MONOTONE COMBINED NUMERICAL SCHEME FOR ANISOTROPIC CHEMOTAXIS-FLUID MODEL

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Abstract. This paper is devoted to the numerical study of a model arising from biology, consisting of chemotaxis equations coupled to viscous incompressible fluid equations through transport and external forcing. A detailed convergence analysis of this chemotaxis-fluid model by means of a suitable combination of the finite volume method and the nonconforming finite element method is investigated. In the case of non positive transmissiblities, a correction of the diffusive fluxes is necessary to maintain the monotonicity of the numerical scheme. Finally, many numerical tests are given to illustrate the behavior of the anisotropic Keller-Segel-Stokes system.

Keywords: Degenerate parabolic equation; Navier-Stokes equations ; heterogeneous and anisotropic diffusion; combined scheme.

1. Introduction

Chemotaxis is a biological process, in which cells move towards a chemically more favorable environment. This behavior enables them to locate nutrients or to avoid predators and the chemical can be produced or consumed by the cells. For example, bacteria often swim towards higher concentration of oxygen to survive. The best studied model of chemotaxis phenomenon in mathematical biology is the Keller-Segel system introduced in [21] and the article of Horstmann [20] has provided a detailed introduction into the mathematics of this model. In nature, cells often live in a viscous fluid and meanwhile the motion of the fluid is under the influence of gravitational forcing generated by aggregation of cells. Unfortunately, chemotaxis systems do neglect the surrounding fluid and they are unable to predict the influence of the fluid on the anisotropic chemotaxis phenomenon. Thus, it is interesting and important to study some phenomenon of chemotaxis on the basis of the coupled cell-fluid model. For that, we investigate in this paper a system consisting of the parabolic Keller-Segel equations with general tensors coupled to Navier-Stokes equations,

$$
\left\{
\begin{array}{rcll}
\partial_t N - \nabla\cdot(S(x)a(N)\nabla N) + \nabla\cdot(S(x)\chi(N)\nabla C) + u\cdot\nabla N &=& f(N), & \\
\partial_t C - \nabla\cdot(M(x)\nabla C) + u\cdot\nabla C &=& h(N,C), & \\
\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla P &=& -N\nabla\phi, & \\
\nabla\cdot u &=& 0, & t>0, x\in\Omega,
\end{array}
\right.
\tag{1.1}
$$

where Ω is a spatial domain where the cells and the fluid move and interact. We assume that Ω is an open bounded domain in $\mathbb{R}^d$, $d = 2$ or 3 with smooth boundary $\partial\Omega$. The experimental set-up corresponds to mixed type boundary conditions. For simplicity we use no-flux conditions for N and C

and zero Dirichlet for u to reflect the no-slip boundary conditions of the flow. Therefore, the system (1.1) is supplemented by the following boundary conditions on $\Sigma_t = \partial\Omega \times (0, T)$,

$$S(x)a(N)\nabla N \cdot \eta = 0,\ M(x)\nabla C \cdot \eta = 0,\ u = 0, \tag{1.2}$$

where η is the exterior unit normal to $\partial\Omega$. The initial conditions on Ω are given by,

$$N(x,0) = N_0(x),\ C(x,0) = C_0(x),\ u(x,0) = u_0(x). \tag{1.3}$$

Here, the unknowns N and C are the concentrations of cells and chemical, respectively and u is the velocity field of a fluid flow governed by the incompressible Navier-Stokes equations with pressure P and viscosity ν. Anisotropic and heterogeneous tensors are denoted by $S(x)$ and $M(x)$. The function $\chi(N)$ is usually written in the form $\chi(N) = N\tilde{h}(N)$ where $\tilde{h}$ is commonly referred to as the chemotactic sensitivity function. Moreover, the density-dependent diffusion coefficient is denoted by $a(N)$. The function $h(N, C)$ describes the rates of production and degradation of the chemical signal (chemoattractant); here, we assume it is of birth-death structure, i.e., a linear function,

$$h(N, C) = \alpha N - \beta C;\ \alpha, \beta \geq 0\,. \tag{1.4}$$

It can be seen in the model (1.1) that the coupling of chemotaxis and fluid is realized through both the transport of cells and chemical substrates $u \cdot \nabla N$, $u \cdot \nabla C$ and the external gravitational force $g = -N\nabla\phi$ exerted on the fluid by cells. In fact, this external force can be produced by different physical mechanisms such as gravity, electric and magnetic forces but in our study, we are only interested in the case of gravitational force $\nabla\phi =$ "$V_b(\rho_b - \rho)g$"z exerted by a bacterium onto the fluid along the upwards unit vector z proportional to the volume of the bacterium V_b, the gravitation acceleration $g = 9,8m/s^2$ and the density of bacteria is ρ_b (bacteria are about 10% denser than water). Furthermore, since the fluid is slow, we can also consider the simplified Chemotaxis-Stokes system taking the following form

$$\begin{cases} \partial_t N - \nabla \cdot \big(S(x)a(N)\nabla N\big) + \nabla \cdot \big(S(x)\chi(N)\nabla C\big) + u \cdot \nabla N &= f(N), \\ \partial_t C - \nabla \cdot (M(x)\nabla C) + u \cdot \nabla C &= h(N, C), \\ \partial_t u - \nu\Delta u + \nabla P &= -N\nabla\phi, \\ \nabla \cdot u &= 0 \qquad t > 0,\ x \in \Omega, \end{cases} \tag{1.5}$$

where compared with (1.1), the nonlinear convective term $(u \cdot \nabla)u$ is ignored in the fluid equation.

The questions of global existence of weak solutions of the model (1.1) and uniqueness of solutions of the model (1.5) have been answered in [7] and thus our model (1.1) is well-posed. Motivated by experiments described in [5, 6] which explain the dynamics of anisotropic chemotaxis models in a fluid at rest ($u = 0$) and interested by numerical issues related to the dynamics of these models coupled to a viscous fluid through transport and gravitational force, we investigate in this paper the numerical analysis of models (1.1) and (1.5). One should also note the experiments given in [8, 26, 17] of bacteria only consuming the chemical with another function $h(N, C) = -k(C)N$ where a cut-off function k is introduced to describe the aggregation of a part of bacteria below an interface between two fluids, while other bacteria are rendered inactive wherever the oxygen concentration has fallen below the threshold of activity. To our knowledge, there are only a few numerical results given for related systems (see [8, 23]). For example, the finite element method has been used to illustrate the behavior of the elliptic-parabolic Keller-Segel-Stokes system with different numerical examples in [23].

In the sequel and for the sake of clarity, we will divide our model (1.1) into two systems:

$$\begin{cases} \partial_t N - \nabla \cdot (S(x)a(N)\nabla N) + \nabla \cdot (S(x)\chi(N)\nabla C) + u \cdot \nabla N &= f(N), \\ \partial_t C - \nabla \cdot (M(x)\nabla C) + u \cdot \nabla C &= h(N, C), \\ S(x)a(N)\nabla N \cdot \eta = 0,\ M(x)\nabla C \cdot \eta &= 0, \\ N(x,0) = N_0(x),\ C(x,0) &= C_0(x)\,, \end{cases} \tag{1.6}$$

and

$$
\left\{
\begin{array}{rcll}
\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla P &=& -N\nabla\phi, & \\
\nabla\cdot u &=& 0 & t>0,\ x\in\Omega, \\
u &=& 0 & x\in\partial\Omega, \\
u(x,0) &=& u_0(x). &
\end{array}
\right.
\tag{1.7}
$$

Assuming a vanishing fluid velocity field u and neglecting the hydrodynamic force f between cells, the system (1.6) is then reduced to an anisotropic Keller-Segel model. A scheme recently developed in the finite volume framework (see [1]) treats the discretization of the isotropic Keller-Segel model in a homogeneous domain where the diffusion tensors are considered to be the identity matrix. In this case, the mesh used for the space discretization is assumed to satisfy the orthogonality condition. However, standard finite volume schemes not permit to handle anisotropic diffusion on general meshes where the orthogonality condition is lost and consequently the consistency of the diffusive flux. A large variety of methods have been proposed to reconstruct a consistent gradient as the hybrid finite volume scheme (also known as SUSHI method) first proposed in [15] and then generalized in [11]. On the other hand, it is well-known that the finite element method allows a very simple discretization of full diffusion tensors and does not impose any restrictions on the meshes, but many numerical instabilities may arise in the convection-dominated case. They were used a lot for the discretization of degenerate parabolic equations (see [3]). A quite intuitive idea is hence used in [5, 6] to discretize the anisotropic Keller-Segel model on general meshes, which is the combination of a piecewise linear nonconforming finite element discretization of the diffusion term with a finite volume discretization of the other terms. In the other hand, it is well known $\big($see [12, 13]$\big)$ that the discrete maximum principle is no more guaranteed in the case of non positive transmissibilities. Therefore, the authors in [5] elaborate, in the spirit of methods described in [4, 22], a general approach to construct a nonlinear correction providing a discrete maximum principle while retaining the main properties of the scheme, in particular coercivity and convergence toward the weak solution of the continuous problem as the mesh size tends to zero.

The ultimate aim of this article is the numerical analysis of the two-sidedly degenerate anisotropic chemotaxis-fluid model (1.1). A combined finite volume-nonconforming finite element method will be used to discretize the system (1.6) and a correction of the diffusive flux will be needed in the case of negative transmissiblities. Clearly, the main difficulty in the system (1.7) comes from the numerical treatment of the incompressibility condition ($\nabla\cdot u = 0$) and it is not possible to approximate the associated functional space by the most simple finite elements where the results are less general and vary according to the dimension (see [24]). Moreover, and simply for a Stokes equation, the authors in [9, 10] were disappointed from the error estimates obtained with the conforming finite elements using polynomials of degree 2 in each triangle of the mesh when the nonconforming finite elements enable them to obtain the same asymptotic error estimates with polynomials of degree 1 in each triangle of the mesh. In addition to that, the nonconforming finite element method violates the inter-element continuity conditions of the velocities. Consequently, we have chosen this convenient method to discretize the Navier-Stokes equation (1.7).

The paper is structured as follows. In section 2, we state the assumptions on the data and we present a weak formulation of the continuous problem. In section 3, we describe the time and space discretizations, we introduce the combined finite volume-nonconforming finite element scheme and we state the main theorem of convergence. In section 4, we present discrete properties of the numerical scheme and we construct estimates in order to prove the convergence. Finally, numerical results will be given in section 5.

2. Preliminaries

Let us now state the assumptions on the data we will use in the sequel. We assume at first that consumption stops when the cell density reaches a maximal value N_m and therefore the chemotactical sensitivity $\chi(N)$ vanishes when N tends to N_m. The effect of a threshold cell density or a volume filling effect has been taken into account in the modeling of chemotaxis phenomenon in [18, 19]. Upon normalization, we can assume that the threshold density is $N_m = 1$ and consequently,

$$\chi : [0,1] \longmapsto \mathbb{R} \text{ is continuous and } \chi(0) = \chi(1) = 0\,. \tag{2.1}$$

A standard example for χ is $\chi(N) = N(1-N)$; $N \in [0,1]$. The positivity of χ means that the chemical attracts the cells; the repellent case is the one of negative χ.

Secondly, we suppose that the density diffusion coefficient $a(N)$ degenerates for $N = 0$ and $N = N_m$. This means that the diffusion vanishes when N approaches values close to the threshold $N_m = 1$ and also in the absence of cell-population. Therefore,

$$a : [0,1] \longmapsto \mathbb{R}^+ \text{ is continuous, } a(0) = a(1) = 0 \text{ and } a(s) > 0 \text{ for } 0 < s < 1\,, \tag{2.2}$$

Next, we require

$$f : [0,1] \longmapsto \mathbb{R}^+ \text{ is a continuous function with } f(0) = 0\,, \tag{2.3}$$

$$\nabla\phi \in (L^\infty(\Omega))^d \text{ and } \phi \text{ is independent of time.} \tag{2.4}$$

The permeabilities S, M: $\Omega \longrightarrow \mathcal{M}_d(\mathbb{R})$ where $\mathcal{M}_d(\mathbb{R})$ is the set of symmetric matrices $d\times d$, verify

$$S_{i,j} \in L^\infty(\Omega),\ M_{i,j} \in L^\infty(\Omega),\ \forall i,j \in \{1,..,d\}\,, \tag{2.5}$$

and there exist $c_S \in \mathbb{R}_+^*$ and $c_M \in \mathbb{R}_+^*$ such that a.e. $x \in \Omega, \forall \xi \in \mathbb{R}^d$,

$$S(x)\xi\cdot\xi \geq c_S|\xi|^2,\ M(x)\xi\cdot\xi \geq c_M|\xi|^2\,. \tag{2.6}$$

Finally, we introduce basic spaces associated to the Navier-Stokes equation,

$$\wp = \{u \in \mathcal{D}(\Omega), \nabla\cdot u = 0\},\ V = \bar{\wp}^{H_0^1(\Omega)} \text{ and } H = \bar{\wp}^{L^2(\Omega)}\,, \tag{2.7}$$

where V and H are the closure of $\wp$ in $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively.

In the following definition, we give a proper notion of a weak solution.

