NOVIKOV HOMOLOGY,

TWISTED ALEXANDER POLYNOMIALS

AND THURSTON CONES

Classical Morse theory. Let M be a closed manifold, $f : M \to \mathbf{R}$ be a C^∞ function on M. A point $p \in M$ is called *critical point of* f if $f'(p) = 0$. A critical point p is called *non-degenerate* if the bilinear form $f''(p)$ is non–degenerate. In this case the number of negative eigenvalues of $f''(p)$ is called *index* of p. f is called *Morse function* if all critical points are non-degenerate.

For a Morse function $f : M \to \mathbf{R}$ let us denote

$$S(f) = \{p \mid f'(p) = 0\}, \qquad S_k(f) = \{p \mid f'(p) = 0, \operatorname{ind} p = k\}$$

$$m(f) = \text{card } S(f), \qquad m_k(f) = \text{card } S_k(f)$$

Theorem. (Morse, 1925) *Let $f : M \to \mathbf{R}$ be a Morse function. Then*

$$m_k(f) \geqslant b_k(M)$$

(where $b_k(M)$ is the rank of $H_k(M)$).

Theorem. *(Pitcher, 1958) Furthermore*

$$m_k(f) \geqslant b_k(M) + q_k(M) + q_{k-1}(M)$$

(where $q_k(M)$ is the minimal number of generators of the torsion submodule of $H_k(M)$).

For the proof Morse shows that M is homotopy equivalent to a CW complex X such that the number of k-cells in X is $m_k(f)$.

In 1982 E.Witten suggested a geometric construction of a chain complex $\mathcal{M}_*(f, v)$ which is associated to each Morse function f and a generic f-gradient v such that

(1) $\mathcal{M}_k(f, v)$ is the free abelian group generated by $S_k(f)$,
(2) $H_*(\mathcal{M}_*(f, v)) \approx H_*(M)$.

Circle-valued Morse theory. Let $f : M \to S^1$ be a C^∞ function. All the local notions of the classical Morse theory are valid in this framework. But the Morse inequalities are not valid as they stand. For example, the number of critical points of the function $\mathrm{id} : S^1 \to S^1$ is zero, but $H_*(S^1) \neq 0$.

The relevant generalization of the Morse theory for the circle-valued case was found by S.P.Novikov (*Many-valued functions and functionals. An analogue of Morse theory*", Doklady AN SSSR, v.260 (1981).)

MORSE THEORY

Morse map $f : M \to \mathbf{R}$ and a generic f-gradient v

$$\downarrow$$

chain complex $\mathcal{M}_*(f,v)$ generated over $\mathbf{Z}$ by $S(f)$

$$\downarrow$$

$$H_*(\mathcal{M}_*(f,v)) \approx H_*(M)$$

NOVIKOV THEORY

Morse map $f : M \to S^1$ and a generic f-gradient v

$$\downarrow$$

chain complex $\mathcal{N}_*(f,v)$ generated over $\widehat{\Lambda} = \mathbf{Z}((t))$ by $S(f)$

$$\downarrow$$

$$H_*(\mathcal{N}_*(f,v)) \approx \widehat{H}_*(\bar{M}) = H_*(\bar{M}) \bigotimes_{\Lambda} \widehat{\Lambda}$$

where

$$\begin{array}{ccc} \bar{M} & \xrightarrow{F} & \mathbf{R} \\ \downarrow \mathbf{Z} & & \downarrow \mathbf{Z} \\ M & \xrightarrow{f} & S^1 \end{array}$$

$$\Lambda = \mathbf{Z}[t, t^{-1}] = \mathbf{Z}[\mathbf{Z}]$$

$$\widehat{\Lambda} = \mathbf{Z}((t)) = \{a_{-n}t^{-n} + \ldots + a_0 + \ldots + a_k t^k + \ldots \mid a_i \in \mathbf{Z}\}$$

Since $\widehat{\Lambda}$ is a principal ring, we deduce the *Novikov inequalities*:

$$m_k(f) \geqslant \widehat{b}_k(M) + \widehat{q}_k(M) + \widehat{q}_{k-1}(M),$$

where $\widehat{b}_k(M) = rk_{\widehat{\Lambda}} \widehat{H}_k(M), \qquad \widehat{q}_k(M) = t.n._{\widehat{\Lambda}} \widehat{H}_k(M).$

