DYNAMICS OF GRADIENT FLOWS IN THE NON-TRANSVERSAL MORSE THEORY

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1. Introduction

This talk is a report on our long-term project with H. Goda concerning the circle-valued Morse theory on 3-manifolds. At first we recall the notions of the classical transversal Morse theory (real-valued and circle-valued). Then we proceed to a new branch: the *non-transversal Morse theory* and explain our results; we discuss possible applications to the Seiberg-Witten equations.

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2. Transversal Morse theory

2.1. **The real-valued Morse theory.** In the classical Morse theory every real-valued Morse function $f : M \to \mathbf{R}$ on a closed manifold M induces a cellular decomposition of M. Namely, if we choose a generic gradient v of f then the stable manifolds of the zeros of v form this cellular decomposition, so that the number of cells of dimension k equals the number $\sharp Crit_k(f)$ of critical points of f of index k. The cellular chain complex $\mathcal{M}_*(f, v)$ (*the Morse complex*) computes the homology of M. Moreover, one can construct a natural chain equivalence

$$\mathcal{M}_*(f, v) \to \Delta_*(M)$$

where $\Delta_*(M)$ is the simplicial chain complex of M. From this point of view the classical Morse theory gives just another way of constructing cellular decompositions of manifolds. This construction can be refined so as to give the *universal Morse complex* $\tilde{\mathcal{M}}_*(f, v)$ which is freely generated over $\mathbf{Z}\pi_1(M)$ by

the critical points of f and computes the homology of the universal covering $\tilde{M}$ of M. There is a chain equivalence

$$\tilde{\mathcal{M}}_*(f,v) \xrightarrow{\sim} \Delta_*(\tilde{M}) \tag{1}$$

which is a simple homotopy equivalence (that is, its torsion vanishes in the group $\mathrm{Wh}(\pi_1(M))$).

Remark 2.1. The description above is actually very schematic; we have omitted several technical details; see [17], Chapter 6 for the details.

2.2. **The Novikov homology.** The case of Morse functions $f : M \to S^1$ is quite different, and the schema above can not be applied directly. The stable manifolds of the critical points in general do not form a cellular decomposition of the underlying manifold; the closure of each such manifold is not necessarily compact, and can well be everywhere dense in the manifold. The appropriate generalisation of the Morse theory to this case was constructed by S.P. Novikov about 1980. In this theory one associates to each Morse map $f : M \to S^1$ and a transversal f-gradient v the *Novikov complex* $\mathcal{N}_*(f,v)$, which is freely generated over $\mathrm{Z}((t))$ by the set of critical points of f, and we have a canonical chain equivalence

$$\phi : \mathcal{N}_*(f,v) \xrightarrow{\sim} \mathrm{Z}((t)) \underset{\mathbf{Z}[t,t^{-1}]}{\otimes} \Delta_*(\bar{M}) \tag{2}$$

where $\bar{M}$ is the infinite cyclic covering induced by f from the covering $\mathrm{R} \to S^1$.

2.3. **The twisted Novikov homology.** Similarly to the case of the real-valued Morse functions one can also construct a universal Novikov complex (see [16], [9]). The base ring of this complex is a rather complicated algebraic object, namely a special completion of the group ring of the fundamental group, and is very difficult to work with, although it has obvious theoretical advantages. In the joint work of H. Goda and the author [6] we have introduced a twisted version of the Novikov homology, which is simpler to compute, and is sufficient for many purposes. Let G denote the fundamental group of M, and let let $\rho : G \to GL(n,\mathrm{Z})$ be a right representation of G (that is, $\rho(ab) = \rho(a)\rho(b)$ for every $a,b \in G$). The twisted Novikov complex $\mathcal{N}_*(f,v;\rho)$ is a free chain complex over $\widehat{\Lambda} = \mathrm{Z}((t))$ which has $n \cdot \sharp\mathrm{Crit}_k(f)$ generators in the

degree k and we have a canonical chain equivalence

$$\phi_\rho : \mathcal{N}_*(f, v; \rho) \xrightarrow{\sim} \widehat{\Lambda}^n \underset{\bar{\rho}}{\otimes} \Delta_*(\tilde{M}) \tag{3}$$

