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Introduction

It is well known that given a Morse function f on a smooth manifold M one can construct a chain complex $C_*(f)$ (the Morse complex), which is a complex of the free abelian groups and the number of free generators in dimension p is exactly the Morse number $m_p(f)$ (i.e. the number of critical points of index p). This complex computes the homology of M itself: $H_*(C_*(f)) \approx H_*(M, \mathbf{Z})$.

One can strengthen this result so as to associate to any regular covering $\widehat{M} \to M$ with the structure group G the free $\mathbf{Z}G$-complex $C_*(f, \widehat{M})$, such that the number $\mu_{\mathbf{Z}G}(C_p(f, \widehat{M}))$ of free $\mathbf{Z}G$-generators of the module $C_p(f, \widehat{M})$ equals $m_p(f)$. The simple homotopy type (over $\mathbf{Z}G$) of this complex appears to be the same as of $C_*(\widehat{M})$. That is a useful tool for some kinds of nonsimply connected surgery ([2], [9]).

Approximately 10 years ago S.P. Novikov [6] proposed that Morse theory should be expanded to the case of Morse forms, that is closed 1-forms which are locally the derivatives of Morse functions. He suggested the construction of the appropriate analogue of the Morse complex. For the case when the homology class of the 1-form ω is integral, this complex is a free complex over a ring $\mathbf{Z}[[t]][t^{-1}]$ of Laurent power series. This ring is a kind

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of completion of the group ring $\mathbf{Z}[\mathbf{Z}]$ and the Novikov complex computes the correspondingly completed homology of the infinite cyclic covering $\overline{M} \xrightarrow{\mathbf{Z}} M$, which corresponds to the homology class $[\omega] \in H^1(M, \mathbf{Z})$.

The existence of such a complex enables one to obtain the lower bounds for the Morse number $m_p(\omega)$ in homotopy invariant terms, and in some cases prove their sharpness, thus pursuing the analogy with usual Morse theory further.

The sharpness of the arising inequalities was proved by Farber for $\pi_1 M^n = \mathbf{Z}$, $n \geq 6$ and by the author for $\pi_1 M^n = \mathbf{Z}^m$, $n \geq 6$ (under some restrictions on the homotopy type of M).

I must note that up to the present there existed no detailed proof of the mentioned properties of Novikov complex. The papers [1] and [7] appealed only to the inequalities which were obtained in this papers without using Novikov complex. But certainly what was underlying these proofs is the Novikov complex.

So in the present paper I suggest the full treatment of Novikov complex for the case $[\omega]$ is integral.

To formulate it here I need some notations. Let ω be a Morse 1-form on a closed manifold M, such that deRham cohomology class $[\omega]$ is nonzero and is up to the multiplication by a constant, an element of $H^1(M, \mathbf{Z})$. Let $\hat{p} : \widehat{M} \to M$ be a regular connected covering, such that ω resolves on $\widehat{M}$, i.e. $\hat{p}^*\omega = d\hat{f}$, $\hat{f} : \widehat{M} \to \mathbf{R}$. Then cohomology class $[\omega] : \pi_1 M \to \mathbf{R}$ is factored uniquely through some homomorphism $\xi : G \to \mathbf{R}$. The Novikov construction (carried to the case of nonabelian groups by J.-C. Sikorav [10]) attributes to any $\xi : G \to \mathbf{R}$ a completion Λ_ξ^- of a group ring $\Lambda = \mathbf{Z}[G]$ (see §1). There is an appropriate notion of a Whitehead group over Λ_ξ^- that is a factor of $K_1(\Lambda_\xi^-)$ by a subgroup of trivial units (see §1). By the appropriate version of the Kupka-Smale theorem (see Appendix A) we can choose the gradient-like vector field v for ω in such a way that all the stable and unstable manifolds of v are transversal.

The Main Theorem (see Theorem 2.2). *To this data there is associated the Novikov complex $C_*(v, \widehat{M})$ which is a free complex of Λ_ξ^--modules, such that the number of free generators $m_{\Lambda_\xi^-}(C_p(v, \widehat{M}))$ is equal to $m_p(\omega)$ and it is simply homotopy equivalent to $C_*(M) \otimes_{\mathbf{Z}[G]} \Lambda_\xi^-$.*

Now we explain the main idea of the proof and the contents of the paper.

The main instrument for proving the similar result for Morse functions (see [5]) is an inductive argument. One uses the construction of a manifold cell by cell using a Morse function and proves by induction that each step of the construction is simply homotopy equivalent to the corresponding part of the Morse complex. The main difficulty in our situation is that although we can work with Morse functions, say f on $\widehat{M}$, there is nothing to begin with, since f, for example, is not bounded from below and the descending discs of the critical points come infinitely downwards.

So we act in the opposite direction. Since we suppose that $[\omega]$ is a multiple of an integer class, there is an infinite cyclic covering $\bar{p} : \overline{M} \xrightarrow{\mathbf{Z}} M$, such that $\bar{p}^*\omega$ resolves: $\bar{p}^*\omega = d\bar{f}$, where $\bar{f}$ is a Morse function. The covering $\hat{p} : \widehat{M} \to M$ factors through $\bar{p}$ like $\widehat{M} \xrightarrow{Q} \overline{M} \xrightarrow{\bar{p}} M$.

Now we invite the reader to look at the picture 4.1. The notations are clear from the picture itself (and also explained in the very beginning of §4). We only indicate that t is a generator of a structure group $\mathbf{Z}$ of a covering, and we assume $\bar{f}(tx) < \bar{f}(x)$.

Let G^- denote the submonoid of G, consisting of all the g's, such that $\xi(g) \leq 0$, and let $\mathbf{Z}G^-$ denote its group ring. The Novikov ring Λ_ξ^- is formed by the power series in the elements of G which are infinite to the direction where ξ descends (a precise definition is given in §1). The same construction for $\mathbf{Z}G^-$ gives what we call $\Lambda_\xi^=$.

Now consider the function $\bar{f}$, restricted to $W(n) = \{x \in \overline{M} \mid 0 \geq \bar{f}(x) \geq -(n+1)a\}$. It is a Morse function and we can define a Morse complex with respect to the covering Q, restricted to $W(n) \subset \overline{M}$ and a gradient-like vector field v, coming from the base. Since v is t-invariant, one easily shows that it is actually a $\mathbf{Z}G^-$-complex, and since the elements $g \in G^-$ with $\xi(g) \leq -(n+1)$ act trivially, it is also defined over $\mathbf{Z}G_n^-$ where $\mathbf{Z}G_n^- = \mathbf{Z}G^-/\{g \in G^- \mid \xi(g) \leq -(n+1)\}$. We denote this complex $C_*^-(v,n)$. It is a free $\mathbf{Z}G_n^-$-complex. There is an obvious map $C_*^-(v,n) \to C_*^-(v,n-1)$, and the inverse limit will be denoted $C_*^-(v)$. By the definition the Novikov complex is $C_*^-(v) \otimes_{\Lambda_\xi^=} \Lambda_\xi^-$.

On the other hand, if we choose the triangulation of M in such a way that $\bar{p}(V)$ is a subcomplex, then there arises a triangulation $\triangle$ of $\overline{M}$, invariant under the action of $\mathbf{Z}$ and such that all the $W(n)$ and W_i are subcomplexes. The complexes $C_*((V^-)\hat{}, (t^{n+1}V^-)\hat{})$, where $\hat{}$ denotes passing to the preimage in $\widehat{M}$, are thus the free $\mathbf{Z}G_n$-complexes. There is an obvious projection map $C_*((V^-)\hat{}, (t^{n+1}V^-)\hat{}) \to C_*((V^-)\hat{}, (t^nV^-)\hat{})$ and $C_*((V^-)\hat{}) \otimes_{\mathbf{Z}G^-} \Lambda_\xi^-$

is exactly the inverse limit of this system.

It suffices to prove the simple homotopy equivalence over $\Lambda_\xi^=$ of the two complexes $C_*^-(v)$ and $C_*((V^-)\widehat{\ }) \otimes_{\mathbf{Z}G^-} \Lambda_\xi^-$.

That is done by comparison of two inverse systems. The ordinary Morse theory gives a homotopy equivalence $h_n : C_*^-(v,n) \to C_*((V^-)\widehat{\ }, (t^{n+1}V^-)\widehat{\ })$ over $\mathbf{Z}H$. We show that this can be improved as to give the homotopy equivalence over $\mathbf{Z}G_n^-$. This homotopy equivalence is by no means unique but one can fix some data from which it is constructed and, these data fixed, it is unique up to chain homotopy. [More precisely this isomorphism is constructed via some new Morse function $\varphi : W(n) \to \mathbf{R}$ which has the same g.-l. vector field v as f and satisfies a) $\varphi(tx) < \varphi(x)$ and b) if c, d are the critical points of φ and ind $c <$ ind d then $\varphi(c) < \varphi(d)$. Such functions will be called t-regular. Choose a sequence $-(n+1)a = a_0 < a_1 < \ldots < a_{n+1} = 0$ of regular values such that all the critical points of index p belong to $\varphi^{-1}(a_p, a_{p+1})$. Then obviously the homology of the part of $W(n)$ lying between a_p and a_{p+1} is zero everywhere except p'th dimension and is a free $\mathbf{Z}G_n^-$-module generated by the descending discs of that critical points c_i of index p, which belong to W_0. Then we show that there is a homotopy equivalence h such that the image of C_p consists of simplices, contained in the subspace $\{\varphi(x) \leq a_{p+1}\}\widehat{\ }$ and after passing to the homology modulo $\{\varphi(x) \leq a_p\}\widehat{\ }$ each critical point is carried to its descending disc. And now the homotopy equivalence satisfying that condition is unique up to a chain homotopy.]

We can also choose these h_n such that they are compatible for the different n up to chain homotopy (the details are similar to the above), and such that h_0 is a simple homotopy equivalence. That implies that we have a map of inverse systems

$$\begin{array}{ccccccc} \cdots \longrightarrow & C_*^-(v,n) & \longrightarrow & C_*^-(v,n-1) & \longrightarrow \cdots \\ & \downarrow h_n & & \downarrow h_{n-1} & \\ \cdots \longrightarrow & C_*((V^-)\widehat{\ }), (t^{n+1}V^-)\widehat{\ }) & \longrightarrow & C_*((V^-)\widehat{\ }, (t^nV^-)\widehat{\ }) & \longrightarrow \cdots \end{array} \tag{I.1}$$

such that all the squares are homotopy commutative. From this one deduces using the telescopic construction that the inverse limits of these systems are chain homotopy equivalent and the condition on h_0 guarantees that the resulting chain homotopy will be simple over $\Lambda_\xi^=$.

Now we expose the plan of the paper.

The definition of a Novikov ring is given in §1. In §2 we state the main theorem. §3 is purely algebraic. Here we first develop some formalism helpful for working with chain complexes endowed with the filtrations similar to these arising from t-regular Morse functions. This formalism (rather simple) answers the question up to which degree the homotopy equivalence of a Morse complex of a function and of the chain complex of the triangulation is unique. Here we also expose what we need about telescopes. In the §4 we start proving the main theorem. There are mainly the explanations and the reduction ot the existence of a diagram (I.1). This existence is proved in §5.

There are two appendices. In Appendix A we expose a proof of a version of the Kupka-Smale theorem which we need. Appendix B contains all the information about Morse functions and Morse complexes for Morse functions. In particular there are the full proofs of the theorems cited in the very beginning of this introduction.

The volume of this paper is rather large as compared with the ideas used in the proof (all of them are more or less exhibited in this introduction). The point here is that the main result of the paper was predicted since long ago and I myself for example always believed it to be true (although I never used it in the proofs). So the principle which I was guided by was the following: if I had a least doubt on some point, I included the full proof. Therefore there is a lot of material, which is not new. Two examples will show what I have in mind. 1) The classical Kupka-Smale theorem as exposed by Peixoto in [8] allows us to perturb any vector field so as to obtain the new one with hyperbolic zeros and transversally intersecting stable and unstable manifolds. The version which I need says that if the field has already only hyperbolic zeros, the perturbation can be chosen as not to change the field in the neighborhoods of zeros. 2) The fact that the differential in the Morse complex as defined by means of paths of steepest descent coincide with the differential in the complex, defined by the relative homology of the corresponding filtration is proved in the classical book [4] in the simply connected situation. I needed it in the general case so I reproduced the argument with the necessary modifications. There are many places like that (of course always with the corresponding references) and the reader familiar with differential topology is of course invited to omit these proofs or to consult the sources or to provide his own idea of proof.

There is one more reason for providing all the details. The author believes that the

Novikov theory, as developed in [6], [1], [7] cannot any more do without considering the Novikov complex directly rather than finding the roundabout ways. The author's project is to develop surgery directly in terms of the Novikov complex as for example is suggested in [4] for Morse functions.

In this connection I mention here that Prof. F. Latour recently announced results which are obviously in this direction. Unfortunately these results are not published until now and the author had no opportunity to attend F. Latour's talk in Orsay last summer.

Here are some results of F. Latour as communicated to me by J.-C. Sikorav. Let M^n be a closed manifold, θ - some closed 1-form, and $p^*\theta = df$ where $p : \widetilde{M} \to M$ is the universal covering. F. Latour introduced a new object – the "Novikov-type" homotopy groups the $\pi_\xi(M^n, \theta)$, which are essentially the sets of homotopy classes of maps $\mathbf{R}^s \xrightarrow{\kappa} \widetilde{M}^n$, such that the neighborhood of infinity in R^s goes to the "lower end" of $\widetilde{M}^n$ (more precisely, if $a_n \in \mathbf{R}^s$, $a_n \to \infty$, then $f(\kappa(a_n)) \to -\infty$). These groups are in the same relation to the usual homotopy as $H_*(\widetilde{M}^n) \otimes \Lambda_\xi^-$ – to the usual homology. The theorem of Latour says that if all the $\pi_s(M^n, \theta)$ vanish then there is an obstruction belonging to $Wh(\pi_1 M^n, [\theta])$ which vanishes if and only if the class $[\theta]$ contains a non-singular Morse form (if $n \geq 6$). We take here the opportunity to note that although the group $Wh(G, \xi)$ is very natural to appear in this general context and was known to some people since long ago, it was F. Latour who first described this group in his talk.

It seems that his theory should contain something similar to the main theorem of this paper. On the other hand, F. Latour seems not to use the notion of Novikov complex at all. In any case the things will be clear when F. Latour's paper is published.

After this paper was finished, I learned from J.-C. Sikorav that he has another idea for the proof of our theorem. Namely: for a Morse form ω consider the Morse form $Cdf + \omega$, where f is any Morse funciont and $C > 0$ is a large positive integer. The Morse complex for it is exactly the Morse complex for f, so it computes not only the completed homology of $\widetilde{M}$, but this homology itself. To get the result about ω we deform it to $Cdf + \omega$ and apply the Floer deformation argument. This idea has the advantage to cope also with irrational Morse forms, although it could present some technical difficulties to realize.

Some words about notations

If $f : X \to \mathbf{R}$ is any function we denote by $\{f(x) \leq a\}$ and $\{a \leq f(x) \leq b\}$ the preimages of $(-\infty, a]$ and $[a, b]$ respectively.

For a covering $\hat{p} : \widehat{M} \to M$ and a subset $Y \subset M$ we let $\hat{Y}$ or $Y^{\frown}$ denote the preimage of Y in $\widehat{M}$. If the covering has a section ξ over Y, the image $\xi(Y)$ is sometimes denoted by ${}^{\frown}Y$, and if Y is a point y, then we simplify the notation to $\hat{y}$. There will be no possibility of mixing these two notations.

The open (resp. closed) disc in $\mathbf{R}^k$ with the centre in a and of radius r is denoted by $B^k(a, r)$ (resp. $D^k(a, r)$).

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1 The Novikov Ring

In this section we recall the definition and the properties of the Novikov ring.

Suppose that G is a (discrete) group, $\xi : G \to \mathbf{R}$ – a homomorphism of the group G to the additive group of real numbers and A – a commutative ring with a unit. The object, which we are going to construct, is a special completion of the group ring $A[G]$ (denoted Λ for short), with respect to ξ.

Namely, denote by $\hat{\hat{\Lambda}}$ the abelian group of all the linear combinations of the type $\lambda = \sum n_g g$, where $g \in G$, $n_g \in A$ and the sum may be infinite. (In other words, $\hat{\hat{\Lambda}}$ is the set of all the functions $G \to A$ and the group operation is the addition). The ring Λ is a subgroup of $\hat{\hat{\Lambda}}$, but the ring structure of Λ does not expand in general to give a ring structure on $\hat{\hat{\Lambda}}$.

For any $\lambda \in \hat{\hat{\Lambda}}$ we denote by $\mathrm{supp}\lambda$ the subset of G, consisting of all the elements g, for which $n_g \neq 0$.

For a real number $c \in \mathbf{R}$ denote by G_c the subset of G, consisting of all the elements g, for which $\xi(g) \geq c$.

Now we denote by Λ_ξ^- the subset of $\hat{\hat{\Lambda}}$, consisting of all the elements $\lambda \in \hat{\hat{\Lambda}}$, such that for any $c \in R$ the set $\mathrm{supp}\lambda \cap G_c$ is finite. It is obvious that Λ_ξ^- is a subgroup of $\hat{\hat{\Lambda}}$, containing Λ.

Moreover, Λ_ξ^- possesses the natural ring structure. Namely, let $\lambda = \sum n_g g$, $\mu = \sum m_h h$ belong to Λ_ξ^-. For any $f \in G$ we set $\ell_f \in R$ to be $\sum_{gh=f} n_g m_h$. Note that this is well defined since the sum is finite and that in turn follows if we let $B \in \mathbf{R}$ be the $\max_{g \in \mathrm{supp}\lambda} \xi(g)$, $C \in \mathbf{R}$ be the $\max_{g \in \mathrm{supp}\mu} \xi(g)$ observe that $gh = f$ implies $\xi(g) + \xi(h) = \xi(f)$, hence $\xi(h) \geq \xi(f) - B$, $\xi(g) \geq \xi(f) - C$; and there is not more than a finite set of h, g satisfying these inequalities by the definition of Λ_ξ^-. Furthermore, the similar argument applies to show that the sum $\nu = \sum \ell_f f$ belongs again to Λ_ξ^-, and we call ν the product of λ, μ. One easily sees that it is an associative algebra over A with the unit $1.\mathbf{L}$, where $\mathbf{L}$ is the neutral element of G, and that $\Lambda = A[G]$ is the subalgebra of Λ_ξ^-.

Note that $\mathrm{supp}(\lambda \cdot \mu) \subset \mathrm{supp}\lambda \cdot \mathrm{supp}\mu$.

The ring Λ_ξ^- is called a Novikov ring. It was introduced by Novikov [6] for the case $G = \mathbf{Z}^m$ and by J.-C. Sikorav [10] in the general case.

The basic example one should have in mind is the following: $G = \mathbf{Z}$ and the homomorphism $\xi : \mathbf{Z} \to \mathbf{R}$ is the identity. If we use the multiplicative notation for the group $\mathbf{Z}$, and denote (-1) by t, then the ring Λ_ξ^- is the ring $\mathbf{Z}[[t]][t^{-1}]$ of all Laurent power series in t.

In general one should think of Λ_ξ^- as of the ring of all the power series with coefficients in A, which are allowed to be infinite to the direction where ξ descends, and are demanded to be finite to the opposite direction.

In the sequel we will use the notion of simple homotopy type of chain complexes over Λ_ξ^-. For that we need the analogue of a Whitehead group. Denote by $U(G,\xi)$ the multiplicative group of units of the ring Λ_ξ^- of the form $\pm g + \lambda$, where $g \in G$ and $\mathrm{supp}\lambda \subset \{\xi(\mathrm{g}) < 0\}$. Now we set $Wh(G,\xi) = K_1(\Lambda_\xi^-)/U(G,\xi)$. As usual two f.g. free Λ_ξ^--complexes C_1, C_2 with fixed bases will be called simply homotopy equivalent if there exists a homotopy equivalence $f : C_1 \to C_2$ such that the torsion of f vanishes in $Wh(G,\xi)$. As we already mentioned in the introduction, the definition of $Wh(G,\xi)$ is due to F. Latour.

2 The Statement of the Main Theorem

Recall that a closed 1-form ω on a manifold M^n is called a Morse form if locally ω is an exterior derivative of a Morse function, defined locally. The zeros of a Morse form ω are isolated and the index of each zero is defined. For any zero c there exists an open neighborhood W_c of c and a diffeomorphism Φ_c of W_c onto the product $B^p(O, 2\varepsilon) \times B^{n-p}(O, 2\varepsilon)$ of the standard open 2ε-balls in $\mathbf{R}^n = \mathbf{R}^p \times \mathbf{R}^{n-p}$, such that the point c is carried by Φ_c to zero, and the form ω – to the form $2(-x_1dx_1 - \cdots - x_pdx_p + y_1dy_1 + \cdots + y_{n-p}dy_{n-p})$. Here x_i are the standard coordinates in R^p, and y_j – the standard coordinates in $\mathbf{R}^{n-p}$. We denote by U_c the preimage of $B^p(O, \varepsilon) \times B^{n-p}(O, \varepsilon)$. We refer to the vectors $x = (x_1, \ldots, x_p) \in \mathbf{R}^p$ and $y = (y_1, \ldots, y_{n-p})$ as to the coordinates of the vector $(x_1, \ldots, x_p, y_1, \ldots, y_{n-p}) = (x, y)$. We enumerate the zeros as $c_1, \ldots, c_i$ and denote, correspondingly, the neighborhoods etc. by $U_i, W_i, \ldots$, etc.