Definition 2.1. *Assume that* $0 \leq N_0 \leq 1$, $C_0 \geq 0$, $C_0 \in L^\infty(\Omega)$, $u_0 \in L^2(\Omega)$ *and* $\nabla\cdot u_0 = 0$. *A triple* (N,C,u) *is said to be a* **weak solution** *of* (1.1)-(1.3) *if*

$$0 \leq N(x,t) \leq 1,\ C(x,t) \geq 0 \ \text{a.e. in } Q_T = \Omega\times[0,T],$$

$$N \in C_w(0,T;L^2(\Omega)),\ \partial_t N \in L^2(0,T;(H^1(\Omega))'),\ A(N) := \int_0^N a(r)dr \in L^2(0,T;H^1(\Omega))\,,$$

$$C \in L^\infty(Q_T)\cap L^2(0,T;H^1(\Omega))\cap C(0,T;L^2(\Omega));\ \partial_t C \in L^2(0,T;(H^1(\Omega))')\,,$$

$$u \in L^\infty(0,T;H)\cap L^2(0,T;V)\cap C_w(0,T;H);\ \frac{du}{dt} \in L^1(0,T;V')\,,$$

and (N,C,u) *satisfy*

$$\int_0^T <\partial_t N,\psi_1>_{(H^1)',H^1} dt + \iint_{Q_T} [S(x)a(N)\nabla N - S(x)\chi(N)\nabla C - Nu]\cdot\nabla\psi_1\, dxdt = -\iint_{Q_T} f(N)\psi_1\, dxdt\,, \tag{2.8}$$

$$\int_0^T <\partial_t C,\psi_2>_{(H^1)',H^1} dt + \iint_{Q_T} [M(x)\nabla C - Cu]\cdot\nabla\psi_2\, dxdt = \iint_{Q_T} h(N,C)\psi_2\, dxdt\,, \tag{2.9}$$

$$\int_0^T < \partial_t u, \psi >_{V',V} dt + \iint_{Q_T} \nabla u \cdot \nabla \psi \, dxdt + \iint_{Q_T} (u \cdot \nabla) u \psi \, dxdt = \iint_{Q_T} -N \nabla \phi \psi \, dxdt = \iint_{Q_T} g \psi \, dxdt \,, \tag{2.10}$$

for all ψ_1, $\psi_2 \in L^2(0,T;H^1(\Omega))$ and $\psi \in C_c^0(]0,T[;V)$, where $C_c^0(]0,T[;V)$ denotes the space of continuous functions with compact support and values in V and $C_w(0,T;L^2(\Omega))$ denotes the space of continuous functions with values in (a closed ball of) $L^2(\Omega)$ indowed with the weak topology.

Remark 2.2. *Since the fluid in the experiment is slow, the Navier-Stokes equation can be simplified to the Stokes equation. In this case, a triple (N,C,u) is said to be a* **weak solution** *in the same sense of Definition 2.1 with better time regularity obtained for u. Due to the linearity of the Stokes equation $\big($the nonlinear convective term $(u\cdot\nabla)u$ is ignored$\big)$, one obtains that $\frac{du}{dt}$ belongs to $L^2(0,T;V')$ and consequently the component u belongs to $C(0,T;H)$.*

3. Combined Finite Volume-Nonconforming Finite Element Scheme

This section is devoted to the formulation of a combined scheme for the anisotropic chemotaxis-fluid model (1.1). First, we describe the space and time discretizations, we define the approximation spaces associated to the Navier-Stokes equations and then we introduce the combined scheme.

3.1. **Space Discretization of Ω.** Recall that Ω be an open bounded subset of $\mathbb{R}^d$ with $d=2$ or 3. In order to discretize the problem (1.6), we consider a family $\mathcal{T}_h$ of meshes of the domain Ω, consisting of disjoint closed simplices such that $\bar{\Omega} = \cup_{K\in\mathcal{T}_h}\bar{K}$ and such that if $K,L\in\mathcal{T}_h$, $K\neq L$, then $K\cap L$ is either an empty set or a common face or edge of K and L. The size of the mesh $\mathcal{T}_h$ is defined by $h{:=}\max_{K\in\mathcal{T}_h} diam(K)$, which has the sense of an upper bound for the maximum diameter of the control volumes in $\mathcal{T}_h$. We denote by $\mathcal{E}_h$ the set of all sides, by $\mathcal{E}_h^{int}$ and $\mathcal{E}_h^{ext}$ the set of all interior and exterior sides, respectively. We also define a geometrical factor, linked with the regularity of the mesh, by making the following shape regularity assumption on the family of triangulations $\{\mathcal{T}_h\}_h$: There exists a positive constant $k_{\mathcal{T}}$;

$$\min_{K\in\mathcal{T}_h} \frac{|K|}{(diam(K))^d} \geq k_{\mathcal{T}},\, \forall h>0\,. \tag{3.1}$$

Assumption (3.1) is equivalent to the more common requirement of the existence of a constant $\theta_{\mathcal{T}}>0$;

$$\max_{K\in\mathcal{T}_h} \frac{diam(K)}{\rho_K} \leq \theta_{\mathcal{T}},\, \forall h>0\,, \tag{3.2}$$

where ρ_K is the diameter of the largest ball inscribed in the simplex K.

We also use a dual partition $\mathcal{D}_h$ of disjoint closed simplices called control volumes of Ω such that $\bar{\Omega} = \cup_{D\in\mathcal{D}_h}\bar{D}$. There is one dual element D associated with each side $\sigma_D = \sigma_{K,L}\in\mathcal{E}_h$. We construct it by connecting the barycenters of every $K\in\mathcal{T}_h$ that contains σ_D through the vertices of σ_D. The point P_D is referred to as the barycenter of the side σ_D. For all $D\in\mathcal{D}_h$, denote by $|D|$ the d-dimensional Lebesgue measure of D and by $\mathcal{N}(D)$ the set of neighbors of the volume D. A generic neighbor of D is often denoted by E. For all $E\in\mathcal{N}(D)$, denote by $\sigma_{D,E}$ the interface between a dual volume D and E, by $d_{D,E} := |P_E - P_D|$ the distance between the centers P_D and P_E and by $\eta_{D,E}$ the unit normal vector to $\sigma_{D,E}$ outward to D. We denote by $K_{D,E}$ the element of $\mathcal{T}_h$ such that $\sigma_{D,E}\subset K_{D,E}$ i.e. $K_{D,E} = \{K\in\mathcal{T}_h;\ \sigma_{D,E}\subset K\}$. For an interface $\sigma_{D,E}$, denote by $|\sigma_{D,E}|$ its $(d-1)$-dimensional measure. As for the primal mesh, $\mathcal{D}_h^{int}$ and $\mathcal{D}_h^{ext}$ denote respectively the set of all interior and exterior dual volumes and we define $\mathcal{F}_h$, $\mathcal{F}_h^{int}$ and $\mathcal{F}_h^{ext}$ as the set of all dual, interior and exterior mesh sides, respectively. For $\sigma_D\in\mathcal{F}_h^{ext}$, the contour of D is completed by the side σ_D itself. We refer to the Figure 1 for the two-dimensional case.

Next, we define the following finite-dimensional spaces:

$$X_h := \{\varphi_h\in L^2(\Omega);\ \varphi_h|_K \text{ is linear } \forall K\in\mathcal{T}_h,\ \varphi_h \text{ is continuous at the points } P_D,\ D\in\mathcal{D}_h^{int}\}\,, \tag{3.3}$$

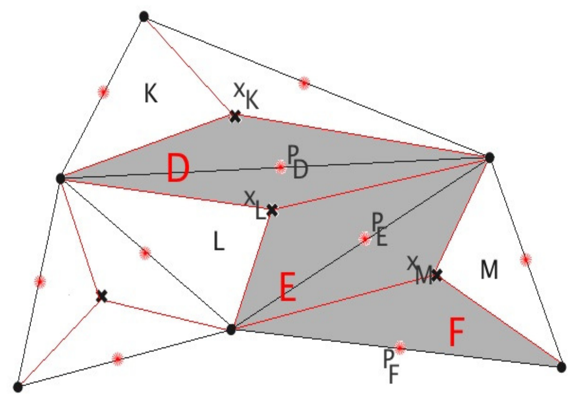

FIGURE 1. Dual volumes associated with edges of the primal mesh.

$$X_h^0 := \{\varphi_h \in X_h;\, \varphi_h(P_D) = 0,\ \forall D \in \mathcal{D}_h^{ext}\}.$$

The basis of X_h is spanned by the shape functions $\varphi_D,\ D \in \mathcal{D}_h$, such that $\varphi_D(P_E) = \delta_{DE},\ E \in \mathcal{D}_h$, δ being the Kronecker delta. We recall that the approximations in these spaces are nonconforming since $X_h \not\subset H^1(\Omega)$. Indeed, only the weak continuity of the solution is provided through the interfaces and therefore the solution may be discontinuous on the faces. We equip X_h^0 with the scalar product

$$((N_h, V_h))_h = \sum_{K\in\mathcal{T}_h} \int_K \nabla N_h \cdot \nabla V_h \, dx \text{ and the seminorm } ||N_h||_{X_h}^2 := \sum_{K\in\mathcal{T}_h} \int_K |\nabla N_h|^2 \, dx$$

becomes a norm on X_h^0. We have the following lemma proved in [13].

Lemma 3.1. *For all $N_h = \sum_{D\in\mathcal{D}_h} N_D \varphi_D \in X_h$, one has*

$$\sum_{\sigma_{D,E}\in\mathcal{D}_h} diam(K_{D,E})^{d-2}(N_E - N_D)^2 \le \frac{d+1}{2d\kappa_\tau}||N_h||_{X_h}^2\,, \tag{3.4}$$

$$\sum_{\sigma_{D,E}\in\mathcal{D}_h} \frac{|\sigma_{D,E}|}{d_{D,E}}(N_E - N_D)^2 \le \frac{d+1}{2(d-1)k_{\mathcal{T}}}||N_h||_{X_h}^2\,. \tag{3.5}$$

3.2. **Time Discretization of** $[0,T]$**.** Let us consider a constant time step $\Delta t \in [0,T]$. A discretization of $[0,T]$ is given by $\tilde{N} \in \mathbb{N}^*$ such that $t_n = n\Delta t$, for $n \in \{0, ..., \tilde{N}+1\}$. The discrete unknowns are denoted by $\left\{w_D^n, D \in \mathcal{D}_h, n \in \{0, ..., , \tilde{N}+1\}\right\}$, the value w_D^n is an approximation of $w(P_D, n\Delta t)$ where $w = N$, C or u.

3.3. **Discretization of the Navier-Stokes' equation.** In this subsection, we state the main tools for the discretization of Navier-Stokes equation by means of nonconforming finite element methods. Due to the incompressibility condition $\nabla \cdot u = 0$, it was shown by M. Fortin [16] and R. Temam [24] that it is not possible to approximate the space V defined in (2.7) by the most simple finite elements, the piecewise linear continuous functions where the results, even for the Stokes equations, are less general and vary according to the dimension since no basis of the approximate space V_h is available. For this reason, the approximation studied in this subsection is certainly very useful for Stokes and

Navier-Stokes problems. Several numerical computations of viscous incompressible flows, using these elements have been performed by F. Thomasset [25]. Let us start by giving in the following Definition, a summary from $\big[$[24], chapter I$\big]$ about the external approximation of a normed space.

Definition 3.2. (1) *An **external approximation of a normed space** V is a set consisting of*

- *a normed space F and a isomorphism $\bar{w}$ of V into F,*
- *a family of triples $\{V_h, p_h, r_h\}_h$ such that for each h, where*

V_h is a normed space, p_h is a linear continuous prolongation mapping of V_h into F and r_h is a restriction mapping of V into V_h.

(2) *An **external approximation** of a normed space V is said to be **stable** if the prolongation operators are stable i.e. if their norms $||p_h|| = \sup_{u_h \in V_h, ||u_h||_{V_h}=1} ||p_h u_h||_F$ can be majorized independently of h.*

(3) *An **external approximation** of a normed space V is said to be **convergent** if: As $h \to 0$,*

- *$\forall u \in V$, $p_h r_h u \to \bar{w}u$ in F,*
- *For each sequence $u_{h'}$ in $V_{h'}$ such that $p_{h'} u_{h'}$ converges to some element Φ in the weak topology of F, we have $\Phi \in \bar{w}V$.*

3.3.1. *Approximation of the space $H_0^1(\Omega)$.* In the sense of Definition 3.2, the family $\{X_h^0\}_h$ defined in (3.3) is a stable and convergent external approximation of $H_0^1(\Omega)$. It suffices to take:

- $F = (L^2(\Omega))^{d+1}$, $\bar{w} : u \in V \to \bar{w}u = \{u, \partial_1 u, ..., \partial_d u\} \in F$,
- $p_h u_h = \{u_h, \partial_1 u_h, .., \partial_d u_h\}$ and $r_h u = u_h \in X_h^0$ with $u_h(P_D) = u(P_D)$.

One can refer to $\big[$[24], ch. I, subsection IV.5$\big]$ for more details.

3.3.2. *Approximation of the space V.* Let V_h be a subspace of the preceding space X_h such that

$$V_h = \{u_h \in X_h,\, div_h(u_h) = 0\}\,, \tag{3.6}$$

where the discrete divergence is defined by the following step function,

$$div_h(u_h) = \sum_{K \in \mathcal{T}_h} \eta_K 1_K;\, \eta_K = \frac{1}{|K|} \int_K \nabla \cdot u_h \, dx\,.$$

The space V_h is a stable and convergent external approximation of the space V defined in (2.7). It suffices to take:

- $F = (L^2(\Omega))^{d+1}$, $\bar{w} : u \in V \to \bar{w}u = \{u, \partial_1 u, ..., \partial_d u\} \in F$,
- $p_h u_h = \{u_h, \partial_1 u_h, .., \partial_d u_h\}$ and $r_h u = u_h$ with

$$u_h(P_D) = \frac{1}{|\sigma|} \int_\sigma u \, d\gamma\,, \tag{3.7}$$

This last integral exists due to the theorem on traces in the space $H^1(K)$ and $r_h u$ belongs to the space X_h (see [2]). Let us show that u_h belongs to V_h. Indeed, since $\nabla \cdot u_h$ is constant on each simplex K, the condition concerning the discrete divergence of u_h in (3.6) is equivalent to

$$\nabla \cdot u_h = 0 \; in \; K, \, \forall K \in \mathcal{T}_h\,. \tag{3.8}$$

Then, it follows from the Green formula and (3.7) that $\forall K \in \mathcal{T}_h$,

$$\int_K \nabla \cdot u_h \, dx = \sum_{\sigma \in \partial K} \int_\sigma u_h \cdot \eta \, d\gamma = \sum_{\sigma \in \partial K} \int_\sigma u \cdot \eta \, d\gamma = \int_K \nabla \cdot u \, dx = 0,$$

and this last integral is zero since $\nabla \cdot u = 0$.

3.3.3. *Approximation of the Navier-Stokes problem.* Using the above approximation of V, we can propose a nonconforming finite element scheme for the approximation of the Navier-Stokes' problem. The approximate problem is then:

$$u_h^0 = \text{ the orthogonal projection of } u_0 \text{ onto } V_h \text{ in } L^2(\Omega), \tag{3.9}$$

and to find $u_h^{n+1} \in V_h$, $\forall n \in \{0, .., \tilde{N}\}$ such that

$$\frac{1}{\Delta t}(u_h^{n+1} - u_h^n, v_h) + \nu((u_h^{n+1}, v_h))_h + b_h(u_h^n, u_h^{n+1}, v_h) = (g^n, v_h),\ \forall v_h \in V_h, \tag{3.10}$$

where $b_h(u_h^n, u_h^{n+1}, v_h)$ is an approximation of the nonlinear term $(u \cdot \nabla)u$.

