EXISTENCE OF TWO-CLUSTER THRESHOLD RESONANCES AND THE N-BODY EFIMOV EFFECT

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ABSTRACT. We prove the existence of two-cluster threshold resonances in N-body problems and study their perturbation by inter-cluster interactions. As application, we construct concrete examples based on Yukawa potentials for which the N-body Efimov effect happens with $N \geq 4$.

1. INTRODUCTION

In [15], the first author of the present work proved the existence of the N-body Efimov effect for $N \geq 4$ which says that if the bottom of essential spectrum is attained by a unique three-cluster decomposition and if the three resulting two-cluster Subhamiltonians possess a threshold resonance at this threshold, then the number of discrete eigenvalues is always infinite even if the interaction potentials decay rapidly at the infinity. This phenomenon is similar to the famous three-body Efimov effect, discovered by the physicist V. Efimov ([4]) and studied in mathematics literature by many authors. See, for example, [8, 9, 10, 11, 17]. For N-body quantum systems with $N \geq 4$, it has been unknown since a long time whether the Efimov effect exists. In [1], R. D. Amado and F. C. Greenwood argued that the zero eigenvalue and zero resonance of $(N-1)$-particle subsystems do not produce an infinite number of eigenvalues in N-body problems when $N \geq 4$. Note that in spite of the title, the result

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of [15] is not in discrepancy with [1], because it corresponds to a different spectral configuration, and that a mathematical proof of the affirmation of [1] is always lacking. After the work [15], a question has then naturally been raised as how to give concrete examples such that all conditions acquired in [15] are satisfied, especially the existence of two-cluster threshold energy resonances with any number of particles. Trivial examples can be constructed from two-body zero resonance by adding some particles with independent variables. The goal of this work is to show the existence of two-cluster threshold resonances and the Efimov effect for a class of non-trivial interactions. We will show that mathematically, one can produce the N-body Efimov effect by adjusting parameters in Yukawa type interactions of the form $-\frac{\alpha e^{-\epsilon\alpha|x|}}{|x|}$, $\alpha, \epsilon > 0$. Yet, even for such concrete potentials, our proof for the existence of two-cluster N-body threshold resonance is abstract. We leave the interested reader make numerical simulation or physical experiment.

Our example will be given in terms of atomic type four-body Schrödinger operators. This leads us to consider generalized N-body Schrödinger operators which include both atomic-type and regular N-body operators. Let P denote a generalized N-body Schrödinger operator of the form $P = -\Delta + \sum_{a\in\mathcal{A}} V_a(x^a)$ in $L^2(\mathbf{X})$, where $\mathcal{A}$ denotes the set of all cluster decompositions of the N-particle system labelled by $\{1, \cdots, N\}$, $x^a \in \mathbf{X}^a$ and $\{\mathbf{X}^a\}$ is a family of linear subspaces of $\mathbf{X}$ satisfying some axioms. See [3]. A typical example of such operators is regular N-body Schrödinger operator obtained by removing the mass-center from the operator

$$-\sum_{j=1}^{N} \frac{1}{2m_j}\Delta_{x_j} + \sum_{1\le i<j\le N} V_{ij}(x_i - x_j), \quad x_j \in \mathbb{R}^3, \tag{1.1}$$

where x_j and m_j denote the position and mass of the j-th particle. P is regarded as a self-adjoint operator in $L^2(\mathbf{X})$, where $\mathbf{X}$ is the $3(N-1)$ dimensional real vectorial space: $\mathbf{X} = \{(x_1, \cdots, x_N) \in \mathbb{R}^{3N}; \sum_{j=1}^N m_j x_j = 0\}$. For $a \in \mathcal{A}$, let $\#a$ denote the number of clusters in a and P^a the Subhamiltonian associated with a. For $i, j \in \{1, \cdots, N\}$, $i \neq j$, we write $(ij) \in a$ if i and j belong to a same cluster in a. If a, b are two cluster decompositions, we write $b \subset a$ if b is a refinement of a. If a be a k-cluster decomposition $a = (a_1, \cdots, a_k)$, then $\mathbf{X}^a = \{x \in \mathbf{X}; \sum_{l \in a_j} m_l x_l = 0, j = 1, \cdots, k\}$ the intra-cluster space and $\mathbf{X}_a = \{x \in \mathbf{X}; x_i = x_j \text{ if } (ij) \in a_m \text{ for some } m \in \{1, \cdots, k\}\}$ the inter-cluster space. $\mathbf{X}^a$ and $\mathbf{X}_a$ give an orthogonal decomposition for $\mathbf{X}$ relative to the quadratic form $q(x) = \sum_j 2m_j |x_j|^2, x \in \mathbf{X}$. For $x \in \mathbf{X}$, we have the orthogonal decomposition: $x = x^a + x_a$ with $x^a \in \mathbf{X}^a$ and $x_a \in \mathbf{X}_a$. The N-body Schrödinger operator, P, obtained above can be written in the form $P = P_0 + V(x)$ where $P_0 = -\Delta$ is the Laplace-Beltrami operator on the Euclidean space $(\mathbf{X}, q)$ and $V(x) = \sum_{a \in \mathcal{A}} V_a(x^a)$ with $V_a(x^a) = V_{ij}(x_i - x_j)$ if a is an $(N-1)$-cluster decomposition and $(ij) \in a$; $V_a(x^a) = 0$, otherwise. The atomic-type N-body operators can also be written in the form of generalized N-body operators. Let $-\Delta^a$ ($-\Delta_a$, resp.) denote the restriction of P_0 on $\mathbf{X}^a$ (on $\mathbf{X}_a$, resp.). For $a \in \mathcal{A}$, denote $P^a = -\Delta^a + \sum_{b \subseteq a} V_b(x^b), P_a = P^a - \Delta_a, I_a(x) = \sum_{b \not\subseteq a} V_b(x^b)$. P^a is the Subhamiltonian associated with the cluster decomposition a and I_a is the sum of all inter-cluster interactions. Assume that $V_a(x^a)$ is real and relatively form-compact with respect to $-\Delta^a$ in $L^2(\mathbf{X}^a)$, satisfying the decay

$$|V_a(x^a)| \leq C\langle x^a \rangle^{-\rho_0}, \quad \rho_0 > 2, \tag{1.2}$$

for x^a outside some compact of $\mathbf{X}^a$. P is regarded as self-adjoint operator in $L^2(\mathbf{X})$ with form domain $H^1(\mathbf{X})$. Let $\mathcal{T} = \cup_{a \in \mathcal{A}, \#a \geq 2} \sigma_p(P^a)$ be the set of thresholds of P. The HVZ theorem says that the bottom of the essential spectrum, $E_0 \triangleq \inf \sigma_{ess}(P)$, of P is given by

$$E_0 = \min_{a \in \mathcal{A}, \#a \geq 2} \inf \sigma(P^a) = \min_{a \in \mathcal{A}, \#a = 2} \inf \sigma(P^a).$$