A vector field v on M^n is called a gradient-like vector field for ω if 1) $\omega(v) > 0$ apart from the zeros of ω and 2) for every zero c the coordinates of v in the standard coordinate system are $(-x_1, \ldots, -x_p, y_1, \ldots, y_{n-p})$. This definition is due to Milnor, [4] for the case of Morse functions and the same argument as given in that book shows that the gradient-like fields exist. From now on we suppose that the manifold M is compact without boundary. We denote by $\varphi(t, v)$ the diffeomorphism of M onto itself, given by the shift by t along the trajectories of v. We denote by B_i^+ the Φ_i-preimage of $B_i^p(O, \varepsilon) \times \{0\}$, by B_i^- the Φ_i-preimage of $\{0\} \times B_i^{n-p}(O, \varepsilon)$. The image $\varphi(t, v)B_i^\pm$ will be denoted by $B_i^\pm(t, v)$, or simply $B_i^\pm(t)$, if no ambiguity is possible. The set $\bigcup_{t>0} B_i^-(t)$ will be denoted by B_i^{un}, and the set $\bigcup_{t<0} B_i^+(t)$ by B_i^{st}. These are injectively immersed manifolds, of dimensions $n - p_i$ and p_i. The gradient-like vector field will be called perfect if for every pair (i, j) the manifolds B_i^{un}, B_j^{st} are transversal. (We sometimes will abbreviate "gradient-like vector field" and "perfect gradient-like vector field" as "g.l. vector field" and "p.g.l. vector field".) The version of the Kupka-Smale theorem, exposed with the proof in Appendix A provides us with a perfect gradient-like vector field for our form ω. (Indeed, pick up any g.l. vector field v_0 for ω. By theorem A.1 of Appendix A there exists a perfect g.l. vector field v, arbitrarily close to v_0, and which coincides with v_0 inside U_i. This v is obviously a p.g.l. vector field for ω.)

Let $[a, b]$ be a segment of the real line, where a may equal to $-\infty$, b – to $+\infty$. We say that a trajectory $\gamma(t)$ of the gradient-like vector field v starts at $x \in M$ (correspondingly, finishes at $y \in M$) if either a is finite and $\gamma(a) = x$ (correspondingly b is finite and $\gamma(b) = y$) or $a = -\infty$, $v(x) = 0$ and $\lim_{t=-\infty} \gamma(t) = x$ (correspondingly, $b = \infty$, $v(y) = 0$ and $\lim_{t\to\infty} \gamma(t) = y$).

A connected regular covering $p : \overline{M} \to M$ with a structure group G will be called ω-resolving if $p^*\omega = d\bar{f}$, where $\bar{f}$ is a $\mathbf{R}$-valued smooth function on M. Since ω is a Morse form, $\bar{f}$ is a Morse function. In the set of all ω-resolving coverings there is one, which is minimal in the sense that any other one factors through it. That is exactly the covering, corresponding to the subgroup $\text{Ker}([\omega] : \pi_1 M \to \mathbf{R})$ (here we consider $[\omega] \in H^1(M, \mathbf{R})$ as a homomorphism of $\pi_1 M$ to $\mathbf{R}$); its structure group is $\pi_1 M/\text{Ker}[\omega]$, which is a finitely generated subgroup of $\mathbf{R}$, hence a f.g. free abelian group. We denote it $\bar{p} : \overline{M}_{[\omega]} \to M$.

In what follows we concentrate only on the forms for which the rank of that group is 1, i.e. the cohomology class $[\omega] \in H^1(M, \mathbf{R})$ is up to a positive constant, an integer cohomology class. In this case the covering $\bar{p} : \overline{M}_\xi \to M$ is an infinite cyclic covering.

Suppose now that $\hat{p} : \widehat{M} \to M$ is any ω-resolving covering with the structure group G. The homomorphism $[\omega] : \pi_1 M \to \mathbf{R}$ is factored uniquely through G and the resulting homomorphism will be denoted by $\xi : G \to \mathbf{R}$. We choose and fix 1) the gradient-like vector field v for ω, 2) for each zero c the lifting $\hat{c}$ of c to $\widehat{M}$, and 3) for each zero c_i the orientation of the stable disc B_i^+. to this data we will associate a finitely generated free based chain complex over Λ_ξ^-, where $\Lambda = \mathbf{Z}G$, which is called Novikov complex and possesses the natural properties, stated below in Theorem 2.1.

Namely, let c_i, c_j be zeros of v and let γ be any trajectory of the field $(-v)$, starting at c_i and finishing at c_j. There is a unique lift of γ to the covering M, starting at c_i and finishing at $c_j \cdot g$, where g is an uniquely determined element of G, which we denote by $g(\gamma)$. Every point of γ is the intersection point of B_i^{st} with B_j^{un}. The tangent space $T_i^{st}(x)$ to B_i^{st} at any point x splits as a sum of the 1-dimensional space, tangent to flow, where we name $(-v)$ to have positive orientation and the direct complement $T_i^{st,0}(x)$, which therefore also obtains the orientation. Suppose now that $p_i = \text{ind} c_i$ is equal to $\text{ind } c_j + 1$. Then $\dim T_i^{st}(x) + \dim T_j^{un}(x) = n + 1$, and since the vector v belongs to both and they are transversal by our assumption, we get that $TM = T_i^{st,0} \oplus T_j^{un}$. Note that since we have oriented all the discs B_i^{st}, we have cooriented the discs B_i^{un}, hence T_j^{un} is cooriented, and there are two orientations on $T_i^{st,0}$, hence the sign $\varepsilon(\gamma)$ is attributed to γ.

Lemma 2.1 *For each two zeros c_i, c_j such that ind c_i = ind $c_j + 1$ and each $g \in G$ there is at most finite number of $(-v)$-trajectories γ, such that $g(\gamma) = g$. Moreover the element $\sum_\gamma \varepsilon(\gamma)g(\gamma)$ of $\hat{\hat{\Lambda}}$ belongs to the Novikov ring Λ_ξ^-.*

This lemma will be proved in §4.

Thus to any pair c_i, c_j of zeros with the indices which differ by 1, we have the incidence coefficient $n(c_i, c_j) \in \Lambda_\xi^-$.

Now let $C_p(v)$ be the free right module over Λ_ξ^-, generated by the zeros c of index p. For every generator c we set

$$\partial_p c = \sum_d d \cdot n(c, d) \tag{2.1}$$

where the sum runs over the zeros of index $(p-1)$. We expand this ∂_p to the whole module $C_p(v)$ as to give the map of the right Λ_ξ^--modules.

Theorem 2.2 *1) $\partial_p \cdot \partial_{p+1} = 0$; hence $(C_*(v), \partial_*)$ is a chain complex.*

2) This chain complex is simply homotopy equivalent to the chain complex $C_*(\widehat{M}) \otimes_\Lambda \Lambda_\xi^-$ (for the notion of simple homotopy equivalence see the end of §1). The proof of this theorem occupies the rest of the present paper.

3 Preliminaries on chain complexes

This section is purely algebraic and is divided into two parts. Part A deals with some special filtration in chain complexes, part B – with chain cylinders and telescopes.

A. A self-indexing Morse function on a manifold M determines the filtration in the singular chain complex of M, and also of any covering of M (for a precise definition of the filtration see Appendix B). This filtration F_n possesses a special property that homology of F_n/F_{n-1} vanishes except in dimension n. The chain complexes with the filtrations like that have some natural and simple properties which we treat in this section.

A general remark to all this section. The chain complexes are supposed to begin from zero dimension, i.e. to be of the form $\{0 \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots\}$, and the filtrations to be indexed by natural numbers, i.e. be of the form $0 \subset F_x^{(1)} \subset F_*^{(2)} \subset \cdots$ and to be exhausting, i.e. $\bigcup_i F_*^{(i)} = C_*$.

Let A be a ring and $C_* = \{0 \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_n \leftarrow \cdots\}$ be a chain complex of right A-modules.

Definition 3.1 *A filtration $0 \subset F_*^{(1)} \subset \cdots \subset F_*^{(n)} \subset \cdots$ of C_* by subcomplexes $F_*^{(i)}$, where $\bigcup_i F_*^{(i)} = C_*$ is called good if $H_i(F_*^{(n)}/F_*^{(n-1)})$ is zero for $c \neq n$.*

Note that for a good filtration $H_i(F_*^{(n)})$ is zero for $n \neq i$.

To any filtration $\{F_*^{(i)}\}$ of some complex C_* we associate a complex F_*^{gr}, setting $F_n^{gr} = H_n(F_*^{(n)}/F_*^{(n-1)})$, and introducing the differential $\partial_n : F_n^{gr} \to F_{n-1}^{gr}$ to be that of the exact sequence of the triple $(F_*^{(n)}, F_*^{(n-1)}, F_*^{(n-2)})$.

An obvious example of a good filtration is the filtration of the complex D_* by the subcomplexes $D_*^{(i)} = \{0 \leftarrow D_0 \leftarrow D_1 \leftarrow \cdots \leftarrow D_i \leftarrow 0 \leftarrow \cdots\}$. The associated complex D_*^{gr} is D_* itself. This filtration will be called trivial.

Lemma 3.2 *Suppose that $F_*^{(n)}$ is a good filtration of a complex C_*. Let $D_* = \{0 \leftarrow D_0 \leftarrow D_1 \leftarrow \cdots\}$ be a chain complex of free right A-modules and $\varphi : D_* \to F_*^{gr}$ be a chain map. Then there exists a chain map $f : D_* \to C_*$, preserving filtrations (we imply that D_* is good-filtered trivially) and inducing the map φ in the graded complexes. This chain map f is unique up to chain homotopy, preserving filtrations.*

Proof: 1) Construction of f.

Suppose by induction that we constructed the maps $f_i : D_i \to C_i$ where $i \leq n-1$, commuting with the differentials ∂_* in C_* and d_* in D_*, preserving filtrations (i.e. Im $f_i \subset C_i^{(i)}$) and inducing φ in the graded groups.

It suffices to define f_n on the free generators of D_n. Let e be a generator of D_n. Let x be any element in $C_n^{(n)}$, representing in the group $F_n^{gr} = H_n(C_n^{(n)}/C_n^{(n-1)})$ the element $\varphi(e)$. Consider the element $z = \partial x - f_{n-1}(de) \in C_{n-1}^{(n-1)}$. Since $\partial x \in C_{n-1}^{(n-1)}$ this z belongs actually to $C_{n-1}^{(n-1)}$ and $\partial z = 0$. The homology class of z in $H_{n-1}(C^{(n-1)}/C^{(n-2)})$ is zero, since the class of $f_{n-1}(de)$ is equal to the boundary of $\varphi(e)$ in F_*^{gr} which is ∂x. Hence $z = \partial u + v$ where $u \in C_n^{(n-1)}$, $v \in C_{n-1}^{(n-2)}$. Note that $dv = 0$ (in C_* itself) hence $v = \partial w$, $w \in C_n^{(n-2)}$ by the remark, following the def. 3.1. Now we put $f_n(e) = x - (u+w)$. The class of $x - (u+w)$ in $C_n^{(n)}/C_n^{(n-1)}$ is equal to that of x and $\partial f_n(e) = f_{n-1}(de)$; all the conditions are satisfied.

2) Chain homotopy uniqueness of f.

Let f, g be the maps of D_* to C_*, satisfying the conditions of the theorem. Suppose that we have constructed the maps $H_i : D_i \to C_{i+1}$, satisfying $\partial_{i+1}H_i + H_{i-1}d_i = f_i - g_i$ for $i \leq n-1$. Let e be a free generator of D_n. It suffices to construct an element $x \in C_{n+1}^{(n)}$, such that $\partial_{n+1}x + H_{n-1}d_n e = f_n(e) - g_n(e)$. Consider the element $z = -H_{n-1}d_n(e) + f_n(e) - g_n(e)$. It is easy to check, using the induction assumption that z is a cycle in $C_n^{(n)}$. Furthermore, the condition of f and g implies that $f_n(e) - g_n(e) = \partial_{n+1}u + v$, where $u \in C_{n+1}^{(n)}$, $v \in C_n^{(n-1)}$. Therefore, $z = -H_{n-1}d_n e + \partial_{n+1}u + v$. Note that z is a cycle, therefore $v - H_{n-1}d_n e$ is also a cycle. But since it belongs to $C_n^{(n-1)}$, it is homologous to zero in $C_*^{(n-1)}$, therefore $z = \partial_{n+1}u + \partial_{n+1}w$, where $u \in C_{n+1}^{(n)}$, $w \in C_{n+1}^{(n-1)}$. Now we put $x = u + w$ and the lemma 3.2 is proven. □

Definition 3.3 *A good filtration $F_*^{(i)}$ of a complex C_* is called nice if every module $H_n(F_*^{(n)}/F_*^{(n-1)})$ is a free right A-module.*

Corollary 3.4 *For a nice filtration $F_*^{(i)}$ of a complex C_* there exists a homotopy equivalence $F_*^{gr} \to C_*$, functorial up to chain homotopy in the category of nicely filtered complexes.*

Proof: The homotopy equivalence $F_*^{gr} \to C_*$ follows from lemma 3.2 if we set $D_* = F_*^{gr}$ and let $f : D_* \to F_*^{gr}$ be the identity.

To prove the functoriality suppose that C_* and D_* are nicely filtered complexes with filtrations $F_*^{(i)}$, $G_*^{(i)}$ and $C_* \to D_*$ is a chain map, preserving filtrations. Denote by $\varphi : F_*^{gr} \to G_*^{gr}$ the chain map, induced by f, and by $g : F_*^{gr} \to C_*$, $h : C_*^{gr} \to D_*$ the chain homotopy equivalences, preserving filtrations (recall that F_*^{gr}, G_*^{gr} are trivially filtered). Then $f \circ g$ and $h \circ \varphi$ are chain maps from F_*^{gr} to D_*, preserving filtrations and inducing the same map in graded homology, namely φ. Hence they are chain homotopic via the homotopy, preserving filtrations. □

Now we pass to the category of based complexes.

We need one more notation. For a given nice filtration $\{F_*^{(n)}\}$ of a complex C_* we denote by $R_*^{(n)}$ the complex, which vanishes in all the dimensions except $* = n$ and is equal to $H_n(F_*^{(n)}/F_*^{(n-1)})$ for $* = n$. By lemma 3.2 there exists a (uniquely defined up to homotopy) homotopy equivalence $\kappa_n : R_*^{(n)} \to F_*^{(n)}/F_*^{(n-1)}$, inducing identity in homology.

The basic ring now is $\mathbf{Z}G$. Let $C_* = \{0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_n \leftarrow 0\}$ be a free f.g. based complex of right $\mathbf{Z}G$-modules.

Let $F_*^{(i)}$ be a filtration of C_*.

Definition 3.5 *The filtration $F_*^{(i)}$ is called perfect if the conditions 1)-4) below hold.*

1. *The complexes $F_*^{(i)}$ and the factor complexes $F_*^{(i)}/F_*^{(i-1)}$ are free f.g. complexes; the filtration $F_*^{(i)}$ is finite.*

2. *All the complexes $F_*^{(i)}$ and the factor complexes $F_*^{(i)}/F_*^{(i-1)}$ are endowed with the classes of preferred bases, compatible in the sense that a preferred base for $F_*^{(i-1)}$ and a preferred base for $F_*^{(i)}/F_*^{(i-1)}$ form a preferred base for $F_*^{(i)}$. The preferred base for the final $F_*^{(i)}$ is a preferred base for C_*.*

3. *$F_*^{(n)}$ is nice and $H_i(F_*^{(n)}/F_*^{(n-1)})$ is endowed with a preferred class of bases.*

4. *The map $\kappa_n : R_*^{(n)} \to F_*^{(n)}/F_*^{(n-1)}$, introduced above is a simple homotopy equivalence.*

Note that if C_* is perfectly filtered, then every $F_*^{(n)}$ inherits from it the perfect filtration; the corresponding graded group is $(F_*^{gr})(n)$.

Lemma 3.6 *For a perfect filtration $\{F_*^{(i)}\}$ of a complex C_* the homotopy equivalence $F_*^{gr} \to C_*$ (existing by the lemma 3.3) is a simple homotopy equivalence.*

Proof: Induction in the length of filtration.

For the filtration of the length 1 the mentioned homotopy equivalence is homotopic to identity and by the pro. 2) of def. 3.5 it is a simple homotopy equivalence.

Suppose that lemma 3.6 is proved for filtrations of length $\leq n-1$, and consider a filtration $\{F_*^{(i)}\}$ of C_* of length n. Denote by $f : F_*^{gr} \to C_*$ the homotopy equivalence, provided by the lemma 3.3. It preserves filtrations, hence carries $(F_*^{gr})^{(n-1)}$ to $F_*^{(n-1)}$. We denote this map by φ. Note that φ preserves filtration and induces identity in graded groups, hence by induction it is a simple homotopy equivalence. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_*^{(n-1)} & \longrightarrow & C_* & \longrightarrow & C_*/F_*^{(n-1)} & \longrightarrow & 0 \\
\uparrow & & \uparrow \varphi & & \uparrow f & & \uparrow g & & \\
0 & \longrightarrow & (F_*^{gr})^{(n-1)} & \longrightarrow & F_*^{gr} & \longrightarrow & R_*^{(n)} & \longrightarrow & 0
\end{array}$$

where g is a factor map. By definition of f the map g induces the identity in homology, hence it is homotopic to κ_n, hence it is a simple homotopy equivalence by property 4). Thus both right and left arrows are s.h.e. and therefore the central one is also s.h.e.

B. In this part we prove a technical result on chain complexes which will be of use in section 4. Let $\{C_*^n\}$ and $\{D_*^n\}$ $(n \geq 0)$ be two inverse systems of chain complexes of right modules over some ring R. Let $h_n : C_*^n \to D_*^n$ be the chain homotopy equivalences, such that all the following diagrams commute up to chain homotopy:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & C_*^0 & \longleftarrow & C_*^1 & \longleftarrow & C_*^2 & \longleftarrow & \cdots \\
 & & \downarrow h_0 & & \downarrow h_1 & & \downarrow h_2 & & \\
0 & \longleftarrow & D_*^0 & \longleftarrow & D_*^1 & \longleftarrow & D_*^2 & \longleftarrow & \cdots
\end{array} \qquad (3.1)$$

□

Let $\Gamma_* = \lim C_*^n$, $\Delta_* = \lim D_*^n$.

Proposition 3.7 *1) There exists a chain complex Z_* and two chain maps $A : Z_* \to \Gamma_*$, $B : Z_* \to \Delta_*$ which induce isomorphisms in homology.*

2) If Γ_* and Δ_* are free chain complexes, then there exists a homotopy equivalence $f : \Gamma_* \to \Delta_*$, such that the diagram

$$\begin{array}{ccc} \Gamma_* & \xrightarrow{f} & \Delta_* \\ \downarrow & & \downarrow \\ C^0_* & \xrightarrow{h_0} & D^0_* \end{array}$$

is homotopy commutative.

Proof: For the proof we need some preliminaries on cylindric and telescopic constructions for maps.

One remark on notations. In this section we will meet many chain complexes. All the differentials will be denoted by a single letter ∂ since there do not occur two differentials on the same complex and thus there is no possibility of confusion. Normally the element of, say, C_* of degree n will be denoted c_n. The sign $\sim$ denotes "homotopy equivalent".

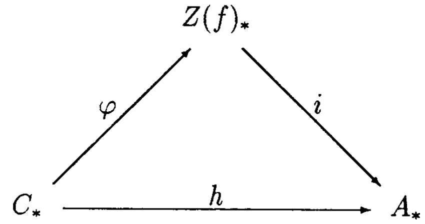
Recall that for a chain map $f : C_* \to D_*$ of chain complexes the cylinder $Z(f)_*$ is a chain complex defined by $Z(f)_n = D_n \oplus C_n \oplus C_{n-1}$, $\partial(d_n, c_n, c_{n-1}) = (\partial d_n - f(c_{n-1}), \partial c_n + c_{n-1}, -\partial c_{n-1})$. Let $i : C_* \to Z(f)_*$, $\pi : Z(f)_* \to D_*$, $j : D_* \to Z(f)_*$ denote the maps, defined by $i(c_n) = (0, c_n, 0)$, $\pi(d_n, c_n, c_{n-1}) = f(c_n) + d_n$, $j(d_n) = (d_n, 0, 0)$. □

Lemma 3.8 *$Z(f)_*$ is indeed a chain complex with that differential. The maps i, π, j are the chain maps and have the following properties: $\pi j = id$, $j\pi \sim id$, $\pi i = f$. The following diagram is strictly commutative*

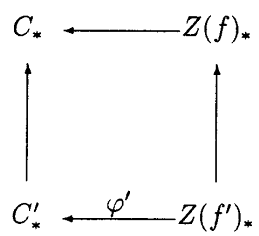
$$\begin{array}{ccccc} & & Z(f)_* & & \\ & {}^{i}\nearrow & & \searrow^{\pi} & \\ C_* & & \xrightarrow{\quad f \quad} & & D_* \end{array}$$

Proof: is trivial and left to the reader. (For example, the homotopy H from j to id is given as $H(d_n, c_n, c_{n-1}) = (0, 0, c_n)$.) $\square$

Lemma 3.9 *1) Let $f : A_* \to B_*$, $g : B_* \to C_*$, $h : A_* \to C_*$ be the chain maps and $gf \sim h$ via homotopy $H : A_* \to C_{*-1}$. Then there is a map $\varphi : Z(f)_* \to C_*$, such that the following diagram is strictly commutative*



2) This map φ is functorial with respect to the data $\{f, g, h, H\}$, i.e. if $f' : A'_* \to B_*$, $h' : A'_* \to C'_*$, $H' : A'_* \to C'_{*-1}$ is another system of chain maps like above, which maps into our first one so that all the diagrams are strictly commutative, then the φ and φ', constructed in p.1, form the commutative diagram

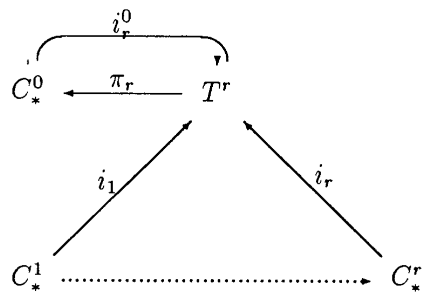


Proof: For $(b_n, a_n, a_{n-1}) \in Z(f)_n$ set $\varphi(b_n, a_n, a_{n-1}) = g(b_n) + h(a_n) + H(a_{n-1})$. Both 1) and 2) are obvious. $\square$

Lemma 3.10 *Let $Z(f)_*$ be the cylinder of $f : C_* \to D_*$ and $\varphi, \psi : D_* \to D_*$ two chain maps, homotopic via h. Let $\Phi, \Psi : Z(f)_* \to Z(f)_*$ be two maps preserving D_*, such that $\Phi|D_* = \varphi$, $\Psi|D_* = \psi$. Then $\Phi \sim \Psi$ via a homotopy H, preserving D_* and such that $H|D_* = h$.*

Proof: Consider the map $\Psi = \varphi \cdot \pi$. That is the chain map $Z(f)_* \to D_*$ which sends D_* to zero, hence is defined on the factor complex $Z(f)_*/D_*$. The latter is contractible, hence the original map $\Phi - \varphi \cdot \pi : Z(f)_* \to Z(F_*)$ is homotopic to zero via the homotopy which sends D_* to zero. Applying the same procedure to $\Psi - \psi \cdot \pi$ we are left to prove that $\varphi \cdot \pi \sim \psi \cdot \pi$ via the homotopy H, preserving D_* and $H|D_* = h$. We put $h(d_n, c_n, c_{n-1}) = h(d_n) + h(f(c_n))$. One easily checks that it is the homotopy sought. □

Now we pass to telescopes. Suppose that $C^0_* \overset{\alpha_1}{\leftarrow} C^1_* \overset{\alpha_2}{\leftarrow} \cdots \overset{\alpha_r}{\leftarrow} C^r_* \leftarrow \cdots$ is an inverse system of chain complexes and maps. We define by induction a sequence of chain complexes T^r_* together with a set of chain maps:



such that 1) $i_0, \ldots, i_r$ are embeddings, 2) $i_0\pi \sim id_{T^r_*}$, $\pi i_0 = id_{C^0_*}$, 3) $\pi i_s \sim \alpha_1 \circ \ldots \circ \alpha_r$.