3.3.4. *Approximation of the pressure.* We want now to present the "approximate" pressure which is implicitly contained in (3.10). The form

$$v_h \to \frac{1}{\Delta t}(u_h^{n+1} - u_h^n, v_h) + \nu((u_h^{n+1}, v_h))_h + b_h(u_h^n, u_h^{n+1}, v_h) - (g^n, v_h)$$

appears as a linear form defined on X_h and vanishes on V_h. Hence introducing the Lagrange multipliers corresponding to the linear constraints (3.8) we find, with the aid of a classical theorem of linear algebra, that there exist numbers $\{\lambda_K\}_{K \in \mathcal{T}_h} \in \mathbb{R}$ such that the following equation holds,

$$\frac{1}{\Delta t}(u_h^{n+1} - u_h^n, v_h) + \nu((u_h^{n+1}, v_h))_h + b_h(u_h^n, u_h^{n+1}, v_h) - (g^n, v_h) = \sum_{K \in \mathcal{T}_h} \lambda_K \Big(\int_K \nabla \cdot v_h \, dx \Big),\ \forall v_h \in X_h.$$

Let 1_K denotes the characteristic function of K, we introduce a function p_h^n which belongs to space of piecewise constant functions Y_h such that:

$$p_h^n = \sum_{K \in \mathcal{T}_h} p_h^n(K) 1_K \text{ with } p_h^n(K) = \frac{\lambda_K}{|K|}\,.$$

We then have

$$\frac{1}{\Delta t}(u_h^{n+1} - u_h^n, v_h) + \nu((u_h^{n+1}, v_h))_h + b_h(u_h^n, u_h^{n+1}, v_h) - (p_h^n, div_h v_h) = (g^n, v_h),\ \forall v_h \in X_h\,. \tag{3.11}$$

Remark 3.3. *We remark that the discretization of the Navier Stokes' equation does not solve completely the problem of numerical approximation of these equations. For the actual computation of the solution, we must have an explicit basis of V_h. The solution of* (3.10) *is not always easy since we only know a simple basis of V_h in dimension two (see* [9]*). One possibility for solving* (3.10) *would be to interpret the Navier-Stokes' problem as a variational problem* (3.11) *with linear constraints* (3.8) *and to solve it with a classical Uzawa algorithm .*

3.4. **Combined Scheme for the system** (1.1)**.** The aim of this subsection is the discretization of the anisotropic chemotaxis-fluid model (1.1) only supposing the shape regularity condition for the primal mesh (3.1). For this purpose, we use the implicit Euler scheme in time and we consider the piecewise linear nonconforming finite element method in space for the discretization of the diffusive terms of the model (1.6) and the Navier-Stokes equation (1.7). The other terms are discretized by means of a finite volume scheme on the dual mesh.

Let us denote the approximation of the flux $S(x)\nabla C \cdot \eta_{D,E}$ on the interface $\sigma_{D,E}$ by $\delta C_{D,E}$. Then, we approximate the numerical flux $S(x)\chi(N)\nabla C \cdot \eta_{D,E}$ by means of the values N_D, N_E and $\delta C_{D,E}$ that are available in the neighborhood of the interface $\sigma_{D,E}$. For that, we use a numerical flux function $G(N_D, N_E, \delta C_{D,E})$ which satisfies the following properties:

- $G(., b, c)$ is non-decreasing for all $b, c \in \mathbb{R}$ and $G(a, ., c)$ is non-increasing for all $a, c \in \mathbb{R}$;
- $G(a, b, c) = -G(b, a, -c)$ for all $a, b, c \in \mathbb{R}$; hence the flux is conservative.
- $G(a, a, c) = \chi(a)c$ for all $a, c \in \mathbb{R}$; hence the flux is consistent.
- there exists $C > 0$, such that for all $a, b, c \in \mathbb{R}$, $|G(a, b, c)| \leq C(|a| + |b|)|c|$;

- $|G(a,b,c) - G(a',b',c)| \leq |c|(|a-a'| + |b-b'|)$ for all $a, b, a', b', c \in \mathbb{R}$.

Moreover, a possibility to construct the numerical flux G has been given in [5].

Next, we denote the approximation of the flux $u \cdot \eta_{D,E}$ on the interface $\sigma_{D,E}$ by $u_{D,E}$. Then, we approximate the flux $Nu \cdot \eta_{D,E}$ by a new numerical convection function $G_1(N_D, N_E, u_{D,E})$ which has the same standard properties of the function G. This convective flux G_1 is the well-known upwind scheme written as,

$$G_1(N_D, N_E, u_{D,E}) = u^+_{D,E} N_D - u^-_{D,E} N_E = u_{D,E} \bar{N}_{D,E}\,, \tag{3.12}$$

where $u^+_{D,E}$ and $u^-_{D,E}$ denote the positive and negative parts of $u_{D,E}$ (i.e. $u^+_{D,E} = max(u_{D,E}, 0)$ and $u^-_{D,E} = max(-u_{D,E}, 0)$). In addition, $\bar{N}_{D,E} = N_D$ if $u_{D,E} \geq 0$ and $\bar{N}_{D,E} = N_E$ if $u_{D,E} \leq 0$.

For all $N_h = \sum_{D \in \mathcal{D}_h} N_D \varphi_D \in X_h$, we define a discrete function of $A(N_h)$ as

$$A_h(N_h) = \sum_{D \in \mathcal{D}_h} A(N_D) \varphi_D\,. \tag{3.13}$$

Finally, a combined finite volume-nonconforming finite element scheme for the discretization of the model (1.1) is given by the following iterative algorithm:

3.4.1. *First step:* Let $\big(N^n_D, C^n_D\big)_{D \in \mathcal{D}_h, n \in \{0,..,\tilde{N}\}}$ be the discrete solution of the model (1.6) given by the system of equations (3.17)-(3.18) defined below at time t_n. We interpret the system (1.7) as a variational problem (3.11) with linear constraints (3.8) and we study the convergence by a classical algorithm d'Uzawa. If the elements $\tilde{N}^n_h = \big\{N^n_D\big\}_{D \in \mathcal{D}_h}$, u^n_h and p^n_h have been computed at time t_n, then we compute $u^{n+1}_h \in X_h$ and $p^{n+1}_h \in Y_h$ as the limits of two sequences of elements

$$u^{n+1,r}_h \in X_h \text{ and } p^{n+1,r}_h \in Y_h,\ r = 0, 1, .., +\infty\,.$$

We start the algorithm with an arbitrary element $p^{n+1,0}_h$. When $u^{n+1,r}_h$ is known, we define $u^{n+1,r+1}_h$ and $p^{n+1,r+1}_h$ by

$$\frac{1}{\Delta t}(u^{n+1,r+1}_h - u^n_h, v_h) + \nu((u^{n+1,r+1}_h, v_h))_h + b_h(u^n_h, u^{n+1,r+1}_h, v_h) - (p^{n,r+1}_h, div_h v_h) = (g^n, v_h),\ \forall v_h \in X_h\,. \tag{3.14}$$

where $g^n = -\tilde{N}^n_h \nabla\phi \in L^2(\Omega)$.

$$(p^{n+1,r+1}_h - p^{n,r}_h, q_h) + \rho\big(div_h(u^{n+1,r+1}_h), q_h\big) = 0,\ \forall q_h \in Y_h\,. \tag{3.15}$$

The existence and the uniqueness of the solution $u^{n+1,r+1}_h$ follow from the projection theorem and we can observe that (3.15) defines $p^{n+1,r+1}_h$ explicitly. Regarding the convergence of this algorithm, we have the following Proposition proved in $\big[$[24], ch. VII, Proposition 6.7 $\big]$,

Proposition 3.4. *If the number ρ satisfies*

$$0 < \rho < \frac{2\nu}{d}$$

then, as $r \to +\infty$, $u^{n+1,r+1}_h$ converges to u^{n+1}_h in X_h and $p^{n+1,r+1}_h$ converges to p^{n+1}_h in $Y_h/\mathbb{R}$.

3.4.2. *Second step:* Given u^{n+1}_h the fluid velocity at time t_{n+1} detailed in the first step. Then, $\forall D \in D_h$,

$$N^0_D = \frac{1}{|D|}\int_D N_0(x)\,dx,\ C^0_D = \frac{1}{|D|}\int_D C_0(x)\,dx, \tag{3.16}$$

and for all $D \in \mathcal{D}_h$, $n \in \{0, 1, ..., \tilde{N}\}$,

$$|D|\frac{N_D^{n+1} - N_D^n}{\Delta t} - \sum_{E \in \mathcal{D}_h} \mathcal{S}_{D,E} A(N_E^{n+1}) + \sum_{E \in \mathcal{N}(D)} G(N_D^{n+1}, N_E^{n+1}; \delta C_{D,E}^{n+1}) + \sum_{E \in \mathcal{N}(D)} G_1(N_D^{n+1}, N_E^{n+1}; u_{D,E}^{n+1}) = f(N_D^{n+1}), \tag{3.17}$$

$$|D|\frac{C_D^{n+1} - C_D^n}{\Delta t} - \sum_{E \in \mathcal{D}_h} \mathcal{M}_{D,E} C_E^{n+1} + \sum_{E \in \mathcal{N}(D)} G_1(C_D^{n+1}, C_E^{n+1}; u_{D,E}^{n+1}) = h(N_D^n, C_D^{n+1}). \tag{3.18}$$

The diffusion matrix $\mathcal{S}$ (resp. $\mathcal{M}$) of elements $\mathcal{S}_{D,E}$ (resp. $\mathcal{M}_{D,E}$) for $D, E \in \mathcal{D}_h$ is the stiffness matrix of the nonconforming finite element method. So that,

$$\mathcal{S}_{D,E} = -\sum_{K \in \mathcal{T}_h} (\mathcal{S}(x)\nabla\varphi_E, \nabla\varphi_D)_{0,K} \text{ and } \mathcal{M}_{D,E} = -\sum_{K \in \mathcal{T}_h} (M(x)\nabla\varphi_E, \nabla\varphi_D)_{0,K}. \tag{3.19}$$

Otherwise, $\delta C_{D,E}^{n+1}$ and $u_{D,E}^{n+1}$ denote the approximation of the fluxes $S(x)\nabla C \cdot \eta_{D,E}$ and $u \cdot \eta_{D,E}$ on the interface $\sigma_{D,E}$, respectively,

$$\delta C_{D,E}^{n+1} = S_{D,E}\big(C_E^{n+1} - C_D^{n+1}\big) \text{ and } u_{D,E}^{n+1} = \int_{\sigma_{D,E}} u_h^{n+1} \cdot \eta_{D,E}\, d\gamma. \tag{3.20}$$

Definition 3.5. *We will define now two approximate solutions by means of the combined finite volume-nonconforming finite element scheme:*

i) A nonconforming finite element solution $(N_{h,\Delta t}, C_{h,\Delta t}, u_{h,\Delta t})$ as a function piecewise linear and continuous in the barycenters of the interior sides in space and piecewise constant in time, such that:

$$\big(N_{h,\Delta t}(x,0),\, C_{h,\Delta t}(x,0),\, u_{h,\Delta t}(x,0)\big) = \Big(N_h^0(x),\, C_h^0(x),\, u_h^0(x)\Big) \text{ for } x \in \Omega,$$

$$\big(N_{h,\Delta t}(x,t),\, C_{h,\Delta t}(x,t),\, u_{h,\Delta t}(x,t)\big) = \Big(N_h^{n+1}(x),\, C_h^{n+1}(x),\, u_h^{n+1}(x)\Big) \text{ for } x \in \Omega \text{ and } t \in]t_n, t_{n+1}],$$

where $N_h^{n+1} = \sum_{D \in \mathcal{D}_h} N_D^{n+1}\varphi_D$, $C_h^{n+1} = \sum_{D \in \mathcal{D}_h} C_D^{n+1}\varphi_D$ *and* $u_h^{n+1} = \sum_{D \in \mathcal{D}_h} u_h^{n+1}(P_D)\varphi_D$.

ii) A finite volume solution $(\tilde{N}_{h,\Delta t}, \tilde{C}_{h,\Delta t}, \tilde{u}_{h,\Delta t})$ defined as piecewise constant on the dual volumes in space and piecewise constant in time, such that:

$$\big(\tilde{N}_{h,\Delta t}(x,0),\, \tilde{C}_{h,\Delta t}(x,0),\, \tilde{u}_{h,\Delta t}(x,0)\big) = \Big(N_D^0,\, C_D^0,\, u_h^0(P_D)\Big) \text{ for } x \in D,\ D \in \mathcal{D}_h,$$

$$\big(\tilde{N}_{h,\Delta t}(x,t),\, \tilde{C}_{h,\Delta t}(x,t),\, \tilde{u}_{h,\Delta t}(x,t)\big) = \Big(N_D^{n+1},\, C_D^{n+1},\, u_h^{n+1}(P_D)\Big) \text{ for } x \in D,\ D \in \mathcal{D}_h,\ t \in]t_n, t_{n+1}].$$

Now, we state a first convergence result of the combined scheme under the assumption that all transmissibilities coefficients are positive:

$$\mathcal{S}_{D,E} \geq 0 \text{ and } \mathcal{M}_{D,E} \geq 0,\ \forall D \in \mathcal{D}_h,\ E \in \mathcal{N}(D). \tag{3.21}$$

Further, we will present a method to overcome this assumption (3.21) by the introduction of a family of monotone schemes.

For the discrete problem (3.9)-(3.10), we will state first a convergence Theorem proved in [24].

Theorem 3.6. *(Convergence of the Navier-Stokes equation)*
Assume that $u_0 \in H$, $\nabla \cdot u_0 = 0$ a.e. on Ω and $g \in L^2(0,T;H)$. Then:
1) There exists a unique discrete solution $u_{h,\Delta t}$ of (3.9)-(3.10) if $d = 2$ and there exists at least one such solution if $d = 3$.

2) The following convergences hold for subsequences, denoted as sequences, as h and Δt tend to zero, $u_{h,\Delta t} \to u \in L^2(Q_T)$ strongly, $u_{h,\Delta t} \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;L^2(\Omega))$ and $p_h u_{h,\Delta t} \rightharpoonup \bar{w}u$ in $L^2(0,T;F)$.