Assume that E_0 is attained by a unique three-cluster threshold, b, with $b = (b_1, b_2, b_3)$.

$$\inf \sigma(P^b) = E_0 = \inf \sigma_{ess}(P). \tag{1.3}$$

Let $a_k = (b_i \cup b_j, b_k)$, where i, j and k take different values in $\{1, 2, 3\}$. Throughout this paper, we suppose that $\dim \mathbf{X}^{a_j} \cap \mathbf{X}_b = 3$, which is always satisfied for the operator obtained from (1.1). By HVZ Theorem, one has $\sigma(P^{a_k}) = [E_0, \infty[$. We assume that these are the only cluster decompositions attaining E_0:

$$\inf \sigma(P^a) = E_0, a \in \{b, a_1, a_2, a_3\}; \quad \inf \sigma(P^a) > E_0, a \notin \{b, a_1, a_2, a_3\}, \#a \geq 2. \tag{1.4}$$

Assume that the inter-cluster interactions related to b are attractive:

$$V_a(x^a) \leq 0, \text{ for } a \not\subseteq b. \tag{1.5}$$

Definition. Let Q be a generalized N-body Schrödinger operator with $N \geq 3$. Let $E_0 = \inf \sigma_{\text{ess}}(Q)$. (a). Q is called *unique two-cluster* if there exits only one two-cluster decomposition a_0 such that $\inf \sigma(Q^{a_0}) = E_0$. (b). E_0 is called a *threshold resonance* of Q if the equation $Qu = E_0 u$ has a solution $u \in H^{1,-s}(\mathbf{X}) \setminus L^2(\mathbf{X})$ for any $s > \frac{1}{2}$.

In our work, Q plays the role of one of the two-cluster Subhamiltonians P^{a_j} and $a_0 = b$. Under the assumptions (1.2) and (1.4), the analysis of threshold resonances made in [14] is valid for P^{a_j}. One knows that if E_0 is a resonance of P^{a_j}, then it is simple and the corresponding resonant state u behaves like

$$u(x^{a_j}) = O(e^{-\delta |x^b|} |x_b^{a_j}|^{-1}), \quad |x^{a_j}| \to \infty, \quad \delta > 0. \tag{1.6}$$

Note the threshold resonances studied in [14] is two-cluster in nature and it is an interesting open question to study the many-cluster threshold resonances.

Theorem 1.1 ([15])**.** *Let $N \geq 4$. Let the conditions (1.2) with $\rho > 2$, (1.4) and (1.5) be satisfied. Assume that each of the two-cluster Subhamiltonians P^{a_j}, $j = 1, 2, 3$, have a threshold resonance at E_0. Let $N(\lambda)$ denote the number of the eigenvalues of P below $\lambda < E_0$. Then, there exists $C_0 > 0$ depending only on the reduced masses of b_1, b_2, b_3 such that*

$$N(\lambda) \geq C_0 |\log|\lambda - E_0||, \lambda \to E_0. \tag{1.7}$$

Assumption (1.4) may be realized when some particles are remarkably distinct from the others. In this work, we study the existence of two-cluster threshold resonances by varying some coupling constant. We will prove the following

Theorem 1.2. *Let Q be an N-body operator with potentials V_c satisfying (1.2) and let a be a two-cluster decomposition. Assume that $V_c \leq 0$ for all $c \not\subseteq a$, $I_a = \sum_{c \not\subseteq a} V_c \neq 0$, $E_0 = \inf \sigma(Q^a) \in \sigma_d(Q^a)$. Denote $Q(g) = Q_a + gI_a$. Let g_0 be the critical coupling constant defined by*

$$g_0 = \sup\{g \geq 0; \inf \sigma(Q(g)) = \inf \sigma_{ess}(Q(g)) = E_0\}. \tag{1.8}$$

Then, if $Q(g_0)$ is unique two-cluster, E_0 is a threshold resonance of $Q(g_0)$.

We will also give a sufficient condition to guarantee the unique two-cluster property of $Q(g_0)$ and study the perturbation of threshold resonance (Theorem 3.1), which is in some sense a converse of Theorem 1.2.

The idea of varying coupling constant limit is used in [7] for the study of absorption of discrete eigenvalues into a unique two-cluster threshold. In [7], the authors deduced their results by splitting some reduced operator into sum of a compact continuous part and a bounded holomorphic part. Our proof of Theorem 1.2 relies on the idea of [14], where the notion of N-body threshold resonance is introduced. In the proof of existence of resonant state (Proposition 2.3), the compactness of the whole reduced operator is essential.

Using Theorem 1.2, we will show that for appropriate constant $\alpha_1 > 0$, $\epsilon > 0$ small enough, there exists $g_j \in]0,1[$, $j = 1,2,3$, such that Theorem 1.1 holds for the atomic-type four-body Schrödinger operator with Yukawa potentials

$$\begin{aligned} P &= -\Delta_{x_1} - \Delta_{x_2} - \Delta_{x_3} - \frac{\alpha_1 e^{-\epsilon\alpha_1|x_1|}}{|x_1|} - g_1\frac{e^{-\epsilon|x_2-x_3|}}{|x_2-x_3|} \\ &\quad -g_2(\frac{e^{-\epsilon|x_2|}}{|x_2|} + \frac{e^{-\epsilon|x_1-x_2|}}{|x_1-x_2|}) - g_3(\frac{e^{-\epsilon|x_3|}}{|x_3|} + \frac{e^{-\epsilon|x_1-x_3|}}{|x_1-x_3|}), \quad x_j \in \mathbb{R}^3. \end{aligned} \tag{1.9}$$

The plan of this work is as follows. In Section 2, we prove the existence of two-cluster threshold resonance. In section 3, we show that if E_0 is a two threshold resonance of $Q(g)$ for $g = g_0$, it is shifted to the left to become a non-degenerate discrete eigenvalue when g increases. These results are applied to the study of the N-body Efimov effect in Section 4: we give more explicit sufficient conditions under which Theorem 1.1 holds and establish a lower bounded on the number of discrete eigenvalues of $P(g)$ as $g \to g_0$ and $g > g_0$, where $P = P(g_0)$ satisfies the assumptions of Theorem 1.1.

