Asymptotic behaviour of resolvent for N-body Schrödinger operators near a threshold

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Abstract

In this paper, we analyze the spectral properties of N-body Schrödinger operators near its first threshold, the bottom of the essential spectrum, and give the asymptotics of the resolvent in weighted Sobolev spaces.

1 Introduction

Let P denote the N-body Schrödinger operator obtained after the removal of the center of mass from the total energy operator

$$-\sum_{j=1}^{N} \frac{1}{2m_j}\Delta_{x_j} + \sum_{1\le i<j\le N} V_{ij}(x_i - x_j), \quad x_j \in \mathbf{R}^3, \tag{1.1}$$

where x_j and m_j denote the position and mass of the j-th particle, V_{ij} is assumed to be real and relatively compact with respect to $-\Delta$ in $L^2(\mathbf{R}^3)$ and satisfies the decay

$$|V_{ij}(y)| \le C_{ij}|y|^{-\rho} \tag{1.2}$$

for $y \in \mathbf{R}^3, |y| > R$ for some $R > 0$. In this paper we assume at least $\rho > 2$ and ρ varies according to the results to obtain. P is regarded as a self-adjoint operator in $L^2(X)$, where X is the $3(N-1)$ dimensional real vectorial space: $X = \{(x_1, \cdots, x_N) \in \mathbf{R}^{3N}; \sum_{j=1}^N m_j x_j = 0\}$. Let $\mathcal{A}$ denote the set of all cluster decompositions of the N-particle system. For $a \in \mathcal{A}$, let $\#a$ denote the number of clusters in a and P^a the Subhamiltonian associated with a. For $i, j \in \{1, \cdots, N\}$, we write $(ij) \subset a$ if i and j belong to a same cluster in a. If a is a k-cluster decomposition $a = (a_1, \cdots, a_k)$, we denote

$$\mathbf{X}^a = \{x \in \mathbf{X}; \sum_{l\in a_j} m_l x_l = 0, j = 1, \cdots, k\}.$$

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and

$$\mathbf{X}_a = \{x \in \mathbf{X}; x_i = x_j \text{ if } (ij) \subset a_m \text{ for some } m \in \{1, \cdots, k\}\}.$$

$\mathbf{X}^a$ and $\mathbf{X}_a$ give an orthogonal decomposition for $\mathbf{X}$ equiped with the quadratic form

$$q(x) = (\sum_j 2m_j |x_j|^2), \qquad x \in \mathbf{X}.$$

For $x \in \mathbf{X}$, we have the corresponding orthogonal decomposition: $x = x^a + x_a$ with $x^a \in \mathbf{X}^a$ and $x_a \in \mathbf{X}_a$. The N-body Schrödinger operator P introduced above can be written in the form

$$P = P_0 + V(x)$$

where P_0 is the Laplace-Beltrami operator on the Euclidean space $(\mathbf{X}, q)$ and

$$V(x) = \sum_{a \in \mathcal{A}} V_a(x^a)$$

with $V_a(x^a) = V_{ij}(x_i - x_j)$ if a is an (N-1)-cluster decomposition and $(ij) \subset a$, 0 otherwise. Remark that the method of this work can be applied to generalized N-body Schrödinger operators and in this case, the results depends on the dimension of the intercluster space. See [5] and [10] for two-body case.

Let $-\Delta^a$ ($-\Delta_a$, resp.) denote the restriction of P_0 on $\mathbf{X}^a$ (on $\mathbf{X}_a$, resp.). For $a \in \mathcal{A}$, denote

$$P^a = -\Delta^a + \sum_{b \subseteq a} V_b(x^b), P_a = P^a - \Delta_a,$$

$$I_a(x) = \sum_{b \not\subseteq a} V_b(x^b).$$

P^a is the Subhamiltonian associated with the cluster decomposition a and I_a is the sum of all intercluster interactions. The detailed expression of P^a depends on the choice of coordinates on $\mathbf{X}^a$ and for those resulting from (1.1), see formulae given in [16] in some special coordinates. Let

$$\mathcal{T} = \cup_{a \in \mathcal{A}, \# a \geq 2} \sigma_p(P^a)$$

be the set of thresholds of P. The well-known HVZ theorem gives the bottom of the essential spectrum $E_0 \equiv \inf \sigma_{ess}(P)$ of P by the formula

$$E_0 = \min_{a \in \mathcal{A}, \# a \geq 2} \inf \sigma(P^a) = \min_{a \in \mathcal{A}, \# a = 2} \inf \sigma(P^a).$$

Under general conditions, the positive eigenvalues of P are absent and its negative eigenvalues can only accumulate at the thresholds ([7],[12],[15]). Less is known about the spectral properties of P near a threshold (see [6]). The purpose of this paper is to study the spectral properties of P near its first threshold E_0. The following assumption is made throghout this work:

$$\textit{there exists a unique decomposition, } a_0 \in \mathcal{A}, \textit{ such that } E_0 = \inf \sigma(P^{a_0}). \tag{1.3}$$

For $N \geq 3$, by the HVZ Theorem, the assumption (1.3) implies that $\# a_0 = 2$, E_0 is in the discrete spectrum of P^{a_0} and there is $\epsilon_0 > 0$ such that

$$\inf \sigma(P^a) \geq E_0 + \epsilon_0, \quad \text{for } a \in \mathcal{A}, \# a \geq 2, a \neq a_0.$$

Let $\varphi_0 \in D(P^{a_0})$ be an eigenfunction of P^{a_0}

$$P^{a_0} \varphi_0 = E_0 \varphi_0, \quad \|\varphi_0\| = 1. \tag{1.4}$$

Let $H^{m,s}$, $m, s \in \mathbf{R}$, be the weighted Sobolev space on $\mathbf{X}$ equiped with the norm

$$||u||_{m,s} = (\int_{\mathbf{X}} |(1+|x_{a_0}|^2)^{s/2}(1-\triangle_x)^{m/2}u|^2 dx)^{1/2}.$$

Note that we use only the weight in intercluster variables x_{a_0} which is weaker than the weight usually used in the limiting absorption principle of P. Sometimes, we denote also by the same symbol $H^{m,s}$ the weighted Sobolev space on $\mathbf{X}_{a_0}$ equiped with the norm

$$||u||_{m,s} = (\int_{\mathbf{X}_{a_0}} |(1+|x_{a_0}|^2)^{s/2}(1-\triangle_{a_0})^{m/2}u|^2 dx_{a_0})^{1/2}.$$

Let $<\cdot,\cdot>$ ($(\cdots,\cdot)_a, <\cdot,\cdot>_a$, resp.) denote the scalar product on $L^2(\mathbf{X})$ (on $L^2(\mathbf{X}^a)$, $L^2(\mathbf{X}_a)$, resp.).

In this paper, we identify $H^{1,-s}, s \geq 0$, to the dual space of $H^{-1,s}$ with the usual L^2 product as the dual product. Let $\mathcal{L}(m,s;m',s') = \mathcal{L}(H^{m,s}, H^{m',s'})$. It will be convenient to regard P as an operator in $\mathcal{L}(1,0;-1,0)$.

Definition. *We say that E_0 is a* resonance *of P if the equation $Pu = E_0 u$ has a solution $u \in H^{1,-s}(\mathbf{X}) \setminus L^2(\mathbf{X})$ for any $s > 1/2$. Following Jensen-Kato ([9]), we call E_0 a regular point of P if E_0 is neither eigenvalue, nor resonance of P; an exceptional point of the first kind (respectively, the second kind; the third kind) of P if it is a resonance but not an eigenvalue (respectively, an eigenvalue but not a resonance; both an eigenvalue and a resonance) of P.*

Let $R(z) = (P-z)^{-1}, z \notin \sigma(P)$. To prove the existence of the asymptotic expansions of $R(z)$ for z near E_0, we need a detailed analysis of the spectral properties of P at E_0. The following result proved in Sections 3 and 4 plays an important role.

Theorem 1.1 *Assume (1.2) with $\rho > 2$ and (1.3). If E_0 is an eigenvalue of P in L^2, its multiplicity is finite. If $u \in H^{1,-s}, \forall s > 1/2$ and $Pu = E_0 u$, then, u is a resonant function of P if and only if*

$$\int_{\mathbf{X}} I_{a_0}(x)u(x)\varphi_0(x^{a_0})\, dx \neq 0. \tag{1.5}$$

Here φ_0 is a normalized eigenfunction of P^{a_0} associated with E_0. The resonance of P at E_0, if it does exist, is simple, which means that the eigenspace of P associated with E_0 is of codimension 1 in the kernel of $P - E_0$ in $H^{1,-s}$, $\forall s > 1/2$.

Let $\delta > 0$ be small. Denote $U_\delta(E_0) = \{z \in \mathbf{C}; \Im z > 0, |z - E_0| < \delta\}$ and $U_\delta = U_\delta(0)$. The main results of this work can be resumed as follows.

Theorem 1.2 *Let $\lambda = z - E_0$ and $(\lambda)^{1/2}$ be the branch such that $\Im(\lambda)^{1/2} > 0$ for $\Im\lambda > 0$.*

(a). Assume that E_0 is a regular point of P. Then, for $k \in \mathbf{N}$, $\rho > \max\{2, 2k+1\}$ and $(2k+1)/2 < s < \rho - (2k+1)/2$ for $k \geq 1$ and $1 < s < \rho - 1$ for $k = 0$, one has

$$R(z) = \sum_{j=0}^{k} \lambda^{j/2} B_j^{(0)} + O(|\lambda|^{k/2+\epsilon}), \tag{1.6}$$

in $\mathcal{L}(-1,s;1,-s)$, for $\lambda \in U_\delta$ and $\epsilon > 0$.

(b). Suppose that E_0 is an exceptional point of the first kind of P. Assume $\rho > \max\{2, 2k-1\}$, $k \in \mathbf{N}$ with $k \geq 1$. We have

$$R(z) = \sum_{j=-1}^{k-2} \lambda^{j/2} B_j^{(1)} + O(|\lambda|^{(k-2)/2+\epsilon}) \tag{1.7}$$

in $\mathcal{L}(-1, s; 1, -s)$ with $s > 1$ if $k = 1$ and $(2k-1)/2 < s < \rho - (2k-1)/2$ if $k \geq 2$. Here

$$B_{-1}^{(1)} = i < \cdot, u > u, \tag{1.8}$$

and u is an E_0 resonant function of P normalized by

$$\frac{1}{2\sqrt{\pi}} \int_{\mathbf{X}} I_{a_0}(x) u(x) \varphi_0(x^{a_0})\, dx = 1. \tag{1.9}$$

(c). Suppose that E_0 is an exceptional point of the second kind (l= 2) or the third kind (l= 3) of P. Assume $\rho > 2k-1$, $k \in \mathbf{N}$ with $k \geq 2$. One has

$$R(z) = \sum_{j=-2}^{k-4} \lambda^{j/2} B_j^{(l)} + O(|\lambda|^{(k-4)/2+\epsilon}), \quad l = 2, 3, \tag{1.10}$$

in $\mathcal{L}(-1, s; 1, -s)$ for $(2k-1)/2 < s < \rho - (2k-1)/2$, $\lambda \in U_\delta$ with $\delta > 0$ small enough and $\epsilon \in]0, 1/2]$ verifies $\epsilon < (2s - 2k + 1)/4$. Here

$$B_{-2}^{(2)} = B_{-2}^{(3)} = -\Pi_{E_0},$$

and Π_{E_0} is the orthogonal projection onto the eigenspace of P associated with the eigenvalue E_0.

Remarks *(a). Under the assumption (1.3), the methods of this paper can be used to establish the asymptotics of the resolvent near the thresholds $E \in \sigma_d(P^{a_0})$ with $E < \min_{a \in \mathcal{A}, \#a \geq 2, a \neq a_0} \inf \sigma(P^a)$. The projection Π used in Section 2 should then be replaced by the spectral projection associated to all eigenvalues less than or equal to E.*
(b). In the case where

$$\min_{a \in \mathcal{A}, \#a = 2} \inf \sigma(P^a) = \min_{a \in \mathcal{A}, \#a \geq 3} \inf \sigma(P^a),$$

the spectral properties of an N-body Schrödinger operator with rapidly decreasing pair potentials are sometimes similar to two-body Schrödinger operators with slowly decreasing potentials ($\rho < 2$). In fact, the Efimov effect in three-body problems shows that in this case, even for compactly supported pair potentials, the discrete spectrum can be infinite. In these cases, the results of the form stated in Theorem 1.2 can not hold.
(c). It is an open question to establish the resolvent expansions near E_0 when it is attained by several two-cluster Subhamiltonians, i.e., when we replace the assumption (1.3) by the condition

$$E_0 = \min_{a \in \mathcal{A}, \#a = 2} \inf \sigma(P^a) < \min_{a \in \mathcal{A}, \#a \geq 3} \inf \sigma(P^a).$$

For an N-body Schrödinger operator with identical particles, if this is only due to the internal symmetries, the method of this paper can still be applied to by fixing a representation. See [19] for the study of the discrete spectrum. More generally, one can also try to establish the asymptotics of resolvent near the two-cluster thresholds which are below all three-cluster thresholds. New ideas are needed in these cases.

For more precisions on the statement of results, see Proposition 3.3 and Theorems 3.10 and 4.11.

The expression of the coefficients $B_j^{(l)}$ depends not only on l which classifies the spectral nature of P at E_0, but also on wether E_0 belongs to the discrete spectrum of an auxiliary operator P' defined in Section 2. For example, in the case $E_0 \notin \sigma_d(P')$, $B_0^{(0)}$ is given by

$$B_0^{(0)} = R'(E_0) + \epsilon_+(E_0)(1 + G_0V_0)^{-1}G_0\epsilon_-(E_0)$$

(see Proposition 3.3) and if $E_0 \in \sigma_d(P')$, the expression of $B_0^{(0)}$ is more complicated and is given by (4.77) in Theorem 4.11 (a). Even though our results are in form similar to those known for two-body Schrödinger operators ([5], [9], [13]), the proof is technically involved. It is only after a tedious calculus that we arrive at the simple leading term in the case $l = 1, 2, 3$.

We expect the results of Theorem 1.2 to be useful in the study of low-energy many-body scattering (see [3], [5], [9], [13] and [20] for two-body problems and [18] for results on many-body scattering matrices) and in the study of the finiteness or infiniteness of discrete spectrum of N-body Schrödinger operators in the presence of two-cluster resonances (see for example [17], [21] for the study of the Efimov effect in three-body problems).

The following is immediate consequences of Theorem 1.2.

(1). The results of Theorem 1.2 imply the absence of embedded eigenvalues in $]E_0, E_0 + \delta[$: Suppose $\rho > 2$ in the cases (a) and (b) and $\rho > 3$ in the cases (c). Then there exists $\delta > 0$ such that the eigenvalues of P are absent in $]E_0, E_0 + \delta[$. This result was already proved by P. Perry in [14] by a different method.
(2). By the representation formulae (3.4) and (4.8) for the resolvent $R(z)$ which are valid for $\rho > 0$, making use of the well-known methods for establishing the limiting absorption principle for two-body Schrödinger operators ([1], [12], [15]), we can show, under rather mild assumptions on the potentials, the existence of the boundary value of the resolvent

$$R(\lambda + i0) : H^{-1,s} \to H^{1,-s}, s > 1/2,$$

for $\lambda \in]E_0, E_0 + \delta[\backslash\sigma_p(P)$. Note that here only the weight in x_{a_0} is used and that the situation is similar to the low-energy scattering in the Born-Oppenheimer approximation ([11]).
(3). Applying Theorem 1.2, we can write down the asymptotic expansion of the derivative of the spectral projection of P near E_0 and obtain the time-decay of the quantum evolution e^{-itP} near E_0. For example, taking $\chi \in C_0^\infty(]E_0 - \delta, E_0 + \delta)$ with $\chi(E_0) = 1$, one has, under appropriate assumptions on ρ, the asymptotic expansion

$$e^{-it(P-E_0)}\chi(P) = \Pi_{E_0} + t^{-1/2}C_1 + t^{-3/2}C_2 + \cdots, \quad t \to \infty.$$

The C_j's can be computed from Theorem 1.2. In particular, if E_0 is a regular point of P, $\Pi_{E_0} = C_1 = 0$ and $C_2 \neq 0$; if E_0 is an exceptional point of the first kind, $\Pi_{E_0} = 0$ and $C_1 = \frac{1}{\pi}B_{-1}^{(1)} \neq 0$.

Let us now sketch the main ideas used in the proof of Theorem 1.2. Under the assumption (1.3), by resolving a Grushin problem associated to $P - z$, we obtain a representation for the resolvent $R(z)$ in the form

$$R(z) = \mathcal{E}(z) - \mathcal{E}_+(z)\mathcal{E}_{+-}(z)^{-1}\mathcal{E}_-(z)$$

for z near E_0, where $\mathcal{E}(z), \mathcal{E}_\pm(z)$ and $\mathcal{E}_{+-}(z)$ are all holomorphic near E_0. $\mathcal{E}_{+-}(\lambda + E_0)$ is of the form

$$\lambda - (-\Delta + V(\lambda)) \quad \text{or} \quad \lambda - \begin{pmatrix} -\Delta + V(\lambda) & C \\ C^* & 0 \end{pmatrix},$$

depending on whether E_0 is an eigenvalue of P' or not. Here Δ is the 3-dimensional Laplacian, $C : \mathbf{C}^m \to H^{1,s}, \forall s > 0$, and $V(\lambda)$ is a family of non-local operators, admitting a convergent expansion of the form

$$V(\lambda) = \sum_{j=0}^{\infty} \lambda^j V_j, |\lambda| < \delta.$$

Note that C may not be injective which implies that the resolvent of the unperturbed operator may have a pole at zero. To prove the existence of the asymptotic expansion of $\mathcal{E}_{+-}(z)^{-1}$, we again resolve an appropriate Grushin problem in weighted Sobolev spaces to obtain a representation of the operators

$$(1 + (-\Delta - \lambda)^{-1} V(\lambda))^{-1} \quad \text{and} \quad \left(1 + \left(\begin{pmatrix} -\Delta & C \\ C^* & 0 \end{pmatrix} - \lambda\right)^{-1} \begin{pmatrix} V(\lambda) & 0 \\ 0 & 0 \end{pmatrix}\right)^{-1}.$$

In the both cases, the singularities of $\mathcal{E}_{+-}(z)^{-1}$ near $z = E_0$ can be calculated from those of the inverse of a holomorphic matrix-valued function $E_{+-}(\lambda)$ for λ near 0. To prove the existence of an asymptotic expansion of $E_{+-}(\lambda)^{-1}$ near $\lambda = 0$, we need detailed informations on eigenfunctions and resonant functions of the operators

$$-\Delta + V_0 \quad \text{and} \quad \begin{pmatrix} -\Delta + V_0 & C \\ C^* & 0 \end{pmatrix}$$

at 0 and to relate them to those of P at E_0. Notice that in the presence of threshold eigenvalue, the operator V_1 intervenes in the leading term. Our approach allows to optimize the decay assumption on ρ. For example, to obtain the leading term we just need $\rho > 2$ in the case of an exceptional point of the first kind and $\rho > 3$ in the case of an exceptional point of the second or the third kind.