Now, our aim is to prove the following main convergence Theorem of this paper.

Theorem 3.7. *(Convergence of the combined scheme)*
Assume (2.1) *till* (2.7)*. Consider $0 \le N_0 \le 1$, $C_0 \in L^\infty(\Omega)$ and $C_0 \ge 0$. Under the assumptions of Theorem 3.6 and* (3.21)*, then:*
1) There exists a solution $(\tilde{N}_{h,\Delta t}, \tilde{C}_{h,\Delta t})$ of the discrete system (3.17)-(3.18) *with initial data* (3.16)*.*
2) Any sequence $(h_m)_m$ decreasing to zero possesses a subsequence such that $(N_{h_m}, C_{h_m}, u_{h_m})$ converges a.e. on Q_T to a solution (N, C, u) of the chemotaxis-fluid system (1.1) *in the sense of Definition 2.1 .*

Remark 3.8. *If assumption* (3.21) *is not satisfied, one can use a nonlinear technique to correct the diffusive flux blocking the discrete maximum principle and to maintain the monotonicity and the convergence of the corrected numerical scheme. One can see* [5] *(section* 4*) for more details.*

4. Proof of Theorem 3.7

We state here the properties and estimates which are satisfied by the combined scheme. We first present some technical Lemmas that show the conservativity of the scheme, the coercivity and the continuity of the diffusion term. Then, we show a priori estimates needed to prove the existence of a discrete solution of (3.16)-(3.18) and to prove the convergence.

4.1. Discrete properties of the scheme.

The first two Lemmas and the details of their proofs are given in [5].

Lemma 4.1 (Nonconforming finite element diffusion matrix)**.** *For all $D \in \mathcal{D}_h$, one has:*

$$\mathcal{S}_{D,D} = -\sum_{E \in \mathcal{N}(D)} \mathcal{S}_{D,E} \text{ and } \mathcal{M}_{D,D} = -\sum_{E \in \mathcal{N}(D)} \mathcal{M}_{D,E}\,. \tag{4.1}$$

Using the fact that $\mathcal{S}_{D,E} \neq 0$ and $\mathcal{M}_{D,E} \neq 0$, only if $E \in \mathcal{N}(D)$ or if $E = D$, we deduce from (4.1),

$$\sum_{E \in \mathcal{D}_h} \mathcal{S}_{D,E} A(U_E^{n+1}) = \sum_{E \in \mathcal{N}(D)} \mathcal{S}_{D,E} A(N_E^{n+1}) + \mathcal{S}_{D,D} A(N_D^{n+1}) = \sum_{E \in \mathcal{N}(D)} \mathcal{S}_{D,E} (A(N_E^{n+1}) - A(N_D^{n+1}))\,, \tag{4.2}$$

$$\sum_{E \in \mathcal{D}_h} \mathcal{M}_{D,E} C_E^{n+1} = \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{D,E} (C_E^{n+1} - C_D^{n+1})\,. \tag{4.3}$$

Consequently, the diffusive flux discretization in (3.17)-(3.18) is **conservative** with respect to the dual mesh $\mathcal{D}_h$. In fact, we remark that $\mathcal{S}_{D,E} = \mathcal{S}_{E,D}$ using the symmetry of the tensor S, this yields an equality up to the sign between two discrete diffusive flux, from D to E and from E to D. In other terms, $\mathcal{S}_{D,E}\big(A(N_E^{n+1}) - A(N_D^{n+1})\big) = -\mathcal{S}_{E,D}\big(A(N_D^{n+1}) - A(N_E^{n+1})\big)$.

Lemma 4.2. *For all $A_h(N_h) = \sum_{D \in \mathcal{D}_h} A(N_D)\varphi_D$ and $C_h = \sum_{D \in \mathcal{D}_h} C_D \varphi_D \in X_h$, then*

The discrete degenerate diffusion operator is **coercive***,*

$$-\sum_{D \in \mathcal{D}_h} A(N_D) \sum_{E \in \mathcal{D}_h} \mathcal{S}_{D,E} A(N_E) \ge C_S ||A_h(N_h)||_{X_h}^2 \text{ and } -\sum_{D \in \mathcal{D}_h} C_D \sum_{E \in \mathcal{D}_h} \mathcal{M}_{D,E} C_E \ge C_M ||C_h||_{X_h}^2\,. \tag{4.4}$$

The discrete degenerate diffusion operator is **continuous***,*

$$|\sum_{D \in \mathcal{D}_h} A(N_D) \sum_{E \in \mathcal{D}_h} \mathcal{S}_{D,E} A(N_E)| \le c_S ||A_h(N_h)||_{X_h}^2 \text{ and } |\sum_{D \in \mathcal{D}_h} C_D \sum_{E \in \mathcal{D}_h} \mathcal{M}_{D,E} C_E| \le c_M ||C_h||_{X_h}^2\,. \tag{4.5}$$

In addition to that, the transmissibilities are bounded

$$|\mathcal{S}_{D,E}| \leq \frac{c_S}{k_{\mathcal{T}}} \frac{(diam(K_{D,E})^{d-2}}{(d-1)^2} \text{ and } |\mathcal{M}_{D,E}| \leq \frac{c_M}{k_{\mathcal{T}}} \frac{(diam(K_{D,E})^{d-2}}{(d-1)^2}, \forall D \in \mathcal{D}_h,\ E \in \mathcal{N}(D)\,. \tag{4.6}$$

Lemma 4.3. *One can write,*

$$\sum_{E\in\mathcal{N}(D)} G_1(N_D^{n+1}, N_E^{n+1}, u_{D,E}^{n+1}) = -\sum_{E\in\mathcal{N}(D)} (u_{D,E}^{n+1})^-(N_E^{n+1} - N_D^{n+1})\,. \tag{4.7}$$

Proof. For all $n \in \{0, .., \tilde{N}\}$, the discrete solution u_h^{n+1} belongs to the space V_h defined in (3.6). Then, $\nabla \cdot u_h^{n+1} = 0$ for all $K \in \mathcal{T}_h$. Consequently,

$$\begin{aligned}\sum_{E\in\mathcal{N}(D)} \left((u_{D,E}^{n+1})^+ - (u_{D,E}^{n+1})^-\right) &= \sum_{E\in\mathcal{N}(D)} u_{D,E}^{n+1} = \sum_{E\in\mathcal{N}(D)} \int_{\sigma_{D,E}} u_h^{n+1} \cdot \eta_{D,E}\, d\gamma = \int_D \nabla \cdot u_h^{n+1}\, dx \\ &= \int_{D\cap K} \nabla \cdot u_h^{n+1}\, dx + \int_{D\cap L} \nabla \cdot u_h^{n+1}\, dx = 0.\end{aligned} \tag{4.8}$$

We still need to add and subtract $(u_{D,E}^{n+1})^- N_D^{n+1}$ to (3.12) and Lemma 4.3 will be a straightforward consequence of (4.8). □

Proposition 4.4. *(Discrete maximum principle)*
Let $\left(N_D^{n+1}, C_D^{n+1}\right)_{D\in\mathcal{D}_h, n\in\{0,..,\tilde{N}\}}$ *be the discrete solution of* (3.16)-(3.18). *Under the assumption* (3.21)*, for all* $D \in \mathcal{D}_h$ *and for all* $n \in \{0, .., \tilde{N}\}$*, this discrete solution satisfies:*

$$0 \leq N_D^{n+1} \leq 1 \text{ and } 0 \leq C_D^{n+1} \leq M,\ \forall D \in \mathcal{D}_h,\ n \in \{0, 1, ..., \tilde{N}\}.$$

Proof. We fix a dual volume D_0 such that $N_{D_0}^{n+1} = max_{D\in\mathcal{D}_h} N_D^{n+1}$. Due to (4.7), one has

$$\begin{aligned}N_{D_0}^n = N_{D_0}^{n+1} &- \frac{\Delta t}{|D_0|} \sum_{E\in\mathcal{N}(D_0)} \mathcal{S}_{D_0,E}(A(N_E^{n+1}) - A(N_{D_0}^{n+1})) + \frac{\Delta t}{|D_0|} \sum_{E\in\mathcal{N}(D_0)} G(N_{D_0}^{n+1}, N_E^{n+1}; \delta C_{D_0,E}^{n+1}) \\ &- \frac{\Delta t}{|D_0|} \sum_{E\in\mathcal{N}(D_0)} (u_{D_0,E}^{n+1})^-(N_E^{n+1} - N_{D_0}^{n+1}) - f(N_{D_0}^{n+1})\,.\end{aligned}$$

One has $0 \leq N_{D_0}^0 \leq 1$ and $C_{D_0}^0 \geq 0$ under the assumptions $0 \leq N_0 \leq 1$ and $C_0 \geq 0$.
We now make use of an induction argument. Let us suppose that $N_{D_0}^n \leq 1$, we will show that $N_{D_0}^{n+1} > 1$. Using the monotonicity of A, the function $G(., ., .)$ is increasing with respect to the first variable and the extension of χ and f by 0 for $N_{D_0}^{n+1} \geq 1$, we deduce that $1 \geq N_{D_0}^n \geq N_{D_0}^{n+1}$. Then, $N_D^{n+1} \leq 1$, $\forall D \in \mathcal{D}_h$ and $\forall n \in \{0, 1, ..., \tilde{N}\}$. Following the same guidelines and with the same arguments, the proof of this Proposition is achieved. □

Proposition 4.5. *If* $u_0 \in L^\infty(\Omega)$ *then*

$$||u_0||_{L^\infty(\Omega)} - (T+1)||\nabla\phi||_{L^\infty(\Omega)} \leq u_h^k \leq ||u_0||_{L^\infty(\Omega)} + (T+1)||\nabla\phi||_{L^\infty(\Omega)},$$

for all $k \in \{0, ..., \tilde{N}\}$ *and* $k\Delta t \leq T$.

Proof. The variational problem (3.10) can be written as:

$$\frac{1}{\Delta t}\int_\Omega (u_h^{n+1} - u_h^n) v_h\, dx + \nu \int_\Omega \nabla u_h^{n+1} \cdot \nabla v_h\, dx + b_h(u_h^{n+1}, u_h^{n+1}, v_h)\, dx = -\int_\Omega \tilde{N}_h^n \nabla\phi v_h\, dx\,. \tag{4.9}$$

First, we introduce $\bar{H}^n = (H_i^n)_{i=1,..,d}$ such that for all $i = \{1, ..., d\}$, the piecewise constant discrete function in space H_i^n is denoted by $(H_D^{n+1})_{D\in\mathcal{D}_h,\, n\in\{0,...,\tilde{N}\}}$:

$$\forall n \in \{0, ..., \tilde{N}+1\},\ \forall D \in \mathcal{D}_h,\ H_D^n \equiv H^n = ||\nabla\phi||_{L^\infty(\Omega)} n\Delta t + ||u_0||_{L^\infty(\Omega)}\,. \tag{4.10}$$

Our aim now is to prove that this function (4.10) is a super-solution of (4.9). Indeed,

$$\left\{\begin{array}{rcl} H_D^0 \equiv H^0 & = & ||u_0||_{L^\infty(\Omega)}, \\ \frac{1}{\Delta t}(H_D^{n+1} - H_D^n) & = & ||\nabla\phi||_{L^\infty(\Omega)} \,. \end{array}\right. \tag{4.11}$$

Let us prove by induction that:

$$u_h^k \leq \bar{H}^k,\ \forall k \in \{0, ..., \tilde{N}+1\}\,.$$

For $k = 0$, $u_h^0 \leq H^0 = ||u_0||_{L^\infty(\Omega)}$. We assume that $u_h^n \leq \bar{H}^n$ and we prove by contradiction that $u_h^{n+1} \leq \bar{H}^{n+1}$. Subtracting equations (4.9) and (4.11), one has:

$$\frac{1}{\Delta}\int_\Omega (u_h^{n+1} - \bar{H}^{n+1})v_h\,dx + \int_\Omega \nabla u_h^{n+1}\cdot\nabla v_h\,dx + b_h(u_h^{n+1}, u_h^{n+1}, v_h)\,dx = \frac{1}{\Delta}\int_\Omega (u_h^n - \bar{H}^n)v_h\,dx +$$

$$\int_\Omega [-\nabla\phi - ||\nabla\phi||_{L^\infty(\Omega)}]v_h\,dx$$

Then, we choose $v_h = (u_h^{n+1} - \bar{H}^{n+1})^+ = \max(u_h^{n+1} - \bar{H}^{n+1}, 0) \in V_h$ as a test function. We ramark that for $u_h^{n+1} > \bar{H}^{n+1}$, $b_h(u_h^{n+1}, u_h^{n+1}, (u_h^{n+1} - \bar{H}^{n+1})^+) = b_h(u_h^{n+1}, (u_h^{n+1} - \bar{H}^{n+1})^+, (u_h^{n+1} - \bar{H}^{n+1})^+) = 0$. Consequently, one can easily deduces that:

$$0 \leq \frac{1}{\Delta t}\int_\Omega \Big((u_h^{n+1} - \bar{H}^{n+1})^+\Big)^2\,dx \leq 0\,.$$

Then $(u_h^{n+1} - H^{n+1})^+ = 0$ and therefore $u_h^{n+1} < \bar{H}^{n+1}$ which is a contradiction. So

$$\sup_{\tilde{N}\in\mathbb{N}}(\max_{0\leq n\leq\tilde{N}} \bar{H}^{n+1}) \leq ||u_0||_{L^\infty(\Omega)} + (T+1)||\nabla\phi||_{L^\infty(\Omega)}\,.$$

Now, it suffices to consider $H_D^n = -||\nabla\phi||_{L^\infty(\Omega)} n\Delta t + ||u_0||_{L^\infty(\Omega)}$ to prove similarly the other inequality of this Proposition. □

Consequently, we obtain a useful estimate needed in the sequel.

Corollary 4.6. *For all $n \in \{0, ..., \tilde{N}\}$,*

$$\big|u_{D,E}^{n+1}\big| \leq \int_{\sigma_{D,E}} \big|u_h^{n+1}\cdot\eta_{D,E}\big| \leq C_u\big|\sigma_{D,E}\big|\,, \tag{4.12}$$

where C_u is a constant depending on the initial data.