Notation. For $s \in \mathbb{R}$ and $k \in \mathbb{Z}$, we denote by $L^{2,s}$ and $H^{k,s}$ the weighted-L^2 and weighted-Sobolev spaces $L^2(\langle x\rangle^{2s}dx)$ and $H^k(\langle x\rangle^{2s}dx)$, respectively, and by $\mathcal{L}(s;s')$ and $\mathcal{L}(k,s;k',s')$ the space of bounded operators from $L^{2,s}$ to $L^{2,s'}$ and from $H^{k,s}$ to $H^{k',s'}$, respectively.

2. Existence of two-cluster threshold resonances

The N-body Hamiltonian $Q = -\Delta + \sum_{c\in\mathcal{A}} V_c$ studied in Sections 2 and 3 corresponds to a two-cluster Subhamiltonian in the Efimov effect with four or more particles. Thus let $N \geq 3$. Let a be a two-cluster decomposition such that $\dim \mathbf{X}_a = 3$. Set $E_0 = \inf \sigma(Q^a)$. Assume that E_0 is in $\sigma_d(Q^a)$. Let φ_a be an associated eigenfunction: $Q^a\varphi_a = E_0\varphi_a$, $\|\varphi_a\| = 1$. We want to show that by varying the coupling constant g in $Q(g) = Q_a + gI_a(x)$, we can find a critical value g_0 such that E_0 is a threshold resonance of $Q(g_0)$. Let Π denote the orthogonal projection in $L^2(\mathbf{X})$ induced by φ_a

$$\Pi u = (\varphi_a, u)_a \otimes \varphi_a, \quad u \in L^2(\mathbf{X}),$$

where $(,\cdot,\cdot)_a$ denotes the scalar product in x^a-variables. Let $\Pi' = 1 - \Pi$. Assume that for any $b \in \mathcal{A}$, $b \not\subseteq a$, $V_b \leq 0$, and V_b is $-\Delta$-form compact and outside some bounded set in $\mathbf{X}^b$,

$$|V_b(x^b)| \leq C|x^b|^{-\rho_0}, \quad \rho_0 > 2. \tag{2.1}$$

Assume that the intercluster interaction I_a is not identically zero. Put

$$Q(g) = Q^a - \Delta_a + gI_a, \quad Q'(g) = \Pi' Q(g)\Pi', \tag{2.2}$$

where $g > 0$ is a coupling constant. It is naturel to expect that for g small, the bottom of the essential spectrum of $Q(g)$ remains to be E_0, while for g large, it may be shifted to the left.

Lemma 2.1. *Let V_b be $-\Delta_{x^b}$-form compact. There exists $\mu_0 > 0$ such that if*

$$g\|(-\Delta_{x^b} + 1)^{-1/2} V_b (-\Delta_{x^b} + 1)^{-1/2}\| \leq \mu_0, \tag{2.3}$$

then $\inf \sigma_{ess}(Q(g)) = E_0$ and a is the unique two-cluster decomposition which attains E_0: $\forall a' \in \mathcal{A}, \#a' = 2$ and $a' \neq a$, $\inf \sigma(Q^{a'}) > E_0$.

Proof. Since E_0 is the discrete spectrum of Q^a, by HVZ theorem, for any $b \in \mathcal{A}$, $b \subset a$, (which means that b is a decomposition finer than a)one has

$$Q^b \geq E_0 + \epsilon_b$$

for some $\epsilon_b > 0$. Let a' be another two-cluster decomposition of the N-body system with $a' \neq a$. Take $b = a \cap a'$. Then, $b \subset a$ and the Subhamiltonian associated to a' can be decomposed as

$$Q_{a'}(g) = Q_b + g \sum_{c \subseteq a', c \nsubseteq a} V_c \tag{2.4}$$

An argument of perturbation shows that if μ_0 is small enough, one has $Q_{a'} \geq E_0 + \epsilon_b/2$. By HVZ Theorem, $\inf \sigma_{ess}(Q(g)) = \min_{\#c=2} \inf \sigma(Q^c(g)) = \inf \sigma(Q^a) = E_0$. ■

For potentials satisfying (2.1), with Lemma 2.1, one can apply the result on finiteness of discrete spectrum in N-body problems ([5, 13]) to conclude that $Q(g)$ has only a finite number of discrete eigenvalues for $g > 0$ small. In particular, if $g > 0$ is small enough, there is no discrete spectrum and $E(g) \triangleq \inf \sigma(Q(g))$ verifies $E(g) = \inf \sigma_{ess}(Q(g)) = E_0$. Set

$$g_0 = \sup\{g > 0; E(g) = E_0, \inf \sigma_{ess}(Q(g)) = E_0\}. \tag{2.5}$$

By Lemma 2.1, one has $g_0 > 0$. Since $I_a \leq 0$ and $I_a \neq 0$, one can show that for g large enough, there exists some Subhamiltonian $Q^{a'}(g), a' \neq a$, such that $\inf \sigma(Q^{a'}(g)) < E_0$. Therefore $g_0 \in]0, +\infty[$ is well-defined. It may happen that when g increases from 0 to g_0, the bottom of the essential spectrum, E_0, of $Q(g_0)$ is attained by several two-cluster Subhamiltonians. In the following, we give the proof of Theorem 1.2, under the condition that $Q(g_0)$ is still unique two-cluster. By Theorem 3.3 (a) of [15], E_0 is not an eigenvalue of $Q(g_0)$. Therefore, to prove Theorem 1.2, we need only to construct a non-trivial solution for the equation $(Q(g_0) - E_0)u = 0$ with $u \in H^{1,-s}$ for any $s > \frac{1}{2}$. Then E_0 is a threshold resonance of $Q(g_0)$. The construction of a resonant state is based on the reduction made in [14].

According to whether or not E_0 is in $\sigma_p(Q'(g))$, the spectral property of $Q(g)$ at E_0 is determined either by an operator of the form $-\Delta_{x_a} + V_0$ or by an operator of the form

$$\begin{pmatrix} -\Delta_{x_a} + V_0 & C \\ C^* & 0 \end{pmatrix}$$

where V_0 is some effective potential and C some finite rank operator. Part (b) of the following Lemma shows that we are actually in the first simpler situation.

Lemma 2.2. *Assume that Q_a is the unique two-cluster Subhamiltonian of $Q(g_0)$ which attains the bottom of its spectrum. There exists $\delta_0 > 0$ such that for $|g-g_0| < \delta_0$, one has*

(a). $\inf \sigma_{ess}(Q(g)) = E_0$*;*

(b). $Q'(g) \geq (E_0 + \epsilon_0)\Pi'$, *for some* $\epsilon_0 > 0$.