The plan of this paper is as follows. In Section 2, we introduce an auxiliary operator P' and prove that it has only discrete spectrum near E_0. In Section 3, under the assumption that E_0 is not an eigenvalue of the operator P', we study two Grushin problems to reduce the study of $R(z)$ to the inverse of a matrix-valued holomorphic function and establish the asymptotic expansion of $R(z)$ according to the spectral properties of E_0 with respect to P. In Section 4, the case where E_0 is an eigenvalue of P' is studied. We analyze the zero eigenvalue and the zero resonance of a matrix-valued operator and establish again the asymptotics of $R(z)$ according to the spectral properties of E_0 with respect to P. The proof of Theorems 1.1 and 1.2 is given in Section 3 or in Section 4 according to the situation E_0 belongs to $\sigma_d(P')$ or not.

2 Preliminaries

Throughout this paper, the conditions (1.2) and (1.3) are assumed. ¿From now on, we denote $a = a_0$. Let φ_0 be an normalized eigenfunction of P^a associated with E_0. Let Π be the projection in $L^2(\mathbf{X})$ defined by

$$\Pi f = \varphi_0 \otimes (f, \varphi_0)_a, \ f \in L^2(\mathbf{X}).$$

Put $\Pi' = 1 - \Pi$ and $P' = \Pi' P \Pi'$.

Lemma 2.1 *There exists $\epsilon_0' > 0$ such that the essential spectrum of $P' = \Pi' P \Pi'$ satisfies*

$$\sigma_{\text{ess}}(P') \subset [E_0 + \epsilon_0', \infty[.$$

Proof. Let $\{J_b; b \in \mathcal{A}, \#b = 2\}$ be a partition of unity such that $\sum J_b(x)2 = 1$ on $\mathbf{X}$, $J_b(x) = 1$ for x outside a compact set and $|x^b| \leq c_0|x|$ for some $c_0 > 0$ and $|\partial^\alpha J_b(x)| \leq C_\alpha < x >^{-|\alpha|}$ on $\mathbf{X}$, for all $\alpha \in \mathbf{N}^{3(N-1)}$. Write

$$P = (-\triangle_b + P^b) + I_b = P_b + I_b.$$

From the IMS-localization formula (cf. Lemma 3.2 in [CFKS]), we have

$$\begin{aligned}
P' &= \Pi'(\sum_{\#b=2} J_b P J_b - \sum_{\#b=2} |\nabla J_b|^2)\Pi' \\
&= \Pi'(\sum_{\#b=2} J_b(P_b + I_b)J_b - \sum_{\#b=2} |\nabla J_b|^2)\Pi' \\
&= \Pi'(\sum_{\#b=2} J_b P_b J_b + \sum_{\#b=2} I_b J_b 2 - \sum_{\#b=2} |\nabla J_b|^2)\Pi' \\
&\equiv \Pi'(\sum_{\#b=2} J_b P_b J_b)\Pi' + K.
\end{aligned}$$

Due to the assumption (1.2), $\Pi'|\nabla J_b|\Pi'$ and $\Pi' I_b J_b 2\Pi'$ are relativement compact perturbation of $\Pi' P_0 \Pi'$ and of P', so is K.

For $b \neq a$, by (1.3), $\inf \sigma(P_b) = \inf \sigma(P^b) \geq E_0 + \epsilon_0$. Therefore,

$$\sum_{\#b=2, b\neq a} \Pi' J_b P_b J_b \Pi' \geq (E_0 + \epsilon_0) \sum_{\#b=2, b\neq a} \Pi' J_b^2 \Pi' \tag{2.1}$$

For $b = a$, we decompose $\Pi' J_a P_a J_a \Pi'$ as

$$\Pi' J_a P_a J_a \Pi' = J_a \Pi' P_a \Pi' J_a + K_1$$

where

$$K_1 = [\Pi', J_a] P_a J_a \Pi' + J_a \Pi' P_a [\Pi', J_a].$$

Note that $[\Pi', J_a] = [\Pi, 1 - J_a]$. Let $A = (1 - J_a)\Pi$. AA^* is of the integral kernel $(1 - J_a(x))\varphi_0(x^a)\varphi_0(y^a)(1 - J_a(y))$. Since φ_0 is exponentially decreasing on $\mathbf{X}^a$ and $1 - J_a(x)$ has support in $\{|x^a| > \delta|x|\}$, $\delta > 0$, AA^* is a Hilbert-Schmidt operator. In particular, A is compact, and so is the commutator $[\Pi', J_a]$. This proves that K_1 is relatively compact with respect to P'. Since E_0 is the lowest eigenvalue of P^a, there exists $\epsilon_1 > 0$ such that

$$\Pi' P_a \Pi' \geq (E_0 + \epsilon_1)\Pi'.$$

We obtain

$$\begin{aligned}
\Pi' J_a P_a J_a \Pi' &\geq (E_0 + \epsilon_1) J_a \Pi' J_a + K_1 \\
&= (E_0 + \epsilon_1)\Pi' J_a^2 \Pi' + K_2,
\end{aligned} \tag{2.2}$$

where

$$K_2 = (E_0 + \epsilon_1)(J_a \Pi' J_a - \Pi' J_a^2 \Pi') + K_1$$

is still a relatively compact perturbation of P' by the arguments used above. Let $\epsilon_0' = \min\{\epsilon_0, \epsilon_1\}$. One deduces from (2.1) and (2.2) that

$$\begin{aligned} P' &= \sum_{\#b=2} \Pi' J_b P_b J_b \Pi' + K \\ &\geq (E_0 + \epsilon_0')\Pi' + K_3 \\ &\geq (E_0 + \epsilon_0') + K_3 \end{aligned}$$

where $K_3 = K_2 + K$ is relatively compact and we have used the fact that $E_0 + \epsilon_0' < 0$, so $(E_0 + \epsilon_0')\Pi \leq 0$. By Weyl's Theorem on essential spectrum, we obtain

$$\sigma_{ess}(P') = \sigma_{ess}(\sum_{\#b=2} \Pi' J_b P_b J_b \Pi') \subset [E_0 + \epsilon_0', \infty[.$$

□

In the following, in order to give the asymptotics of the resolvent $R(z) = (P - z)^{-1}$ for $z \in U_\delta(E_0)$, we shall distinguish the cases $E_0 \in \sigma_d(P')$ and $E_0 \notin \sigma_d(P')$.

Let $R_0(z) = (-\Delta_a - z)^{-1}$, $\Im z > 0$. Since $\dim \mathbf{X}_a = 3$, $R_0(z)$ can be continuously extended to $\overline{\mathbf{C}}_+$ as operator in $\mathcal{L}(-1, s; 1, -s')$, $s, s' > 1/2, s + s' > 2$ and has an asymptotic expansion near 0 of the form

$$R_0(\lambda) = \sum_{j=0} k\lambda^{j/2} G_j + O(\lambda^{k/2}), \text{ in } \mathcal{L}(-1, r'; 1, -r), r', r > k + 1/2, \tag{2.3}$$

where

$$G_0 \in \mathcal{L}(-1, s'; 1, -s), s, s' > 1/2, \; s + s' > 2,$$
$$G_j \in \mathcal{L}(-1, s'; 1, -s), \quad s, s' > j + 1/2, \text{ for } j \geq 1.$$

The G_j's are integral operators whose kernel can be computed explicitly. Remark that if $a = (a_{0,1}, a_{0,2})$ and x' denotes the relative position of the two mass centers, then

$$\Delta_a = (\frac{1}{2M_{a_{0,1}}} + \frac{1}{2M_{a_{0,2}}})\Delta_{x'},$$

$M_{a_{0,j}} = \sum_{k \in a_{0,j}} m_k$. It is more convenient to use the orthonormal coordinates x_0 on $\mathbf{X}_a$ so that the kernel of G_j is just given by

$$K_j(x_0, y_0) = \frac{i^j}{4\pi j!} |x_0 - y_0|^{j-1}. \tag{2.4}$$

The following lemma is useful to optimize remainder estimates.

Lemma 2.2 *For $s > 1/2$, $0 < r < 2s - 1, 0 < r \leq 1$, we have*

$$|| < x >^{-s} R_0(\lambda + i0) < x >^{-s} || \leq C_{s,r} |\lambda|^{-1+r}, \; \lambda \in]0, 1]. \tag{2.5}$$

More generally, let $s > k + 1/2$ for $k \in \mathbf{N}^*$ and $s > 1$ for $k = 1$. One has,

$$|| < x >^{-s} (R_0(\lambda + i0) - \sum_{j=0}^{k} \lambda^{j/2} G_j) < x >^{-s} || \leq C_{s,s',r} |\lambda|^{k/2+\epsilon}, \; \lambda \in]0, 1], \tag{2.6}$$

for $0 < \epsilon < \min\{1/2, (2s - 2k - 1)/4\}$. ϵ can be taken to be $1/2$ if $(2s - 2k - 1)/4 > 1/2$. Here $x \in \mathbf{R}^3$ and $\| \cdot \|$ denotes the norm on $\mathcal{L}(L^2(\mathbf{R}^3))$.

Lemma 2.2 can be easily deduced by dilation and complex interpolation. In fact, since $R_0(1+i0) \in \mathcal{L}(-1,s;1,-s)$, $s > 1/2$, we deduce by a dilation

$$|| < \lambda^{1/2}x >^{-s} R_0(\lambda + i0) < \lambda^{1/2}x >^{-s} || \leq C_s|\lambda|^{-1}. \tag{2.7}$$

For $s > 1/2$,

$$|| < x >^{-s} R_0(\lambda + i0) < x >^{-s} || \leq C_s|\lambda|^{-1}, 0 < \lambda \leq 1.$$

On the other hand, we have, for $s > 1$,

$$|| < x >^{-s} R_0(\lambda + i0) < x >^{-s} || \leq C_s, \tag{2.8}$$

for $0 < \lambda \leq 1$. (2.5) follows from a complex interpolation. Using (2.3), (2.6) can be proved similarly.

3 The case $E_0 \notin \sigma_d(P')$

In this section, we suppose that $E_0 \notin \sigma_d(P')$ and $\delta > 0$ is chosen small so that $\sigma(P') \cap \{|z - E_0| < \delta\} = \emptyset$. In this case, we can reduce the N-body problem to a generalized two-body operator with non-linear spectral parameter.

3.1 A reduction.

We begin with the study of the Grushin problem

$$\mathcal{P}(z) = \begin{pmatrix} P - z & \varphi_0 \otimes \cdot \\ (\cdot, \varphi_0)_a & 0 \end{pmatrix} : D(P) \times L^2(\mathbf{X}_a) \to L^2(\mathbf{X}) \times L^2(\mathbf{X}_a), \tag{3.1}$$

where $D(P)$ is the domain of the operator P. We want to prove that $\mathcal{P}(z)$ is invertible in a small neighbourhood of E_0. To see this, let

$$P' = \Pi' P \Pi', \quad \text{and} \quad R'(z) = (\Pi' P \Pi' - z)^{-1}\Pi',$$

where

$$\Pi = \varphi_0 \otimes (\cdot, \varphi_0)_a \ : L^2(\mathbf{X}) \to L^2(\mathbf{X})$$

and $\Pi' = 1 - \Pi$. Let $\mathcal{Q}(z)$ be defined by

$$\mathcal{Q}(z) = \begin{pmatrix} R'(z) & \varphi_0 \otimes \cdot \\ (\cdot, \varphi_0)_a & z - (P(\varphi_0 \otimes \cdot), \varphi_0)_a \end{pmatrix}.$$

Then, in $L^2(\mathbf{X}) \times L^2(\mathbf{X}_a)$,

$$\begin{aligned} \mathcal{P}(z)\mathcal{Q}(z) &= \begin{pmatrix} (P - z)R'(z) + \Pi & P(\varphi_0 \otimes \cdot) - \Pi P \Pi(\varphi_0 \otimes \cdot) \\ 0 & 1 \end{pmatrix} \\ &= 1 + \begin{pmatrix} \Pi I_a R'(z) & \Pi' I_a(\varphi_0 \otimes \cdot) \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Let

$$\mathcal{B} = \begin{pmatrix} \Pi I_a R'(z) & \Pi' I_a(\varphi_0 \otimes \cdot) \\ 0 & 0 \end{pmatrix}.$$

It is easy to check that $\mathcal{B}3 = 0$. Therefore, $1+\mathcal{B}$ is invertible and therefore $\mathcal{P}(z)$ is invertible on the right. Similarly, one can show that $\mathcal{P}(z)$ is invertible on the left. Consequently, $\mathcal{P}(z)$ is invertible and one has

$$\mathcal{P}(z)^{-1} = \mathcal{Q}(z)(1-\mathcal{B}+\mathcal{B}2). \tag{3.2}$$

Writing the inverse of $\mathcal{P}(z)$ in the form

$$\mathcal{P}(z)^{-1} = \begin{pmatrix} \epsilon(z) & \epsilon_+(z) \\ \epsilon_-(z) & \epsilon_{+-}(z) \end{pmatrix}, \tag{3.3}$$

we can compute that

$$\begin{aligned}
\epsilon(z) &= R'(z), \\
\epsilon_+(z) &= -R'(z)I_a(\varphi_0\otimes\cdot) + (\varphi_0\otimes\cdot), \\
\epsilon_-(z) &= (\cdot,\varphi_0)_a - (I_aR'(z)\cdot,\varphi_0)_a, \\
\epsilon_{+-}(z) &= (z-E_0) - (-\triangle_{x_a} + \tilde{I}_a(x_a)) + (I_aR'(z)I_a(\varphi_0\otimes\cdot),\varphi_0)_a,
\end{aligned}$$

where

$$\tilde{I}_a(x_a) = (I_a\varphi_0,\varphi_0)_a.$$

Making use of the relations $\mathcal{P}(z)\mathcal{P}(z)^{-1} = 1$ and $\mathcal{P}(z)^{-1}\mathcal{P}(z) = 1$, we obtain the following identities

$$\begin{aligned}
(P-z)\epsilon(z) + \varphi_0\otimes(\epsilon_-(z)\cdot) &= 1 \\
(P-z)\epsilon_+(z) + \varphi_0\otimes(\epsilon_{+-}(z)\cdot) &= 0 \\
\epsilon(z)(P-z) + \epsilon_+(z)(\cdot,\varphi_0)_a &= 1 \\
\epsilon_-(z)(P-z) + \epsilon_{+-}(z)(\cdot,\varphi_0)_a &= 0.
\end{aligned}$$

As usual (see *e.g.* [8]), we deduce from the above relations the expression of the resolvent $R(z)$

$$R(z) = \epsilon(z) - \epsilon_+(z)\epsilon_{+-}(z)^{-1}\epsilon_-(z). \tag{3.4}$$

The formula (3.4) is valid for any z with $\Im z > 0$ without the assumption that $E_0 \notin \sigma(P')$. With the assumption $E_0 \notin \sigma(P')$, $R'(z)$ is holomorphic in a small neighbourhood of E_0, so are $\epsilon_+(z), \epsilon_{+-}(z)$ and $\epsilon_-(z)$. The asymptotic behaviour of $R(z)$ near E_0 is then determined by that of $\epsilon_{+-}(z)^{-1}$. To study the resonance of P at E_0, we remark that by the method of commutators, the relation (3.4) initially established in L^2 remains valid in any weighted-L^2 space.

Set $\lambda = z - E_0$ and write $\epsilon_{+-}(z)$ as

$$\epsilon_{+-}(z) = \lambda - (-\triangle_{x_a} + V(\lambda)),$$

where

$$V(\lambda) = \tilde{I}_a(x_a)) - (I_aR'(\lambda+E_0)I_a(\varphi_0\otimes\cdot),\varphi_0)_a \tag{3.5}$$

is holomorphic near $\lambda = 0$. $V(\lambda)$ has a convergent expansion near $\lambda = 0$

$$V(\lambda) = \sum_{j=0}^{\infty} \lambda^j V_j$$

with

$$\begin{aligned}
V_0 &= \tilde{I}_a(x_a)) - (I_aR'(E_0)I_a(\varphi_0\otimes\cdot),\varphi_0)_a \\
V_j &= -(I_a(R'(E_0))^{j+1}I_a(\varphi_0\otimes\cdot),\varphi_0)_a, \quad j\geq 1.
\end{aligned}$$

Note that for $j \geq 1$, $V_j = (I_a(R'(E_0))^{j+1} I_a(\varphi_0 \otimes \cdot), \varphi_0)_a \in \mathcal{L}(0, s; 0, s+2\rho)$, $\forall s \in \mathbf{R}$. In this work, it is important to note that

$$V_1 = -(I_a(R'(E_0))^2 I_a(\varphi_0 \otimes \cdot), \varphi_0)_a \text{ is a negative operator on } L^2(\mathbf{X}_a). \tag{3.6}$$

Making use of the relation

$$\epsilon_{+-}(z)^{-1} = -(1 + R_0(\lambda)V(\lambda))^{-1} R_0(\lambda),$$

we are reduced to establishing the asymptotic expansion of $(1 + R_0(\lambda)V(\lambda))^{-1}$ in suitable spaces.

Proposition 3.1 *For $k \in \mathbf{N}$, $\rho > \max\{2, 2k+1\}$ and $(2k+1)/2 < s < \rho - (2k+1)/2$, one has for $W(\lambda) = 1 + R_0(\lambda)V(\lambda)$*

$$W(\lambda) = \sum_{j=0}^{k} \lambda^{j/2} W_j + O(|\lambda|^{k/2+\epsilon}), \tag{3.7}$$

in $\mathcal{L}(1, -s; 1, -s)$, for $\lambda \in U_\delta$ and $\epsilon \in]0, 1/2]$ verifies $\epsilon < (2s - 2k - 1)/4$. Here

$$W_0 = 1 + G_0 V_0, W_1 = G_1 V_0,$$

$$W_2 = G_2 V_0 + G_0 V_1, W_3 = G_3 V_0 + G_1 V_1.$$

Proposition 3.1 follows from the results of Section 2 and the expansion of $V(\lambda)$ near 0. From (3.4), we can deduce the following asymptotic expansion of the resolvent in the case where E_0 is a regular point of P.

Proposition 3.2 *Assume that E_0 is a regular point for P. Then, for $k \in \mathbf{N}$, $\rho > \max\{2, 2k+1\}$, $s > 1$ if $k = 0$ and $(2k+1)/2 < s < \rho - (2k+1)/2$ if $k \geq 1$, one has*

$$R(z) = \sum_{j=0}^{k} \lambda^{j/2} B_j^{(0)} + O(|\lambda|^{k/2+\epsilon}), \tag{3.8}$$

in $\mathcal{L}(-1, s; 1, -s)$, for $\lambda = z - E_0 \in U_\delta$ and $\epsilon > 0$ is the same as in Proposition 3.2. Here

$$B_0^{(0)} = R'(E_0) + \epsilon_+(E_0) C_0 G_0 \epsilon_-(E_0), \tag{3.9}$$

$$B_1^{(0)} = \epsilon_+(E_0)(C_0 G_1 + C_1 G_0)\epsilon_-(E_0), \tag{3.10}$$

with $C_0 = (1 + G_0 V_0)^{-1}$ and $C_1 = -C_0 G_1 V_0 C_0$.

