CIRCLE VALUED MORSE FUNCTIONS

AND DYNAMICS OF GRADIENT FLOWS

Classical Morse theory. Let M be a closed manifold, $f : M \to \mathbf{R}$ be a C^∞ function on M. A point $p \in M$ is called *critical point of* f if $f'(p) = 0$. A critical point p is called *non-degenerate* if the bilinear form $f''(p)$ is non–degenerate. In this case the number of negative eigenvalues of $f''(p)$ is called *index* of p. f is called *Morse function* if all critical points are non-degenerate.

For a Morse function $f : M \to \mathbf{R}$ let us denote

$$S(f) = \{p \mid f'(p) = 0\}, \qquad S_k(f) = \{p \mid f'(p) = 0, \operatorname{ind} p = k\}$$

$$m(f) = \text{card } S(f), \qquad m_k(f) = \text{card } S_k(f)$$

Theorem. (Morse, 1925) ***Let $f : M \to \mathbf{R}$ be a Morse function. Then***

$$m_k(f) \geqslant b_k(M)$$

(where $b_k(M)$ is the rank of $H_k(M)$).

Theorem. *(Pitcher, 1958) Furthermore*

$$m_k(f) \geqslant b_k(M) + q_k(M) + q_{k-1}(M)$$

(where $q_k(M)$ is the minimal number of generators of the torsion submodule of $H_k(M)$).

For the proof Morse shows that M is homotopy equivalent to a CW complex X such that the number of k-cells in X is $m_k(f)$.

(It is much more difficult to prove that M is *homeomorphic* **to a CW complex having $m_k(f)$ cells.)**

Witten's proof of Morse inequalities

(”Supersymmetry and Morse theory”, J. Diff. Geom., v. 17, 1982.)

Let $f : M \longrightarrow \mathrm{R}$ be a Morse function and v a generic Riemannian gradient of f. Witten constructs a chain complex $\mathcal{M}_*(f, v)$ such that

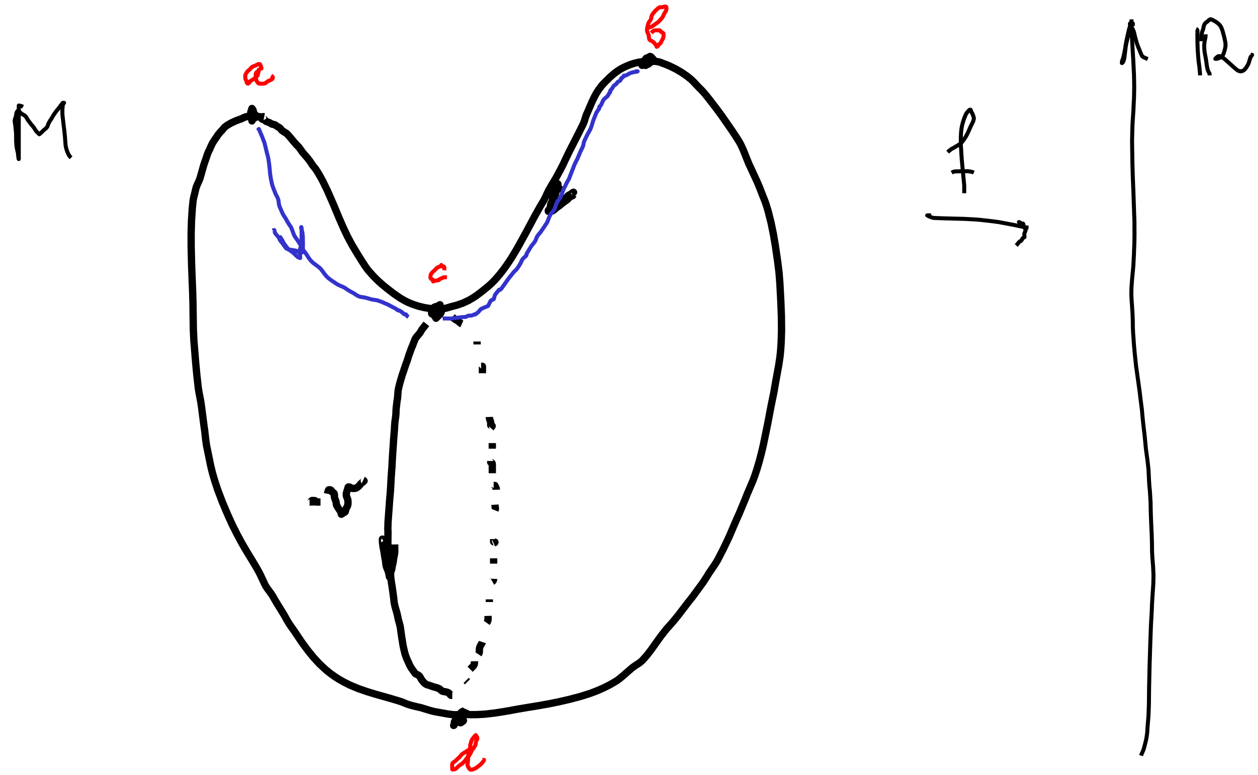
1) $\mathcal{M}_k(f, v)$ is a free abelian group generated by $S_k(f)$

2) $H_*(\mathcal{M}_*(f, v)) \approx H_*(M)$.

To define the boundary operators in $\mathcal{M}_*(f, v)$ put

$$d_k p = \sum_{q \in S_{k-1}(f)} n(p, q; v) \cdot q, \quad \text{where} \quad n(p, q; v) \in \mathrm{Z} \quad \text{is}$$

the *algebraic number of flow lines of v joining p with q.*



$n(a,c) = \pm 1, \quad n(b,c) = \pm 1, \quad n(c,d) = 0$

flow line joining p with q =an integral curve γ of v, such that $\lim_{t\to\infty}\gamma(t) = p$, and $\lim_{t\to-\infty}\gamma(t) = q$.

number : if v is a generic f-gradient then the number of flow lines joining p with q is finite.

algebraic : to each flow line of v joining p and q one associates a sign, and $n(p, q; v)$ is the sum of these signs.

$$d_k p = \sum_{q\in S_{k-1}(f)} n(p, q; v) \cdot q,$$

Theorem. $d_k \circ d_{k-1} = 0$ *and* $H_k(\mathcal{M}_*(f, v)) = H_k(M)$.

Remarks

1. The genericity condition in the theorem above is as follows: for every $p, q \in S(f)$ we must have:

$$D(p, v) \pitchfork D(q, -v)$$

where

$$D(p, v) = \{x | \gamma(x, t; v) \xrightarrow[t \to \infty]{} p\} \tag{1}$$

$$D(q, -v) = \{x | \gamma(x, t; v) \xrightarrow[t \to -\infty]{} q\} \tag{2}$$

2. One can construct a canonical chain equivalence between the Morse complex and the singular chain complex of the manifold:

$$\mathcal{E}_*^{Morse} : \mathcal{M}_*(f, v) \xrightarrow{\sim} \mathcal{S}_*(M)$$

<u>Circle-valued Morse theory.</u> **Let $f : M \to S^1$ be a C^∞ function. All the local notions of the classical Morse theory are valid in this framework. But the Morse inequalities are not valid as they stand. For example, the number of critical points of the function id : $S^1 \to S^1$ is zero, but $H_*(S^1) \neq 0$.**

The relevant generalization of the Morse theory for the circle-valued case was found by S.P.Novikov (*Many-valued functions and functionals. An analogue of Morse theory"***, Doklady AN SSSR, v.260 (1981).)**

