Pointwise Semiclassical Asymptotics for Total Cross Sections in N-body Problems

Didier ROBERT and Xue Ping WANG
Département de Mathématiques, URA CNRS 758
Université de Nantes
44072 Nantes Cedex, FRANCE

Abstract

Under a non-trapping assumption on the energy for the classical Hamiltonian and all the classical sub- Hamiltonians, we establish the pointwise semiclassical asymptotics, for each incident direction, of total cross sections with initial two-cluster channels in N-body problems.

1 INTRODUCTION

The total scattering cross section is a physically important quantity in quantum scattering theory, because it is observable in laboratories. The semi-classical asymptotics of total scattering cross sections in two-body quantum scattering theory were studied by several authors. See for example [10, 11, 13, 16, 18] and the references quoted therein for earlier works. In a recent work [6], Ito and Tamura studied total cross sections in three-body scattering theory and obtained a leading term, in the sense of distribution in energy and incident directions, for total cross sections with initial two-cluster scattering channels. In this work, we shall give a pointwise semiclassical asymptotic formula for total cross sections in general N-body problems, under a non-trapping condition on the classical Hamiltonian and all classical sub-Hamiltonians.

Let us now introduce some notations. In this work, we treat generalized N-body Schrödinger operators introduced as follows (see Agmon [1]). Let X be an d-dimensional Euclidean space equiped with a positive definite quadratic form $q(\cdot)$. Let $\mathcal{A}$ denote the set of all cluster decompositions of an N-body system labelled by $\{1, 2, \cdots, N\}$. To each $a \in \mathcal{A}$, it is assigned a subspace X_a of X with $X_{a_{\min}} = X$ for some $a_{\min} \in \mathcal{A}$ and $\cap_{a\in\mathcal{A}} X_a = \{0\}$. Let $\mathcal{A}$ be partially ordered by

$$a \subset b \text{ iff } X_b \subset X_a.$$

Assume also that for $a,\ b \in \mathcal{A}$, the union of a and b, $a \cup b \in \mathcal{A}$, is defined so that

$$X_a \cap X_b = X_{a\cup b}.$$

For the definition of $a \cup b$ in physical N-body Schrödinger operators, we refer to [12]. For each a, we denote X^a the orthogonal complement of X_a in X. We write the corresponding orthogonal decomposition of coordinates x as:

$$x = x^a + x_a.$$

With these notations, the N-body Schrödinger operators we are interested in are of the form:

$$P(h) = -h^2\Delta + \sum_{a \in \mathcal{A}} V_a(x^a), \tag{1.1}$$

where $h > 0$ is a small parameter, Δ is the Laplace-Beltrami operator associated with q on X. It is always assumed that $V_a = 0$ if $a = a_{\min}$. To simplified notations, we fix an orthonormal (with respect to q) basis of X so that X is identified with $\mathbf{R}^d$ and Δ is the usual Laplacian on $\mathbf{R}^d$. Unless otherwise stated (see Section 4), we assume throughout this paper that V_a is a smooth short range potential satisfying:

$$|\partial_y^\alpha V_a(y)| \le C_\alpha \langle y \rangle^{-\rho-|\alpha|}, \tag{1.2}$$

for some $\rho > 1$. Here $\langle y \rangle = (1 + |y|^2)^{1/2}$.

For each $a \in \mathcal{A}$, we denote $\#a$ the number of clusters in a and $P^a(h)$ the cluster Hamiltonian:

$$P^a(h) = -h^2\Delta^a + \sum_{b \subseteq a} V_b(x^b),$$

where Δ^a is the Laplacian in x^a-variables. Put

$$I_a(x) = \sum_{b \not\subseteq a} V_b(x^b), \;\; P_a(h) = P^a(h) - h^2\Delta_a,$$

where Δ_a is the Laplacian in x_a-variables. Then one has: $P(h) = P_a(h) + I_a(x)$ for any cluster decomposition a.

Let a be a non trivial cluster decomposition (with the number of clusters $\#a \ge 2$). We denote by $\alpha = (a, E_{j(\alpha)}, \phi_\alpha)$ a scattering channel, where $E_{j(\alpha)}$ is an eigenvalue of $P^a(h)$ and ϕ_α is a normalized eigenfunction associated to $E_{j(\alpha)}$. In the case a is an N-cluster decomposition, we use the convention $\alpha = (a, 0, 1)$. In this paper, we mainly study the semiclassical asymptotics of total cross sections for initial scattering channels α associated with a two-cluster decomposition a and an eigenvalue $E_j \in \sigma_d(P^a(h))$. In this case, ϕ_α is exponentially decreasing in x^a-variables. To define the total cross sections without having to study the scattering amplitudes, we use an idea of [4]. Recall that in two-body case Enss and Simon defined for each given incident direction ω the total scattering cross section as a distribution in energy in the following way (see [4]): Let $g \in C_0^\infty(]0, \infty[)$ and let S be the scattering operator. Then for any $\omega \in \mathbb{S}^{n-1}$ the total scattering section $\sigma(., \omega) \in \mathcal{D}'(\mathbf{R}_+)$ is defined by:

$$\int \sigma(\lambda, \omega)|g(\lambda)|^2 d\lambda = \|(S-1)g_\omega\|^2, \tag{1.3}$$

where $\|.\|$ denotes the norm in L^2 and

$$g_\omega(x) = (2\pi)^{-1/2} \int e^{i\lambda\omega\cdot x} g(\lambda) d\lambda, x \in \mathbf{R}^n.$$

Here $\omega \cdot x$ is the scalar product of ω and x in $\mathbf{R}^n$. g_ω is not in L^2, but one can prove that $\|(S-1)g_\omega\|$ is finite if the potential decays like $O(\langle x \rangle^{-\rho})$ with $\rho > (n+1)/2$. In many body problems, they also defined in a similar way the total cross sections with initial two-cluster scattering channels. One can verify (see Section 2) that, with a suitable rescaling in energy, the total cross section defined in this way is equivalent with the usual one defined from the scattering amplitude. See for example [2, 6, 11]. One of the advantages of this definition is that it makes sense for *each* incident directions. This is particularly useful if we would distinguish some directions. See [15] where the finiteness of total cross sections from an arbitrary channel to a two-cluster channel is studied.

The plan of this work is as follows: In Section 2, we give the definition of total cross sections and state the main result of this work. Then we establish in Section 3 an optical theorem and prove that the total cross section defined as a distribution in λ is in fact a continuous function in (λ, ω) for λ outside the set of thresholds. In Section 4, we compute the classical limit of the bottom of the essential spectrum of $P(h)$, an easy result which we think is of independent interest. In Section 5, we give the proof for the main result. With the semiclassical resolvent estimates established in [14], we can follow the arguments of [11] to produce the leading term in semiclassical limit.

N0TATIONS: In the following, we shall denote $\mathbb{S}_a$ the unit sphere in X_a, $\|\cdot\|_a$ and $<.,.>_a$ the norm and scalar product in $L^2(X_a)$.