Circle valued Morse theory for knots and links in S^3. **Let L be an oriented link in S^3. We consider Morse maps $f : S^3 \setminus L \to S^1$, satisfying an additional restriction nearby L:**

f is called *regular*, if there is an orientation preserving diffeomorphism $\phi : L \times D^2 \to \mathrm{Tub}\, L$ such that $f \circ \phi(l, z) = \frac{z}{|z|}$.

Definition. **$\mathcal{MN}(L) = \min m(f)$ where the minimum is taken over the set of all regular Morse functions $S^3 \setminus L \to S^1$.**

(C.Weber, L.Rudolph, A.P. *Morse-Novikov number for knots and links*, **Algebra i Analiz, v. 13, 2001).**

A link L is fibred if and only if $\mathcal{MN}(L) = 0$. To obtain lower bounds for $\mathcal{MN}(L)$ we apply the Novikov inequalities:

$$m_i(f) \geqslant \widehat{b}_i(S^3 \setminus L) + \widehat{q}_i(S^3 \setminus L) + \widehat{q}_{i-1}(S^3 \setminus L).$$

One can prove that $\widehat{b}_i = \widehat{q}_i = 0$ for $i = 0, 2, 3$ and that $\widehat{b}_1 = \widehat{b}_2$. Thus $\mathcal{MN}(L) \geqslant 2\widehat{b}_1(S^3 \setminus L) + 2\widehat{q}_1(S^3 \setminus L)$.

It turns out that for knots we have $\widehat{b}_1 = 0$ and the inequalities are reduced to the following:

$$\mathcal{MN}(K) \geqslant 2\widehat{q}_1(S^3 \setminus K).$$

Proposition. *Let K be a knot. Then the Novikov homology $\widehat{H}_*(S^3 \setminus K) = H_*(\overline{S^3 \setminus K}) \underset{\Lambda}{\otimes} \widehat{\Lambda}$ vanishes if and only if the Alexander polynomial $\Delta_K(t)$ of K is monic (that is, the principal term of $\Delta_K(t)$ equals ± 1).*

Twisted Novikov homology. In a joint work with H.Goda (*Twisted Novikov homology and circle-valued Morse theory for knots and links*, Osaka J.Math, v.42 N.3, 2005) we have introduced the twisted Novikov homology, which allows to give better lower estimates for $\mathcal{MN}(L)$.

The twisted Novikov homology is defined for any topological space X with respect to an epimorphism $\xi : \pi_1(X) \to \mathbf{Z}$ and a right representation $\rho : \pi_1(X) \to \mathrm{GL}(n, \mathbf{Z})$ (that is, $\rho(ab) = \rho(b)\rho(a)$).

Put $G = \pi_1(X)$ and consider ξ as a homomorphism

$$\xi : G \to (\mathbf{Z}[\mathbf{Z}])^\times \xrightarrow[\approx]{I} \big(\mathbf{Z}[t, t^{-1}]\big)^\times = \mathrm{GL}(1, \Lambda) \hookrightarrow \mathrm{GL}(1, \widehat{\Lambda})$$

where I sends the generator 1 of the group $\mathbf{Z}$ to t^{-1}.

Form the tensor product of the two representations ρ and ξ of G, to obtain a right representation $\widehat{\rho}_\xi : G \to \mathrm{GL}(n, \widehat{\Lambda})$.

Definition. The $\widehat{\Lambda}$-module $H_*(\widehat{\Lambda}^n \bigotimes_{\mathbf{Z}G} C_*(\widetilde{X}))$ is called *the twisted Novikov homology of* X and denoted $\widehat{H}_*(X, \rho, \xi)$

(here $\widetilde{X}$ is the universal covering of X and $\widehat{\Lambda}^n$ is a $\mathbf{Z}G$-module via the representation $\widehat{\rho}_\xi : G \to \mathrm{GL}(n, \widehat{\Lambda}.)$

To apply these constructions to knots and links observe that there is a canonical homomorphism $\xi : \pi_1(S^3 \setminus L) \to \mathbf{Z}$. The twisted Novikov homology of $S^3 \setminus L$ with respect to this homomorphism and a right representation $\rho : \pi_1(S^3 \setminus L) \to \mathrm{GL}(n, \mathbf{Z})$ will be denoted $\widehat{H}_*(S^3 \setminus L, \rho)$.