4.2. **Discrete a priori estimates.**

Proposition 4.7. (*A priori estimate under assumption* (3.21))
Let $(N_D^{n+1}, C_D^{n+1})_{D\in\mathcal{D}_h, n\in\{0,\ldots,\tilde{N}\}}$ be a solution of the scheme (3.16)-(3.18). *Then, there exists a constant $C > 0$, depending on $||C_0||_\infty$, α, d and the constant of the bound of G such that*

$$\frac{1}{2}\sum_{D\in\mathcal{D}_h}|D|\big|C_D^{\tilde{N}+1}\big|^2 + C_M\sum_{n=0}^{\tilde{N}-1}\Delta\big|\big|C_h^{n+1}\big|\big|_{X_h}^2 \leq C\,. \tag{4.13}$$

Furthermore, there exists a constant $C > 0$ such that,

$$\sum_{n=0}^{\tilde{N}-1}\Delta t\big|\big|A_h(N_h^{n+1})\big|\big|_{X_h}^2 \leq C\,, \tag{4.14}$$

Consequently, there exists a constant $C > 0$, depending on Ω, T, $||C_0||_\infty$, α, d and the constant from the fourth property of G such that

$$\sum_{n=0}^{\tilde{N}-1}\Delta t\sum_{D\in\mathcal{D}_h}\sum_{E\in N(D)} S_{D,E}|A(N_D^{n+1}) - A(N_E^{n+1})|^2 + \sum_{n=0}^{\tilde{N}-1}\Delta t\sum_{D\in\mathcal{D}_h}\sum_{E\in N(D)} S_{D,E}|C_D^{n+1} - C_E^{n+1}|^2 \leq C\,. \tag{4.15}$$

Proof. First, let us prove (4.13). We multiply (3.18) by $\Delta t C_D^{n+1}$ and we sum for all $D \in \mathcal{D}_h$ and $n \in \{0, ..., \tilde{N}\}$. We obtain $E_{2,1} + E_{2,2} + E_{2,3} = E_{2,4}$.
From the following inequality $(a-b)a \geq \frac{a^2-b^2}{2}$, for all $a, b \in \mathbb{R}$, we have

$$E_{2,1} = \sum_{n=0}^{\tilde{N}-1} \sum_{D\in\mathcal{D}_h} |D|\big(C_D^{n+1} - C_D^n\big)C_D^{n+1}$$

$$\geq \frac{1}{2}\sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} |D|\big(|C_D^{n+1}|^2 - |C_D^n|^2\big) = \frac{1}{2}\sum_{D\in\mathcal{D}_h} |D|\big(|C_D^{\tilde{N}+1}|^2 - |C_D^0|^2\big).$$

Using the estimate (4.4), one has

$$E_{2,2} = -\sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} C_D^{n+1} \sum_{E\in\mathcal{D}_h} \mathcal{M}_{D,E} C_E^{n+1} \geq C_M \sum_{n=0}^{\tilde{N}-1} \Delta t ||C_h^{n+1}||^2_{X_h}.$$

Next,

$$E_{2,3} = \sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} C_D^{n+1} \sum_{E\in\mathcal{N}(D)} u_{D,E}^{n+1}\bar{C}_{D,E}^{n+1} = \sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} C_D^{n+1} \sum_{E\in\mathcal{N}(D)} G_1(C_D^{n+1}, C_E^{n+1}; u_{D,E}^{n+1}) \,.$$

Therefore,

$$E_{2,3} \leq MC_u \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} |\sigma_{D,E}|\big|C_D^{n+1} - C_E^{n+1}\big| \leq C_u \frac{M}{\varepsilon}\Big(\sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} |\sigma_{D,E}| d_{D,E}\Big)$$

$$+C_u \frac{M}{\varepsilon}\Big(\sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} \frac{|\sigma_{D,E}|}{d_{D,E}}\big|C_D^{n+1} - C_E^{n+1}\big|^2\Big) \leq C_u \frac{M}{\varepsilon}|\Omega| + C_u M\varepsilon ||C_h||^2_{X_h} \,.$$

Finally, it follows from the form (1.4) of h and the Proposition 4.4 that

$$E_{2,4} = -\sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} |D| h(N_D^n, C_D^{n+1}) C_D^{n+1} \leq \alpha M T |\Omega| \,.$$

Collecting the previous inequalities we readily deduce (4.13).

To obtain (4.15), we multiply (3.17) by $\Delta t A(N_D^{n+1})$ and we sum for all $D \in \mathcal{D}_h$ and $n \in \{0, ..., \tilde{N}\}$. We obtain $E_{1,1} + E_{1,2} + E_{1,3} + E_{1,4} = E_{1,5}$.

From the convexity of the function $\mathcal{B}(s) = \int_0^s A(r)dr$ ($\mathcal{B}''(s) = a(s) \geq 0$), we have the following inequality: $(a-b)A(a) \geq \mathcal{B}(a) - \mathcal{B}(b)$. Then,

$$E_{1,1} = \sum_{n=0}^{\tilde{N}-1} \sum_{D\in\mathcal{D}_h} |D|(N_D^{n+1} - N_D^n)A(N_D^{n+1}) \tag{4.16}$$

$$\geq \sum_{n=0}^{\tilde{N}-1} \sum_{D\in\mathcal{D}_h} |D|(\mathcal{B}(N_D^{n+1}) - \mathcal{B}(N_D^n)) = \sum_{D\in\mathcal{D}_h} |D|(\mathcal{B}(\tilde{N}_D^{N+1}) - \mathcal{B}(N_D^0)) \,.$$

The discrete property (4.2) yields

$$E_{1,2} = -\sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} A(N_D^{n+1}) \sum_{E\in\mathcal{D}_h} \mathcal{S}_{D,E} A(N_E^{n+1}) = \sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} \mathcal{S}_{D,E}\Big(A(N_E^{n+1}) - A(N_D^{n+1})\Big)^2 . \tag{4.17}$$

It follows from the coercivity property (4.4) that

$$E_{1,2} \geq C_S \sum_{n=0}^{\tilde{N}-1} \Delta t ||A_h(N_h^{n+1})||_{X_h}^2 \,. \tag{4.18}$$

Finally, using the fact that the numerical flux G is conservative, we integrate by parts to obtain:

$$E_{1,3} = \sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} A(N_D^{n+1}) \sum_{E\in\mathcal{N}(D)} G(N_D^{n+1}, N_E^{n+1}; \delta C_{D,E}^{n+1})$$

$$= \frac{1}{2}\sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} G(N_D^{n+1}, N_E^{n+1}; \delta C_{D,E}^{n+1})\Big(A(N_D^{n+1}) - A(N_E^{n+1})\Big) \,,$$

and consequently from the bound of the numerical flux G and from (3.20), we have

$$|E_{1,3}| \leq \frac{1}{2}\sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} |\mathcal{S}_{D,E}|\big|C_D^{n+1} - C_E^{n+1}\big|\big|A(N_D^{n+1}) - A(N_E^{n+1})\big| \,.$$

By a weighted Young's inequality and due to the positivity of the transmissibilities coefficients $\mathcal{S}_{D,E}$,

$$|E_{1,3}| \leq \frac{\varepsilon}{2}\sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} \mathcal{S}_{D,E}(C_D^{n+1} - C_E^{n+1})^2 + \frac{1}{2\varepsilon}\sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} \mathcal{S}_{D,E}\Big(A(N_D^{n+1}) - A(N_E^{n+1})\Big)^2 ,$$

Due to the inequalities (3.5) and (4.6),

$$|E_{1,3}| \leq \frac{C}{\varepsilon}||C_h^{n+1}||_{X_h}^2 + C\varepsilon||A_h(N_h^{n+1})||_{X_h}^2 \,, \tag{4.19}$$

for some positive constant C. Otherwise,

$$E_{1,4} = \sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} A(N_D^{n+1}) \sum_{E\in\mathcal{N}(D)} u_{D,E}^{n+1}\bar{N}_{D,E}^{n+1}$$

$$= \frac{1}{2}\sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} G_1(N_D^{n+1}, N_E^{n+1}; u_{D,E}^{n+1})(A(N_D^{n+1}) - A(N_E^{n+1})) \,.$$

Under the fact of Proposition 4.4 and (4.12), one has

$$|E_{1,4}| \leq \frac{C_u}{2}\sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} |\sigma_{D,E}||A(N_D^{n+1}) - A(N_E^{n+1})|$$

$$\leq \varepsilon\frac{C_u}{2}\Big(\sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} \frac{|\sigma_{D,E}|}{d_{D,E}}|A(N_D^{n+1}) - A(N_E^{n+1})|^2\Big) + \frac{C_u}{\varepsilon}\Big(\sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} |\sigma_{D,E}|d_{D,E}\Big)$$

It follows from Lemma 3.5 that

$$|E_{1,4}| \leq \varepsilon C||A_h(N_h^{n+1})||_{X_h}^2 + C' \,. \tag{4.20}$$

The boundedness of the last term is a consequence of the Lipschitz continuity of f and $N_D^{n+1} \leq 1$,

$$|E_{1,5}| = \Big|\sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} |D|N_D^{n+1} f(N_D^{n+1})\Big| \leq L_f T|\Omega| \,. \tag{4.21}$$

We can readily deduce (4.14) by collecting (4.16), (4.18), (4.19), (4.20) and (4.21). Consequently, one can also easily deduce the estimate (4.15). $\square$

4.3. **Existence of a discrete solution.** The existence of a discrete solution for the combined scheme is given in the following proposition.

Proposition 4.8. *The discrete problem* (3.16)-(3.18) *has at least one solution.*

Proof. Denote by $N_h^n = \{N_D^n\}$ and $C_h^n = \{C_D^n\}$. We will show the existence of a discrete solution by induction on n. Assume that the couple (N_h^n, C_h^n) exists and show the existence of (N_h^{n+1}, C_h^{n+1}). The discrete system (3.18) is a finite dimensional linear system with respect to the unknowns $C_D^{n+1}, D \in \mathcal{D}_h$. The resulting matrix of this system is symmetric and definite positive, thus C_h^{n+1} is the unique solution of (3.18).

As $A(.)$ is strictly monotone then $A(.)$ is invertible. We can rewrite the scheme (3.17) in terms of w_h^i with $N_h^i = A^{-1}(w_h^i)$, $i \in [0, \tilde{N}]$. Assume that w_h^n and C_h^{n+1} exist. We choose the componentwise product $[\cdot,\cdot]$ as the scalar product on $\mathbb{R}^{\mathcal{T}}$. We define the mapping $\mathcal{M}$, that associates to the vector $\mathcal{W} = (W_D^{n+1})_{D\in\mathcal{D}_h}$, the following expression:

$$\mathcal{M}(\mathcal{W}) = \Big(|D|\frac{A^{-1}(W_D^{n+1}) - A^{-1}(W_D^n)}{\Delta t} - \sum_{E\in\mathcal{N}(D)} \mathcal{S}_{D,E}(W_E^{n+1} - W_D^{n+1})$$

$$+ \sum_{E\in\mathcal{N}(D)} G\big(A^{-1}(W_D^{n+1}), A^{-1}(W_E^{n+1}); \delta C_{D,E}^{n+1}\big) + \sum_{E\in\mathcal{N}(D)} G_1\big(A^{-1}(W_D^{n+1}), A^{-1}(W_E^{n+1}); u_{D,E}^{n+1}\big)$$

$$-f(A^{-1}(W_D^{n+1}))\Big)_{D\in\mathcal{D}_h}.$$

We multiply by W_D^{n+1} and we sum over all the volumes $D \in \mathcal{D}_h$. We shall use this further estimate,

$$\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)} G_1(A^{-1}(W_D^{n+1}), A^{-1}(W_E^{n+1}); u_{D,E}^{n+1})\Big(A(N_E^{n+1}) - A(N_D^{n+1})\Big)$$

$$\leq \sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)} |u_{D,E}^{n+1}|\Big|A(N_E^{n+1}) - A(N_D^{n+1})\Big|$$

$$\leq C_u\Big(\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)} \frac{|\sigma_{D,E}|}{d_{D,E}}\big(A(N_E^{n+1}) - A(N_D^{n+1})\big)^2\Big)^{\frac{1}{2}}\Big(\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)} |\sigma_{D,E}|d_{D,E}\Big)^{\frac{1}{2}}$$

$$\leq C_u|\Omega|^{\frac{1}{2}}\big\|\mathcal{W}\big\|_{X_h}.$$

We will also use the estimate (4.15) and the Young inequality to obtain,

$$[\mathcal{M}(\mathcal{W}), \mathcal{W}] \geq C|\mathcal{W}|^2 - C'|\mathcal{W}| - C'' \geq 0 \text{ for } |\mathcal{W}| \text{ large enough},$$

for some constants $C, C^{'}, C^{''} > 0$. This implies that,

$$[\mathcal{M}(\mathcal{W}), \mathcal{W}] > 0 \text{ for } |\mathcal{W}| \text{ large enough},$$

and therefore we obtain the existence of $\mathcal{W}$ such that,

$$\mathcal{M}(\mathcal{W}) = 0\,.$$

$\square$

4.4. **Compactness estimates on discrete solutions.** In this subsection we derive estimates on differences of space and time translates necessary to apply Kolmogorov's compactness theorem which will allow us to pass to the limit.