By definition $T^0_* = C^0_*$. If T^r_* is defined we set $T^{r+1}_* = Z(i_r \circ \alpha_{r+1})$. Note that the inclusion $I_r : T^r_* \subset T^{r+1}_*$ has the homotopy inverse $P_{r+1} : T^{r+1}_* \to T^r_*$ by lemma 3.8. Note also that P_r is a retraction onto a subcomplex. Let now $i^0_{r+1} = I_r \circ i^0_r$, $\pi_{r+1} = \pi_r \circ P_r$. One easily checks that 1), 2) and 3) above are satisfied. We get the sequence

$$C^0_* \underset{P_0}{\overset{I_0}{\rightleftarrows}} T^1_* \underset{I_1}{\overset{P_0}{\rightleftarrows}} T^2_* \cdots \underset{I_r}{\overset{P_r}{\rightleftarrows}} T^r_*$$

where all the I_r are inclusions, P_r – retractions. We set $T_* = \bigcup_{r=1}^{\infty} T^r_* = \varinjlim T^r_*$. Since P_r are retractions, the map $P : T_* \to C^0_*$, defined as P_r on each T^r_* is a well-defined retraction. There are also the embeddings $i_r : C^r_* \to T_*$, such that $i_r \circ \alpha_{r+1} \sim i_{r+1}$. The composition $P \circ i_r$ is exactly $\alpha_1 \circ \ldots \circ \alpha_r$. The composition $P \circ i_0$ is identity. The composition $i_0 \circ P$ is homotopic to identity. (The homotopy H_r from $id_{T^r_*}$ to $P|T^r_*$ is constructed by induction on r, using lemma 3.10. By this lemma $H_{r+1}|T^r_*$ is exactly H_r, hence the union of the maps H_r gives the homotopy H sought.)

We will use telescopes for turning a sequence of homotopy commutative diagrams to a sequence of strictly commutative ones. Namely, let

$$\begin{array}{ccccccccc}
C^1_* & \xleftarrow{\alpha_2} & C^2_* & \xleftarrow{\alpha_3} & C^3_* & \longleftarrow & \cdots & \xleftarrow{\alpha_r} & C^r_* & \longleftarrow \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & & & \downarrow{\scriptstyle f_r} & \\
D^1_* & \xleftarrow{\beta_2} & D^2_* & \xleftarrow{\beta_3} & D^3_* & \longleftarrow & \cdots & \xleftarrow{\beta_r} & D^r_* & \longleftarrow
\end{array} \tag{3.2}$$

be a sequence of chain complexes and maps, such that every square is homotopy-commutative. Denote by ${}^rT_*$ the telescope of $(C^r_* \xleftarrow{\alpha_{r+1}} C^{r+1}_* \longleftarrow \cdots)$ and by $P_r : {}^rT_* \to C^r_*$ the corresponding projection. By construction of ${}^rT_*$ there is an obvious embedding ${}^{r+1}T_* \subset_{j_{r+1}} {}^rT_*$, and a strictly commutative sequence

$$\begin{array}{ccccccc}
{}^1T_* & \overset{j_2}{\supset} & {}^2T_* & \overset{j_3}{\supset} & \cdots & \supset & {}^rT_* \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & & & \downarrow{\scriptstyle p_r} \\
C^1_* & \xleftarrow{\alpha_2} & C^2_* & \longleftarrow & \cdots & \xleftarrow{\alpha_r} & C^r_*
\end{array} \tag{3.3}$$

Lemma 3.11 *For every r there exists the map ${}^rT_* \xrightarrow{F_r} D^r_*$, such that 1) $\beta_r F_r = F_{r-1} j_r$. 2) $f_r \circ p_r \sim F_r$. 3) $F_r | C^r_* = f_r$.*

Proof: Note first that 3) implies 2). To construct the system of maps F_r, satisfying 1), 3) we argue as follows: denote by ${}^rT^s_*$, $s \geq r$ the s-stage of telescope for $\{C^r \xleftarrow{\alpha_{r+1}} \cdots\}$. Namely ${}^rT^s_*$ is the result of successive constructions of telescopes for α_i, $i \leq s-1$. We get the finite sequence ${}^1T^s_* \supset {}^2T^s_* \supset \cdots \supset {}^sT^s_* = C^s_*$. Suppose by induction in s that we have constructed the maps

$$\begin{array}{cccccccc}
{}^1T^s_* & \supset & {}^2T^s_* & \supset & \cdots & \supset & {}^sT^s_* & = C^s_* \\
\downarrow{\scriptstyle F^s_1} & & \downarrow{\scriptstyle F^s_2} & & & & \downarrow{\scriptstyle F^s_s} & \\
D^1_* & \xleftarrow{\beta_1} & D^2_* & \longleftarrow & \cdots & \xleftarrow{\beta_s} & D^s_* &
\end{array} \tag{3.4}$$

such that the squares are strictly commutative, and such that F^s_j, restricted to the original stage C^j_* of the (finite) telescope ${}^jT^s_*$ is exactly f_j.

This implies in particular that the map F^s_j restricted to the top of the telescope

${}^jT^s$ is $\beta_j \dots \beta_{s-1}\beta_s f_s$. To construct the sequence 3.4 for $(s+1)$ we choose and fix a chain homotopy h from $f_s\alpha_{s+1}$ to $\beta_{s+1}f_{s+1}$. For every j the composition $C_*^{s+1} \xrightarrow{\alpha_{s+1}} C_*^s \subset {}^jT_*^s \xrightarrow{F_j^s} D_*^j$ is exactly $\beta_j \cdots \beta_s f_s \alpha_{s+1}$ which is homotopic to $\beta_j \cdots \beta_s \beta_{s+1} f_{s+1}$ via $\beta_j \cdots \beta_s \cdot h$. The diagram

$$\begin{array}{ccccccc} C_*^{s+1} & \xrightarrow{\alpha_{s+1}} & C_*^s & \subset & {}^jT_*^s & \xrightarrow{F_j^1} & D_*^j \\ \| & & \| & & \cap & & \downarrow \\ C_*^{s+1} & \xrightarrow{\alpha_{s+1}} & C_*^s & \subset & {}^{j-1}T_*^s & \xrightarrow{F_{j-1}^1} & D_*^{j-1} \end{array}$$

is strictly commutative and applying now the lemma 3.9 we get the maps $F_j^{s+1} : {}^jT_*^{s+1} = Z(\alpha_{s+1})$ and the diagram (4) for the next number $(s+1)$, expanding the (4) for s. Passing to the direct limit we get the $F_r = \cup_{s \geq r} F_r^s : T_*^r \to D_*^r$ satisfying 1). The property 2) goes by construction. □

Proof of prop. 3.7: 1) We consider the telescopic inductive system for $\{C_*^r\}$ as in 3.3 and the map of this system to $\{D_*^r\}$ existing by lemma 3.10. Note that maps F_r are homotopy equivalences by the property 2) of lemma 3.10. Let Z_* be the $\varprojlim {}^rT_*$. Passing to the inverse limits, we get now the maps $A : Z_* \to \Gamma_*$, $B : Z_* \to \Delta_*$ which are homology equivalences, since all the maps in the systems were. □

To prove 2) we need one more lemma.

Lemma 3.12 *Suppose that* $\Gamma_* = \{\Gamma_1 \leftarrow \Gamma_2 \leftarrow \cdots,\}$ $C_* = \{C_1 \leftarrow C_2 \leftarrow \cdots\}$ *and* $\alpha : \Gamma_* \to C_*$ *is a map of chain complexes which induces isomorphism in homology. Suppose that* α *is epimorphic and that* C_* *is free. Then there is a chain map* $\beta : C_* \to \Gamma_*$, *such that* $\alpha\beta = id$.

Proof: Let $\Delta_* \subset \Gamma_*$ be $\ker \alpha$. That is a subcomplex with zero homology. Suppose by induction that we constructed the maps $\beta_* : C_* \to \Gamma_*$ for $* \leq s$, such that $\alpha_1\beta_1 = id$ and β_* commute with differential. It suffices to choose for each generator e of C_{s+1} the element $b(e)$ in Γ_{s+1}, such that $\alpha b(e) = e$ and $\partial b(e) = \beta_s(\partial e)$. Consider $z = \partial b(e) - \beta_s(\partial e) \in \Gamma_s = C_s \oplus \Delta_s$, and denote by z_1, z_2 its components in C_s and Δ_s. By induction assumption $\partial z_2 = 0$ in Δ_s but since $H_*(\Delta_*) = 0$, $z_2 = \partial \tilde{z}_2$, where $\tilde{z}_2 \in \Delta_{s+1}$. Now we set $\beta_{s+1}(e) = b(e) - \tilde{z}_2$ and the lemma is proven. □

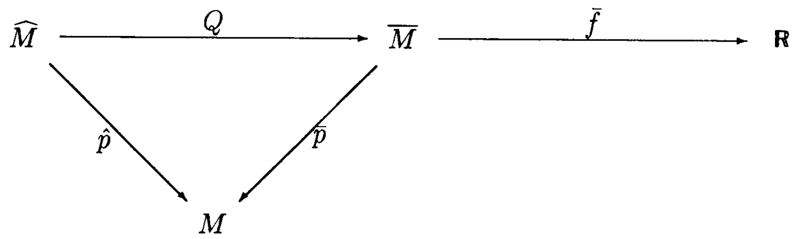
Now we return to the proof of prop. 3.7. We now have the diagram

$$
\begin{array}{ccccc}
 & & \overset{\beta}{\curvearrowright} & & \\
\Gamma_* & \xleftarrow{\;A\;} & Z_* & \xrightarrow{\;B\;} & \Delta_* \\
{\scriptstyle p_0''}\downarrow & & {\scriptstyle p_0}\downarrow & & {\scriptstyle p_0'}\downarrow \\
C^0_* & \xleftarrow{\;\pi_0\;} & T^0_* & \xrightarrow{\;F_0\;} & D^0_* \\
 & & \underset{h_0}{\curvearrowright} & &
\end{array}
\qquad (3.5)
$$

where two squares are strictly commutative, $A\beta = id$, A, B are homology equivalences, $F_0 \sim h_0\pi_0$. Hence $B\beta$ is the homology isomorphism (Γ_*, Δ_* are free, hence a homotopy equivalence) such that $p_0' \circ B\beta \sim h_0 \circ p_0''$. □

4 The Proof of the Main Theorem. 1

Recall that ω stands for a Morse form on a closed manifold M and the rank of $\mathrm{Im}([\omega] : \pi_1 M \to \mathbf{R})$ equals 1. We fix any ω-resolving covering $\hat{p} : \widehat{M} \xrightarrow{G} M$ and denote by $\bar{p} : \overline{M} \xrightarrow{\mathbf{Z}} M$ the minimal ω-resolving covering, which is infinite cyclic. There exists a commutative triangle



corresponding to an epimorphism of groups $q : G \to \mathbf{Z}$. Q is a covering with the structure group $H = \ker q$. Let $\bar{f} : \overline{M} \to \mathbf{R}$ be the corresponding Morse function; $d\bar{f} = \bar{p}^*\omega$, and denote $\bar{f} \circ Q$ by $\hat{f}$.

Let t be a generator of $\mathbf{Z}$, uniquely determined by the condition $\bar{f}(tx) < \bar{f}(x)$. Choose and fix any element $\theta \in G$, such that $q(\theta) = t$.

Recall that we denoted by ξ the homomorphism $G \to \mathbf{R}$, induced by $[\omega] : \pi_1 M \to \mathbf{R}$. The class $[\omega]$ factors also through $\mathbf{Z}$ and we denote $\bar{\xi} : \mathbf{Z} \to \mathbf{R}$ the corresponding homomorphism.

Choose now any regular value of $\bar{f}$. Since we can change $\bar{f}$ by adding a constant, we can assume that this value is zero. Let a denote $\bar{f}(x) - \bar{f}(tx)$. The preimage $\bar{f}^{-1}([-a, 0])$ is a compact manifold W with boundary $\partial W = V \sqcup tV$, where V stands for $\bar{f}^{-1}(0)$, and $\bar{f}$ is a Morse function on a cobordism W. Let V^- denote the space $\{\bar{f}(x) \leq 0\}$, V^+ - the space $\{\bar{f}(x) \geq 0\}$, and by W_n - the preimage $f^{-1}([-(n+1)a, -na])$, so $W_0 = W$. Let $W(n)$ denote the preimage $f^{-1}([-(n+1)a, 0])$, that is $W(n) = \bigcup_{i=1}^{n} W_i$. Note that for any n the restriction $Q|Q^{-1}(W(n)) \to W(n)$ is a regular covering with the structure group H.

If v is a gradient-like vector field for ω on the manifold M, then the lifting of v to $\overline{M}$, which will be denoted by the same letter is a t-invariant vector field on $\overline{M}$, which provides a gradient-like vector field for a Morse function $\bar{f}|W(n)$ for every n. If $c \in M$ is a zero of v and $\bar{c}$ is some lifting of c to M then the liftings of the stable and unstable manifolds

$B^{st}(c)$, $B^{un}(c)$ to $\overline{M}$, such that center lifts to $\bar{c}$ are the stable and unstable manifolds for a critical point $\bar{c}$ of the function $\bar{f}$. Hence if $B^{st}(c_i) \pitchfork B^{un}(c_j)$ for two zeros c_i, c_j of v, then $B^{st}(\bar{c}_i) \pitchfork B^{un}(\bar{c}_j)$ where $\bar{c}_i, \bar{c}_j$ are the arbitrary liftings of c_i and c_j.

Proof of the lemma 2.1: We are given two zeros c_i, c_j of ω of indexes $p, p-1$ and some liftings $\hat{c}_i, \hat{c}_j$ to $\widehat{M}$. Denote by $\bar{c}_i, \bar{c}_j$ the projections $Q(\hat{c}_i), Q(\hat{c}_j)$.

Let $\alpha \in \mathbf{R}$ and let γ be any $(-v)$-trajectory starting at c_i, finishing at c_j, with $\xi(h(\gamma)) \geq \alpha$. We are to prove that there is at most a finite number of such γ's. The lifting of γ to $\widehat{M}$, starting at $\hat{c}_i$, finishes at $\hat{z} = \hat{c}_j \cdot h(\gamma)$, hence $\hat{f}(\hat{z}) \geq \hat{f}(\hat{c}_j) + \xi(h(\gamma)) \geq \bar{f}(\bar{c}_j) + \alpha$. This implies that the projection of γ to $\overline{M}$ is the v-trajectory, which starts at the point $\bar{c}_i$ of index p, finishes at the point $\bar{c}_j$ of the index $(p-1)$ and stays all the time in the domain $f^{-1}([-Na, Na])$ for some large N. Since v satisfies the transversality condition, there is at most a finite number of such trajectories (Lemma B.3), hence at most finite number of their liftings to $\widehat{M}$. □

Proof of theorem 2.2: The proof will occupy the rest of this section and the following one.

First we shall reduce the proof to the study of the Morse function f, restricted to V^-. For that we need some more notations.

Let $\mathbf{Z}G^-$ and $\Lambda_\xi^=$ denote the subrings of $\mathbf{Z}G$ (and, correspondingly, of Λ_ξ^-), consisting of finite linear combinations (correspondingly, power series) with the supports contained in $\{\xi(g) \leq 0\}$. Note that those $x \in \mathbf{Z}G^-$ (corresp. $x \in \Lambda_\xi^=$) for which suppx $\in \{\xi(\mathrm{g}) < 0\}$ form the double-sided ideal of $\mathbf{Z}G^-$ (corr. $\Lambda_\xi^=$), which equals $\theta \cdot \mathbf{Z}G^-$ (corr. $\theta \cdot \Lambda_\xi^=$), and that $\mathbf{Z}G^-/\theta \cdot \mathbf{Z}G^-$, $\Lambda_\xi^=/\theta\Lambda_\xi^=$ are isomorphic to ZH (as $\mathbf{Z}G^-$ – and, correspondingly $\Lambda_\xi^=$-modules).

Let $U(G,\xi)^-$ denote the multiplicative group of power series $x \in \Lambda_\xi^=$ of the form $x = \pm g + x'$, where suppx$' \subset \{\xi(\mathrm{g}) < 0\}$; g then belongs necessarily to H. We denote by $Wh^-(G,\xi)$ the group $K_1(\Lambda_\xi^=)/U(G,\xi)^-$. The projection $\pi : \Lambda_\xi^= \to ZH$ sends $U(G,\xi)^-$ to $\{\pm h\}$ and hence determines the homomorphism $\pi_* : Wh^-(G,\xi) \to Wh(H)$. If C_* is a free f.g. complex over $\Lambda_\xi^=$ then $C_*/\theta C_* = C_* \otimes_{\Lambda_\xi^=} \mathbf{Z}H$ is a free f.g. ZH-complex. If C_*, D_* are the free f.g. complexes over $\Lambda_\xi^=$ and $h : C_* \to D_*$ is a homotopy equivalence, then $h : C_*/\theta C_* \to D_*/\theta D_*$ is also a homotopy equivalence and $\pi(\tau(h)) = \tau(\bar{h})$. □

Lemma 4.1 *$h_* : Wh^-(G,\xi) \to Wh(H)$ is an isomorphism.*

Proof: Surjectivity is obvious. Suppose next that for some matrix A over $\Lambda_\xi^=$ the matrix $A \otimes_{\Lambda_\xi^=} \mathbf{Z}H$ is equivalent to $(\pm h) \oplus E$ where $h \in H$, $(\pm h)$ represents a matrix in $GL(1)$, E stands for a unit matrix via several elementary transformations. Making the same elementary transformations over $\Lambda_\xi^=$ (note that $ZH \subset \Lambda_\xi^=$) we get that A is equivalent to the matrix B, where $b_{00} = \pm h + \lambda_{00}, \ldots, b_{ii} = 1 + \lambda_{ii}, \ldots$ and all the λ_{ii} have the support contained in $\{\xi(g) < 0\}$. Since the elements b_{ii} are invertible in $\Lambda_\xi^=$ we can make the elementary transformations with B so as to get the matrix, equivalent to $(\pm h) + E$ over $\Lambda_\xi^=$. □

Corollary 4.2 *The chain homotopy equivalence $h : C_* \to D_*$ of the free f.g. $\Lambda_\xi^=$-complexes is simple if and only if $\bar{h}$ is simple.*

Return now to the Novikov complex.

We recall that to define the complex we must fix a g.-l. vector field v satisfying the transversality assumption, a lifting of all the zeros to the covering M and the orientations of descending discs. Note that if we change the liftings of the zeros, the matrices ∂_p will change by multiplication of some of the rows and the columns by elements of G. That does not influence the equality $\partial_p \circ \partial_{p+1} = 0$, neither the simple homotopy type of the resulting complex. So we can assume henceforth that the liftings c_i for the ω-zeros c_i belong to $Q^{-1}(W_0)$. Let c_i denote the projection $Q\hat{c}_i \in W_0$. Note that $\hat{c}_i \cdot g$ belongs to W_k if and only if $\xi(g) = -ka$, hence for this choice of liftings all the incidence coefficients $n(c_i, c_j)$ belong to $\Lambda_\xi^=$. This implies that the set of maps ∂_p is defined over $\Lambda_\xi^=$, and we denote the corresponding object $(C_*^-(v), \partial_*^-)$. Note next that if we choose the smooth triangulation of M in such a way that V is a simplicial subcomplex that induces the smooth triangulation of $\overline{M}$ such that $V_i, W_i, W(i), V^-, t^i V^-$ are subcomplexes. We denote by the sign $\hat{}$ their preimages in M. The chain complex of $\widehat{V}^-$ is a free f.g. chain complex over $\mathbf{Z}G^-$ and $C_*(\widehat{V}^-)/\theta C_*(\widehat{V}^-) = C_*(\widehat{V}^-) \otimes_{\mathbf{Z}G^-} \mathbf{Z}H$ is exactly $C_*(\widehat{W}_0, \widehat{V})$.

We claim that in order to prove the theorem 2.2 it is enough to prove the following:

Theorem 4.3 *1) $\partial_p \cdot \partial_{p+1} = 0$*

2) the complex $(C_*^-(v), \partial^-)$ is homotopy equivalent to $C_*(\widehat{V}^-) \otimes_{\mathbf{Z}G^-} \Lambda_\xi^=$ via homotopy equivalence h.