Theorem. *Let L be a link in S^3, and $f : S^3 \setminus L \to S^1$ a regular Morse function, v a generic f-gradient, $\rho : \pi_1(S^3 \setminus L) \to \mathrm{GL}(n, \mathbf{Z})$ a right representation. Then there is a chain complex $\mathcal{N}_*^\rho(f, v)$ over $\widehat{\Lambda}$ such that*

(1) *$\mathcal{N}_k^\rho(f, v)$ is a free $\widehat{\Lambda}$-module with $n \cdot m_k(f)$ generators.*

(2) *$H_*(\mathcal{N}_*^\rho(f, v)) \approx \widehat{H}_*(S^3 \setminus L, \rho)$.*

Corollary. *The Morse-Novikov number of any link L satisfies*

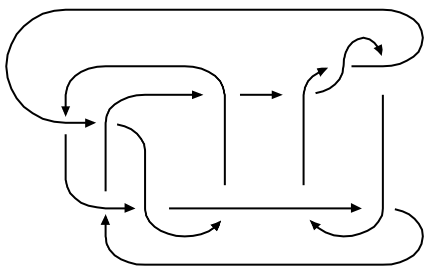
$$\mathcal{MN}(L) \geqslant \frac{2}{n}\Big(\widehat{b}_1(S^3 \setminus L) + \widehat{q}_1(S^3 \setminus L)\Big)$$

where $\widehat{b}_1, \widehat{q}_1$ are resp. rank and the torsion number of the $\widehat{\Lambda}$-module
$\widehat{H}_1(S^3 \setminus L, \rho)$.

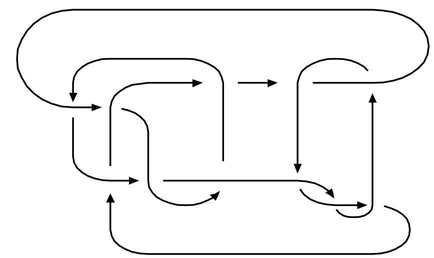
Theorem. *Let $\mathfrak{C}$ be the Conway knot. For every m we have*

$$2m \geqslant \mathcal{MN}(m\mathfrak{C}) \geqslant \frac{2m}{5}, \quad \text{where} \quad m\mathfrak{C} = \underbrace{\mathfrak{C} \# \ldots \# \mathfrak{C}}_{m \text{ times}}.$$

The same holds for the Kinoshita-Terasaka knot $\mathfrak{KT}$.



Conway knot



Kinoshita-Terasaka knot

One can show that for every knot K the limit

$$\lim_{m\to\infty} \frac{\mathcal{MN}(mK)}{m}$$

exists. This is an interesting invariant of a knot, called *asymptotic Morse-Novikov number of the knot.*

Fibering of 3-manifolds over S^1. (*Novikov homology, twisted Alexander polynomials and Thurston cones*, A.P., math.GT/0406498.)

Let M be a C^∞ manifold, $\omega \in \Omega^1(M)$ a closed 1-form on M. We say that ω is a Morse form, if locally it is the differential of a Morse function. In particular each real-valued, resp. circle-valued Morse function f gives rize to the Morse form df. The Morse-Novikov theory generalizes to this setting as follows.