Proof. By the assumption and Proposition 3.1, 0 is a regular point for $\epsilon_{+-}(E_0) = -\Delta + V_0$. In this case, $C_0 \in \mathcal{L}(1, -s; 1, -s)$ for $1/2 < s < \rho - 1/2$ and one has

$$(1 + R_0(\lambda)V(\lambda))^{-1} = C_0 + \lambda^{1/2} C_1 + \cdots$$

where C_0, C_1 are as above. Consequently, one has in suitable spaces

$$\begin{aligned} -\epsilon_{+-}^{-1}(\lambda) &= (1 + R_0(\lambda)V(\lambda))^{-1} R_0(\lambda) \\ &= C_0 G_0 + \lambda^{1/2}(C_0 G_1 + C_1 G_0) + \cdots \end{aligned}$$

Proposition 3.2 follows from (3.4). □

3.2 Spectral properties of P at the threshold

Definition. We say that 0 is a *resonance* of $\epsilon_{+-}(E_0)$ if the equation $\epsilon_{+-}(E_0)v = 0$ has a solution $v \in H^{1,-s}(\mathbf{X}_a)$ for any $s > 1/2$ and $v \notin L^2(\mathbf{X}_a)$. As for P, we call 0 a regular point or an exceptional point of the first, the second or the third kind of $\epsilon_{+-}(E_0)$ according to the spectral properties of 0 with respect to $\epsilon_{+-}(E_0)$.

It is well-known that E_0 is an eigenvalue of P if and only if 0 is an eigenvalue of $\epsilon_{+-}(E_0)$. As to resonance, we have the similar result.

Proposition 3.3 *E_0 is a resonance of P if and only if 0 is a resonance of $\epsilon_{+-}(E_0)$.*

Proof. Using (3.3) for $\mathcal{P}(E_0)^{-1}$, we obtain the following equations

$$(P - E_0)\epsilon_+(E_0) + \varphi_0 \otimes \epsilon_{+-}(E_0) = 0,$$

$$\epsilon_-(E_0)(P - E_0) + \epsilon_{+-}(E_0)(\cdot, \varphi_0)_a = 0.$$

These relations are still valid in $L^{2,s}$, $s \in \mathbf{R}$.

Assume that zero is a resonance of $\epsilon_{+-}(E_0)$. Let v be a resonant function

$$v \in H^{1,-s}(\mathbf{X}_a), \forall s > 1/2, \text{ and } \epsilon_{+-}(E_0)v = 0.$$

Then,

$$u \equiv \epsilon_+(E_0)v = -R'(E_0)(I_a(\varphi_0 \otimes v)) + \varphi_0 \otimes v$$

is in $H^{1,-s}(\mathbf{X}), \forall s > 1/2$, and satisfies $(P - E_0)u = 0$. We claim that $u \notin L^2(\mathbf{X})$. In fact, $\varphi_0 \otimes v \in L^{2,-s} \setminus L^2$ and since $I_a(\varphi_0 \otimes v) \in L^{2,\rho-s}$, so does $R'(E_0)(I_a(\varphi_0 \otimes v))$. So, $u \in L^{2,-s} \setminus L^2$.

Conversely, if E_0 is a resonance of P and u an associated resonant function, set $v = (u, \varphi_0)_a$. $v \in H^{1,-s}, \forall s > 1/2$, and $\epsilon_{+-}(E_0)v = 0$. $v \notin L^2$, because by the relation

$$\epsilon(z)(P - z) + \epsilon_+(z)(\cdot, \varphi_0)_a = I,$$

$u = \epsilon_+(E_0)v$. If $v \in L^2$, u would also be in L^2, which contradicts the fact that u is a resonant function, which shows that 0 is a resonance of $\epsilon_{+-}(E_0)$. □

The proof of Proposition 3.3 shows that the dimensions of resonant spaces are the same. Theorem 1.1 in the case $E_0 \notin \sigma_d(P')$ is derived from Proposition 3.3.

Proof of Theorem 1.1 in the case $E_0 \notin \sigma_d(P')$. Remark that $\epsilon_{+-}(E_0) = -(-\Delta_a + V_0)$ is a two-body Schrödinger operator. Although V_0 is non-local, the analysis of [9] is still valid in this case. In particular, one can show that the kernel of $\epsilon_{+-}(E_0)$ in $H^{1,-s}$, $\forall s > 1/2$, is of finite dimension and that if v is a resonant function of $\epsilon_{+-}(E_0)$, then

$$< V_0 v, 1 >_a \neq 0.$$

As a consequence, the 0 resonance of $\epsilon_{+-}(E_0)$ is at most simple, so is the E_0 resonance of P, by Proposition 3.3. It follows that if E_0 is an eigenvalue of P, its multiplicity is finite. By the proof of Proposition 3.3, u is resonant function of P if and only if it can be written in the form $u = \epsilon_+(E_0)v$, where v is a resonant function of $\epsilon_{+-}(E_0)$. Since $u \in H^{1,-s}$, $\forall s > 1/2$, $\rho > 2$ and φ_0 is rapidly decreasing on $\mathbf{X}_a$, $I_a u \varphi_0 \in L1(\mathbf{X})$. This gives

$$\int I_a u \varphi_0 \, dx = \int I_a(\epsilon_+(E_0)v)\varphi_0 \, dx$$

$$= \int [(I_a - I_a R'(E_0) I_a)(\varphi_0 \otimes v)]\varphi_0 \, dx =< V_0 v, 1 >_a \neq 0,$$

□

3.3 Asymptotic expansions of $(1+R_0(\lambda)V(\lambda))^{-1}$

In the case E_0 is not an eigenvalue of P (*i.e.* it may be a regular point or an exceptional point of the first kind of P), the asymptotic expansion of $(1+R_0(\lambda)V(\lambda))^{-1}$ can be obtained by method of perturbation based on the known results for two-body Schrödinger operators. If E_0 is an eigenvalue of P, or equivalently, 0 is an eigenvalue of $\epsilon_{+-}(E_0) = -\Delta + V_0$, due to the presence of the term λV_1 in $V(\lambda)$, the method of perturbation does not work. In the rest of this section, we present a self-contained proof in this general case, using once again the Grushin's method.

Recall that

$$G_0 = (-\Delta)^{-1} \in \mathcal{L}(-1, s'; 1, -s), \ \ s+s' > 2, \ s, s' > \frac{1}{2},$$

and

$$(1 + G_0 V_0) \in \mathcal{L}(1, -s; 1, -s)$$

for $1/2 < s < \rho - 1/2$. Let

$$K = \text{ Ker } (1 + G_0 V_0) \subset H^{1,-s}, K^* = \text{ Ker } (1 + G_0 V_0)^* \subset H^{-1,s}.$$

As in [JK], one can show that K is independent of $s > 1/2$ and is spanned by the resonant states and the eigenfunctions of $-\Delta + V_0$. The resonance, if it exists, is always simple and the resonant function ϕ is characterized by the relation

$$< V_0\phi, 1 >_a = \int (V_0\phi)(x)dx \neq 0.$$

One can easily check that V_0 is injective from K into K^* and $V_0^* = V_0$ is injective from K^* into K. Consequently, V_0 is bijective from K onto K^*. This shows that $\dim K = \dim K^*$ and the form

$$\phi \to < \phi, -V_0\phi >_a$$

is non-degenerate on K. Using the positivity of $-\Delta$, one can prove that this form is definite positive. Let

$$\mu = \dim K.$$

We can choose a basis $\{\phi_1, \cdots, \phi_\mu\}$ of K such that

$$< \phi_i, -V_0\phi_j >_a = \delta_{ij}. \tag{3.11}$$

Recall that the 0-resonance, if it does exist, is simple for $-\Delta + V_0$. Consequently, we can choose the basis $\{\phi_j\}$ such that $\phi_j, j \geq 2$, are eigenfunctions. Denote

$$\phi_j^* = -V_0\phi_j, j = 1, \cdots, \mu, \tag{3.12}$$

which form a basis of K^*.

In order to get the expansion of $(1+R_0(\lambda)V(\lambda))^{-1}$, we consider

$$A(\lambda) = \begin{pmatrix} W(\lambda) & T \\ S & 0 \end{pmatrix} : H^{1,-s} \times \mathbf{C}^\mu \to H^{1,-s} \times \mathbf{C}^\mu,$$

where $s > 1/2$, operators T and S are defined as follows:

$$T \begin{pmatrix} c_1 \\ \vdots \\ c_\mu \end{pmatrix} = \sum_{j=1}^{\mu} c_j\phi_j, \ \begin{pmatrix} c_1 \\ \vdots \\ c_\mu \end{pmatrix} \in \mathbf{C}^\mu,$$

and

$$Sf = \begin{pmatrix} < f, -V_0\phi_1 >_a \\ \vdots \\ < f, -V_0\phi_\mu >_a \end{pmatrix}, \quad f \in H^{1,-s}.$$

Define $Q : H^{1,-s} \to H^{1,-s}$ by

$$Qf = \sum_{j=1}^{\mu} < f, \phi_j^* >_a \phi_j.$$

Then,

$$TS = Q \text{ on } H^{1,-s} \text{ and } ST = I_\mu \text{ on } \mathbf{C}^\mu.$$

Lemma 3.4 *One has the decomposition*

$$H^{1,-s} = K \oplus \text{ Range } (1 + G_0V_0). \tag{3.13}$$

Q is the projection from $H^{1,-s}, s > 1/2$, onto K with Range $Q =$ Range $(1 + G_0V_0)$.

Proof. For $u \in K \cap$ Range $(1 + G_0V_0)$, we have, for some v, $(1 + G_0V_0)u = 0$ and $u = (1 + G_0V_0)v$. Thus, $u = -G_0V_0u$ and

$$< -V_0u, u >_a = < -V_0u, v + G_0V_0v >_a = - < V_0(1 + G_0V_0)u, v >_a = 0.$$

Since $-V_0$ defines a positive definite form on K, this implies $u = 0$. This proves that $K \cap$ Range $(1 + G_0V_0) = \{0\}$. Since G_0V_0 is compact on $H^{1,-s}$, Range $(1 + G_0V_0)$ is closed and is equal to (Ker $(1 + G_0V_0)^*)^\perp$. For any $u \in H^{1,-s}$, we write $u = Qu + (u - Qu)$ and we can check that $u - Qu \in$ (Ker $(1 + G_0V_0)^*)^\perp$. This proves $H^{1,-s} = K \oplus$ Range $(1 + G_0V_0)$.

It is easy to verify that Q is the projection onto K corresponding to this decomposition of $H^{1,-s}$. □

Denote $F =$ Range $(1 + G_0V_0)$. Then, $Q' = 1 - Q$ is a projection from $H^{1,-s}$ onto F.

Proposition 3.5 *$Q'(1 + G_0V_0)Q'$ is bijective on F. For $\lambda \in U_\delta$, $\delta > 0$ small enough, $Q'W(\lambda)Q'$ is invertible on F. Let*

$$D(\lambda) = (Q'W(\lambda)Q')^{-1}Q'.$$

Then if $\rho > 2$, one has

$$D(\lambda) = D_0 + O(|\lambda|^\epsilon), \tag{3.14}$$

in $\mathcal{L}(1, -s; 1, -s), s > 1/2$. Here $0 < \epsilon < \min\{1/2, (2s - 1)/4\}$. More generally, if $k \geq 1$ and $\rho > 2k + 1$, one has

$$D(\lambda) = \sum_{j=0}^{k} \lambda^{j/2} D_j + O(|\lambda|^{k/2+\epsilon}), \tag{3.15}$$

in $\mathcal{L}(1, -s; 1, -s), (2k + 1)/2 < s < \rho - (2k + 1)/2$. Here $\epsilon > 0$ is the same as in Lemma 2.2 and

$$D_0 = (Q'(1 + G_0V_0)Q')^{-1}Q', D_1 = -D_0W_1D_0.$$

Proof. For $u \in F$, such that $Q'(1 + G_0V_0)Q'u = 0$, we have $Q'u = u$ and $0 = Q'(1 + G_0V_0)Q'u = (1 + G_0V_0)u - Q(1 + G_0V_0)u = (1 + G_0V_0)u$. This means $u \in K$ and $u = 0$, by Lemma 3.4. This proves that $Q'(1 + G_0V_0)Q'$ is injective on F. Since $Q'G_0V_0Q'$ is compact on F, the Fredholm's alternative theorem shows that $Q'(1 + G_0V_0)Q'$ is also surjective on F. Consequently,

$$D_0 = (Q'(1 + G_0V_0)Q')^{-1}Q', \tag{3.16}$$

belongs to $\mathcal{L}(1,-s;1,-s), s > 1/2$. The second part of the Proposition follows from Lemma 2.2 and Proposition 3.1 . $\square$

Using the operator $D(\lambda)$, we can construct an approximate inverse of $A(\lambda)$. Set

$$B(\lambda) = \begin{pmatrix} D(\lambda) & T \\ S & -SW(\lambda)T \end{pmatrix} : H^{1,-s} \times \mathbf{C}^\mu \to H^{1,-s} \times \mathbf{C}^\mu, s > 1/2.$$

Proposition 3.6 *For $\lambda \in U_\delta$, the operator $A(\lambda)$ is invertible from $H^{1,-s} \times \mathbf{C}^\mu$ to $H^{1,-s} \times \mathbf{C}^\mu$. The inverse $A(\lambda)^{-1}$ can be written in the form*

$$A(\lambda)^{-1} = \begin{pmatrix} E(\lambda) & E_+(\lambda) \\ E_-(\lambda) & E_{+-}(\lambda) \end{pmatrix},$$

where

$$E(\lambda) = D(\lambda), \qquad E_+(\lambda) = T - D(\lambda)Y, \tag{3.17}$$

$$E_-(\lambda) = S - SX, \qquad E_{+-}(\lambda) = -SW(\lambda)T + SXY, \tag{3.18}$$

with

$$X = QW(\lambda)Q'D(\lambda), Y = Q'W(\lambda)T.$$

Proof. By direct calculus, we have, in $\mathcal{L}(H^{1,-s} \times \mathbf{C}^\mu, H^{1,-s} \times \mathbf{C}^\mu)$,

$$A(\lambda)B(\lambda) = \begin{pmatrix} W(\lambda)D(\lambda) + TS & W(\lambda)T - TSW(\lambda)T \\ SD(\lambda) & ST \end{pmatrix}. \tag{3.19}$$

Since $ST = 1$ and $TS = Q$, we can simplify the above expression to obtain

$$A(\lambda)B(\lambda) = 1 + \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}. \tag{3.20}$$

Here X and Y are defined as above. We can check that $X2 = 0$ and $Y2 = 0$. So the left hand-side of the last equation is invertible. This proves $A(\lambda)$ has a right inverse. Similarly, we can show that it has also a left inverse. $A(\lambda)$ is thus invertible and

$$A(\lambda)^{-1} = B(\lambda)\left(1 + \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}\right)^{-1}. \tag{3.21}$$

An easy calculus gives the desired result. $\square$

As before, we obtain from Proposition 3.6 an expression for the inverse of $W(\lambda)$:

$$W(\lambda)^{-1} = E(\lambda) - E_+(\lambda)E_{+-}(\lambda)^{-1}E_-(\lambda). \tag{3.22}$$

Remark also that if $\rho > 2k+1$, $E_\pm(\lambda)$ have an asymptotic expansion of the form

$$E_\pm(\lambda) = \sum_{j=0}^{k} \lambda^{j/2} E_{\pm,j} + O(|\lambda|^{k/2+\epsilon}), \tag{3.23}$$

in $\mathcal{L}(\mathbf{C}^\mu; H^{1,-s})$ or $\mathcal{L}(H^{1,-s}; \mathbf{C}^\mu)$, $s > k + 1/2$, where

$$E_{-,0} = S, \quad E_{-,1} = -SW_1D_0,$$

and

$$E_{+,0} = T, \quad E_{+,1} = -D_0 W_1 T.$$

To study the asymptotics of $W(\lambda)^{-1}$ near 0, we are led to the study of that for $E_{+-}(\lambda)^{-1}$ near 0, which is just a $\mu \times \mu$-matrix valued function.

Lemma 3.7 *Assume $\rho > 2k-1$ and $k \geq 2$. One has:*

$$E_{+-}(\lambda) = \sum_{j=1}^{k} \lambda^{j/2} E_{+-,j} + O(|\lambda|^{k/2+\epsilon}), \tag{3.24}$$

for $0 < \epsilon < \min\{1/2, (\rho - 2k+1)/4\}$. Here

$$\begin{aligned} E_{+-,1} &= -SW_1T, & (3.25)\\ E_{+-,2} &= -SW_2T + SW_1D_0W_1T, & (3.26)\\ E_{+-,3} &= -SW_3T + S(W_1D_1W_1 + W_2D_0W_1 + W_1D_0W_2)T & (3.27) \end{aligned}$$

and $\epsilon > 0$ is the same as in Proposition 3.5. In addition, if 0 is an exceptional point of the first kind (in this case $\mu = 1$),

$$E_{+-,1} = c_0 := \frac{i}{4\pi} | < V_0\phi_1, 1 >_a |^2 \neq 0. \tag{3.28}$$

If 0 is an exceptional point of the second kind, then,

$$E_{+-,1} = 0, \quad E_{+-,2} = (< (1-V_1)\phi_i, \phi_j >_a)_{1 \leq i,j \leq \mu}. \tag{3.29}$$

If 0 is both an eigenvalue and a resonance, $E_{+-,1}$ and $E_{+-,2}$ can be written in the form:

$$E_{+-,1} = \begin{pmatrix} c_0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{3.30}$$

$$E_{+-,2} = \begin{pmatrix} v_1 & v_2 & \cdots & v_\mu \\ \bar{v}_2 & < (1-V_1)\phi_2, \phi_2 >_a & \cdots & < (1-V_1)\phi_2, \phi_\mu >_a \\ \vdots & \vdots & \ddots & \vdots \\ \bar{v}_\mu & < (1-V_1)\phi_\mu, \phi_2 >_a & \cdots & < (1-V_1)\phi_\mu, \phi_\mu >_a \end{pmatrix}. \tag{3.31}$$

Here $c_0 \neq 0$ has the same expression as in (3.28) and $v_j \in \mathbf{C}$.

Proof. From Proposition 3.6, one has

$$E_{+-}(\lambda) = -SW(\lambda)T + SW(\lambda)D(\lambda)W(\lambda)T,$$

where $W(\lambda) = 1 + R_0(\lambda)V(\lambda)$. $-SW(\lambda)T = (< W(\lambda)\phi_i, V_0\phi_j >_a)_{1 \leq i,j \leq \mu}$. Note that in general, $\phi_j \in H^{1,-s}$ for any $s > 1/2$ and $\phi_j \in H^{1,0}$ if it is an eigenfunction. For $\rho > k+1$, we have an expansion of the form (3.24) for $-SW(\lambda)T$ and if 0 is not a resonance, we just need $\rho > k + 1/2$ to obtain the same result. Remark that

$$S(1 + G_0V_0) = (1 + G_0V_0)T = 0.$$

$SW(\lambda)$ and $W(\lambda)T$ have asymptotic expansions beginning with $\lambda^{1/2}$. In order to obtain an expansion of $SW(\lambda)D(\lambda)W(\lambda)T$ up to the oder $\lambda^{k/2}$, we just need to expand each factor up to the oder $\lambda^{(k-1)/2}$. By Proposition 3.1 and Proposition 3.5, the condition $\rho > 2(k-1)$ is sufficient. This proves (3.24).

