I: Chromatic basics

contains everything: ccs, $H\mathbb{F}_p$!, algebra...

$\exists X$ ...

$\mathcal{S}p$ model category with w.eq. = π_*-isos, cofibs + fibs

goal: study stable homotopy category $SHC = Ho(\mathcal{S}p)$

$\rightsquigarrow$ invert "W-equivalences" instead of π_*-isos

e.g. E_*-isos, rational htpy theory

break up into more manageable pieces: CHT

machinery for inverting:

W class of maps in $\mathcal{S}p$:

- X is <u>W-local</u> if $\forall f: A \to B$ in W, $f^*: [B, X] \xrightarrow{\cong} [A, X]$
- $f: C \to D$ <u>W-equivalence</u> if $f^*: [D, X] \xrightarrow{\cong} [C, X]$ $\forall X$ W-local.

<u>N.B:</u> $W \subseteq W\text{-equivalences} \underset{=}{\cancel{\subseteq}} W\text{-}(W\text{-equivalences})$ | W-acyclic: $[Z, X] = 0$ $\forall$ local X

also, π_*-isos $\subseteq$ W-equivalences

<u>Def:</u> W-local model structure on $\mathcal{S}p$ (if exists): $\boxed{L_W \mathcal{S}p}$

weak equiv. = W-equivalences

cofibs = old cofibs

fibrations = what they have to be $\rightsquigarrow$ form $\underbrace{Ho(L_W \mathcal{S}p)}_{W\text{-local } SHC}$

<u>Properties</u> (if exists)

- $\mathrm{id}: \mathcal{S}p \rightleftarrows L_W \mathcal{S}p : \mathrm{id}$ Quillen adjunction
- fibrant replacement gives $\underset{W\text{-equivalence}}{X \longrightarrow} \underbrace{L_W X}_{\text{local object}}$
- universal property:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ L_W X \arrow[ur, dashed] & \end{tikzcd}$$ $\leftarrow$ W-local

need W to be a set, find suitable set J_E

$\boxed{\text{Good news:}}$ For E a homology theory, $L_E \mathcal{S}p$ with w.eq. $= E_*$-isos $= \pi_*(E \wedge \_)$-isos exists!

<u>Examples:</u> $E = H\mathbb{Q}$: $S^0 \longrightarrow L_{H\mathbb{Q}} S^0 = H\mathbb{Q}$ $\leftarrow$ something with $\pi_*(X)$ already rational

π_*-iso after smashing with $H\mathbb{Q}$ and $H\mathbb{Q}$ detects $H\mathbb{Q}$-isos

PTO

X $H\mathbb{Q}$-local $\Leftrightarrow$ $[B, X] \xrightarrow{\cong} [A, X]$ for $A \to B$ $H\mathbb{Q}$-equiv.

$\Leftrightarrow$ $[B, X \wedge H\mathbb{Q}] \cong [A, X \wedge H\mathbb{Q}]$ as $X = X \wedge H\mathbb{Q}$

$\Leftrightarrow$ $[B \wedge H\mathbb{Q}, X] \cong [A \wedge H\mathbb{Q}, X]$ $\leftarrow$ true as $A \wedge H\mathbb{Q} \to B \wedge H\mathbb{Q}$ w.eq.

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Example p-localisation: $E = H\mathbb{Z}_{(p)}$

(similar to $H\mathbb{Q}$) $L_{H\mathbb{Z}_{(p)}} X =: X_{(p)} = X \wedge H\mathbb{Z}_{(p)}$

$X \to X \wedge H\mathbb{Z}_{(p)}$ is $H\mathbb{Z}_{(p)}$-iso + $X_{(p)} \to X_{(p)} \wedge H\mathbb{Z}_{(p)}$ π_*-iso

$\pi_*(X_{(p)}) = \pi_*(X) \otimes \mathbb{Z}_{(p)}$ (universal coefficient s. ex. seq.)

in particular, $S^0_{(p)} = H\mathbb{Z}_{(p)}$, $X_{(p)} = X \wedge S^0_{(p)}$ ⇝ useful to study SHC and $\pi_* S^0$

Def: A localisation is smashing if $L_E X = X \wedge L_E S^0$.

- smashing localisations preserve compact objects
- $L_E S^0$ is a compact generator of $Ho(L_E Sp)$
- L_E commutes with coproducts

⋮ and much more

Example p-completion $E = H\mathbb{Z}/p$

$L_{H\mathbb{Z}/p} X =: X^\wedge_p$

If $\pi_n(X)$ is finitely generated, then $\pi_n(X^\wedge_p) = \pi_n(X) \otimes \mathbb{Z}^\wedge_p$

$(S^0)^\wedge_p = Map(\Sigma^{-1} H(\mathbb{Z}/p^\infty), S^0)$ ⇝ not smashing.