where the tensor product in the right hand side refers to the structure of a right $\mathbf{Z}G$-module on $\widehat{\Lambda}^n$ defined by the representation

$$\bar{\rho} = \rho \otimes \xi : G \to GL(n, \widehat{\Lambda}).$$

Here ξ is the homotopy class of f in $[M, S^1] \approx \mathrm{Hom}(G, \mathbf{Z})$ which is considered as the following composition of group homomorphisms:

$$G \to \mathbf{Z} = (\mathbf{Z}[\mathbf{Z}])^\times = (\mathbf{Z}[t, t^{-1}])^\times \subset \widehat{\Lambda}^\times = GL(1, \widehat{\Lambda}).$$

2.4. **Closed orbits of the gradient flow.** The gradient flow of the circle-valued Morse function can have closed orbits (contrarily to the case of real-valued Morse functions). It turns out that the Lefschetz zeta function counting these orbits can be expressed in terms of invariants of the simple homotopy type of the Novikov complex. This was discovered in the work of M. Hutchings and Y.-J. Lee [2]. To explain our results let us first recall the notion of *torsion*.

Definition 2.2. (1) For an acyclic based chain complex C_* over $\widehat{\Lambda} = \mathbf{Z}((t))$ one defines an element

$$\tau(C_*) \in \mathbf{Z}[[t]] = \bar{K}_1(\widehat{\Lambda})/\{t^{\pm 1}\}$$

(the *torsion* of C_*). For the chain complex

$$C_* = \{0 \longleftarrow C_1 \xleftarrow{\phi} C_2 \longleftarrow 0\}$$

where both C_1, C_2 are isomorphic to $\widehat{\Lambda}$ and the homomorphism ϕ is the multiplication by a unit α of the ring $\mathbf{Z}[[t]]$ we have $\tau(C_*) = \alpha$.

(2) For a chain equivalence $\phi : C_* \to D_*$ of two based chain complexes over $\widehat{\Lambda}$ one defines the torsion $\tau(\phi)$ as the torsion of the chain cone $Cone_*(\phi)$.

We have the following result ([13], [15]) concerning the torsion of the canonical chain equivalence from Subsection 2.2 (valid for generic gradients v of the circle-valued function $f : M \to \mathbf{R}$):

$$\tau(\phi) = \prod_{PrCl(-v)} \left(1 - \nu(\gamma) t^{n(\gamma)}\right)^{-\mu(\gamma)} \in \mathbf{Z}[[t]]. \tag{4}$$

The right hand side is known as *the Lefschetz zeta function of* $(-v)$ and denoted by $\zeta_{(-v)}$; here the product is over the set $\boldsymbol{PrCl}(-v)$ of all prime closed orbits γ of $(-v)$ and $\nu(\gamma), \mu(\gamma) = \pm 1, n(\gamma) \in \mathbf{Z}$ are certain dynamical invariants of a closed orbit γ (see [17], Ch. 13 for details). If the Morse function f has no critical points the formula above is equivalent to the classical Milnor's formula [11] relating the torsion of the mapping torus of a continuous map and the Lefschetz zeta function of the map.

There is a generalisation of the formula (4) to the case of the twisted Novikov homology; this is a joint result of H. Goda and A. Pajitnov (in preparation):

$$\tau(\phi_\rho) = \prod_{\boldsymbol{PrCl}(-v)} \det\Big(1 - \nu(\gamma)\bar{\rho}([\gamma])\Big)^{-\mu(\gamma)} \in \mathbf{Z}[[t]]. \tag{5}$$

The right hand side of this formula is known as the *twisted zeta function* of the flow; we will denote it by $\zeta_{(-v,\rho)}$. It was introduced in the work of Boju Jiang and Shicheng Wang [7], where the authors studied the twisted invariants of maps of manifolds. Any such map gives rise to a circle-valued Morse function without critical points on the mapping torus of the map, so that the formula above can be considered as a generalisation of their result.