2 THE MAIN RESULT

Let $\alpha = (a, E_{j(\alpha)}, \phi_\alpha)$ be an arbitrary scattering channel with $E_{j(\alpha)} \in \sigma_d(P^a)$. In the case $\#a = N$, we put $E_{j(\alpha)} = 0$ and $\phi_\alpha = 1$. We write $P(h)$ as:

$$P(h) = -h^2\Delta_a + P^a(h) + I_a(x). \tag{2.1}$$

We note: $\mathbf{I}_\alpha =]E_{j(\alpha)}, \infty[$ and $\mathcal{J}_\alpha : L^2(X_a) \to L^2(X)$ the channel identification $(\mathcal{J}_\alpha f)(x) = f(x_a)\phi_\alpha(x^a)$. Then under the assumption (1.2) for $\rho > 1$, the channel wave operators

$$W_\alpha^\pm = \text{s-}\lim_{t\to\pm\infty} e^{itP(h)}e^{-itP_a(h)}\mathcal{J}_\alpha, \text{ on } L^2(X_a)$$

exist and are complete ([12]). Put $S_{\beta\alpha} = W_\beta^{+*}W_\alpha^- : L^2(X_a) \to L^2(X_b)$, where β is a channel associated with the cluster decomposition b. Let $\mathbb{S}_a$ be the unit sphere in X_a. For any $\omega \in \mathbb{S}_a$, we define the total cross section $\sigma_{\beta\alpha}(\lambda, \omega; h)$ from a two cluster channel α to any channel β by

$$\int \sigma_{\beta\alpha}(\lambda, \omega)|g(\lambda)|^2 d\lambda = \|(S_{\beta\alpha} - \delta_{\beta\alpha})g_\omega\|_b^2, \tag{2.2}$$

for any $g \in C_0^\infty(\mathbf{I}_\alpha)$. Here $g_\omega(x_a) = \hat{g}(x_a\cdot\omega)$ ($x_a\cdot\omega$ being the scalar product of x and ω induced by q which is now the standard one in $\mathbf{R}^d$) and

$$\hat{g}(\mu) = \frac{1}{2(\pi h)^{1/2}} \int e^{i\mu n(\lambda)/h} \frac{g(\lambda)}{n(\lambda)^{1/2}} d\lambda$$

with $n(\lambda) = \sqrt{(\lambda - E_{j(\alpha)})}$. Note that our definition is slightly different from that in [4] and the normalizing constant is chosen so that $\|\hat{g}\| = \|g\|$.

REMARK 1. The definition (2.2) is equivalent with the usual one used, for example, in [2, 6, 10, 11, 13, 16]. In [4], Enss and Simon gave an argument of scattering into cones to justify this point. See [2] for more details on scattering into cones. Here we present another argument to justify this fact. Let $T_{\beta\alpha}(\lambda)$ denote the scattering matrices from α to β:

$$T_{\beta\alpha}(\lambda) = F_\beta(\lambda) T_{\beta\alpha} F_\alpha(\lambda)^*,$$

where $T_{\beta\alpha} = S_{\beta\alpha} - \delta_{\beta\alpha}$ and $F_\alpha(\lambda)$, $F_\beta(\lambda)$ are the spectral representation for the cluster Hamiltonians $P_a \mathcal{J}_\alpha$ and $P_b \mathcal{J}_\beta$, respectively. Then the total cross section $\sigma_{\beta\alpha}(\lambda, \omega)$ (with $h = 1$) is usually defined by (c.f. [2]):

$$\sigma_{\beta\alpha}(\lambda, \omega) = N_1(\lambda) \int_{\$_b} |t_{\beta\alpha}(\lambda; \omega', \omega)|^2 d\omega', \tag{2.3}$$

where $t_{\beta\alpha}(\lambda; ., .)$ is the kernel of $T_{\beta\alpha}(\lambda)$ and $N_1(\lambda)$ is a constant of normalization. To see that the definition (2.2) is equivalent with (2.3), we calculate, at a formal level, the decomposition of g_ω in the spectral representation of $P_a \mathcal{J}_\alpha$:

$$\begin{aligned} (F_\alpha(\lambda) g_\omega)(\lambda, \theta) &= C_\alpha(\lambda) \int e^{-in(\lambda)\theta \cdot x_a} g_\omega(x_a) dx_a \\ &= C_\alpha(\lambda) \int\int e^{i(n(\mu)\omega - n(\lambda)\theta)\cdot x_a} \frac{g(\mu)}{2\pi^{1/2} n(\mu)^{1/2}} d\mu dx_a \\ &= N_\alpha(\lambda) \delta(\omega - \theta) g(\lambda). \end{aligned}$$

Here $C_\alpha(\lambda)$ and $N_\alpha(\lambda)$ only depend on λ. This gives, again at a formal level,

$$\begin{aligned} \|T_{\beta\alpha} g_\omega\|^2 &= \int \|T_{\beta\alpha}(\lambda) F_\alpha(\lambda) g_\omega\|^2_{L^2(\$_b)} \, d\lambda \\ &= \int N_\alpha^2(\lambda) |g(\lambda)|^2 \int_{\$_b} | \int_{\$_a} t_{\beta\alpha}(\lambda; \omega', \theta) \delta(\theta - \omega) d\theta|^2 \, d\omega' d\lambda \\ &= \int N_2(\lambda) |g(\lambda)|^2 \int_{\$_b} |t_{\beta\alpha}(\lambda; \omega', \omega)|^2 \, d\omega' d\lambda. \end{aligned}$$

Therefore, (2.2) is formally equivalent with (2.3) within a normalization in energy. In two- or three-body problems, our definition coincides with that used in [6, 11] with the same constant of normalization.

Let us single out that g_ω is not in L^2. To make a sense out of (2.2) we need to study the limit: $\lim_{R\to\infty} \|(S_{\beta\alpha} - \delta_{\beta\alpha}) h_R g_\omega\|_b^2$, where $h_R = h_R(x_a)$ is an appropriate family of cut-off functions in $x_a - (x_a \cdot \omega)\omega$ variables. For example, we can take as in [4]

$$h_R(x_a) = e^{-(x_a^2 - (x_a\cdot\omega)^2)/R}.$$

If α is a non-threshold two-cluster scattering channel, it is proved that the right hand side of (2.2) is finite if (1.2) is satisfied with $\rho > (n_a + 1)/2$, where $n_a = \dim X_a$. See [4].

Let now α be a two-cluster channel with non-threshold energy. The total scattering cross section $\sigma_\alpha(\lambda, \omega; h)$ for an initial two-cluster channel α is obtained by summing up $\sigma_{\beta\alpha}$ over all final channels β:

$$\int \sigma_\alpha(\lambda, \omega) |g(\lambda)|^2 d\lambda = \sum_\beta \|(S_{\beta\alpha} - \delta_{\beta\alpha}) g_\omega\|_b^2, \tag{2.4}$$

for any $g \in C_0^\infty(\mathbf{I}_\alpha)$. We shall show in the next Section that σ_α defined as distribution on $\mathbf{I}_\alpha$ is in fact a continuous function in $(\lambda, \omega) \in (\mathbf{I}_\alpha \setminus \mathcal{T}) \times S_a$. Here $\mathcal{T}$ is the set of all thresholds: $\mathcal{T} = \cup_a \sigma_{pp}(P^a)$. Thus we can study the pointwise asymptotics of $\sigma_\alpha(\lambda, \omega; h)$ as $h \to 0$ for (λ, ω) in the above set. To do this we need as in [11] a non-trapping condition.

