Discrete Parabolic Anderson Model with Heavy Tailed Potential

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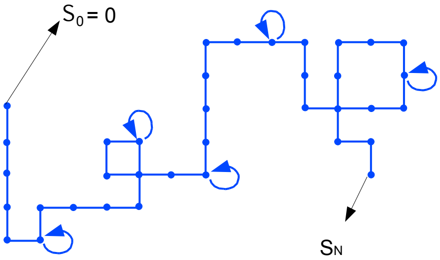
1 The Model

1.1) Configurations.

Let $d \in \mathbb{N} \setminus \{0\}$ and for $x \in \mathbb{Z}^d$ set $|x| = |x_1| + \cdots + |x_d|$.

Let $S = \{S_k\}_{k \geq 0}$ be a (lazy) nearest-neighbor RW on $\mathbb{Z}^d$, i.e.,

- $S_0 = 0$,
- $(S_{k+1} - S_k)_{k \geq 0}$ is i.i.d.,
- $\{x \in \mathbb{Z}^d \colon P(S_1 = x) > 0\} = \{0\} \cup \{y \in \mathbb{Z}^d \colon |y| = 1\}$



1.2) The medium.

Pick $\alpha > d$ and let $(\xi(x))_{x\in\mathbb{Z}^d}$ be an i.i.d. field of Pareto-distributed random variables, i.e.,

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2.3) Hamiltonian.

Given $\xi \in \mathbb{R}^{\mathbb{Z}^d}$, each RW trajectory S is associated with the Hamiltonian

$$H_N^\xi(S) = -\sum_{i=1}^N \xi(S_i) = -\sum_{x\in\mathbb{Z}^d} l_N(S,x)\,\xi(x).$$

1.4) Perturbed measure.

Given $\xi \in \mathbb{R}^{\mathbb{Z}^d}$, we let P_N^ξ be the perturbed law in size N, i.e.,

$$\frac{dP_N^\xi}{dP}(S) = \frac{\exp\Big(- H_N^\xi(S)\Big)}{Z_N^\xi},$$

and denote by p_N^ξ the law of S_N, i.e.,

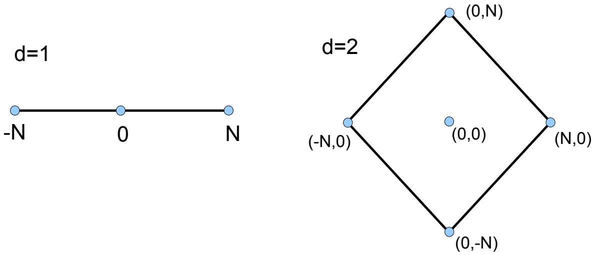
$$p_N^\xi(x) = P_N^\xi(S_N = x) \quad \forall x \in \mathbb{Z}^d.$$

2 Goals and former results

2.1) Challenges.

Pick $N \in \mathbb{N}$, $\xi \in \mathbb{R}^{\mathbb{Z}^d}$ and notice that

$$\mathcal{B}_N = \{x \in \mathbb{Z}^d \colon p_N^{\xi}(x) > 0\} = \{x \in \mathbb{Z}^d \colon |x| \leq N\}.$$



- Determine the smallest $\mathcal{A}_N^\xi \subset \mathcal{B}_N$ such that $\mathbb{P}$-a.s. in ξ,

$$\lim_{N\to\infty} \sum_{x\in\mathcal{A}_N^\xi} p_N^\xi(x) = 1,$$

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- Find a narrowed $\mathcal{W}_N^\xi \subset \{S\colon S_N \in \mathcal{A}_N^\xi\}$ which still satisfies that $\mathbb{P}$-a.s. in ξ,

$$\lim_{N\to\infty} P_N^\xi(S \in \mathcal{W}_N^\xi) = 1.$$

2.2) Two sites localization in continuous time.

With a continuous time random walk on $\mathbb{Z}^d$

Theorem (Konig, Lacoin, Morters and Sidorova (2008))

Let $\alpha > d$. For all $t > 0$ and $\xi \in \mathbb{R}^{\mathbb{Z}^d}$ there exist $z_{t,\xi}^{(1)}, z_{t,\xi}^{(2)} \in \mathbb{Z}^d$ such that

$$\lim_{t\to\infty} p_t^\xi(z_{t,\xi}^{(1)}) + p_t^\xi(z_{t,\xi}^{(2)}) = 1 \qquad \mathbb{P}\text{-a.s. in } \xi.$$

Super-balistic localization :

$$|z_{t,\xi}^{(1)}|, |z_{t,\xi}^{(2)}| \sim (t/\log t)^{1+q},$$

with $q = d/(\alpha - d) > 0$.

3 Single site localization of the endpoint

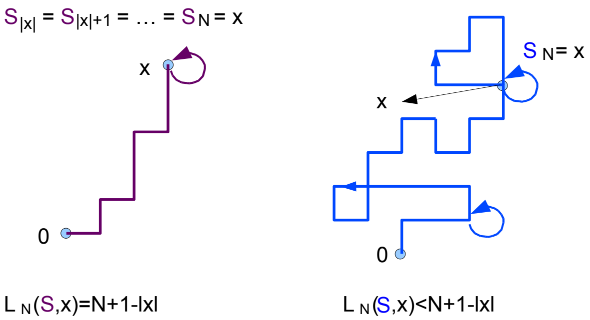
3.1) Modified field.