To precise the leading term, we remark that in the case of an exceptional point of the first kind, the 0 resonance is simple and the resonant function ϕ_1 is characterized by the condition

$$< V_0\phi_1, 1 >_a \neq 0.$$

Since G_1 is an integral operator with the constant integral kernel $\frac{i}{4\pi}$,

$$-SW_1T =< G_1V_0\phi_1, V_0\phi_1 >_a = \frac{i}{4\pi} | < V_0\phi_1, 1 >_a |^2 \neq 0.$$

In the case of an exceptional point of the second kind , we have $\phi_j \in L^2$ for all j and thus, $< V_0\phi_j, 1 >_a = 0, j = 1, \cdots, \mu$. The same calculus as above shows $E_{+-,1} = 0$. For the same reason, we have also $SW_1D_0W_1T = 0$. Consequently,

$$E_{+-,2} = -S(G_2V_0 + G_0V_1)T = (< (G_2V_0 + G_0V_1)\phi_i, V_0\phi_j >_a)_{i,j=1,\cdots,\mu}$$

Moreover, since $\phi_j \in L^2$ and $< V_0\phi_j, 1 >_a = 0, j = 1, \cdots, \mu$, we can check that

$$< G_2V_0\phi_i, V_0\phi_j >_a =< \phi_i, \phi_j >_a .$$

In fact, writing

$$G_2 = \lambda^{-1}(R_0(\lambda) - G_0 - \lambda^{1/2}G_1) + O(|\lambda|^\epsilon),$$

one has

$$G_2V_0\phi_i = -R_0(\lambda)\phi_i + O(|\lambda|^\epsilon)V_0\phi_i,$$

and

$$< G_2V_0\phi_i, V_0\phi_j >_a = - < \phi_i, R_0(\overline{\lambda})V_0\phi_j >_a + O(|\lambda|^\epsilon).$$

$R_0(\overline{\lambda})V_0\phi_j = -\phi_j - \overline{\lambda}R_0(\overline{\lambda})\phi_j$. Since $\phi_j \in L^2$, for $\lambda = i\gamma$ with $\gamma \in \mathbf{R}_+$, $\gamma R_0(-i\gamma)$ converges to 0 in the strong operator topology on L^2. Therefore, taking the limit $\lambda = i\gamma \to 0$, $\gamma \in \mathbf{R}_+$, we obtain

$$< G_2V_0\phi_i, V_0\phi_j >_a =< \phi_i, \phi_j >_a .$$

It is clear that $< G_0V_1\phi_i, V_0\phi_j >_a = - < V_1\phi_i, \phi_j >_a$. This proves (3.29).

The case of an exceptional point of the third kind can be calculated in the same way. Note in particular that in this case,

$$< V_0\phi_1, 1 >_a \neq 0, \quad < V_0\phi_j, 1 >_a = 0, \text{ for } j = 2, \cdots, \mu,$$

which implies that $W_1\phi_j = 0$ for $j \geq 2$. This allows to show that the (i, j)-entry, $i \geq 2, j \geq 2$, of $E_{+-,1}$ is zero, while that of $E_{+-,2}$ is $< (1 - V_1)\phi_i, \phi_j >_a$. □

Remark If 0 is an exceptional point of the first kind, under the assumption $\rho > 2$, one can obtain the leading term in (3.24). In fact, $W(\lambda)T = ((R_0(\lambda) - G_0)V(\lambda) + G_0V_1(\lambda))T$, where $V_1(\lambda) = V(\lambda) - V_0 = O(\lambda < x >^{-2\rho})$. Applying (2.6) with $k = 0$ and $s = \rho/2$

$$W(\lambda)T = O(\lambda^r)$$

in $\mathcal{L}(\mathbf{C}; H^{1,-\rho/2})$ for any $r < (\rho - 1)/4$. Similarly, one has $SW(\lambda) = O(\lambda^r)$ in $\mathcal{L}(H^{1,-\rho/2}; \mathbf{C})$. This gives $SW(\lambda)D(\lambda)W(\lambda)T = O(\lambda^{2r})$. Since $\rho > 2$, we can take r such that $2r > 1/2$. We obtain from the proof of Lemma 3.7 that

$$E_{+-}(\lambda) = \lambda^{1/2}E_{+-,1} + O(|\lambda|^{1/2+\epsilon}).$$

This will allow us to obtain the leading term of the asymptotics of the resolvent in the case of an exceptional point of the first kind under the assumption $\rho > 2$.

Note that in the case of an exceptional point of the second or third kind, the span of $\{\phi_1, \cdots, \phi_\mu\}$ or $\{\phi_2, \cdots, \phi_\mu\}$ is just the 0-eigenspace of $-\Delta + V_0$. Let Π_0 be the spectral projection in L^2 onto the 0-eigenspace of $-\Delta + V_0$.

Proposition 3.8 *(a). Suppose that* 0 *an exceptional point of the first kind. For* $\rho > 2k-1$, $k \geq 2$, $(2k-1)/2 < s < \rho - (2k-1)/2$, *we have*

$$(1 + R_0(\lambda)V(\lambda))^{-1} = \sum_{j=-1}^{k-2} \lambda^{j/2} C_j^{(1)} + O(|\lambda|^{(k-2)/2+\epsilon}) \tag{3.32}$$

where

$$C_{-1}^{(1)} = -c_0^{-1} Q. \tag{3.33}$$

If $\rho > 2$, *one has* $(1 + R_0(\lambda)V(\lambda))^{-1} = -\lambda^{-1/2} c_0^{-1} Q + O(\lambda^{-1/2+\epsilon})$.

(b). Suppose that 0 *is an exceptional point of the second kind. For* $\rho > 2k-1$, $k \in \mathbf{N}$ *with* $k \geq 2$, $(2k-1)/2 < s < \rho - (2k-1)/2$, *we have*

$$(1 + R_0(\lambda)V(\lambda))^{-1} = \sum_{j=-2}^{k-4} \lambda^{j/2} C_j^{(2)} + O(|\lambda|^{(k-4)/2+\epsilon}) \tag{3.34}$$

where

$$C_{-2}^{(2)} = (\Pi_0(1 - V_1)\Pi_0)^{-1}\Pi_0 V_0, \ C_{-1}^{(2)} = -C_{-2}^{(2)} G_3 V_0 C_{-2}^{(2)}. \tag{3.35}$$

(c). Suppose that 0 *is an exceptional point of the third kind. For* $\rho > 2k-1$, $k \in \mathbf{N}$ *with* $k \geq 2$, $(2k-1)/2 < s < \rho - (2k-1)/2$, *we have*

$$(1 + R_0(\lambda)V(\lambda))^{-1} = \sum_{j=-2}^{k-4} \lambda^{j/2} C_j^{(3)} + O(|\lambda|^{(k-4)/2+\epsilon}) \tag{3.36}$$

where

$$C_{-2}^{(3)} = (\Pi_0(1 - V_1)\Pi_0)^{-1}\Pi_0 V_0, \tag{3.37}$$
$$C_{-1}^{(3)} = -C_{-2}^{(3)} G_3 V_0 C_{-2}^{(3)} + c_0^{-1}(1 + A)T_1 T_1^*(1 + A^*)V_0 \tag{3.38}$$

with $A = C_{-2}^{(3)} W_2$.
Here $\lambda \in U_\delta$ *with* $\delta > 0$ *small and* $\epsilon \in]0, 1/2]$ *verifies* $\epsilon < (2s - 2k + 1)/4$ *and the expansions are valid in* $\mathcal{L}(1, -s; 1, -s)$.

Proof. (a). If 0 is an exceptional point of the first kind, then, Lemma 3.7 shows that for $\rho > 2k-1, k \geq 2$,

$$E_{+-}(\lambda) = c_0 \lambda^{1/2}(1 + \sum_{j=1}^{k-1} \lambda^{j/2} c_0^{-1} E_{+-,j+1} + O(|\lambda|^{(k-1)/2+\epsilon})).$$

Since $c_0 \neq 0$, $E_{+-}(\lambda)$ is invertible for λ sufficiently small and $E_{+-}(\lambda)^{-1}$ has an expansion up to the order $\lambda^{(k-2)/2}$ with the leading term $c_0^{-1}\lambda^{-1/2}$. To obtain (3.32), we apply (3.22) and Proposition 3.6 to the order $k-2$ and (3.23) to the order $k-1$.

The case $\rho > 2$ follows from the Remark after Lemma 3.7.

(b). In this case, for $\rho > 2k-1, k \geq 2$, one has

$$E_{+-}(\lambda) = \lambda E_{+-,2} + \sum_{j=3}^{k} \lambda^{j/2} E_{+-,j} + O(|\lambda|^{k/2+\epsilon}).$$

Let $T^* : L^2 \to \mathbf{C}^\mu$ be the adjoint of T. So $S = -T^* V_0$. By Lemma 3.7, $E_{+-,2} = T^*(1-V_1)T$. Since $V_1 \leq 0$, $T^*(1-V_1)T$ is invertible. So, $E_{+-}(\lambda)$ is invertible for λ small and we have an asymptotic expansion for $E_{+-}(\lambda)^{-1}$ to the order $\lambda^{(k-4)/2}$ with the leading term $\lambda^{-1}(T^*(1-V_1)T)^{-1}$. Then, (3.34) follows from (3.22) and Proposition 3.6 to the order $k-4$ and (3.23) to the order $k-2$.

Since $S = -T^* V_0$, we have

$$C_{-2}^{(2)} = -T(T^*(1-V_1)T)^{-1} S = T(T^*(1-V_1)T)^{-1} T^* V_0.$$

To express $C_{-2}^{(2)}$ in terms of the spectral projection Π_0, we remark that

$$\Pi_0 = T(TT^*)^{-1}T^*, \Pi_0 T = T, T^*\Pi_0 = T^*.$$

It is then easy to check

$$(\Pi_0(1-V_1)\Pi_0)T(T^*(1-V_1)T)^{-1}T^* = T(TT^*)^{-1}T^* = \Pi_0$$

and

$$T(T^*(1-V_1)T)^{-1}T^*(\Pi_0(1-V_1)\Pi_0) = T(TT^*)^{-1}T^* = \Pi_0.$$

Therefore, $T(T^*(1-V_1)T)^{-1}T^* = (\Pi_0(1-V_1)\Pi_0)^{-1}\Pi_0$ and

$$C_{-2}^{(2)} = (\Pi_0(1-V_1)\Pi_0)^{-1}\Pi_0 V_0.$$

For $C_{-1}^{(2)}$, we note that $E_{+-,3}$ is reduced to $-SW_3T$ and we have

$$\begin{aligned} & E_{+-}(\lambda)^{-1} \\ = \ & \lambda^{-1}(T^*(1-V_1)T)^{-1} + \lambda^{-1/2}(T^*(1-V_1)T)^{-1}SW_3T(T^*(1-V_1)T)^{-1} + \cdots \end{aligned}$$

Hence,

$$\begin{aligned} C_{-1}^{(2)} &= -(T^*(1-V_1)T)^{-1}SW_3T(T^*(1-V_1)T)^{-1}S \\ &= -C_{-2}^{(2)} W_3 C_{-2}^{(2)} = -C_{-2}^{(2)} G_3 V_0 C_{-2}^{(2)} \end{aligned}$$

(c). In the case where 0 is both an eigenvalue and a resonance, both $E_{+-,1}$ and $E_{+-,2}$ are non nul. To prove the existence of an asymptotic expansion, we need to study $\lambda^{1/2}E_{+-,1} + \lambda E_{+-,2}$. Note that $\rho > 2k-1, k \geq 2$, one has

$$E_{+-}(\lambda) = \lambda^{1/2}E_{+-,1} + \lambda E_{+-,2} + \sum_{j=3}^{k} \lambda^{j/2} E_{+-,j} + O(|\lambda|^{k/2+\epsilon}).$$

$\lambda^{1/2}E_{+-,1} + \lambda E_{+-,2}$ can be written in the form

$$\lambda^{1/2}E_{+-,1} + \lambda E_{+-,2} = \begin{pmatrix} \lambda^{1/2}c_0 + \lambda v_1 & \lambda v'^* \\ \lambda v' & \lambda M \end{pmatrix},$$

where $v' \in \mathbf{C}^{\mu-1}$ and $M = (< (1-V_1)\phi_i, \phi_j >_a)_{2 \leq i,j \leq \mu}$. Since $V_1 \leq 0$, M is invertible. We can compute

$$(\lambda^{1/2}E_{+-,1} + \lambda E_{+-,2}) \begin{pmatrix} \lambda^{-1/2}c_0^{-1} & 0 \\ 0 & \lambda^{-1}M^{-1} \end{pmatrix} = 1 + \begin{pmatrix} \lambda^{1/2}c_0^{-1}v_1 & v'^*M^{-1} \\ \lambda^{1/2}c_0^{-1}v' & 0 \end{pmatrix}$$

The right hand side of the above equation is invertible for λ small and the inverse is given by

$$\begin{pmatrix} b(\lambda) & -b(\lambda)v'^*M^{-1} \\ -\lambda^{1/2}c_0^{-1}b(\lambda)v' & 1+c_0^{-1}b(\lambda)v'v'^*M^{-1} \end{pmatrix},$$

where $b(\lambda) = (1+\lambda^{1/2}c_0^{-1}(v_1 - v'^T M^{-1}v'))^{-1}$. This shows that $\lambda^{1/2}E_{+-,1}+\lambda E_{+-,2}$ is invertible for λ small and its inverse is given by

$$\begin{aligned} \Psi(\lambda) &\equiv (\lambda^{1/2}E_{+-,1}+\lambda E_{+-,2})^{-1} \\ &= \lambda^{-1}\begin{pmatrix} 0 & 0 \\ 0 & M^{-1} \end{pmatrix} + \lambda^{-1/2}c_0^{-1}b(\lambda)\begin{pmatrix} 1 & -v'^*M^{-1} \\ -M^{-1}v' & M^{-1}v'v'^*M^{-1} \end{pmatrix}. \end{aligned}$$

Therefore, $E_{+-}(\lambda)$ is invertible for $\lambda \in U_\delta$ and

$$\begin{aligned} & E_{+-}(\lambda)^{-1} \\ &= \Psi(\lambda)(1+(\sum_{j=3}^{k}\lambda^{j/2}E_{+-,j}+O(|\lambda|^{k/2+\epsilon}))\Psi(\lambda))^{-1} \\ &= \Psi(\lambda)(1-(\sum_{j=3}^{k}\lambda^{j/2}E_{+-,j}+O(|\lambda|^{k/2+\epsilon})\Psi(\lambda))+\cdots) \\ &= \lambda^{-1}\begin{pmatrix} 0 & 0 \\ 0 & M^{-1} \end{pmatrix} + \lambda^{-1/2}c_0^{-1}\begin{pmatrix} 1 & -v'^*M^{-1} \\ -M^{-1}v' & M^{-1}v'v'^*M^{-1} \end{pmatrix} \\ &\quad -\lambda^{-1/2}\begin{pmatrix} 0 & 0 \\ 0 & M^{-1} \end{pmatrix} E_{+-,3}\begin{pmatrix} 0 & 0 \\ 0 & M^{-1} \end{pmatrix} + O(1) \end{aligned}$$

The existence of the expansion (3.36) is deduced as in the first two cases. Remark that $C_{-2}^{(3)}$ and $C_{-1}^{(3)}$ are determined by

$$-TE_{+-}(\lambda)^{-1}S = \lambda^{-1}C_{-2}^{(3)} + \lambda^{-1/2}C_{-1}^{(3)} + \cdots$$

To compute them, write

$$T = (T_1, T'), \quad S = \begin{pmatrix} S_1 \\ S' \end{pmatrix},$$

where T_1 and S_1 are just the first components in T and S. As in (b), we have,

$$T'M^{-1}T'^* = (\Pi_0(1-V_1)\Pi_0)^{-1}\Pi_0.$$

So $C_{-2}^{(3)} = (\Pi_0(1-V_1)\Pi_0)^{-1}\Pi_0 V_0$. Compared with $C_{-1}^{(2)}$, $C_{-1}^{(3)}$ has only an additional term from the 0-resonance

$$\begin{aligned} & -c_0^{-1}T\begin{pmatrix} 1 & -v'^*M^{-1} \\ -M^{-1}v' & M^{-1}v'v'^*M^{-1} \end{pmatrix}S \\ &= c_0^{-1}(T_1T_1^* - T_1v'^*M^{-1}T'^* - T'M^{-1}v'T_1^* + T'M^{-1}v'v'^*M^{-1}T'^*)V_0. \end{aligned}$$

Note that $v' = S'W_2T_1 = -T'^*V_0W_2T_1$ and $T'M^{-1}T'^*V_0 = C_{-2}^{(3)}$. The above equation is equal to

$$= c_0^{-1}(T_1T_1^* + T_1T_1^*W_2^*C_{-2}^{(3)*} + C_{-2}^{(3)}W_2T_1T_1^* + C_{-2}^{(3)}W_2T_1T_1^*W_2^*C_{-2}^{(3)*})V_0.$$

This proves the formula for $C_{-1}^{(3)}$. □

Note that if $V_1 = 0$, one can choose suitably the resonant function ϕ_1 such that $v' = 0$.

3.4 Asymptotic expansions of $R(z)$ when $E_0 \notin \sigma_d(P')$

Now we are ready to deduce the asymptotics of the resolvent $R(z)$ near E_0. This is based on the following representation and Proposition 3.8.

$$\begin{aligned} R(z) &= \epsilon(z) - \epsilon_+(z)\epsilon_{+-}(z)^{-1}\epsilon_-(z) \\ &= \epsilon(z) + \epsilon_+(z)(1 + R_0(\lambda)V(\lambda))^{-1}R_0(\lambda)\epsilon_-(z), \end{aligned} \tag{3.39}$$

with $\lambda = z - E_0$.