MORSE THEORY	**NOVIKOV THEORY**
Morse map $f : M \to \mathbf{R}$ **and a generic** f**-gradient** v	**Morse map** $f : M \to S^1$ **and a generic** f**-gradient** v
$\downarrow$	$\downarrow$
chain complex $\mathcal{M}_*(f, v)$ **generated over** $\mathbf{Z}$ **by** $S(f)$	**chain complex** $\mathcal{N}_*(f, v)$ **generated over** $\widehat{\Lambda} = \mathbf{Z}((t))$ **by** $S(f)$
$\downarrow$	$\downarrow$
$H_*(\mathcal{M}_*(f, v)) \approx H_*(M)$	$H_*(\mathcal{N}_*(f, v)) \approx \widehat{H}_*(\bar{M}) = H_*(\bar{M}) \underset{\Lambda}{\otimes} \widehat{\Lambda}$

where

$$\begin{array}{ccc} \bar{M} & \xrightarrow{F} & \mathbf{R} \\ \Big\downarrow \mathbf{Z} & & \Big\downarrow \mathbf{Z} \\ M & \xrightarrow{f} & S^1 \end{array}$$

$$\Lambda = \mathbf{Z}[t, t^{-1}] = \mathbf{Z}[\mathbf{Z}]$$

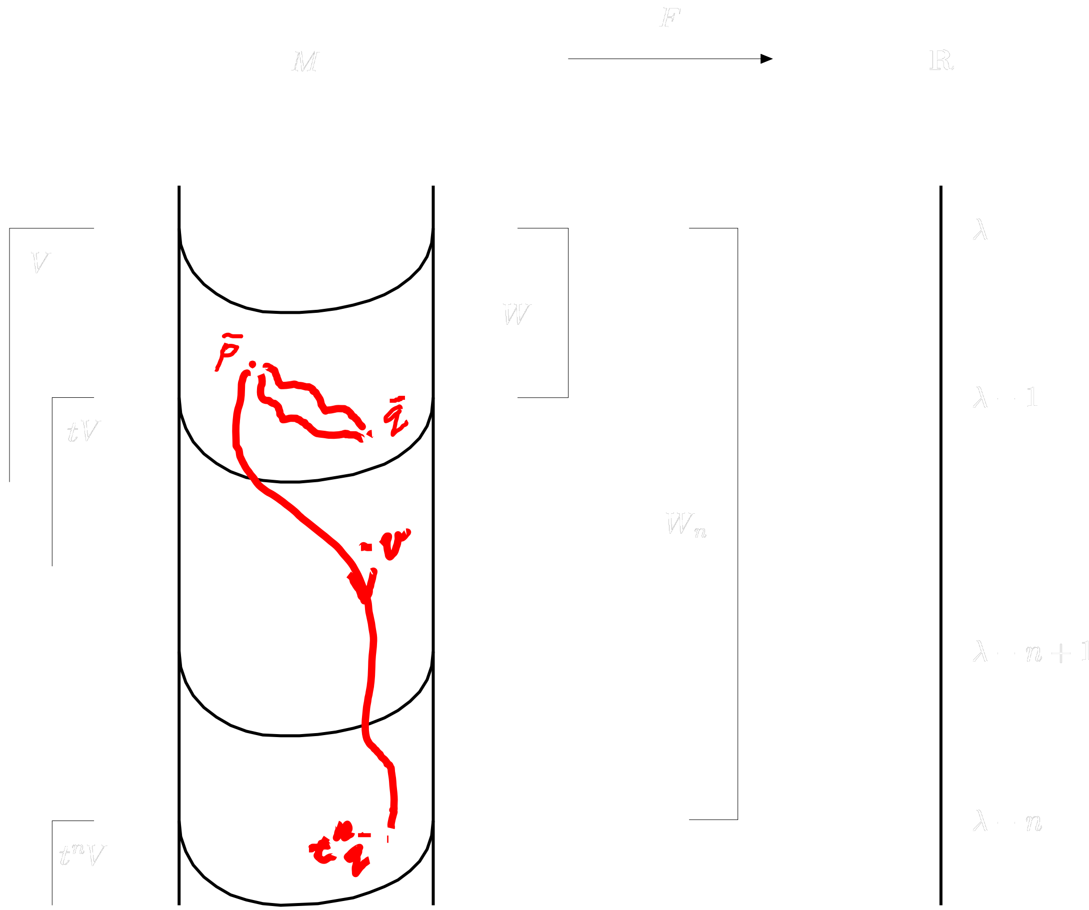
$$\widehat{\Lambda} = \mathbf{Z}((t)) = \{a_{-n}t^{-n} + \ldots + a_0 + \ldots + a_k t^k + \ldots \mid a_i \in \mathbf{Z}\}$$

The construction of the Novikov complex. Define $\mathcal{N}_k(f,v)$ to be the free $\widehat{\Lambda}$-module freely generated by $S_k(f)$. The boundary operator $d_k : \mathcal{N}_k(f,v) \to \mathcal{N}_{k-1}(f,v)$, is necessarily of the form

$$d_k(p) = \sum_{q \in S_{k-1}(f)} N(p,q;v)q \quad \text{with} \quad N(p,q;v) \in \widehat{\Lambda}.$$

Write $N(p,q;v) = \sum_{k \in \mathrm{N}} n_k(p,q;v)t^k$ with $n_k(p,q;v) \in \mathrm{Z}$ and define the integer $n_k(p,q;v)$ as follows.

For each critical point $p \in S(f)$ choose a lift $\bar{p}$ of p to the infinite cyclic covering $\bar{M}$. Then all the lifts of p are of the form $t^k\bar{p}$ where t is the downward generator of the structural group of the covering and $k \in \mathrm{Z}$. Let $n_k(p,q;v)$ be the algebraic number of flow lines of v joining $\bar{p}$ with $t^k\bar{q}$.



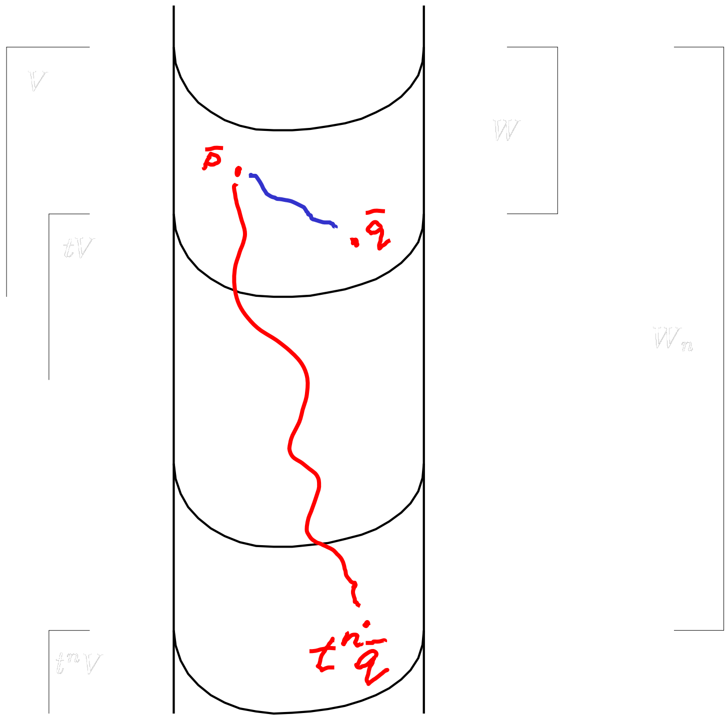
F
M
$\mathbb{R}$
V
tV
t^nV
W
W_n
λ
$\lambda - 1$
$\lambda - n + 1$
$\lambda - n$

Theorem. *Let $f : M \to S^1$ be a Morse function, and v a generic f-gradient. Endowed with the boundary operator above $\mathcal{N}_*(f,v)$ is a chain complex and*

$$H_*(\mathcal{N}_*(f,v)) \approx H_*(\bar{M}, \mathbf{Z}) \bigotimes_{\Lambda} \widehat{\Lambda}.$$

We shall briefly outline the proof (A.P., 1991). Assuming that $\partial_k \circ \partial_{k-1} = 0$ we shall construct a chain equivalence $\mathcal{N}_* \to \widehat{\mathcal{S}}_*(\bar{M}) = \mathcal{S}_*(\bar{M}) \bigotimes_{\Lambda} \widehat{\Lambda}$.