Definition. *Let H be a free abelian group, and $\xi : H \to \mathbf{R}$ a homomorphism. Put $\Lambda = \mathbf{Z}H$ and define the Novikov ring as follows:*

$$\widehat{\Lambda}_\xi = \{\lambda = \textstyle\sum_{g\in H} n_g g \mid \textit{for every } C \in \mathbf{R} \\ \textit{the set } \ \xi^{-1}([C,\infty[) \ \cap \ \operatorname{supp} \lambda \textit{ is finite}\}$$

The ring $\Lambda = \mathbf{Z}H$ is isomorphic to the Laurent polynomial ring in several variables, and $\widehat{\Lambda}_\xi$ can be considered as a special completion of this ring. For example, if $H = \mathbf{Z}$ and the map $\xi : H \to \mathbf{R}$ is the standard inclusion $\mathbf{Z} \hookrightarrow \mathbf{R}$ we obtain the Novikov ring $\mathbf{Z}((t))$.

The notion of *twisted Novikov homology* generalizes immediately to this setting. Namely, let X be a finite CW complex, and put $H = H_1(X)/Tors$. Let $\xi : H \to \mathbf{R}$ be a homomorphism. Denote $\pi_1(X)$ by G and let $\rho : G \to \mathrm{GL}(n, \mathbf{Z})$ be a right representation. To these data one associates the *twisted Novikov homology*

$$\widehat{H}_*(X, \rho, \xi) = H_*\Big(\widehat{\Lambda}^n_\xi \bigotimes_{\mathbf{Z}G} C_*(\widetilde{X})\Big)$$

Here the tensor product is with respect to the structure of the right $\mathbf{Z}G$-module on $\widehat{\Lambda}_\xi^n$ induced by the right representation

$$\widehat{\rho}_\xi = \rho \otimes \xi : G \to \mathrm{GL}(n, \widehat{\Lambda}_\xi)$$

and ξ is considered as a representation

$$G \to (\mathbf{Z}[H])^\times \xrightarrow{\approx} \mathrm{GL}(1, \Lambda) \hookrightarrow \mathrm{GL}(1, \widehat{\Lambda}_\xi).$$

Returning to geometry, let ω be a Morse form on a closed manifold M. Let ξ denote the De Rham cohomology class of ω; then ξ can be also considered as a homomorphism $H = H_1(X)/Tors \to \mathrm{R}$.

We have a twisted version of the Novikov complex $\mathcal{N}_*^\rho(\omega, v)$, with the following properties:

(1) $\mathcal{N}_*^\rho(\omega, v)$ is a free $\widehat{\Lambda}_\xi$-module with $n \cdot m_k(\omega)$ generators in degree k.

(2) $H_*(\mathcal{N}_*^\rho(\omega, v)) \approx \widehat{H}_*(M, \rho, \xi)$.

In particular if ω has no zeros, the Novikov homology $\widehat{H}^*(M, \rho, \xi)$ vanishes for $\xi = [\omega]$, and any right representation ρ.

<u>A geometric problem:</u> For a given closed manifold M^3 describe the subset $\mathcal{V}(M) \subset H^1(M, \mathrm{R})$ of all cohomology classes ξ containing a 1-form without zeros. A Thurston's theorem (1979) says that $\mathcal{V}(M)$ is an open polyhedral conical subset.

Theorem. *Let M^3 be a manifold with $\chi(M) = 0$. Let $\rho : \pi_1(M) \to \mathrm{GL}(n, \mathbf{Z})$ be a right representation. The set $\mathcal{V}_\rho(M)$ of all $\xi \in H^1(M, \mathbf{R})$ such that the twisted Novikov homology $\widehat{H}_*(M, \rho, \xi)$ vanishes is an open conical polyhedral subset of $H^1(M, \mathbf{R})$.*