3) this h can be chosen so as to give simple homotopy equivalence when tensored with $\mathbb{Z}H$.

Proof of the theorem 2.2 from the theorem 4.3: The map $\partial_p : C_p(v) \to C_{p-1}(v)$ is as we have seen the tensor product $\partial_p^- \otimes_{\Lambda_\xi^=} \Lambda_\xi^-$, hence 1) of 4.3 implies 1) of 2.2, and $C_*(v) = C_*^-(v) \otimes_{\Lambda_\xi^=} \Lambda_\xi^-$. Now $C_*(\widehat{M}) = C_*(\widehat{V}^-) \otimes_{\mathbb{Z}G^-} \mathbb{Z}G$, hence $C_*(\widehat{M}) \otimes_{\mathbb{Z}G} \Lambda_\xi^- = C_*(\widehat{V}^-) \otimes_{\mathbb{Z}G^-} \Lambda_\xi^- = C_*(\widehat{V}^-) \otimes_{\mathbb{Z}G^-} \Lambda_\xi^= \otimes_{\Lambda_\xi^=} \Lambda_\xi^-$. Therefore p.2) of 4.3 provides a homotopy equivalence $h \otimes_{\Lambda_\xi^=} \Lambda_\xi^- : C_*(v) \to C_*(\widehat{M}) \otimes_{\mathbb{Z}G} \Lambda_\xi^-$. Now h is simple by corollary 4.2, which implies that $\tau(h)$ is contained in the image of $U^-(G,\xi)$ in $K_1(\Lambda_\xi^=)$, hence $\tau(h \otimes_{\Lambda_\xi^=} \Lambda_\xi^-)$ vanishes in $Wh(G,\xi)$.

Passing to the proof of Th. 4.3 we need still more notations. Let I_n be the two-sided ideal of $\mathbb{Z}G^-$, consisting of all the elements $\lambda \in \mathbb{Z}G^-$ with $\mathrm{supp}\lambda \subset \{\xi(\mathrm{g}) \leq -(\mathrm{n}+1)\mathrm{a}\}$, and let $\mathbb{Z}G_n$ be $\mathbb{Z}G^-/I_n$. $\mathbb{Z}G_n$ is a ring and a $\mathbb{Z}G^-$-bimodule, isomorphic to $\mathbb{Z}G^-/\theta^n\mathbb{Z}G^-$ as a right $\mathbb{Z}G^-$-module. Note that $\mathbb{Z}G_0 \approx \mathbb{Z}H$. Note further that $\mathbb{Z}G_n^-$ is a free right $\mathbb{Z}H$-module with the generators $1, \theta, \ldots, \theta^n$. Denote, further, by $\mathbb{Z}G(m,n)$ where $n \geq m$ the ring I_m/I_n; it has no unit. The left multiplication by θ^m determines the isomorphism of right $\mathbb{Z}G^-$-modules $ZG_{n-m} \to I_m/I_n$. The ring $\Lambda_\xi^=$ is by definition the completion of $\mathbb{Z}G^-$ with respect to the system of ideals $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$, i.e. the inverse limit of the system $\to \mathbb{Z}G_n \to \mathbb{Z}G_{n-1} \to \cdots \to \mathbb{Z}H$. For any free f.g. $\mathbb{Z}G^-$-module F the module $F \otimes \Lambda_\xi^-$ is the inverse limit of $\cdots \to F/\theta^n F \to F/\theta^{n-1}F \to \cdots \to F/\theta F$, and, vice versa, a free f.g. module $\mathcal{F}$ over $\Lambda_\xi^=$ is the inverse limit of the modules $\mathcal{F}/I_n\mathcal{F} = \mathcal{F} \otimes_{\Lambda_\xi^=} ZG_n^-$.

Next we consider the cobordism $W(n)$, $\partial W(n) = V \cup t^{n+1}V$ and a Morse function $\bar{f} : W(n) \to [-(n+1)a, 0]$ with the g.-l. vector field v. All the zeros of v are of the form $\bar{c}_i t^k$, where $0 \leq k \leq n$. We lift these points to the covering $\widehat{W(n)} \to W(n)$, choosing for $\bar{c}_i t^k$ the lifting $c_i\theta^k$. We have oriented the descending discs already, therefore we have the data for constructing the Morse complex $C_*(\widehat{W(n)}, v)$, which is a free f.g. complex over $\mathbb{Z}H$. The generator, corresponding to $\hat{c}_i\theta^k$, will be denoted $e_{i,k}$. Due to our special choice of liftings, there is a right action of $\mathbb{Z}G_n$ on it. Indeed, every element of $C_*(\widehat{W(n)}, v)$ (correspondingly of $\mathbb{Z}G^-$) is a $\mathbb{Z}$-linear combination of the elements $e_{i,k}h$, where $h \in H$ (correspondingly of $\theta^s h$, where $0 \leq s \leq n-1$, $h' \in H$), hence it suffices to define the action for such pairs. We set $(e_{i,k}\cdot h)(\theta^s h') = 0$ if $k+s > n-1$ and $(e_{i,k}\cdot h)(\theta^s h') = e_{i,k+s}(\theta^{-s}h\theta^s h')$. One easily checks associativity, and we obtain the $\mathbb{Z}G^-$-action, which expands the $\mathbb{Z}H$-

action. Note that differential ∂ of the Morse complex respects this action. (Indeed, the differential of $e_{i,k}h$ is by definition $\sum e_{j,s}\cdot n(\hat{c}_i\theta^k, \hat{c}_j\theta^s)\cdot h$ where the sum runs over $0 \le n-1$ and c_j – over critical points of index, less by a unit. The differential of $(e_{i,k}\cdot h)(\theta^q h')$ is $\sum e_{\ell,r} n(\hat{c}_i\theta^{k+q}, \hat{c}_\ell\theta^r)\cdot\theta^{-q}h\theta^q h'$.

But v is θ-invariant, therefore $n(\hat{c}_i\theta^{k+q}, \hat{c}_i\theta^{k+q}, \hat{c}_j\theta^{s+q}) = \theta^{-q}\cdot n(\hat{c}_i\theta^k, \hat{c}_j\theta^s)\cdot\theta^{+q}$ and hence $\partial[(e_{i,k}h)(\theta^q h')] = \sum e_{j,s}\cdot\theta^{-q}n(\hat{c}_i\theta^k, \hat{c}_j\theta^s)\cdot h\theta^q h'$, which is by definition $[\partial(e_{i,k}\cdot h)]\cdot\theta^q h'$.)

It is obvious that $C_*(\widehat{W(n)}, v)$ is a free $\mathbb{Z}G_n^-$-module with the generators $e_{i,o} = \hat{c}_i$ and so $(C_*(\widehat{W(n)}, v), \partial)$ is a free f.g. $\mathbb{Z}G_n$-complex. Note that it is also the complex of $\mathbb{Z}G^-$- (and $\Lambda_\xi^{\mp}$)-modules via epimorphism $\mathbb{Z}G^- \to \mathbb{Z}G_n^-$ (corr. $\Lambda_\xi^{\mp} \to \mathbb{Z}G_n$) and that a natural projection $C_*(W(\widehat{n+1}), v) \to C_*(\widehat{W(n)}, v)$ commutes with this $\mathbb{Z}G^-$-action, as well as with $\Lambda_\xi^{\mp}$-action. This implies that the inverse system

$$\cdots \to C_*(W(\widehat{n+1}), v) \to C_*(\widehat{W(n)}, v) \to \cdots \tag{4.1}$$

has an inverse limit which bears the structure of $\Lambda_\xi^{\mp}$-module.

Consider now the $\mathbb{Z}G_n^-$-module $C_*^-(v,n) = C_*^-(v) \otimes_{\Lambda_\xi^{\mp}} \mathbb{Z}G_n^-$. The generators of this module are the v-zeros c_i and the map $\partial_p : C_p^-(v,n) \to C_{p-1}^-(v,n)$ is given by the formula (2.1). where the elements g of $\mathbb{Z}G^-$ with $\xi(g) < -na$ are neglected.

It is obvious now that there is ZG_n^--isomorphism $C_*^-(v,n) \to C_*(\widehat{W(n)}, v)$, sending c_i to $e_{i,o} = \hat{c}_i$, which commutes with differentials. This implies that an inverse system

$$\cdots \to C_*^-(v,n) \to C_*^-(v,n-1) \to \cdots \tag{4.2}$$

of $\Lambda_\xi^{\mp}$-modules is isomorphic to the system (4.1) and thus the direct limit of (4.1) is $\Lambda_\xi^{\mp}$-isomorphic to the $C_*^-(v)$.

Lemma 4.4 *There exists a diagram*

$$\begin{array}{ccccccc}
\cdots \to & C_*(\widehat{W(n)}, v) & \longrightarrow & C_*(W(\widehat{n-1}), v) & \to \cdots \\
& \downarrow^{h_n} & & \downarrow^{h_{n-1}} & \\
\cdots \to & C_*^\Delta(\hat{V}^-, (t^{n+1}V^-)\hat{}) & \longrightarrow & C_*^\Delta(\hat{V}^-, (t^n V^-)\hat{}) & \to \cdots
\end{array} \tag{4.3}$$

with the following properties

1. *the maps h_n are the chain homotopy equivalences over $\Lambda_\xi^=$*

2. *the diagrams (4.3) are homotopy commutative*

3. *the map h_0 is a simple homotopy equivalence over $\mathbf{Z}H$*

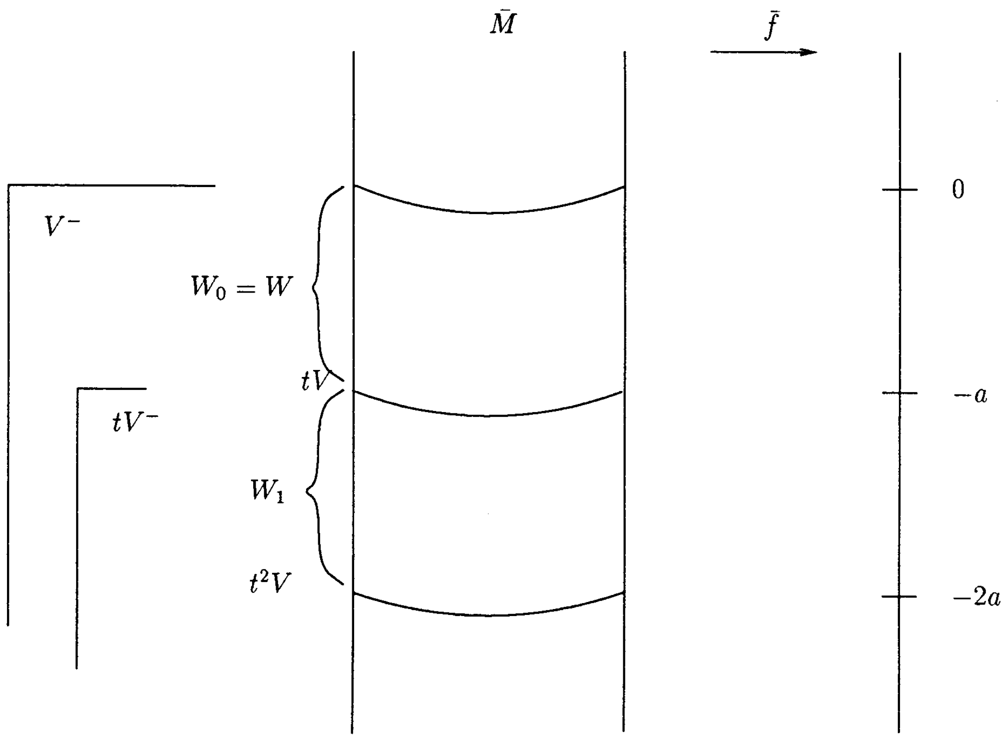
(here C_*^Δ stands for a chain simplicial complex for a triangulation, introduced above).

Proof of the theorem 4.3 from lemma 4.4: From the proposition 3.7 we deduce that there is a $\Lambda_\xi^=$-homotopy equivalence f such that the following diagram is homotopy commutative

$$\begin{array}{ccccccc} C_*^-(v) & \approx & \underline{\lim} C_*(\widehat{W(n)}, v) & \xrightarrow{f} & \lim C_*(\widehat{V}^-, (t^n\widehat{V}^-)) & \approx & C_*^-(\widehat{V}^-) \otimes_{\mathbf{Z}GZ} G^-\Lambda_x \\ & \pi_1 \searrow & \downarrow & & \downarrow & \swarrow \pi_2 & \\ & & C_*(W, v) & \xrightarrow{h_0} & C_*(\widehat{V}^-, (t\widehat{V}^-)) & & \end{array}$$

Both slant arrows π_1, π_2 are tensor products with $\mathbf{Z}H$ over $\Lambda_\xi^=$, hence we get $h_0 \cdot \pi_1 \sim f \otimes_{\Lambda_\xi^=} \mathbf{Z}H$ and since any $\Lambda_\xi^=$-map of a free $\Lambda_\xi^=$-module F to $\mathbf{Z}H$-module G factors uniquely through $F \otimes_{\Lambda_\xi^=} \mathbf{Z}H$, we get also $h_0 \sim f \otimes_{\Lambda_\xi^=} \mathbf{Z}H$, which means that f is a simple homotopy equivalence, and the theorem 4.3 is proved. □

Lemma 4.4 will be proved in the next section.



Picture 4.1

5 The proof of the main theorem. 2.

Lemma 5.1 *There exists a regular Morse function $F : W(n) \to \mathbf{R}$, such that v is a gradient-like vector field for F, F is constant on V and on $t^{n+1}V$, and if x, $tx \in W(n)$ then $F(tx) < F(x)$.*

Proof: We need some notations. Let N be the number of different indices $p_1, \dots, p_N$ of zeros of v; $0 \leq N \leq n$. We assume $N \geq 2$. Let $Cr(q) \subset M$ denote the set of all zeros of index q, $Cr_i := Cr(p_i)$, $Cr := \cup_i Cr_i$. Denote by $\overline{Cr_i}$, $\overline{Cr(q)}$, $\overline{Cr}$ the preimages of these sets in $\overline{M}$.

Our original function $\bar{f}$ is a Morse function, when restricted to W. By [4], §4, we can change $\bar{f}$ to a function $\bar{f}_1 : W \to [-c, 0]$, with the same g.-l. vector field v, and such that $\bar{f}_1 | Cr_i = -c + \frac{c}{N}(i - \frac{1}{2})$ for every i.

Let $\pi : \mathbf{R} \to \mathbf{R}/c\mathbf{Z} = S^1_c$ be the natural projection. Since $\bar{f}_1(tx) = \bar{f}_1(x) - c$, $\bar{f}_1$ determines correctly the map $\varphi_1 : M \to S^1_c$, which is a Morse map. (Recall that a map $\varphi : M \to S^1$ is called a Morse map, if the differential $d\varphi$, considered as a differential 1-form via the standard trivialization of TS^1, is a Morse form.) The field v is a g.-l. vector field for φ. The φ_1-image of its critical set is the set Γ, consisting of N points, equidistantly placed on S^1_c, the zero $0 \in S^1_c$ being in the middle of one of the segments. We will call a Morse map $g : M \to S^1_c$ admissible, if 1) v is a g.-l. vector field for g; 2) $g(Cr) = \Gamma$, for every i the map $g|Cr_i$ is constant and $g|Cr_i \neq g|Cr_j$ for $i \neq j$.

Note that 2) implies that for $\gamma \in \Gamma$ the index ind_γ is defined.

Now we describe an operation by which one changes an admissible map.

Suppose that $g : M \to S^1_c$ is an admissible map, $\bar{g} : \overline{M} \to \mathbf{R}$ – some lifting of g and $\gamma_1, \gamma_2 \in \Gamma$, such that $\gamma_2 = \gamma_1 + \frac{c}{N}$ and $\mathrm{ind}\ \gamma_2 < \mathrm{ind}\ \gamma_1$. This means that γ_2 is a critical value, next to γ_1. For a small enough ε let N denote $g^{-1}([\gamma_1 - \varepsilon, \gamma_2 + \varepsilon])$, $\partial N = g^{-1}(\gamma_2 + \varepsilon) \cup g^{-1}(\gamma_1 - \varepsilon)$. The map $g|N$ is a Morse function to $[\gamma_1 - \varepsilon, \gamma_2 + \varepsilon]$, having two critical values $\gamma_2 > \gamma_1$ and $\mathrm{ind}\ \gamma_2 < \mathrm{ind}\ \gamma_1$. Then construction of [4], §4, provides us with the Morse function $r : N \to [\gamma_1 - \varepsilon, \gamma_2 + \varepsilon]$, having v as a g.-l. vector field and such that $r|g^{-1}(\gamma_2) = \gamma_1$, $r|g^{-1}(\gamma_1) = \gamma_2$. Set $g_1|M \backslash N = g$, $g_1|N = r$. That is obviously again an admissible Morse map. Choose the lifting $\bar{g}_1$ in such a way that $\bar{g}_1 = \bar{g}$ everywhere except $\bar{p}^{-1}(N)$. To be precise, $\bar{g}(x) = \bar{g}_1(x)$ if $\bar{g}(x)$ does not belong to one of the segments

$[\gamma_1 - \varepsilon + nc, \gamma_2 + \varepsilon + nc]$ and if $\bar{g}(x) \in [\gamma_1 - \varepsilon + nc, \gamma_2 + \varepsilon + nc]$ then $\bar{g}_1(x) = r(x) + nc$, which belongs to the same segment. We call this the elementary modification of index (ind γ_2, ind γ_1). The ind γ_2 is called the lower index of the modification, the ind γ_1 – the upper index. The result of this operation on the $\bar{g}|\overline{Cr}$ is the following

$$\begin{cases} \bar{g}_1(x) = \bar{g}(x) + \frac{c}{N} & \text{for} \quad x \in \overline{Cr}(\text{ind}\gamma_1) \\ \bar{g}_1(x) = \bar{g}(x) - \frac{c}{N} & \text{for} \quad x \in \overline{Cr}(\text{ind}\gamma_2) \\ \bar{g}_1(x) = \bar{g}(x) & \text{for} \quad x \in \overline{Cr}(g), \quad q \neq \text{ind}\gamma_1, \quad q \neq \text{ind}\gamma_2. \end{cases}$$

Note that if $y, z \in \overline{Cr}$ and $\text{ind}(y) < \text{ind}(z)$ and $\bar{g}(y) < \bar{g}(z)$ then for any elementary operation as above we have again $\bar{g}_1(y) < \bar{g}_1(z)$. (Indeed if at least one of y, z is not carried by $g : \overline{M} \to S^1$ neither to γ_1 nor to γ_2, that is obvious. That is obvious also if y, z both belong to different bricks of $g^{-1}([\gamma_1 - \varepsilon, \gamma_2 + \varepsilon])$, and if they both belong to the same brick $\bar{g}^{-1}([\gamma_1 - \varepsilon + nc, \gamma_2 + \varepsilon + nc])$ the original assumption does not hold.) □

Lemma 5.2 *Let A be a finite subset of the set $\overline{Cr}$ of critical points of $\bar{f}_1$. Then there exists a finite series of elementary modifications of $\bar{f}_1$ resulting with an admissible function $g : M \to S^1_c$ such that if $x, y \in A$ and ind $x <$ ind y then $\bar{g}(x) < \bar{g}(y)$.*

Proof: Denote by $\mathcal{A}_g \subset A \times A \backslash \Delta$ the set of all the pairs $(x, y) \in A \times A$, such that ind $x <$ ind y and $\bar{g}(x) > \bar{g}(y)$. By the remark above the set $\mathcal{A}_{g_1}$ is contained in $\mathcal{A}_g$. So to prove lemma 5.2 it is enough for any admissible g to construct a series of elementary modifications with the resulting $\mathcal{A}_{g_n} \subsetneqq \mathcal{A}_g$. □

For that we need an extra lemma.

Lemma 5.3 *Let $x, y \in \overline{Cr}$, ind $x <$ ind y, $\bar{g}(x) > \bar{g}(y)$. There is a sequence of elementary modifications with lower indices $\leq$ ind x, such that for the resulting admissible function $f : M \to S^1$ we have $\bar{g}(y) \leq \bar{f}(y)$, $\bar{f}(x) < \bar{g}(x)$.*

Proof: Induction in $n = (\bar{g}(x) - \bar{g}(y))\frac{c}{N}$ (that is a natural number since g is admissible).

1) $n = 1$. We apply an elementary modification to the pair $(\gamma_1, \gamma_2) = (\pi\bar{g}(x), \pi\bar{g}(y))$ and are over.

2) induction step. Suppose that $n = (\bar{g}(x) - \bar{g}(y))\frac{c}{N} > 1$. We distinguish two cases.

A) For all the points $w \in \overline{Cr}$ such that $\bar{g}(y) < \bar{g}(w) < \bar{g}(x)$ we have ind $w \leq$ ind x.

In this case pick up some $w \in \overline{Cr}$ such that $\bar{g}(w) - \bar{g}(y) = c/N$. Apply to this pair the elementary modification. We obtain a new function $\bar{f}$, $\bar{f}(y) = \bar{g}(w) = \bar{f}(y) + \frac{c}{N}$. The upper index of this modification is ind $y >$ ind x, hence $\bar{f}(x) \leq \bar{g}(x)$. Now $\bar{f}(x) - \bar{f}(y) \leq \bar{g}(x) - \bar{g}(y) - \frac{c}{N}$ and we refer to the induction assumption.

B) There exists a point $w \in \overline{Cr}$ such that ind $w >$ ind x and $\bar{g}(y) < \bar{g}(w) < \bar{g}(x)$. Then we apply the induction assumption to the pair w, x, and get as a result a function $\bar{f}$, such that $\bar{g}(w) \leq \bar{f}(w)$, $\bar{f}(x) < \bar{g}(x)$. The lower indices of elementary modifications are $\leq$ ind x, ind y hence the value of y had not decreased, and $\bar{g}(y) \leq \bar{f}(y)$, and we are again in the situation of induction assumption.