Let a be a non trivial cluster decomposition. We denote by p and p^a the h-symbol of $P(h)$ and $P^a(h)$, respectively. We shall say $\lambda > 0$ is a non-trapping energy for a classical Hamiltonian q if

$$\lim_{t\to\infty} |e^{tH_q}(x, \xi)| = +\infty, \text{ for any } (x, \xi) \in q^{-1}(\lambda). \tag{2.5}$$

Here H_q is the Hamiltonian vector field of the classical Hamiltonian q. In this work, we shall use the following:

$$\lambda \text{ is non trapping for } p \text{ and all sub-Hamiltonians } p^a. \tag{2.6}$$

Let a be a k-cluster decomposition of the $N-$body system with $k = 1, 2, \cdots, N-1$. We define Σ_{cl}^a by

$$\Sigma_{cl}^a = \min_{b\subset a, \#b=k+1} \inf_x \{\sum_{c\subseteq b} V_c(x^c)\}. \tag{2.7}$$

When $\#a = N-1$, we have $\Sigma_{cl}^a = 0$. With these notations the main result of this work can be stated as follows:

THEOREM 2.1 *Let a be a 2-cluster decomposition. Denote $n_a = dim X_a$. Assume*

$$\text{for any } b \in \mathcal{A},\ V_b \text{ satisfies (1.2)} \tag{2.8}$$

for some $\rho > (n_a+1)/2$. Let $\alpha = \alpha(h)$ be a family of 2-cluster scattering channels: $\alpha = (a, E_{j(\alpha)}(h), \phi_\alpha(h))$ with $E_{j(\alpha)}(h) \le \Sigma_{cl}^a - \epsilon_0$ for some $\epsilon_0 > 0$ uniformly in $h > 0$. Let $\lambda_0 > 0$ satisfy the non-trapping condition (2.6). Then there exist a small interval $\mathbf{I}$ around λ_0 and some $\delta_0 > 0$ such that for any $\omega \in \mathbb{S}_a$ and $\lambda \in \mathbf{I}$, one has

$$\sigma_\alpha(\lambda, \omega; h) = 4\int_{\Pi_\omega} \sin^2\{\frac{1}{4hn(\lambda)} \int_{\mathbf{R}} I_0(u_a + s\omega) ds\} du_a + O(h^{-(n_a-1)/(\rho-1)+\delta_0}), \tag{2.9}$$

where Π_ω is the hyperplane in X_a defined by $\Pi_\omega = \{u_a \in X_a; u_a \cdot \omega = 0\}$. Here $I_0(x_a) = I_a(x)|_{x^a=0}$ and the remainder estimate is uniform in $\omega \in \mathbb{S}_a$ and $\lambda \in \mathbf{I}$.

REMARKS 2. (a). In physical situations, one has $n_a \equiv dim X_a = 3$ if $\#a = 2$ and if the position of each particle is described by a point in $\mathbf{R}^3$. So our result holds in this case if (2.8) is satisfied with $\rho > 2$.

(b).In a recent work ([6]), Ito and Tamura established a semiclassical asymptotic formula for $\sigma_\alpha(\lambda, \omega; h)$ in the sense of distributions in $\mathbf{R}_+ \times \mathbb{S}^2$ (with $n_a = 3$) for potential satisfying (1.2) with $\rho > 5/2$. Remark that they do not need the non-trapping condition (2.6). But for pointwise semiclassical asymptotics, the necessity of (2.6) will become clear in Section 5. See the result in [14].

(c). The assumption $E_{j(\alpha)}(h) < \Sigma_{cl}^a - \epsilon_0$ in Theorem 2.1 is only used to ensure that

$$\int \langle x^a \rangle^{N_0} |\phi_\alpha(h)|^2 \, dx^a \le C, \tag{2.10}$$

for some large N_0, uniformly in $h > 0$. The result of Theorem 2.1 remains true for two-cluster channels α satisfying (2.10). This is in particular the case if $a = (C_1, C_2)$,

$E_{j(\alpha)} = E^{C_1}(h) + E^{C_2}(h)$, with $E^{C_j}(h) < \Sigma_{cl}^{C_j} - \epsilon_0, j = 1, 2$, uniformly in $h > 0$. See lemma 4.1.

(d). If $I_a(0, x_a) = \Phi(x_a/|x_a|)|x_a|^{-\rho} + O(|x_a|^{-\rho-\epsilon_0})$, $\epsilon_0 > 0$, as $|x_a| \to +\infty$, we can prove as in Remark 1.3 of [6] that $\sigma_\alpha(\lambda, \omega; h) = c_0(\lambda, \Phi) h^{-(n_a-1)/(\rho-1)}(1 + O(h^{\epsilon'}))$, $\epsilon' > 0$. Here $c_0(\lambda, \Phi) \neq 0$ if $\Phi \not\equiv 0$.

3 CONTINUITY OF TOTAL CROSS SECTION

In this Section, $h > 0$ is fixed. Let α be a two-cluster scattering channel: $\alpha = (a, E, \phi_\alpha)$ with ϕ_α satisfying (2.10). Then for any $b \not\subseteq a$, $\#b \neq N$, one has:

$$|x^a| + |x^b| \geq c|x|, \ \forall x \in X.$$

Consequently, under the condition (2.8), one has:

$$\langle x \rangle^r I_a(x) \phi_\alpha \in L^2(X),$$

for any $1/2 < r < \rho - n_a/2$. We shall frequent use this estimate.

In this Section, we shall prove that $\sigma_\alpha(.,.;h)$ is a continuous function on $\mathbf{R}_+ \times \mathbb{S}_a$. To simplify notations, we denote: $n = n_a$, where $n_a = dim X_a$. The following result is important for studying the pointwise semiclassical asymptotics of total cross sections.

THEOREM 3.1 (Optical Theorem) *Let $g \in C_0^\infty(\mathbf{I}_\alpha)$ with $supp g \cap \mathcal{T} = \emptyset$. Then under the assumption (2.8), one has:*

$$\int \sigma_\alpha(\lambda, \omega; h)|g(\lambda)|^2 d\lambda = \int \frac{1}{hn(\lambda)} \Im < R(\lambda + i0, h) I_a e_\alpha, I_a e_\alpha > |g(\lambda)|^2 d\lambda. \tag{3.1}$$

Here $R(\lambda \pm i0, h)$ are the boundary values of the resolvent $(P(h) - z)^{-1}$ and e_α is defined by:

$$e_\alpha(x, \lambda, \omega; h) = \phi_\alpha(x^a) e^{in(\lambda) x_a \cdot \omega / h},$$

with $n(\lambda) = (\lambda - E)^{1/2}$. In particular, $\sigma_\alpha(\lambda, \omega; h)$ defined by (2.3) is a continuous function:

$$\sigma_\alpha(\lambda, \omega; h) = \frac{1}{hn(\lambda)} \Im < R(\lambda + i0, h) I_a e_\alpha, I_a e_\alpha >, \tag{3.2}$$

for $(\lambda, \omega) \in (\mathbf{I}_\alpha \setminus \mathcal{T}) \times \mathbb{S}_a$.