3. Non-transversal Morse theory

The results of the preceding sections rely heavily on the transversality condition for the gradient flow. It is quite surprising that the analogs of these results still hold for some classes of non-transversal gradients, as was first shown by T.Mark [10]. We will consider only 3-dimensional manifolds here, although the theory can be extended to manifolds of any dimension. For a Morse function $f : W \to [a, b]$ on a 3-dimensional cobordism and an f-gradient v let $(-v)^{\rightsquigarrow}$ denote the gradient descent map from the upper boundary to the lower boundary. If there are no critical points of indices 0 and 3 this map is defined on an open and dense subset of $\partial_1 W$ and maps it diffeomorphically to an open and dense subset of $\partial_0 W$.

Definition 3.1. Let $f : W \to [a, b]$ be a Morse function on a 3-dimensional cobordism W. We say that (f, v) is a *smooth descent Morse pair* if

(1) f does not have critical points of indices 0 or 3.
(2) We have $Crit_2(f) = \{q_1, \dots, q_k\}$, $Crit_1(f) = \{p_1, \dots, p_k\}$; with $f(p_i) > f(q_j)$ for every i, j, and there are two flow lines joining p_i and q_i for every i.
(3) The map $(-v)^{\rightsquigarrow}$ can be extended to a diffeomorphism of $\partial_1 W$ onto $\partial_0 W$.

Observe that if v is a *transversal* gradient, and the set of critical points is non-empty, the map $(-v)^{\rightsquigarrow}$ can never be extended to a diffeomorphism $\partial_1 W \to \partial_0 W$. [†]

Let us proceed to the circle-valued Morse theory.

Definition 3.2. Let $f : M \to S^1$ be a Morse function on a closed 3-dimensional manifold M, and v be an f-gradient. We say that v is *half-transversal* if

(1) There is a regular value λ of f such that cutting M along the regular level surface $f^{-1}(\lambda)$ we obtain a smooth descent Morse pair on the corresponding cobordism.
(2) All stable and unstable manifolds of dimension 2 of v are transversal to each other.

Observe that in the half-transversal case there is a well-defined monodromy map

$$H : f^{-1}(\lambda) \xrightarrow{\approx} f^{-1}(\lambda),$$

derived from the gradient descent $(-v)^{\rightsquigarrow}$. This simplifies the computations for the half-transversal case as compared with the transversal one.

One can define the Novikov complex for the half-transversal gradients. There are only two non-zero terms, namely $\mathcal{N}_1$ and $\mathcal{N}_2$ and the boundary operator from $\mathcal{N}_2$ to $\mathcal{N}_1$ is defined by the usual counting procedure of flow lines (which works here since the stable manifolds of the critical points of index 2 are transversal to the unstable manifolds of the critical points of index 1). The computational advantage of the half-transversal case over the transversal one, is that both Novikov complex and the Lefschetz zeta function are computable in terms of the

[†]The class of gradients introduced by T. Mark in [10] is somewhat smaller: in our terms his defintion requires the existence of an involution I of the cobordism W sending v to $-v$, and f to $-f$.

monodromy diffeomorphism H. As in the transversal case, we have the a chain equivalence

$$\phi : \mathcal{N}_*(f, v) \xrightarrow{\sim} \mathbf{Z}((t)) \underset{\mathbf{Z}[t,t^{-1}]}{\otimes} \Delta_*(\bar{M}) \tag{6}$$

such that its torsion equals the zeta function of the gradient flow:

$$\tau(\phi) = \prod_{PrCl(-v)} \left(1 - \nu(\gamma) t^{n(\gamma)}\right)^{-\mu(\gamma)} \in \mathbf{Z}[[t]]. \tag{7}$$

This is a generalisation of the T. Mark's result [10]; T. Mark proves it for symmetric gradients in the case when the Novikov complex is acyclic over $\mathbf{Q}$. He uses an approximation argument, reducing the proof to the case of transversal gradients. Our proof proceeds by a direct construction of a cellular decomposition of the infinite cyclic covering of the manifold in question.

