HOMOTOPY GROUPS OF SOME EMBEDDING SPACES

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Based on the joint work with Peter Teichner (MPIM Bonn)
https://arxiv.org/abs/2105.13032

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Motivation

Spaces of embeddings

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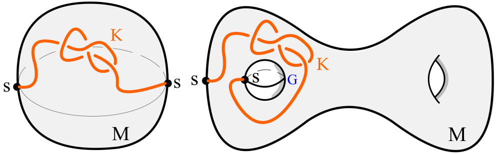

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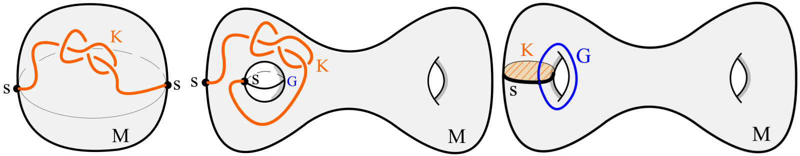

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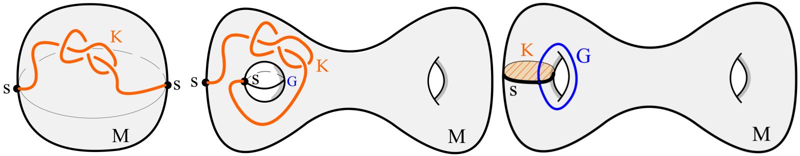

- For $V = \mathbb{D}^k$, the setting with a dual: if there exists $G\colon \mathbb{S}^{d-k} \hookrightarrow \partial M$, such that G has trivial normal bundle and $G \pitchfork \mathbf{s} = \{pt\}$. Like pictures 2 and 3!

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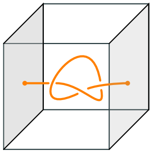

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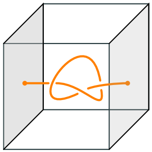

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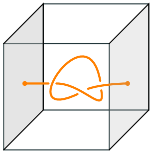

- Recently, intensively studied is the set of (long) 2-knots in a 4-manifold M:

$$\pi_0 \,\mathrm{Emb}_\partial(\mathbb{D}^2, M)$$

  This can be huge – for example, "spinning" a classical knot gives a 2-knot in $\pi_0 \,\mathrm{Emb}_\partial(\mathbb{S}^2, \mathbb{R}^4) \cong \pi_0 \,\mathrm{Emb}_\partial(\mathbb{D}^2, \mathbb{D}^4)$.

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Recall that setting with a dual means: we have a d-manifold M and embedding $\mathbf{s} = \partial \mathrm{U} \colon \mathbb{S}^{k-1} \hookrightarrow \partial M$, such that there exists $G \colon \mathbb{S}^{d-k} \hookrightarrow \partial M$ with trivial normal bundle and such that $G \pitchfork \mathbf{s} = \{pt\}$.

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For any $1 \leq k \leq d$, in a setting with a dual, any choice of $\mathrm{U}\colon \mathbb{D}^k \hookrightarrow M$ leads to an (explicit) homotopy equivalence

$$\mathsf{Emb}_\partial(\mathbb{D}^k, M) \simeq \Omega\, \mathsf{Emb}^\varepsilon_\partial(\mathbb{D}^{k-1}, X).$$

where $X := M \cup_{\nu G} h^{d-k+1}$.

Superscript ε means embedded disks are equipped with "push-offs"...

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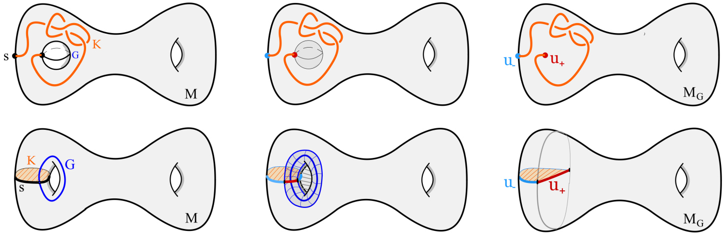

The main result today, and applications

How to compute homotopy groups?

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- We make this more explicit, and compute many classes of examples in K' 21.

The rest of the talk: We give some applications of the two theorems, and then discuss Theorem 2.

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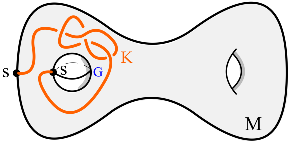

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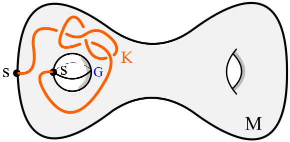

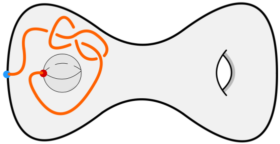

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- Moreover, we get an (unexpected) group structure on $\pi_0\,\mathrm{Emb}_\partial(\mathbb{D}^2, M)$! It is usually nonabelian!