It follows from this lemma that for $x, y \in \overline{Cr}$, ind $x <$ ind y there is a sequence of elementary modifications such that for a resulting function f we get $\bar{f}(x) < \bar{f}(y)$, and that is precisely what we needed to prove the lemma 5.2. □

The proof of lemma 5.1: We apply the lemma 5.2 to the set A which is the set of all zeros of v contained in the domain $\{-(n+1) \leq \bar{f}(x) \leq 0\}$. As a result we get a Morse map $g : M \to S^1_c$, belonging to the homology class of $[\omega] \in H^1(M, R)$, which has v as a gradient-like field and such that if $x, y \in A$ then $\bar{g}(x) < \bar{g}(y)$ where $\bar{g}$ is any lifting of g to the map $\bar{M} \to \mathbf{R}$. Note that $\bar{g}(xt) = \bar{g}(x) - c$.

Restrict now the function $\bar{g}$ to $W(n)$. It is not constant neither on V nor on $t^{n+1}V$. To make it constant there we apply to g the upper and lower damping introduced in Appendix B. Recall that the resulting function $F = \overline{\underline{\mathbf{g}}}$ enjoys the following properties (here $\overline{C}$ is some small neighborhood of V and $\underline{C}$ - some small neighborhood of $t^{n+1}V$):

1. $F(x) = \bar{g}(x)$ except for $x \in \underline{C} \cup \overline{C}$.

2. $F(x) \geq \bar{g}(x)$ for $x \in \overline{C}$, $F(x) \leq \bar{g}(x)$ for $x \in \underline{C}$.

3. F is constant on V and on $t^{n+1}V$.

4. F has the same critical points as $\bar{g}$ and v is also a g.-l. vector field for F.

I claim that F satisfies the conditions of lemma 5.1. Indeed we are only to show that if x, $tx \in W(n)$ then $F(x) > F(tx)$. If x and tx both do not belong to $\underline{C} \cup \overline{C}$, then

$F(x) = \bar{g}(x) > \bar{g}(tx) = F(tx)$. If one of these points belongs to $\underline{C} \cup \overline{C}$ or $tx \in \underline{C}$. In the first case, for example, we get $F(x) \geq g(x) > g(tx) = F(tx)$, the latter equality is due to the fact that $tx \in \underline{C}$. The case $tx \in \underline{C}$ is treated similarly. The lemma 5.1 is proved.

Now we pass to the proof of lemma 4.4. To construct the isomorphisms h_n we apply the machinery, exposited in Appendix B. We need a bit more than that, since we want the homotopy equivalence I of the Morse complex and the simplicial one to commute with θ. For that we use the regular function F from lemma 5.1.

More precisely, let $f : W(n) \to \mathbf{R}$ be a regular Morse function, constant on V and on $t^{n+1}V$, having v as a gradient-like vector field, and such that if x, $tx \in W(n)$ then $f(tx) < f(x)$. Such functions will be called t-regular; they exist by the above. We can assume that $a = f(t^{n+1}V) = \min_{x \in W(n)} f(x)$, $b = f(V) = \max_{x \in W(n)} f(x)$. Now we choose a sequence of regular values $a = a_0 < a_1 < \cdots < a_n < a_{n+1} = b$, such that all the critical points of index p belong to $\{a_p < f(x) < a_{p+1}\}$. Consider the filtration Q_i of the pair $(V^-, t^{n+1}V^-)$ given by $Q_i = (t^{n+1}V^- \cup \{f(x) \leq a_i\},\ t^{n+1}V^-)$. By our condition on f this filtration is preserved by the action of t. Consider the filtration F^i_*, induced by Q_i on the singular chain complex: $F^i_* = C^s_*((t^{n+1}V^- \cup \{f(x) \leq a_i\})\hat{}, t^{n+1}V^-)\hat{})$. That is invariant under t, therefore, this filtration is a filtration by $\mathbf{Z}G^-_n$-modules. This filtration is good, since f is regular and it is nice if considered over $\mathbf{Z}H$ by appendix B. Moreover this filtration is nice over $\mathbf{Z}G^-_n$. To prove that denote $c_1, \ldots, c_N$ the critical points of f (that is, the zeros of v) of index p, contained in W and let $\hat{c}_1, \ldots, \hat{c}_N$ be the arbitrary liftings of their to $\widehat{W}$.

The free $\mathbf{Z}H$-generators of $H_p(\widehat{Q}_{p+1}, \widehat{Q}_p)$ are then given by the liftings of the descending discs $s_{ij} = (D^p_{i,j}, S^{p-1}_{i,j})$, where the center of this lifting is $\hat{c}_i \cdot \theta^j$, and $S^{p-1}_{i,j}$ belongs to $\{f(x) \leq a_p\}\hat{}$. Note that the result of the action of θ^k on $s_{i,j}$ is again the lifting of the descending disc of the point $c_i t^{j+k}$ with the origin in the $\hat{c}_i \theta^{j+k}$ (by the t-invariance of v) which means that the homology class $[s_{i,j}]\theta^k$ is the same as $[s_{i,j+k}]$. (To be more precise one applies the lemma B.13 as follows. Consider the manifold $W' = W_{-1} \cup W(n+1)$ and expand f to W to be a Morse function which is more than d on W_{-1} and less than c on $W(n+1)$. Consider $\varphi : W(n+1) \to \mathbf{R}$ to be defined as $\varphi(x) = f(tx)$. Apply now the lemma B.13 to $W = W(n+1)$ and the functions f, φ and $\alpha = \alpha' = a_p$ and S^+_c is the boundary of the stable disc of $c_i \cdot t^{j+k}$.)

That implies that the homomorphism of the free $\mathbf{Z}G_n$-module, generated by $\hat{c}_i$, to $H_p(\hat{Q}_{p+1}, \hat{Q}_p)$, which sends c_i to $D_{i,o}$ is an isomorphism.

This map is exactly the composition of $\mathcal{J}_f : C_p(\widehat{W(N)}, v) \to H_p(\{f(x) \leq a_{p+1}\}\hat{}, \{f(x) \leq a_p\}\hat{})$ with excision isomorphism. Hence we get the isomorphism of chain complexes, preserving bases, which we again denote by $\mathcal{J}_f$:

$$\mathcal{J}_f : C_*(\widehat{W(n)}, v) \to F_*^{gr}.$$

By the corollary 3.4 we obtain a $\mathbf{Z}G_n^-$-chain homotopy equivalence $H(n,f) : C_*(\widehat{W(n)}, v) \to C_*^s((V^-)\hat{}, (t^{n+1}V^-)\hat{})$, which respects the filtration F_*, induces $\mathcal{J}_f$ in the graded groups and is uniquely determined up to chain homotopy by these properties. □

Lemma 5.4 *For two t-regular functions f, g the chain maps $H(n,f), H(n,g)$ are chain homotopy equivalent.*

Proof: We cannot just refer to lemma B.14 since we need not only $\mathbf{Z}H$- but a $\mathbf{Z}G^-$-homotopy. But the proof evidently is a simple modification of that of lemma B.14. Let $f : W(n) \to [a,b]$, $g : W(n) \to [a', b']$ be the regular Morse functions as above and $a = a_0 < a_1 < \cdots < a_{n+1} = b$, $a' < a_1 < \cdots < a_{n+1} = b'$ the sequences of the regular values of f.g. separating the zeros of v of different indices. Let Q_i, Q_i' be the corresponding filtrations of $(V^-, t^{n+1}V^-)$. Let P_i denote $Q_i \cup Q_i'$; the space P_i is invariant under t and as it is proved in lemma B.14 the inclusions $Q_i \subset P_i \subset Q_i'$ induce the isomorphisms in homology, hence P_i is a nice filtration, and both $H(n,f)$ and $H(n,f')$ respect it. □

All what is left to prove is that every generator c_i of $C_*(\widehat{W(n)}, v)$ goes to the same element in $H_*(\hat{P}_i, \hat{P}_{i-1})$. That is provided by lemma B.13.

Lemma 5.5 *For any n and any t-regular functions $f : W(n) \to \mathbf{R}, g : W(n-1) \to \mathbf{R}$ the square*

$$\begin{array}{ccc} C_*(\widehat{W(n)}, v) & \xrightarrow{p_n} & C_*(W(\widehat{n-1}), v) \\ H(n,f) \Big\downarrow & & H(n-1,g) \Big\downarrow \\ C_*^s((V^-)\hat{}\,) & \xrightarrow{\pi_n} & C_*^s((V^-)\hat{}\,, (t^n V^-)\hat{}\,) \end{array} \quad (5.1)$$

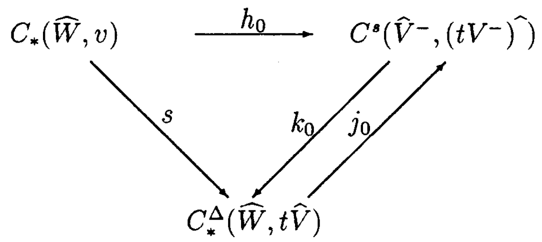
is homotopy commutative.

Proof: This is again an obvious modification of the proof of lemma B.15. By the lemma 5.4 we can choose any g we like. Consider the function $f : W(n) \to [a, b]$, restricted to $W(n-1)$. It is a regular Morse function, constant on the upper boundary. Take now a lower damping $\underline{f}$ which will be denoted g. This is a t-regular Morse function on $W(n-1)$. Choose a sequence $a = a_0 < \cdots < a_{n+1} = b$ separating the points with different indices. This sequence is also separating the points for $\underline{f}$ and if we let P_i and Q_i denote the corresponding filtrations of the pairs $((V^-)\hat{}\,, (t^{n+1}V^-)\hat{}\,)$ and $((V^-)\hat{}\,, (t^n V^-)\hat{}\,)$, then $P_i \subset Q_i$, which means that π_n in the diagram (5.1) preserves filtrations. Hence to prove that $H(n-1, g)p_n \sim \pi_n H(n, f)$ it is enough to prove that these maps induce the same homomorphism in the graded homology. This follows from the lemma B.13 applied to the cobordism $W = W(n)$, the descending disc of some zero c of v and two functions, one of which is f and another which is g on $W(n-1)$ and continued to some Morse function the whole $W(n)$ in such a way that $\varphi|W_n$ is less than $g|t^n V$. □

To get the lemma 5.1 finally we consider the canonical imbeddings $j_n : C_*^\Delta((V^-)\hat{}\,, (t^{n+1}V^-)\hat{}\,) \to C_*^s((V^-)\hat{}\,, (t^{n+1}V^-)\hat{}\,)$ which are $\mathbb{Z}G_n$-homotopy equivalences and commute obviously with π_n. Choosing an arbitrary homotopy inverse k_n for j_n we get the diagram (4.3) sought.

To treat the zero step we choose a regular Morse function $f : W \to [a, b]$ with a g.-l. vector field v, a sequence of values $a = a_0 < a_1 < \cdots < a_n < a_{n+1} = b$ separating the critical points of different indices, and a smooth triangulation Δ for which all the submanifolds $f^{-1}(a_i)$, $f^{-1}([a_i, a_j])$ and all the descending discs are the subcomplexes.

We recall from the theorem 8.4 that there exists a simple homotopy equivalence $s : C_*(\widehat{W}, v) \to C_*^\Delta(\widehat{W}, t\widehat{V})$, preserving the simplicial filtration of the latter complex and sending the zeros of v to the corresponding descending discs. This implies that in the diagram below



we have $j_0 s \sim h_0$ (by corollary 3.4), hence $k_0 h_0 \sim s$ and we get the point 3) of lemma 4.4. $\square$

A Appendix

In this appendix we expose the proof of a (simplified) version of the Kupka-Smale theorem, following [8]. The discussion can be found in the introduction and now we proceed to the necessary definitions.

Let M^n be a compact smooth (i.e. C^∞)-manifold. A vector field v on M will be called regular, if all the zeros of v are hyperbolic and for any zero c of v of index p, there exists a neighborhood W_c of c and a diffeomorphism $\Phi_c : W_c \to B^p(0, 2\varepsilon) \times B^{n-p}(0, 2\varepsilon) \subset \mathbf{R}^p \times \mathbf{R}^{n-p}$, such that the point c is carried by φ to zero and the field v to the field with the coordinates $(-x_1, \dots, -x_p, y_1, \dots, y_{n-p})$, where x_i are the standard coordinates in R^p and y_j – in R^{n-p}. We denote by U_c the preimage of $B^n(0, \varepsilon) \times B^{n-p}(0, \varepsilon)$. We assume henceforth that the closures W_c are disjoint for different c. We enumerate all zeros c like $c_1, \dots, c_k$ and, correspondingly, denote $U_{c_i}, W_{c_i}, \dots$ by $U_i, W_i, \dots$. We suppose that the field v comes equipped with the neighborhoods W_c and diffeomorphisms Φ_c, so that these are the part of the definition. The examples of such fields are given by gradient-like fields of Morse forms and these examples give us the motivation of this definition. The Φ_{c_i}-preimages of the discs $B^p(0, \varepsilon) \times \{0\}$, $D^p(0, \varepsilon) \times \{0\}$, $\{0\} \times B^{n-p}(0, \varepsilon)$, $\{0\} \times D^{n-p}(0, \varepsilon)$ (lying in U_c) will be denoted by B_i^+, D_i^+, B_i^-, D_i^-. Sometime we indicate their dimension and also the vector field v, like $B_i^{+,p}(v)$. The boundaries of these discs will be denoted by S_i^+, S_i^-. For every $\lambda \le \varepsilon$ we introduce the λ-fence $\sum_i^+(\lambda)$ over S_i^+, setting it to be the Φ_i-preimage of $S_i^+ \times B^{n-p}(0, \lambda)$. The $\sum_i^+(\lambda)$ is an open codimension 1 submanifold of M^n, transversal to the flow.

We recall that the space $\mathcal{V}$ of all the C^∞-vector fields on M has a natural topology, generated by the family of seminorms and that convergence with respect to this topology is exactly the convergence together with all partial derivatives in any chart. This space is a complete metric space, hence it has the Baire property. The space $\mathcal{D}$ of all C^∞-diffeomorphisms of M also has a natural topology generated by a notion of convergence with all partial derivatives. For each t the t-shift along the trajectories of a field determines the map $S_t : \mathcal{V} \to \mathcal{D}$, which is continuous in the topologies above. We also will need the topology on the space of all maps from N to M, where N is a manifold (not necessarily compact). That is the topology of convergence (together with all the partial derivatives) on compact subsets.

Let v be a regular vector field. Choose and fix some riemannian metric $\langle\cdot,\cdot\rangle$ on M, such that it is euclidean in the neighborhoods W_i with respect to coordinates x_i, y_i. We call a vector field $\tilde{v}$ a regular perturbation of v if $\tilde{v}$ coincides with v in all the neighborhoods U_i and $\inf_{x\in M\setminus\cup_i U_i}\langle\tilde{v}(x),\tilde{v}(x)\rangle \geq \inf_{x\in M\setminus\cup_i U_i}\langle v(x),v(x)\rangle$. (Then $\tilde{v}$ is obviously regular). Note that "being a perturbation of ..." is a transitive relation. For a given v the space of all regular perturbations of v will be denoted $\mathcal{V}_v$. That is also a complete metric space.

For a field v and a real number t we denote by $\varphi(t,v)$ the diffeomorphism of the time t-shift along the trajectories. If no confusion is possible we omit v in this notation. For a regular vector field v and a zero c_i of v we denote by $B_i^+(t)$ the set $\varphi(t,v)B_i^+$ (similarly $B_i^-(t)$, $D_i^-(t)$ etc.). Note that if $t_1 \leq t_2$ then $B_i^+(t_2) \subset B_i^+(t_1), B_i^-(t_2) \supset B_i^-(t_1)$. We denote by B_i^{st} the union $\bigcup_{t\in\mathbf{R}} B_i^+(t) = \bigcup_{t\leq 0} B_i^+(t)$, by B_i^{un} - the union $\bigcup_{t\in\mathbf{R}} B_i^-(t) = \bigcup_{t\geq 0} B_i^-(t)$. These are injectively immersed manifolds of dimensions p_i and $n-p_i$.

The regular field v will be called perfect if for every pair i,j (also for $i=j$) the manifolds B_i^{un} and B_j^{st} are transversal.

Theorem A.1 *In any neighborhood $\mathcal{U}$ of a given regular vector field v there exists a regular perturbation $\tilde{v}$ of v, such that $\tilde{v}$ is perfect.*

The main ingredient in the proof is the following lemma.

Lemma A.2 *Let v be a regular vector field, c_i, c_j - two zeros of v (i may be equal to j) and t be a positive real number. Then in any neighborhood $\mathcal{U}\subset\mathcal{V}$ of v there is a regular perturbation $\tilde{v}$ of v, belonging to $\mathcal{U}$, such that $D_i^+(-t,\tilde{v})$ is transversal to $B_j^-(t,\tilde{v})$.*

First of all we deduce the theorem A.1 from lemma A.2.

For that note that if for some regular vector field v and two zeros c_i, c_j of v the assertion of lemma A.2 holds, it also holds for every close enough regular perturbation v of v, these two zeros and any $t_0 < t$. (Indeed, $D_i^+(-t)\pitchfork B_j^-(t)$ is equivalent to $D_i^+\pitchfork B_j^-(2t)$. For a close enough regular perturbation $\tilde{v}$ of v the manifold $B_j^-(2t,\tilde{v}) = \varphi(2t,v)(B_j^-)$ is C^∞-close to $B_j^-(2t,v) = \varphi(2t,v)(B_j^-)$ and, therefore, also transversal to D_i^+ on any compact subset, for example on $D_j^-(t+t_0,\tilde{v})$. That implies $D_i^+(-t,\tilde{v})\pitchfork B_j^-(t_0,\tilde{v})$.)

Note further that if v is regular, then $(-v)$ is also regular with the same zeros c_i and the same neighborhoods U_i, W_i and diffeomorphisms Φ_i. The stable and unstable discs for $(-v)$ satisfy $B_i^+(t;-v) = B_i^-(-t,v)$, $D_i^+(t,v) = D_i^-(-t,v)$.

Apply now the lemma A.2 to the field v, zeros c_i, c_j and the number $t+1$. We get the regular perturbation $\tilde{v}$, such that $D_i^+(-t-1,\tilde{v}) \pitchfork B_j^-(+t+1,\tilde{v})$. Next we apply the same lemma to the vector field $(-\tilde{v})$, zeros c_i, c_j and the number t and get the perturbation w of $(-\tilde{v})$, arbitrarily close to $(-\tilde{v})$, such that $D_i^+(-t,w) \pitchfork B_j^-(t,w)$, which is equivalent to $D_i^-(t,-w) \pitchfork B_j^+(-t,-w)$. Note that $(-w)$ is a regular perturbation of $\tilde{v}$ and if we choose w to be close enough to $(-\tilde{v})$ the transversality $D_i^-(t+1,-v) \pitchfork B_j^+(-t-1,-v)$ is preserved on a smaller segment by the argument above, i.e. $D_i^-(-t,w) \pitchfork B_j^+(t,w)$, which means that $(-w)$ is an (arbitrary small) regular perturbation of v, such that $D_i^{\pm}(\pm t,w) \pitchfork B_j^{\pm}(\pm t,w)$.

Applying this argument several times, we finally get that the set $\mathcal{V}_{t,v}$ of all regular perturbations of v, such that for given t the manifolds $D_i^{\pm}(\pm t)$ are transversal to $B_j^{\pm}(\pm t)$ for all i, j, is dense in $\mathcal{V}_v$. It follows also that some open neighborhood of $\mathcal{V}_{t,v}$ belongs to $\mathcal{V}_{t-1,v}$. Denote that neighborhood by $\overline{\mathcal{V}}_{t,v}$. Applying to the system $\overline{\mathcal{V}}_{t,v}$ the Baire theorem we get the theorem A.1.

Now we proceed to the proof of the lemma A.2.

Let $0 < \lambda \leq \varepsilon$ and T_1, T_2 be positive real numbers. We denote by $\Psi(\lambda; T_1, T_2)$ the map of $(-T_1, T_2) \times \sum_i^+(\lambda)$ to M, defined as follows. Consider the trajectory γ of v, defined on the interval $(-T_1, T_2)$, such that $\gamma(0) = x$, where $x \in \sum_i^+(\lambda)$. Then the Ψ-image of (τ, x) is $\gamma(\tau)$. Since the $\gamma(t)$ does not have selfintersections if $x \in S_i^+$, for any given (T_1, T_2) there exists λ so small that $\Psi(\lambda; T_1, T_2)$ is injective. From the definition of $\sum_i^+(\lambda)$ it is obvious that the image of $(-\theta, \theta) \times \sum_i^+(\lambda)$ is open for θ small enough and, therefore $\Psi(\lambda; T_1, T_2)$ is a diffeomorphism onto its image $L(\lambda; T_1, T_2)$ which is the open set in M.

We choose T_2 so big that $\varphi(T_2, v)D_i^+ = D_i^+(T_2)$ does not intersect with $\varphi(t,v)D_j^- = D_j^-(t)$, T_1 to be $t+1$ and λ to be so small that $\Psi(\lambda; T_1, T_2)$, which will be abbreviated as Ψ, is an embedding and that $\varphi(T_2, v) \sum_i^+(\lambda)$ does not intersect with $D_j^-(t)$.

Note that the Ψ-inverse image of v is just the field $(-1, 0)$. Note also that the manifold $\Psi(\{\tau\} \times \sum_i^+(\lambda))$ is the codimension 1 submanifold of M, transversal to the flow v, and therefore to the submanifold $B_j^-(\tau')$ (where $\tau, \tau' \in \mathbf{R}$), so their intersection is a submanifold, denoted by $I(\tau, \tau')$.

We will denote $\varphi(-t, v) \sum_i^+(\lambda)$ by $\widehat{\sum}_i^+(\lambda)$ and $\varphi(T_2, v) \sum_i^+(\lambda)$ by $\underline{\sum}_i^+(\lambda)$.