Pick $x \in \mathcal{B}_N$ and S: $S_N = x$. The contribution of $\xi(x)$ to $H_N^{\xi}(S)$ is

$$l_N(S,x)\,\xi(x) \leq (N+1-|x|)\,\xi(x) := (N+1)\,\psi_N(x)$$

with

$$\psi_N(x) := \left(1 - \frac{|x|}{N+1}\right)\xi(x)\,.$$



3.2) Order statistics of the modified field.

Set

$$z_N^{(1)} = \operatorname{argmax}\{\psi_N(x) \colon x \in \mathcal{B}_N|\},$$
$$z_N^{(k)} = \operatorname{argmax}\left\{\psi_N(x) \colon x \in \mathcal{B}_N \setminus \{z_N^{(1)}, \dots, z_N^{(k-1)}\}\right\},$$

such that

$$\psi_N(z_N^{(1)}) > \psi_N(z_N^{(2)}) > \cdots > \psi_N(z_N^{(|\mathcal{B}_N|)}),$$

is the order statistics of the field $\{\psi_N(x)\}_{x \in \mathcal{B}_N}$.

3.3) Localization.

For all $\alpha > d$, $\xi \in \mathbb{R}^{\mathbb{Z}^d}$ and $N \geq 1$ set

$$w_{N,\xi} \;=:\; \text{argmax}\{p_N^\xi(x) : \; x \in \mathcal{B}_N\}.$$

Theorem (One-site localization)

It comes that $\mathbb{P}$-*a.s. in* ξ

$$\lim_{N\to\infty} p_N^\xi(w_{N,\xi}) \;=\; 1,$$

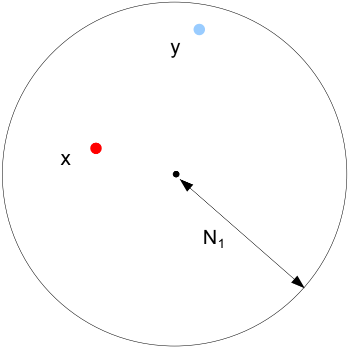
and

$$\lim_{N\to\infty} \mathbb{P}\left(w_{N,\xi} = z_{N,\xi}^{(1)}\right) \;=\; 1\,.$$

Moreover,

$$\mathbb{P}\left(w_{N,\xi} = z_{N,\xi}^{(2)} \textit{ for infinitely many } N\right) \;=\; 1\,.$$

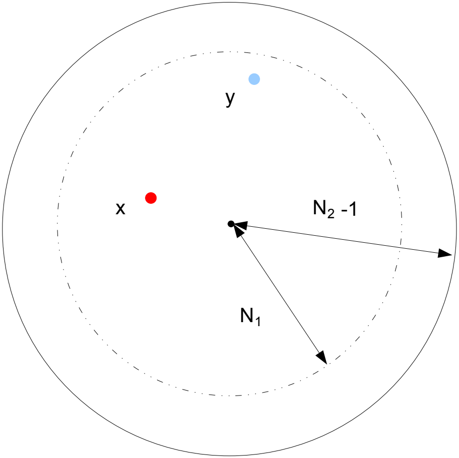
3.4) Behavior of $(z_N^{(1)}, z_N^{(2)})$.



$$\xi(Y) > \xi(X) \quad \text{but} \quad \left(1 - \tfrac{|Y|}{N_1+1}\right)\xi(Y) < \left(1 - \tfrac{|X|}{N_1+1}\right)\xi(X)$$

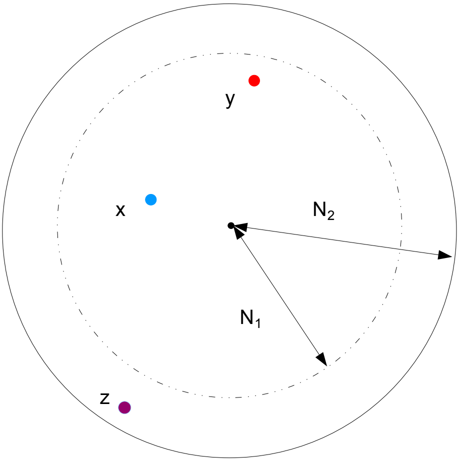
$$\text{such that} \qquad \psi_{N_1}(Y) < \psi_{N_1}(X)$$