Let us choose the lifts of critical points to $\bar{M}$ in such a way that they are all in the fundamental cobordism $W = F^{-1}([\lambda - 1, \lambda])$. Then the Laurent series $N(p,q,v)$ are in $\mathbf{Z}[[t]]$



and the Novikov complex is obtained by the ring extension of the chain complex $\mathcal{N}_*^-$ defined over the ring $\widehat{P} = \mathbf{Z}[[t]]$, where $P = \mathbf{Z}[t]$.

Let $\mathcal{S}_*^- = \mathcal{S}_*(V^-)$ so that $\widehat{\mathcal{S}}_*(\bar{M})$ is the ring extension of the chain complex

$$\widehat{\mathcal{S}}_*^- = \mathcal{S}_*^- \bigotimes_P \widehat{P}$$

We shall construct a chain equivalence $\mathcal{N}_* \longrightarrow \widehat{\mathcal{S}}_*^-$. The main observation is that

$$\mathcal{N}_*^- / t^n \mathcal{N}_*^- \approx \mathcal{M}_*(F|W_n, v|W_n)$$

so that we have chain equivalences

$$\mathcal{E}_n : \mathcal{N}_*^-/t^n\mathcal{N}_*^- \approx \mathcal{M}_*(F|W_n, v|W_n) \xrightarrow{\mathcal{E}^{Morse}} \mathcal{S}_*(W_n, \partial_0 W_n)$$

$$\xrightarrow{\sim} \mathcal{S}_*(V^-, t^nV^-) = \mathcal{S}_*(V^-)/t^n\mathcal{S}_*(V^-)$$

One can deduce from the functoriality of the Morse chain equivalence that in the following infinite diagram

$$\begin{array}{ccccccccc}
\mathcal{N}_*^-/t\mathcal{N}_*^- & \leftarrow & \mathcal{N}_*^-/t^2\mathcal{N}_*^- & \leftarrow & \ldots\ldots & \leftarrow & \mathcal{N}_*^-/t^k\mathcal{N}_*^- & \leftarrow & \mathcal{N}_*^-/t^{k+1}\mathcal{N}_*^- \ldots \\
\downarrow \mathcal{E}_1 & & \downarrow \mathcal{E}_2 & & \ldots & & \downarrow \mathcal{E}_k & & \downarrow \mathcal{E}_{k+1} \\
\mathcal{S}_*^-/t\mathcal{S}_*^- & \leftarrow & \mathcal{S}_*^-/t^2\mathcal{S}_*^- & \leftarrow & \ldots\ldots & \leftarrow & \mathcal{S}_*^-/t^k\mathcal{S}_*^- & \leftarrow & \mathcal{S}_*^-/t^{k+1}\mathcal{S}_*^- \ldots
\end{array}$$

all the squares are homotopy commutative. Observe that $\mathcal{N}_*^- \approx \varprojlim \mathcal{N}_*^-/t^k\mathcal{N}_*^-$ and $\widehat{\mathcal{S}}_* \sim \varprojlim \mathcal{S}_*^-/t^k\mathcal{S}_*^-$.

Now one can prove by a purely algebraic argument that the existence of such homotopy commutative diagram imply a chain homotopy equivalence between the inverse limits of the lines. The theorem follows.

Remarks.

1. The genericity condition on the vector field is the same as for the Morse complex: the stable and unstable manifolds of critical points must have transversal intersection.

2. As in the case of Morse functions one can show that there is a canonical chain equivalence

$$\phi : \mathcal{N}_*(f, v) \longrightarrow \widehat{\mathcal{S}}_*(\bar{M}) = \mathcal{S}_*(\bar{M}) \underset{\Lambda}{\otimes} \widehat{\Lambda}$$

Since $\widehat{\Lambda}$ is a principal ring, we deduce the *Novikov inequalities*:

$$m_k(f) \geqslant \widehat{b}_k(M) + \widehat{q}_k(M) + \widehat{q}_{k-1}(M),$$

where $\widehat{b}_k(M) = rk_{\widehat{\Lambda}} \widehat{H}_k(M), \qquad \widehat{q}_k(M) = t.n._{\widehat{\Lambda}} \widehat{H}_k(M).$

Theorem (Farber, 1984) If $\dim M \geqslant 6$, and $\pi_1(M) \approx \mathrm{Z}$ then there is a Morse map $f : M \longrightarrow S^1$, such that $[f] \in [M, S^1] = H^1(M, \mathrm{Z}) \approx \mathrm{Z}$ is a generator of this group and all the inequalities above are equalities.

Novikov's exponential growth conjecture

Let $f : M \to S^1$ be a Morse map, v an f-gradient, $p \in S_{k+1}(f), q \in S_k(f)$. The Novikov incidence coefficient $N(p,q;v) = \sum_k n_k(p,q;v)t^k$ is a Laurent series.

CONJECTURE (Novikov, about 1982) There are $A, B > 0$ such that

$$|n_k(p,q;v)| \leqslant Ae^{Bk} \quad \text{for every} \quad k$$

The first published version appeared in the paper of V. I. Arnold "Dynamics of Intersections", 1989:

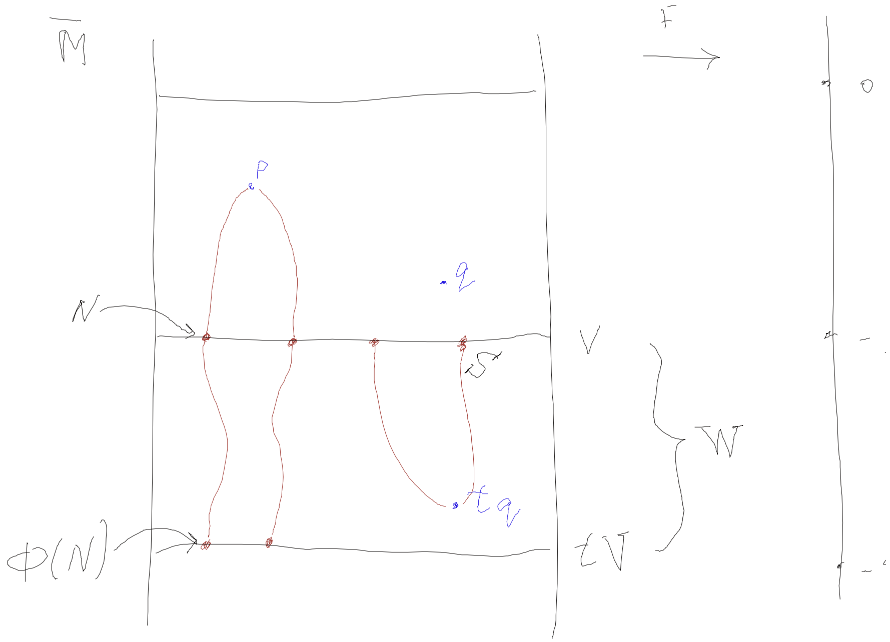
"The author is indebted to S. P. Novikov who has communicated the following conjecture, which was the starting point of the present paper. Let $p : \widetilde{M} \to M$ be a covering of a compact manifold M with fiber $\mathbf{Z}^n$, and let α be a closed 1-form on M such that $p^*\alpha = df$, where $f : M \to \mathbf{R}$ is a Morse function.