The proof of Lemma A.2 will be over if we find a vector field $\tilde{v}$, which is close enough to v and satisfy the three following conditions:

1) The support of $\tilde{v}-v$ is contained in the Ψ-image of some compact subset, belonging to $(-\varepsilon,0)\times\sum_i^+(\lambda/3)$ and on the set supp($\tilde{\text{v}}-\text{v}$) the number $\langle\tilde{v}(x),\tilde{v}(x)\rangle$ is not less than ε.

2) the Ψ^{-1}-image of the vector field $\tilde{v}$ (we keep for that image the same notation) has the first coordinate equal to $(+1)$ (by the first coordinate we mean the projection $(-T_1,T_2)\times\sum_i^+(\lambda)\to(-T_1,T_2)$).

3) The trajectories of $(-\tilde{v})$, starting at $\{0\}\times\sum_i^+(\lambda/3)$ stay in $[-t,0]\times\sum_i^+(\lambda/2)$ for the values of parameter not bigger than t and the manifold $\varphi(+t,-v)S_i^+$ is transversal to the intersection $I(-t,t)=B_j^-(t)\cap\varphi(-t,v)\sum_i^+(\lambda)$.

Indeed, assume that we have constructed a vector field $\tilde{v}$, close to v and satisfying 1)-3). We show now that it satisfies the conclusion of the lemma A.2.

Note that $\tilde{v}=v$ for $x\in U_i$. Further, the infimum $\langle v(x),v(x)\rangle$ for x belonging to ∂U_i is equal to ε, so the condition 1) implies that $\tilde{v}$ is a regular perturbation of v. Consider now the intersection $\varphi(-t,\tilde{v})D_i^+\cap B_j^-(t,\tilde{v})$. The set $\varphi(-t,\tilde{v})D_i^+$ is the union $\cup\varphi(\tau,\tilde{v})S_i^+$ where $\tau\in[-t,+\infty)$, which is the union of $\cup_{\tau\geq T_2}\varphi(\tau,v)S_i^+=\varphi(T_2)D_i^+$ and $\cup_{-t\leq\tau<T_2}\varphi(\tau,\tilde{v})S_i^+$. The first one does not intersect with $D_j^-(t)$ by the choice of T_2, hence if $\tilde{v}$ is close enough to v, the first one does not intersect with $D_j^-(t)$ either. The second one is contained in L by 3). So we are to prove that at any point x of $\varphi(-t,\tilde{v})D_i^+\cap B_j^-(t,\tilde{v})$, belonging to L, the intersection is transversal. Consider the trajectory γ of $(-\tilde{v})$, beginning at x. By the property 3) it lies in L until it meets $\Psi(-t,\sum_i^+(\lambda))$ at the point x, which also belongs to this intersection. Note that if some v-trajectory γ, defined on $[a,b]$, joining the points x and y is contained in $B_j^-(t,\tilde{v})$ then the $T_gB_j^-(t,\tilde{v})=D\varphi(b-a,\tilde{v})T_xB_j^-(t,\tilde{v})$. The same for D_i^+, so it is enough to prove the transversality sought at the point x. For that, in turn it is enough to prove the transversality of $\varphi(-t,\tilde{v})S_i^+$ and $B_j^-(t,\tilde{v})\cap\widehat{\sum}_i^+(\lambda)$ (since $\widehat{\sum}_i^+(\lambda)$ is transversal to the flow which has not been changed in its neighborhood). For that in turn it is enough to prove that $B_j^-(t,\tilde{v})\cap\widehat{\sum}_i^+(\lambda)$ is the same with that of the initial field: $B_j^-(t,v)\cap\widehat{\sum}_i^+(\lambda)$.

To prove that we demand the vector field v to be so close to v that any $\tilde{v}$-trajectory beginning at the point belonging to $[0,-\varepsilon]\times\overline{\sum_i^+(\lambda/3)}$ stays inside $[0,-\varepsilon]\times\sum_i^+(\lambda/2)$ until the first coordinate becomes bigger than zero. (Recall that v just preserves the second coordinate.)

Suppose now that there is $x \in B_j^-(t,\tilde{v}) \cap \widehat{\sum}_i^+(\lambda)$ but $x \notin B_j^-(t,v) \cap \widehat{\sum}_i^+(\lambda)$. Consider the v-trajectory γ such that $\gamma(0) = x$ and let $(-\infty, t_0)$ be the maximal interval, such that $\gamma(-\infty, t_0) \subset B_j^-(t,\tilde{v})$. Note that this trajectory must have crossed the set $\Psi([0,-\varepsilon] \times \sum_i^+(\lambda/3))$, for some $t_1 < 0$. By the remark above the v-trajectory coming from $\gamma(t_1)$ stays inside $[-\varepsilon, 0] \times \sum_i^+(\lambda/2)$ until the first coordinate becomes greater than zero and then (we are now inside the set $\mathbf{R} \times \sum_i^+(\lambda)$, having applied Ψ^{-1}) the second coordinate is preserved until we reach the set $\{T_2\} \times \sum_i^+(\lambda/2) \subset \sum_i^+(\lambda)$. Note that this must happen at a moment greater than $t_0 > 0$, if $\tilde{v}$ is close enough to v, since $\overline{\sum_i^+(\lambda)} \cap D_j^-(t,\tilde{v}) = \emptyset$ and hence for the small $\tilde{v} - v$ the set $\overline{\sum_i^+(\lambda)} \cap D_j^-(t,\tilde{v}) = \emptyset$. On the other hand there is no point $x = \gamma(0)$ inside the part of L with the first coordinate greater than $(-\varepsilon)$. Contradiction.

Now we proceed as to find the vector field v with the properties 1)-3).

First of all note that the condition on the length of $\tilde{v}(x)$ on p.1) follows from the other conditions of p.1). Indeed, the set $\sum_i^+(\lambda/3)$ is a subset of U_i. The direct calculation (using our restriction $\varepsilon < 1/3$) shows that the set of all τ-shifts of $\sum_i^+(\lambda/3)$ along the trajectories of $(-v)$ for $0 < \tau < \varepsilon$ is the subset of $W_i \backslash U_i$. The metric there is euclidean and the $\langle v(x), v(x) \rangle > \varepsilon$ for all $x \in W_i \backslash U_i$, hence if we change the vector field v on the compact subset of $\Psi((-\varepsilon, 0) \times \sum_i^+(\lambda/3)) \subset W_i \backslash U_i$ and the perturbation is small enough, this property is preserved.

Now we construct our field $\tilde{v}$ at last. To do so we will work inside L and change v to the field $\tilde{v}$, which differs from v only on the compact subset inside L, so that $\tilde{v}$ inside L and v outside L glue together and form the field, defined on the whole M.

The space L is by definition the product $(-T_1, T_2) \times B^q(0,\lambda) \times S_i^+$ (where q is the index of the zero c_i). The subspace $\widehat{\sum}_i^+(\lambda)$ is in these notations $\{-t\} \times B^q(0,\lambda) \times S_i^+$. There is a submanifold $C = B_j^-(t) \cap \widehat{\sum}_i^+(\lambda)$ inside $\widehat{\sum}_i^+(\lambda)$, which is a smooth manifold with countable base. Let us denote the projections of L onto its factors by p_1, p_2, p_3. Applying the Sard's lemma to the map p_2, restricted to C we get that there exist arbitrarily small vectors $u \in \mathbf{R}^q$ (in particular $|u| < \lambda/2$) such that u is a regular value of $p_2|C$, which implies that C is transversal to the submanifold $\{-t\} \times \{u\} \times S_i^+$. (That is a usual Thom transversality, but we preferred to give a direct proof since some of the manifolds involved are non-compact.)

Now we choose a C^∞-function θ, defined on $B^q(0,\lambda)$, which equals to 1 on the ball $B^q(0,\lambda/6)$ and vanishes outside the ball $B^q(0,\lambda/3)$ and put $w_u(r)=\theta(r)\cdot u$ where $r\in B^q(0,\lambda)$). The field w_u is equal to the constant field u inside $B^q(0,\lambda/6)$, in particular in the neighborhood of the segment $\{tu|t\in[0,1]\}$, vanishes outside $B^q(0,\lambda/3)$, is proportional to u at any point and becomes arbitrarily small together with the vector u. Choose next the C^∞-function $\psi(t)\geq 0$ on the interval $(-T_1,T_2)$ with the support, contained in $[-\varepsilon/3,-2\varepsilon/3]$, and such that $\int_{-T_1}^{T_2}\psi(t)dt=1$.

Now we define the vector field $\delta(u)$ on L to be the field which in the point $(t,a,s)\in(-T_1,T_2)\times B^q(0,\lambda)\times S_i^+$ has the coordinates $(0,-\psi(t)w_u,0)$. Finally we set $\tilde{v}=v+\delta(u)$.

Next we check that this field satisfies 1)-3). The first part of 1) is satisfied by the definition of δ and the second follows automatically as discussed above. The condition 2) follows since the first projection of δ is zero and the first projection of v was $(+1)$ by definition. The first part of condition 3), concerning the behaviour of $\varphi(t,-\tilde{v})(\{0\}\times\sum_i^+(\lambda/3))$ follows from the property that δ can be chosen to be arbitrarily small so that the property $\varphi(t,-v)(\{0\}\times\overline{\sum_i^+(\lambda/3)})\subset\widehat{\sum}_i^+(\lambda/2)$ is reserved after replacing v by $\tilde{v}$.

Now we are to find $\varphi(t,-\tilde{v})S_i^+$. Since the third coordinate of $\tilde{v}$ is zero it is enough to find the trajectories of the vector field W on the space $(-T_1,T_2)\times B^q(0,\lambda)$, defined in the point (τ,r) by the formula $(-1,\psi(\tau)\cdot\theta(r)\cdot u)$. We need the trajectories, starting from $(0,0)$. It is easy to check that the curve $\gamma(\tau)=(-\tau,u\cdot\int_{-\tau}^{0}\psi(\mu)d\mu)$ satisfies the equation $\dot{\gamma}=W(\gamma)$, and that $\gamma(t)$ equals $(-t,u)$. So, the trajectory of $(-v)$, beginning at $(0,0,X)\in S_i^+$ ends at $(-t,u,x)\in\widehat{\sum}_i^+$ and the image of S_i^+ under $\varphi(-t,\tilde{v})$ is exactly $\{-t\}\times\{u\}\times S_i^+$, hence transversal to C. □

B Appendix

Morse complex of a Morse function

Let $(W; V_0, V_1)$ be an n-dimensional manifold with boundary $\partial W = V_0 \sqcup V_1$. Let $f : W \to \mathbf{R}$ be a Morse function such that $Imf = [a, b]$, $f^{-1}(a) = V_0$, $f^{-1}(b) = V_1$ and all the zeros of f' belong to $W = W \backslash \partial W$.

Pick up any gradient-like vector field v for f. We will keep all the notations and conventions introduced in §2 for the neighborhoods of critical points (where we understand that ω is df), namely U_c, W_c, B_i^+, B_i^-.

One essential difference from §2 here is that W is not closed and therefore the diffeomorphisms $\varphi(t, v)$ are no longer defined. So we denote by B_i^{un} the set of all points $x \in \overset{\circ}{W}$, such that the trajectory γ of the field v, starting at x converges to c_i when $t \to \infty$. The definition of B_i^{st} is the same except that we change v for $(-v)$. One easily checks that B_i^{st} is the image of an open conical set in $\mathbf{R}^p$, containing the zero, under the injective immersion. This open set will be denoted U_i^{st}.

Now we make the *transversality assumption*: for any two zeros c_i, c_j the manifolds B_i^{un} and B_j^{st} are transversal.

Lemma B.1 *There exists the g.l. vector field v for f, satisfying the transversality assumption.*

Proof: (an obvious modification of that of theorem 5.2 in [4]). Enumerate the critical values of f as $a_1, \ldots, a_k$, $a_i > a_{i-1}$. We will proceed by induction.

Suppose by induction assumption that we constructed a g.l. vector field v_i such that 1) $v_i = v$ for $\{f(x) \geq a_i - \varepsilon\}$, ε is small enough. 2) For any pair of zeros c_k, c_m belonging to $\{f(x) \leq a_i\}$ the B_k^{st} is transversal to B_m^{un}. (Note that if c_k, $c_m \in \{f(x) \leq a_i\}$ then $B_k^{st} \cap B_m^{un} \subset \{f(x) \leq a_i\}$.)

Choose now any value $d \in (a_i, a_{i+1})$. For any zero c_k, lying below d we denote by $\overline{B_k}^{un}$ the intersection $B_k^{un} \cap f^{-1}(d)$. This set is the image under the injective immersion of a submanifold of U_k^{st}, defined by the equation $f(z) = d$, and denoted by $V_k^{st}(d)$. Consider now all the stable discs descending from critical points with value a_{i+1} and denote by $K(v_i)$ the intersection of this set with $\{f(x) = d\}$. That is a compact submanifold of $f^{-1}(d)$ with trivial normal fibration. Now, proceeding exactly as in [4], §5, lemma 5.3, we find a small isotopy φ of K such that $\varphi(K)$ is transversal to all the $V_k^{st}(d)$ and applying [4], §4, lemma 4.6, we get the new vector field v_{i+1}, which equals v_i everywhere except $\varphi^{-1}[d, d+\varepsilon]$ and such that $K(v_{i+1})$ is exactly $\varphi(K)$.

Now the condition $B_k^{st} \pitchfork B_m^{un}$ for $c_k, c_m \in [a, a_i]$ still holds, since $v_{i+1} = v_i$ if $f(x) \in [a, a_i]$. Further, the intersection of $B_k^{un} \cap f^{-1}(d)$ also did not change for $c_k \in [a, a_i]$, hence the transversality $B_m^{st} \pitchfork B_k^{un}$ holds also for $f(c_m) = a_{i+1}$ and we get the induction step.

After we get the inductive proposition for $i = k$ the lemma is proved. □

Note that for the principal text we do not need to apply this lemma since the g.-l. vector field satisfying transversality assumption is provided by the g.-l. vector field for the form ω satisfying the assumption. We give here the proof for completeness.

In what follows the vector field v, satisfying the transversality assumption is fixed.

Lemma B.2 *There exists a Morse function $g : W \to [c, d]$, such that v is a gradient-like vector field for g and that all the critical points of the same index have the same value, increasing with the index.*

Proof: Apply several times the [4], th. 4.1. Such functions will be called regular with respect to φ. □

Lemma B.3 *Let c_1 and c_2 be the zeros of v of the index p and $p-1$. Then there is at most a finite set of trajectories of $(-v)$, starting at c_1 and finishing at c_2.*

Proof: Choose the Morse function g so as to satisfy the conditions of Lemma B.2. Let d be any value in the interval $(g(c_1), g(c_2))$. The set of $(-v)$-trajectories going from c_1 to c_2 is in $(1-1)$-correspondence with the set of points in the intersection of $(B_1^{st} \cap g^{-1}(d))$ and $(B_2^{un} \cap g^{-1}(d))$. These manifolds both are compact submanifolds of $g^{-1}(d)$, transversal to each other and the sum of dimensions is $n-1$. □

Now we will describe two versions of Morse complex and prove that these are isomorphic. To define the first version we need to fix the following data:

1. The g.-l. vector field v for the function f.

2. For each critical point c_k of f the orientation of the manifold B_k^{st}.

3. The regular covering $\bar{p} : \overline{W} \to W$ with a structure group G (It is not supposed to be connected).

4. For each zero c_k the lifting $\bar{c}_k$ to the covering $\overline{W}$.

With these data fixed we proceed as follows. For any $0 \leq p \leq n$ form a free right $\mathbf{Z}G$-module $C_p(v)$ where the zeros of v of index p serve as a free generators.

Note that since B_k^{st} is oriented, the complementary disc B_k^{un} is cooriented. Now suppose that γ is a trajectory of $(-v)$ coming from c_1 to c_2 where c_1 is a zero of index p, and c_2 - of index $(p-1)$. Let x belong to γ. Choose the positive base $\{v, e_1, \ldots, e_{p-1}\}$ in $TB_1^{st}(x)$. The vectors $e_1, \ldots, e_{p-1}$ form the base in the normal fibration to TB_2^{un}. The latter is cooriented and the transition sign is denoted by $\varepsilon(x)$. It is easy to see that it does not depend on the choice of the base $\{e_i\}$ and on the choice of a particular point x on γ. So there is a sign $\varepsilon(\gamma)$. Further, if we lift a trajectory γ to $\overline{M}$ so as to start at the point $\bar{c}_1$ then it will end at a point $\bar{c}_2 \cdot g$, where g is a uniquely determined element of G. We call it $g(\gamma)$.

Now for every critical point c_i of index p and a critical point c_n of index $(p-1)$ we put $n(c_i, c_k) = \sum_\gamma \varepsilon(\gamma) g(\gamma)$ where the sum runs over all the $(-v)$-trajectories, coming from c_i to c_k. For a critical point c of index p we set $\partial_p c = \sum d \cdot n(c, d)$, where the sum runs over all the critical points of index $(p-1)$. Expanding by linearity we get a map $\partial_p : C_p(v) \to C_{p-1}(v)$.

Remark about the notations. Recall that for $X \subset W$ we denote by ^{-}X some lifting of X to W, i.e. some section of $\overline{W} \to W$, defined on X. The lifting of a point x is denoted by $\bar{x}$. The full preimage of some $Y \subset W$ is denoted by $\overline{Y}$ or Y^{-}.

Theorem B.4 $\partial_p \partial_{p-1} = 0$ *and the resulting complex is simply homotopy equivalent to the chain complex of* $(\overline{W}, \overline{V}_0)$*, induced by any smooth triangulation of the pair* (W, V_0).

Remark B.5 *Note that the complex depends only on the vector field* v*, but not on the choice of the particular Morse function* f*, for which* v *is a g.-l. vector field. The full notation would be* $C_*(v, \overline{W}, \mathcal{O}, \ell)$ *where* $\mathcal{O}$ *is the orientation choice,* ℓ *is the lifting choice. The change of* $\mathcal{O}$ *changes by sign some of the lines and rows in the matrices* ∂_p *and the change of* ℓ *- by multiplication of some lines and rows by elements of* G.

We shall begin the proof after we define the second version of Morse complex, which will be used in the proof.

Choose any Morse function $\varphi : W \to [\alpha, \beta]$ for which v is a g.-l. vector field and such that there exists a sequence $\alpha < \alpha_1 < \cdots < \alpha_n < \beta$ for which all the critical points of φ of index p lie inside $\{\alpha_p < \varphi(x) < \alpha_{p+1}\}$. (We do not suppose that all the critical points of the same index lie on the same level.)

Lemma B.6 *1) For every critical point c_i of φ of index p the pair $(B_i^{st} \cap \{\varphi(x) \geq \alpha_p\}$, $B_i^{st} \cap \{\varphi(x) = \alpha_p\})$ is diffeomorphic to (D^p, S^{p-1}). Denote it by $(D^p(c_i), S^{p-1}(c_i))$.*

2) The pair $\{\varphi(x) \leq \alpha_p\} \cup [\cup D^p(c_i)]$ where the union runs over the critical points of index p is a deformation retract of $\{\varphi(x) \leq \alpha_{p+1}\}$.

Proof: If all the critical points of index p were lying on the same level of φ then the assertion is contained in [4], the end of §3. If not, then we apply [4], th. 4.1, to change the function φ inside $\{\alpha_p + \varepsilon \leq \varphi(x) \leq \alpha_p - \varepsilon\}$ so as to get the new one, possessing this property. Neither the discs $D(c_i)$, nor the sets $\{\varphi(x) \leq \alpha_p\}$, $\{\varphi(x) \leq \alpha_{p+1}\}$ changed. □

Now fix the liftings $\bar{c}_i$ of the critical points to $\overline{M}$. Then there exist the unique liftings ${}^{-}D(c_i)$ of $D(c_i)$ such that c_i goes to $\bar{c}_i$ (by covering homotopy), and the cells ${}^{-}D(c_i) \cdot g$ where $g \in G$ form the cell decomposition for the pair $(\{\varphi(x) \leq \alpha_{p+1}\}^{-}, \{\varphi(x) \leq \alpha_p\}^{-})$. That implies that if we choose an orientation for $D(c_i)$, then the homology $H_*(\{\varphi(x) \leq \alpha_{p+1}\}, \{\varphi(x) \leq \alpha_p\})$ vanishes for $* \neq p$ and is a free $\mathbf{Z}G$-module for $* = p$ with base elements $[{}^{-}D(c_i)]$ corresponding to c_i. We denote this module by $C_p(\varphi, \overline{W})$ (or simply by $C_p(\varphi)$) and denote by $d_p : C_p(\varphi) \to C_{p-1}(\varphi)$ the differential of the exact sequence of the triad $(\{\varphi(x) \leq \alpha_{p+1}\}, \{\varphi(x) \leq \alpha_p\}, \{\varphi(x) \leq \alpha_{p-1}\})$. This gives rise to the free finitely generated $\mathbf{Z}G$-complex $C_*(\varphi)$.

Suppose now that the choices of liftings $\bar{c}_i$ and orientations of $D(c_i)$ in the definitions of $(C_*(v), \partial)$ and $(C_*(\varphi), d)$ are the same. There is a natural identification of the free base of $C_p(v)$ and of $C_*(\varphi)$, given by $c_i \to [{}^{-}D(c_i)]$. Expand it to the isomorphism $\mathcal{J}$ of right $\mathbf{Z}G$-modules.