$$(z_{N_1}^{(1)}, z_{N_1}^{(2)}) = (X, Y)$$



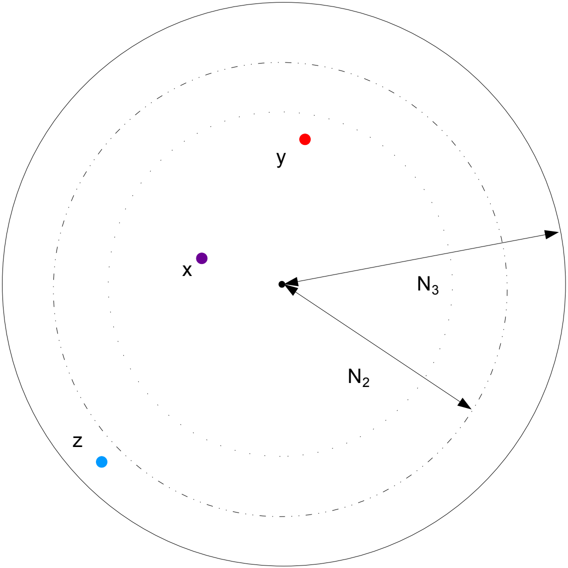
$$N_2 = \inf\{n \geq N_1 : \psi_n(X) < \psi_n(Y)\}$$

$$(z^{(1)}_{N_1}, z^{(2)}_{N_1}) = (X, Y) \text{ and } (z^{(1)}_{N_2-1}, z^{(2)}_{N_2-1}) = (X, Y)$$



$$(z^{(1)}_{N_2-1}, z^{(2)}_{N_2-1}) = (X, Y) \text{ and } (z^{(1)}_{N_2}, z^{(2)}_{N_2}) = (Y, X)$$

$$\xi(Z) > \xi(Y) > \xi(X) \text{ but } \psi_{N_2}(Z) < \psi_{N_2}(X) < \psi_{N_2}(Y)$$



$$(z^{(1)}_{N_2}, z^{(2)}_{N_2}) = (Y, X) \text{ and } (z^{(1)}_{N_3}, z^{(2)}_{N_3}) = (Y, Z)$$

$$\psi_{N_3}(X) < \psi_{N_3}(Z) < \psi_{N_3}(Y)$$

3.5) Heuristic.

Set for all S

$$x_N(S) = \text{argmax}\{\xi(x) \colon l_N(S,x) > 0\}.$$

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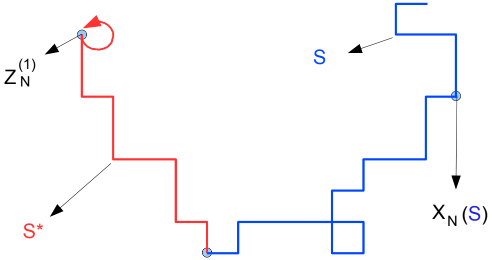
and pick S such that $x_N(S) \notin \{z_N^{(1)}, z_N^{(2)}\}$.



Remark : $\mathbb{P}$-a.s. in ξ and for N large enough,

$$H_N^{\xi}(S^*) \geq (N+1)\ \psi_N(z_N^{(1)}) \geq u_N = \frac{N^{1+d/\alpha}}{(\log \log N)^{1/\alpha}}.$$

Set $v_N = \frac{N^{d/\alpha}}{(\log N)^{1/\alpha}} = o(\frac{u_N}{N})$ and then

$$H_N^{\xi}(S) \leq \sum_{x:\, \xi(x) \geq v_N} l_N(x)\, \xi(x) + \sum_{x:\, \xi(x) \leq v_N} l_N(x)\, \xi(x)$$

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Since S visits at least $|x_N(S)|$ distinct sites it comes

$$l_N(\{x\colon \xi(x) \ge v_N\}) \le N - |x_N(S)| + |\{x \in \mathcal{B}_N : \xi(x) \ge v_N\}|.$$

Remark : $\mathbb{P}$-a.s. in ξ and for N large enough,

$$|\{x \in \mathcal{B}_N : \xi(x) \geq v_N\}| \leq (\log N)^2.$$

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Thus, for N large

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Therefore

$$H_N^\xi(S) \leq (N+1)\psi_N(x_N(S)) + o(u_N),$$

and $x_N(S) \notin \{z_N^{(1)}, z_N^{(2)}\}$, then

$$H_N^\xi(S) \leq (N+1)\psi_N(z_N^{(3)}) + o(u_N).$$

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Therefore, for every $S : x_N(S) \notin \{z_N^{(1)}, z_N^{(2)}\}$

$$H_N^{\xi}(S) = o\big(H_N^{\xi}(S^*)\big).$$

4 Path properties

For all S, let $\tau_{N,\xi} := \inf\{n \geq 0\colon\ S_n = w_{N,\xi}\}$. Set

$$\begin{aligned}\mathcal{C}_{N,\xi} := \Big\{S\colon\ & S \text{ is injec. on } [0, \tau_{N,\xi}],\\ & \tau_{N,\xi} \leq |w_{N,\xi}| + o(N),\\ & \xi(S) < \xi(w_{N,\xi}) \text{ on } [0, \tau_{N,\xi}],\\ & S = w_{N,\xi} \text{ on } [\tau_{N,\xi}, N]\Big\},\end{aligned}$$

We then have the following result.

Theorem

It comes that $\mathbb{P}$-a.s. in ξ,

$$\lim_{N\to\infty} \mathbb{P}_{N,\xi}(\mathcal{C}_{N,\xi}) = 1.$$