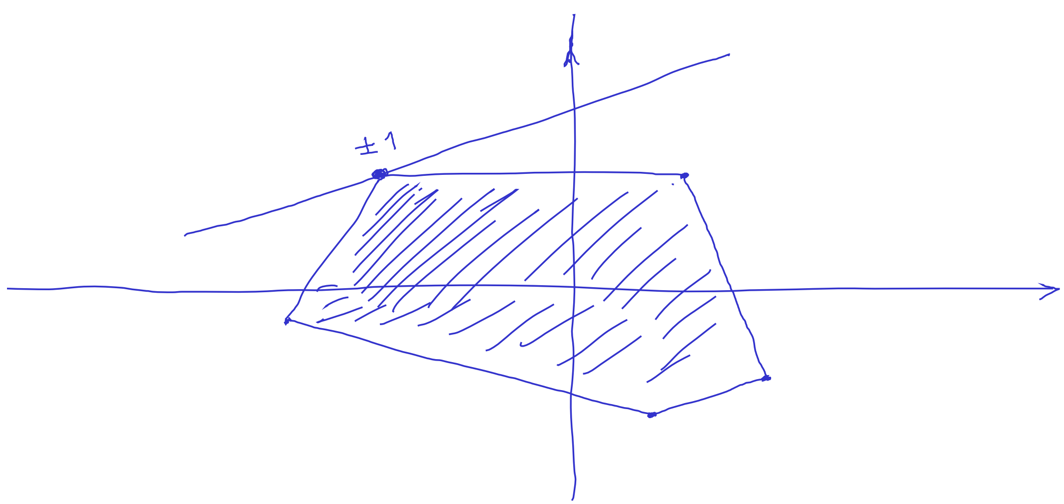
For every ρ we have $\mathcal{V}(M) \subset \mathcal{V}_\rho(M)$ so that the conical subset $\mathcal{V}_\rho(M)$ provides an upper bound for $\mathcal{V}(M)$ which is computable in homological terms, actually as we shall see later in terms of the twisted Alexander polynomials.

Problem: Is it true that

$$\mathcal{V}(M) = \bigcap_\rho \mathcal{V}_\rho(M) \ ?$$

M.Wada's construction associates to each representation $\lambda : G \to \mathrm{GL}(n, \mathbf{Z})$ of a finitely presented group G the *twisted Alexander polynomial* Δ_λ. In the case when the rank of the group $H = H_1(G)/Tors$ is $\geqslant 2$ the twisted Alexander polynomial is an element of $\Lambda = \mathbf{Z}H$.

Write $\Delta_\lambda = \sum_{g \in H} n_g g$, where $n_g \in \mathbf{Z}$. Let us say that Δ_λ is ξ-monic, if there is only one vertex $g \in H$ of the Newton polytope P of Δ_λ such that $\xi | P$ reaches its maximal value, and for this g we have $n_g = \pm 1$.



Theorem. *Let M be a 3-dimensional manifold with $\chi(M) = 0$ and $b_1(M) \geqslant 2$. Let ρ be a right representation of $\pi_1(M)$. Then $\mathcal{V}_\rho(M)$ is equal to the set of all $\xi \in H^1(M, \mathbf{R})$ such that $\Delta_{\bar{\rho}}$ is ξ-monic.*

(here $\bar{\rho}$ denotes the conjugate representation for thee right representation ρ).

When $\operatorname{rk} H_1(G) = 1$ the twisted Alexander polynomial associated to a representation $\lambda : G \to \mathrm{GL}(n, \mathbf{Z})$ is not a polynomial, but rather a rational function of the form $\frac{P(t)}{Q(t)}$, where $P, Q \in \mathbf{Z}[t]$ and the first and the last terms of $Q(t)$ are equal to ± 1.

Such rational function will be called *monic* if the first coefficient of P is equal to ± 1, and *strictly monic* if it is equal to 1.

A theorem of H.Goda, T.Kitano and T.Morifuji (2002) says that if a knot K is fibred, then for every representation $\lambda : \pi_1(S^3 \setminus K) \to SL(n, F)$ where F is a field the twisted Alexander polynomial Δ_λ associated to λ is strictly monic.

Theorem. *Let M be a 3-manifold with $b_1(M) = 1$ and $\chi(M) = 0$. Let λ be a representation of $\pi_1(M) \to \mathrm{GL}(n, \mathbf{Z})$. Then the twisted Alexander polynomial Δ_λ is monic if and only if the twisted Novikov homology $\widehat{H}_*(M, \xi, \bar{\lambda})$ associated to $\xi : H \xrightarrow{\approx} \mathbf{Z}$ and $\bar{\lambda}$ vanishes.*

Corollary. *Let L be a link in S^3. Let $\lambda : \pi_1(S^3 \setminus L) \to \mathrm{GL}(n, \mathbf{Z})$ be a representation. If L is fibred, then the twisted Alexander polynomial associated with λ is monic.*