Lemma B.7 *$\mathcal{J} \circ \partial = d \circ \mathcal{J}$, hence $\partial^2 = 0$ and the chain complexes $C_*(v)$ and $C_*(\varphi)$ are isomorphic with isomorphism $\mathcal{J}$ preserving bases.*

Proof: Nothing of the data changes if we change the function φ (preserving the g.-l. vector field) inside the set $\{\alpha_p + \varepsilon \leq \varphi(x) \leq \alpha_{p+1}\}$ for some p. So, proving that $\mathcal{J} \circ \partial \mid C_p = d \circ \mathcal{J} \mid C_p$ we suppose that φ is constant on the set of critical points of index p and take there a value, say β; $\alpha_p < \beta < \alpha_{p+1}$. □

We need one more lemma. To formulate it we introduce some more notations. Let $Q^p \subset \{\alpha_p \leq \varphi(x) \leq \alpha_{p+1}\}$ be any oriented compact manifold without boundary, belonging to $\{\varphi(x) = \alpha_{p+1}\}$. We denote by $S_1^{n-1-p}, \dots, S_k^{n-1-p}$ the intersections of the unstable manifolds $B_1^{un}, \dots, B_k^{un}$ of the critical points $c_1, \dots, c_k$ of index p with $\{\varphi(x) = \alpha_{p+1}\}$. These are indeed spheres of dimension $n-1-p$. We assume that Q^p is transversal to each of them. Denote by $q_1, \dots, q_r \in Q^p$ the intersection points, by $\varepsilon_1, \dots, \varepsilon_r$ the orientation signs, by γ_i the trajectory of $(-v)$, beginning at q_i, and by D_i – the small closed p-disc in Q^p around q_i. We denote by $c(i)$ the critical point of φ (i.e. one of $c_1, \dots, c_k$) which is the end of γ_i.

Lemma B.8 *There exists a $\gamma > 0$, such that the imbedding $id : Q^p \subset \{\alpha_p \leq \varphi(x) \leq \alpha_{p+1}\}$ is homotopic via a homotopy F_t, $0 \leq t \leq 1$ to a map $\psi = F_1 : Q^p \to \{\alpha_p \leq \varphi(x) \leq \alpha_{p+1}\}$, satisfying the following properties*

1. $\psi(Q^p) \subset \{\varphi(x) \leq \beta\}$.
2. $\psi(Q^p \setminus \cup_i D_i) \subset \{\varphi(x) \leq \beta - \delta\}$.
3. *The preimage of c_i is the set of all q_s, such that $c(s) = c_i$.*
4. *For some small neighborhood U_i of the origin q_i in D_i the map $\psi \mid U_i$ is a diffeomorphism onto some neighborhood of $c(i)$ in the $B_{c(i)}^{st}$. The sign of this diffeomorphism is ε_i and $\psi^{-1}(\psi(U_i)) \cap D_i = U_i$.*
5. *The image of the curve $F_t(q_i)$ is $\gamma_i \cup \{c(i)\}$.*

Proof: The homotopy F_t will be just a restriction of the deformation retraction described in the proof of the theorem 3.14 if [4].

For the sake of completeness we describe here the retraction and deduce the properties 1)-5). Let c be any critical point of φ; c is also a critical point of f. The function φ is obtained from f by a series of standard modifications, described in [4] §4. In the small neighborhoods of zeros these modifications reduce to adding a constant; so there exists $\bar{\varepsilon} < \varepsilon$, such that for any zero c the diffeomorphism $\Phi_c : U_c \to B^p(0, \bar{\varepsilon}) \times B^{n-p}(0, \bar{\varepsilon})$ reduced to $\Phi_c^{-1}(B^p(0, \bar{\varepsilon}) \times B^{n-p}(0, \bar{\varepsilon}))$, takes φ to the function $\varphi(c) - \sum_{i=1}^p x_i^2 + \sum_{j=1}^{n-p} y_j^2$.

We are interested in the critical points of φ of index p, which all lie on the level β. We will assume for simplicity that there is one critical point on the level β. Nothing changes in the general case except the notations. For any $\nu \leq \bar{\varepsilon}$ we denote by V_ν the subspace $\{\varphi(x) = \varphi(c) - \nu^2\}$ and by D_ν the closed disc $|\mathbf{x}| \leq \nu; D_\nu \subset B^p(0, \bar{\varepsilon}) \times \{0\} \subset U_c$ (we do not distinguish the points in U_c and their Φ_c-images for brevity.

Denote by C_ν the cylinder $\{|\mathbf{x}| \leq \sqrt{2}\nu,\ |\mathbf{y}| \leq \nu, -|x|^2 + |y|^2 \geq -\nu^2\} \subset B^p(0, \bar{\varepsilon}) \times B^{n-p}(0, \bar{\varepsilon})$ (see the picture B.1., we assume naturally that $\sqrt{2}\nu < \bar{\varepsilon}$). The boundary ∂C_ν consists of two parts, one is called $E_\nu = \{|\mathbf{y}| = \nu\}$, another – $G_\nu = \{-|x|^2 + |y|^2 = -\nu^2\}$. Denote by F_ν the boundary of $\{\varphi(x) \leq \beta - \nu^2\} \cup C_\nu$, that is $F_\nu = (V_\nu \backslash C_\nu) \cup E_\nu$.

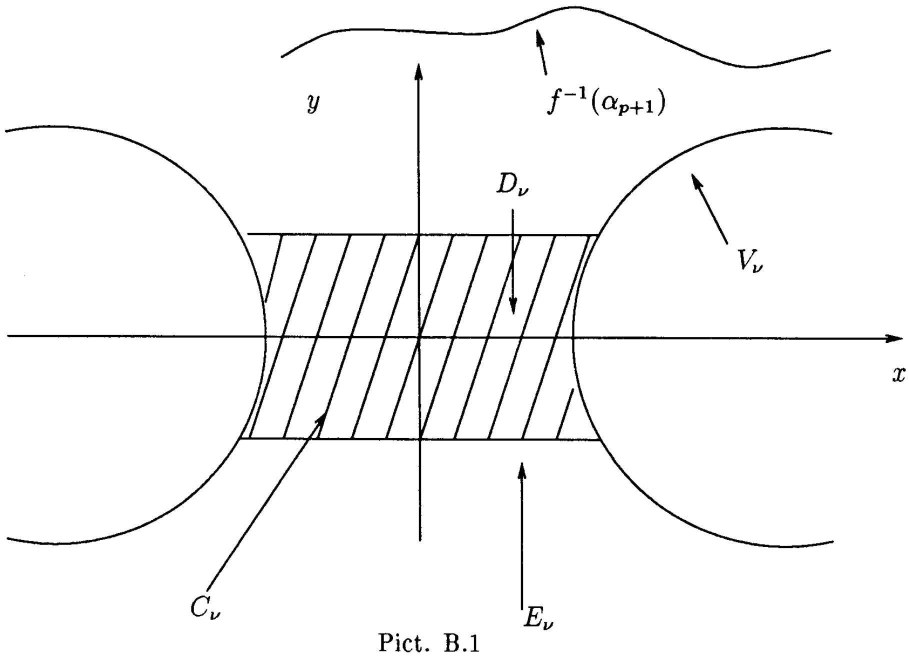
Now for any point $z \in \varphi^{-1}(\alpha_{p+1})$ we denote by γ_z the trajectory of $(-v)$ starting at z. Since the vector field v is separated from zero inside $\{\beta - \nu^2 \leq \varphi(x) \leq \alpha_{p+1}\} \backslash C_\nu$, the trajectory γ_z reaches F_ν at some moment $t(z)$ and at some point $v(z) \in F_\nu$. It is easy to show that $t(z)$ and $w(z)$ are continuous in z and that the map $z \to w(z)$ determines the homeomorphism h_1 of $\varphi^{-1}(\alpha_{p+1})$ to F_ν which is a final map of a continuous isotopy h_t, $0 \leq t \leq 1$, taking place in $\{\varphi(x) \leq \alpha_{p+1}\}$.

Now we define the deformation retraction h_t, $1 \leq t \leq 2$, of C_ν to $G_\nu \cup D_\nu$, which is going along the paths $\{x = \text{const}, \mathbf{y}/|\mathbf{y}| = \text{const}$, as drawn in the picture B.2.

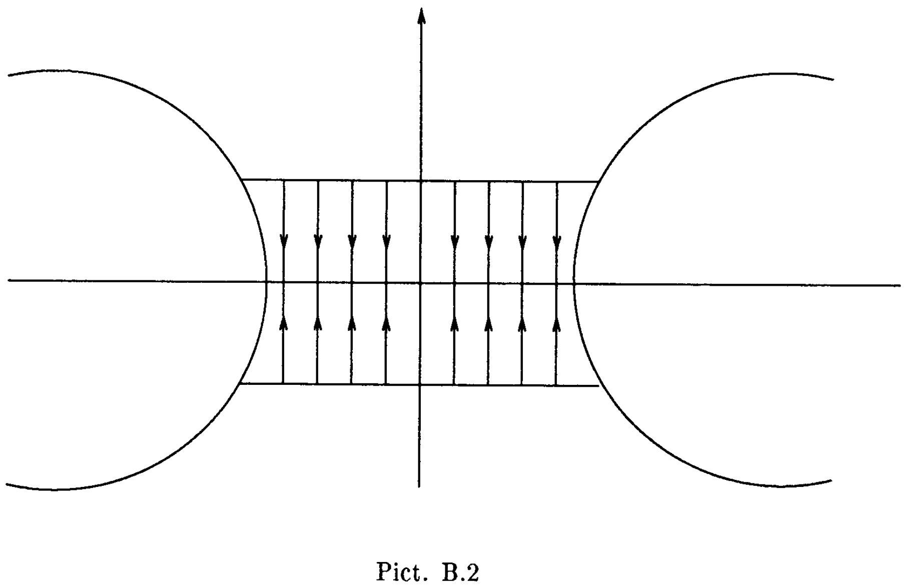
The resulting homotopy will be a composition of those two and the map will be denoted by $h : \varphi^{-1}(\alpha_{p+1}) \to V_\nu \cup D_\nu$. Note that it is not smooth.

For the case of several critical points, belonging to the level β of φ, the retraction h is constructed in the same way and the final image of h is $V_\nu \cup \cup_{i=1}^k D_{\nu,i}$ (the additional index i refers to the number of the critical point).

Now we restrict h to our submanifold Q^p (keeping the same notation h). The preimage of any critical point c_i consists of all the points $z \in Q^p$, such that the trajectory $\gamma(z)$ of $(-v)$ ends at c_i. These are exactly the intersection points $Q^p \cap S_i^{n-1-p}$. For any such point q_s there is a neighborhood A_s of q_s in $\varphi^{-1}(\alpha_{p+1})$, for which the restriction $h|A_s$ is *smooth*. (Indeed it is a composition of h_1 and h_2. The first is smooth since 1) $h_1(z)$ for z close



Pict. B.1



Pict. B.2

to q_s is an intersection point of a $(-v)$-trajectory starting at z with a smooth manifold $C_{\nu,i}$ transversal to a flow, 2) $h_2(u)$ for u having the x-coordinate small enough is just the projection to the coordinate x.) Furthermore, the derivative of $(h|Q^p)$ is epimorphic by the transversality condition. Hence there exists a small closed ball D_s in Q^p around q_s such that the map $(h|D_s)$ is a diffeomorphism onto some neighborhood of $c_i = c(s)$ in $D_{\nu,i}$. Note that there exists a neighborhood P_i of c_i such that the h-preimage of P_i is contained in the union of D_s. (Indeed, if not, then there is a sequence of points in $V_\nu \cup D_\nu$, converging to c_i, such that their h-preimages do not belong to $\cup D_s$; choose the subsequence of these preimages, which converges in Q^p and get the contradiction).

Now we choose δ to be so small that all the h-images of ∂D_s belong to $\{\varphi(x) \leq \beta - \delta\}$. Now it is obvious that $F_1 = h$ satisfies the conditions of the lemma.

Remark B.9 The restriction $\psi|D_s$ is a diffeomorphism of this disc onto the neighborhood of zero in $B^p_{c(s)}$ and ∂D_s is carried to the subset $|x| \geq \sqrt{\delta}$. The standard arguments show that the map of pairs $\psi : (D_s, \partial D_s) \to (|x| \leq \bar{\varepsilon}, \sqrt{\delta} \leq |x| \leq \bar{\varepsilon})$ carries the fundamental class $i \in H_p(D_s, \partial D_s)$ to $\varepsilon_i \cdot \mathbf{1}$, where $\mathbf{1}$ is the identity generator of $H_p(B^p(0, \bar{\varepsilon}), B^p(0, \bar{\varepsilon}) \backslash B^p(0, \sqrt{\delta}))$.

The proof of lemma B.7. Let c be a critical point of index $(p+1)$. To compute the differential of $[^-D(c)] \in H_{p+1}(\{\varphi(x) \leq \alpha_{p+2}\}^-, \{\varphi(x) \leq \alpha_{p+1}\}^-)$ it is enough to compute the homology class of $\partial^- D(c)$ in the group $H_p(\{\varphi(x) \leq \alpha_{p+1}\}^-, \{\varphi(x) \leq \alpha_p\}^-)$. Consider the manifold $\partial D(c) \in \{\varphi(x) = \alpha_{p+1}\}$ and denote by $q_1, \ldots, q_s$ the points of intersection of $\partial D(c)$ with the spheres $S_1^{n-1-p}, \ldots, S_k^{n-1-p}$ in $\{\varphi(x) = \alpha_{p+1}\}$ (here $s_i^{n-1-p} = B_i^{un} \cap \{\varphi(x) = \alpha_{p+1}\}$). The point q_i is exactly the point of intersection with $\{\varphi(x) = \alpha_{p+1}\}$ of the $(-v)$-trajectory γ_i joining c with $c(i)$.

By lemma B.8 the inclusion id of $S^p = \partial D(c)$ into $\{\varphi(x) = \alpha_{p+1}\}$ is homotopic to the map $\Psi : S^p \to \{\varphi(x) \leq \alpha_{p+1}\}$, satisfying the conclusions of Lemma B.8. Now we lift the homotopy from id to ψ to M in such a way that the initial sphere S^p is lifted as $\partial^- D(c)$. Then ψ lifts to a map which takes to $\{\varphi(x) \leq \beta - \delta\}^-$ all the S^p except the neighborhoods of q_i which are carried diffeomorphically to the neighborhoods of $\bar{c}_i \cdot g(\gamma_i)$. Applying the remark B.9 we get that the homology class of $\partial^- D(c)$ in $H_p(\{\varphi(x) \leq \alpha_{p+1}\}^-$, $\{\varphi(x) \leq \beta - \delta\}^-)$ is exactly the $\sum_{i=1}^{s} [^-D(c(i))] \cdot \varepsilon_i \cdot g(\gamma_i)$.

Now we note that the pair $(\{\varphi(x) \leq \alpha_{p+1}\}, \{\varphi(x) \leq \beta - \delta\})$ is a deformation retract of $(\{\varphi(x) \leq \alpha_{p+1}\}, \{\varphi(x) \leq \alpha_p\})$ and that the deformation (along the $(-v)$-trajectories) carries $D(c(i))$ to itself. Lifting that homotopy to M, we obtain the formula sought. $\square$

Proof of theorem B.4. First of all we note that the simple homotopy type of $C_*(\overline{W}, \overline{V}_0)$ does not indeed depend on the triangulation. If W is connected, then denote by $\widehat{W}$ some connected component of $\overline{W}$. The elementary theory of covering spaces shows that $\widehat{W}$ is a regular covering with a structure group $G_0 \subset G$, such that $\widehat{W} \times_{G_0} G = \overline{W}$. Hence $C_*(\overline{W}, \overline{V}_0) = C_*(\widehat{W}, \widehat{V}_0) \otimes_{\mathbf{Z}G_0} \mathbf{Z}G$. In turn the complex $C_*(\widehat{W}, \widehat{V}_0)$ equals $C_*(\widetilde{W}, \widetilde{V}_0) \otimes_{\mathbf{Z}\pi_1 W} \mathbf{Z}G_0$ where $\widetilde{W} \to W$ is a universal covering, and the simple homotopy type of $C_*(\widetilde{W}, \widetilde{V}_0)$ over $\mathbf{Z}\pi_1 W$ is independent of triangulation.If W is non-connected it is a disjoint sum of connected components W_i, and $C_*(\overline{W}, \overline{V}_0)$ is the same as $\oplus_i C_*(\overline{W}_i, \overline{V}_{0,i})$, each of the latter ones independent of triangulation.

By lemma B.2 we choose a Morse function $\varphi : W \to [a, b]$ such that v is a g.-l. vector field for φ, $\varphi_{-1}(a) = V_0$, $\varphi^{-1}(b) = V_1$ and there is a sequence $a = a_0 < a_1 < \cdots < a_n < a_{n+1} = b$ of regular values of φ, such that all the critical points of index i lie in the domain $\{a_i < \varphi(x) < a_{i+1}\}$.

We choose a smooth triangulation of W in such a way that all the $\varphi^{-1}(a_i)$ and all the $\varphi^{-1}([a_i, a_{i+1}])$ are the subcomplexes. Furthermore, we demand that for every critical point c_i of index p the disc $B_i^{st} \cap \{\varphi(x) \geq a_p\}$ and the boundary of the disc is the subcomplex of this triangulation. All that is possible by [3].

Consider now the filtration in the chain complex of this triangulation and the corresponding filtration in the chain complex of the corresponding triangulation of $(\overline{W}, \overline{V}_0)$, namely $F_i = C_*(\{\varphi(x) \leq a_{i+1}\}^-)$. That is the nice filtration of the $\mathbf{Z}G$-complex $C_*(\overline{W}, \overline{V}_0)$ in the sense of def. 3.3 of §3. Indeed, by elementary Morse theory the pair $(\{a_i \leq \varphi(x) \leq a_{i+1}\}, \{\varphi(x) = a_i\})$ is homotopy equivalent to the relative CW-complex $(\{\varphi(x) = a_i\} \cup D_1^i \cup \ldots \cup D_k^i, \{\varphi(x) = a_i\})$. We choose the generators for $H_n(F_n, F_{n-1})$ to be the chosen liftings of the descending discs of the pair with the chosen orientation. The complex F_*^{gr} is then isomorphic to the Morse complex $C_*(v)$ and the isomorphism preserves bases (by lemma B.7). This implies that $C_*(v)$ is homotopy equivalent to $C_*(\overline{W}, \overline{V}_0)$ over $\mathbf{Z}G$ by corollary 3.4.

To get the simple homotopy equivalence $F_*^{gr} \to C_*(\overline{W}, \overline{V}_0)$ we proceed as follows.

By lemma 3.6 it is enough to show that for each i the map λ_n of the complex $R_n = \{0 \leftarrow \cdots 0 \leftarrow H_n(F_*^{(n)}/F_*^{(n-1)}) \leftarrow 0 \leftarrow \cdots\}$ to the complex $F_*^{(n)}/F_*^{(n-1)}$ inducing identical isomorphism in homology is a simple homotopy equivalence (such map exists and is unique up to homotopy by the lemma 3.2. That is of course essentially known by [5], §9, and we only are to make a reduction to the argument of that paper (i.e. to cope with the disconnectedness).

1) $n = 0$. Denote by $V_1, \ldots, V_s$ the connected components of $\{\varphi(x) = a_0\}$. The pair $(\{\varphi(x) \leq a_1\}, \{\varphi(x) = a_0\})$ is diffeomorphic to the disjoint union of $(V_i \times [0, a_0], V_i \times 0)$ and some disjoint n-discs corresponding to the points of index zero. So the chain complex of the covering is the direct sum of corresponding complexes. The chain complex of $(V_i \times [0, a_0], V_i \times 0)^-$ is by the above remark equivalent to a $C_*^{(i)} \otimes_{\mathbf{Z}G_0} \mathbf{Z}G$ where $C^{(i)}$ is a chain complex for some connected covering of $(V_i \times [0, a_i], V_i \times 0)$ and is therefore simply homotopy contractible.

2) $n > 0$. For a map $\lambda_n : R_n \to F_*^{(n)}/F_*^{(n-1)}$ we can take any map carrying every element x in R_n to the cycle, represented by x. We choose λ_n to carry the generator $[c_i]$, corresponding to a critical point c_i to the simplicial cycle, representing the generator in the pair $(D_i, \partial D_i)$, where D is the lifting of the descending disc of the point c_i with the zero lifted to $\bar{c}_i$. That is possible by our choice of triangulation.

Now we denote by $(U_1, V_1), \ldots, (U_n, V_n)$ the connected components of $(\{a_n \leq \varphi(x) \leq a_{n+1}\}, \{\varphi(x) = a_n\})$, where V_i is $\partial U_i \cap \{\varphi(x) = a_i\}^-$. The complex $C_*(\{\varphi(x) \leq a_{i+1}\}^-, \{\varphi(x) = a_i\}^-)$ is a direct sum of $C_*((U_i, V_i)^-)$. Note that each critical point belongs together with its descending disc to one of these components, since $n > 0$, and our map λ_n carries each generator to corresponding direct summand. Hence it suffices to consider the case of the connected manifold $(U, V) = (\{a_n \leq \varphi(x) \leq a_{n+1}\}, \{\varphi(x) = a_n\})$. By our choice of λ_i its image belongs to the chain complex of one of the connected components $\overline{U}_0$ of the covering $\overline{U}$, and by our remark above the natural embedding $C_*(\overline{U}_0) \to C_*(\overline{U})$ generates the $\mathbf{Z}G$-isomorphism $C_*(\overline{U}_0) \otimes_{\mathbf{Z}G_0} \mathbf{Z}G \to C_*(\overline{U})$, where G_0 is a structure group of the covering $\overline{U}_0 \to U$. Hence it is enough to prove our result for the connected covering, and therefore for the universal covering.

It is an easy algebraic exercise to show that the map $\lambda_n : R_n \to F_*^{(n)}/F_*^{(n-1)}$ is a simple homotopy equivalence if and only if the chosen basis in $H(F_*^{(n)}/F_*^{(n-1)})$ gives the zero torsion of $F_*^{(n)}/F_*^{(n-1)}$.

Now for a connected manifold (U, v), and a Morse function $f : U \to [a, b]$, $f^{-1}(a) = V$ with a critical point of only one index the vanishing of this torsion is proved in [5], lemma 9.4. □

For our further needs we state here a more precise version.