Proof. (a). By the unique two-cluster assumption for $Q(g_0)$, one has for any two-cluster decomposition $b \neq a$ $Q_b(g_0) \geq (E_0 + \epsilon_b)$ for some $\epsilon_b > 0$. The same argument as in Lemma 2.1 shows that

$$\inf \sigma_{ess}(Q(g) = \inf_{\#b=2} \sigma(Q^b(g)) = \inf \sigma(Q^a) = E_0$$

for g near g_0.

(b). We note that $E_0 \notin \sigma_p(Q'(g_0))$. In fact, if $E_0 \in \sigma_p(Q'(g_0))$, the associated eigenfunction ψ with $\|\psi\| = 1$, would verify $\Pi'\psi = \psi$ and

$$E_0 =< Q'(g_0)\psi, \psi >=< Q(g_0)\psi, \psi > .$$

Since $Q(g_0) \geq E_0$, this would imply $Q(g_0)\psi = E_0\psi$, which is impossible because by (a) of Theorem 3.3 [15], E_0 is not an eigenvalue of $Q(g_0)$. By Lemma 2.1 of [14], one has

$\inf \sigma_{ess}(Q'(g_0)) \geq (E_0 + \delta)$ for some $\delta > 0$. Since $Q'(g_0) \geq E_0\Pi'$ and $E_0 \notin \sigma_p(Q'(g_0))$, one sees that $Q'(g_0) \geq (E_0 + \epsilon)\Pi'$ for some $\epsilon > 0$. An easy argument of perturbation shows that for g sufficiently near g_0, $Q'(g) = Q'(g_0) + (g - g_0)\Pi' I_a \Pi' \geq (E_0 + \epsilon/2)\Pi'$. ■

Lemma 2.2 (a) shows that for $g \in]g_0, g_0 + \delta_0[$, $\delta_0 > 0$ small, $E(g)$ is in the discrete spectrum of $Q(g)$. It is clear that $E(g) \to E_0$ as $g \to g_0$.

With Lemma 2.2 (b), we can apply the reduction of Section 3 in [14] which says that for E near E_0, E is an eigenvalue (resp., threshold resonance) of $Q(g)$ if and only if 0 is an eigenvalue (resp., threshold resonance) of operator $\epsilon_{+-}(\lambda)$

$$\epsilon_{+-}(\lambda, g) = \lambda - (-\Delta_{x_a} + V(\lambda, g)) \tag{2.6}$$

where $\lambda = E - E_0$, $R'(z, g) = (Q'(g) - z)^{-1}\Pi'$ and

$$V(\lambda, g) = ([gI_a - g^2 I_a R'(\lambda + E_0, g) I_a](\varphi_a \otimes \cdot), \varphi_a)_a.$$

Note that $I_a \leq 0$ and $R'(\lambda + E_0, g) \geq 0$ for λ near 0. $V(\lambda, g)$ is a negative self-adjoint operator, analytic in λ near 0 and g near g_0. Set

$$K(\lambda, g) = (-\Delta_{x_a} - \lambda)^{-1} V(\lambda, g). \tag{2.7}$$

Then, $K(\lambda, g) : H^{1,-s} \to H^{1,-s}$ is compact operator-valued for $1/2 < s < (\rho_0 - 1)/2$ and is continuous in λ near 0 and g near g_0. Clearly, u belongs to the kernel of $\epsilon_{+-}(\lambda, g)$ in $H^{1,-s}$ if and only if

$$(1 + K(\lambda, g))u = 0 \tag{2.8}$$

Proposition 2.3. *One has:* $\dim \ker(1 + K(0, g_0))_{H^{1,-s}} = 1$.

Proof. By the assumption on $Q(g_0)$, for any $g > g_0$ sufficiently close to g_0, $E(g)$ is an eigenvalue of $Q(g)$ with $E(g) < E_0$ and $E(g) \to E_0$ as $g \to g_0$. Therefore, there exists a family of functions $u(g) \in H^1$ such that

$$u(g) = -K(\lambda(g), g)u(g), \quad \|u(g)\|_{H^{1,-s}} = 1, \quad \lambda(g) = E(g) - E_0.$$

The compactness of $K(\lambda(g), g)$ allows to extract a subsequence $\{u(g_k)\}$ of $\{u(g)\}$ such that $u(g_k) = -K(\lambda(g_k), g_k)u(g_k)$ converges (strongly) in $H^{1,-s}$. Let u_0 denote its limit. It is clear that $u_0 = -K(0, g_0)u_0$, $\|u_0\|_{H^{1,-s}} = 1$. This proves that 0 is an eigenvalue of $1 + K(0, g_0)$. Theorem 3.3 (a) of [15] implies that 0 is not an eigenvalue of $-\Delta_{x_a} + V_0$, where $V_0 = V(0, g_0)$. Therefore $u_0 \notin H^1$. Since the threshold resonance of $-\Delta_{x_a} + V_0$ at zero is at most simple ([6, 14]), one sees that $\ker(1 + K(0, g_0))$ in $H^{1,-s}$ is of dimension one and is spanned by u_0. ■

Proof of Theorem 1.2. Since $(-\Delta_{x_a} + V_0)u_0 = 0$, a direct calculation shows that

$$\psi_0 = \varphi_a \otimes u_0 - g_0 R'(E_0, g_0) I_a(\varphi_a \otimes u_0) \neq 0$$

verifies $(Q(g_0) - E_0)\psi_0 = 0$. See [14]. Since $u_0 \in H^{1,-s}(\mathbf{X}_a) \setminus H^1(\mathbf{X}_a)$, $\psi_0 \in H^{1,-s}(\mathbf{X}) \setminus H^1(\mathbf{X})$ for any $s > 1/2$. E_0 is thus a threshold resonance of $Q(g_0)$. ■

Corollary 2.4. *Let $Q = Q_a + I_a$, where a is a two-cluster decomposition, $I_a = \sum_{b \not\subseteq a} V_b$, $V_b \leq 0$, such that $E_0 \triangleq \inf \sigma(Q^a) \in \sigma_d(Q^a)$ and that $Q^{a'} > E_0$ for any other two-cluster decompositions $a' \neq a$. Assume in addition that Q has at least one discrete eigenvalue strictly below E_0. Then, E_0 is a threshold resonance of $Q(g_0)$.*

Proof. It suffices to remark that g_0 defined by (2.5) is given by

$$g_0 = \inf\{g \in]0, 1]; \inf \sigma(Q(g)) < E_0\}, \tag{2.9}$$

and $g_0 \in]0, 1[$. $Q(g_0)$ is still unique two-cluster, because for any $a' \in \mathcal{A}$ with $\#a' = 2$ and $a' \neq a$, (2.4) shows that the Subhamiltonian of $Q(g_0)$ associated with a' satisfies $Q^{a'}(g_0) \geq Q^{a'}(1) > E_0$ for $g_0 \in]0, 1[$. Theorem 1.2 gives the desired result. ■

As an application of Corollary 2.4, we give a concrete example for the existence of two-cluster threshold resonance for three-body atomic-type Schrödinger operator with Yukawa-type potentials. Similar examples can be constructed for any number of particles.