Theorem 3.9 *(a). Assume $\rho > \max\{2, 2k-1\}$ for some $k \in \mathbf{N}$ with $k \geq 1$. Suppose that E_0 is an exceptional point of the first kind of P. We have*

$$R(z) = \sum_{j=-1}^{k-2} \lambda^{j/2} B_j^{(1)} + O(|\lambda|^{(k-2)/2+\epsilon}) \tag{3.40}$$

where

$$B_{-1}^{(1)} = i < \cdot, u > u, \tag{3.41}$$

with u an E_0 resonant function of P satisfying

$$\frac{1}{2\sqrt{\pi}} \int_{\mathbf{X}} I_a(x)u(x)\varphi_0(x^a)\, dx = 1. \tag{3.42}$$

(b). Assume $\rho > 2k-1$ for some $k \in \mathbf{N}$ with $k \geq 2$. Suppose that E_0 is an exceptional point of the second kind of P. One has

$$R(z) = \sum_{j=-2}^{k-4} \lambda^{j/2} B_j^{(2)} + O(|\lambda|^{(k-4)/2+\epsilon}) \tag{3.43}$$

where

$$B_{-2}^{(2)} = -\Pi_{E_0}, \tag{3.44}$$

$$B_{-1}^{(2)} = \epsilon_+(E_0)(C_{-1}^{(2)}G_0 + C_{-2}^{(2)}G_1)\epsilon_-(E_0). \tag{3.45}$$

(c). Assume $\rho > 2k-1$ for some $k \in \mathbf{N}$ with $k \geq 2$. Suppose that E_0 is an exceptional point of the third kind of P. One has

$$R(z) = \sum_{j=-2}^{k-4} \lambda^{j/2} B_j^{(3)} + O(|\lambda|^{(k-4)/2+\epsilon}) \tag{3.46}$$

where

$$B_{-2}^{(3)} = -\Pi_{E_0}, \tag{3.47}$$

$$B_{-1}^{(3)} = \epsilon_+(E_0)(C_{-1}^{(3)}G_0 + C_{-2}^{(3)}G_1)\epsilon_-(E_0). \tag{3.48}$$

Π_{E_0} denotes the spectral projection of P onto its E_0-eigenspace. The expansions are valid in $\mathcal{L}(-1, s; 1, -s)$ for $(2k-1)/2 < s < \rho - (2k-1)/2$ if $k \geq 2$, for $1 < s < \rho - 1$ if $k = 1$, $\lambda \in U_\delta$ with $\delta > 0$ small enough and $\epsilon \in]0, 1/2]$ verifies $\epsilon < (2s - 2k + 1)/4$.

Proof. Since $\epsilon(z), \epsilon_+(z)$ and $\epsilon_-(z)$ are holomorphic near E_0, making use of (3.39), the existence of the expansions (3.40) follows from Lemma 2.2 and Proposition 3.8. We need only to give the precision on the leading terms.

(a). By Proposition 3.8 (a), we have

$$\begin{aligned} B_{-1}^{(1)} &= -c_0^{-1}\epsilon_+(E_0)QG_0\epsilon_-(E_0) \\ &= -c_0^{-1} < G_0\epsilon_-(E_0)\cdot, -V_0\phi_1 >_a \epsilon_+(E_0)\phi_1. \end{aligned}$$

The proof of Proposition 3.3 shows that $u_0 = \epsilon_+(E_0)\phi_1$ is a resonant function of P. Since $\epsilon_-(E_0)^* = \epsilon_+(E_0)$ and $G_0V_0\phi_1 = -\phi_1$.

$$< G_0\epsilon_-(E_0)\cdot, -V_0\phi_1 >_a = < \cdot, \epsilon_+(E_0)\phi_1 > .$$

So $B_{-1}^{(1)} = -c_0^{-1} < \cdot, u_0 > u_0$. Let

$$u = \frac{2\sqrt{\pi}}{< V_0\phi_1, 1 >_a}\epsilon_+(E_0)\phi_1.$$

Then u is a resonant function of P and $B_{-1}^{(1)} = i < \cdot, u > u$. To verify the condition (3.42) on u, put

$$\phi = \frac{2\sqrt{\pi}}{< V_0\phi_1, 1 >_a}\phi_1.$$

Then, $u = \epsilon_+(E_0)$ and

$$\frac{1}{2\sqrt{\pi}} < V_0\phi, 1 >_a = 1.$$

As in the proof of Corollary 3.4, we have

$$\frac{1}{2\sqrt{\pi}}\int_{\mathbf{X}} I_a(x)u(x)\varphi_0(x^a)\, dx = \frac{1}{2\sqrt{\pi}} < V_0\phi, 1 >_a = 1.$$

(b). In this case,

$$\begin{aligned} B_{-2}^{(2)} &= \epsilon_+(E_0)(\Pi_0(1-V_1)\Pi_0)^{-1}\Pi_0V_0G_0\epsilon_-(E_0). \\ &= -\epsilon_+(E_0)(\Pi_0(1-V_1)\Pi_0)^{-1}\Pi_0\epsilon_-(E_0). \end{aligned}$$

Denote $v_j = \epsilon_+(E_0)\phi_j$. Then $\{v_j, j = 1, \cdots, \mu\}$ is a basis of the E_0-eigenspace of P. Let $U = \epsilon_+(E_0)T$. Then, $T^*\epsilon_-(E_0) = U^*$. We can calculate

$$\begin{aligned} U^*U &= T^*(1 - (R'(E_0)I_a(\varphi_0 \otimes \cdot), \varphi_0)_a \\ &\quad -(I_aR'(E_0)(\varphi_0 \otimes \cdot), \varphi_0)_a + (I_aR'(E_0)^2I_a(\varphi_0 \otimes \cdot), \varphi_0)_a)T \\ &= T^*(1-V_1)T, \end{aligned}$$

because $(R'(E_0)I_a(\varphi_0 \otimes \cdot), \varphi_0)_a = (I_aR'(E_0)(\varphi_0 \otimes \cdot), \varphi_0)_a = 0$. As seen in the proof of Proposition 3.8,

$$B_{-2}^{(2)} = -\epsilon_+(E_0)T(T^*(1-V_1)T)^{-1}T^*\epsilon_-(E_0)$$

can be simplified to $-U(U^*U)^{-1}U^* = -\Pi_{E_0}$.

The case (c) can be proved in the same way. The details are omitted. $\square$

4 The case $E_0 \in \sigma_d(P')$

In this section, we suppose that $E_0 \in \sigma_d(P')$. Let m be its multiplicity and $\psi_1, \cdots, \psi_m$ be the orthonormal eigenfunctions of P':

$$P'\psi_j = E_0\psi_j, \quad j = 1, \cdots, m.$$

We denote

$$\Pi_1 = \Pi,$$

$$\Pi_2 u = \sum_{j=1}^{m} < u, \psi_j > \psi_j, \quad u \in L^2(\mathbf{X}).$$

Recall that Π is defined in Section 3. Let $\Pi'_j = 1 - \Pi_j, j = 1, 2$, and $\Pi'' = 1 - \Pi_1 - \Pi_2$. Since the ranges of Π_1 and Π_2 are orthogonal, Π'' is an orthogonal projection and

$$\Pi'' = \Pi'_1\Pi'_2 = \Pi'_2\Pi'_1.$$

Let $P'' = \Pi'' P \Pi''$ and $R''(z) = (P'' - z)^{-1}\Pi''$. $R''(z)$ is well-defined in a small neighbourhood of E_0 and is holomorphic there.

Denote $\mathcal{H}^{r,s} = H^{r,s}(\mathbf{X}_a) \times \mathbf{C}^m$ equiped with the norm $(\|u\|^2_{r,s} + |v|^2)^{1/2}$ for $(u, v) \in H^{r,s}(\mathbf{X}_a) \times \mathbf{C}^m$. Let $\mathcal{H} = L^2(\mathbf{X}_a) \times \mathbf{C}^m$. Consider the operator

$$\mathcal{P}(z) = \begin{pmatrix} P - z & \mathcal{E}0_+ \\ \mathcal{E}^0_- & 0 \end{pmatrix} : D(P) \times \mathcal{H} \to L^2(\mathbf{X}) \times \mathcal{H}, \tag{4.1}$$

where

$$\mathcal{E}^0_+ = \begin{pmatrix} \mathcal{E}^0_{+,1} & \mathcal{E}^0_{+,2} \end{pmatrix} : \mathcal{H} \to L^2(\mathbf{X}) \tag{4.2}$$

is defined by

$$\begin{aligned} \mathcal{E}^0_{+,1} f &= \varphi_0 \otimes f, \quad f \in L^2(\mathbf{X}_a), \\ \mathcal{E}^0_{+,2} c &= \sum_{j=1}^{m} c_j \psi_j, \quad c = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \in \mathbf{C}^m, \end{aligned}$$

and

$$\mathcal{E}^0_- = \begin{pmatrix} \mathcal{E}^0_{-,1} \\ \mathcal{E}^0_{-,2} \end{pmatrix} = \mathcal{E}^{0*}_+ : L^2(\mathbf{X}) \to \mathcal{H}.$$

One has

$$\mathcal{E}^0_+ \mathcal{E}^0_- = \Pi_1 + \Pi_2 \quad \text{on } L^2(\mathbf{X}), \quad \mathcal{E}^0_- \mathcal{E}^0_+ = 1 \quad \text{on } \mathcal{H}. \tag{4.3}$$

As in Section 3, we can construct the inverse of $\mathcal{P}(z)$ by first considering an approximate one. Let

$$\mathcal{R}_0(z) = \begin{pmatrix} R''(z) & \mathcal{E}0_+ \\ \mathcal{E}^0_- & \mathcal{E}^0_{+-} \end{pmatrix}, \tag{4.4}$$

where

$$\mathcal{E}^0_{+-}(z) = \mathcal{E}^0_-(z - P)\mathcal{E}^0_+.$$

A direct calculus gives

$$\begin{aligned}
\mathcal{P}(z)\mathcal{R}_0(z) &= \begin{pmatrix} (P-z)R''(z)+\Pi_1+\Pi_2 & (P-z)\mathcal{E}0_+ + \mathcal{E}^0_+\mathcal{E}^0_{+-}(z) \\ \mathcal{E}^0_- R''(z) & 1 \end{pmatrix} \\
&= 1+\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
A &= (\Pi_1 I_a + \Pi_2 P)R''(z), \\
B &= \Pi'' P\mathcal{E}^0_+.
\end{aligned}$$

Note that $\Pi_2\Pi'' = \Pi_2$ and $\Pi_1'\Pi'' = \Pi''$. $\Pi_2 P\Pi'' = \Pi_2 P'\Pi'' = E_0\Pi_2\Pi'' = 0$. This gives

$$\begin{aligned}
A &= \Pi_1 I_a R''(z), && (4.5)\\
B &= \Pi'' P\mathcal{E}^0_+. && (4.6)
\end{aligned}$$

Since $A2 = B2 = 0$, $1+\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ is invertible. This proves that $\mathcal{P}(z)$ admits a right inverse. Similarly, we can show that $\mathcal{P}(z)$ admits also a left inverse. So, $\mathcal{P}(z)$ is invertible and we can calculate

$$\begin{aligned}
\mathcal{P}(z)^{-1} &= \mathcal{R}_0(z)\left(1+\begin{pmatrix} -A & AB-B \\ 0 & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} \mathcal{E}(z) & \mathcal{E}_+(z) \\ \mathcal{E}_-(z) & \mathcal{E}_{+-}(z) \end{pmatrix}, \qquad (4.7)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{E}(z) &= R''(z)(1-A) = R''(z), \\
\mathcal{E}_+(z) &= R''(z)(AB-B)+\mathcal{E}^0_+ = (1-R''(z)\Pi'' P)\mathcal{E}^0_+, \\
\mathcal{E}_-(z) &= \mathcal{E}^0_-(1-A), \\
\mathcal{E}_{+-}(z) &= \mathcal{E}^0_{+-}(z)+\mathcal{E}^0_-(AB-B).
\end{aligned}$$

¿From the above formulae, we derive the representation for the resolvent $R(z)$

$$R(z) = \mathcal{E}(z) - \mathcal{E}_+(z)\mathcal{E}_{+-}(z)^{-1}\mathcal{E}_-(z). \qquad (4.8)$$

$\mathcal{E}(z), \mathcal{E}_+(z), \mathcal{E}_-(z)$ and $\mathcal{E}_{+-}(z)$ are all holomorphic near E_0. Decomposing $\mathcal{E}_\pm 0$ into

$$\mathcal{E}^0_+ = (\mathcal{E}^0_{+,1}, \mathcal{E}^0_{+,2}), \quad \mathcal{E}^0_- = \begin{pmatrix} \mathcal{E}^0_{-,1} \\ \mathcal{E}^0_{-,2} \end{pmatrix},$$

we have

$$\begin{aligned}
\mathcal{E}_{+-}(z) &= \mathcal{E}^0_{+-}(z)+\mathcal{E}^0_-(AB-B) \\
&= \mathcal{E}^0_-(z-P+(\Pi_1 I_a R''(z)-1)\Pi'' P)\mathcal{E}^0_+ \\
&= \begin{pmatrix} \mathcal{E}^0_{-,1}(z-P+\Pi_1 I_a R''(z) I_a)\mathcal{E}^0_{+,1} & -\mathcal{E}^0_{-,1}P\mathcal{E}^0_{+,2} \\ -\mathcal{E}^0_{-,2}P\mathcal{E}^0_{+,1} & \mathcal{E}^0_{-,2}(z-P)\mathcal{E}^0_{+,2} \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} \mathcal{E}^0_{-,1}(z-P+I_aR''(z)I_a)\mathcal{E}^0_{+,1} & -\mathcal{E}^0_{-,1}I_a\mathcal{E}^0_{+,2} \\ -\mathcal{E}^0_{-,2}I_a\mathcal{E}^0_{+,1} & (z-E_0) \end{pmatrix} \\
&\equiv \begin{pmatrix} P_e(z) & -C \\ -C^* & z-E_0 \end{pmatrix}, \qquad (4.9)
\end{aligned}
$$

where

$$
\begin{aligned}
P_e(z) &= \mathcal{E}^0_{-,1}(z-P+I_aR''(z)I_a)\mathcal{E}^0_{+,1} \\
&= z-E_0-(-\Delta_a+(I_a\varphi_0,\varphi_0)_a-(I_aR''(z)I_a(\varphi_0\otimes\cdot),\varphi_0)_a)
\end{aligned}
$$

and $C=\mathcal{E}^0_{-,1}I_a\mathcal{E}^0_{+,2}:\mathbf{C}^m\to L^2(\mathbf{X}_a)$ and $C^*=\mathcal{E}^0_{-,2}I_a\mathcal{E}^0_{+,1}:L^2(\mathbf{X}_a)\to\mathbf{C}^m$ are given by

$$
Cv = \sum_{j=1}^m v_j\phi_j, \quad \phi_j=(I_a\psi_j,\varphi_0)_a, j=1,\cdots,m, \qquad (4.10)
$$

$$
C^*\phi = \begin{pmatrix} < I_a\varphi_0\phi,\psi_1 > \\ \vdots \\ < I_a\varphi_0\phi,\psi_m > \end{pmatrix}. \qquad (4.11)
$$

Note that $\psi_j, j=1,\cdots,m$, are rapidly decreasing on $\mathbf{X}$. C^* extends to an operator from $H^{1,-s}$ to $\mathbf{C}^m$, $\forall s$.

Lemma 4.1 *Assume that rank $\{\phi_1,\cdots,\phi_m\}=m_1<m$. Then E_0 is an eigenvalue of P with multiplicity at least equal to $m-m_1$.*

Proof. Let $(c_1,\cdots,c_m)\in\mathbf{C}^m\setminus\{0\}$ be such that

$$
c_1\phi_1+\cdots+c_m\phi_m=0.
$$

Put $\psi=\sum_{j=1}^m c_j\psi_j$. Then,

$$
(I_a\psi,\varphi_0)_a=0 \text{ and } P'\psi=E_0\psi.
$$

One can check that

$$
(P\psi,\varphi_0)_a=((-\Delta_a+P^a+I_a)\psi,\varphi_0)_a=(-\Delta_a+E_0)((\psi,\varphi_0)_a)=0
$$

because ψ is in the range of Π_1', which implies $(\psi,\varphi_0)_a=0$ in $L^2(\mathbf{X}_a)$. The above equation shows that $P\psi$ is in the range of Π_1' and that $P\psi=\Pi_1'P\psi=P'\psi=E_0\psi$. ψ is an eigenfunction of P associated with E_0. The claim on the multiplicity is easy to verify. □

As before, set $\lambda=z-E_0$. We write $\mathcal{E}_{+-}(z)$ in the form

$$
\mathcal{E}_{+-}(z)=\lambda-\begin{pmatrix} -\Delta+V(\lambda) & C \\ C^* & 0 \end{pmatrix} \qquad (4.12)
$$

where

$$
V(\lambda)=(I_a\varphi_0,\varphi_0)_a-(I_aR''(\lambda+E_0)I_a(\varphi_0\otimes\cdot),\varphi_0)_a
$$

is holomorphic in λ near 0. We shall often expand it in the form

$$
V(\lambda)=V_0+\lambda V_1+\lambda 2V_2+\cdots,\lambda\to 0. \qquad (4.13)
$$

In the following Δ denotes Δ_a. Put

$$
\mathbf{P}(\lambda)=\begin{pmatrix} -\Delta+V(\lambda) & C \\ C^* & 0 \end{pmatrix}, \quad \mathbf{P}_0=\begin{pmatrix} -\Delta & C \\ C^* & 0 \end{pmatrix} \qquad (4.14)
$$

We shall regard $\mathbf{P}_0$ as the unperturbed operator and treat $\mathbf{P}(\lambda)$ as a perturbation of $\mathbf{P}_0$.

4.1 Asymptotics of $(\mathbf{P}_0 - \lambda)^{-1}$

Lemma 4.2 *Let Q be a self-adjoint operator in $L^2(\mathbf{X}_a)$. Let $\lambda \in U_\delta$. Then, Ker $(C^*(Q-\lambda)^{-1}C) =$ Ker C.*

Proof. If $v \in \mathbf{C}^m$ is in Ker $C^*(Q-\lambda)^{-1}C$,

$$< (Q-\lambda)^{-1}Cv, Cv >= 0.$$

Note that $\Im(Q-\lambda)^{-1} > 0$ for $\Im\lambda > 0$. This gives $Cv = 0$. So $v \in$ Ker C. This shows that Ker $C^*(Q-\lambda)^{-1}C \subseteq$ Ker C. The inclusion in the other sense is trivial. □

Let π_0 be the orthogonal projection from $\mathbf{C}^m$ onto Ker C and $\pi_0' = 1 - \pi_0$. π_0' is the projection onto Range C^*. Lemma 4.2 implies that $C^*(Q-\lambda)^{-1}C$ is bijective from Range C^* onto Range C^* and

$$(\pi_0' C^*(Q-\lambda)^{-1}C\pi_0')^{-1}\pi_0 \text{ exists.} \tag{4.15}$$

In the following, we shall use this result with $Q = -\Delta$. In this case, Lemma 4.2 remains true for $\lambda = 0$. In fact, if $C^*G_0Cv = 0$, $v \in \mathbf{C}^m$, then $< G_0Cv, Cv >_a= 0$ which implies $Cv = 0$ in $H^{1,s}$. So $v \in$ Ker C. The implication in the other sense is trivial. This proves that Ker $C^*G_0C =$ Ker C.