Lemma 4.9. *(Time and space translate estimate)*

(1) *There exists a constant $c>0$ depending on Ω, T and A such that:*

$$\iint_{\Omega\times[0,T-\tau]}\Big(A(\tilde N_h(t+\tau,x))-A(\tilde N_h(t,x))\Big)^2\,dxdt\le C(\tau+\Delta t),\forall\tau\in[0,T]\,.$$

(2) *There exists a positive constant $c'>0$ depending on Ω, T, A and ξ such that:*

$$\iint_{\Omega\times[0,T]}\Big(A(\tilde N_h(t,x+\xi))-A(\tilde N_h(t,x))\Big)^2\,dxdt\le c'|\xi|(|\xi|+h),\forall\xi\in\mathbb{R}^d\,.$$

Proof. Comparing to [5], it remains just to check the new convective term for the time translate estimate,

$$\mathcal{B}_2(t):=-\sum_{t\le n\Delta t\le t+\tau}\Delta t\int_0^{T-\tau}\chi(n,t)\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}G_1(N_D^{n+1},N_E^{n+1};u_{D,E}^{n+1})\big(A(U_D^{n_1(t)})-A(U_D^{n_0(t)})\big)\tag{4.22}$$

$$=-\sum_{t\le n\Delta t\le t+\tau}\Delta t\int_0^{T-\tau}\chi(n,t)\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\Big(G_1(N_D^{n+1},N_E^{n+1};u_{D,E}^{n+1})\big(A(U_E^{n_1(t)})-A(U_D^{n_1(t)})\big)$$

$$+G_1(N_D^{n+1},N_E^{n+1};u_{D,E}^{n+1})\big(A(U_E^{n_0(t)})-A(U_D^{n_0(t)})\big)\Big)\,.$$

We use Young's inequality, $|G_1(a,b,c)|\le C(|a|+|b|)|c|$, (4.12) and the Proposition 4.4 to deduce that

$$\mathcal{B}_2(t)\le C'(\mathcal{B}_3(t)+\mathcal{B}_4(t))$$

for some constant $C'>0$, with

$$\mathcal{B}_3(t)=C_u\sum_{n=0}^{\tilde N-1}\Delta t\int_0^{T-\tau}\chi(n,t)\sum_{\sigma_{D,E}\in\mathcal{F}_h^{int}}|\sigma_{D,E}|\Big|A(N_E^{n_1(t)})-A(N_D^{n_1(t)})\Big|dt\,,$$

$$\mathcal{B}_4(t)=C_u\sum_{n=0}^{\tilde N-1}\Delta t\int_0^{T-\tau}\chi(n,t)\sum_{\sigma_{D,E}\in\mathcal{F}_h^{int}}|\sigma_{D,E}|\Big|A(N_E^{n_0(t)})-A(N_D^{n_0(t)})\Big|dt\,.$$

Consequently,

$$\mathcal{B}_3(t)\le C_u\sum_{n=0}^{\tilde N-1}\Delta t\int_0^{T-\tau}\chi(n,t)\Big(\sum_{\sigma_{D,E}\in\mathcal{F}_h^{int}}\frac{|\sigma_{D,E}|}{d_{D,E}}\big(A(N_E^{n_1(t)})-A(N_D^{n_1(t)})\big)^2\Big)^{\frac12}\Big(\sum_{\sigma_{D,E}\in\mathcal{F}_h^{int}}|\sigma_{D,E}|d_{D,E}\Big)^{\frac12}dt\,,$$

$$\le\tau C_uC'|\Omega|^{\frac12}=\tau C\,.$$

Reasoning along the same lines yields $\mathcal{B}_4(t)\le\tau C$ for some constant $C>0$. This concludes the proof of this Lemma. ∎

4.5. **Convergence.** This subsection is mainly devoted to the proof of the strong $L^2(Q_T)$ convergence of approximate solutions, using the estimates proved in the previous subsection and Kolmogorov's compactness criterion for the convergence. Then, we prove that the limit is a weak solution to the continuous problem. We start by giving the following Lemma proved in [5].

Lemma 4.10. *The sequence* $\Big(A_h(N_{h,\Delta t}) - A(\tilde{N}_{h,\Delta t})\Big)_{h,\Delta t}$ *converges strongly to zero in* $L^2(Q_T)$ *as* h, $\Delta t \to 0$.

Theorem 4.11. *(Strong $L^2(Q_T)$-Convergence)*
There exists a subsequence of $\big(A_h(N_{h,\Delta t})\big)_{h,\Delta t}$ *which converges strongly in* $L^2(Q_T)$ *to some function* $\Gamma = A(N) \in L^2(0,T;H^1(\Omega))$.

Proof. The a priori estimate (4.14) and Lemma 4.9 imply that $\big(A(\tilde{N}_{h,\Delta t})\big)_{h,\Delta t}$ satisfies the assumptions of the Kolmogorov's compactness criterion, and consequently $\big(A(\tilde{N}_{h,\Delta t})\big)_{h,\Delta t}$ is relatively compact in $L^2(Q_T)$. This implies the existence of subsequences of $\big(A(\tilde{N}_{h,\Delta t})\big)_{h,\Delta t}$ which converges strongly to some function $\Gamma \in L^2(Q_T)$. Due to Lemma 4.10, $\big(A_h(N_{h,\Delta t})\big)_{h,\Delta t}$ converges strongly to some function $\Gamma \in L^2(Q_T)$. Using the monotonicity of A, we get $\Gamma = A(N)$. Moreover, due to the space translate estimate of Lemma 4.9, $\big[$[14], Theorem 3.10$\big]$ gives that $A(N) \in L^2(0,T;H^1(\Omega))$. □

As A^{-1} is well defined and continuous, applying the L^∞ bound on $\tilde{N}_{h,\Delta t}$ and the dominated convergence theorem of Lebesgue to $\tilde{N}_{h,\Delta t} = A^{-1}(A(\tilde{N}_{h,\Delta t}))$, there exist subsequences $\tilde{N}_{h,\Delta t}$ and $N_{h,\Delta t}$, which have the same notation of the sequences and converges strongly in $L^2(Q_T)$ and a.e. in Q_T to the same function N.

We will prove now that the limit couple (N, C, u) is a weak solution of the continuous problem. We introduce

$$\Psi := \left\{\psi \in C^{2,1}(\bar{\Omega} \times [0,T]),\ \psi(.,T) = 0\right\}. \tag{4.23}$$

We then multiply (3.17) by $\Delta t \psi(P_D, t_{n+1})$ and we sum the result over $D \in \mathcal{D}_h$, $n \in \{0, ..., \tilde{N}-1\}$, to obtain:

$$T_T + T_D + T_C + \tilde{T}_C = T_R,$$

We successively search for the limit of each of these terms as h and Δt tend to zero.
Time evolution, Diffusion and Chemoattractant Convection terms. We refer to [5] for a proof of the limits

$$T_T = \sum_{n=0}^{N-1} \sum_{D\in\mathcal{D}_h} |D|(N_D^{n+1} - N_D^n)\psi(P_D, t_{n+1}) \overset{h,\Delta t\to 0}{\longrightarrow} -\iint_{Q_T} \tilde{N}_{h,\Delta t}\frac{\partial \psi}{\partial t}\, dxdt - \int_\Omega \tilde{N}_{0,h}(0,x)\psi(0,x)\, dx,$$

$$T_D = \sum_{n=0}^{N-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{D}_h} \mathcal{S}_{D,E} A(N_E^{n+1})\psi(P_D, t_{n+1}) \overset{h,\Delta t\to 0}{\longrightarrow} \iint_{Q_T} S(x)\nabla A(N)\cdot\nabla\psi\, dxdt,$$

$$T_C = \sum_{n=0}^{N-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} G(N_D^{n+1}, N_E^{n+1}, \delta C_{D,E}^{n+1})\psi(P_D, t_{n+1}) \overset{h,\Delta t\to 0}{\longrightarrow} -\iint_{Q_T} S(x)\chi(N)\nabla C\cdot\nabla\psi\, dxdt\,.$$

It remains to check the convergence of the other terms.
Fluid Convection term. Let us prove that

$$\tilde{T}_C = \sum_{n=0}^{N-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} G_1(N_D^{n+1}, N_E^{n+1}, u_{D,E}^{n+1})\psi_D^{n+1} \overset{h,\Delta t\to 0}{\longrightarrow} -\iint_{Q_T} N(x,t)u(x,t)\cdot\nabla\psi(x,t)\, dxdt\,.$$

For each couple of neighbor volumes D and E, we introduce

$$N_{D,E}^{n+1} = \min(N_D^{n+1}, N_E^{n+1})\,, \tag{4.24}$$

and using the consistency of the flux G_1, we introduce

$$\tilde{T}_C^* = \sum_{n=0}^{N-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} N_{D,E}^{n+1} u_{D,E}^{n+1} \psi_D^{n+1}\,.$$

The diamond constructed from the neighbor edge centers P_D, P_E and the interface $\sigma_{D,E}$ of the dual mesh is denoted by $T_{D,E} \subset K_{D,E}$. Then, we introduce

$$\overline{N_h}\Big|_{]t_n,t_{n+1}]\times T_{D,E}} := \max(N_D^{n+1}, N_E^{n+1}),\ \underline{N_h}\Big|_{]t_n,t_{n+1}]\times T_{D,E}} := \min(N_D^{n+1}, N_E^{n+1})\,,$$

By the monotonicity of A and thanks to estimates (3.4) and (4.14), we have

$$\int_0^T \int_\Omega |A(\overline{N_h}) - A(\underline{N_h})|^2 \le \sum_{n=0}^{N-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} |T_{D,E}| \big|A(N_D^{n+1}) - A(N_E^{n+1})\big|^2$$

$$\le \sum_{n=0}^{N-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} (diam(K_{D,E}))^2 \big|A(N_D^{n+1}) - A(N_E^{n+1})\big|^2 \le Ch^{4-d} \overset{h\to 0}{\longrightarrow} 0\,,$$

with $d = 2$ or 3. Because A^{-1} is continuous, up to extraction of another subsequence, we deduce

$$|\overline{N_h} - \underline{N_h}| \overset{h\to 0}{\longrightarrow} 0 \text{ a.e. on } Q_T. \tag{4.25}$$

In addition, $\underline{N_h} \le \tilde{N}_h \le \overline{N_h}$; moreover, $\tilde{N}_h \overset{h\to 0}{\longrightarrow} N$ a.e. on Q_T. Let us first prove that

$$\tilde{T}_C^* \longrightarrow \sum_{n=0}^{N-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} \int_{\sigma_{D,E}} N u_h^{n+1} \cdot \eta_{D,E} \psi(x, t_{n+1}) d\gamma(x)\,. \tag{4.26}$$

We add and we subtract $N\psi(P_D, t_{n+1}) u_{D,E}^{n+1}$ and $N_{D,E}^{n+1} \int_{\sigma_{D,E}} u_h^{n+1} \cdot \eta_{D,E} \psi(x, t_{n+1})$, we obtain

$$T_C = T_{C_1} + T_{C_2} + T_{C_3}\,,$$

with,

$$T_{C_3} = \sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} N\psi(x, t_{n+1}) \sum_{E\in\mathcal{N}(D)} u_{D,E}^{n+1} = 0 \text{ due to (4.8)},$$

$$T_{C_2} = \sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} N_{D,E}^{n+1} \int_{\sigma_{D,E}} u_h^{n+1} \cdot \eta_{D,E} \psi(x, t_{n+1})\, d\gamma(x) = 0\,,$$

indeed, we need just to denote $u_{\psi;D,E} = \int_{\sigma_{D,E}} u_h^{n+1} \cdot \eta_{D,E} \psi(x, t_{n+1})\, d\gamma(x)$ to remark that each interior side appears twice in T_2 and $u_{\psi;D,E} = -u_{\psi;E,D}$.

$$T_{C_1} = \sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} \big(N_{D,E}^{n+1} - N\big)\Big(\psi(P_D, t_{n+1}) u_{D,E}^{n+1} - \int_{\sigma_{D,E}} u_h^{n+1} \cdot \eta_{D,E} \psi(x, t_{n+1})\, d\gamma(x)\Big)\,.$$

Consequently, the Cauchy-Schwarz inequality implies that $T_{C_1}^2 \le T_{C_4} T_{C_5}$ with

$$T_{C_4} = \iint_{Q_T} \big(\underline{N_h} - N\big)\, dxdt \overset{h,\Delta t\to 0}{\longrightarrow} 0 \text{ due to the dominated convergence theorem of Lebesgue,}$$

and

$$T_{C_5}=\sum_{n=0}^{\tilde{N}-1}\Delta t\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\Big(\int_{\sigma_{D,E}}u_h^{n+1}\cdot\eta_{D,E}\Big(\psi(P_D,t_{n+1})-\psi(x,t_{n+1})\Big)\,d\gamma(x)\Big)^2$$

$$\Rightarrow |T_{C_5}|\le C_{2,\psi}^2h^2C_u^2\sum_{n=0}^{N-1}\Delta t\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}|\sigma_{D,E}|^2\le C_{2,\psi}^2h^2C_u^2|\Omega|T\overset{h,\Delta t\to 0}{\longrightarrow}0\,.$$

Consequently, we have

$$T_{C_1}\overset{h,\Delta t\to 0}{\longrightarrow}0\,.$$

Using the divergence theorem and the property $\nabla\cdot(f\vec{A})=f(\nabla\cdot\vec{A})+\vec{A}\cdot\nabla f$, one has

$$\sum_{n=0}^{\tilde{N}-1}\Delta t\sum_{D\in\mathcal{D}_h}N\sum_{E\in\mathcal{N}(D)}\int_{\sigma_{D,E}}u_h^{n+1}\cdot\eta_{D,E}\psi(x,t_{n+1})\,d\gamma(x)=\sum_{n=0}^{\tilde{N}-1}\Delta t\sum_{D\in\mathcal{D}_h}\int_D\nabla\cdot(Nu_h^{n+1})\psi(x,t_{n+1})\,dx\,.$$

$$=\sum_{n=0}^{\tilde{N}-1}\Delta t\sum_{D\in\mathcal{D}_h}\int_D Nu_h^{n+1}\cdot\nabla\psi(x,t_{n+1})\,dx+\sum_{n=0}^{\tilde{N}-1}\Delta t\sum_{D\in\mathcal{D}_h}N\int_D(\nabla\cdot u_h^{n+1})\psi(x,t_{n+1})\,dx\,.$$

As $\nabla\cdot u_h^{n+1}=0$, it suffices now to prove that

$$\sum_{n=0}^{\tilde{N}-1}\Delta t\sum_{D\in\mathcal{D}_h}\int_D Nu_h^{n+1}\cdot\nabla\psi(x,t_{n+1})\,dx\overset{h,\Delta t\to 0}{\longrightarrow}-\iint_{Q_T}N(x,t)u(x,t)\cdot\nabla\psi(x,t)\,dxdt\,. \tag{4.27}$$

We introduce

$$T_{C_6}=\int_0^T\int_\Omega Nu_h^{n+1}(x)\cdot\big(\nabla\psi(x,t_{n+1})-\nabla\psi(x,t)\big)\,dxdt\to 0\,.$$

Indeed, $|\nabla\psi(x,t_{n+1})-\nabla\psi(x,t)|\le g(\Delta t)$ and $|T_{C_6}|\le g(\Delta t)C_uh$. And

$$T_{C_7}=\int_0^T\int_\Omega N(x,t)\big(u_h^{n+1}(x)-u(x,t)\big)\cdot\nabla\psi(x,t)\,dxdt\to 0\,,$$

due to the weak convergence of u_h^{n+1} to u, N belongs to $L^2(Q_T)$ and $|\nabla\psi(x,t)|\le C_{2,\psi}$. Gathering (4.26) and (4.27), one obtains that

$$\tilde{T}_C^*\overset{h,\Delta t\to 0}{\longrightarrow}-\iint_{Q_T}N(x,t)u(x,t)\cdot\nabla\psi(x,t)\,dxdt\,.$$