Bousfield squares

pullbacks

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p X^\wedge_p \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & L_{\mathbb{Q}}(\prod_p X^\wedge_p) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & X_{(P_1)} \arrow[d] \\ X_{(P_2)} \arrow[r] & L_{\mathbb{Q}} X \end{tikzcd}$$

In algebra, this is "enough" - in homotopy theory, one can do more!

⇝ chromatic homotopy theory

MU complex cobordism

$MU_* = \mathbb{Z}[x_1, x_2, \dots]$ $|x_i| = 2i$ "universal"

$MU_{(p)} = \bigvee_{\dots} \Sigma^{\dots} BP$, $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ $|v_i| = 2p^i - 2$

⇝ from now on, everything is p-local

...nson-Wilson spectra $E(n)$ with $E(n)_* = \mathbb{Z}[v_1, \dots, v_n, v_n^{-1}]$
... Morava-K-theories $K(n)$ with $K(n)_* = \mathbb{Z}/p[v_n, v_n^{-1}]$

$\boxed{E(n):}$ start with $BP \overset{\text{kill hpy}}{\rightsquigarrow} BP\langle n\rangle_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$

$\overset{\text{invert } v_n^{-1}}{\rightsquigarrow} E(n) = \operatorname{colim}\left(BP\langle n\rangle \xrightarrow{v_n} \Sigma^{-|v_n|} BP\langle n\rangle \xrightarrow{v_n} \dots \right)$

$\boxed{K(n):}$ $BP \overset{\text{kill hpy}}{\rightsquigarrow} k(n)_* = \mathbb{Z}/p[v_n] \overset{\text{invert}}{\rightsquigarrow} K(n) = \operatorname{colim}(k(n) \xrightarrow{v_n} \dots)$

Convention: $E(0) = K(0) = H\mathbb{Q}$

$\boxed{n=1:}$ K-theory $\overset{?}{\rightsquigarrow} E(1)$ $\quad E(1)_* = \mathbb{Z}_{(p)}[v_1^{\pm 1}]$

$\qquad\qquad\quad\ \overset{?}{\rightsquigarrow} K(1)$ $\quad K(1)_* = \mathbb{Z}/p[v_1^{\pm 1}]$

X is KU-local $\Leftrightarrow$ X is KO-local

exact triangle $\Sigma KO \xrightarrow{\eta \wedge 1} KO \longrightarrow KU \longrightarrow \Sigma^2 KO$

$\Rightarrow$ $KO \wedge X = 0 \Rightarrow KU \wedge X = 0$

Assume $KU \wedge X = 0 \Rightarrow \Sigma KO \wedge X \xrightarrow{\eta \wedge 1 \wedge X} KO \wedge X$ iso

but $\eta^4 = 0$, so $KO \wedge X = 0$.

$\Rightarrow L_{K_{(p)}} = L_{KO_{(p)}} = L_{KU_{(p)}} = L_{E(1)} =: L_1$

[Adams]: $K^*_{(p)}(X) = \bigoplus_{0 \le i \le p-2} \Sigma^{2i} G^*(X)$ "Adams summand" $G = E(1)$

$KU_{(p)*} = \mathbb{Z}_{(p)}[\beta^{\pm 1}] \underset{\text{at } p=2}{\overset{\sim}{=}} E(1)_* \Rightarrow KU_{(2)} = E(1)$

$|\beta| = 2$

$L_{K(1)} X = L_{E(1) \wedge M\mathbb{Z}/p} X = (L_{E(1)} X)^\wedge_p$

Recall: need set J_E s.th. J_E-equivalences = E_*-isos

For K-theory, just need one map.

$M = M\mathbb{Z}/p$ has a v_1-selfmap $v_1 : \Sigma^{2p-2} M \longrightarrow M \quad (p>2)$

$v_1^4 : \Sigma^8 M \longrightarrow M \quad (p=2)$

c] X $E(1)$-local $\iff$ $[M,X]_* \xrightarrow{(v_1)^4} [M,X]_{*+2p-2}$ iso

$\rightsquigarrow L_1 X = L_{\{v_1\}} X$

Iso: K-localisation is smashing, i.e. $L_1 X = X \wedge L_1 S^0$
$E(1)$'

$\rightsquigarrow \pi_* L_1 S^0 = ?$

Relationship between $E(1)$ and $E(0)$:
If X is rational, then it is K-local (obvs, not the other way.)