To give the asymptotics of $\mathbf{R}_0(\lambda) = (\mathbf{P}_0 - \lambda)^{-1}$, we look at the system:

$$\begin{pmatrix} -\Delta & C \\ C^* & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{4.16}$$

$u = (u_1, u_2), v = (v_1, v_2)$ are in $\mathcal{H}$. This is equivalent with

$$\begin{aligned} u_1 &= (-\Delta - \lambda)^{-1}[-Cu_2 + v_1], \\ v_2 &= C^*(-\Delta - \lambda)^{-1}[-Cu_2 + v_1] - \lambda u_2. \end{aligned}$$

$C^*(-\Delta-\lambda)^{-1}C + \lambda$ is invertible for $\lambda \in U_\delta$. To see this, we note that $C^*(-\Delta-\lambda)^{-1}C + \lambda$ can be decomposed as

$$C^*(-\Delta-\lambda)^{-1}C + \lambda = \lambda\pi_0 + \pi_0'(C^*(-\Delta-\lambda)^{-1}C + \lambda)\pi_0'$$

and has an asymptotic expansion up to any order given by

$$C^*G_0C + \lambda^{1/2}C^*G_1C + \lambda(1 + C^*G_2C) + \cdots \tag{4.17}$$

Since, by Lemma 4.2 with $Q = -\Delta$ and $\lambda = 0$, $\pi_0'C^*G_0C\pi_0'$ is invertible on the range of π_0', $\pi_0'(C^*(-\Delta-\lambda)^{-1}C + \lambda)\pi_0'$ is also invertible there for λ small and we obtain

$$(C^*(-\Delta-\lambda)^{-1}C + \lambda)^{-1} = \lambda^{-1}\pi_0 + (\pi_0'(C^*(-\Delta-\lambda)^{-1}C + \lambda)\pi_0')^{-1}\pi_0'. \tag{4.18}$$

The solution of (4.16) is given by

$$u_1 = (-\Delta - \lambda)^{-1}[-Cu_2 + v_1], \tag{4.19}$$

$$u_2 = (C^*(-\Delta-\lambda)^{-1}C + \lambda)^{-1}[C^*(-\Delta-\lambda)^{-1}v_1 - v_2]. \tag{4.20}$$

The expressions for u_1 and u_2 in terms of v_1 and v_2 give the following

Proposition 4.3 *For $\lambda \in U_\delta$,*

$$\mathbf{R}_0(\lambda) = \begin{pmatrix} (-\Delta-\lambda)^{-1} - D(\lambda)F(\lambda)D^\dagger(\lambda) & D(\lambda)F(\lambda) \\ F(\lambda)D^\dagger(\lambda) & -\lambda^{-1}\pi_0 - F(\lambda) \end{pmatrix} \tag{4.21}$$

where

$$\begin{aligned} D(\lambda) &= (-\Delta-\lambda)^{-1}C \\ D^\dagger(\lambda) &= C^*(-\Delta-\lambda)^{-1} \\ F(\lambda) &= (\pi_0'(C^*(-\Delta-\lambda)^{-1}C+\lambda)\pi_0')^{-1}\pi_0'. \end{aligned}$$

All these operators admit a complete expansion in the powers of $\lambda^{1/2}$ in suitable spaces. In particular,

$$F(\lambda) = F_0 + \lambda^{1/2}F_1 + \lambda F_2 \cdots : \mathbf{C}^m \to \mathbf{C}^m$$

with

$$\begin{aligned} F_0 &= (\pi_0' C^* G_0 C \pi_0')^{-1}\pi_0', \\ F_1 &= -F_0 C^* G_1 C F_0, \\ F_2 &= -F_0(1+C^*G_2C)F_0 + (F_0C^*G_1C)2F_0. \end{aligned}$$

4.2 Spectral properties of P at the threshold

Let

$$\mathbf{P} = \mathbf{P}(0) = \begin{pmatrix} -\Delta + V_0 & C \\ C^* & 0 \end{pmatrix}.$$

We shall say that 0 is a resonance of $\mathbf{P}$ if the equation

$$\begin{pmatrix} -\Delta + V_0 & C \\ C^* & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \tag{4.22}$$

has a solution for $u = (u_1, u_2) \in \mathcal{H}^{1,-s} = H^{1,-s}(\mathbf{X}_a) \times \mathbf{C}^m$ for any $s > 1/2$ and $u_1 \notin L^2$.

As in Section 3, one can show the following

Lemma 4.4 *E_0 is an eigenvalue (resonance, resp.) of P if and only if 0 is an eigenvalue (resonance, resp.) of $\mathbf{P}$ and the dimension of the eigenspaces (the resonant spaces, resp.) is the same.*

(4.22) is equivalent with

$$\begin{cases} (-\Delta + V_0)u_1 + Cu_2 = 0 \\ C^*u_1 = 0 \end{cases} \tag{4.23}$$

or

$$\begin{cases} (1 + G_0V_0)u_1 + G_0Cu_2 = 0 \\ C^*u_1 = 0 \end{cases} \tag{4.24}$$

This shows that

$$u_1 \in \text{ Ker } C^*, C^*G_0Cu_2 = -C^*G_0V_0u_1.$$

Write $u_2 = v_1 + v_2$ with $v_1 \in \text{Ker } C$ and $v_2 \in \text{Range } C^*$. Since C^*G_0C is invertible on the range of C^*, v_2 is uniquely determined by u_1

$$v_2 = -(\pi_0'C^*G_0C\pi_0')^{-1}\pi_0'C^*G_0V_0u_1 = -F_0C^*G_0V_0u_1. \tag{4.25}$$

Therefore, u_1 satisfies

$$\begin{cases} (-\Delta + V_0 - CF_0C^*G_0V_0)u_1 = 0 \\ C^*u_1 = 0 \end{cases} \tag{4.26}$$

Set

$$\pi_1 = C(\pi_0'C^*G_0C\pi_0')^{-1}\pi_0'C^*G_0. \tag{4.27}$$

Then $\pi_1 2 = \pi_1$ and π_1 is the projection of $H^{-1,s}$ onto Range (C) with the kernel

$$\text{Ker } \pi_1 = \text{ Ker } (C^*G_0).$$

Let $\pi_1' = 1 - \pi_1$. π_1' is the projection $H^{-1,s} \to$ Ker C^*G_0. Put

$$V_0^{\#} = \pi_1'V_0.$$

Summing up, we have proved the following

Proposition 4.5 *Let $u = (u_1, u_2) \in \mathcal{H}^{1,-s} = H^{1,-s}(\mathbf{X}_a) \times \mathbf{C}^k$. Then,*

$$\begin{pmatrix} -\Delta + V_0 & C \\ C^* & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \tag{4.28}$$

if and only if u_1 satisfies

$$\begin{cases} (-\Delta + V_0^{\#})u_1 = 0 \\ C^*u_1 = 0 \end{cases} \tag{4.29}$$

and then u_2 is given by

$$u_2 = v_1 + v_2, \quad v_1 \in \text{ Ker } C, v_2 = -\pi_1V_0u_1. \tag{4.30}$$

In the following, we shall say that u_1 is an eigenvalue or a resonance of $-\Delta + V_0^{\#}$ if there exists $u_1 \in H^{1,-s}$, s suitable, satisfying (4.29). The spectral nature of 0 with respect to $-\Delta + V_0^{\#}$ is classified as for P. Although $V_0^{\#}$ is non-local and non-selfadjoint, we can still mimic the proof of Lemma 3.3 of [9] for two-body Schrödinger operators to derive the following

Corollary 4.6 *The kernel of $\mathbf{P}$ in $\mathcal{H}^{1,-s}, \forall s > 1/2$, is of finite dimension. The resonance of $\mathbf{P}$ at 0 (hence the resonance of P at E_0) is at most simple and if $u_1 \in H^{1,-s}, \forall s > 1/2$, is a solution of (4.29), then $u_1 \in L^2$ if and only if*

$$< V_0^{\#}u_1, 1 >= 0$$

and in this case, (u_1, u_2) with u_2 given by (4.30) is an eigenfunction of $\mathbf{P}$.

Theorem 1.1 in the case $E_0 \in \sigma_d(P')$ can be derived from Proposition 4.5 and its Corollary.

Proof of Theorem 1.1 in the case $E_0 \in \sigma_d(P')$. Note that $u = (u_1, u_2)$ is a solution of (4.28) if and only if $w = \mathcal{E}_+(E_0)u$ is in the kernel of $P - E_0$ in $H^{1,-s}$ and that $w \in L^2$ if and only if $u \in \mathcal{H}$. Making use of the expression of $\mathcal{E}_+(E_0)$, one has

$$\begin{aligned} &\int_{\mathbf{X}} I_a(x)w(x)\varphi_0(x^a)\, dx \\ &= \; < (I_a(\mathcal{E}_{+,1,0}u_1 + \mathcal{E}_{+,2,0}u_2), \varphi_0)_a, 1 >_a \\ &= \; < (1 - CF_0C^*G_0)V_0u_1, 1 >_a = < V_0^{\#}u_1, 1 >_a = 0. \end{aligned}$$

This finishes the proof of Theorem 1.1. □

As in Section 3, the main step in obtaining an asymptotic expansion of the resolvent $R(z)$ is to establish that for $\mathcal{E}_{+-}(z)^{-1}$. As before, we shall apply the method of Grushin to reduce to a holomorphic family of matrices whose behaviour near 0 depends on the threshold properties of P. We write

$$\mathbf{P}(\lambda) = \mathbf{P}_0 + \mathbf{V}(\lambda), \text{ where } \mathbf{V}(\lambda) = \begin{pmatrix} V(\lambda) & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.31}$$

Then,

$$-\mathcal{E}_{+-}(z)^{-1} = (1 + \mathbf{R}_0(\lambda)\mathbf{V}(\lambda))^{-1}\mathbf{R}_0(\lambda). \tag{4.32}$$

Using Proposition 4.3, we can compute

$$\mathbf{T}(\lambda) \equiv \mathbf{R}_0(\lambda)\mathbf{V}(\lambda) = \begin{pmatrix} A(\lambda) & 0 \\ B(\lambda) & 0 \end{pmatrix}$$

where

$$A(\lambda) = ((-\Delta - \lambda)^{-1} - D(\lambda)F(\lambda)D^{\dagger}(\lambda))V(\lambda) \tag{4.33}$$

$$B(\lambda) = F(\lambda)D^{\dagger}(\lambda)V(\lambda). \tag{4.34}$$

An elementary calculus shows that

$$(1 + \mathbf{T}(\lambda))^{-1} = \begin{pmatrix} (1 + A(\lambda))^{-1} & 0 \\ -B(\lambda)(1 + A(\lambda))^{-1} & 1 \end{pmatrix} \tag{4.35}$$

Set

$$\mathbf{T}_0 = \mathbf{T}(0) = \begin{pmatrix} (G_0 - D(0)F_0D^{\dagger}(0))V_0 & 0 \\ F_0D^{\dagger}(0)V_0 & 0 \end{pmatrix}.$$

$\mathbf{T}_0$ is a compact operator on $\mathcal{H}^{1,-s}$, $1/2 < s < \rho - 1/2$. The kernel, $\mathbf{K}$, of $1 + \mathbf{T}_0$ in $\mathcal{H}^{1,-s}$ is related to that of $\mathcal{P}$.

Let $(u, v) \in \text{Ker } (1 + \mathbf{T}_0)$. Then,

$$\begin{cases} (1 + (G_0 - D(0)F_0D^{\dagger}(0))V_0)u = 0 \\ F_0D^{\dagger}(0)V_0u + v = 0 \end{cases}$$

Therefore, v is uniquely determined by u

$$v = -F_0D^{\dagger}(0)V_0u. \tag{4.36}$$

It is clear that $v \in \text{range } C^*$. The equation for u shows that

$$C^*u = -(C^*G_0 - C^*G_0CF_0C^*G_0)V_0u$$

As in the proof of Proposition 4.5, we can show that

$$C^*G_0CF_0 = C^*G_0C(\pi_0'C^*G_0C\pi_0')^{-1}\pi_0' = C^*G_0C(C^*G_0C)^{-1}\pi_0' = \pi_0',$$

where π_0' is the projection onto RangeC^* with Ker $\pi_0' =$ Ker C. This proves

$$C^*u = -(C^*G_0 - \pi_0'C^*G_0)V_0u = 0$$

Consequently, $u \in \text{Ker } C^*$ and u satisfies the equation

$$(-\Delta + V_0 - CF_0C^*G_0V_0)u = 0.$$

We have proved that $(u, v) \in$ Ker $(1 + \mathbf{T}_0)$ if and only if

$$(-\Delta + V_0^{\#})u = 0, \quad v = -F_0C^*G_0V_0u. \tag{4.37}$$

This shows that (u, v) is in Ker $\mathbf{P}$ with $v \in \text{Range } C^*$. These arguments can be reversed. The relation between $\mathbf{T}_0$ and $A(0)$ gives immediately the following

Proposition 4.7 *The kernel of* $1+\mathbf{T}_0$ *in* $\mathcal{H}^{1,-s}$ *is given by (4.37) and coincides with that of* $\mathbf{P}$ *in* $H^{1,-s}\times$ *Range* C^*. $u \in$ *Ker* $(1+A(0))$ *if and only if* $(u,v) \in$ *Ker* $(1+\mathbf{T}_0)$ *with* v *given by* $v = -F_0 C^* G_0 V_0 u$.

Note that we can decompose

$$\text{Ker } \mathbf{P} = \text{ Ker } (1+\mathbf{T}_0) \oplus (\{0\} \times \text{ Ker } C).$$

Note that the part $\{0\} \times$ Ker C of the kernel of $\mathbf{P}$ is already taken into account in $\mathbf{R}_0(\lambda)$. This separation is necessary for our method to work.

Proposition 4.8 *Let* $A_0 = A(0)$. *Then one has*
(a). $\pi_1^* = 0, \quad \pi_1'^* = 1$ *on Ker* $(1+A_0)$.
(b). $G_0\pi_1 = \pi_1^* G_0$, $G_0\pi_1' = \pi_1'^* G_0$ *and* $C^* G_0 \pi_1' = 0$.
(c). $< G_2 V_0^{\#} u, V_0^{\#} v >=< u, v >$ *for* $u, v \in$ *Ker* $(1+A_0)\cap L^2$.

Proof. Recall that $\pi_1' = 1-\pi_1$ and $\pi_1 = CF_0C^*G_0$. Since Ker $(1+A_0) \subset$ Ker C^* by proposition 4.7, (a) is evident.

To see (b), we remark

$$G_0\pi_1 = G_0 CF_0C^*G_0 = \pi_1^* G_0$$

and

$$C^*G_0\pi_1 = C^*G_0CF_0C^*G_0 = \pi_0' C^* G_0 = C^* G_0.$$

Therefore, $C^*G_0\pi_1' = 0$.

(c). It can be proved as in Lemma 3.8. □

4.3 Asymptotic expansions of $(1+A(\lambda))^{-1}$

The asymptotic expansion of $(1+A(\lambda))^{-1}$ can be obtained as in Section 3.3 by the method of Grushin, but the calculus is more complicated. We begin with the following preparatory result.

Lemma 4.9 *Let* $\rho > \max\{2, 2k+1\}$, $k \in \mathbf{N}$.

$$A(\lambda) = \sum_{j=0}^{k} \lambda^{j/2} A_j + O(\lambda^{k/2+\epsilon}), \lambda \to 0, \tag{4.38}$$

in $\mathcal{L}(1,-s;1,-s)$ *for* $1 < s < \rho - 1$ *if* $k=0$ *and* $(2k+1)/2 < s < \rho - (2k+1)/2$ *if* $k \geq 1$. *Here* $\epsilon > 0$ *can be chosen as in Lemma 2.2 and*

$$A_0 = G_0\pi_1' V_0 \tag{4.39}$$
$$A_1 = G_1\pi_1' V_0 + G_0 M_1 V_0 \tag{4.40}$$
$$A_2 = G_0\pi_1' V_1 + G_0 M_2 V_0 + G_1 M_1 V_0 + G_2 \pi_1' V_0 \tag{4.41}$$

with

$$M_1 = -CF_0C^*G_1\pi_1',$$
$$M_2 = -CF_2C^*G_0 - CF_1C^*G_1 - CF_0C^*G_2.$$

Proof. We can write

$$A(\lambda) = R_0(\lambda)(1 - CF(\lambda)C^* R_0(\lambda))V(\lambda)$$

Notice that $C^* : H^{1,-s} \to \mathbf{C}^m$ is well defined for any $s \in \mathbf{R}$. Therefore, for any $k \in \mathbf{N}$ and $s > (2k+1)/2$,

$$1 - CF(\lambda)C^* R_0(\lambda) = \sum_{j=0}^{k} \lambda^{j/2} M_j + O(\lambda^{k/2+\epsilon}), \text{ in } \mathcal{L}(-1, s; -1, s), \tag{4.42}$$

where

$$M_0 = 1 - CF_0C^*G_0 = \pi_1',$$

π_1' being the projection onto Ker C^*G_0 and

$$M_1 = -CF_1C^*G_0 - CF_0C^*G_1,$$

$$M_2 = -CF_2C^*G_0 - CF_1C^*G_1 - CF_0C^*G_2.$$

Note that $F_1 = -F_0C^*G_1CF_0$ and $CF_0C^*G_0 = \pi_1$. one has

$$M_1 = CF_0C^*G_1CF_0C^*G_0 - CF_0C^*G_1 = CF_0C^*G_1(\pi_1 - 1) = -CF_0C^*G_1\pi_1'.$$

(4.38) then follows from the asymptotic expansions of $R_0(\lambda)$ and $V(\lambda)$ in suitable spaces. □

Let $\nu = \dim\ \mathrm{Ker}\ (1 + A(0)) = \dim\ \mathrm{Ker}\ (1 + \mathbf{T}_0)$ and $\theta = \dim\ \mathrm{Ker}\ C$. Then,

$$\nu + \theta = \dim\ \mathrm{Ker}\ \mathbf{P}. \tag{4.43}$$

By Proposition 4.8, $\pi_1'^* = 1$ on Ker $(1 + A_0)$. The map

$$(\ \mathrm{Ker}\ (1 + A_0))2 \ni (u, v) \to < u, -V_0v >_a = < u, -V_0^{\#}v >_a \in \mathbf{C}$$

is a Hermitian form and is positive, since $< -V_0^{\#}u, u >_a = < -\Delta u, u >_a$ is positive on Ker $(1+A_0)$. As in Section 3, we can show that it is non degenerate. Consequently, we can choose a basis $\{u_1, \cdots, u_\nu\}$ of Ker $(1 + A_0)$ such that

$$< u_i, -V_0u_j >_a = \delta_{ij}, \tag{4.44}$$

and if the 0-resonance of $-\Delta + V_0^{\#}$ exists, we take $\{u_j\}$ such that u_1 is a resonant function and $u_j, j \geq 2$, are eigenfunctions.