Our method allows also to generalize the results to the twisted case. Namely, the twisted Novikov complex for half-transversal gradients is chain equivalent to the chain complex computing the twisted Novikov homology:

$$\phi_\rho : \mathcal{N}_*(f, v; \rho) \xrightarrow{\sim} \widehat{\Lambda}^n \underset{\bar{\rho}}{\otimes} \Delta_*(\tilde{M}) \tag{8}$$

and the torsion of ϕ_ρ equals the zeta function of the gradient flow:

$$\tau(\phi_\rho) = \prod_{PrCl(-v)} \det\left(1 - \nu(\gamma)\bar{\rho}([\gamma])\right)^{-\mu(\gamma)} \in \mathbf{Z}[[t]]. \tag{9}$$

4. Complements of knots

The corresponding analogs of the theorems of the previous sections hold also for the manifolds with boundary. One of the most interesting cases here is the case of complements of knots. For a knot K in the 3-sphere let $M = S^3 \setminus \mathrm{Int}\, N(K)$ be the complement to the open tubular neighborhood of the knot. Let $f : M \to S^1$ be a Morse function such that its restriction to the boundary of $N(K)$ is the fibration corresponding to the trivialisation of the normal bundle to K. In this case the torsion of the infinite cyclic covering is easily computed from

the Alexander polynomial of the knot; we have the following formula:

$$\tau(\mathcal{N}_*) \cdot \zeta_{(-v)} = \frac{A(t)}{1-t},$$

where $A(t)$ is the Alexander polynomial of the knot, and $\zeta_{(-v)}$ is the zeta function of the flow $(-v)$.

It is difficult to compute the Novikov complex and the zeta function for gradient flows, and in the transversal case there are not much examples available. In the half-transversal case the computation becomes possible using H. Goda's technique of Hegaard splittings of 3-manifolds (see [3], [4]). For the twisted knots K_n we have

$$\zeta_{(-v)} = (1-t)^3, \qquad \tau(\mathcal{N}_*) = \frac{-n + (2n-1)t - nt^2}{(1-t)^4}.$$

For the pretzel knot of type $(5,5,5)$ we have

$$\zeta_{(-v)} = (1-t)^3, \quad \tau(\mathcal{N}_*) = \frac{19 - 37t + 19t^2}{(1-t)^6}.$$

(see the paper of H. Goda, H. Matsuda and the author [5]). We are computing the twisted invariants for these knots (in preparation).

5. Concluding remarks: relations with the Seiberg-Witten invariants

The work of T.Mark was motivated by the relation of the circle-valued Morse theory with Seiberg-Witten invariants of 3-manifolds and the Meng-Taubes theorem. The Meng-Taubes theory implies that for a closed three-manifold M with $b_1(M) > 0$ the Seiberg-Witten invariant of M is essentially the Milnor-Turaev torsion of M. M.Hutchings and Yi-Jin Lee developed another approach to the Meng-Taubes theorem, which is based on the Morse-Novikov theory. To explain very briefly their idea, recall that the Seiberg-Witten theory associates to every $Spin^{\mathrm{C}}$-structure α an integer $SW(\alpha)$ defined via counting of solutions of the *Seiberg-Witten equations* (see [18]). The Meng-Taubes theorem implies

$$\sum_{\alpha} SW(\alpha) t^{c_1(\alpha)\cdot\xi} = \tau(\bar{M}_\xi) \tag{10}$$

where ξ is a non-zero element of $H^1(M, \mathbf{Z})$, and $\bar{M}_\xi$ is the infinite cyclic covering corresponding to this cohomology class.

The Hutchings-Lee theorem says that

$$\tau(\mathcal{N}_*(f,v))\cdot\zeta_{(-v)}=\tau(\bar{M}_\xi) \tag{11}$$

where $\mathcal{N}_*(f,v)$ is the Novikov complex corresponding to any Morse function $f:M\to S^1$ in the homotopy class ξ and an f-gradient v. An alternative proof of the equality

$$\tau(\mathcal{N}_*(f,v))\cdot\zeta_{(-v)}=\sum_\alpha SW(\alpha)t^{c_1(\alpha)\cdot\xi} \tag{12}$$

would imply the Meng-Taubes theorem. This project was realized in the paper [10], where T.Mark gives the proof of (12) along the lines of the TQFT for Seiberg-Witten equations developed by S. Donaldson (see [1]).

Our results provide in particular the twisted version of (11) and indicate therefore that there should be a twisted version of the formula (10). We hope that further study of the twisted dynamical zeta functions of gradient flows will contribute to the realization of this idea.

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