$L_1 X = L_1 L_{H\mathbb{Q}} X = X \wedge L_{H\mathbb{Q}} S^0 \wedge L_1 S^0 = X \wedge L_{H\mathbb{Q}} S^0 = X$ $\blacksquare$

<u>What about the higher n?</u>

$n=2 \rightsquigarrow$ elliptic cohomology theories

but in general, the interaction between the levels / $K(n)$ / $E(n)$ is interesting in itself.

~~smashing~~ smashing? $L_n := L_{E(n)}$ is smashing [Ravenel]

$L_{K(n)}$ is not smashing: take a $K(n)$-local spectrum E s.th. $L_{H\mathbb{Q}} E \neq *$.

Assume $L_{K(n)}$ was smashing: $E = E \wedge L_{K(n)} S^0$

$L_{K(n)} H\mathbb{Q} = H\mathbb{Q} \wedge L_{K(n)} S^0 \simeq *$

$\rightsquigarrow 0 \neq \pi_*(E \wedge H\mathbb{Q}) = \pi_*(E \wedge \underbrace{L_{K(n)} S^0 \wedge H\mathbb{Q}}_{= *}) = 0$ $\lightning$

(Does such an E exist? yes - $E = E_n$.)

Landweber exactness M_* BP_*-module $\rightsquigarrow$ $\forall n$: $(p, v_1, v_2, \ldots, v_n)$ regular sequence for M, i.e. v_i not zero-divisor for $M/(p, \ldots, v_{i-1})M$

explicit algebraic conditions s.th.

$\rightsquigarrow$ $M_*(X) := BP_*(X) \otimes_{BP_*} M_*$ is a homology theory.

$E(n)_*$ is Landweber exact

$K(n)_*$ is not.

Künneth iso $K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y) \cong K(n)_*(X \wedge Y)$ ($K(n)$ graded field)

but nothing like that for $E(n)$.

nce [Hopkins-Smith]

$f: X \longrightarrow Y$ smash nilpotent $\Leftrightarrow$ $K(n)_* f = 0$, $0 \le n \le \infty$

f nilpotent $\Leftrightarrow$ $K(n)_* f = 0$, $0 \le n < \infty$, $(K(\infty) = H\mathbb{F}_p)$

periodicity Are there any maps on a spectrum X that never die?

Let n be the largest integer s.th, $K(m)_*(X) = 0$, $m < n$

$\Rightarrow$ X has a v_n-self map $\alpha: \Sigma^d X \longrightarrow X$, i.e.

- $K(n)_* \alpha$ is mult. by v_n^k for some k
- $K(m)_* \alpha = 0$ for $m > n$.

tion between $E(n)$ and $K(n)$

Theorem [Ravenel] $L_n = L_{(K(0)\vee K(1)\vee \dots \vee K(n))}$

Corollary $E(n+1)_*(X)=0 \Rightarrow E(n)_*(X)=0$

X is $E(n)$-local $\Rightarrow$ X is $E(n+1)$-local (remember: rational $\Rightarrow$ K-local)

$\rightsquigarrow$ nat. trf. $L_{n+1} \to L_n$

$E(n)_*(X)=0 \Rightarrow K(n)_*(X)=0$

X $K(n)$-local $\Rightarrow$ X $E(n)$-local $\rightsquigarrow$ nat. trf. $L_n \to L_{K(n)}$

$\rightsquigarrow$ homotopy pullback square

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] & L_{K(n)} X \arrow[d] \\ L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X \end{tikzcd}$$

Chromatic convergence [Ravenel]

X p-localisation of finite CW-spectrum

$\Rightarrow$ $X \simeq \operatorname{holim}(L_0 X \leftarrow L_1 X \leftarrow L_2 X \leftarrow \dots\dots)$

X needs to be finite: $L_n HG = HG_{\mathbb{Q}}$

Thick Subcategory Theorem

$\mathcal{F}$ thick subcat. of triangulated category $\mathcal{T}$: full subcat. closed under Δ and retracts.