Example. Consider an atomic-type three-body Schrödinger operator with Yukawa potentials

$$Q(g) = -\Delta_{x_1} - \Delta_{x_2} - \frac{\alpha_1 e^{-\epsilon\alpha_1|x_1|}}{|x_1|} - g(\frac{\alpha_2 e^{-\epsilon\alpha_2|x_2|}}{|x_2|} + \frac{\alpha_3 e^{-\epsilon\alpha_3|x_1-x_2|}}{|x_1-x_2|}) \tag{2.10}$$

where the nucleus is fixed at the origine, $x_j \in \mathbb{R}^3$, α_i, ϵ and g are suitable positive constants with $\alpha_1 > \max\{\alpha_2, \alpha_3\}$. We want to show Corollary 2.4 can be applied with $a = \{(01), (2)\}$. Consider first the two-body operator

$$Q(\alpha, \epsilon) = -\Delta - \frac{\alpha e^{-\alpha\epsilon|x|}}{|x|}, \quad x \in \mathbb{R}^3,$$

with $\alpha, \epsilon > 0$. By the change of variables $y = \alpha x$, one sees that $Q(\alpha, \epsilon)$ is unitarily equivalent with $\alpha^2 Q_1(\epsilon)$ where

$$Q_1(\epsilon) = -\Delta_y - \frac{e^{-\epsilon|y|}}{|y|}.$$

Using the perturbation theory around $Q_1(0) = -\Delta_y - \frac{1}{|y|}$, one sees that for $\epsilon > 0$ small,

$$\inf \sigma(Q_1(\epsilon)) = -\frac{1}{4} + c_0\epsilon + O(\epsilon^2).$$

This shows that there exists $\epsilon_0 > 0$ such that

$$\inf \sigma(Q(\alpha, \epsilon)) = \alpha^2(-\frac{1}{4} + c_0\epsilon + O(\epsilon^2)) < 0 \tag{2.11}$$

for all $0 < \epsilon < \epsilon_0$, $\alpha > 0$. For $g = 1$, let P_j denote the three two-body Subhamiltonians of $Q(1)$ associated with the cluster decompositions $\{(01), 2\}$, $\{(02), (1)\}$, $\{(12), (0)\}$, respectively. Using (2.11), one sees that their respectives bottoms of spectrum are given by

$$E_j = \alpha_j^2(-\frac{1}{4} + O(\epsilon)), j = 1, 2, \quad E_3 = \alpha_3^2(-\frac{1}{8} + O(\epsilon)),$$

which are all negative if $\epsilon > 0$ is small enough. So $Q(1)$ is unique two-cluster, since $\inf \sigma_{ess}(Q(1)) = E_1 < \min\{E_2, E_3\}$. By minimax principle, one can easily check that $Q(1)$ has at least one discrete eigenvalue e_0 with $e_0 \le E_1 + E_2 < E_1$. By Corollary 2.4, there exists some $g_0 \in]0, 1[$ such that $E_0 = E_1$ is a two-cluster threshold resonance of $Q(g_0)$.

3. Perturbation of two-cluster threshold resonances

In this Section, we want to show that under inter-cluster perturbation, a two-cluster threshold resonance may be shifted to the left to produce discrete eigenvalue of the N-body system. Let $a \in \mathcal{A}$ be a two-cluster decomposition of an N-body system such that $\dim \mathbf{X}_a = 3$. With the notation of Section 2, assume that

- $E_0 \triangleq \inf \sigma(Q^a) \in \sigma_d(Q^a)$;
- For any $b \not\subseteq a$, $V_b(x^b) = O(\langle x^b \rangle^{-\rho_0})$, $\rho_0 > 2$, and $V_b \le 0$ with $I_a \triangleq \sum_{b \not\subseteq a} V_a \neq 0$;
- Let $Q(g) = Q_a + gI_a$. Assume that there exists $g_0 > 0$ such that $Q(g_0) \ge 0$, $Q(g_0)$ is unique two-cluster at E_0 and E_0 is a threshold resonance of $Q(g_0)$.

Note that different from Section 2, here we do not assume that g_0 is the critical coupling constant such that $\inf \sigma(Q(g)) < E_0$ for any $g > g_0$.

Theorem 3.1. *Under the above assumptions, there exists $\delta_0 > 0$ such that for $g \in]g_0, g_0 + \delta_0[$, $Q(g)$ possesses only one eigenvalue $E(g) \in]E_0 - \delta_0, E_0[$ which is non-degenerated and satisfies*

$$E(g) = E_0 + E_1(g - g_0)^2 + o((g - g_0)^2), \tag{3.1}$$

as $g \to g_0$*, where* $E_1 < 0$.

Proof. We use the idea of [14] and only sketch the proof. Under the assumptions of Theorem 3.1, as in Section 2, one has $Q'(g) \geq (E_0 + \epsilon_0/2)\Pi'$ for g near g_0. We can then apply the reduction made in Section 3 of [14] which implies that for $\delta_0 > 0$ small enough such that $E \in]E_0 - \delta_0, E_0 + \delta_0[$, $|g - g_0| < \delta_0$, E is an eigenvalue or threshold resonance of $Q(g)$ if and only if 0 is an eigenvalue or threshold resonance of $\epsilon_{+-}(\lambda, g)$ with the same multiplicity, where $\lambda = E - E_0$ and $\epsilon_{+-}(\lambda, g)$ is given by (2.6). Let

$$G_0 = \lim_{z \to 0, \Im z > 0} (-\Delta_{x_a} - z)^{-1} : H^{-1,s} \to H^{1,-s'}, s, s' > 1/2, s + s' > 2.$$