The Novikov conjecture states that, "generically" the number of the trajectories of the vector field $-\mathrm{grad} f$ on M starting at a critical point x of the function f of index k and connecting it with the critical points y having index $k-1$ and satisfying $f(y) \geqslant f(x) - n$, grows in n more slowly than some exponential, e^{an}."

Here is the version of the conjecture, published in 1993 by S. P. Novikov:

"CONJECTURE. For any closed quantized analytic 1-form $\boldsymbol{\omega}$, the boundary operator $\boldsymbol{\partial}$ in the Morse complex has all coefficients $\boldsymbol{a_{pq}^{(i)} \in K}$ with positive part convergent in some region $\boldsymbol{|t| \leqslant r_1^*(\omega) \leqslant r_1}$. (If we replace $\boldsymbol{S}$ by $\boldsymbol{(-S)}$ and $\boldsymbol{t}$ by $\boldsymbol{t^{-1}}$, we have to replace $\boldsymbol{r_1}$ by $\boldsymbol{r_1^{-1}}$.) In particular, $\boldsymbol{r_1^* \neq 0}$ and there will be coefficients $\boldsymbol{a_{pq}^{(i)}}$ with radius of convergence not greater than $\boldsymbol{r_1(M, [\omega])}$ if at least one jumping point really exists."

Motivation:



Put $N = D(p, v) \cap V$,
$S = D(tq, -v) \cap V$.
Then $n_1(p, q; v) = N \# S$

$$n_2(p, q; v) = \Phi(N) \# tS = \zeta(N) \# S$$

where $\zeta = t^{-1} \circ \Phi$, and $\Phi : V \to tV$ is the gradient descent map (not everywhere defined).

Similarly, $n_{r+1} = \zeta^r(N) \# S$, and $\zeta^r(N)$ is the result of the r th iteration of one and the same application ζ.

Arnold's theory of dynamics of intersection

Let N, S be compact submanifolds of a closed manifold V, and $\dim N + \dim S = \dim V$. Let $\zeta : M \to M$ be a diffeomorphism, such that $\zeta^r(N) \pitchfork S$ for each r. Then for a generic S we have:

$$|\zeta^r(N) \cap S| \leqslant A \cdot e^{rB} \quad \text{for every} \quad r$$

For the case when $\dim N + \dim S > \dim M$ one can obtain an exponential estimate for the growth of the Betti numbers and the volume of the intersection $\zeta^r(N) \cap S$.

C^0-generic rationality of the Novikov complex

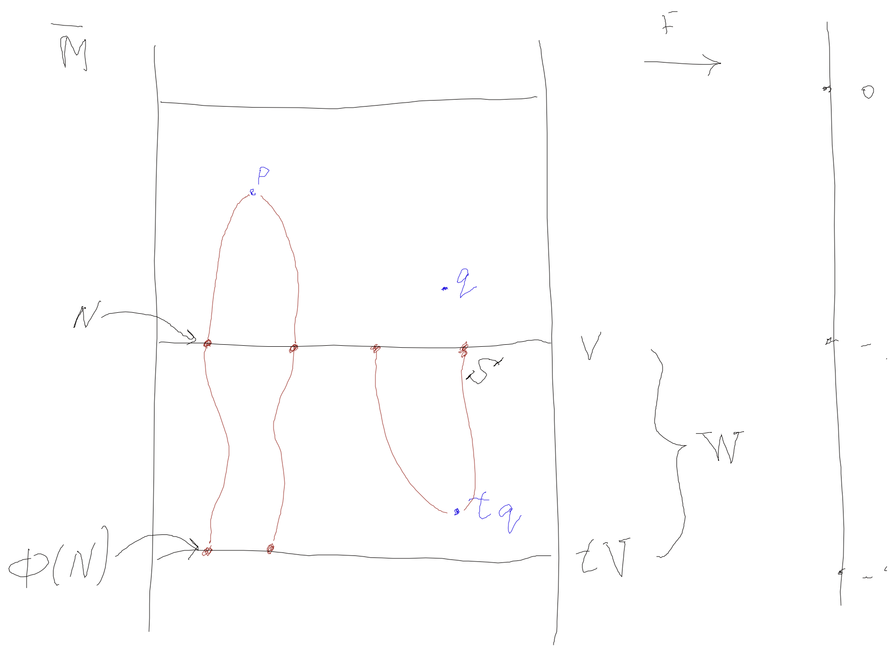
Theorem (A.P., 1995) Let M be a closed manifold, $f : M \to S^1$ be a Morse map. Let $G_T(f)$ denote the set of all transverse f-gradients. Then there is a subset $G_0(f) \subset G_T(f)$ such that

1. $G_0(f)$ is open and dense in $G_T(f)$ with respect to the C^0-topology.
2. For every $v \in G_0(f)$ and every $p \in S_{k+1}(f), q \in S_k(f)$ we have

$$N(p, q; v) = \frac{P(t)}{t^n(1 + tQ(t))} \quad \text{with} \quad P(t), Q(t) \in \mathbf{Z}[t]$$

(In particular this proves the Novikov exponential growth conjecture for a C^0-open and dense subset in the space of all transverse gradients.)

The main idea of the proof is in the regularization of the gradient descent map. Recall that we have the formula $n_{r+1}(p,q;v) = \zeta^r(N) \# S$, where $\zeta = t^{-1} \circ \Phi : V \to V$ is a non-everywhere-defined map of the level surface V to itself.



It turns out that for a C^0-generic gradient v there is a filtration $V^1 \subset V^2 \subset ...V^k \subset ...$ of V by compact submanifolds V^k with boundary, such that for every k the quotient V^k/V^{k-1} has the homotopy type of a wedge of k-dimensional spheres, and

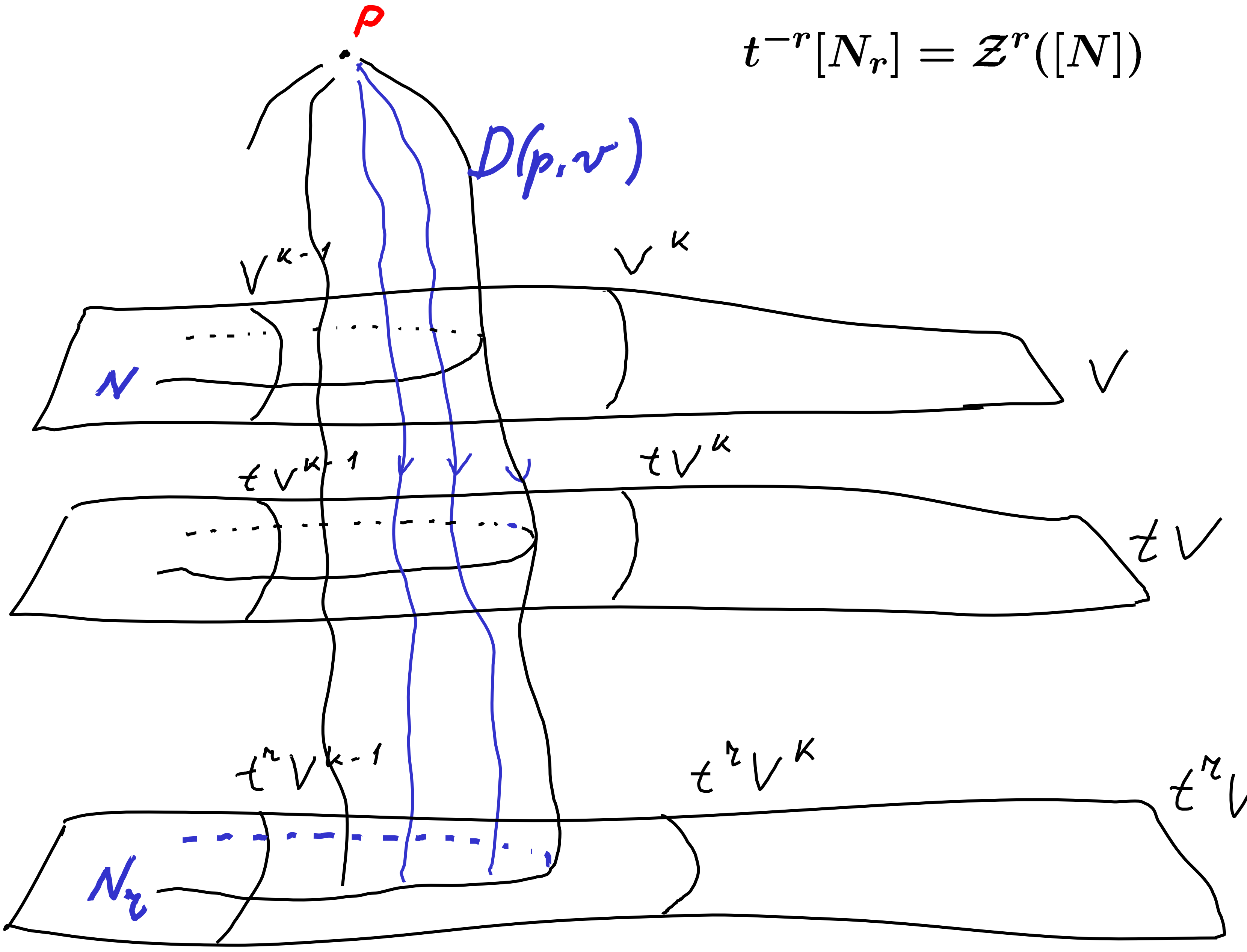
1. The map Φ gives rize to a family of continuous maps $v\downarrow: V^k/V^{k-1} \to tV^k/tV^{k-1}$.