The nontrivial thick subcat. of $Ho(\mathcal{S}p_{(p)})^{\omega}$ are the

$\mathcal{F}_n = \{X \text{ finite } p\text{-local},\ K(n-1)_*(X)\} = 0.$

$\rightsquigarrow$ 'atomic pieces'

more generally:

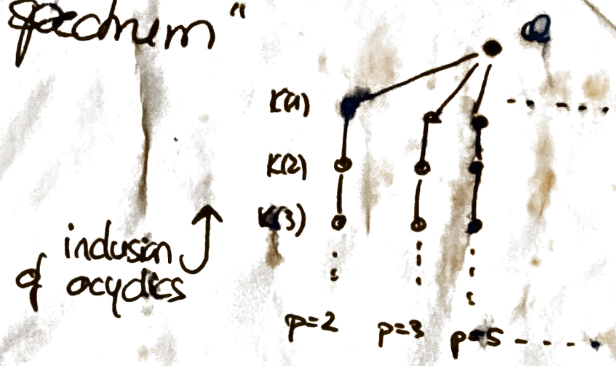
$\mathcal{T}$ $\otimes$-category (tensor-triangulated, e.g. $Ho(\mathcal{C})$ $\leftarrow$ stable monoidal model cat)

$\mathcal{F}$ thick subcategory is an ideal if $X \in \mathcal{T}$, $Y \in \mathcal{F} \Rightarrow X \wedge Y \in \mathcal{F}$

prime ideal: $X \wedge Y \in \mathcal{F} \Rightarrow X \in \mathcal{F}$ or $Y \in \mathcal{F}$

the $K(n)$-acyclics, for each p, form the thick prime ideals of $Ho(Sp)^\omega$ 6

$\rightsquigarrow$ "Balmer spectrum"



G compact Lie

can study Balmer spectrum of other tt-categories, e.g. $Ho(G\text{-}Sp)^\omega$:

$$\mathcal{P}(H, p, n) = \{ X \in Ho(G\text{-}Sp)^\omega \mid K(n)_*(\Phi^H(X)) = 0 \}$$

subgp of G — prime — geometric fixed points

<u>return to chromatic square:</u>

$$\begin{array}{ccc} L_n S^0 & \longrightarrow & L_{K(n)} S^0 \\ \downarrow & & \downarrow \\ L_{n-1} S^0 & \longrightarrow & L_{n-1} L_{K(n)} S^0 \end{array}$$

$L_{K(n)} S^0\text{-mod}$
$\downarrow$

$\rightsquigarrow$ $L_n S^0\text{-mod} \underset{Q}{\simeq} (L_{n-1} Sp \longrightarrow L_{n-1} L_{K(n)} S^0\text{-mod})$

$= L_n Sp$

with homotopy limit model structure [Balchin - Greenlees]

$$C_0 \xrightarrow{F_0} C_{01} \xleftarrow{F_1} C_1$$ $\leftarrow$ col. objects

s.th. in htpy category $F_0(X_0) \cong F_1(X_1)$

- rigidity + exotic objects
- adelic rigidity

In general,

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \Lambda_w \mathcal{C} \\ \downarrow & & \downarrow \\ L_w \mathcal{C} & \longrightarrow & (\ldots) \end{array}$$

OR

<u>blue shift</u> G finite group, K, H p-groups, $s = \log_p(|H/K|)$

$\rightsquigarrow \mathcal{P}(K,p,n+s) \subseteq \mathcal{P}(H,p,n)$

What is the minimal i with $\mathcal{P}(K,p,n+i) \subseteq \mathcal{P}(H,p,n)$?

"n^{th} blue shift"

<u>$L_{K(2)}S^0$</u>

$$\ldots \to KO_{(2)} \xrightarrow{\psi^3 - 1} KO_{(2)} \longrightarrow L_1 S^0 \longrightarrow \Sigma KO_{(2)} \to \ldots$$

Adams op (+ $\underbrace{\text{complete version}}$)

$$\left(\ldots \to K_{(p)} \xrightarrow{\psi^5 - 1} K_{(p)} \longrightarrow L_1 S^0 \longrightarrow \Sigma K_{(p)} \to \ldots \right)$$

$\rightsquigarrow$ can calculate $\pi_* L_1 S^0$ from l.ex. seq. and ASS

Morava-E-theories / Lubin-Tate spectra:

E_n $K(n)$-local spectrum with $(E_n)_* = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$

$\circlearrowleft \mathbb{G}_n$ Morava stabilizer group $\qquad |u_i| = 0 \quad |u| = -2$

$\rightsquigarrow$ cofiber sequence $L_{K(1)} S^0 \simeq E_1^{h\mathbb{G}_1} \longrightarrow \underbrace{E_1^{hC_2}}_{= KO_2^\wedge} \longrightarrow E_1^{hC_2}$