Denote $V_0 = V(0, g_0)$. The zero resonance of $\epsilon_{+-}(0, g_0)$ is simple ([6, 14]) and Theorem 3.3 (a) of [15] says that 0 is not an eigenvalue of $\epsilon_{+-}(0, g_0)$. Therefore the kernel of $\epsilon_{+-}(0, g_0)$ in $H^{1,-s}$ is of dimension one. Let u_0 be the resonant state of $\epsilon_{+-}(0, g_0)$ normalized by $< u_0^*, u_0 >_a = 1$, where $u_0^* = -V_0 u_0$. Let $W(\lambda, g) = 1 + K(\lambda, g)$ with $K(\lambda, g)$ defined as in Section 2. Study the Grushin problem associated with the operator

$$\mathcal{P}(\lambda, g) = \begin{pmatrix} W(\lambda, g) & T \\ & \\ S & 0 \end{pmatrix} : H^{1,-s} \times \mathbb{C} \to H^{1,-s} \times \mathbb{C},$$

where $s > 1/2$, $T : \mathbb{C} \ni c \to cu_0 \in H^{1,-s}$ and $S : H^{1,-s} \ni f \to < u_0^*, f > \in \mathbb{C}$. Define $Q : H^{1,-s} \to H^{1,-s}$ by $Qf = < u_0^*, f >_a u_0$. Denote $F =$ Range $(1 + G_0 V_0)$. Then, $Q' = 1 - Q$ is a projection from $H^{1,-s}$ onto F and $Q'(1 + G_0 V_0)Q'$ is invertible on F. For λ near 0 and g near g_0, one has

$$\begin{aligned} & Q'W(\lambda, g)Q'(Q'(1 + G_0V_0)Q')^{-1}Q' \\ = \; & Q' + Q'(K(\lambda, g) - K(0, g_0))Q'(Q'(1 + G_0V_0)Q')^{-1}Q' \qquad (3.2) \\ = \; & Q' + O(|\lambda|^\epsilon + |g - g_0|), \quad \epsilon > 0, \end{aligned}$$

in $\mathcal{L}(1,-s;1,-s)$. We deduce that $Q'W(\lambda,g)Q'$ is invertible on F for $|\lambda|$ small and g near g_0. Let

$$D(\lambda,g) = (Q'W(\lambda,g)Q')^{-1}Q'.$$

Then for any $s > 1/2$, one has

$$D(\lambda,g) = (Q'W(0,g_0)Q')^{-1}Q' + O(|\lambda|^{\epsilon} + |g-g_0|), \tag{3.3}$$

as operator on $H^{1,-s}$. Using $D(\lambda,g)$ to construct an approximate inverse, one can show that for λ near 0 and g near g_0, the operator $\mathcal{P}(\lambda,g)$ is invertible from $H^{1,-s}\times\mathbb{C}$ to $H^{1,-s}\times\mathbb{C}$ with the inverse given by

$$\mathcal{P}(\lambda,g)^{-1} = \begin{pmatrix} E(\lambda,g) & E_+(\lambda,g) \\ \\ E_-(\lambda,g) & E_{+-}(\lambda,g) \end{pmatrix},$$

where

$$E(\lambda,g) = D(\lambda,g), \qquad E_+(\lambda,g) = T - D(\lambda,g)Y, \tag{3.4}$$

$$E_-(\lambda,g) = S - SX, \qquad E_{+-}(\lambda,g) = -SW(\lambda,g)T + SXY, \tag{3.5}$$

with

$$X = QW(\lambda,g)Q'D(\lambda,g), Y = Q'W(\lambda,g)T.$$

Note that $E_{+-}(\lambda,g)$ is a scalar and that 0 is an eigenvalue of $1+K(\lambda,g)$ if and only if

$$E_{+-}(\lambda,g) =< V_0u_0, (1-W(\lambda,g)Q'D(\lambda,g)Q')W(\lambda,g)u_0 >= 0.$$

One has the asymptotic expansion

$$(-\Delta_a - \lambda)^{-1} = G_0 + i\lambda^{1/2}G_1 + O(|\lambda|^{1/2+\delta})$$

as operator from $H^{-1,s}$ to $H^{1,-s}$ with $s > 3/2$, for some $\delta > 0$. Set $\gamma = g - g_0$, $V_0 = V(0,g_0)$. One has

$$V(\lambda,g) = V_0 + \gamma B_1 + \lambda C_1 + O((|\gamma| + |\lambda|)^2)$$

as operator from $H^{1,s}$ to $H^{1,s+\rho_0}$, where

$$B_1 = ([I_a - 2g_0 I_a R'(E_0, g_0) I_a](\varphi_a \otimes \cdot), \varphi_a)_a \tag{3.6}$$

$$+(g_0^2 I_a R'(E_0, g_0)) I_a R'(E_0, g_0) I_a(\varphi_a \otimes \cdot), \varphi_a)_a \leq 0 \tag{3.7}$$

as a sum of three negative self-adjoint operators in x_a-variables. Since $u_0 \in H^{1,-s}$ for any $s > 1/2$, $V_0 u_0 \in H^{-1,s_0}$ for some $3/2 < s_0 < \rho_0 - 1/2$. By a direct calculation, one obtains

$$E_{+-}(\lambda, g) = b_1\gamma + c_1 i\lambda^{1/2} + O(\gamma^2 + \gamma|\lambda|^{1/2} + |\lambda|^{1/2+\delta_0}) \tag{3.8}$$

where

$$b_1 = -< u_0, B_1 u_0 >_a, \quad c_1 = \frac{1}{4\pi}| < V_0(u_0), 1 >_a |^2.$$

$c_1 > 0$ because u_0 is a resonant state so that $< V_0(u_0), 1 >_a \neq 0$ ([14]), and

$$b_1 \geq - < I_a(\varphi_a \otimes u_0), (\varphi_a \otimes u_0) >> 0,$$

because $I_a \leq 0$ and is not identically zero. It follows from (3.8) that for each $\gamma > 0$ small enough, $\lambda \to E_{+-}(\lambda, g_0 + \gamma)$ has a zero $\Lambda(g_0 + \gamma) < 0$ satisfying

$$\Lambda(g_0 + \gamma) = E_1\gamma^2 + O(\gamma^{2+\epsilon}), \tag{3.9}$$

where $E_1 < 0$. This shows that the generalized eigenvalue problem $(Q(g) - E)u = 0$, $E \in]E - \delta_0, E + \delta_0[$, $u \in H^{1,-s}$, has only one solution $(E(g), u(g))$ with $E(g) = E_0 + \Lambda(g)$. Since $E(g) < \inf \sigma_{ess}(Q(g))$, by the method of a priori energy estimate, one can show that $u(g) \in H^1$. Thus, $E(g)$ is the unique eigenvalue of $Q(g)$ in $]E - \delta_0, E + \delta_0[$ and $E(g)$ is simple. This proves Theorem 3.1. ■

If g_0 be defined as in Section 2, under the conditions of Theorem 1.2, one can apply Theorem 3.1 to conclude that $E(g) = \inf \sigma(Q(g))$ satisfies

$$E(g) = E_0 + E_1(g - g_0)^2 + o((g - g_0)^2), \tag{3.10}$$

as $g \to g_0$ and $g > g_0$. (3.10) is known for C_0^∞ potentials ([7]). If all potentials are dilation analytic, one can show that $E_{+-}(\lambda, g)$ has homomorphic extension for $\lambda \in \mathbb{C}$ in a small rotated half-disk around 0 and for g near g_0. From (3.8), we can derive that for $g < g_0$ and g near g_0, $\lambda \to E_{+-}(\lambda, g)$ has a zero $\Lambda(g) \in \mathbb{C}$ with $\Re\Lambda(g) > 0$ and $\Im\Lambda(g) < 0$. $E_0 + \Lambda(g)$ is then a resonance of $Q(g)$ as pole of its resolvent.