Proof. Let h_R be the cut-off function introduced in Section 2. As in [4], we can give a sense to $\|(S_{\beta\alpha} - \delta_{\beta\alpha}) g_\omega\|_b$ by defining it as:

$$\|(S_{\beta\alpha} - \delta_{\beta\alpha}) g_\omega\|_b = \lim_{R \to \infty} \|(S_{\beta\alpha} - \delta_{\beta\alpha}) h_R g_\omega\|_b.$$

Then (2.3) is interpreted as:

$$\int \sigma_\alpha(\lambda, \omega; h)|g(\lambda)|^2 d\lambda = \sum_\beta \lim_{R \to \infty} \|T_{\beta\alpha} h_R g_\omega\|_b^2$$

where $T_{\beta\alpha} = S_{\beta\alpha} - \delta_{\beta\alpha}$. Now $h_R g_\omega$ is in $L^2(X_a)$. By the completeness of wave operators, we have:

$$\sum_\beta \|T_{\beta\alpha} h_R g_\omega\|_b^2 = -2\Re < T_{\alpha\alpha} h_R g_\omega, h_R g_\omega >_a . \tag{3.3}$$

By the orthogonality of channel wave operators, one has:

$$T_{\alpha\alpha} = W_\alpha^{+*}(W_\alpha^- - W_\alpha^+).$$

Since ϕ_α is rapidly decreasing in x^a, we can check the following equalities for any u, $v \in \mathcal{S}(X_a)$ such that the Fourier transforms of u and v are supported away from 0:

$$< T_{\alpha\alpha} u, v >_a = - < W_\alpha^{+*} \int U(t)^* i I_a U_a(t) \mathcal{J}_\alpha u, v >_a dt$$

$$= - \int < \mathcal{J}_\alpha^* U_a(t)^* i W_\alpha^{+*} I_a U_a(t) \mathcal{J}_\alpha u, v >_a dt$$

$$= - \int < i \mathcal{J}_\alpha^* U_a(t)^* \{ I - i \int_{\mathbf{R}_+} U_a(s)^* I_a U(s) ds \} I_a U_a(t) \mathcal{J}_\alpha u, v >_a dt.$$

Here $U(t)$, $U_a(t)$ are unitary groups generated by P and P_a, respectively. Applying the above equalities to $u = v = h_R g_\omega$, we obtain:

$$\Re < T_{\alpha\alpha} h_R g_\omega, h_R g_\omega >_a$$

$$= -\Re \int \int_{\mathbf{R}_+} < \mathcal{J}_\alpha^* U_a(t+s)^* I_a U(s) I_a U_a(t) \mathcal{J}_\alpha u, v >_a dt\ ds.$$

Note that $P_a e_\alpha = \lambda e_\alpha$. Under the assumption (2.8), we can verify that $I_a \mathcal{J}_\alpha U_a(t) g_\omega \in L^2(X)$ and by the choice of h_R,

$$\lim_{R\to\infty} I_a \mathcal{J}_\alpha U_a(t) h_R g_\omega = \frac{1}{2(\pi h)^{1/2}} \int e^{-it\lambda} I_a e_\alpha g(\lambda) \frac{d\lambda}{n(\lambda)^{1/2}}, \tag{3.4}$$

in $L^2(X)$. This gives:

$$\begin{aligned} &\Re < T_{\alpha\alpha} g_\omega, g_\omega >_a \\ &= \ -\frac{1}{4\pi h} \int_{\mathbf{R}} \int_{\mathbf{R}_+} \Re < \int_{\mathbf{R}_\lambda} U(s) e^{-it\lambda} f(\lambda) d\lambda, \int_{\mathbf{R}_\mu} e^{-i(t+s)\mu} f(\mu) d\mu > \ dt ds, \end{aligned} \tag{3.5}$$

where

$$f(\lambda) = \frac{1}{\sqrt{n(\lambda)}} I_a e_\alpha(\lambda, \omega) g(\lambda).$$

Applying the generalized Plancherel formula to the integration in t in (3.5), we obtain:

$$\begin{aligned} &\Re < T_{\alpha\alpha} g_\omega, g_\omega >_a \\ &= \ -\frac{1}{2h} \int_{\mathbf{R}_\mu} \int_{\mathbf{R}_+} \Re < e^{is\mu} U(s) I_a e_\alpha(\mu, \omega), I_a e_\alpha(\mu, \omega) > \frac{|g(\mu)|^2}{n(\mu)} ds\ d\mu. \end{aligned} \tag{3.6}$$

By introducing a convergence factor $e^{-\epsilon s}, \epsilon > 0$ in (3.6) and making $\epsilon \to 0$ after having integrated with respect to s, we finally obtain:

$$\begin{aligned} &\Re < T_{\alpha\alpha} g_\omega, g_\omega >_a \\ &= \ -\frac{1}{2h} \int_{\mathbf{R}_\mu} \Re i^{-1} < R(\mu + i0, h) I_a e_\alpha(\mu, \omega), I_a e_\alpha(\mu, \omega) > \frac{|g(\mu)|^2}{n(\mu)}\ d\mu. \end{aligned}$$

This proves that for each $\omega \in S_a$, one has:

$$\sigma_\alpha(\lambda, \omega; h) = \frac{1}{hn(\lambda)} \Im < R(\lambda + i0, h) I_a e_\alpha(\lambda, \omega), I_a e_\alpha(\lambda, \omega) >$$

in the sense of distributions on $\mathbf{I}_\alpha \setminus \mathcal{T}$.

Let $1/2 < s < \rho - n/2$. The maps $\lambda \to \langle x \rangle^{-s} R(\lambda + i0, h) \langle x \rangle^{-s} \in \mathcal{L}(L^2(X))$ and $(\lambda, \omega) \to \langle x \rangle^s I_a e_\alpha(\lambda, \omega) \in L^2(X)$ are Hölder continuous for $(\lambda, \omega) \in (\mathbf{I}_\alpha \setminus \mathcal{T}) \times \mathbb{S}_a$. Therefore $\sigma_\alpha(\lambda, \omega; h)$ extends to a continuous function in this set. $\Box$

Let us single out that $\mathcal{T}$ is h-dependent. But for potentials satisfying (1.2), the positive eigenvalues and thresholds of $P(h)$ are absent. This means: $\mathcal{T} \subset]-\infty, 0]$. So Theorem 3.1 shows in particular that $\sigma_\alpha(\lambda, \omega; h)$ is a continuous function on $\mathbf{R}_+ \times \mathbb{S}_a$ for any $h > 0$. Since $e^{i\lambda\omega\cdot x}$ is a generalized eigenfunction of $-\Delta$ with eigenvalue λ^2, from the computation made in the proof of Theorem 3.1, one can verify that the total cross sections defined in [4] at the energy λ are related to the boundary value of resolvent $R(\lambda^2 + i0)$.