It remains now to prove that

$$\lim_{h\to 0}\big|\tilde{T}_C-\tilde{T}_C^*\big|=0\,.$$

By properties of G_1, we have:

$$\big|G_1(N_D^{n+1},N_E^{n+1},u_{D,E}^{n+1})-N_{D,E}^{n+1}u_{D,E}^{n+1}\big|=\big|G_1(N_D^{n+1},N_E^{n+1},u_{D,E}^{n+1})-G_1(N_{D,E}^{n+1},N_{D,E}^{n+1},u_{D,E}^{n+1})\big|$$

$$\le 2\big|N_D^{n+1}-N_{D,E}^{n+1}\big||u_{D,E}^{n+1}|\le 2C_u\big|\overline{N_h}-\underline{N_h}\big||\sigma_{D,E}|\,.$$

Therefore, it follows from (4.25) and the theorem of dominated convergence of Lebesgue that,

$$\big|\tilde{T}_C-\tilde{T}_C^*\big|\le 2C_u\sum_{n=0}^{N-1}\Delta t\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\big|\overline{N_h}-\underline{N_h}\big||\sigma_{D,E}|\big|\psi_E^{n+1}-\psi_D^{n+1}\big|$$

$$\le 2C_uC_\psi h^2\iint_{Q_T}\big|\overline{N_h}-\underline{N_h}\big|\,dxdt\longrightarrow 0\,.$$

Source term: Let us now prove that

$$T_R = \sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} |D| f(N_D^{n+1})\psi(Q_D,t_{n+1}) \overset{h,\Delta t\to 0}{\longrightarrow} \int_0^T \int_\Omega f(N(x,t))\psi(x,t)\,dxdt\,.$$

For this purpose, we introduce

$$T_{R_1} = \sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} f(N_D^{n+1}) \int_D \big(\psi(Q_D,t_{n+1}) - \psi(x,t)\big)\,dxdt \overset{h,\Delta t\to 0}{\longrightarrow} 0\,,$$

indeed, $\big|\psi(Q_D,t_{n+1})-\psi(x,t)\big| \le C_{3,\psi}(h+\Delta t)$ and $|T_{R_1}| \le C_{3,\psi}L_f(h+\Delta t)|\Omega|T$.

$$T_{R_2} = \sum_{n=0}^{\tilde{N}-1} \Delta t \sum_{D\in\mathcal{D}_h} \int_D \big(f(N_D^{n+1}) - f(N)\big)\psi(x,t)\,dxdt \overset{h,\Delta t\to 0}{\longrightarrow} 0\,,$$

in fact, $|T_{R_2}| \le C_{1,\psi}L_f\int_0^T\int_\Omega |\tilde{N}_{h,\Delta t}(x,t)-N(x,t)|\,dxdt$ and $\tilde{N}_{h,\Delta t}$ converges strongly to N in $L^2(Q_T)$.

Reasoning along the same lines for the chemo-attractant and fluid equations, we conclude that the limit couple (N,C,u) is a weak solution of the continuous problem in the sense of Definition 2.1 by using the density of the set Ψ in $\mathcal{W} = \{\phi\in L^2(0,T;H^1(\Omega)), \frac{\partial\phi}{\partial t}\in L^2(Q_T), \phi(.,T)=0\}$.

5. Numerical experiments

In this section, we present many numerical tests to show the dynamics of solutions of the following chemotaxis-fluid system:

$$\left\{\begin{array}{rcl} \partial_t N - \nabla\cdot\big(S(x)a(N)\nabla N\big) + \nabla\cdot\big(S(x)\chi(N)\nabla C\big) + c_1(u\cdot\nabla N) &=& 0,\\ \partial_t C - d\nabla\cdot(M(x)\nabla C) + c_2(u\cdot\nabla C) &=& \alpha N - \beta C,\\ \partial_t u - \nu\Delta u + \nabla P &=& -N\nabla\phi,\\ \nabla\cdot u &=& 0, \end{array}\right. \tag{5.1}$$

with $A(N) = D\int_0^N a(N)\,dx = D\int_0^N N(1-N)\,dx = D(\frac{N^2}{2}-\frac{N^3}{3})$, $\chi(N)=cN(1-N)^2$ and D, α, β, c, c_1, c_2, d are positive constants. This system is discretized by the combined method along the algorithm detailed in the section 3.4.

5.1. **Test 1: Influence of the gravitational force.** We start by this numerical test in order to show the influence of the gravitation exerted by the cells on the fluid. We suppose no-slip boundary conditions for the velocity of the fluid (homogeneous Dirichlet). The simulations are done on a mesh given in Figure 3(b) and the initial conditions are defined by regions in Figure 2(a). The initial density is defined by $N_0(x,y)=0.5$ in the square $(x,y)\in\big([0.45,0.55]\times[0.65,0.75]\big)$ and 0 otherwise. The initial concentration of the chemo-attractant is defined by $C_0(x,y)=10$ in the square $(x,y)\in\big([0.45,0.55]\times[0.25,0.35]\big)$ and 0 otherwise. The initial components of the velocity are neglected. In addition to that, we consider isotropic diffusive tensors ($S(x)=M(x)=Id$) in this test. Next, we choose $dt=0.0005$, $\alpha=0.01$, $\beta=0.05$, $D=0.05$, $c=0.5$, $c_1=20$, $c_2=0$, $d=10^{-4}$, $\nu=10^{-2}$ and $\nabla\phi=(0,100)$. In the Figure 2, we clearly observe the diffusion a part of cells towards the chemo-attractant and the other part fall down under the influence of the flux created by the gravitation force proportional to the density of cells.

5.2. **Test 2: Driven cavity.** In this test, we consider the standard problem of fluid flow in a two-dimensional square cavity where the speed at the top wall remains constant $\begin{pmatrix}1\\0\end{pmatrix}$ and this requires movement of the fluid at that constant speed over the edge at the top of the domain. This is one of the most important tests in fluid mechanics and this is mainly due to two reasons: the simplicity of its geometry and the multitude physical phenomena observed in this flow.

5.2.1. *Stokes problem.* In fact, we do not know the analytical solution of the following problem:

$$\left\{\begin{array}{rcll} \partial_t u - \nu\Delta u + \nabla P & = & 0 & \text{dans } \Omega, \\ \nabla\cdot u & = & 0 & \text{dans } \Omega, \\ u & = & \begin{pmatrix} u_1 \\ 0 \end{pmatrix} & \text{dans } \partial\Omega\,, \end{array}\right. \tag{5.2}$$

with a viscosity $\nu = 5\times 10^{-3}$ and

$$u_1(x,y) = \left\{\begin{array}{ll} 1 & \text{if } y = 1, \\ 0 & \text{if not .} \end{array}\right.$$

For that, we discretize the system using nonconforming finite elements, we consider a non-structured mesh of a square unit and a dual mesh associated to the primal mesh (see Figure 3(a)). Moreover, the initial pressure is neglected and $dt = 0.0005$ is chosen as a constant step in time. The characteristics of the mesh are given in Table 1 and the velocity evolution in time is clearly shown in Figures 3(c) and 3(d). We observe the rotational motion of the fluid and its attachment to the upper wall.

5.2.2. *Anisotropic chemotaxis in a driven cavity.* We are now interested in the dynamics of anisotropic behavior of chemotactic cells, modeled by (5.1) system, in a driven cavity. For this, we consider the following tensors:

$$S = \left[\begin{array}{cc} 8 & -7 \\ -7 & 20 \end{array}\right],\; M = Id\,.$$

Simulations of this test is carried out on the mesh given in Figure 3(a) and initial conditions are defined by regions in Figure 4(a). The initial density is defined by $N_0(x,y) = 0.1$ in the square $(x,y) \in \big([0.6,0.7]\times[0.6,0.7]\big)$. The initial concentration of chemo-attractant is defined by $C_0(x,y) = 20$ in the square $(x,y) \in \big([0.25,0.35]\times[0.25,0.35]\big)$. The initial velocity and pressure are supposed neglected. Then, we choose $dt = 0.0005$, $\alpha = 0.01$, $\beta = 0.05$, $D = 0.001$, $c = 0.1$, $c_1 = 1$, $c_2 = 0$, $d = 2\times 10^{-4}$, $\nu = 5\times 10^{-3}$ and $\nabla\phi = (0,1)$. In the Figure 4, we remark the evolution in time of the cell density and the profils of the chemo-attractant and the velocity of the fluid. At time $t = 0.5$, the anisotropic diffusion starts according to the tensor S. Then, they are influenced and transported by the velocity field at time $t = 4$. At a certain time and under the influence of chemical signals, a part of cells are attracted to the area of chemoattractant and the other part remains carried by the fluid.

5.3. **Anisotropic chemotaxis in an oblique fluid.** Motivated by the dynamics of the cell population in a fluid that carries both the cells and chemical, we consider the system (5.1) with nonzero constants c_1 and c_2. First, we consider the following tensors:

$$S = \left[\begin{array}{cc} 1 & 0 \\ 0 & 5 \end{array}\right],\; M = Id\,.$$

Simulations of this test are done on the mesh given in Figure 3(b) and initial conditions are defined by regions in Figure 5(a). The initial density is defined by $N_0(x,y) = 0.1$ in the square $(x,y) \in \big([0.45,0.45]\times[0.45,0.45]\big)$ and 0 otherwise. The initial concentration of chemo-attractant is defined by $C_0(x,y) = 20$ in the square $(x,y) \in \big([0.2,0.3]\times[0.2,0.3]\big) \cup \big([0.7,0.8]\times[0.7,0.8]\big)$. The initial components of the velocity are defined by $u_1(x,y) = 5$ and $u_2(x,y) = 5$ in the square $(x,y) \in \big([0.1,0.6]\times[0.1,0.6]\big)$. Next, we choose $dt = 0.0005$, $\alpha = 0.01$, $\beta = 0.05$, $D = 0.001$, $d = 10^{-3}$, $\nu = 5\times 10^{-3}$ and $\nabla\phi = (0,10)$. As a first case, we assume that convection fluid is greater than the attraction by the chemical. For this, we select $1 = c < c_1 = 4$ and $c_2 = 10^{-4}$. Therefore, we observe in Figure 5 that cells are quickly transported by the fluid to the upper region of chemoattractant before their convection to the lower region. For the second case, we consider $c = 4$ and $c_1 = c_2 = 0.01$. We remark in this last case (Figure 6) that first the cells are attracted to the lower area of the chemoattractant to swim together in the fluid towards the upper region of chemical substrates.

Meshes	Number of triangles	Number of diamonds	$\max(diam(D))$	$\max(diam(K))$
Figure 3(a)	224	352	3.75×10^{-3}	5.47×10^{-3}
Figure 3(b)	3584	5440	2.3×10^{-4}	3.41×10^{-4}

TABLE 1. Characteristics of the meshes .

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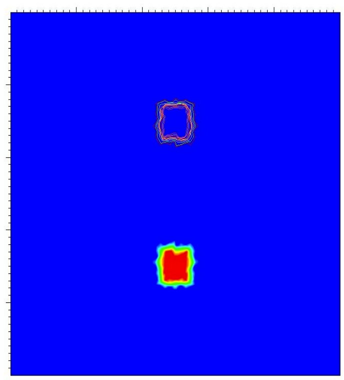
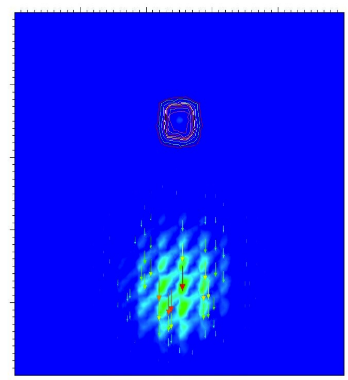
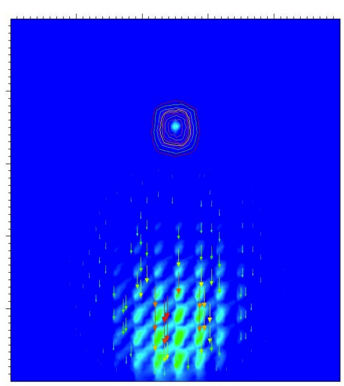


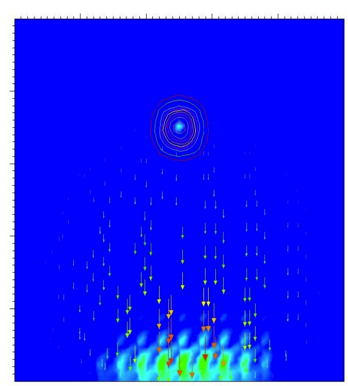
(a) Conditions initiales: $N_0(x,y) = 0.5$ pour $(x,y) \in \{0.45, 0.55\} \times \{0.65, 0.75\}$ et $C_0(x,y) = 10$ pour $(x,y) \in \{0.45, 0.55\} \times \{0.25, 0.35\}$.



(b) $0 \leq N(t = 0.75) \leq 0.2329$, $0.87 \leq C(t = 0.75) \leq 8.72$.



(c) $0 \leq N(t = 1.5) \leq 0.1974$, $0.8220 \leq C(t = 1.5) \leq 8.220$.



(d) $0 \leq N(t = 4) \leq 0.3141$, $0.6039 \leq C(t = 4) \leq 6.039$.

FIGURE 2. Test 0- Effect of the gravitational force ($\nabla\phi = (0, 100)$).