Proposition 4.10 *(a). Suppose that 0 is a regular point for $-\Delta + V_0^{\#}$ (in this case, $\nu = \theta = 0$). Then for $\rho > \max\{2, 2k+1\}$, $k \in \mathbf{N}$, we have*

$$(1 + A(\lambda))^{-1} = \sum_{j=0}^{k} \lambda^{j/2} A_j^{(0)} + O(|\lambda|^{k/2+\epsilon}) \tag{4.45}$$

in $\mathcal{L}(1, -s; 1, -s)$ for $1 < s < \rho - 1$ when $k = 0$ and $(2k+1)/2 < s < \rho - (2k+1)/2$ when $k \geq 1$. Here

$$A_0^{(0)} = (1 + A_0)^{-1}. \tag{4.46}$$

(b). Suppose that 0 is an exceptional point of the first kind of $-\Delta + V_0^{\#}$ (in this case, $\nu = 1, \theta = 0$). For $\rho > \max\{2, 2k-1\}$, $k \geq 1$, we have

$$(1 + A(\lambda))^{-1} = \sum_{j=-1}^{k-2} \lambda^{j/2} A_j^{(1)} + O(|\lambda|^{(k-2)/2+\epsilon}) \tag{4.47}$$

in $\mathcal{L}(1,-s;1,-s)$ *for* $1 < s < \rho - 1$ *when* $k = 1$ *and* $(2k-1)/2 < s < \rho - (2k-1)/2$ *when* $k \geq 2$, *where*

$$A_{-1}^{(1)} = -c_0^{-1} < \cdot, -V_0 u_1 >_a u_1 \textit{ with } c_0 = \frac{i}{4\pi} | < V_0^{\#} u_1, 1 >_a |^2. \tag{4.48}$$

(c). Suppose that 0 *is an exceptional point of the second (* $l = 2$ *) or the third (* $l = 3$*) kind of* $-\Delta + V_0^{\#}$. *For* $\rho > 2k-1$, $k \in \mathbf{N}$ *with* $k \geq 2$, $(2k-1)/2 < s < \rho - (2k-1)/2$, *we have*

$$(1 + A(\lambda))^{-1} = \sum_{j=-2}^{k-4} \lambda^{j/2} A_j^{(l)} + O(|\lambda|^{(k-4)/2+\epsilon}) \tag{4.49}$$

where

$$A_{-2}^{(l)} = (\Pi_0(1 - V_1 + W^* W)\Pi_0)^{-1} \Pi_0 V_0, \quad l = 2, 3. \tag{4.50}$$

Here Π_0 *is the orthogonal projection in* L^2 *onto the* 0 *eigenspace of* $-\Delta + V_0^{\#}$ *and*

$$W = F_0 C^* G_0 V_0.$$

The above expansions are valid in $\mathcal{L}(1,-s;1,-s)$ *for* $\lambda \in U_\delta$ *with* $\delta > 0$ *small and* $\epsilon > 0$ *is the same as in proposition 3.9.*

Proof. The method is the same as in Subsection 3.3. We just sketch the main steps for (b) and (c).

In order to get the expansion of $(1 + A(\lambda))^{-1}$, we consider

$$\mathcal{F}(\lambda) = \begin{pmatrix} 1 + A(\lambda) & \mathcal{T} \\ \mathcal{S} & 0 \end{pmatrix} : H^{1,-s} \times \mathbf{C}^\nu \to H^{1,-s} \times \mathbf{C}^\nu,$$

where $s > 1/2$, operators $\mathcal{T}$ and $\mathcal{S}$ are defined by

$$\mathcal{T} \begin{pmatrix} c_1 \\ \vdots \\ c_\nu \end{pmatrix} = \sum_{j=1}^{\nu} c_j u_j, \quad \begin{pmatrix} c_1 \\ \vdots \\ c_\nu \end{pmatrix} \in \mathbf{C}^\nu,$$

and

$$\mathcal{S} f = \begin{pmatrix} < f, -V_0 u_1 >_a \\ \vdots \\ < f, -V_0 u_\nu >_a \end{pmatrix}, \ f \in H^{1,-s}.$$

Let $\mathcal{Q} f = \sum_{j=1}^{\nu} < f, -V_0 u_j >_a u_j$. Then,

$$\mathcal{T}\mathcal{S} = \mathcal{Q} \text{ on } H^{1,-s} \text{ and } \mathcal{S}\mathcal{T} = I_\nu \text{ on } \mathbf{C}^\nu.$$

$\mathcal{Q}$ is the projection from $H^{1,-s}$ onto the range of $\mathcal{T}$ with Ker $\mathcal{Q}$ = Ker $\mathcal{S}$. The following relations can be easily verified:

$$\mathcal{S}(1 + A_0) = 0, \qquad (1 + A_0)\mathcal{T} = 0, \tag{4.51}$$

$$\mathcal{S}\mathcal{Q} = \mathcal{S}, \qquad \mathcal{Q}\mathcal{T} = \mathcal{T}. \tag{4.52}$$

Let $\mathcal{Q}' = 1 - \mathcal{Q}$. Then one can show that

$$\mathcal{D}(\lambda) = (\mathcal{Q}'(1 + A(\lambda))Q')^{-1} Q'.$$

is well-defined near $\lambda = 0$ and

$$\mathcal{D}(\lambda) = \sum_{j=0}^{k} \lambda^{j/2}\mathcal{D}_j + O(|\lambda|^{k/2+\epsilon}), \tag{4.53}$$

in $\mathcal{L}(1, -s; 1, -s), (2k+1)/2 < s < \rho - (2k+1)/2$ and

$$\mathcal{D}_0 = (\mathcal{Q}'(1 + A_0)\mathcal{Q}')^{-1}\mathcal{Q}', \mathcal{D}_1 = -\mathcal{D}_0 A_1 \mathcal{D}_0.$$

In the case Ker $(1 + A_0) = \{0\}$, $\nu = 0$ and $\mathcal{Q}' = 1$. In this case, (a) follows from (4.53).

In the general case, we construct the inverse of $\mathcal{F}(\lambda)$ and give a representation formula for $(1 + A(\lambda))^{-1}$.

Using the operator $\mathcal{D}(\lambda)$, we can first construct an approximate inverse $\mathcal{G}(\lambda)$,

$$\mathcal{G}(\lambda) = \begin{pmatrix} \mathcal{D}(\lambda) & \mathcal{T} \\ \mathcal{S} & -\mathcal{S}(1 + A(\lambda))\mathcal{T} \end{pmatrix} : H^{1,-s} \times \mathbf{C}^\nu \to H^{1,-s} \times \mathbf{C}^\nu, s > 1/2.$$

of $\mathcal{F}(\lambda)$ to show that $\mathcal{F}(\lambda)$ is invertible and then to calculate explicitly the inverse

$$\mathcal{F}(\lambda)^{-1} = \begin{pmatrix} E(\lambda) & E_+(\lambda) \\ E_-(\lambda) & E_{+-}(\lambda) \end{pmatrix},$$

where

$$\begin{aligned} E(\lambda) &= \mathcal{D}(\lambda), & (4.54)\\ E_+(\lambda) &= \mathcal{T} - \mathcal{D}(\lambda)\mathcal{Q}'(1 + A(\lambda))\mathcal{T}, & (4.55)\\ E_-(\lambda) &= \mathcal{S} - \mathcal{S}\mathcal{Q}(1 + A(\lambda))\mathcal{Q}'\mathcal{D}(\lambda), & (4.56)\\ E_{+-}(\lambda) &= \mathcal{S}(-1 + (1 + A(\lambda))\mathcal{Q}'\mathcal{D}(\lambda)\mathcal{Q}')(1 + A(\lambda))\mathcal{T}. & (4.57) \end{aligned}$$

As before, we obtain from this inverse an expression for the inverse of $1 + A(\lambda)$:

$$(1 + A(\lambda))^{-1} = E(\lambda) - E_+(\lambda)E_{+-}(\lambda)^{-1}E_-(\lambda). \tag{4.58}$$

Remark also that if $\rho > 2k + 1$, $E_\pm(\lambda)$ have an asymptotics expansion up to an error $O(|\lambda|^{k/2+\epsilon})$.

$$E_\pm(\lambda) = \sum_{j=0}^{k} \lambda^{j/2} E_{\pm,j} + O(|\lambda|^{k/2+\epsilon}), \tag{4.59}$$

in $\mathcal{L}(\mathbf{C}^\mu; H^{1,-s})$ or $\mathcal{L}(H^{1,-s}; \mathbf{C}^\mu)$, $s > k + 1/2$, where

$$E_{-,0} = \mathcal{S}, \quad E_{-,1} = -\mathcal{S}A_1\mathcal{D}_0,$$

and

$$E_{+,0} = \mathcal{T}, \quad E_{+,1} = -\mathcal{D}_0 A_1 \mathcal{T}.$$

For $E_{+-}(\lambda)$, we have a similar result of Lemma 3.7. Assume $\rho > 2k - 1$ and $k \geq 2$. One has:

$$E_{+-}(\lambda) = \sum_{j=1}^{k} \lambda^{j/2} E_{+-,j} + O(|\lambda|^{k/2+\epsilon}). \tag{4.60}$$

Here

$$\begin{aligned} E_{+-,1} &= -\mathcal{S}A_1\mathcal{T}, & (4.61)\\ E_{+-,2} &= -\mathcal{S}A_2\mathcal{T} + \mathcal{S}A_1\mathcal{D}_0A_1\mathcal{T}, & (4.62)\\ E_{+-,3} &= -\mathcal{S}A_3\mathcal{T} + \mathcal{S}(A_1\mathcal{D}_1A_1 + A_2\mathcal{D}_0A_1 + A_1\mathcal{D}_0A_2)\mathcal{T} & (4.63) \end{aligned}$$

and $\epsilon > 0$ is the same as in Proposition 3.5.

We should calculate the leading terms of $E_{+-}(\lambda)$ to show that $E_{+-}(\lambda)^{-1}$ admits an asymptotic expansion. We carry out the computation in the most general case, *i.e.*, 0 is an eigenvalue and a resonance of $-\Delta + V_0^{\#}$.

Since $A_1 = G_1 V_0^{\#} + G_0 M_1 V_0$,

$$\begin{aligned}
E_{+-,1} &= (< (G_1 V_0^{\#} + G_0 M_1 V_0) u_i, V_0 u_j >_a)_{i,j=1,\cdots,\nu} \\
&= (< G_1 V_0^{\#} u_i, V_0 u_j >_a - < G_0 C F_0 C^* G_1 V_0^* u_i, u_j >_a)_{i,j=1,\cdots,\nu} \\
&= (< \pi_1'^* G_1 V_0^{\#} u_i, V_0 u_j >_a)_{i,j=1,\cdots,\nu} \qquad (4.64)
\end{aligned}$$

G_1 being an integral operator with the constant integral kernel $i/(4\pi)$,

$$< G_1 V_0^{\#} u_i, V_0^{\#} u_j >_a = \frac{i}{4\pi} < V_0^{\#} u_i, 1 >_a \overline{< V_0^{\#} u_j, 1 >_a}$$

By the characterization of resonant functions, $< V_0^{\#} u_i, 1 >_a \neq 0$ only when $i = 1$. So we obtain

$$E_{+-,1} = \begin{pmatrix} c_0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad (4.65)$$

with

$$c_0 = \frac{i}{4\pi} | < V_0^{\#} u_1, 1 >_a |^2 \neq 0$$

Write $E_{+-,2}$ as

$$E_{+-,2} = (< A_2 u_i, V_0 u_j >_a - < A_1 \mathcal{D}_0 A_1 u_i, V_0 u_j >_a)_{i,j=1,\cdots,\nu}.$$

Note that

$$A_1 = G_1 V_0^{\#} - G_0 C F_0 C^* G_1 V_0^{\#} = \pi_1'^* G_1 V_0^{\#}.$$

Making use of the fact $< V_0^{\#} u_j, 1 >_a = 0$ for $j \geq 2$, we can compute

$$\begin{aligned}
&< A_1 \mathcal{D}_0 A_1 u_i, V_0^{\#} u_j >_a \\
&= \; < G_1 V_0^{\#} \mathcal{D}_0 \pi_1'^* G_1 V_0^{\#} u_i, V_0^{\#} u_j >_a \\
&= \; 0, \quad \text{if } (i,j) \neq (1,1)
\end{aligned}$$

Set

$$b_{11} = < G_1 V_0^{\#} \mathcal{D}_0 \pi_1'^* G_1 V_0^{\#} u_1, V_0^{\#} u_1 >_a .$$

Then,

$$(< A_1 \mathcal{D}_0 A_1 u_i, V_0^{\#} u_j >_a)_{i,j=1,\cdots,\nu} = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \qquad (4.66)$$

The operator A_2 is given by

$$A_2 = G_0 \pi_1' V_1 + G_2 V_0^{\#} + G_1 M_1 V_0 + G_0 M_2 V_0$$

with

$$\begin{aligned}M_1 &= -CF_0C^*G_1\pi_1',\\ M_2 &= = -CF_2C^*G_0 - CF_1C^*G_1 - CF_0C^*G_2.\end{aligned}$$

Using the formulae of F_1 and F_2, we can calculate

$$\begin{aligned}G_0M_2V_0 &= -G_0C(F_2C^*G_0 + F_1C^*G_1 + F_0C^*G_2)V_0\\ &= -G_0C((-F_0(1 + C^*G_2C)F_0 + (F_0C^*G_1C)^2F_0)C^*G_0\\ &\quad -F_0C^*G_1CF_0C^*G_1 + F_0C^*G_2)V_0\\ &= G_0C(F_02C^*G_0 - F_0C^*G_2\pi_1' + F_0C^*G_1CF_0C^*G_1\pi_1')V_0\\ &= (G_0CF_02C^*G_0 - \pi_1^*G_2\pi_1' + \pi_1^*G_1CF_0C^*G_1\pi_1')V_0.\end{aligned}$$

We can put A_2 into a more explicit form

$$A_2 = G_0\pi_1'V_1 + G_0CF_02C^*G_0V_0 + \pi_1'^*G_2\pi_1'V_0 - \pi_1'^*G_1CF_0C^*G_1\pi_1'V_0. \tag{4.67}$$

The contribution to $E_{+-,2}$ of each term of A_2 can be calculated as follows. Since $G_0\pi_1' = \pi_1'^*G_0$,

$$\begin{aligned}< G_0\pi_1'V_1u_i, V_0u_j >_a &= < \pi_1'^*G_0V_1u_i, V_0u_j >_a\\ =< V_1u_i, G_0V_0^\# u_j >_a &= - < V_1u_i, u_j >_a .\end{aligned}$$

Let $W = F_0C^*G_0V_0$. Then

$$< G_0CF_02C^*G_0V_0u_i, V_0u_j >_a =< W^*Wu_i, u_j >_a .$$

By Proposition 4.8 (c), one has

$$\begin{aligned}< \pi_1'^*G_2V_0^\# u_i, V_0u_j >_a &= < G_2V_0^\# u_i, V_0^\# u_j >_a\\ &= < u_i, u_j >_a\end{aligned}$$

for $i, j = 2, \cdots, \nu$. By the characterization of resonant function,

$$- < \pi_1'^*G_1CF_0C^*G_1V_0^\# u_i, V_0u_j >_a = - < G_1CF_0C^*G_1V_0^\# u_i, V_0^\# u_j >_a = 0,$$

for $(i, j) \neq (1, 1)$.

In the basis $(u_1, \cdots, u_\nu)$, $E_{+-,2}$ can be written in the form

$$E_{+-,2} = \begin{pmatrix} a_1 & a_2 & \cdots & a_\nu\\ \bar{a}_2 & e_{22} & \cdots & e_{2\nu}\\ \vdots & \vdots & \ddots & \vdots\\ \bar{a}_\nu & e_{\nu 2} & \cdots & e_{\nu\nu}\end{pmatrix} \tag{4.68}$$

Here $a_j \in \mathbf{C}$ and

$$e_{ij} =< (1 - V_1 + W^*W)u_i, u_j >_a . \tag{4.69}$$

It is important to notice that $V_1 \leq 0$. Consequently, the matrix

$$e = \begin{pmatrix} e_{22} & \cdots & e_{2\nu}\\ \vdots & \ddots & \vdots\\ e_{\nu 2} & \cdots & e_{\nu\nu}\end{pmatrix}$$

is invertible.

The expressions for $E_{+-,1}$ and $E_{+-,2}$ are similar to those obtained in Section 3. Following the discussions in the proof of Proposition 3.9, we can show that $\lambda^{1/2}E_{+-,1} + \lambda E_{+-,2}$ is invertible for λ near 0 and that $E_{+-}(\lambda)^{-1}$ admits an asymptotic expansion near $\lambda = 0$. ¿From (4.58), we deduce the existence of the asymptotic expansion of $(1 + A(\lambda))^{-1}$ near $\lambda = 0$. The leading term in the expansions and the condition on ρ can be checked as in Proposition 3.9. The details are left to the reader. □

4.4 Asymptotic expansions of $R(z)$ when $E_0 \in \sigma_d(P')$.

To establish the expansion of $R(z)$ we use

$$R(z) = \mathcal{E}(z) + \mathcal{E}_+(z)(1 + \mathbf{R}_0(\lambda)\mathbf{V}(\lambda))^{-1}\mathbf{R}_0(\lambda)\mathcal{E}_-(z). \tag{4.70}$$

Making use of the formulae

$$(1 + \mathbf{T}(\lambda))^{-1} = \begin{pmatrix} (1 + A(\lambda))^{-1} & 0 \\ -B(\lambda)(1 + A(\lambda))^{-1} & 1 \end{pmatrix} \tag{4.71}$$

and

$$\mathbf{R}_0(\lambda) = \begin{pmatrix} (-\Delta - \lambda)^{-1} - D(\lambda)F(\lambda)D^\dagger(\lambda) & D(\lambda)F(\lambda) \\ F(\lambda)D^\dagger(\lambda) & -\lambda^{-1}\pi_0 - F(\lambda) \end{pmatrix}, \tag{4.72}$$

we can compute explicitly $\mathbf{M}(\lambda) \equiv (1 + \mathbf{R}_0(\lambda)\mathbf{V}(\lambda))^{-1}\mathbf{R}_0(\lambda)$

$$\mathbf{M}(\lambda) = \begin{pmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix} \tag{4.73}$$

where

$$\begin{aligned} m_{11}(\lambda) &= (1 + A(\lambda))^{-1}(G_0 - G_0CF_0C^*G_0 + O(\lambda^{1/2})) \\ &= (1 + A(\lambda))^{-1}(G_0\pi_1' + O(\lambda^{1/2})), \\ m_{12}(\lambda) &= (1 + A(\lambda))^{-1}(G_0CF_0 + O(\lambda^{1/2})), \\ m_{21}(\lambda) &= -(F_0C^*G_0V_0 + O(\lambda^{1/2}))m_{11}(\lambda) + F_0C^*G_0 + O(\lambda^{1/2}), \\ m_{22}(\lambda) &= -(F_0C^*G_0V_0 + O(\lambda^{1/2}))m_{12}(\lambda) - \lambda^{-1}\pi_0 - F_0 + O(\lambda^{1/2}). \end{aligned}$$

Let $k \in \mathbf{N}$. Under suitable assumptions on ρ and s depending on k and on the nature of the threshold, we have

$$m_{ij}(\lambda) = \sum_{l=-2}^{k} \lambda^{l/2} m_{ij}^{(l)} + O(\lambda^{k/2+\epsilon}).$$

$\mathbf{M}(\lambda)$ has an asymptotic expansion of the form

$$\mathbf{M}(\lambda) = \sum_{j=-2}^{k} \lambda^{j/2}\mathbf{M}_j + O(\lambda^{k/2+\epsilon}), \tag{4.74}$$

where

$$\mathbf{M}_j = \begin{pmatrix} m_{11}^{(j)} & m_{12}^{(j)} \\ m_{21}^{(j)} & m_{22}^{(j)} \end{pmatrix}$$