4 BOTTOM OF THE ESSENTIAL SPECTRUM

As another preparation for the proof of the main result, we give here the quantum implication of the classical quantities Σ_{cl}^a. We shall see thatΣ_{cl}^a is equal to the classical limit of the bottom of the essential spectrum for $P^a(h)$ as $h \to 0$. Let us first prove that the eigenfunctions ϕ_α used in Theorem 2.1 are exponentially decreasing in x^a and h. In this Section, we just need a weaker condition on V_{ij}:

$$V_{ij} \text{ is continuous and satisfying for some } \rho' > 0\text{: } |V_{ij}(y)| \le C\langle y \rangle^{-\rho'}. \tag{4.1}$$

LEMMA 4.1 *Let $\alpha = (a, E(h), \phi(h))$ be a family of scattering channels with $E(h) \le \Sigma_{cl}^a - \delta_0$ for some $\delta_0 > 0$. Then for any $\epsilon > 0$, there exist $R_\epsilon > 0$ and a continuous function $d \ge 0$ with*

$$d(x^a) = \begin{cases} 0, & \text{for } |x^a| \le R_\epsilon, \\ (\sqrt{\Sigma_{cl}^a - E(h) - \epsilon})|x^a|, & \text{for } |x^a| > 2R_\epsilon \end{cases}$$

such that

$$h\|\nabla(e^{d/h}\phi(h))\| + \|e^{d/h}\phi(h)\| \le C_\epsilon, \tag{4.2}$$

uniformly in $0 < h \le h_0$.

Proof. We only prove the result in the case $\#a = 1$. In this case, $P^a(h) = P(h)$ and we denote $\Sigma^{cl} = \Sigma_{cl}^a$. The general case can be treated by exactly the same arguments. Let $\Omega_a(\eta) = \{x \in X; |x^a| < \eta|x|, |x^b| > \eta^2|x|, \forall b \not\subset a\}$. Then for $\eta > 0$ small enough, we have a covering for X:

$$X = \cup_{\#a=2} \Omega_a(\eta).$$

For $x \in \Omega_a(\eta)$ with $\#a = 2$, we have:

$$V(x) = \sum_{b \subseteq a} V_b(x^b) + I_a(x)$$

$$\geq \Sigma^{cl} + O(\langle x\rangle^{-\rho}) \geq \Sigma^{cl} - \epsilon,$$

for $|x| > R_\epsilon(a) >> 1$. This proves that

$$V(x) \geq \Sigma^{cl} - \epsilon, \text{ for } |x| \geq R_\epsilon. \tag{4.3}$$

Let d be a smooth function verifying the conditions of Lemma 4.1 and $|\nabla d(x)|^2 \leq \Sigma^{cl} - E(h) - \epsilon$. Then we have the following identity:

$$\begin{aligned}&\int \{h^2|\nabla(e^{d/h}u)|^2 + (V - E(h) - |\nabla d|^2)|e^{d/h}u|^2\}dx\\ &= \Re \int e^{2d/h}((P(h) - E(h))u)\bar{u}dx\end{aligned}$$

for any $u \in H^2(X)$ with compact support. (4.1) follows from (4.3) in approximating ϕ_h by $\chi_R\phi_h$, where $\chi_R \in C_0^\infty$ is a family of appropriate cut-off functions equal to 1 for $|x| < R$. The details are omitted here. See also [1] and [5]. □

Lemma 4.1 shows that the eigenfunctions of $P^a(h)$ associated with eigenvalues below Σ^a_{cl} are exponentially decreasing both in x^a and $h > 0$. This can be regarded as an easy extension of the result of [1] (for $h = 1$) and that of [5] (for x in compact domain). The following result is not needed in this paper. We include it just to show the quantum meaning of Σ^a_{cl}.

LEMMA 4.2 *For any cluster decomposition a with $\#a = 1, 2, \cdots, N-1$, put $\Sigma^a_{ess}(h) = \inf \sigma_{ess}(P^a(h))$. Then we have*

$$\lim_{h\to 0} \Sigma^a_{ess}(h) = \Sigma^a_{cl}, \tag{4.4}$$

where Σ^a_{cl} is defined in Section 2.

Proof. We just prove (4.4) for $P(h)$, since $P^a(h)$ is a many body operator with fewer particles. Denote in this case $\Sigma^{cl} = \min_{\#a\geq 2}\Sigma^a_{cl}$. By the HVZ Theorem (see [9]),

$$\inf \sigma_{ess}(P(h)) = \min_{\#a\geq 2} \inf \sigma(P^a(h)).$$

It is clear that we always have:

$$\inf \sigma(P^a(h)) \geq \Sigma^a_{cl} \equiv \inf_x \sum_{b\subseteq a} V_b(x^b),$$

for any $h > 0$. On the other hand, by the continuity of potentials, for any $\epsilon > 0$ we can find an $x_0 \in X^a$ and a δ -neighbourhood around it such that:

$$\Sigma^a_{cl} > \sum_{b\subseteq a} V_b(x^b) - \epsilon,$$

for x^a in this neighbourhood. Take $\psi \in C_0^\infty(X^a)$ such that $\psi \geq 0$, supp$\psi \subset \{|x^a| < 1\}$ and $\|\psi\| = 1$. Put:

$$\psi_h(x^a) = h^{-N_a/4}\psi((x^a - x_0)/h^{1/2}),$$

where $N_a = dim\ X^a$. Then for $h > 0$ small enough, we have:

$$< P^a(h)\psi_h, \psi_h > \leq Ch + \Sigma^a_{cl} + \epsilon.$$

This proves by the spectral properties of self-adjoint operators:

$$\inf \sigma(P^a(h)) \leq < P^a(h)\psi_h, \psi_h > \leq Ch + \Sigma_{cl}^a + \epsilon.$$

Taking the minimum over a with $\#a \geq 2$, we obtain: $\Sigma^{cl} \leq \inf \sigma_{ess}(P(h)) \leq \Sigma^{cl} + Ch + \epsilon$. The desired result follows easily. $\square$

5 PROOF OF THEOREM 2.1

Our proof for Theorem 2.1 is based on the representation formula (3.2) for $\sigma_\alpha(\lambda, \omega; h)$. From (3.2), it is clear that in order to obtain a pointwise semiclassical asymptotics for $\sigma_\alpha(\lambda, \omega; h)$, we have to control the resolvent $R(\lambda + i0, h)$ with respect to $h > 0$. In this connection, recall that under the assumption (2.6) for λ_0, it is proved in [14] that if the potentials satisfy (1.2) for some $\rho' > 0$, then there exists $\epsilon > 0$ such that for any $\lambda \in \mathbf{I} \equiv [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$ and for any $s > 1/2$, one has:

$$\|\langle x\rangle^{-s} R(\lambda \pm i0, h)\langle x\rangle^{-s}\| \leq C_s h^{-1}, \tag{5.1}$$

for $h > 0$ small and $\lambda \in \mathbf{I}$ and

$$\lambda \rightarrow \langle x\rangle^{-s} R(\lambda \pm i0, h)\langle x\rangle^{-s}$$

is norm-continuous for $\lambda \in \mathbf{I}$. See [10, 11] in 2-body case and [3] in 3-body case. We indicate in particular that the non-trapping assumption on all sub-Hamiltonians is necessary. In fact, it is proved in [14] that the condition (2.6) is equivalent with (5.1). In the following, we shall only use (2.6) in its equivalent form (5.1).

5.1 Bound on Total Cross Sections

The proof of Theorem 2.1 will be divided into several steps. We shall first introduce several localizations and show that some pieces do not contribute to the leading term in semiclassical asymptotics. This allows us in particular to deduce a pointwise bound on $\sigma_\alpha(\lambda, \omega; h)$ at the end of this subsection. In the following, we shall always assume that $\lambda \in \mathbf{I}$ so that (5.1) is satisfied uniformly in λ.