2. For every critical point $p \in S_{k+1}(f)$ the manifold $N = D(p,v) \cap V$ is in V^k and it has its fundamental class $[N]$ in $H_k(V^k, V^{k-1})$.

3. The manifold $N_r = D(p,v) \cap t^r V$ is in $t^r V^k$ and has its fundamental class $[N_r] \in H_k(t^s V^r / t^s V^{r-1})$. This fundamental class satisfies

$$t^{-r}[N_r] = \mathcal{Z}^r([N]) \quad \text{where} \quad \mathcal{Z} = t^{-1}\Big(v\downarrow\Big)_*$$

$$t^{-r}[N_r] = \mathcal{Z}^r([N])$$



4. Let $\widehat{V}^k$ denote the dual filtration of V, that is, $\widehat{V}^k = \overline{V \setminus V^{m-2-k}}$ where $m = \dim M = \dim V + 1$.

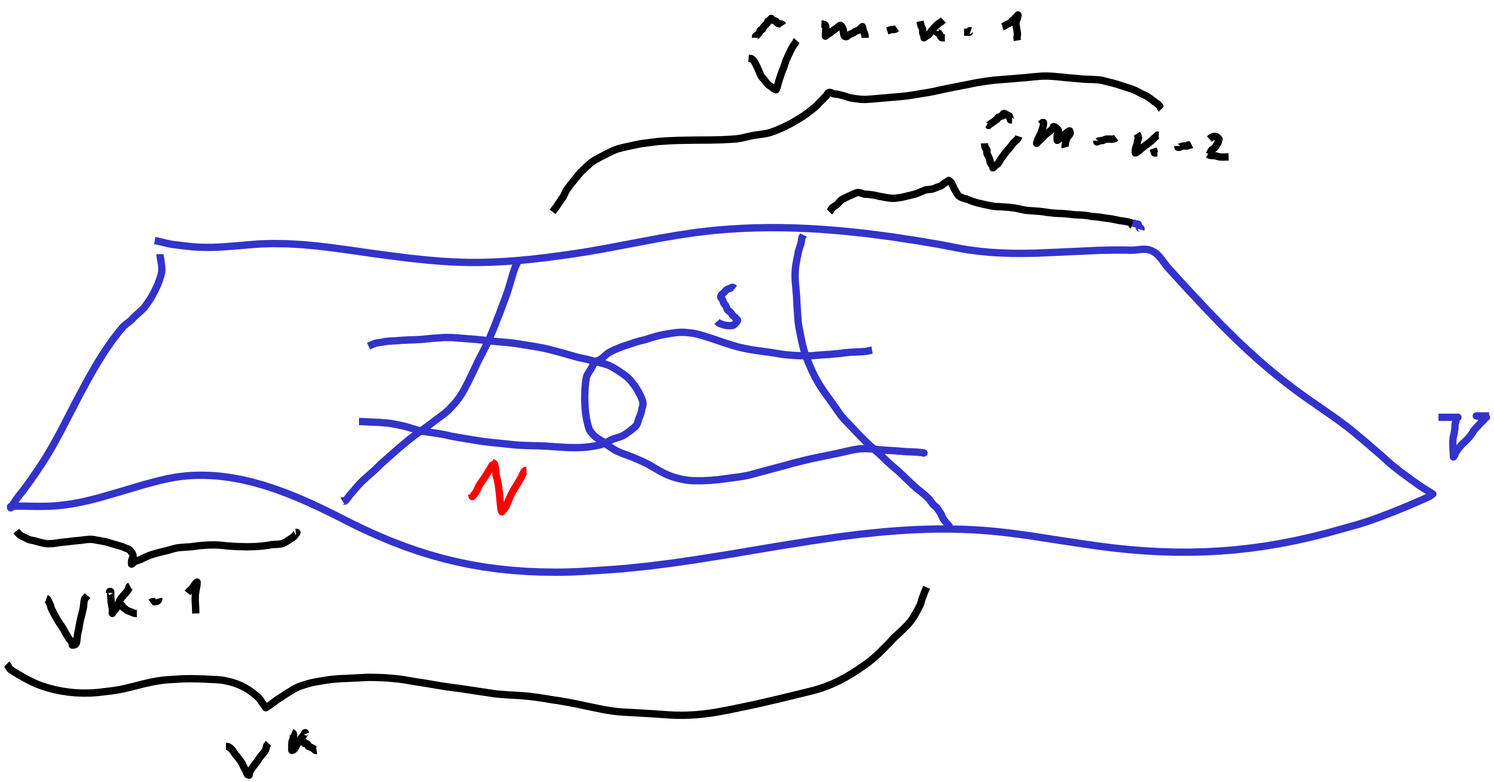
Then for every critical point $q \in S_k(f)$ the manifold $N = D(tq, -v) \cap V$ is in $\widehat{V}^{m-k-1}$ and it determines a homology class $[S]$ in $H_{m-1-k}(\widehat{V}^{m-k-1}, \widehat{V}^{m-k-2})$.

4. We have the intersection pairing

$$\langle \cdot , \cdot \rangle : H_k(V^k, V^{k-1}) \times H_{m-1-k}(\widehat{V}^{m-k-1}, \widehat{V}^{m-k-2}) \to \mathbf{Z}$$

and the Novikov incidence coefficients satisfy

$$n_{r+1}(p, q; v) = \Big\langle \mathcal{Z}^r_*([N]), [S] \Big\rangle.$$



Such gradients are called *cellular* since the gradient descent map induced by these gradient resembles a cellular map between CW complexes. One can prove that the set of cellular transverse gradients is open and dense in the set of all transverse gradients with repect to C^0-topology.