4. Applications to the N-body Efimov effect

Let $\mathcal{P} = -\Delta + \sum_a V_a(x^a)$ be an N-body Schrödinger operators, $N \geq 4$. Let $b \in \mathcal{A}$ with $\#b = 3$, $E_0 = \inf \sigma(\mathcal{P}^b)$. Assume that

- (1.5) is satisfied and $\inf \sigma(\mathcal{P}^a) > E_0$, $a \notin \{b, a_1, a_2, a_3\}$;
- each $\mathcal{P}^{a_j}$ admits a discrete eigenvalue strictly below E_0.

Note that I_b can be decomposed as $I_b = I_1 + I_2 + I_3$ with $I_j = \sum_{c \not\subseteq b, c \subseteq a_j} V_c(x^c)$.

Theorem 4.1. *Under the above conditions, let* $\mathcal{P}^{a_j}(g) = \mathcal{P}^b - \Delta_{x_b^{a_j}} + gI_j$, *where* $-\Delta_{x_b^{a_j}} = -\Delta|_{\mathbf{X}^{a_j} \cap \mathbf{X}_b}$. *Then there exists* $g_j \in]0, 1[$ *such that* E_0 *is a threshold resonance of* $P^{a_j} = \mathcal{P}^{a_j}(g_j)$, $j = 1, 2, 3$. *In particular, Theorem 1.1 holds for the* N*-body operator* $P = \mathcal{P}_b + \sum_{j=1}^3 g_j I_j$.

Proof. Theorem 4.1 follows easily from Corollary 2.4. In fact, $\mathcal{P}^b$ can be regarded as a two-cluster Subhamiltonian of $\mathcal{P}^{a_j} = \mathcal{P}^b - \Delta_{x_b^{a_j}} + I_j$, $j = 1, 2, 3$. The assumptions made on $\mathcal{P}^b$ imply that $\mathcal{P}^b$ is the only two-cluster Subhamiltonian of $\mathcal{P}^{a_j}$ such that $\inf \sigma(\mathcal{P}^c) = \inf \sigma_{\text{ess}}(\mathcal{P}^{a_j})$, $c \subset a_j$, $c \neq a_j$, that is, $\mathcal{P}^{a_j}$ is unique two-cluster. Since $\mathcal{P}^{a_j}$ admits at least one discrete eigenvalue strictly below $E_0 = \inf \sigma(P^b)$, Corollary 2.4 with $Q = \mathcal{P}^{a_j}$, $j = 1, 2, 3$, says that there exits $g_j \in]0, 1[$ such that $P^{a_j} = \mathcal{P}^{a_j}(g_j) = \mathcal{P}^b - \Delta_{x_b^{a_j}} + g_j I_j$ satisfies that $\inf \sigma(P^{a_j}) = E_0$ and that E_0 is a threshold resonance of P^{a_j}. Now, for the N-body Hamiltonian $P = \mathcal{P}_b + \sum_{j=1}^3 g_j I_j$, one has $\inf \sigma(P^c) \geq \inf \sigma(\mathcal{P}^c) > E_0$ for $c \notin \{b, a_1, a_2, a_3\}$. Therefore, (1.4) is satisfied for P. It is clear that all conditions of Theorem 1.1 are verified for P. ■

Let us give an example of atomic-type four-body Schrödinger operator with Yukawa potentials such that all conditions of Theorem 1.1 are satisfied. Consider an atomic-type four-body quantum system labelled by $\{0,1,2,3\}$, where the particle 0 denotes the nucleus with position at the origine in $\mathbb{R}^3$. Let $x_j \in \mathbb{R}^3$ denote the relative position between the j-th electron and the nucleus. Consider the four-body Schrödinger operator with Yukawa potentials

$$P(\alpha,\gamma,\epsilon) = \sum_{j=1}^{3}(-\Delta_{x_j} + V_j(x_j)) + \sum_{1\le j<k\le 3} V_{jk}(x_j - x_k), \tag{4.1}$$

where

$$V_\beta(y) = -\frac{\gamma_\beta \alpha_\beta e^{-\epsilon_\beta \alpha_\beta |y|}}{|y|}$$

with $\alpha = (\alpha_\beta), \gamma = (\gamma_\beta), \epsilon = (\epsilon_\beta)$, the subscript β taking values in i or (jk) for $i,j,k \in \{1,2,3\}$, $j<k$. Here $\alpha_\beta, \epsilon_\beta > 0$ and $\gamma_\beta \in]0,1]$. Let $b = \{(01),(2),(3)\}$ be the three-cluster decomposition of the four-body system. The three resulting two-cluster decompositions are

$$a_1 = \{(01),(23)\}, \quad a_2 = \{(012),(3)\}, \quad a_3 = \{(013),(2)\}.$$

Let $Q(\alpha)$ be the four-body Schrödinger operator with Coulomb potentials

$$Q(\alpha) = \sum_{j=1}^{3}(-\Delta_{x_j} + W_j(x_j)) + \sum_{1\le j<k\le 3} W_{jk}(x_j - x_k) \tag{4.2}$$

where

$$W_\beta(y) = -\frac{\alpha_\beta}{|y|}.$$

Let $E' = \min\{\inf(\sigma(Q^c(\alpha)); c \in \mathcal{A}, \#c = 2, c \not\supseteq b\}$. E' is independent of α_1. Take

$$\alpha_1 > 2\sqrt{-E'} \text{ and } \gamma_1 = 1. \tag{4.3}$$

Denote

$$P^b = -\Delta_{x_1} - \frac{\alpha_1 e^{-\epsilon_1 \alpha_1 |x_1|}}{|x_1|}, E_0 = \inf \sigma(P^b). \tag{4.4}$$