Assume the condition (1.2) with $\rho > (n+1)/2$, $n = \dim X_a$. Fix two positive constants γ and β by

$$\gamma = \frac{1}{\rho - 1}, \ \beta = (1 + \delta)\gamma, \ \delta > 0.$$

In the following $\delta > 0$ will be chosen to be small enough. Introduce an h- dependent partition of unity on X_a:

$$\sum_{j=1}^{3} \chi_j(x_a, h) \equiv 1 \text{ on } X_a$$

such that

- $\text{supp}\chi_1 \subset \{x_a; |x_a| < 2h^{-\gamma}\}$ with $\chi_1 = 1$ if $|x_a| < h^{-\gamma}$;
- $\text{supp}\chi_2 \subset B_{\gamma\beta} \equiv \{x_a; h^{-\gamma} < |x_a| < 3h^{-\beta}\}$ with $\chi_2 = 1$ if $2h^{-\gamma} < |x_a| < 2h^{-\beta}$;

- $\mathrm{supp}\chi_3 \subset \{x_a; |x_a| > 2h^{-\beta}\}$.

We can construct these χ_j so that

$$|\partial_y^\alpha \chi_j(y)| \le C_\alpha \langle y\rangle^{-|\alpha|}, \forall \alpha \in \mathbf{N}^n, \tag{5.2}$$

uniformly in $h > 0$. Recall that $n = \dim X_a$. By Theorem 3.1, one has:

$$\sigma_\alpha(\lambda, \omega; h) = \frac{1}{hn(\lambda)} \sum_{j,k=1}^{3} \sigma_{jk}(\lambda, \omega; h), \tag{5.3}$$

where

$$\sigma_{jk} = \Im < R(\lambda + i0, h)\chi_j I_a e_\alpha, \chi_k I_a e_\alpha > .$$

In the following, all estimates are uniform with respect to $\omega \in \mathbb{S}_a$ and $\lambda \in \mathbf{I}$. In fact, the uniformity in λ follows from the resolvent estimate (5.1) and the uniformity in ω is a consequence of the local uniformity in ω and the compactness of $\mathbb{S}_a$.

LEMMA 5.1 *There exists some $\epsilon_0 > 0$ such that:*

$$\sigma_{jk}(\lambda, \omega; h) = O(h^{1-(n-1)\gamma+\epsilon_0})$$

for $j = 3$ or $k = 3$.

Proof. We only prove the result for σ_{13}. The other terms can be treated in the same way. Recall that e_α is a generalized eigenfunction for $P_a(h)$. We can compute:

$$(P(h) - z)(\chi_1 e_\alpha) = (I_a - i\epsilon)\chi_1 e_\alpha + [\chi_1, h^2\Delta_a]e_\alpha,$$

for $z = \lambda + i\epsilon$. From this, it follows:

$$R(\lambda + i0)(\chi_1 I_a e_\alpha) = \chi_1 e_\alpha + R(\lambda + i0, h)[\chi_1, h^2\Delta_a]e_\alpha. \tag{5.4}$$

Inserting (5.4) into σ_{13} and noting that the piece corresponding to $\chi_1 e_\alpha$ is real, we obtain:

$$\sigma_{13} = \Im < R(\lambda + i0; h)[\chi_1, h^2\Delta_a]e_\alpha, \chi_3 I_a e_\alpha > . \tag{5.5}$$

Let $1/2 < s < \rho - n/2$. Applying Lemma 4.1 and (5.1) and taking notice of the choice of χ_j, we derive the estimate:

$$\begin{aligned} |\sigma_{13}| &\le O(h^{-1})\|\langle x\rangle^s[\chi_1, h^2\Delta_a]e_\alpha\|\|\langle x\rangle^s \chi_3 I_a e_\alpha\| \\ &\le O(h^{-1})O(h^{\gamma(-s+1-n/2)+1})O(h^{\beta(\rho-n/2-s)}). \end{aligned}$$

Since $\gamma(-s+1-n/2) + \beta(\rho - n/2 - s) = 1 - \gamma(n-1) + \gamma(1-2s) + \delta(\rho - n/2 - s)$ and $\delta > 0$, we can choose s close to $1/2$ so that $\gamma(1-2s) + \delta(\rho - n/2 - s) > 0$. The desired result follows. $\square$

By Lemma 5.1, we have:

$$\sigma_\alpha(\lambda, \omega; h) = \frac{1}{hn(\lambda)} \sum_{1\le j,k\le 2} \sigma_{jk}(\lambda, \omega; h) + O(h^{-(n-1)\gamma+\epsilon_0}), \tag{5.6}$$

for some $\epsilon_0 > 0$. We shall use (5.4) to treat the terms related with χ_1. Note that $\Im < [\chi_1, \Delta_a]e_\alpha, \chi_1 e_\alpha >= 0$. So σ_{11} can be written as:

$$\sigma_{11} = \Im < R(\lambda + i0, h)[\chi_1, h^2\Delta_a]e_\alpha, [\chi_1, h^2\Delta_a]e_\alpha > .$$

We can easily compute:

$$[\chi_1, h^2\Delta_a]e_\alpha = \{-i2hn(\lambda)\omega \cdot \nabla_a\chi_1 - h^2(\Delta_a\chi_1)\}e_\alpha.$$

Define f_1 and f_2 by

$$f_1 = -i2hn(\lambda)\omega \cdot \nabla_a\chi_1, \tag{5.7}$$

$$f_2 = \chi_2 I_0 \text{ with } I_0(x_a) = I_a(0, x_a). \tag{5.8}$$

LEMMA 5.2 *Put* $s_{jk} = \Im < R(\lambda+i0, h)f_j e_\alpha, f_k e_\alpha >$. *Then there exists* $\epsilon_0 > 0$ *such that* $\sigma_{jk} - s_{jk} = O(h^{1-(n-1)\gamma+\epsilon_0})$.

Proof. We only prove the lemma for $j = 1$ and $k = 2$. By (5.4), one has:

$$\sigma_{12} - s_{12} = \Im\{< R(\lambda + i0, h)(f_1 + r_1)e_\alpha, r_2 e_\alpha > + < R(\lambda + i0, h)r_1 e_\alpha, f_2 e_\alpha >\}.$$

Here $r_1 = -h^2\Delta_a\chi_1$ and $r_2 = \chi_2(I_a(x) - I_0(x_a))$. Let us estimate $< R(\lambda+i0, h)f_1 e_\alpha, r_2 e_\alpha >$:

$$\begin{aligned} & | < R(\lambda + i0, h)f_1 e_\alpha, r_2 e_\alpha > | \\ = \; & O(h^{-1})\|\langle x\rangle^s f_1 e_\alpha\|\|\langle x\rangle^s r_2 e_\alpha\| \\ = \; & O(h^{-1})O(h^{1+(1-n/2-s)\gamma})O(h^{(\rho+1-s)\gamma-\beta n/2}), \end{aligned}$$

for any $s > 1/2$. Here we have used (5.1) and Lemma 4.1 which gives $\|(I_a - I_0)e_\alpha\|_{L^2(X^a)} = O(\langle x_a\rangle^{-\rho-1})$, uniformly in h. Since $(1 - n/2 - s)\gamma + (\rho + 1 - s)\gamma - \beta n/2 = 1 - (n - 1)\gamma + (2 - 2s - \delta n/2)\gamma$, we can take $s > 1/2$ close to $1/2$ and $\delta > 0$ small enough so that $2 - 2s - \delta n/2 > 0$. The other terms can be estimated similarly. □