The following computation shows the rationality of the incidence coefficients in case when v is cellular:

$$N(p,q;v) = n_0(p,q;v) + \sum_{r\geqslant 0} n_{r+1}(p,q;v)t^{r+1} =$$

$$n_0 + t\sum_{r\geqslant 0}\langle \mathcal{Z}^r([N]),[S]\rangle t^r = n_0 + t\Big\langle \sum_{r\geqslant 0} t^r \mathcal{Z}^r([N]),[S]\Big\rangle$$

$$= n_0 + t\Big\langle \Big(1-(t\mathcal{Z})^{-1}\Big)([N]),[S]\Big\rangle.$$

Remark.

The question whether the exponential growth conjecture holds for any gradient in some C^{∞}-dense subset in the set of all transverse gradients remains open.

First let us recall some basic facts about the Whitehead torsion.

• Let R be a ring with unit. Denote by $GL(\infty, R)$ the union of the increasing sequence

$$GL(1, R) \subset GL(2, R) \subset ... \subset GL(n, R) \subset ...$$

and put $K_1(R) = GL(\infty, R)/[GL(\infty, R), GL(\infty, R)]$.

Put $\bar{K}_1(R) = K_1(R)/[\pm 1]$. A construction due to J.H.C.Whitehead associates to each chain equivalence ϕ of free finitely generated based chain complexes over R an element $\tau(\phi) \in \bar{K}_1(R)$.

For example, if $C_* = 0$ and $D_* = \{0 \longleftarrow R^n \overset{\alpha}{\longleftarrow} R^n \longleftarrow 0\}$ an acyclic complex, then the torsion of the chain equivalence $\phi = 0$ equals the class of the isomorphism α in the group $\bar{K}_1(R)$.

• If X, Y are finite CW complexes and $\phi : X \to Y$ a homotopy equivalence then the chain map $\phi_* : \Delta_*(\widetilde{X}) \to \Delta_*(\widetilde{Y})$ is a chain equivalence over $\mathrm{Z}G$ where $G = \pi_1(X)$ and the torsion $\tau(\phi) \in \bar{K}_1(\mathrm{Z}G)$ is defined.

The bases in the complexes $\Delta_*(\widetilde{X}), \Delta_*(\widetilde{Y})$ are provided by the lifts of the simplices of X, resp. Y to the coverings, and therefore are defined up to the left action of G.

Thus $\tau(\phi)$ is well defined in the group $\mathrm{Wh}(G) = \bar{K}_1(\mathrm{Z}G)/\pm G$. A theorem due to Chapman says that if ϕ is a homeomorphism, then its torsion $\tau(\phi)$ equals zero.

• Let us return to the Morse theory. We have a canonical chain equivalence of based complexes over $\mathbf{Z}[1]$

$$\mathcal{M}_*(f,v) \xrightarrow{\mathcal{E}^{Morse}} \mathcal{S}_*(M) \sim \Delta_*(M).$$

Its torsion vanishes since $Wh(1) = 0$.

There is also a refined version of the Morse complex $\widetilde{\mathcal{M}}_*(f,v)$ which is a free chain complex over $\mathbf{Z}G$ with $G = \pi_1(M)$, and a chain equivalence $\widetilde{\mathcal{M}}_*(f,v) \to \Delta_*(\widetilde{M})$ of $\mathbf{Z}G$-complexes. One can prove that its torsion also vanishes.

Now let us turn to the Morse-Novikov theory. Let $f : M \to S^1$ be a Morse map and v a transverse f-gradient. We have a canonical chain equivalence

$$\phi : \mathcal{N}_*(f, v) \longrightarrow \mathbf{\Delta}_*(\bar{M}) \underset{\mathbf{\Lambda}}{\otimes} \widehat{\mathbf{\Lambda}},$$

where $\mathbf{\Delta}_*(\bar{M})$ is the simplicial chain complex associated with any C^1-triangulation of M.

These two complexes are based free chain complexes over $\widehat{\mathbf{\Lambda}}$. The bases are defined only up to the action of the group $\mathbf{Z}$ of the covering and the torsion invariant $\tau(\phi)$ is well defined in the group $\bar{K}_1(\widehat{\mathbf{\Lambda}})/\{\pm t^n\}$, which is called *the Whitehead group of* $\widehat{\mathbf{\Lambda}}$ and denoted $\mathrm{Wh}(\widehat{\mathbf{\Lambda}})$.

We have the determinant map det : $K_1(\widehat{\mathbf{\Lambda}}) \to \widehat{\mathbf{\Lambda}}^\bullet$ and proceeding to the quotients we obtain a map det : $\mathrm{Wh}(\mathbf{\Lambda}) \to \mathcal{W}$ where $\mathcal{W}$ is the multiplicative group of all power series of the form $\{1 + a_1 t + a_2 t^2 + ... | a_i \in \mathbf{Z}\}$

Theorem. Let $f : M \to S^1$ be a Morse map, v be a generic f-gradient. Then for the torsion of the canonical chain equivalence $\phi : \mathcal{N}_*(f,v) \to \widehat{\boldsymbol{\Delta}}_*(\bar{M})$ we have:

$$\det \tau(\phi) = (\zeta_L(-v))^{-1}$$

where ζ_L is the Lefschetz zeta function counting the closed orbits of the gradient flow $-v$ (defined below).

The theorem was proved by M.Hutchings and Y.J.Lee (1996) for a particular case when the Novikov complex $\mathcal{N}_*(f,v)$ becomes acyclic after tensoring with Q,

in the case when there are no restrictions on the Novikov complex but the gradient v is cellular by A.P in 1997,

the general case done by D.Schütz and A.P. about 2001 by reduction to the case of cellular gradients.

Definition of $\zeta_L(-v)$

Let γ be a closed orbit of $-v$. Assuming that v is generic, we can associate to γ its *Poincaré index* $\varepsilon(\gamma)\ \pm 1$, and also the multiplicity $m(\gamma) \in \mathbf{N}$ and the winding number

$n(\gamma) = -deg(f \circ \gamma) : S^1 \to S^1$.

$$\eta_L(-v) = \sum_{\gamma} \frac{\varepsilon(\gamma)}{m(\gamma)} t^{n(\gamma)}, \qquad \zeta_L(-v) = \exp(\eta_L(-v)).$$

I will comment only one one point of the proof: the computation of the zeta function for a cellular gradient.

Recall that for every cellular gradient v we have an a filtration $V^1 \subset V^2 \subset ... V^k \subset ...$ of $V = F^{-1}(\lambda)$ by compact submanifolds V^k with boundary,

and the gradient descent from the level surface $V = F^{-1}(\lambda)$ to the level surface $tV = F^{-1}(\lambda - 1)$ gives rize to a family of continuous maps $t^{-1}v\downarrow : V^k/V^{k-1} \to V^k/V^{k-1}$.

Closed orbits of $(-v)$ can be identified with periodic points of the maps $t^{-1}v\downarrow$. The periodic points of this maps can be counted with the help of the Lefschetz trace formula, and one can prove that

$$\zeta_L(-v) = \prod_k \det\left(\mathrm{Id} - t\mathcal{Z}_k\right)^{(-1)^{k+1}}$$

where $\mathcal{Z}_k$ stands for the homomorphism in homology induced by the map $t^{-1}v\downarrow : V^k/V^{k-1} \to V^k/V^{k-1}$.

Circle valued Morse theory for knots and links in S^3. Let L be an oriented link in S^3. We consider Morse maps $f : S^3 \setminus L \to S^1$, satisfying an additional restriction nearby L:

f is called *regular*, if there is an orientation preserving diffeomorphism $\phi : L \times D^2 \to \mathrm{Tub}\, L$ such that $f \circ \phi(l, z) = \frac{z}{|z|}$.

Definition. $\mathcal{MN}(L) = \min m(f)$ where the minimum is taken over the set of all regular Morse functions $S^3 \setminus L \to S^1$.