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$\implies$ We classify isotopy classes of 2-disks in 4-manifolds in the setting with a dual.

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Open problem

Is $\pi_0\,\mathrm{Diff}^+_\partial(\mathbb{D}^4)$ trivial? Compute it.

See Budney-Gabai, Gay, Watanabe for some candidate diffeomorphisms.

Metastable homotopy groups

A generic smooth immersion $V^\ell \looparrowright X^d$ has transverse self-intersections only of multiplicity $n \leq \frac{d}{d-\ell}$.

Stable, metastable, meta^{2}stable...(?)

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  - Use homotopy theoretic tools to study $\mathrm{P}_n(V, X)$.

About the lowest degree in the metastable range

- Therefore, part 1) in Theorem 2, which said

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    It turns out this is given as the image of a certain homomorphism

    $$\mathsf{dax}\colon \pi_{d-\ell}X \to \mathbb{Z}[\pi_1 X \setminus 1].$$

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About the lowest degree in the metastable range

Theorem [Dax '72]

There is an isomorphism $\pi_{d-2\ell-1}(\mathrm{Imm}(V,X), \mathrm{Emb}(V,X), u) \cong \Omega_0(\mathcal{C}_u; \theta_u)$, the degree 0 normal bordism group of a certain space $\mathcal{C}_u$ with a stable normal bundle θ_u over it.

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i.e. F is smooth and its track

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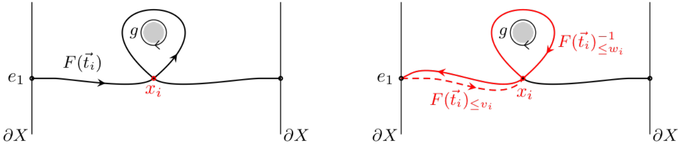

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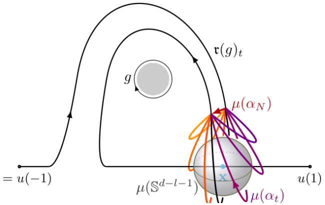

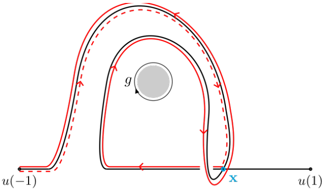

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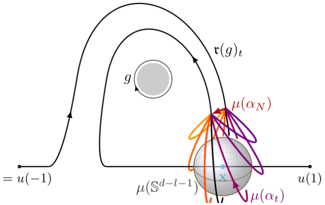
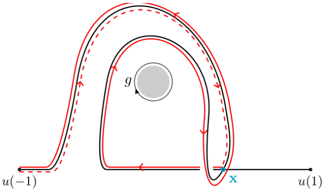


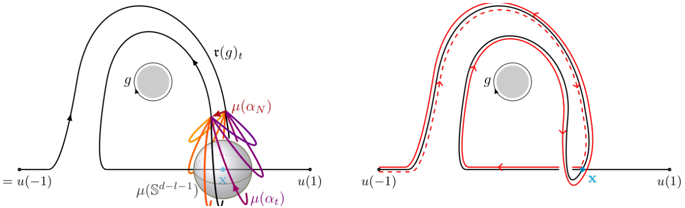


- Finally, for $V = \mathbb{D}^\ell$ we can describe $\mathsf{im}(\delta_{\mathsf{Imm}})$ as $\langle 1\rangle \oplus \mathsf{im}(\mathsf{dax})$ where

$$\mathsf{dax}\colon \pi_{d-\ell}X \to \mathbb{Z}[\pi_1 X \setminus 1], \quad \mathsf{dax}(a) = \mathsf{Dax}(\widetilde{A}),$$

where we represent $a \in \pi_{d-\ell}X$ by a map $A\colon \mathbb{I}^{d-2\ell} \times \mathbb{D}^\ell \to X$.

- Moreover, the inverse of Dax can be made explicit: for $g \in \pi_1 X \setminus 1$ the relative homotopy class $\partial\mathfrak{r}(g)$ is given by



- Finally, for $V = \mathbb{D}^\ell$ we can describe $\mathsf{im}(\delta_{\mathsf{Imm}})$ as $\langle 1 \rangle \oplus \mathsf{im}(\mathsf{dax})$ where

$$\mathsf{dax}\colon \pi_{d-\ell} X \to \mathbb{Z}[\pi_1 X \setminus 1], \quad \mathsf{dax}(a) = \mathsf{Dax}(\widetilde{A}),$$

where we represent $a \in \pi_{d-\ell} X$ by a map $A\colon \mathbb{I}^{d-2\ell} \times \mathbb{D}^\ell \to X$.
- We can compute this in many classes of examples! See [K '21].

Thank you!