$\mathbf{M}_{-2} = \mathbf{M}_{-1} = 0$ if E_0 is a regular point of P. $\mathbf{M}_{-2} = 0$ if E_0 is an exceptional point of the first kind. Note that $\mathcal{E}(z)$, $\mathcal{E}_\pm(z)$ are holomorphic near $z = E_0$. Thus we can write

$$\begin{aligned} \mathcal{E}(z) &= R''(z) = \sum_{j=0}^{\infty} \lambda^j \mathcal{E}_j, \\ \mathcal{E}_+(z) &= = (1 - R''(z)\Pi''P)\mathcal{E}_+^0 = \sum_{j=0}^{\infty} \lambda^j \mathcal{E}_{+,j}, \\ \mathcal{E}_-(z) &= \mathcal{E}_-^0(1 - \Pi_1 I_a R''(z)) = \sum_{j=0}^{\infty} \lambda^j \mathcal{E}_{-,j} \end{aligned}$$

As for $\mathcal{E}_{\pm}^0$, we put

$$\mathcal{E}_{+,j} = (\mathcal{E}_{+,1,j}, \mathcal{E}_{+,2,j}), \quad \mathcal{E}_{-,j} = \begin{pmatrix} \mathcal{E}_{-,1,j} \\ \mathcal{E}_{-,2,j} \end{pmatrix}.$$

It follows that in suitable spaces $R(z)$ has an asymptotic expansion of the form

$$R(z) = \sum_{j\geq -2} \lambda^{j/2}(\mathcal{E}_{j/2} + \sum_{l_1+l_2+l_3=j} \mathcal{E}_{+,l_1/2}\mathbf{M}_{l_2}\mathcal{E}_{-,l_3/2}). \tag{4.75}$$

Here $\mathcal{E}_{j/2}, \mathcal{E}_{\pm,j/2}$ are set to 0 if $j/2 \notin \mathbf{N}$. More precisely, one has the following

Theorem 4.11 *(a). Assume that E_0 is neither eigenvalue, nor resonance of P. Then, for $k \in \mathbf{N}$, $\rho > \max\{2, 2k+1\}$ and $(2k+1)/2 < s < \rho - (2k+1)/2$ for $k \geq 1$ and $1 < s < \rho - 1$ for $k = 0$, one has*

$$R(z) = \sum_{j=0}^{k} \lambda^{j/2} B_j^{(0)} + O(|\lambda|^{k/2+\epsilon}), \tag{4.76}$$

in $\mathcal{L}(-1, s; 1, -s)$. Here

$$\begin{aligned} B_0^{(0)} \quad = \quad & R''(E_0) + C_1(G_0\pi_1'\mathcal{E}_{-,1,0} + C_2^*) - C_2V_0(1 + G_0V_0^{\#})^{-1}C_2^* \\ & + C_2(1 - V_0(1 + G_0V_0^{\#})^{-1}G_0\pi_1'\mathcal{E}_{-,1,0} - K_0 \end{aligned} \tag{4.77}$$

with

$$\begin{aligned} \mathcal{E}_{-,1,0} &= ((1 - I_aR''(E_0))\cdot, \varphi_0)_a, \\ C_1 &= (1 - R''(E_0)I_a)(\varphi_0 \otimes ((1 + G_0V_0^{\#})^{-1}\cdot)), \\ C_2 &= K_0(\varphi_0 \otimes G_0\cdot) \quad \textit{and} \\ K_0 &= (\Pi_2 I_a\mathcal{E}_{+,1}^0 G_0\mathcal{E}_{-,1}^0 I_a\Pi_2)^{-1}\Pi_2. \end{aligned}$$

(b). Suppose that E_0 is an exceptional point of the first kind of P. Assume $\rho > \max\{2, 2k-1\}$, $k \in \mathbf{N}$ with $k \geq 1$. We have

$$R(z) = \sum_{j=-1}^{k-2} \lambda^{j/2} B_j^{(1)} + O(|\lambda|^{(k-2)/2+\epsilon}) \tag{4.78}$$

in $\mathcal{L}(-1, s; 1, -s)$ with $s > 1$ if $k = 1$ and $(2k-1)/2 < s < \rho - (2k-1)/2$ if $k \geq 2$. Here

$$B_{-1}^{(1)} = i < \cdot, u > u, \tag{4.79}$$

and u is an E_0 resonant function of P satisfying

$$\frac{1}{2\sqrt{\pi}} \int_{\mathbf{X}} I_a(x)u(x)\varphi_0(x^a)\, dx = 1.$$

(c). Suppose that E_0 is an exceptional point of the second or the third kind of P. Assume $\rho > 2k-1$, $k \in \mathbf{N}$ with $k \geq 2$. One has

$$R(z) = \sum_{j=-2}^{k-4} \lambda^{j/2} B_j^{(l)} + O(|\lambda|^{(k-4)/2+\epsilon}), \quad l = 2, 3, \tag{4.80}$$

in $\mathcal{L}(-1, s; 1, -s)$ for $(2k-1)/2 < s < \rho - (2k-1)/2$, $\lambda \in U_\delta$ with $\delta > 0$ small enough and $\epsilon \in]0, 1/2]$ verifying $\epsilon < (2s - 2k + 1)/4$. Here

$$B_{-2}^{(2)} = B_{-2}^{(3)} = -\Pi_{E_0},$$

with Π_{E_0} the orthogonal projection onto the eigenspace of P associated with the eigenvalue E_0.

Proof. We just give the symbolic calculus of the leading term in each case and leave to the reader the care to check the condition on ρ and s.

(a). In the case where E_0 is a regular point of P, Ker $(1+A_0)=\{0\}$, $\pi_0=0$ and $\mathbf{M}_{-2}=\mathbf{M}_{-1}=0$. The entries of $\mathbf{M}_0$ are given by

$$\begin{aligned} m_{11}^{(0)} &= (1+G_0V_0^{\#})^{-1}G_0\pi_1', \\ m_{12}^{(0)} &= (1+G_0V_0^{\#})^{-1}G_0CF_0, \\ m_{21}^{(0)} &= F_0C^*G_0(1-V_0m_{11}^{(0)}), \\ m_{22}^{(0)} &= -F_0C^*G_0V_0m_{12}^{(0)}-F_0. \end{aligned}$$

Here $F_0=(C^*G_0C)^{-1}$. Recall that Note that

$$\begin{aligned} &\mathcal{E}_{+,1,0}=(1-R''(E_0)I_a\Pi_1)\mathcal{E}_{+,1}^0, \qquad \mathcal{E}_{+,2,0}=\mathcal{E}_{+,2}^0, \\ &\mathcal{E}_{-,1,0}=\mathcal{E}_{-,1}^0(1-\Pi_1I_aR''(E_0)), \qquad \mathcal{E}_{-,2,0}=\mathcal{E}_{-,2}^0. \end{aligned}$$

Therefore

$$\begin{aligned} B_0^{(0)} &= R''(E_0)+\mathcal{E}_{+,0}\mathbf{M}_0\mathcal{E}_{-,0} \\ &= R''(E_0)+\sum_{i,j=1}^{2}\mathcal{E}_{+,i,0}m_{ij}^{(0)}\mathcal{E}_{-,j,0} \\ &= R''(E_0)+C_1(G_0\pi_1'\mathcal{E}_{-,1,0}+C_2^*)+C_2(1-V_0(1+G_0V_0^{\#})^{-1}G_0\pi_1'\mathcal{E}_{-,1,0} \\ &\quad -C_2V_0(1+G_0V_0^{\#})^{-1}C_2^*-\mathcal{E}_{+,2,0}F_0\mathcal{E}_{-,2,0} \end{aligned}$$

with

$$\begin{aligned} C_1 &= \mathcal{E}_{+,1,0}(1+G_0V_0^{\#})^{-1}, \\ C_2 &= \mathcal{E}_{+,2,0}F_0C^*G_0. \end{aligned}$$

Let $K_0=\mathcal{E}_{+,2,0}F_0\mathcal{E}_{-,2,0}$ and $K_1=\mathcal{E}_{+,1}^0G_0=\varphi_0\otimes G_0\cdot : H^{-1,s}(\mathbf{X}_a)\to H^{1,-s}(\mathbf{X})$. Then,

$$K_0=(\Pi_2I_a\mathcal{E}_{+,1}^0G_0\mathcal{E}_{-,1}^0I_a\Pi_2)^{-1}\Pi_2$$

and

$$C_2=K_0K_1.$$

This proves (a).

(b). Suppose that E_0 is an exceptional point of the first kind of P. Then $\pi'=0$ and dim Ker $(1+A_0)=1$. In this case, $B_{-2}^{(1)}=0$ and $B_{-1}^{(1)}$ is given by

$$B_{-1}^{(1)}=\mathcal{E}_{+,0}\mathbf{M}_{-1}\mathcal{E}_{-,0}.$$

To give the desired formula for $B_{-1}^{(1)}$, we write down the the entries of $\mathbf{M}_{-1}$

$$\begin{aligned} m_{11}^{(-1)} &= A_{-1}^{(1)}G_0\pi_1', \\ m_{12}^{(-1)} &= A_{-1}^{(1)}G_0CF_0, \\ m_{21}^{(-1)} &= -F_0C^*G_0V_0m_{11}^{(-1)}, \\ m_{22}^{(-1)} &= -F_0C^*G_0V_0m_{12}^{(-1)}. \end{aligned}$$

$$m_{11}^{(-1)}=-c_0^{-1}<G_0\pi_1'\cdot,-V_0u_1>_a u_1=-c_0^{-1}<\cdot,u_1>_a u_1,$$

where u_1 is a resonant function normalized by $< u_1, -V_0 u_1 >_a = 1$.

$$m_{12}^{(-1)} = -c_0^{-1} < G_0 C F_0 \cdot, -V_0 u_1 >_a u_1 = -c_0^{-1} < \cdot, v_1 >_a u_1,$$

where

$$v_1 = -F_0 C^* G_0 V_0 u_1.$$

It follows that

$$m_{21}^{(-1)} = -c_0^{-1} < \cdot, u_1 >_a v_1,$$

and

$$m_{22}^{(-1)} = -c_0^{-1} < \cdot, v_1 >_a v_1.$$

Note that according to Proposition 4.7, $w = (u_1, v_1) \in$ Ker $(1 + \mathbf{T}_0)$ is a resonant function of $\mathbf{P}$. This proves that $\mathbf{M}_{-1} : \mathcal{H}^{-1,s} \to \mathcal{H}^{1,-s}$ can be written in the form

$$\mathbf{M}_{-1} = -c_0^{-1} < \cdot, w > w.$$

Let

$$u = \frac{2\sqrt{\pi}}{< V_0^{\#} u_1, 1 >_a} \mathcal{E}_{+,0} w = \frac{2\sqrt{\pi}}{< V_0^{\#} u_1, 1 >_a} (\mathcal{E}_{+,1,0} u_1 + \mathcal{E}_{+,2,0} v_1). \tag{4.81}$$

Then, u is a resonant function of P and

$$B_{-1}^{(1)} = \mathcal{E}_{+,0} \mathbf{M}_{-1} \mathcal{E}_{-,0} = i < \cdot, u > u.$$

The normalizing condition on u becomes now

$$\begin{aligned}
&\int_{\mathbf{X}} I_a(x) u(x) \varphi_0(x^a)\, dx \\
&= \frac{2\sqrt{\pi}}{< V_0^{\#} u_1, 1 >_a} < (I_a(\mathcal{E}_{+,1,0} u_1 + \mathcal{E}_{+,2,0} v_1), \varphi_0)_a, 1 >_a \\
&= \frac{2\sqrt{\pi}}{< V_0^{\#} u_1, 1 >_a} < V_0 u_1 + C v_1, 1 >_a \\
&= \frac{2\sqrt{\pi}}{< V_0^{\#} u_1, 1 >_a} < (1 - C F_0 C^* G_0) V_0 u_1, 1 >_a \\
&= \frac{2\sqrt{\pi}}{< V_0^{\#} u_1, 1 >_a} < \pi_1' V_0 u_1, 1 >_a = 2\sqrt{\pi}.
\end{aligned}$$

This proves (b).

(c). Suppose that E_0 is an exceptional point of the second kind. Then,

$$B_{-2}^{(2)} = \mathcal{E}_{+,0} \mathbf{M}_{-2} \mathcal{E}_{-,0},$$

where the the entries of $\mathbf{M}_{-2}$ are given by

$$\begin{aligned}
m_{11}^{(-2)} &= X V_0 G_0 \pi_1' = -X, \\
m_{12}^{(-2)} &= X V_0 G_0 C F_0 = X W^*, \\
m_{21}^{(-2)} &= -F_0 C^* G_0 V_0 m_{11}^{(-2)} = -W X, \\
m_{22}^{(-2)} &= -F_0 C^* G_0 V_0 m_{12}^{(-2)} - \pi_0 = -W X W^* - \pi_0,
\end{aligned}$$

with

$$\begin{aligned}
X &= (\Pi_0 (1 - V_1 + W^* W) \Pi_0)^{-1} \Pi_0, \\
W &= F_0 C^* G_0 V_0.
\end{aligned}$$

Suppose first $\pi_0 = 0$. Then, $B_{-2}^{(2)}$ can be written in the form

$$B_{-2}^{(2)} = -\mathcal{E}_{+,0} \begin{pmatrix} 1 \\ -W \end{pmatrix} \cdot X \cdot (1, -W^*)\mathcal{E}_{-,0}.$$

In the case $\pi_0 = 0$, Ker $(1 + \mathbf{T}_0) =$ Ker $\mathbf{P}$ and $\mathcal{E}_{0,+}$: Ker $\mathbf{P} \to$ Ker $(P - E_0)$ is bijective. Note also that

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \text{ Ker } (1 + \mathbf{T}_0) \text{ if and only if } u \in \text{ Ker } (1 + A_0) \text{ and } v = -Wu.$$

This shows that

$$\mathcal{K} \equiv \mathcal{E}_{+,0} \begin{pmatrix} 1 \\ -W \end{pmatrix} : \text{ Range } \Pi_0 \to \text{ Ker } (P - E_0)$$

is bijective. $(1, -W^*)\mathcal{E}_{-,0} = \mathcal{K}^*$. We can calculate

$$\mathcal{K} = \mathcal{E}_{+,1,0} - \mathcal{E}_{+,2,0}W$$

and

$$\begin{aligned} \mathcal{K}^*\mathcal{K} &= \mathcal{E}_{-,1,0}\mathcal{E}_{+,1,0} - W^*\mathcal{E}_{-,2,0}\mathcal{E}_{+,1,0} - \mathcal{E}_{-,1,0}\mathcal{E}_{+,2,0}W + W^*\mathcal{E}_{-,2,0}\mathcal{E}_{+,2,0}W \\ &= \mathcal{E}_{-,1,0}\mathcal{E}_{+,1,0} + W^*\mathcal{E}_{-,2,0}\mathcal{E}_{+,2,0}W. \end{aligned}$$

Making use of the formulae for $\mathcal{E}_{\pm,j,0}, j = 1, 2$, one derives that

$$\begin{aligned} \mathcal{E}_{-,1,0}\mathcal{E}_{+,1,0} &= \mathcal{E}_{-,1}^0(1 + \Pi_1 I_a R''(E_0) 2 I_a)\mathcal{E}_{+,1}^0 = 1 - V_1 \\ W^*\mathcal{E}_{-,2,0}\mathcal{E}_{+,2,0}W &= W^*W. \end{aligned}$$

Therefore, $-B_{-2}^{(2)} = \mathcal{K}X\mathcal{K}^*$ satisfies

$$(\mathcal{K}X\mathcal{K}^*)2 = \mathcal{K}X(1 - V_1 + W^*W)X\mathcal{K}^* = \mathcal{K}X\mathcal{K}^*.$$

Therefore, $\mathcal{K}X\mathcal{K}^*$ is the projection onto the range of $\mathcal{K}\Pi_0$ with the kernel equal to Ker $\Pi_0\mathcal{K}^*$. $\mathcal{K}X\mathcal{K}^*$ is thus the orthogonal projection onto the range of $\mathcal{K}\Pi_0$. Since the range of $\mathcal{K}\Pi_0$ is just the eigenspace of P associated with E_0, we conclude

$$\mathcal{K}X\mathcal{K}^* = \Pi_{E_0}.$$

This proves (c) for $l = 2$ in the case $\pi_0 = 0$.

In the case $\pi_0 \neq 0$,

$$\begin{aligned} B_{-2}^{(2)} &= -\mathcal{E}_{+,0} \begin{pmatrix} 1 \\ -W \end{pmatrix} X(1, -W^*)\mathcal{E}_{-,0} - \mathcal{E}_{+,2,0}\pi_0\mathcal{E}_{-,2,0} \\ &= -\mathcal{K}X\mathcal{K}^* - \mathcal{E}_{+,2,0}\pi_0\mathcal{E}_{-,2,0}. \end{aligned}$$

The fact that $\pi_0 W = 0$ implies that

$$\mathcal{K}^*\mathcal{E}_{+,2,0}\pi_0 = -W^*0\pi_0 = 0$$

and

$$\pi_0\mathcal{E}_{-,2,0}\mathcal{K} = -\pi_0 W = 0.$$

This allows to verify

$$\begin{aligned}(-B^{(2)}_{-2})2 &= (\mathcal{K}X\mathcal{K}^* + \mathcal{E}_{+,2,0}\pi_0\mathcal{E}_{-,2,0})2 \\ &= (\mathcal{K}X\mathcal{K}^*)2 + \mathcal{K}X\mathcal{K}^*\mathcal{E}_{+,2,0}\pi_0\mathcal{E}_{-,2,0} + \mathcal{E}_{+,2,0}\pi_0\mathcal{E}_{-,2,0}\mathcal{K}X\mathcal{K}^* + \mathcal{E}_{+,2,0}\pi_0\mathcal{E}_{-,2,0} \\ &= \mathcal{K}X\mathcal{K}^* + \mathcal{E}_{+,2,0}\pi_0\mathcal{E}_{-,2,0} = -B^{(2)}_{-2}.\end{aligned}$$

This shows that $-B^{(2)}_{-2}$ is a projection. Note that

$$\mathcal{E}_{+,0}: \text{ Ker } \mathbf{P} = \text{ Ker } (1+\mathbf{T}_0) \oplus (\{0\} \times \text{ Range } \pi_0) \to \text{ Ker } (P - E_0)$$

is bijective. One can check that the range of $-B^{(2)}_{-2}$ is equal to Range $\mathcal{K}\Pi_0 \oplus$ Range $\mathcal{E}_{+,2,0}\pi_0$ which is equal to Ker $(P-E_0)$ and the kernel of $-B^{(2)}_{-2}$ is equal to Ker $(\Pi_0\mathcal{K}^*) \cap$ Ker $(\pi_0\mathcal{E}^*_{0,+})$, which is orthogonal to Range $\mathcal{K}\Pi_0 \oplus$ Range $\mathcal{E}_{+,2,0}\pi_0$. This proves that $-B^{(2)}_{-2} = \Pi_{E_0}$ for $l=2$ in the general case.

The case of an exceptional point of the third kind ($l=3$) can be treated exactly in the same way as for $l=2$, since by Proposition 4.10, the resonance at E_0 does not contribute to the leading term. □

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