(C.Weber, L.Rudolph, A.P. *Morse-Novikov number for knots and links*, Algebra i Analiz, v. 13, 2001).

A link L is fibred if and only if $\mathcal{MN}(L) = 0$. To obtain lower bounds for $\mathcal{MN}(L)$ we apply the Novikov inequalities:

$$m_i(f) \geqslant \widehat{b}_i(S^3 \setminus L) + \widehat{q}_i(S^3 \setminus L) + \widehat{q}_{i-1}(S^3 \setminus L).$$

One can prove that $\widehat{b}_i = \widehat{q}_i = 0$ for $i = 0, 3$, and $\widehat{q}_2 = 0$ and that $\widehat{b}_1 = \widehat{b}_2$. Thus $\mathcal{MN}(L) \geqslant 2\widehat{b}_1(S^3 \setminus L) + 2\widehat{q}_1(S^3 \setminus L)$.

It turns out that for knots we have $\widehat{b}_1 = 0$ and the inequalities are reduced to the following:

$$\mathcal{MN}(K) \geqslant 2\widehat{q}_1(S^3 \setminus K).$$

Proposition. *Let K be a knot. Then the Novikov homology $\widehat{H}_*(S^3 \setminus K) = H_*(\overline{S^3 \setminus K}) \underset{\Lambda}{\otimes} \widehat{\Lambda}$ vanishes if and only if the Alexander polynomial $\Delta_K(t)$ of K is monic (that is, the principal term of $\Delta_K(t)$ equals ± 1).*

Twisted Novikov homology. In a joint work with H.Goda (*Twisted Novikov homology and circle-valued Morse theory for knots and links*, Osaka J.Math, v.42 N.3, 2005) we have introduced the twisted Novikov homology, which allows to give better lower estimates for $\mathcal{MN}(L)$.

The twisted Novikov homology is defined for any topological space X with respect to an epimorphism $\xi : \pi_1(X) \to \mathbf{Z}$ and a right representation $\rho : \pi_1(X) \to \mathrm{GL}(n, \mathbf{Z})$ (that is, $\rho(ab) = \rho(b)\rho(a)$).

Put $G = \pi_1(X)$ and consider ξ as a homomorphism

$$\xi : G \to (\mathbf{Z}[\mathbf{Z}])^\times \xrightarrow[\approx]{I} \left(\mathbf{Z}[t, t^{-1}]\right)^\times = \mathrm{GL}(1, \Lambda) \hookrightarrow \mathrm{GL}(1, \widehat{\Lambda})$$

where I sends the generator 1 of the group $\mathbf{Z}$ to t^{-1}.

Form the tensor product of the two representations ρ and ξ of G, to obtain a right representation $\widehat{\rho}_\xi : G \to \mathrm{GL}(n, \widehat{\Lambda})$.

Definition. The $\widehat{\Lambda}$-module $H_*(\widehat{\Lambda}^n \bigotimes_{\mathbb{Z}G} C_*(\widetilde{X}))$ is called *the twisted Novikov homology of* X and denoted $\widehat{H}_*(X, \rho, \xi)$

(here $\widetilde{X}$ is the universal covering of X and $\widehat{\Lambda}^n$ is a $\mathbb{Z}G$-module via the representation $\widehat{\rho}_\xi : G \to \mathrm{GL}(n, \widehat{\Lambda}.)$

To apply these constructions to knots and links observe that there is a canonical homomorphism $\xi : \pi_1(S^3 \setminus L) \to \mathbb{Z}$. The twisted Novikov homology of $S^3 \setminus L$ with respect to this homomorphism and a right representation $\rho : \pi_1(S^3 \setminus L) \to \mathrm{GL}(n, \mathbb{Z})$ will be denoted $\widehat{H}_*(S^3 \setminus L, \rho)$.

Theorem. *Let L be a link in S^3, and $f : S^3 \setminus L \to S^1$ a regular Morse function, v a generic f-gradient, $\rho : \pi_1(S^3 \setminus L) \to \mathrm{GL}(n, \mathbf{Z})$ a right representation. Then there is a chain complex $\mathcal{N}^\rho_*(f, v)$ over $\widehat{\Lambda}$ such that*

(1) *$\mathcal{N}^\rho_k(f, v)$ is a free $\widehat{\Lambda}$-module with $n \cdot m_k(f)$ generators.*

(2) *$H_*(\mathcal{N}^\rho_*(f, v)) \approx \widehat{H}_*(S^3 \setminus L, \rho)$.*

Corollary. *The Morse-Novikov number of any link L satisfies*

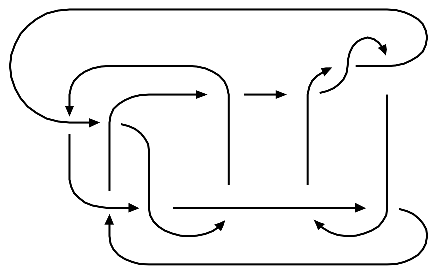
$$\mathcal{MN}(L) \geqslant \frac{2}{n}\Big(\widehat{b}_1(S^3 \setminus L) + \widehat{q}_1(S^3 \setminus L)\Big)$$

where $\widehat{b}_1, \widehat{q}_1$ are resp. rank and the torsion number of the $\widehat{\Lambda}$-module
$\widehat{H}_1(S^3 \setminus L, \rho)$.

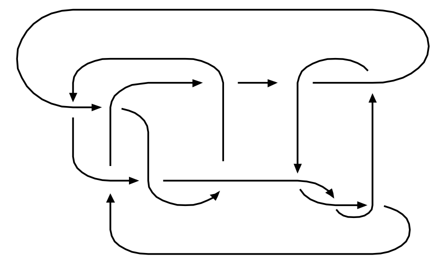
Theorem. *Let $\mathfrak{C}$ be the Conway knot. For every m we have*

$$2m \geqslant \mathcal{MN}(m\mathfrak{C}) \geqslant \frac{2m}{5}, \quad \text{where} \quad m\mathfrak{C} = \underbrace{\mathfrak{C} \# \ldots \# \mathfrak{C}}_{m \text{ times}}.$$

The same holds for the Kinoshita-Terasaka knot $\mathfrak{KT}$.



Conway knot



Kinoshita-Terasaka knot

One can show that for every knot K the limit

$$\lim_{m\to\infty} \frac{\mathcal{MN}(mK)}{m}$$

exists. This is an interesting invariant of a knot, called *asymptotic Morse-Novikov number of the knot.*

Fibering of 3-manifolds over S^1. (*Novikov homology, twisted Alexander polynomials and Thurston cones,* **A.P., math.GT/0406498.**)

Let M be a C^∞ manifold, $\omega \in \Omega^1(M)$ a closed 1-form on M. We say that ω is a Morse form, if locally it is the differential of a Morse function. In particular each real-valued, resp. circle-valued Morse function f gives rize to the Morse form df. The Morse-Novikov theory generalizes to this setting as follows.

Definition. *Let H be a free abelian group, and $\xi : H \to \mathbf{R}$ a homomorphism. Put $\Lambda = \mathbf{Z}H$ and define the Novikov ring as follows:*

$$\widehat{\Lambda}_\xi = \{\lambda = \sum_{g\in H} n_g g \mid \textit{for every } C \in \mathbf{R} \ \textit{the set} \quad \xi^{-1}([C,\infty[) \ \cap \ \operatorname{supp} \lambda \ \textit{is finite}\}$$

The ring $\Lambda = \mathbf{Z}H$ is isomorphic to the Laurent polynomial ring in several variables, and $\widehat{\Lambda}_\xi$ can be considered as a special completion of this ring. For example, if $H = \mathbf{Z}$ and the map $\xi : H \to \mathbf{R}$ is the standard inclusion $\mathbf{Z} \hookrightarrow \mathbf{R}$ we obtain the Novikov ring $\mathbf{Z}((t))$.