[Goerss et al]

$$L_{K(2)} S^0 \longrightarrow E_2^{hG_{24}} \longrightarrow \ldots \longrightarrow \ldots \longrightarrow \ldots \longrightarrow \Sigma^{48} E_2^{hG_{24}}$$

5 terms s.th. $L_{K(2)} S^0 = \lim(\ldots)$ $\qquad (p=3)$

$\rightsquigarrow$ extending this range [Goerss-Henn, Bezh, Stojanoska, Bobkova]

Chromatic square

$$\begin{tikzcd} L_n S^0 \arrow[r] \arrow[d] & L_{K(n)} S^0 \arrow[d] \\ L_{n-1} S^0 \arrow[r] & L_{n-1} L_{K(n)} S^0 \end{tikzcd}$$

[Balchin-Greenlees] ~~Take $\mathcal{K} \subseteq \mathcal{C}$ set of compact objects~~

$$L_n Sp = L_n S^0\text{-mod} \underset{\text{Quillen}}{\simeq} \left(\begin{tikzcd} & L_{K(n)} S^0\text{-mod} \arrow[d] \\ L_{n-1} S^0\text{-mod} \arrow[r] & L_{n-1} L_{K(n)} S^0\text{-mod} \end{tikzcd} \right)$$

with homotopy limit model structure

$$\mathcal{C}_0 \xrightarrow{F_0} \mathcal{C}_{01} \xleftarrow{F_1} \mathcal{C}_1$$

cofibrant objects: $F_0(X_0) \cong F_1(X_1)$ in htpy cat

$\mathcal{K}$ set of compact objects:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \Lambda_{\mathcal{K}} \mathcal{C} \arrow[d] \\ L_{\mathcal{K}} \mathcal{C} \arrow[r] & (\ldots.) \end{tikzcd}$$

if $\mathcal{C} = \mathbb{1}\text{-mod}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \Lambda_{\mathcal{K}} \mathbb{1}\text{-mod} \arrow[d] \\ L_{\mathcal{K}} \mathbb{1}\text{-mod} \arrow[r] & L_{\mathcal{K}} \Lambda_{\mathcal{K}} \mathbb{1}\text{-mod} \end{tikzcd}$$

Rigidity questions $\mathcal{C} \simeq \mathcal{D} \Rightarrow Ho(\mathcal{C}) \cong Ho(\mathcal{D})$, but not necessarily "$\Leftarrow$". $\rightsquigarrow$ $Ho(\mathcal{C})$ "rigid"

Ex: $Ho(K(n)\text{-mod}) \underset{\Delta}{\simeq} \mathcal{D}(K(n)_*\text{-mod})$

[Schwede] $\mathrm{Ho}(\mathcal{S}p) \underset{\Delta}{\cong} \mathrm{Ho}(\mathcal{D}) \implies \mathcal{S}p \underset{QE}{\simeq} \mathcal{D}$

[R] $(p=2)$ $\mathrm{Ho}(L_1 \mathcal{S}p) \cong \mathrm{Ho}(\mathcal{D}) \implies L_1 \mathcal{S}p \underset{QE}{\simeq} \mathcal{D}$

$(p>2)$: not true [Franke, Patchkoria – Pstrągowski]

Idea: $\mathrm{Ho}(\mathcal{S}p) = \mathrm{End}(S^0)\text{-mod}$

$\mathrm{Ho}(\mathcal{D}) = \mathrm{End}(X)\text{-mod}$

compact gen.

$\pi_* \mathrm{End}(S^0) = \pi_* \mathrm{End}(X)$

$\implies (\dots)$ $\mathrm{End}(X) = S^0$

and K-locally $\mathrm{End}(X) = L_1 S^0$

$\rightsquigarrow$ difficulties for $\mathrm{Ho}(L_2 \mathcal{S}p)$

but: <u>Pre-Theorem</u> [Balchin - R - Williamson]

(tt-rigid: ask for tt-equivalence $\mathrm{Ho}(\mathcal{C}) \cong \mathrm{Ho}(\mathcal{D})$)

unitally tt-rigid: — " — , and the QE $F: \mathcal{C} \to \mathcal{D}$ sends unit to unit)

- $L_n \mathcal{S}p$ is unitally tt-rigid $\iff$ $L_{K(i)} \mathcal{S}p$ unitally tt-rigid for $1 \le i \le n$.
- If $L_x \mathcal{C}$ and $\Lambda_x \mathcal{C}$ are unitally tt-rigid, then so is $\mathcal{C}$.