Proposition B.10 *Let $f : W \to [a, b]$ be a Morse function on a cobordism W with a gradient-like field v. Let $\varphi : W \to [a, b]$ be any regular Morse function with the same g.-l. vector field, and let $a = a_0 < a_1 < \cdots < a_{n-1} = a_n = b$ the sequence of regular values of φ, separating the points of different indices. Let $\bar{p} : \overline{W} \xrightarrow{G} W$ be a regular covering with a structure group G and Δ be a smooth triangulation of W, such that all the spaces $\varphi^{-1}(a_i), \varphi^{-1}([a_i, a_{i+1}])$ as well as the descending discs are the subcomplexes. Then there exists a simple homotopy equivalence $\mathcal{J}_\varphi : C_*(v) \to C_*(\overline{W}, \overline{V}_0; \Delta)$, which preserves the filtrations (where $C_*(\overline{W}, \overline{V}_0; \Delta)$ is filtered by $\bar{p}^{-1}(f^{-1}([a, a_i]))$ and in the graded groups it sends the generator c of index i of $C_*(v)$ to the homology class of the descending disc D_c in $H_*((fp)^{-1}([a_{i+1}, a_{i+2}], (fp)^{-1}(a_{i+1}))$. The homotopy class of $\mathcal{J}_\varphi$ is uniquely determined by these two conditions.*

Denote by C_*^s the group of singular chains. For a regular φ we get the nice filtration on $C_*^s(\overline{W}, \overline{V}_0)$. Composing the map $\mathcal{J}_\varphi$ with the natural ${}_{\mathbb{Z}}G$-homotopy equivalence $C_*(\overline{W}, \overline{V}_0; \Delta) \to C_*^s(\overline{W}, \overline{V}_0)$ we get the following.

Proposition B.11 *In the notations of proposition B.10 there exists a $\mathbb{Z}G$-homotopy equivalence $\mathcal{J}_\varphi^s : C_*(v) \to C_*^s(W, V_0)$, preserving the filtrations and sending in the graded groups the generator c to the homology class of the corresponding lifting of the descending disc D_c. Under these properties $\mathcal{J}_\varphi^s$ is chain homotopy unique.*

Note that neither $C_*(v)$, nor $C_*^s(\overline{W}, \overline{V}_0)$ do not depend on φ, but $\mathcal{J}_\varphi^s$ does. However it occurs that $\mathcal{J}_\varphi^s \sim \mathcal{J}_\psi^s$ for two different regular functions and we now proceed as to prove it. First we need a simple lemma.

Suppose $f : W \to [a, b]$ is any Morse function on the cobordism W, $\alpha \in [a, b]$ is any regular value for f, v is a gradient-like vector field for f. Denote by U the set of all the points $w \in W$, such that either $w \in \{f(x) \le \alpha\}$, or the $(-v)$-trajectory γ, starting at

x crosses $\{f(x) = \alpha\}$ at some moment. Since v is a g.-l. vector field for f this moment is unique, hence there is a function $\tau : U \to \mathbf{R}$ and a map $\eta : U \to f^{-1}(\alpha)$, where if $x \in f^{-1}([a, \alpha])$ then $\tau(x) = 0$ and $\eta(x) = x$; and if not, then $\tau(x)$ is the moment of intersection of γ with $f^{-1}(\alpha)$ and $\eta(x)$ is the point of intersection. Sometimes we indicate α and f in notations as follows: $\tau_{\alpha,f}; \eta_{\alpha,f}; U_{\alpha,f}$.

Lemma B.12 *The set U is an open set in W; the map η is smooth and the function τ is continuous on U and the set $\{f(x) \leq \alpha\}$ is the deformation retraction of U, the final map of the retraction H_t being η.*

Proof: 1) U is open: Suppose that $u \in U$. We are seeking for an open set $V \subset U$, containing u. If $u \in \{f(x) < \alpha\}$, V exists obviously, if $f(u) = \alpha$ it exists straightening theorem for the vector field v, if $f(u) > \alpha$, then – by the theorem on the smooth dependence of trajectories on the initial values.

2) τ is continuous and η is smooth: For $u \in \{f(x) < \alpha\}$ that is obvious. For $f(u) = \alpha$ we find a neighborhood V of u, and a diffeomorphism of V onto the product $(-\varepsilon, \varepsilon) \times B^{n-1}(0, \varepsilon) \subset \mathbf{R}^n$, carrying v to the vector field with coordinates $(1, 0, \ldots)$ and the set $f^{-1}(\alpha) \cap V$ – to the $\{0\} \times B^{n-1}(0, \varepsilon)$. In this coordinate the map η is the projection onto $B^{n-1}(0, \varepsilon)$ and the function τ is the absolute value of the first coordinate.

For $f(u) > \alpha$ let γ be a $(-v)$-trajectory coming from u to $w \in f^{-1}(\alpha)$, meeting w at a moment τ_0. It is well known that there is a small neighborhood V of $\gamma([0, \tau_0])$ and a diffeomorphism Φ of V to $(-\varepsilon, \tau_0 + \varepsilon) \times B^{n-1}(0, \varepsilon) \subset \mathbf{R}^n$, such that Φ carries v to the vector field with coordinates $(1, 0, \ldots, 0)$, u – to the point $(\tau_0, 0, \ldots, 0)$ and $f^{-1}(\alpha) \cap V$ – to $\tau_0 \times B^{n-1}(0, \varepsilon)$. Again we see that Φ carries τ to the absolute value of the first coordinate, and η – to the projection to the $0 \times \mathbf{R}^{n-1}$.

3) Now we define the deformation H_t of U onto $\{f(x) \leq \alpha\}$ as follows. If $f(u) \leq \alpha$ then $H_t(u) = u$. If $f(u) > \alpha$, then we denote by $\gamma(u, t)$ the $(-v)$-trajectory, starting at u and we set $H(t, u) = \gamma(u, t \cdot \tau(u))$. Note that for $f(u) > \alpha$ we have $H(1, u) = \gamma(u, \tau(u)) = \eta(u)$, so $H(1, u) = \eta(u)$ for any $u \in U$. The function $H : [0, 1] \times U \to \{f(x) \leq \alpha\}$ is continuous on $[0, 1] \times \{f(x) \leq \alpha\}$ obviously and on $[0, 1] \times \{f(x) \geq \alpha\}$ by continuity of τ, hence H is continuous on the whole domain.

Next we proceed to compare the descending discs of a critical point as defining the elements of the relative homotopy groups in the filtrations, defined by different Morse functions.

Namely, let $f : W \to [a,b]$, $g : W \to [a',b']$ be two Morse functions with the same gradient-like field v. Let $\alpha \in [a,b]$, $\alpha' \in [a',b']$ be the regular values of f, correspondingly g. Let c be a critical point of f and g, having index p and lying above the f-level α and above the g-level α'. Choose a standard neighborhood W_ε of c so small as to belong to the intersection $\{f(x) > \alpha\} \cap \{g(x) > \alpha'\}$.

Assume that S_c^+ belongs to $U_{\alpha,f}$. Then we can define the map F of the unit disc D^p to $\{f(x) \geq \alpha\}$, such that $\gamma|(\partial D^p) \subset \{f(x) = \alpha\}$ as follows. On the disc $D^p(0,\varepsilon)$ it is the standard embedding, provided by the standard coordinate system. On the complement $D^p(0,1)\backslash B^p(0,\varepsilon)$ we define it as follows. For θx, where $x \in S^+$, $\theta \in [1,\varepsilon^{-1}]$ we set $F(\theta x) = H(\frac{(\theta-1)\varepsilon}{1-\varepsilon}, x)$. The resulting map is obviously continuous and sends the boundary $\partial D^p = \{\theta x \mid \theta = \varepsilon^{-1}\}$ to $\{f(x) = \alpha\}$.

For example if f is a regular Morse function, c is a critical point of index p and α is a regular value, separating the critical points of index p from the critical points of index $(p-1)$, then the map $F : (D^p, \partial D^p) \to (\{f(x) \geq \alpha\}, \{f(x) = \alpha\})$ is homotopic to the imbedding of what we called above the descending disc $(D^p(c), S^{p-1}(c))$.

Assume now that S_c^+ belongs to $U_{\alpha',g}$ also. Then we define similarly the map $G : (D^p, \partial D^p) \to (\{g(x) \geq \alpha\}, \{g(x) = \alpha\})$.

For the intersection $U_{\alpha',g} \cap U_{\alpha,f}$ define a continuous function T which is minimum of $\tau_{\alpha',g}$ and $\tau_{\alpha,f}$. (Geometric meaning of $T(x)$ is obviously the first moment when the $(-v)$-trajectory, starting af x, crosses either $\{f(x) = \alpha\}$ or $\{g(x) = \alpha'\}$.) Consider now a new imbedding A of (D^p, S^{p-1}) to $(W, \{g(x) \geq \alpha'\} \cup \{f(x) \geq \alpha\})$ which is again the identity on D^p_ε and if $x \in S^+$, $\theta \in [1,\varepsilon^{-1}]$ the $A(\theta x)$ is $\gamma(x, \frac{(\theta-1)\varepsilon}{1-\varepsilon}\varepsilon \cdot T(x))$. It is obviously continuous, since $x \in U_{\alpha',g} \cap U_{\alpha,f}$.

Lemma B.13 *The maps F, G are homotopic to A if all the three are considered as maps of (D^p, S^{p-1}) to $(\{g(x) \leq \alpha'\} \cup \{f(x) \leq \alpha\} \cup ImA, \{g(x) \leq \alpha'\} \cup \{f(x) \leq \alpha\})$.*

Proof: Obviously it is enough to prove that $A \sim F$. Note that T and $\tau_{\alpha,f}$ are the continuous functions on S^+, $T \leq \tau_{\alpha,f}$ and therefore a function $R : [0,1] \times S^+ \to \mathbf{R}$ given by $R(t,x) = T \cdot t + \tau_{\alpha,f}(1-t)$ is continuous and $T(x) \leq R(t,x) \leq \tau_{\alpha,f}(x)$ for all t. Now we define a homotopy H sought as be the identity on the disc B_+ and

$H(t, \theta x) = \gamma(x, \frac{\theta-1}{1-\varepsilon}\varepsilon \cdot R(t,x))$, where, as above, $x \in S^+$, $\theta \in [1, \varepsilon^{-1}]$. □

Now everything is ready to compare the isomorphisms $\mathcal{I}^s_\varphi, \mathcal{I}^s_\psi$ for two different regular Morse functions $\varphi : W \to [a,b]$, $\psi : W \to [a',b']$. We choose the sequences $a = a_0 < a_1 < \cdots < a_n < a_{n+1} = b$, $a' = a'_0 < a'_1 < \cdots < a'_n < a'_{n+1} = b'$ of regular values of φ and ψ, such that a_i (corresp. a'_i) separates the zeros of v of index $\geq i$ from those of index $\leq (i-1)$.

Lemma B.14 *The chain maps $\mathcal{J}^s_\varphi, \mathcal{J}^s_\psi : C_*(v) \to C^s_*(W, V_0)$ are chain homotopic.*

Proof: Consider two filtrations in the space W, namely $W_i = \{\varphi(x) \leq a_{i+1}\}$, $W'_i = \{\psi(x) \leq a'_{i+1}\}$. These filtrations give rise to the filtrations in $C^s_*(\overline{W}, \overline{V}_0)$, denoted $F^{(i)}_*, G^{(i)}_*$. These latter filtrations are nice and the chain maps $\mathcal{J}^s_\varphi, \mathcal{J}^s_\psi$ preserve these filtrations.

Now we introduce a new filtration of W by the spaces $Y_i = W_i \cup W'_i$, and denote $L^{(i)}_*$ the corresponding filtration in $C^s_*(W, V_0)$.

Note that $L^{(i)}_*$ is a nice filtration. Indeed, the intersection $K_i = \{\varphi(x) \geq a_{i+1}\} \cap \{\psi(x) \leq a'_{i+1}\}$ is a compact set, not containing any zero of v. Hence $|v|$ is separated from zero on K_i, which means that any $(-v)$-trajectory, starting at some $k \in K_i$, eventually quit K_i. Moreover, since this trajectory stays forever at $\{\psi(x) \leq a'_{i+1}\}$, it must cross $\{\varphi(x) = a_{i+1}\}$. Therefore $Y_i \subset U_{\varphi, a_{i+1}}$. Further, Y_i is invariant under the flow $(-v)$, hence the deformation of $U_{\varphi, a_{i+1}}$ onto W_i, constructed in the lemma B.12, leaves Y_i invariant, hence $W_i \subset Y_i$ is a homotopy equivalence, preserving V_0, hence $F^{(i)}_* \subset L^{(i)}_*$ induces an isomorphism on homology.

The maps $\mathcal{J}^s_\varphi, \mathcal{J}^s_\phi$ both preserve the filtration Y_i. Therefore, to prove that they are chain homotopic, it is enough to prove that they induce the same map in graded homology. But that is the content of lemma B.13. □

Our next subject is to compare the Morse complexes and equivalences $\mathcal{J}$ for a Morse complex and its part, lying above some fixed level c. For that we need one more notion.

Suppose that W is a compact cobordism, $\partial W = V_0 \sqcup V_1$, $f : W \to \mathbf{R}$ is a Morse function, v – a gradient-like field for f, such that f has no critical points on the boundary

∂W and v is transversal to ∂W and points inwards W on V_0 and outwards on V_1. We *do not* suppose that f is constant on V_0 or on V_1.

For a small ε consider the diffeomorphism F_0 of $V_0 \times [0, \varepsilon]$ onto the neighborhood of V_0 in W, defined by $F_0(x,t) = \gamma(x,t)$, where $\gamma(x,t)$ stands for the v-trajectory starting at x, evaluated at the moment t. Similarly consider the diffeomorphism F_1 of $V_1 \times [0, \varepsilon]$ onto the neighborhood of V_1 in W given by $F_1(x,t) = \bar{\gamma}(x, -t)$ where $\bar{\gamma}$ is a $(-v)$-trajectory. Denote $C' = ImF_1$, $C = ImF_0$.

We will now define two new functions f and f, satisfying the folowing properties

1. $\bar{f}(x) \geq f(x)$ and $x \notin \bar{C} \Rightarrow \bar{f}(x) = f(x)$

   $\underline{f}(x) \leq f(x)$ and $x \notin \underline{C} \Rightarrow \underline{f}(x) = f(x)$.

2. $\bar{f}$ and $\underline{f}$ are Morse functions and v is a gradient-like vector field for both.

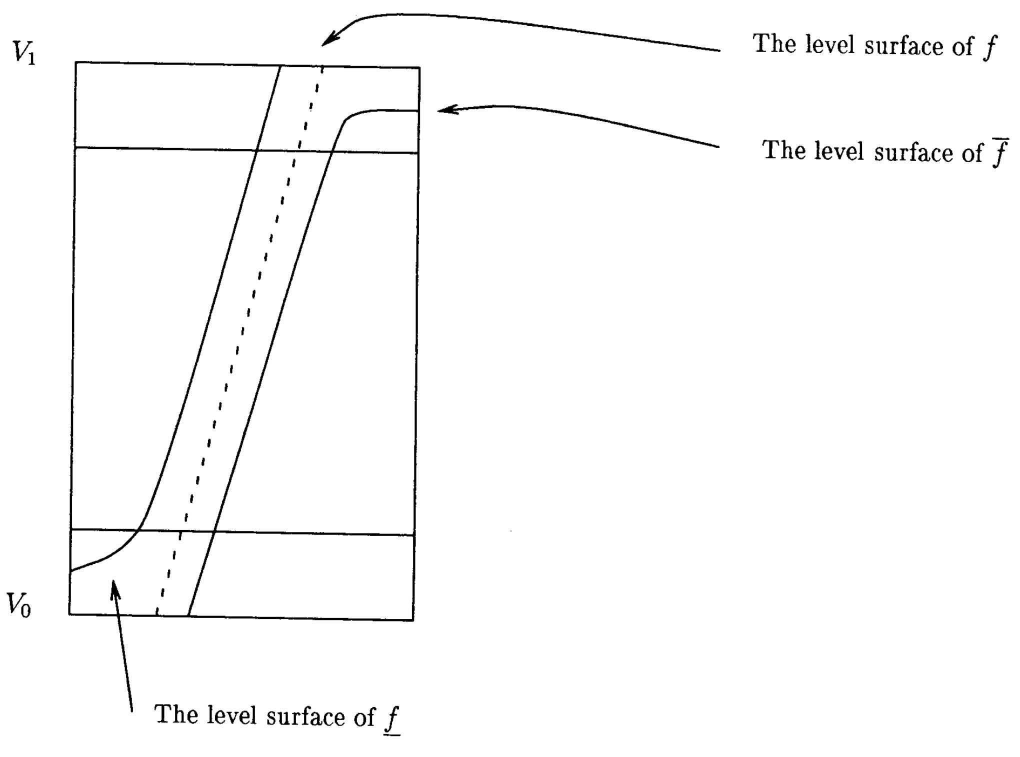
3. $\bar{f}$ is constant on V_1, $\bar{f}|V_1 = \max_{x \in V_1} f(x)$;

4. $\underline{f}$ is constant on V_0, $\underline{f}|V_0 = \min_{x \in V_0} f(x)$.

We call f the upper damping and f the lower damping.

The result of two consequent operations $^-$ and $_-$, applied to the function f will be denoted by $\underline{\bar{f}}$ (obviously it does not depend on the order of applying of the brackets), and called the damping of f. It depends on additional data (such ad ε or functions α, β below) but we never consider more than one damping at a moment so no possibility of confusion occurs. The choice of the term "damping" is justified by the following picture.

We construct for example the lower damping. We can suppose that $f(x) > 0$ on W because adding a constant does not change the situation. Take the composition $f \circ F_0$ which we denote again by the same letter $f : V_0 \times [0, \varepsilon] \to \mathbf{R}$. It is enough to construct a function $g : V_0 \times [0, \varepsilon] \to \mathbf{R}$, $g \leq f$, which equals f near $V_0 \times \{\varepsilon\}$, for which $\frac{\partial g}{\partial t} > 0$ everywhere on $V_0 \times [0, \varepsilon]$. Choose two smooth nonnegative functions $\alpha, \beta : [0, \varepsilon] \to \mathbf{R}$ such that $\alpha + \beta = 1$, α is zero in a small neighborhood of $0, \beta$ is zero in a small neighborhood of ε. Denote D the $\min_{x \in V_0} f(x, 0)$. Then we set $g(x,t) = [\alpha(t) + \frac{D}{f(x,0)}\beta(t)]f(x,t)$ and one easily checks the properties sought.


V_1
The level surface of f
The level surface of $\overline{f}$
V_0
The level surface of $\underline{f}$

Picture B.3

Now we can achieve the aim mentioned.

Namely let W be a cobordism, $\partial W = V_0 \sqcup V_1$, $f : W \to [a,b]$ a Morse function, $f^{-1}(a) = V_0$, $f^{-1}(b) = V_1$. Let $d \in (a,b)$ be any regular value for f and set $V_2 = f^{-1}(d)$, $W_0 = f^{-1}([a,d])$, $W_1 = f^{-1}([d,b])$. Let v be any gradient-like vector field for f. Any regular Morse function $\varphi : W \to [a,b]$ with the same gradient-like vector field v gives rise to a homotopy equivalence $\mathcal{J}^s_\varphi : C_*(v) \to C^s_*(\overline{W}, \overline{V}_0)$, respecting the filtration induced by φ. Any regular Morse function $\psi : W \to [d,b]$ induces a homotopy equivalence $\mathcal{J}^s_\psi : C^1_*(v) \to C^s_*(\overline{W}, \overline{V}_0)$ where $C^1_*(v)$ stands for a Morse complex of the field v restricted to W_1. Note that $C^1_*(v)$ is naturally a factor complex of $C_*(v)$ and the projection map will be denoted π.

Lemma B.15 *The following diagram is chain commutative*

$$\begin{array}{ccccc}
C_*(v) & \xrightarrow{\mathcal{J}^s_\varphi} & C^s_*(\overline{W}, \overline{V}_0) & & \\
 & & & \searrow^{i_0} & \\
\pi \downarrow & & & & C^s_*(\overline{W}, \overline{W}_0) \\
 & & & \nearrow_{i_1} & \\
C^1_*(v) & \xrightarrow{\mathcal{J}^s_\psi} & C^s_*(\overline{W}_1, \overline{V}_2) & &
\end{array}$$

Proof: Consider the Morse function $\varphi | W_1$ and denote by g the lower damping $\underline{\varphi}$. The functions $\varphi | W_1$ and $\underline{\varphi}$ coincide everywhere except small neighborhoods of $V_2 \sqcup V_1$, hence $\underline{\varphi}$ is a regular Morse function on W_1, constant on V_2 and on V_1. The map $\mathcal{J}^s_\psi$ is chain homotopic to $\mathcal{J}^s_\varphi$ by the lemma B.14, hence we can assume that $\psi = \underline{\varphi}$. Let $a = a_0 < a_1 < \cdots < a_{n+1} = b$ the sequence of regular values of φ, such that all the critical points of index i belong to $\varphi^{-1}([a_i, a_{i+1}])$. Then a_i are also the regular values for $\underline{\varphi}$ and those of a_i, who belong to the segment $[d,b]$ form the separating sequence for $\underline{\varphi}$, which will be denoted $d = a_s < a_{s+1} < \cdots < a_{n+1} = b$. Consider the filtration W_i of W by the subspaces $W_i = W_0 \cup \psi^{-1}([d, a_i])$ $(i \geq s)$. This filtration gives rise to a filtration of $C^s_*(\overline{W}, \overline{W}_0)$ which is obviously nice. Now the map $i_0 \circ \mathcal{J}^s_\varphi$ respects this filtration, since $\mathcal{J}^s_\varphi$ respects the filtration generated by $\varphi^{-1}([a, a_i])$ and for $x \in W_1$ we have $\psi(x) = \underline{\varphi}(x) \leq \varphi(x)$. The map $\mathcal{J}^s_\psi$ respects this filtration by definition and all what is left to show is that the images of any generator $c \in C_*(v)$ under $(i_0 \circ \mathcal{J}^s_\varphi)^{gr}$ and $(i_1 \circ \mathcal{J}^s_\psi \pi)^{gr}$ are the same. That follows from the lemma B.13, applied to two functions on the cobordism

W which are $g = \varphi$, f is equal to $\underline{\varphi}$ on W_1 and is expanded on W arbitrarily with the condition $f(x) < \underline{\varphi}|V_2$ for $x \in (W_0 \backslash V_2)$. □

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