The notion of *twisted Novikov homology* generalizes immediately to this setting. Namely, let X be a finite CW complex, and put $H = H_1(X)/Tors$. Let $\xi : H \to \mathbf{R}$ be a homomorphism. Denote $\pi_1(X)$ by G and let $\rho : G \to \mathrm{GL}(n, \mathbf{Z})$ be a right representation. To these data one associates the *twisted Novikov homology*

$$\widehat{H}_*(X, \rho, \xi) = H_*\Big(\widehat{\Lambda}^n_\xi \underset{\mathbf{Z}G}{\otimes} C_*(\widetilde{X})\Big)$$

Here the tensor product is with respect to the structure of the right $\mathbf{Z}G$-module on $\widehat{\Lambda}_\xi^n$ induced by the right representation

$$\widehat{\rho}_\xi = \rho \otimes \xi : G \to \mathrm{GL}(n, \widehat{\Lambda}_\xi)$$

and ξ is considered as a representation

$$G \to (\mathbf{Z}[H])^\times \xrightarrow{\approx} \mathrm{GL}(1, \Lambda) \hookrightarrow \mathrm{GL}(1, \widehat{\Lambda}_\xi).$$

Returning to geometry, let ω be a Morse form on a closed manifold M. Let ξ denote the De Rham cohomology class of ω; then ξ can be also considered as a homomorphism $H = H_1(X)/Tors \to \mathrm{R}$.

We have a twisted version of the Novikov complex $\mathcal{N}_*^{\rho}(\omega, v)$, with the following properties:

(1) $\mathcal{N}_*^{\rho}(\omega, v)$ is a free $\widehat{\Lambda}_\xi$-module with $n \cdot m_k(\omega)$ generators in degree k.

(2) $H_*(\mathcal{N}_*^{\rho}(\omega, v)) \approx \widehat{H}_*(M, \rho, \xi)$.

In particular if ω has no zeros, the Novikov homology $\widehat{H}^*(M, \rho, \xi)$ vanishes for $\xi = [\omega]$, and any right representation ρ.

Problem: **For a given closed manifold M^3 describe the subset $\mathcal{V}(M) \subset H^1(M, \mathrm{R})$ of all cohomology classes ξ containing a 1-form without zeros. A Thurston's theorem (1979) says that $\mathcal{V}(M)$ is an open polyhedral conical subset.**

Theorem. *Let M^3 be a manifold with $\chi(M) = 0$. Let $\rho : \pi_1(M) \to \mathrm{GL}(n, \mathbf{Z})$ be a right representation. The set $\mathcal{V}_\rho(M)$ of all $\xi \in H^1(M, \mathbf{R})$ such that the twisted Novikov homology $\widehat{H}_*(M, \rho, \xi)$ vanishes is an open conical polyhedral subset of $H^1(M, \mathbf{R})$.*

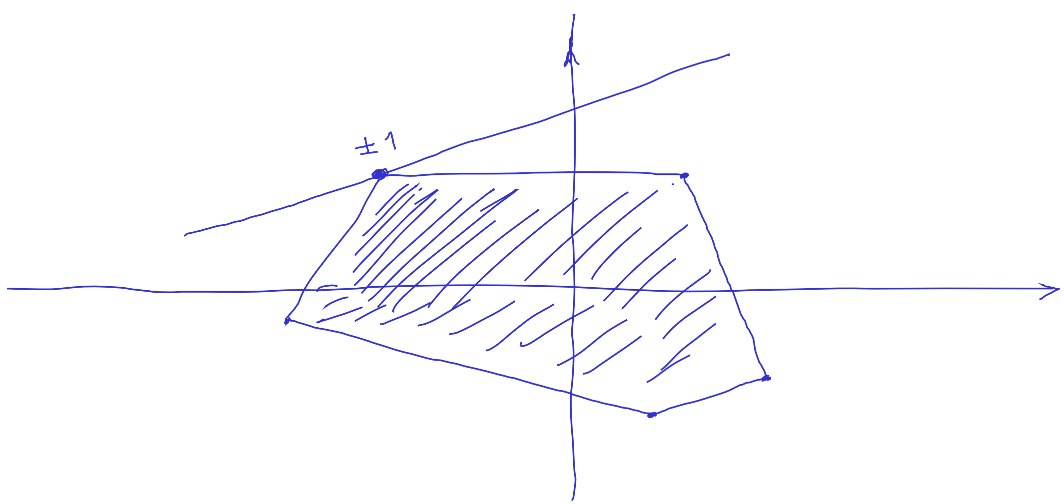
For every ρ we have $\mathcal{V}(M) \subset \mathcal{V}_\rho(M)$ so that the conical subset $\mathcal{V}_\rho(M)$ provides an upper bound for $\mathcal{V}(M)$ which is computable in homological terms, actually as we shall see later in terms of the twisted Alexander polynomials.

Problem: Is it true that

$$\mathcal{V}(M) = \bigcap_{\rho} \mathcal{V}_\rho(M) \ ?$$

M.Wada's construction associates to each representation $\lambda : G \to \mathrm{GL}(n, \mathbf{Z})$ of a finitely presented group G the *twisted Alexander polynomial* Δ_λ. **In the case when the rank of the group $H = H_1(G)/Tors$ is $\geqslant 2$ the twisted Alexander polynomial is an element of $\Lambda = \mathbf{Z}H$.**

Write $\Delta_\lambda = \sum_{g \in H} n_g g$, where $n_g \in \mathbf{Z}$. Let us say that Δ_λ is ξ-monic, if there is only one vertex $g \in H$ of the Newton polytope P of Δ_λ such that $\xi|P$ reaches its maximal value, and for this g we have $n_g = \pm 1$.



Theorem. *Let M be a 3-dimensional manifold with $\chi(M) = 0$ and $b_1(M) \geqslant 2$. Let ρ be a right representation of $\pi_1(M)$. Then $\mathcal{V}_\rho(M)$ is equal to the set of all $\xi \in H^1(M, \mathrm{R})$ such that $\Delta_{\bar\rho}$ is ξ-monic.*

(here $\bar\rho$ denotes the conjugate representation for thee right representation ρ).

When $\operatorname{rk} H_1(G) = 1$ the twisted Alexander polynomial associated to a representation $\lambda : G \to \mathrm{GL}(n, \mathrm{Z})$ is not a polynomial, but rather a rational function of the form $\frac{P(t)}{Q(t)}$, where $P, Q \in \mathrm{Z}[t]$ and the first and the last terms of $Q(t)$ are equal to ± 1.

Such rational function will be called *monic* if the first coefficient of P is equal to ± 1, and *strictly monic* if it is equal to 1.

A theorem of H.Goda, T.Kitano and T.Morifuji (2002) says that if a knot K is fibred, then for every representation $\lambda : \pi_1(S^3 \setminus K) \to SL(n, F)$ where F is a field the twisted Alexander polynomial Δ_λ associated to λ is strictly monic.

Theorem. *Let M be a 3-manifold with $b_1(M) = 1$ and $\chi(M) = 0$. Let λ be a representation of $\pi_1(M) \to \mathrm{GL}(n, \mathbf{Z})$. Then the twisted Alexander polynomial Δ_λ is monic if and only if the twisted Novikov homology $\widehat{H}_*(M, \xi, \bar{\lambda})$ associated to $\xi : H \xrightarrow{\approx} \mathbf{Z}$ and $\bar{\lambda}$ vanishes.*

Corollary. *Let L be a link in S^3. Let $\lambda : \pi_1(S^3 \setminus L) \to \mathrm{GL}(n, \mathbf{Z})$ be a representation. If L is fibred, then the twisted Alexander polynomial associated with λ is monic.*