From Lemma 5.2 and (5.6), it follows

$$\sigma_\alpha(\lambda, \omega; h) = \frac{1}{hn(\lambda)} \sum_{1\le j,k\le 2} s_{jk}(\lambda, \omega; h) + O(h^{-(n-1)\gamma+\epsilon_0}) \tag{5.9}$$

with s_{jk} given in Lemma 5.2. Note that f_1 and f_2 are supported in $B_{\gamma\beta} = \{x_a \in X_a; h^{-\gamma} < |x_a| < 3h^{-\beta}\}$. Let Π_ω be the hyperplane defined by $x_a \cdot \omega = 0$. Let π^ω the orthogonal projection from X_a onto Π_ω. Decompose for $x_a \in B_{\gamma\beta}$: $x_a = y + (x_a \cdot \omega)\omega$ with $y = \pi^\omega x_a \in \Pi_\omega$. y is called impact parameter in the direction ω. See for example [9]. We shall now distinguish two cases: if $|y|$ is small, we shall show the corresponding contribution is negligible; if $|y|$ is effectively large, the classical trajectories behave like the free ones and in this case we can construct an approximation in terms of free trajectories. Let κ be fixed by

$$\kappa = (1 - 2\delta)\gamma, \tag{5.10}$$

where $\delta > 0$ is a small constant as before. Let $\{\theta_1, \theta_2\}$ be a partition of unity on X_a with $0 \le \theta_j \le 1$ and $\theta_1 = 1$ if $|x_a| \le 1$; 0 if $|x_a| > 2$.

LEMMA 5.3 *With the above notations, put:* $f_{jk}(x_a) = \theta_j(h^\kappa y) f_k(x_a)$ *with* $y = \pi^\omega x_a \in \Pi_\omega$. *Then one has, if* $j = 1$ *or* $l = 1$,

$$< R(\lambda + i0, h) f_{jk} e_\alpha, f_{lm} e_\alpha >= O(h^{1-(n-1)\gamma+\epsilon_0})$$

for some $\epsilon_0 > 0$ *and for* $k, m = 1, 2$.

Proof. We note that for $s > 1/2$ and for all j, k,

$$\|\langle x\rangle^s f_{jk} e_\alpha\| \le C_\epsilon h^{(\rho-s-n/2-\epsilon)\gamma}$$

for any $\epsilon > 0$. If $j = 1$, the volume of the support of f_{1k} is of order $O(h^{-\beta/2-(n-1)\kappa})$. So in this case, $\|\langle x\rangle^s f_{1k} e_\alpha\| = O(h^{(\rho-s)\gamma-\beta/2-(n-1)/2\kappa})$. Since $n \ge 2$, $\beta = (1+\delta)\gamma$ and $\kappa = (1-2\delta)\gamma$, we can verify that

$$(\rho - \frac{n}{2} - s - \epsilon)\gamma + (\rho - s)\gamma - \frac{\beta + (n-1)\gamma}{2} = 2 - (n-1)\gamma + (1 - 2s - \epsilon + (n - \frac{3}{2})\delta)\gamma.$$

If $s - 1/2$ and $\epsilon > 0$ are small enough, $(1 - 2s - \epsilon + (n - \frac{3}{2})\delta)\gamma > 0$. This ends the proof. □

Let now f be one of the functions f_{2k}, $k = 1, 2$. For $x_a \in$supp$f$, one has: $x_a = y + s\omega$ with $|y| > h^{-\kappa}$. We construct now as in [11] an approximation for $R(\lambda + i0, h) f e_\alpha$.

For such an f, we define $F(x_a, t)$ and g by

$$F(x_a, t) = f(x_a - n'(\lambda) t\omega) e^{-ih^{-1}\int_0^t I_0(x_a - n'(\lambda)(t-s)\omega)ds}, \tag{5.11}$$

$$g(x_a) = \int_0^{+\infty} F(x_a, t) dt. \tag{5.12}$$

Here $n'(\lambda) = 2n(\lambda)$. Note that supp$f \subset B_{\gamma\beta} \cap \{|y| > h^{-\kappa}\}$, where $y \in \Pi_\omega$. So for each x_a, the integral in (5.12) is only taken over a finite interval and g is a smooth function. We can verify that *suppg* $\subset \{|y| > h^{-\kappa}\}$ and, for any $\epsilon > 0$, $|g(x_a)| \le C_\epsilon \langle y\rangle^{-(\rho-1-\epsilon)}$ uniformly in h. Let $\chi_4 = \zeta(h^\beta x_a \cdot \omega)$ be a cut-off function in $x_a \cdot \omega$ with support in $\{|x_a \cdot \omega| < M h^{-\beta}\}$ and $\chi_4 = 1$ for $|x_a \cdot \omega| < (M-1) h^{-\beta}$ for some $M >> 1$.

By a direct computation, we obtain:

$$2n(\lambda)(\omega \cdot \nabla_a) g + ih^{-1} I_0 g = f. \tag{5.13}$$

It follows that

$$(P(h) - \lambda)(ih^{-1}\chi_4 g e_\alpha) = \chi_4 f e_\alpha + r e_\alpha,$$

where

$$r = -ih\chi_4(\Delta_a g) + ih^{-1}[-h^2\Delta_a, \chi_4] g + ih^{-1}(I_a - I_0)\chi_4 g.$$

Consequently, taking notice that $\chi_4 = 1$ on supp$f$, we obtain an approximation for $R(\lambda + i0, h) f e_\alpha$:

$$R(\lambda + i0, h)(f e_\alpha) = ih^{-1}\chi_4 g e_\alpha - R(\lambda + i0, h) r e_\alpha. \tag{5.14}$$

Now for each $f = f_{2k}$, $k = 1, 2$, we construct g_k and r_k in the above manner with f, g and r replaced by f_{2k}, g_k and r_k, respectively, so that (5.13) and (5.14) hold.

LEMMA 5.4 *With the above notations, we have:*

$$< R(\lambda + i0, h) f_{2j} e_\alpha, f_{2k} e_\alpha >=< ih^{-1} g_j, f_{2k} >_a + O(h^{1-(n-1)\gamma+\epsilon_0})$$

for some $\epsilon_0 > 0$

Proof. We just estimate the remainder term corresponding to $I_a - I_0$ in r_j. The other terms can be treated as in [11]. Since g is bounded by $O(h^{\kappa(\rho-1-\epsilon)})$, we can derive, making use of Lemma 4.1,

$$\begin{aligned} \|\langle x\rangle^s (I_a - I_0)\chi_4 g e_\alpha\| &\le C\|\langle x_a\rangle^{s-\rho-1}\chi_4 g\|_a \\ &= O(h^{\kappa(2\rho-n/2-s-\epsilon)}), \end{aligned}$$

for any $\epsilon > 0$. It follow from (5.1) that

$$h^{-1} < R(\lambda + i0, h)(I_a - I_0)\chi_4 g e_\alpha, f_{2k} e_\alpha >= O(h^{-2+\kappa(2\rho-n/2-s-\epsilon)+\gamma(\rho-n/2-s-\epsilon)}).$$

The desired result follows if we take $\delta > 0$ and $s - 1/2 > 0$ small enough. □

By (5.9) and Lemmas 5.2-5.4, we obtain

$$\sigma_\alpha(\lambda, \omega; h) = \frac{1}{hn(\lambda)} \Im < ih^{-1}(g_1 + g_2), f_{21} + f_{22} >_a + O(h^{-(n-1)\gamma+\epsilon_0}), \tag{5.15}$$

for some $\epsilon_0 > 0$. As a consequence of (5.15), we obtain by a direct computation the following semiclassical bound on $\sigma_\alpha(\lambda, \omega; h)$:

THEOREM 5.5 *Under the assumptions of Theorem 2.1, we have:* $\sigma_\alpha(\lambda, \omega; h) = O(h^{-(n-1)/(\rho-1)})$ *uniformly in* $\lambda \in \mathbf{I}$ *and* $\omega \in \mathbb{S}_a$.

