(f)$ equals $b_p(M, \gamma) + q_p(M, \gamma) + q_{p-1}(M, \gamma)$.

This theorem was proved in [Pa3] under the additional restriction of 4-connectedness of the universal cover $\tilde{M}^n$. This was due to the methods, which I used in the proofs.

4. We have seen in p.3 that for the case $\pi_1 M^n = \mathbb{Z}^m$ the Novikov numbers, which are in this case the only obstructions to fibering, are construct in every connected component of complement $H^1(M, \mathbb{R}) \setminus \bigcup_i \Delta_i$. That is the reason for the following conjecture:

<u>Conjecture 9.6</u>. Let $\pi_1 M$ be abelian. Then there exists a finite number of hyperplanes $\Delta_i \subset H^1(M, \mathbb{Z})$, and a finite number of components U_j of the complement $H^1(M, \mathbb{R}) \setminus \bigcup \Delta_i$, such that the integer class $\xi \in H^1(M, \mathbb{Z}) \setminus \bigcup_i \Delta_i$ is represented by a fibration if and only if ξ belongs to one of U_j.

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