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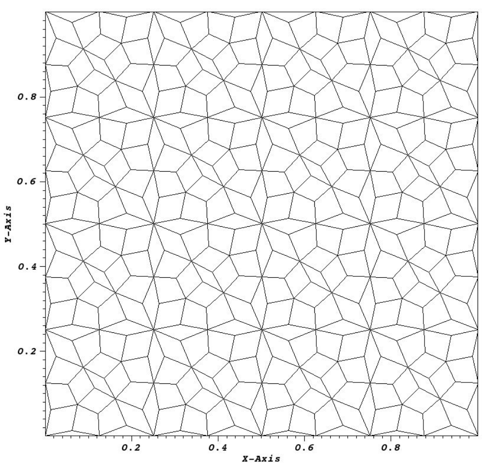
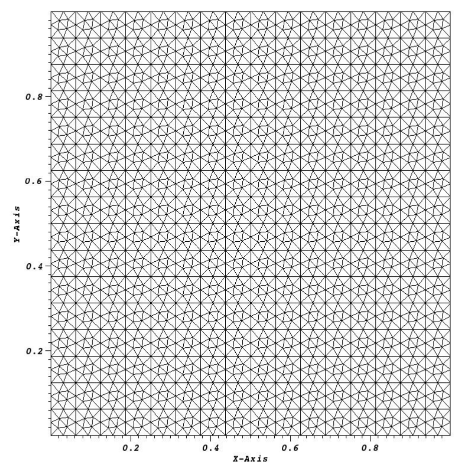
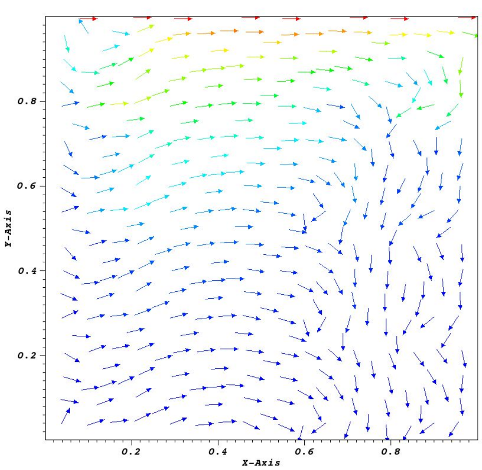


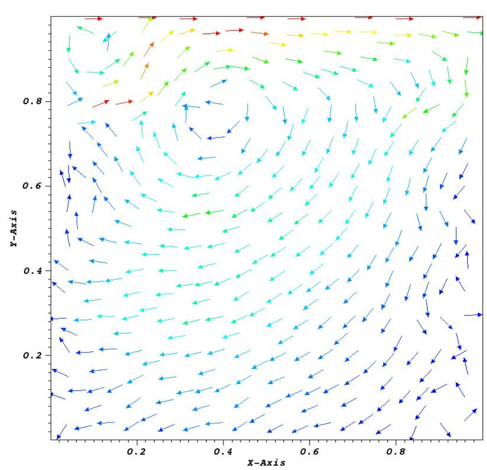
(a) Mesh of the space domain.



(b) Mesh of type Boyer (3584 triangles).



(c) Velocity field at time $t = 20$.



(d) Velocity field at time $t = 50$.

FIGURE 3. Test 2- Driven Cavity.

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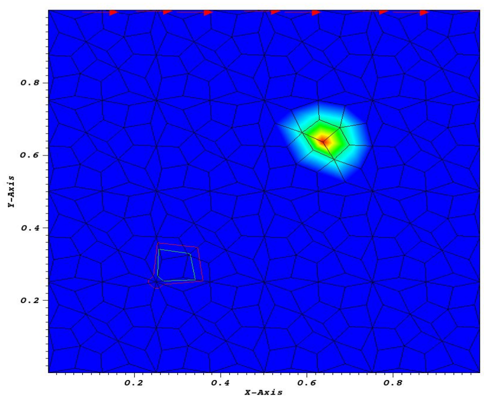
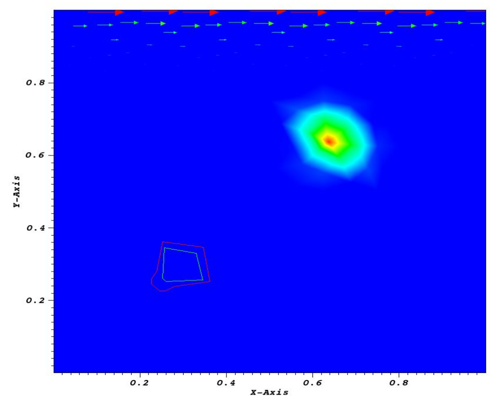
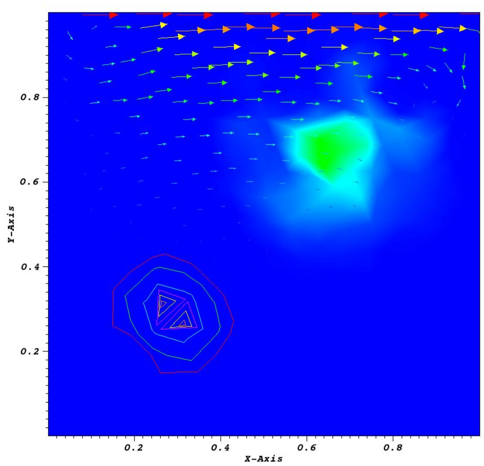


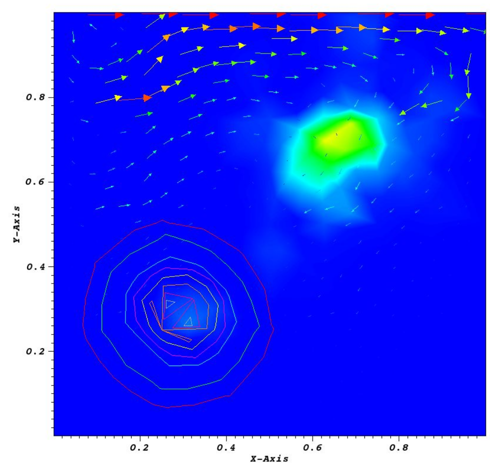
(a) Initial conditions.



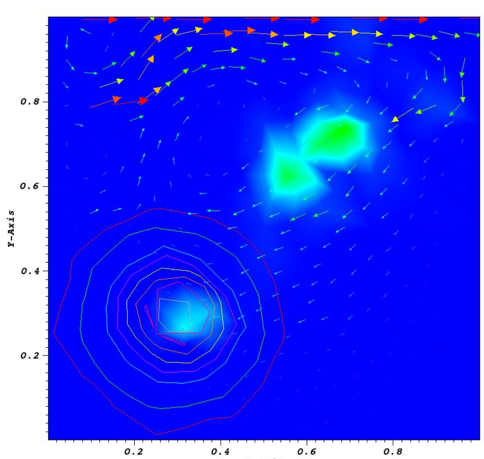
(b) $0 \leq N(t = 0.5) \leq 0.077$.



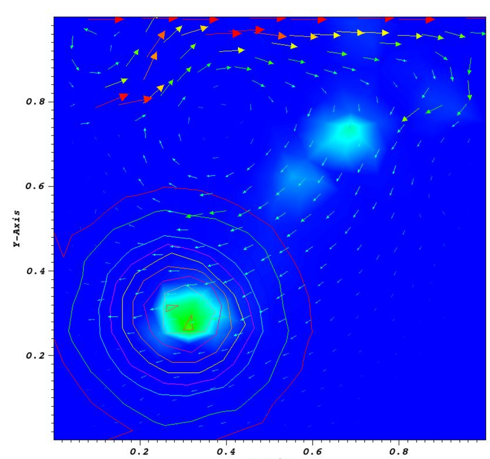
(c) $0 \leq N(t = 4) \leq 0.01$.



(d) $0 \leq N(t = 10) \leq 0.034$.

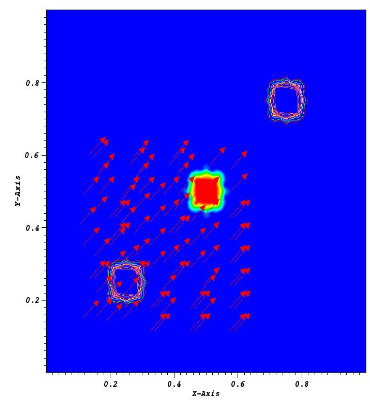


(e) $0 \leq N(t = 25) \leq 0.028$.

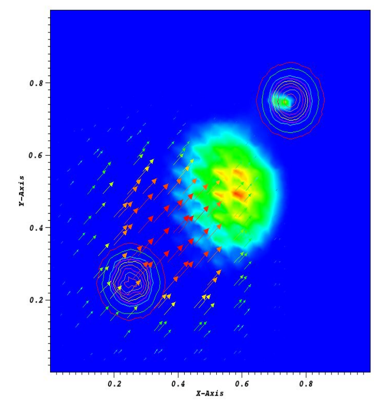


(f) $0 \leq N(t = 35) \leq 0.036$.

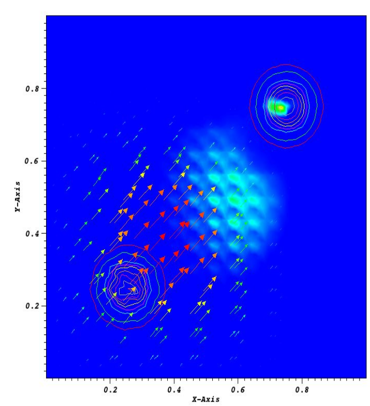
FIGURE 4. Test 2- Evolution in time of cell density in a driven cavity.



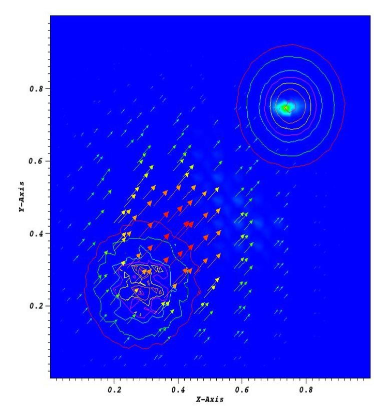
(a) Test 3- Initial conditions.



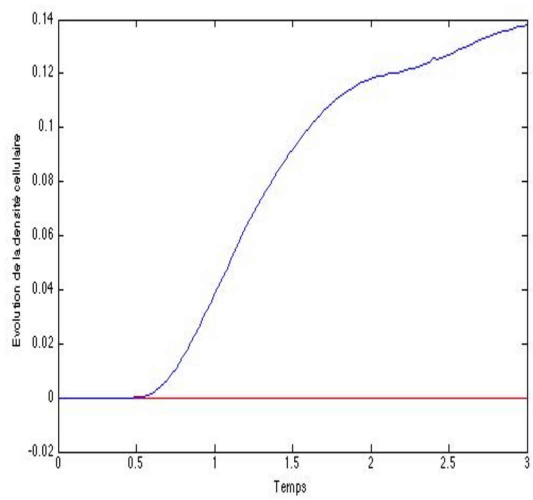
(b) $0 \leq N(t = 0.75) \leq 0.021, \quad 0.411 \leq C(t = 0.75) \leq 5.671$.



(c) $0 \leq N(t = 1) \leq 0.06, \;\; 0 \leq C(t = 1) \leq 4.537$.

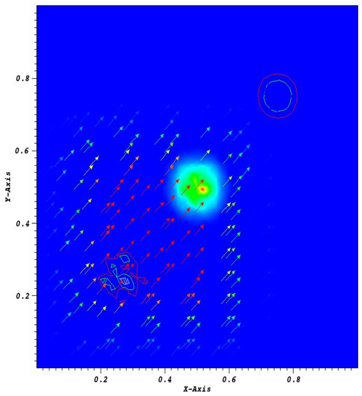


(d) $0 \leq N(t = 3) \leq 0.44, \; 0 \leq C(t = 3) \leq 2.334$.

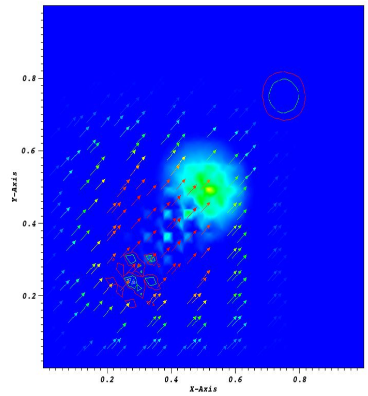


(e) Cell density evolution in time at point $(0.26, 0.29)$ (red curve) and at point $(0.71, 0.82)$ (blue curve).

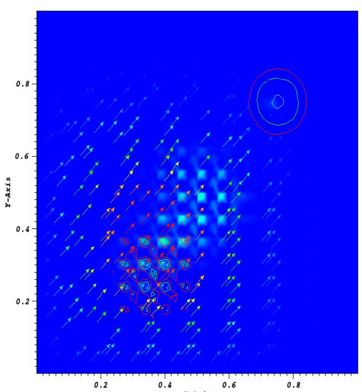
FIGURE 5. Test 3- First case ($c < c_1$).



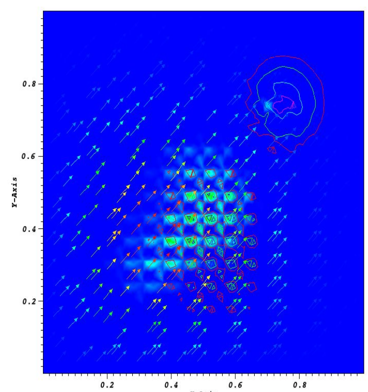
(a) $0 \leq N(t = 0.25) \leq 0.12, \quad 0 \leq C(t = 0.5) \leq 27.34.$



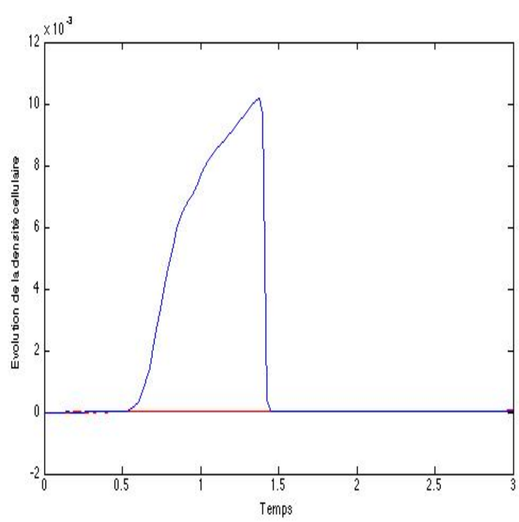
(b) $0 \leq N(t = 0.5) \leq 0.06, \quad 0 \leq C(t = 0.75) \leq 21.16.$



(c) $0 \leq N(t = 1) \leq 0.17, \quad 1.08 \leq C(t = 1) \leq 10.4.$



(d) $0 \leq N(t = 2) \leq 0.08,\ 0.51 \leq C(t = 3) \leq 5.13.$



(e) Cell density evolution in time at point $(0.26, 0.29)$ (red curve) and at point $(0.71, 0.82)$ (blue curve).

FIGURE 6. Test 3- Second case ($c < c_1 = c_2$).