5.2 Computation of the Leading Term

Now we can complete the proof of Theorem 2.1 in using the representation formula (5.15). It is clear that by the expressions for g_j and f_{2k}, we can compute explicitly the leading term. In order to establish an asymptotic formula with no artificial cut-off functions, we shall reinsert some negligible terms. The arguments are often the same as in [11] and we shall omit the details.

Proof of Theorem 2.1. We first note that by the arguments used in the preceeding subsection, one can show that

$$\Im < ih^{-1}(g_1 + g_1), f_{1j} >_a = O(h^{1-(n-1)\gamma+\epsilon_0})$$

and

$$\Im < ih^{-1}(g_1 + g_2), \chi_3 I_0 >_a = O(h^{1-(n-1)\gamma+\epsilon_0}).$$

Since $f_1 = in(\lambda)\omega \cdot \nabla_a \chi_1$, we can derive from (5.13) by an integration by parts that

$$\sigma_\alpha(\lambda, \omega; h) = \frac{1}{hn(\lambda)} \Im < ih^{-1}(g_1 + g_2), I_0 >_a + O(h^{-(n-1)\gamma+\epsilon_0}). \tag{5.16}$$

Let $\tilde{g}_1, \tilde{g}_2$ be defined by (5.11) and (5.12) with f replaced by $\chi_1 I_0$ and $\chi_2 I_0$. Then as in [11], one can show

$$\sigma_\alpha(\lambda, \omega; h) = \frac{1}{hn(\lambda)} \Im < ih^{-1}(\tilde{g}_1 + \tilde{g}_2), I_0 >_a + O(h^{-(n-1)\gamma+\epsilon_0}).$$

It is clear that $\tilde{g}_j$ also satisfies the equation (5.13) with f replaced by $\chi_j I_0$. So

$$\Im < ih^{-1}(\tilde{g}_1+\tilde{g}_2), I_0 >_a = \Im \int_{X_a} -2n(\lambda)\omega\cdot\nabla(\tilde{g}_1+\tilde{g}_2)dx_a.$$

Making use of the decomposition: $x_a = y + s\omega$ with $y \in \Pi_\omega$ and $s \in \mathbf{R}$, we can calculate:

$$\begin{aligned}
&\Im\{\int_{X_a} -2n(\lambda)\omega\cdot\nabla(\tilde{g}_1+\tilde{g}_2)dx_a\}\\
= \quad & \Im\{\int_{\Pi_\omega}\int_{\mathbf{R}}((\chi_1+\chi_2)I_0)(y+s\omega)e^{\frac{i}{2hn(\lambda)}\int_{-\infty}^{s} I_0(y+\tau\omega)d\tau}dyds\}\\
= \quad & \int_{\Pi_\omega}\int_{\mathbf{R}}((\chi_1+\chi_2)I_0)(y+s\omega)\sin(\frac{1}{2hn(\lambda)}\int_{-\infty}^{s} I_0(y+t\omega)dt)dsdy\\
= \quad & \int_{\Pi_\omega}\int_{\mathbf{R}} I_0(y+s\omega)\sin(\frac{1}{2hn(\lambda)}\int_{-\infty}^{s} I_0(y+t\omega)dt)dsdy + O(h^{1-(n-1)\gamma+\epsilon_0}).
\end{aligned}$$

Theorem 2.1 follows from a direct calculation. □

By the methods of this paper, under the assumption (1.1) with $\rho > n_a$, we can also study the semiclassical asymptotics of forward scattering amplitudes for channel scattering operators $S_{\beta\alpha}$, where α and β satisfy both the assumptions of Theorem 2.1. The details will not be given here.

References

[1] S. Agmon, Lectures on Exponential Decay of Solutions to Second Order Elliptic Equations, Princeton Univ. Press, Princeton, NJ, 1982

[2] W. Amrein, J. Jauch and K. Sinha, Scattering Theory in Quantum Mechanics, W.A. Benjamin, 1977

[3] C. Gérard, Semiclassical resolvent estimates for two and three body Schrödinger operators, Commun. in P.D.E., 15(1990), 1161-1178

[4] V. Enss and B. Simon, Finite total cross section in non relativistic quatum mechanics, Commun. in Math. Phys., 76(1980), 177-209

[5] B. Helffer and J. Sjöstrand, Multiple wells in the semiclassical limit, I., Commun. in P. D. E., 9(1984), 337-408

[6] H.T. Ito and H. Tamura, Semi-classical asymptotics for total scattering cross sections of 3-body systems, preprint 1991

[7] P. Perry, I.M. Sigal and B. Simon, Spectral analysis of N-body Schrödinger operators, Ann. of Math., 114(1981), 519-567

[8] M. Reed and B. Simon, Methods of Modern Mathematical Physics, III. Scattering Theory, Academic Press, New York, 1979

[9] M. Reed and B. Simon, Methods of Modern Mathematical Physics, IV. Analysis of Operators, Academic Press, New York, 1978

[10] D. Robert and H. Tamura, Semiclassical bounds for resolvents of Schrödinger operators and asymptotics for scattering phase, Comm. in P. D. E., 9(1984), 1017-1058

[11] D. Robert and H. Tamura, Semiclassical estimates for resolvents and asymptotics for total scattering cross sections, Ann. Inst. H. Poincaré, 46(1987), 415-442

[12] I.M. Sigal and A. Soffer, The N-particle scattering problem: asymptotic completeness for short range systems, Ann. of Math. 126(1987), 35-108

[13] A.V. Sobolev and D.R. Yafaev, On the quasiclassical limit of the total scattering cross-section in non-relativistic quantum mechanics, Ann. Inst. H. Poincaré, 44(1986), 195-210

[14] X.P. Wang, Semiclassical resolvent estimates for N-body Schrödinger operators, J. of Funct. Analysis, 97(1991), 466-483

[15] X.P. Wang, Total scattering cross sections in N-body problems: Finiteness and high energy asymptotics, preprint 1992, Univ. Nantes

[16] D.R. Yafaev, The eikonal approximation and the asymptotics of the total cross-section for the Schrödinger equation, Ann. Inst. H. Poincaré, 44(1986), 397-425

[17] D.R. Yafaev, Radiation conditions and scattering theory for N-particle Hamiltonians, preprint 1992, Univ. Nantes

[18] K. Yajima, The quasiclassical limit of scattering amplitude - L^2-approach for short-range potentials-, Japan J. Math., 13(1987), 77-126