Let $\epsilon_\beta > 0$ be small enough such that the infimum of the spectrum of each two-particle Subhamiltonian

$$-\Delta_y - \frac{\alpha_\beta e^{-\epsilon_\beta \alpha_\beta |y|}}{|y|}$$

is a negative eigenvalue. Then

$$E_0 = \inf \sigma(P^b) = \frac{\alpha_1^2}{4}(-1 + O(\epsilon_1)) \tag{4.5}$$

for ϵ_1 small enough. Since $P^a(\alpha, \gamma, \epsilon) \geq Q^a(\alpha)$ for any $a \in \mathcal{A}$ and for any $\gamma_\beta \in]0, 1]$, one has

$$\inf \sigma(P^b) < E' \leq \min\{\inf(\sigma(P^c(\alpha, \gamma, \epsilon)); c \in \mathcal{A}, \#c = 2, c \not\supseteq b\}. \tag{4.6}$$

This shows that for any $\gamma_\beta \in]0, 1]$, $\beta \neq 1$, $\inf \sigma_{ess}(P(\alpha, \gamma, \epsilon)) = E_0$ and that it can only be attained by the Subhamiltonians associated with $a \in \{b, a_1, a_2, a_3\}$. Note that $P^b(\alpha, \gamma, \epsilon)$ can be regarded as a two-cluster Subhamiltonian of $P^{a_j}(\alpha, \gamma, \epsilon)$. By the example given in Section 2, for $k = 2, 3$, there exists $\gamma_k = \gamma_{1k} = g_k \in]0, 1[$ such that

$$P^{a_k} = -\Delta_{x_1} - \Delta_{x_k} - \frac{\alpha_1 e^{-\epsilon_1 \alpha_1 |x_1|}}{|x_1|} - g_k(\frac{\alpha_k e^{-\epsilon_k \alpha_k |x_k|}}{|x_k|} + \frac{\alpha_{1k} e^{-\epsilon_{1k} \alpha_{1k} |x_1 - x_k|}}{|x_1 - x_k|})$$

satisfies

$$\inf \sigma(P^{a_k}) = \inf \sigma(P^b) = E_0 \text{ and } E_0 \text{ is a threshold resonance of } P^{a_k},\ k = 2, 3. \tag{4.7}$$

Using $y = x_2 - x_3$ as part of intra-cluster coordinates, one can write P^{a_1} as

$$P^{a_1} = -\Delta_{x_1} - 2\Delta_y - \frac{\alpha_1 e^{-\epsilon_1 \alpha_1 |x_1|}}{|x_1|} - \frac{\gamma_{23} \alpha_{23} e^{-\epsilon_{23} \alpha_{23} |y|}}{|y|}.$$

This is a particular case of the example of Section 2. Let g_1 be defined

$$g_1 = \sup\{\gamma_{23} > 0; -2\Delta_y - \frac{\gamma_{23} \alpha_{23} e^{-\epsilon_{23} \alpha_{23} |y|}}{|y|} \geq 0\}.$$

Then, (4.7) is also satisfied for $k = 1$ with $\gamma_{23} = g_1$. All assumptions of Theorem 1.1 are thus satisfied and Theorem 1.1 affirms that the Efimov effect takes place for $P(\alpha, \gamma, \epsilon)$ with the parameters adjusted as above. In particular, if $\alpha_\beta = 1$ for $\beta \neq 1$, and $\epsilon_\beta = \epsilon$, our result claims that for any $\alpha_1 > 2\sqrt{-E'}$ and for any $\epsilon > 0$ small enough, there exists $g_j \in]0,1[$, $j = 1,2,3$, such that the Efimov effect happens at the energy $E_0 = -\frac{\alpha_1^2}{4} + O(\epsilon)$ for the atomic-type four-body Schrödinger operator

$$\begin{aligned} P &= -\Delta_{x_1} - \Delta_{x_2} - \Delta_{x_3} - \frac{\alpha_1 e^{-\epsilon\alpha_1|x_1|}}{|x_1|} - g_1\frac{e^{-\epsilon|x_2-x_3|}}{|x_2-x_3|} \\ & \quad -g_2\left(\frac{e^{-\epsilon|x_2|}}{|x_2|} + \frac{e^{-\epsilon|x_1-x_2|}}{|x_1-x_2|}\right) - g_3\left(\frac{e^{-\epsilon|x_3|}}{|x_3|} + \frac{e^{-\epsilon|x_1-x_3|}}{|x_1-x_3|}\right). \end{aligned} \tag{4.8}$$

As application of Theorem 3.1, we give the following result which is an improvement of Theorem 5.1 of [15], where the additional condition $V_c \in C_0^\infty$ for $c \not\subseteq b$ was needed.

Theorem 4.2. *Let $b \in \mathcal{A}$ with $\#b = 3$ and $P(g) = P_b + gI_b$. Assume the conditions of Theorem 1.1 satisfied for $P = P(g_0)$. Then for $g > g_0$ and g near g_0, the total number of discrete eigenvalues, $\mathcal{N}(g)$, of $P(g)$ is finite and satisfies*

$$\mathcal{N}(g) \geq 2C_0|\log(g-g_0)| - C_1, \tag{4.9}$$

as $g \to g_0$. Here $C_0 > 0$ is the same as in Theorem 1.1 and C_1 is a constant independent of g.

With the results of Sections 2 and 3, one has for $g > g_0$ and g near g_0,

$$\Sigma_3(g) \triangleq \min_{\#c=3} \inf\ \sigma(P^c(g)) = E_0$$

and

$$\Sigma_2(g) \triangleq \min_{\#c=2} \inf\ \sigma(P^c(g)) = \min_{1\leq j\leq 3} \inf\ \sigma(P^{a_j}(g)).$$

Applying theorem 3.1 to $P^{a_j}(g)$ with P^a replaced by P^b, one has

$$\inf\ \sigma(P^{a_j}(g)) = E_0 + E_{1,j}(g-g_0)^2 + o((g-g_0)^2), \quad E_{1,j} < 0.$$

This shows that

$$\Sigma_2(g) = E_0 + E_1(g - g_0)^2 + o((g - g_0)^2) < \Sigma_3(g)$$

where $E_1 = \min_j\{E_{1,j}\} < 0$. Since $\rho_0 > 2$, the number of discrete eigenvalues of $P(g)$ is finite. Noticing that $P(g) \leq P(g_0)$ for $g > g_0$, one has $\mathcal{N}(g) \geq N(\Sigma_2(g))$. Theorem 1.1 gives the desired result.

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